THE APPLICATION OF MATHEMATICAL PROGRAMMING TO A CONTAINERISATION PROBLEM

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Thesis submitted for the degree of

Doctor of Philosophy, December 1971.

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SUMMARY

The movement of containers in the Australia-Europe container shipping system is formulated as a network flow model. Under the assumption that cargo requirements are known, the minimization of the cost of the movement of the containers and the total container inventory is expressed as a minimum-cost network flow problem, with the variables restricted to take integral values. This is a two-commodity problem because it is necessary to distinguish two types of container - general, used for dry cargo, and insulated, used primarily for refrigerated cargo but also for dry cargo. Near-optimal solutions to the two-commodity problem are obtained using an efficient heuristic method which utilizes the structure of the problem and guarantees integral solutions. Computer programs to implement the heuristic solution procedure have been written and used to give realistic answers to many questions about the Australia-Europe system. Several model applications using realistic data are given.

For practical purposes, the heuristic procedure provides a completely adequate integral solution to the two-commodity problem. But it is of theoretical interest to examine the nature of the optimal solutions to the two-commodity problem when the variables are not restricted to take integral values. Certain classes of linear programs have the property of always possessing an optimal solution which is integral. This "integer property" is possessed by one-commodity network flow problems, but in general the optimal solutions to multi-commodity flow problems are non-integral. In a number of special cases it is proved that the two-commodity problem has an integer property. In fact it is shown that there is a class of multi-commodity network flow problems which may be converted to equivalent one-commodity problems. However in general the two-commodity problem has fractional optimal solutions. Simple examples based on the Australia - Europe network are given to demonstrate these theoretical aspects.

SIGNED STATEMENT

I hereby declare that this thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of my knowledge and belief, it contains no material previously published by another person, except where due reference is made in the text of the thesis.

K.J. Noble

ACKNOWLEDGEMENTS

I am greatly indebted to my supervisor, Professor R.B. Potts, for his encouragement and advice during all stages of the development and presentation of this thesis.

Chapter 2 of the thesis reports some of the results of a project sponsored by the Australian Commonwealth Department of Trade and Industry, and I acknowledge the continued interest and help of the officers of this Department. A major portion of the data used was supplied by Associated Container Transportation (Australia) Ltd., and Overseas Containers (Australia) Ltd., and these firms were generous with their assistance in many ways.

I would like to thank the C.S.I.R.O., whose financial support, in the form of a C.S.I.R.O. Postgraduate Studentship, made this work possible. Also my thanks are due to Miss D.J. Potter for doing an excellent job of typing, and Mrs. E. McDonough for running off copies. Finally, I wish to express my gratitude to my wife Jenny for her encouragement and understanding during the course of the thesis.

CHAPTER 1. INTRODUCTION

During the past few years, shipping throughout the world has undergone a dramatic change. In large part, regular scheduled integrated container shipping services have replaced the irregular and haphazard conventional shipping services. While there is a considerable volume of literature on economic aspects of containerisation [2], [33], relatively few papers have applied the techniques of operations research to specific problems concerned with operational aspects of a container shipping system. The Matson Navigation Company was one of the pioneers in the containerisation field, and the classic paper of Weldon [49] is one of the earliest scientific papers in this area. More recently, Matson have tackled the problem of scheduling their freighter fleet [32], but they do not consider in [32] the problem of determining container inventory and empty container movements. The latter problem has been considered for the Australia-Europe container shipping system by Noble and Potts [31], and Chapter 2 of this thesis is based on the material in [31].

The analysis in [31] is applicable to other container shipping systems, and also to similar problems in other areas. For a containerised mail system, the problem of determining optimal container inventory and routing has been treated by Horn [20], who generalizes the work of Samuel and

Ullmann [43]. Several papers have considered the problem of scheduling the movement of empty freight cars in a railroad system [5], [10], [11]. In each of these applications, the problem is formulated using the concept of network flow, but the treatment in [31] differs from the others by considering the flows of two different commodities on the network.

In the literature, many practical problems have been formulated as multi-commodity flow problems on directed or undirected networks, and many papers have considered various theoretical aspects associated with such problems. A comprehensive bibliography is given by Jewell in [28]. While very efficient algorithms exist for solving one-commodity network flow problems [9], the solution of multi-commodity problems is considerably more difficult. Various solution procedures have been proposed, among which are those of Ford and Fulkerson [8], Jewell [28] and Saigal [41], but there appears to be a lack of computational experience for all of them. The nature of the optimal solutions to multi-commodity problems has been investigated by Jewell [27], and other approaches include those of Bellmore et al [4], Cremeans et al [6], Grinold [12], Haley [14]-[17], Rothfarb and Frisch [34] - [36] and Sakarovitch [42]. Tomlin [47] and Jarvis [26] have related the proposal of Ford and Fulkerson [8] to the decomposition method of

Dantzig and Wolfe (ch.23, [7]).

A number of special results have been obtained for two-commodity problems on undirected networks. Hu [21] - [23] has proved a two-commodity max-flow min-cut theorem, which is analogous to the one-commodity theorem of Ford and Fulkerson [9], and various other results have been obtained by Rothschild and Whinston [37] - [40], Tang [44] - [46], Arinal [1] and Hakimi [13]. Some computer times are given in [40], but these are not very encouraging. In any case, the special properties of two-commodity problems on undirected networks do not hold for directed networks. The problem formulated in Chapter 2 of this thesis involves two-commodity flow on a directed network.

Since this problem has a multi-stage structure, it might be thought that use of Dantzig-Wolfe decomposition [7] would provide an efficient solution method. Indeed Bellmore et al have proposed a special decomposition algorithm for a one-commodity multi-period problem [3]. But in their problem, each stage is linked only to the immediately preceding and succeeding stages, whereas in the Chapter 2 problem, the linkage of stages is far more complicated, and decomposition is far less palatable.

Fortunately, it has been possible to obtain near-optimal solutions to the Chapter 2 two-commodity problem using an efficient heuristic procedure, which is described in section 2.5. In numerous applications of the network flow model, the heuristic procedure has proved capable of giving completely adequate solutions to the two-commodity problem. Realistic applications from the Australia - Europe container system are discussed.

Chapter 3 of the thesis is primarily concerned with theoretical aspects of the two-commodity problem. In particular, conditions are established under which the optimal solution to the two-commodity problem is integral. The results of Heller and Tompkins [18], Hoffman and Kruskal [19] and Veinott and Dantzig [48] are used to establish a class of two-commodity problems which have integral optimal solutions. In general, the optimal solution to the two-commodity problem is fractional, as shown by Jewell [30]. Fractional examples based on the Australia - Europe container network are given.

CHAPTER 2. NETWORK FLOW MODEL OF THE AUSTRALIA-EUROPE CONTAINER SERVICE

2.1 Australia-Europe Container Shipping System

In March 1969, the two consortia Associated Container Transportation (ACT) and Overseas Containers Ltd. (OCL) inaugurated a container shipping service between the United Kingdom and Australia. The service was designed to provide a regular schedule of cellular container ships with one port of call - Tilbury - in the UK, and three ports of call - Sydney, Melbourne and Fremantle - in Australia (Fig. 1). With a speed of about 22 knots, the container ships complete a round voyage in about 70 days, and adhere closely to the following schedule:

	port	node	day
leave	Tilbury	1	1
arrive	Fremantle	2	23
arrive	Sydney	3	28
arrive	Melbourne	5	33
arrive	Fremantle	2	39
arrive	Tilbury	1	62
leave	Tilbury	1	71.

It is a feature of container ship operations that the time a ship spends in port is minimal - less than half a day in

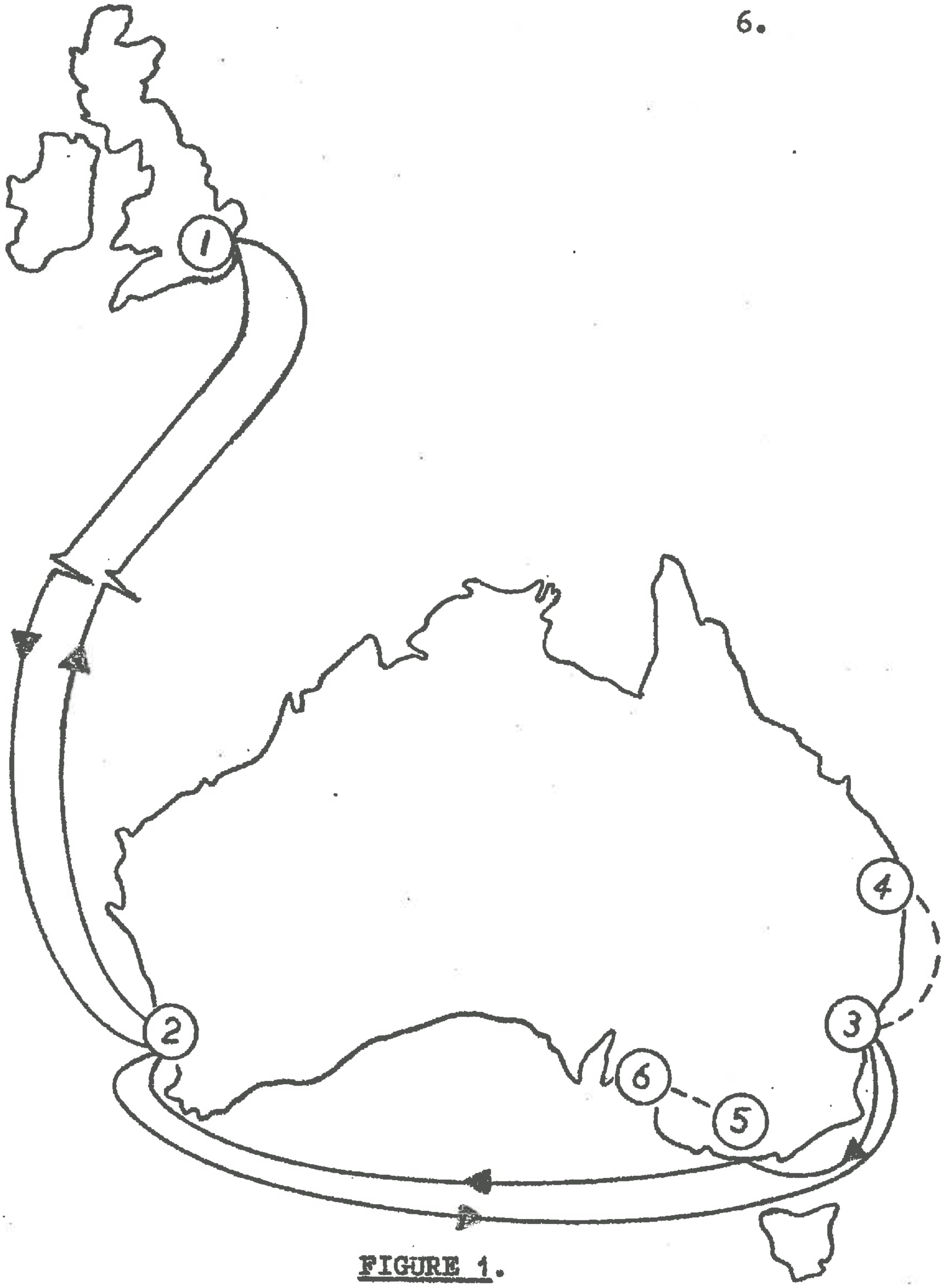

FIGURE 1.

Australia-Europe container shipping system. Nodes 1,2,3,4,5,6 represent Tilbury, Fremantle, Sydney, Brisbane, Melbourne and Adelaide respectively. The solid lines represent the container ship movements. The dashed line from node 3 to node 4 represents a feedership service between Sydney and Brisbane, and the dashed line from node 5 to node 6 represents a rail service between Melbourne and Adelaide.

Fremantle at each Southbound and Northbound call, about three days in each of Sydney and Melbourne, and about nine days at Tilbury.

As new ships have been phased into the service, the frequency of sailings has been increased and in 1971 was one every five or six days, with thirteen container ships in operation. The original schedules have been modified and now include several European ports of call, while often the Northbound call at Fremantle is omitted.

The containers used in the service are mainly standard 20 ft. × 8 ft. × 8 ft. containers of two types - general containers used for dry cargo, and insulated containers designed especially for refrigerated cargo (or reefer cargo as it is commonly called) but used also for dry cargo. Other types of containers are present in insignificant numbers and will be ignored. Each cellular ship was designed to have a capacity of about 1300 containers, including about 350 insulated containers, although subsequent modifications have enabled these capacities to be increased.

In Australia, there are terminal facilities for unloading and loading containers at Fremantle, Sydney and Melbourne, where depots for packing and unpacking of containers are also provided. In addition, Brisbane has a terminal-depot served by a coastal feedership to Sydney, and Adelaide has a depot with a direct rail link to the Melbourne terminal.

Empty containers can be stockpiled at terminals or depots, and because there are local and international imbalances in full container movements, interstockpile empty container movements are required. Typical of these movements of empty containers (empties) are: Sydney to UK by container ship; Sydney to Brisbane by feedership; and Melbourne to Adelaide by rail.

The service provided by the container ships - regular, periodic sailings with few ports of call - is a marked contrast to the conventional shipping service which uses many ports of call in Australia and the UK and which does not adhere to regular schedules. It is the basic simplicity of the container service which makes it readily amenable to mathematical and computer analysis.

2.2 The Problem

This thesis is concerned with just one area of operation of the container system - the problem of how many containers there should be in the system and how the containers should be moved. More precisely, the problem considered is the minimization of the cost of the container inventory and the movement of containers, under the assumption that there are always sufficient empty containers on hand for the packing of available cargo. Shortages are not allowed.

As noted in section 2.1, there are two types of container - general and insulated - and two broad classes of cargo - dry and reefer. But some dry cargo is not suitable for packing in insulated containers and so there are actually three types of cargo:

<u>type 1 cargo</u> - must be packed in general containers;

<u>type 2 cargo</u> (<u>reefer cargo</u>) - must be packed in insulated containers;

<u>type 3 cargo</u> - may be packed in either general or insulated containers.

Thus the problem consists of three inter-related parts:

determining full container movements (essentially this involves allocating type 3 cargo between general and insulated containers);

determining empty container movements for general and insulated containers;

determining container inventory for general and insulated containers.

In the next section it will be shown that this problem may be expressed as a two-commodity network flow problem.

2.3 Network for Container Movements

The movement of full and empty containers may be represented as flow in a network of nodes and links; and the efficient control of this movement may be expressed as the problem of minimizing the cost of flow in this network. The nodes of the network generally represent stockpiles for containers, and the links of the network represent transport modes between the stockpiles. When a container is packed with export cargo, an empty container is consumed and a full container is produced. Conversely, when an imported full container is unpacked, a full container is consumed and an empty container is produced. This production and consumption of full and empty containers is assumed to occur only at nodes of the network, and the containers (full or empty) may be transported only along the network links.

The detail with which the network is chosen depends on the uses to be made of the network model. For many applications, it is adequate to regard each port of call as a container stockpile, represented by a single node. Thus in Fig. 2, node 5 represents the Melbourne stockpile while the link from node 3 to node 1 represents the possibility of a container movement from the Sydney stockpile to the Tilbury stockpile. For other purposes, a more detailed network, as indicated in Fig. 3, is necessary. Each port of call is now represented by two nodes, one representing

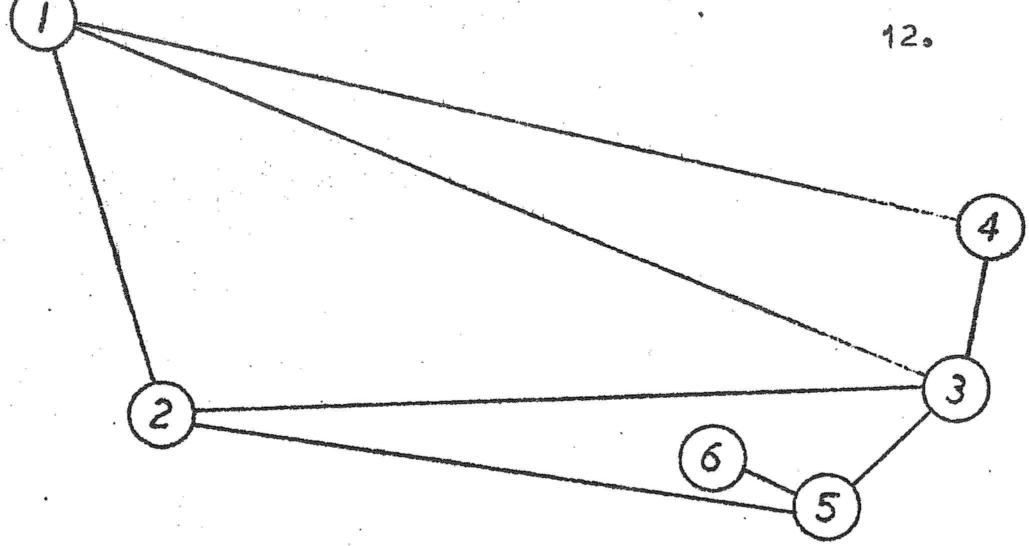

FIGURE 2.

Australia-Europe container network. Nodes 1,2,3,4,5,6 represent container stockpiles at Tilbury, Fremantle, Sydney, Brisbane, Melbourne and Adelaide respectively. Links between nodes represent possible container movements (for clarity only a few links have been drawn).

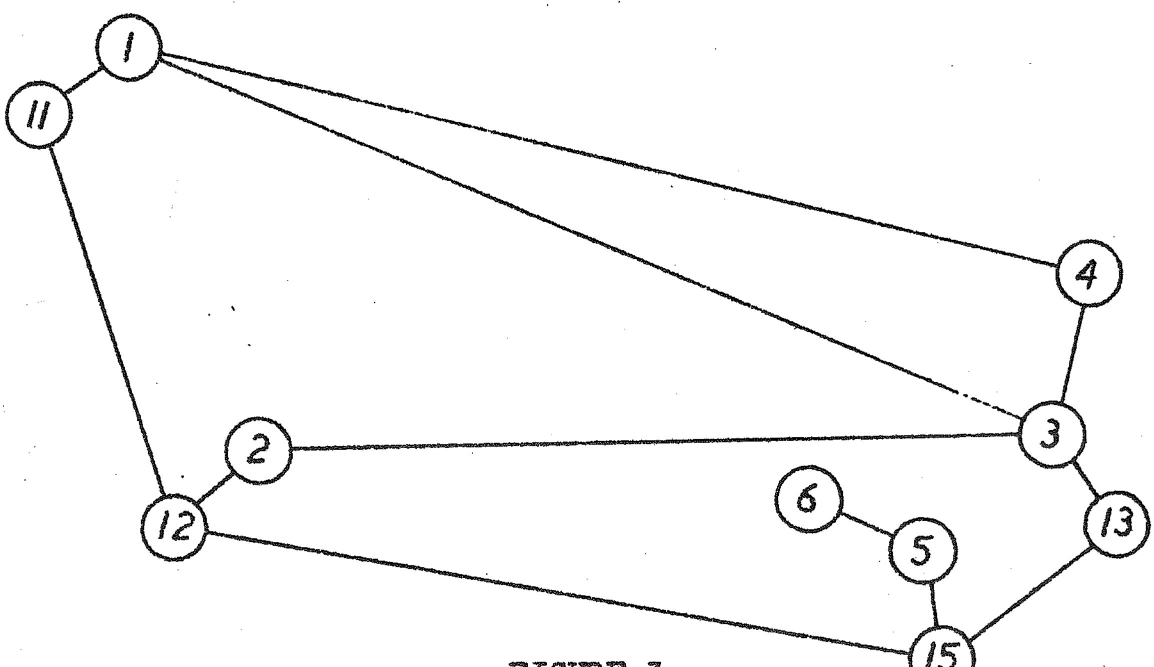

FIGURE 3.

Detailed Australia-Europe container network. Nodes 1,2,3,4,5,6 represent container stockpiles at Tilbury, Fremantle, Sydney, Brisbane, Melbourne and Adelaide respectively. Nodes 11,12,13,15 represent the container ship in port at Tilbury, Fremantle, Sydney and Melbourne respectively.

the container ship in port and the other the container stockpile. This detail helps to distinguish between containers which are loaded and unloaded at a port and those which are simply in transit.

To represent the dynamic flow of containers throughout the network, it is necessary to attach to each link a travel time. Because of the inherent regular periodic structure of the container service it is convenient to measure the travel time in units corresponding to the interval between successive ships. Thus a travel time of 3 units for a weekly container service means a travel time of 3 weeks. For convenience of description, it is assumed that the service is weekly.

When used in reference to full container movements, the term "travel time" is interpreted as follows. The process involved in sending cargo from Tilbury to Fremantle (say) is quite inflexible, except that it may be necessary to choose whether to use a general or an insulated container. A suitable empty container is taken from the Tilbury stockpile, is packed with the cargo, and is loaded on the container ship at Tilbury. When the ship arrives at Fremantle, the imported full container is unpacked and the resultant empty is placed on the Fremantle stockpile. The time occupied by the complete process is called the "travel time" for the full container movement from Tilbury to

Fremantle, and might be 5 weeks compared with (say) 3 weeks for the Tilbury - Fremantle empty movement.

The movement of containers over time is represented as flow in a dynamic network. For example, Fig. 4 is the 10-week dynamic version of Fig. 2, with links illustrating possible container movements. Since the travel time for full containers from Tilbury to Fremantle is 5 weeks, the complete dynamic version of the network contains a link from node 1 in week 1 to node 2 in week 6, from node 1 in week 2 to node 2 in week 7, and so on; these links are shown as solid lines in Fig. 4. Similarly the travel time for empty containers from Sydney to Brisbane is 1 week, and the complete dynamic network contains a link from node 3 in week 1 to node 4 in week 2, from node 3 in week 2 to node 4 in week 3, and so on; and these links are shown as dashed lines in Fig. 4. To allow for containers being held at Melbourne from week 8 to week 9, there is a link (dotted line) from node 5 in week 8 to node 5 in week 9; and the dotted line from node 6 in week 4 to node 6 in week 5 allows for containers being held at Adelaide from week 4 to week 5. Thus it is possible to represent as flows along links of the dynamic network not only full and empty container movements, but also the holding of containers at stockpiles.

In formulating the dynamic network, some judgement is needed in interpreting container movements as occurring

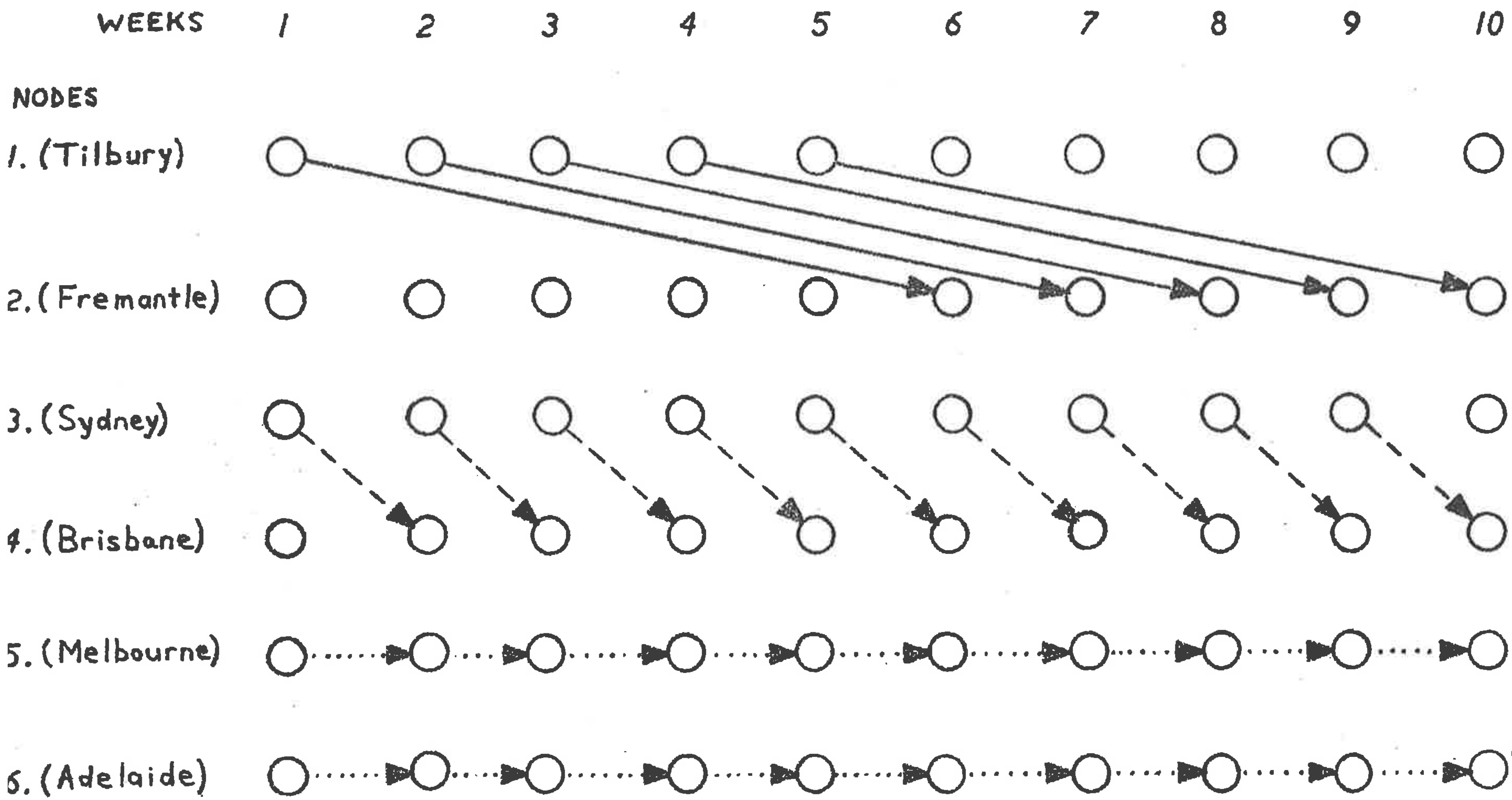

FIGURE 4.

10-period dynamic version of Fig.2. Solid lines represent full container movements, dashed lines represent empty container movements, and dotted lines represent stockholding at stockpiles. Only a few links are shown.

from one week to another week, but the sensitivity of the results to the assumptions made can easily be tested. A more accurate day-to-day network would be useful for problems of detailed container control, but would be far too complex for the applications considered here.

Container inventory and movement costs are represented in the dynamic network by assigning costs to the links, and cargo requirements are represented by assigning capacities to the full container movement links (full links). There is one set of costs and capacities for general containers, and another set for insulated containers.

The container inventory cost reflects the weekly cost of owning a container, and so for example the unit inventory cost assigned to a link with travel time 5 weeks is 5 times the weekly inventory cost per container. Once each link has been assigned a unit inventory cost, the cost on each empty movement link is increased by the appropriate unit empty movement cost. Thus in Fig. 4 the overall unit cost (for general containers) assigned to the Sydney-Brisbane empty movement link is the sum of the weekly inventory cost per general container and the unit empty movement cost from Sydney to Brisbane. The overall unit costs assigned to full links are derived similarly.

Associated with each full link in the dynamic network, there are three cargo quantities which are supposed known. These are the quantities of each of the three types of cargo (see section 2.2) which must be moved along the full link. The full link is assigned a lower capacity for general containers which is equal to the quantity of type 1 cargo, and is assigned a lower capacity for insulated containers which is equal to the quantity of type 2 cargo. But this is not sufficient to ensure that type 3 cargo is carried. The latter is achieved by constraining the total flow of general and insulated containers on the full link to be equal to the total amount of cargo (of all three types) which must be carried.

Non-negativity of flows on links which represent empty container movements and the holding of containers at stockpiles is ensured by assigning zero lower capacities to these links.

Thus the problem of minimizing the cost of container inventory and movements is represented as the problem of determining minimum-cost flows in the dynamic network, subject to capacity restrictions. This is a two-commodity network problem because there are two types of container. The flow of these two commodities is interdependent because type 3 cargo may be packed in either type of container.

2.4 Mathematical Formulation

In the following list of the mathematical notation to be used, subscripts i,j take values $1,2,3,\ldots$ and refer to nodes or stockpiles i,j. The variable t, taking values $1,2,3,\ldots$ refers to the week number. The superscript k signifies the two container types, $k=1$ for general and $k=2$ for insulated containers. The superscript k is also used to signify the three cargo types and in this use only, k also takes the value 3. No confusion should arise, because the superscript 3 will be shown explicitly. Otherwise it is assumed that $k=1$ or $k=2$, and that $\sum_k$ means $\Sigma_{k=1}^{2}$.

The following integer quantities are assumed given:

$a_{ij}^{k}(t)$ = number of container loads of type k $(k=1,2,3)$ cargo to be dispatched in week t from stockpile i to stockpile j (1)

s_{ij}, t_{ij} = number of weeks (travel time) for an empty, full container movement from i to j (2)

α^{k} = unit weekly inventory cost for containers of type k (3)

γ_{ij}, δ_{ij} = unit cost for moving an empty, full container from i to j (4)

$u_{ij}(t)$ = maximum number of empties (general + insulated) which can be dispatched in week t from i to j. (5)

The following integer variables are to be determined:

$e^k_{ij}(t), f^k_{ij}(t)$ = number of empty, full containers of type k to be dispatched in week t from i to j (6)

$n^k(t)$ = total inventory of containers of type k in the system in week t (7)

$g^k_i(t)$ = number of containers of type k added to the system at stockpile i in week t (8)

C = total cost of container inventory and full and empty movements. (9)

It is convenient to interpret $e^k_{ii}(t)$ as the number of containers of type k held at stockpile i during week t (after arrivals from and departures to other stockpiles) and take $s_{ii} = 1$. The total number of containers in week t is then given by those held at stockpiles and those moving empty plus those moving full in week t, so that

$$n^k(t) = \sum_{i,j} \sum_{\tau} e^k_{ij}(t-\tau) + \sum_{i,j} \sum_{\tau}{}' f^k_{ij}(t-\tau). \qquad (10)$$

Here the summation is taken over links of the dynamic network (such as in Fig. 4) with τ varying over the range τ=0 to $s_{ij}-1$ in the summation $\sum_{\tau}$, and τ varying over the range τ=0 to $t_{ij}-1$ in the summation $\sum_{\tau}'$. The summation in (10) is readily interpreted on the dynamic network as the sum over all links which originate in week t or earlier and terminate in week (t+1) or later.

The problem of minimizing the total cost of container flows in the dynamic network can be formulated as the following two-commodity problem:

$$\text{Minimize } C = \sum_t \sum_k [\sum_{i,j} \gamma_{ij} e^k_{ij}(t) + \sum_{i,j} \delta_{ij} f^k_{ij}(t) + \alpha^k n^k(t)] \tag{11}$$

subject to

$$\sum_j \{e^k_{ij}(t) + f^k_{ij}(t)\} - \sum_j \{e^k_{ji}(t-s_{ji}) + f^k_{ji}(t-t_{ji})\} = g^k_i(t) \tag{12}$$

$$f^k_{ij}(t) \geqslant a^k_{ij}(t) \tag{13}$$

$$\sum_k f^k_{ij}(t) = \sum_k a^k_{ij}(t) + a^s_{ij}(t) \tag{14}$$

$$e^k_{ij}(t) \geqslant 0, \; g^k_i(t) \geqslant 0 \tag{15}$$

$$\sum_k e^k_{ij}(t) \leqslant u_{ij}(t) \tag{16}$$

and the additional requirement that all variables be integers.

In obtaining the objective function (11), in which $n^k(t)$ is given by (10), the usual and adequate assumption of linear costs is made.

In general each variable $e^k_{ij}(t)$ occurs in $n^k(\tau)$ for the values $\tau=t$ to $\tau=t+s_{ij}-1$, and each variable $f^k_{ij}(t)$ occurs in $n^k(\tau)$ for the values $\tau=t$ to $\tau=t+t_{ij}-1$. When this is the case, substitution of (10) into (11) and collection of terms in $e^k_{ij}(t)$, $f^k_{ij}(t)$ gives

$$C = \sum_t\sum_k[\sum_{i,j}(\gamma_{ij} + s_{ij}\alpha^k)e^k_{ij}(t) + \sum_{i,j}(\delta_{ij} + t_{ij}\alpha^k)f^k_{ij}(t)]$$

$$= \sum_t\sum_k[\sum_{i,j}(\gamma_{ij}+s_{ij}\alpha^k)e^k_{ij}(t)+\sum_{i,j}t_{ij}\alpha^k f^k_{ij}(t)]+\sum_t\sum_{i,j}\delta_{ij}\sum_k f^k_{ij}(t)$$

where by virtue of (14), the right-hand term in the last equation is a constant. So minimizing C is equivalent to minimizing

$$C' = \sum_t\sum_k[\sum_{i,j}c^k_{ij}(t)e^k_{ij}(t) + \sum_{i,j}d^k_{ij}(t)f^k_{ij}(t)] \tag{17}$$

where

$$c^k_{ij}(t) = \gamma_{ij} + s_{ij}\alpha^k \tag{18}$$

and

$$d^k_{ij}(t) = t_{ij}\alpha^k. \tag{19}$$

The quantities $g^k_i(t)$ have been introduced principally to allow the possibility of adding containers to the system to cope with a growth in container trade. In practice containers are added to the system every few months (say) and then only at certain stockpiles, so that most of the $g^k_i(t)$ are zero. In many applications it has been possible to set all the $g^k_i(t)$ equal to zero.

The constraint equation (16) can be interpreted in different ways. For example, $u_{ii}(t)$ could represent the capacity of the stockpile at i and would be independent of t unless changes in the stockpile were made. Between one port of call i and the next port of call j, $u_{ij}(t)$ could represent the number of empties which can be carried on the container ship on the leg i to j. The space for empties on each leg could be calculated from the given full container flows and the ship capacities. Although reefer (type 2) cargo can be carried only in certain cells of the ship, there is no restriction on the placement of empties. In many applications of the model, it has been possible to ignore the $u_{ij}(t)$.

The above formulation does not specify initial and terminating conditions. For many applications, it is convenient and adequate to impose a cyclic boundary condition, as used by Horn [20]. One year is a natural cycle period. In general let

$$T = \text{number of weeks in the optimization period.} \qquad (20)$$

The variable t then assumes the values $1,2,\ldots,T$ and, for example, $u_{ij}(-2)$ is taken equal to $u_{ij}(T-2)$. The cyclic boundary condition means, in effect, that the T-period dynamic network is wrapped around a cylinder so that weeks T and 1 are adjacent, as indicated in Fig.5. When the cyckic boundary condition is used, summation in (12) over all values of t and i gives

$$\sum_t \sum_i g_i^k(t) = 0$$

and so all the $g_i^k(t)$ are zero.

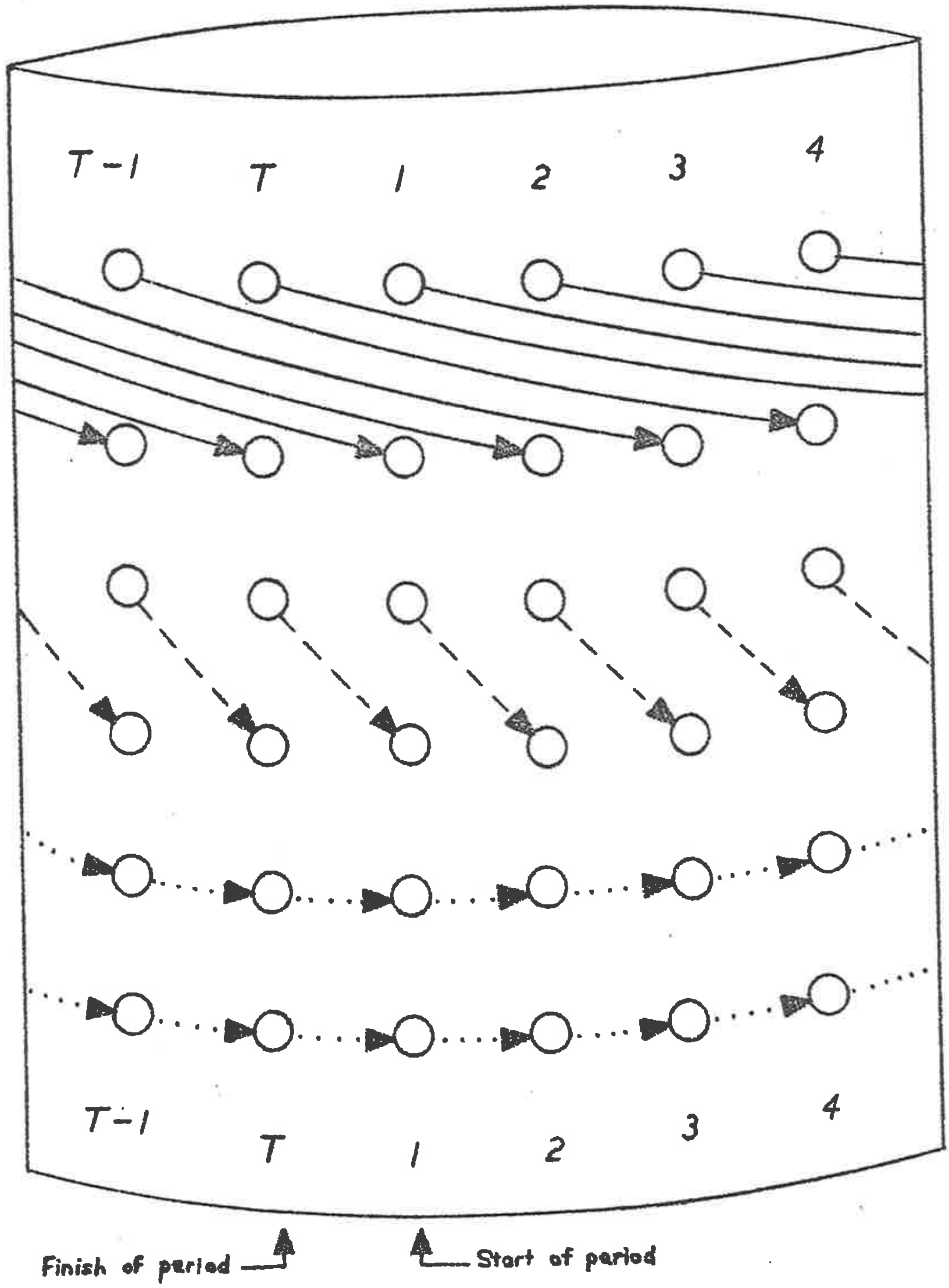

FIGURE 5.

T-period dynamic network with cyclic boundary condition. Network can be thought of as drawn on the surface of a cylinder, so that week T is adjacent to week 1.

For other applications, particularly in situations of rapid growth, a non-cyclic boundary condition is preferable. Again the variable t assumes the values $1,2,\ldots,T$ but it is tacitly assumed in (10) and (12) that any variable $e^k_{ij}(\tau)$ or $f^k_{ij}(\tau)$ with $\tau < 1$ is suppressed. Then it follows in (12) that

$$g^k_i(1) = \sum_j \{e^k_{ij}(1) + f^k_{ij}(1)\}$$

and so

$$\sum_i g^k_i(1) = \sum_i \sum_j \{e^k_{ij}(1) + f^k_{ij}(1)\}$$

$$= \sum_{i,j} e^k_{ij}(1) + \sum_{i,j} f^k_{ij}(1).$$

Thus from (10) it follows that

$$n^k(1) = \sum_i g^k_i(1). \tag{21}$$

Consequently when the non-cyclic boundary condition is used, the variables $g^k_i(1)$, and possibly variables $g^k_i(t)$ for $t \geqslant 2$, will be non-zero.

Finally, in the non-cyclic case, (18) and (19) must be modified to read

$$c^k_{ij}(t) = \gamma_{ij} + \{\min(s_{ij}, T-t+1)\}\alpha^k \tag{22}$$

and $$d^k_{ij}(t) = \{\min(t_{ij}, T-t+1)\}\alpha^k. \tag{23}$$

This is because links which originate in week t and terminate after week T incur inventory costs only in weeks t to T.

2.5 Heuristic Solution

No efficient solution techniques are known for solving general two-commodity flow problems on large networks, but an heuristic procedure is available which reduces the problem (12) - (17) to two one-commodity problems. Each of these can be solved using an algorithm such as the efficient out-of-kilter algorithm [9].

There are two phases in the heuristic solution procedure. In Phase I, type 1 cargo and general containers are excluded and attention focussed on the movement of insulated containers. All of the type 2 cargo and some of the type 3 cargo will be carried in insulated containers. In Phase II, the movement of general containers is considered assuming the Phase I insulated container movements.

The two phases of the heuristic procedure can be described mathematically as follows.

Phase I

Solve the following one-commodity problem:

$$\text{Minimize } \sum_t [\sum_{i,j} c_{ij}^2(t) e_{ij}^2(t) + \sum_{i,j} x_{ij}(t) f_{ij}^2(t)] \qquad (24)$$

subject to

$$\sum_j \{e_{ij}^2(t)+f_{ij}^2(t)\}-\sum_j \{e_{ji}^2(t-s_{ji})+f_{ji}^2(t-t_{ji})\}=g_i^2(t) \qquad (25)$$

$$a_{ij}^2(t) \leqslant f_{ij}^2(t) \leqslant a_{ij}^2(t) + a_{ij}^3(t) \qquad (26)$$

$$0 \leqslant e_{ij}^2(t) \leqslant u_{ij}(t), g_i^2(t) \geqslant 0 \qquad (27)$$

where $x_{ij}(t)$ = heuristic unit cost for dispatching a full insulated container from i to j in week t. (28)

Phase II

Fix $e^2_{ij}(t)$, $f^2_{ij}(t)$, $g^2_i(t)$ at their Phase I values (which will be integers [9]) and solve (12)-(17) to determine $e^1_{ij}(t)$, $f^1_{ij}(t)$, $g^1_i(t)$. Note that (13) and (14) simplify to

$$f^1_{ij}(t) = \sum_k a^k_{ij}(t) + a^3_{ij}(t) - f^2_{ij}(t)$$

and (15) and (16) simplify to

$$0 \leq e^1_{ij}(t) \leq u_{ij}(t) - e^2_{ij}(t), \quad g^1_i(t) \geq 0$$

so that Phase II also involves solving a one-commodity problem, and results in integer values for $e^1_{ij}(t)$, $f^1_{ij}(t)$, $g^1_i(t)$. The final solution at the end of Phase II is a feasible integer solution of the two-commodity problem (12)-(17).

Some choices of the heuristic costs $x_{ij}(t)$ may result in near-optimal heuristic solutions to the two-commodity problem. If the $x_{ij}(t)$ are chosen to be large positive numbers, the optimal solution to Phase I will have (almost surely) $f^2_{ij}(t) = a^3_{ij}(t)$ i.e. no type 3 cargo will be carried in insulated containers. On the other hand, if the $x_{ij}(t)$ are chosen to be large negative numbers, the optimal solution to Phase I will have (almost surely)

$f^2_{ij}(t) = a^2_{ij}(t) + a^3_{ij}(t)$ i.e. all type 3 cargo will be carried in insulated containers. These are two fairly obvious heuristics, but neither is particularly suitable for the Australia-Europe system.

The cargo imbalances in the Australia-Europe system provide the motivation for the choice of the heuristic costs $x_{ij}(t)$. Australia exports much more reefer (type 2) cargo than she imports, and imports more dry (type 1, type 3) cargo than she exports. During 1970 approximate figures per voyage were:

Import	Dry Cargo	Reefer Cargo
Europe-Australia	1200	10
Export		
Australia-Europe	800	210.

Virtually all of the Australian dry export cargo is type 1 while perhaps 50% of her dry import cargo is type 3. Thus for simplicity the breakdown of dry cargo into type 1 and type 3 will be assumed to be:

	Type 1	Type 3
Europe-Australia	600	600
Australia-Europe	800	0.

If type 3 cargo is not packed in Southbound insulated containers, 210-10 = 200 insulated containers travel Southbound empty, and because of the dry cargo imbalance, 1200-800 = 400 general containers must travel empty Northbound. But if all the type 3 cargo is packed in Southbound insulated containers,400(=600+10-210) more insulated containers reach Australia than are needed for reefer exports. 400 insulated containers must travel empty Northbound; and since Australian type 1 exports exceed type 1 imports, 200(=800-600) general containers must travel empty Southbound.

These empty movements in opposite directions are wasteful - a good heuristic solution should try to eliminate empty movements in one direction. Since the total Australian cargo import of 1210 is greater than her export of 1010, there must be a net Northbound flow of 200 empty containers. Because insulated containers are about twice as expensive as general containers, general containers should move empty Northbound. Thus a good heuristic is to fill 200 Southbound insulated containers with type 3 cargo. The remaining 1000 container loads of Southbound dry cargo are packed in general containers, and 200(=1000-800) general containers travel empty Northbound.

What values should be assigned the $x_{ij}(t)$ to achieve this heuristic solution? For i an Australian port and j a European port, let $x_{ij}(t)$ be a large positive number (or equivalently, set $f^?_{ij}(t) = a^?_{ij}(t)$). For i a European port and j an Australian port, choose

$x_{ij}(t)$ in the range

$$0 < x_{ij}(t) < c^2_{ij}(t).$$

The idea is to encourage insulated containers to come to Australia full rather than empty (hence $x_{ij}(t) < c^2_{ij}(t)$), but not to the extent that they come to Australia full and return to Europe empty.

This choice of heuristic costs $x_{ij}(t)$ has been used to compare the heuristic solution to the exact optimal solution for a three stockpile 24-week problem. The stockpiles represented were Tilbury, Sydney and Melbourne, and cargo demands were based on 1970 figures. The exact optimal solution to the two-commodity problem was obtained using a linear programming package. (Even for this relatively small example, the constraint matrix has 240 rows and 480 columns.) The overall costs of container inventory and empty movements were as follows:

Heuristic Solution	$1,902,000
Optimal Solution	$1,825,000.

Thus the heuristic solution is $77,000 (about 4%) more costly than the exact optimal solution.

The heuristic described above works well for the Australia - Europe system. In other container systems, different heuristics may be required.

2.6 Container Movement and Cost Data

Container Flows

An analysis of historical data over the period February-August 1970 and covering 20 voyages yielded the average full container flows and major empty flows listed in Tables 1,2. Because these figures relate to a comparatively short period during the growth of the container service, they should not be taken as an indication of later operations. In particular, the empty movements were atypical because in the period analysed many containers were being positioned in Australia for other trades. More typically, larger numbers of containers would be returned empty to Tilbury.

Table 1 illustrates the cargo imbalances for both dry and reefer cargo. The imbalance at Melbourne, for example, is a net import of 90 container loads of dry cargo and a net export of 107 container loads of reefer cargo. Table 2 shows the major flows of empty general containers. There are no major flows of empty insulated containers because they are imported to Australia packed with dry cargo.

Ship Capacity for Empty Containers

The ship capacity for empty containers can be calculated in the following way. Assume an 'average' ship with a capacity of 1300 containers and carrying the

TABLE 1

AVERAGE CARGO FIGURES (CONTAINER LOADS) PER VOYAGE

OVER PERIOD FEBRUARY - AUGUST 1970.

	Dry Cargo	Reefer Cargo
<u>Import</u>		
Tilbury-Sydney	500	2
Tilbury-Melbourne	480	3
Tilbury-Adelaide	65	1
Tilbury-Brisbane	85	1
Tilbury-Fremantle	80	1
	1210	8
<u>Export</u>		
Sydney-Tilbury	180	20
Melbourne-Tilbury	390	110
Adelaide-Tilbury	70	20
Brisbane-Tilbury	90	20
Fremantle-Tilbury	70	50
	800	220

TABLE 2

MAJOR FLOWS OF EMPTY GENERAL CONTAINERS (AVERAGE NUMBER PER VOYAGE) OVER PERIOD FEBRUARY-AUGUST 1970.

Link	Number
Sydney-Tilbury	100
Melbourne-Tilbury	40
Sydney-Brisbane	5
Melbourne-Adelaide	5

TABLE 3

SHIP CAPACITY FOR EMPTY CONTAINERS (USING DATA IN TABLE 1)

Ship leg	No. of fulls on board	Ship Capacity for empties
Tilbury-Fremantle	1210	90
Fremantle-Sydney	1130	170
Sydney-Melbourne	855	445
Melbourne-Fremantle	900	400
Fremantle-Tilbury	1020	280

full container loads listed in Table 1 (except that for simplicity we ignore the import of any reefer cargo). Calculate the number of full containers carried on each leg of the voyage and hence deduce the capacities for empties. The average ship carries 1210 full containers on the Tilbury-Fremantle leg and therefore 1210-80 = 1130 full containers on the Fremantle-Sydney leg (Fremantle exports are loaded when the ship calls Northbound at Fremantle). The situation at Sydney is a little more complicated. The number of full containers discharged is 500 for Sydney and 85 for Brisbane and the number loaded is 180+20 = 200 from Sydney and 90+20 = 110 from Brisbane. The number of full containers on the Sydney-Melbourne leg is therefore 1130-585+310 = 855. A similar calculation gives 900 full containers on the Melbourne-Fremantle leg and 1020 full containers on the Fremantle-Tilbury leg. These results are summarised in Table 3 together with the ship capacities. It will be noted that there is little space for empties on the Tilbury-Fremantle leg, for a load factor exceeding 90% is achieved on Southbound voyages. In the reverse direction, the load factor is nearly 80%.

Cost Data

Inventory costs and empty movement costs are needed as inputs to the model, and the following estimates (in Australian dollars) were used:

<u>inventory costs</u> (per week per container)

for general containers $5

for insulated containers $10

<u>empty movement costs</u> (per container)

between any two ports via container ship $40

between Melbourne and Adelaide via rail $30

between Sydney and Brisbane via feeder ship $80.

There are alternative transport modes (e.g. rail, feeder ship) between ports in Australia but these are much more costly than the container ship and are used only in emergencies.

<u>Network Considerations</u>

For a proper representation of ship capacities <u>and</u> empty movement costs, the detail in the network illustrated in Fig. 3 is necessary.

Consider Fig. 2. The movement by container ship of empty containers from Sydney (node 3) to Tilbury (node 1) may be represented as flow along the link (3,1). But this movement actually occurs via Melbourne and Fremantle and is subject to empty capacity restrictions on the three ship legs: Sydney-Melbourne, Melbourne-Fremantle, Fremantle-Tilbury. So to represent ship capacities it is simplest to consider the Sydney-Tilbury <u>empty</u> movement as flow along the link (3,5), followed by flow along the link (5,2), followed by flow along the link (2,1). But then the Fig.2

network is inadequate because the Sydney-Tilbury empty movement costis not the sum of the Sydney-Melbourne, Melbourne-Fremantle and Fremantle-Tilbury empty movement costs.

In fact the $40 unit cost for any port to port transport of an empty container is comprised of a loading charge of $20 at the origin port and an unloading charge of $20 at the destination port. Thus in Fig. 3 links such as (1,11), (3,13) representing loading and unloading are ascribed unit costs of $20 while links such as (13,15) representing sea legs are given zero costs. The Sydney-Tilbury empty movement is represented as flow along the links (3,13), (13,15), (15,12), (12,11) and (11,1), and incurs the correct cost of $40.

2.7 Computer Programs

Computer programs to implement the heuristic solution procedure described in section 2.5 have been written for the University of Adelaide CDC 6400 computer; and computer runs have been conducted using a remote terminal in the Mathematics Department. The out-of-kilter algorithm is used to solve both the Phase I and Phase II one-commodity network problems.

An important feature of the out-of-kilter algorithm is that it may use the optimal solution to a problem as a good starting point for a new problem in which only link costs and link capacities are altered from the original problem. If only cost and capacity alterations are involved, a Phase I solution may be used as a good starting point for a new Phase I problem, and similarly for Phase II. Consequently it is possible to make rapid investigations of the sensitivity of the heuristic solution to variations in

empty movement costs
inventory costs
available cargo
ship capacities

and also to variations in the heuristic costs themselves. The same is true when the network geography is altered by deleting nodes and links. For a node may be deleted by deleting all the links which originate or terminate at that node; and the deletion of links may effectively be accomplished by assigning them upper and lower capacities of zero.

The computer programs also allow the network geography to be altered by adding nodes and links, or by changing travel times for existing links; then a previously obtained optimal solution cannot be used.

The great value of the heuristic method - as implemented by the computer programs - is the speed with which it obtains a near-optimal solution to the two-commodity problem. Only about two minutes of central processor (CP) time are needed to obtain the heuristic solution to a 52 week problem; and very little time is required to test the sensitivity of this solution to variations in most of the input parameters. The speed of the heuristic method, and the convenience provided by the remote terminal, have allowed as many as 30 computer runs in a single day. In all, over 200 production runs - involving a total CP time of only a few hours - have been conducted.

The heuristic method involves determining Phase I and Phase II optimal solutions, but provides only a near-optimal solution to the two-commodity problem. If computer time could be greatly reduced by terminating Phase I and Phase II with sub-optimal solutions which provided a satisfactory two-commodity solution, it would be reasonable to ask whether it was worthwhile to determine optimal Phase I and Phase II solutions. However, as noted above, these optimal solutions are obtained so rapidly that earlier termination need not be considered.

2.8 Model Applications

The network flow model has been applied successfully to answer many questions about the Australia-Europe system. Extensive discussions with the container companies have shown that the model gives realistic answers concerning container inventory and empty movement patterns. Furthermore, use of the model has given considerable insight into the operation of the container system. This insight has enabled the author to obtain reasonably adequate answers to many questions by performing simple hand calculations.

Some of the model applications will now be considered.

Empty Container Movement Patterns

The model has been used to determine empty container movement patterns for the Australia-Europe system. In section 2.5, it was suggested that there should be no significant movement of empty insulated containers between Australia and Europe, and that there should be a considerable movement of empty general containers Northbound. Table 4 summarises the results of a computer run using estimates of containerised cargo for 1970. The empty movement trends are as expected. Movements of empty insulated containers are negligible, but movements of empty general containers are considerable. Sydney, a

TABLE 4

SUMMARY RESULTS FOR RUN OF COMPUTER PROGRAM

USING 1970 CARGO ESTIMATES

	General Containers	Insulated Containers
Container Inventory	14,250	4,150
Cost of Empty Movements	$600,000	$10,000
Average Weekly Empty Movements:		
Sydney - Tilbury	160	negligible
Sydney - Melbourne	65	"
Sydney - Fremantle	40	"
Melbourne - Adelaide	35	"
Sydney - Brisbane	10	"

large producer of empties, supplies Brisbane, Melbourne and Fremantle with the empties they consume, and the remainder are returned to Tilbury. Empties from Melbourne satisfy the Adelaide demand.

<u>Packing Dry Cargo in Insulated Containers</u>

The model has also been used to compare the heuristic solution obtained above with the solution which results when no dry cargo is packed in insulated containers. If the same ship capacities (of 1300 containers) are assumed in each case, then in the latter case there is essentially no feasible solution. Even when the effect of ship capacities is removed entirely, the comparison is dramatic. The results are shown in Table 5 and indicate an increase in total cost from \$6.25m. to \$7.46m. Note that the insulated container inventory decreases slightly when insulated containers are not packed with dry cargo, because then insulated containers travel to Australia empty rather than full. (The travel time for an empty is less than the "travel time" for a full container.) But the general container inventory and empty movement costs increase markedly.

The variation in total cost with the percentage of dry cargo which is suitable for packing in insulated containers has been determined by the model. The total cost varies from \$7.46m. to \$6.25m. over the range 0% to

TABLE 5

COST COMPARISON BETWEEN HEURISTIC SOLUTION AND SOLUTION WHERE NO DRY CARGO IS PACKED IN INSULATED CONTAINERS

	Heuristic Solution	No Dry Cargo In Insulated Containers
Container Inventory		
General	14,250	16,950
Insulated	4,150	4,100
Container Inventory Cost (For 50 Weeks)		
General	$3,562,500	$4,237,500
Insulated	$2,075,000	$2,050,000
Empty Movement Cost		
General	$600,000	$740,000
Insulated	$10,000	$430,000
Total Cost	$6,247,500	$7,457,500

25%. Once the 25% level is reached, there is sufficient dry cargo to pack all Southbound insulated containers, and there is no further cost improvement.

Sensitivity to Cost and Travel Time Data

Table 5 illustrates what proves to be a very significant feature of the cost structure of the container flows in the Australia-Europe system. The inventory costs are very much greater, in total, than the costs of moving empties. This helps simplify the analysis of different situations and suggests, for example, that the container flow patterns do not depend sensitively on the assumed unit costs, a fact which can be verified readily.

At any one time, only a small fraction of the total container inventory is moving empty. Consequently the container inventory is fairly insensitive to changes in empty movement travel times. On the other hand, if average travel times for full containers are increased from eight weeks to nine, the container inventory increases by almost one eighth.

Other Applications

Typical of some of the questions which the model has answered, but which will not be discussed here, are the following:

(i) What savings would accrue if the container consortia pooled containers?

(ii) Should containers which would otherwise be travelling empty around the Australian coast be used to carry domestic cargo?

(iii) When should additional containers be injected into the system to cope with a growth in trade?

(iv) What stocks of containers should be held at stockpiles to meet unexpected demands?

(v) What would be the effects on container flows in Australia if inland stockpiles were established?

CHAPTER 3. MATHEMATICAL ASPECTS OF THE TWO-COMMODITY PROBLEM

3.1 The Two-Commodity Problem

The two-commodity problem which is considered in this chapter is a slightly simplified version of the problem (12)-(17) in section 2.4. Two simplifications are made. Firstly, the constraint (16), representing ship capacities, is omitted. Secondly, the variables $g^k_i(t)$ are all set to zero, except when a non-cyclic boundary condition is used. In this case, the variables $g^k_i(1)$ are allowed to be non-zero, but all other $g^k_i(t)$ are set to zero, and the notation is simplified by defining

$$e^k_{ii}(0) = g^k_i(1).$$

When these simplifications are made, together with the change of variable $f^k_{ij}(t)$ to $f^k_{ij}(t) + a^k_{ij}(t)$, the following two-commodity problem results:

$$\text{Minimize} \quad \Sigma_{t=1}^{T} \sum_k [\sum_{i,j} c^k_{ij}(t) e^k_{ij}(t) + \sum_{i,j} d^k_{ij}(t) f^k_{ij}(t)] \tag{29}$$

subject to

$$\sum_j \{e^k_{ij}(t) + f^k_{ij}(t)\} - \sum_j \{e^k_{ji}(t-s_{ji}) + f^k_{ji}(t-t_{ji})\}$$

$$= \sum_j a^k_{ji}(t-t_{ji}) - \sum_j a^k_{ij}(t) \tag{30}$$

$$\sum_k f^k_{ij}(t) = a^s_{ij}(t) \tag{31}$$

$$e^k_{ij}(t) \geqslant 0, \; f^k_{ij}(t) \geqslant 0 \tag{32}$$

and the additional requirement that all variables be integers.

The dual linear program, obtained by assigning multipliers $\pi_i^k(t)$ to (30) and multipliers $\theta_{ij}(t)$ to (31), is:

$$\text{Maximize} \quad \Sigma_{t=1}^{T}[\sum_{k}\sum_{i}\{\sum_{j} a_{ji}^{k}(t-t_{ji})-\sum_{j} a_{ij}^{k}(t)\}\pi_i^k(t)] + \Sigma_{t=1}^{T}[\sum_{i,j} a_{ij}^{s}\theta_{ij}(t)] \tag{33}$$

subject to

$$\pi_i^k(t) - \pi_j^k(t+s_{ij}) \leqslant c_{ij}^k(t) \tag{34}$$

$$\pi_i^k(t) - \pi_j^k(t+t_{ij}) + \theta_{ij}(t) \leqslant d_{ij}^k(t) \tag{35}$$

$$\pi_i^k(t),\ \theta_{ij}(t) \ \text{unrestricted.} \tag{36}$$

As in section 2.4, t assumes the values $1,2,\ldots,T$ and when the non-cyclic boundary condition is used, variables with time quantity $\tau < 1$ or $\tau > T$ are suppressed, except $e_{ii}^k(0)$. In the non-cyclic case, there is an additional dual constraint associated with the variable $e_{ii}^k(0)$:

$$-\pi_i^k(1) \leqslant 0, \tag{37}$$

and the link costs $c_{ij}^k(t)$ and $d_{ij}^k(t)$ are given by (22) and (23). When the cyclic boundary condition is used, $c_{ij}^k(t)$ and $d_{ij}^k(t)$ are given by (18) and (19).

For some purposes, it is convenient to represent the two-commodity problem (29)-(32) and its dual (33)-(37)

in matrix form as follows:

$$\text{Minimize} \quad \underset{\sim}{c}^1\underset{\sim}{e}^1 + \underset{\sim}{d}^1\underset{\sim}{f}^1 + \underset{\sim}{c}^2\underset{\sim}{e}^2 + \underset{\sim}{d}^2\underset{\sim}{f}^2 \tag{38}$$

subject to

$$E\,\underset{\sim}{e}^1 + F\,\underset{\sim}{f}^1 = -F\underset{\sim}{a}^1 \tag{39}$$

$$E\,\underset{\sim}{e}^2 + F\,\underset{\sim}{f}^2 = -F\underset{\sim}{a}^2 \tag{40}$$

$$\underset{\sim}{f}^1 + \underset{\sim}{f}^2 = \underset{\sim}{a}^3 \tag{41}$$

$$\underset{\sim}{e}^1 \geqslant \underset{\sim}{0}, \quad \underset{\sim}{f}^1 \geqslant \underset{\sim}{0}, \quad \underset{\sim}{e}^2 \geqslant \underset{\sim}{0}, \quad \underset{\sim}{f}^2 \geqslant \underset{\sim}{0}. \tag{42}$$

$$\text{Maximize} \quad \underset{\sim}{\pi}^1(-F\underset{\sim}{a}^1) + \underset{\sim}{\pi}^2(-F\underset{\sim}{a}^2) + \underset{\sim}{\theta}\,\underset{\sim}{a}^3 \tag{43}$$

subject to

$$\underset{\sim}{\pi}^1 E \leqslant \underset{\sim}{c}^1 \tag{44}$$

$$\underset{\sim}{\pi}^1 F + \underset{\sim}{\theta} \leqslant \underset{\sim}{d}^1 \tag{45}$$

$$\underset{\sim}{\pi}^2 E \leqslant \underset{\sim}{c}^2 \tag{46}$$

$$\underset{\sim}{\pi}^2 F + \underset{\sim}{\theta} \leqslant \underset{\sim}{d}^2 \tag{47}$$

$$\underset{\sim}{\pi}^1, \ \underset{\sim}{\pi}^2, \ \underset{\sim}{\theta} \ \text{unrestricted.} \tag{48}$$

In (38) - (48), $\underset{\sim}{e}^k = \{e^k_{ij}(t)\}$, $\underset{\sim}{f}^k = \{f^k_{ij}(t)\}$ and $\underset{\sim}{a}^k = \{a^k_{ij}(t)\}$ are column vectors; and $\underset{\sim}{c}^k = \{c^k_{ij}(t)\}$, $\underset{\sim}{d}^k = \{d^k_{ij}(t)\}$, $\underset{\sim}{\pi}^k = \{\pi^k_i(t)\}$ and $\underset{\sim}{\theta} = \{\theta_{ij}(t)\}$ are row vectors. E and F are matrices associated with the dynamic network on which the container flows are occurring.

In fact when the cyclic boundary condition is used, the matrix [E F] is the node-link incidence matrix of the dynamic network. (See Appendix for definitions and results which are used in the following discussion.) E and F are the node-link incidence matrices of the subnetworks (of the dynamic network) which comprise only the empty links, and only the full links, respectively.

When the non-cyclic boundary condition is used, the matrix [E F] is not strictly a node-link incidence matrix, because some columns contain just one non-zero entry. The columns corresponding to variables $e^k_{i1}(0)$ contain a -1 (but no +1); and the columns corresponding to variables $e^k_{ij}(t)$ for $t+s_{ij}>T$, and $f^k_{ij}(t)$ for $t+t_{ij}>T$, contain a +1 (but no -1). Here the matrix [E F] is essentially the matrix which remains when one row of a node-link incidence matrix is deleted.

The constraints (39) - (42) may be written in the form $A\underset{\sim}{y} = \underset{\sim}{b}$, $\underset{\sim}{y} \geqslant \underset{\sim}{0}$ with

$$A = \begin{bmatrix} E & F & 0 & 0 \\ 0 & 0 & E & F \\ 0 & I & 0 & I \end{bmatrix} \tag{49}$$

(where I is a unit matrix, and 0 is a zero matrix)

$$\underset{\sim}{y} = \begin{bmatrix} \underset{\sim}{e}^1 \\ \underset{\sim}{f}^1 \\ \underset{\sim}{e}^2 \\ \underset{\sim}{f}^2 \end{bmatrix} \tag{50}$$

$$\text{and} \quad \underset{\sim}{b} = \begin{bmatrix} -F\underset{\sim}{a}^1 \\ -F\underset{\sim}{a}^2 \\ \underset{\sim}{a}^3 \end{bmatrix}. \qquad (51)$$

The matrix A and the vector $\underset{\sim}{b}$ are integral, and the vector $\underset{\sim}{y}$ is required to be integral.

<u>Lemma 1</u>

Let E and F be arbitrary matrices (not necessarily node-link incidence matrices) and let the matrix A be given by (49). Then the rows of A are linearly independent (ℓ.i.) if and only if the rows of E are ℓ.i.

<u>Proof of Lemma 1</u>

(i) Suppose that the rows of E are ℓ.i. Then $\underset{\sim}{\lambda}E = \underset{\sim}{0}$ implies $\underset{\sim}{\lambda} = \underset{\sim}{0}$.

Now if

$$[\underset{\sim}{\lambda}_1 \ \underset{\sim}{\lambda}_2 \ \underset{\sim}{\lambda}_3] \begin{bmatrix} E & F & 0 & 0 \\ 0 & 0 & E & F \\ 0 & I & 0 & I \end{bmatrix} = [\underset{\sim}{0} \ \underset{\sim}{0} \ \underset{\sim}{0} \ \underset{\sim}{0}]$$

this gives $\underset{\sim}{\lambda}_1 E = \underset{\sim}{0}$ (which implies $\underset{\sim}{\lambda}_1 = \underset{\sim}{0}$)

$\underset{\sim}{\lambda}_1 F + \underset{\sim}{\lambda}_3 = \underset{\sim}{0}$

$\underset{\sim}{\lambda}_2 E = \underset{\sim}{0}$ (which implies $\underset{\sim}{\lambda}_2 = \underset{\sim}{0}$)

$\underset{\sim}{\lambda}_2 F + \underset{\sim}{\lambda}_3 = \underset{\sim}{0}$

and consequently $\underset{\sim}{\lambda}_1 = \underset{\sim}{\lambda}_2 = \underset{\sim}{\lambda}_3 = \underset{\sim}{0}$.

The rows of A are ℓ.i.

(ii) Suppose that E has linearly dependent (ℓ.d.) rows. Then there exists $\underset{\sim}{\lambda} \neq \underset{\sim}{0}$ such that $\underset{\sim}{\lambda}E = \underset{\sim}{0}$. Now

$[\underset{\sim}{\lambda}\ \underset{\sim}{\lambda}\ -\underset{\sim}{\lambda}F]$ is a non-zero vector and

$$[\underset{\sim}{\lambda}\ \underset{\sim}{\lambda}\ \ -\underset{\sim}{\lambda}F]\begin{bmatrix} E & F & O & O \\ O & O & E & F \\ O & I & O & I \end{bmatrix} = [\underset{\sim}{0}\ \underset{\sim}{0}\ \underset{\sim}{0}\ \underset{\sim}{0}].$$

If the rows of E are ℓ.d., the rows of A are ℓ.d. Thus if A has ℓ.i. rows, E has ℓ.i. rows. #

When the cyclic boundary condition is used, the node-link incidence matrix E has ℓ.d. rows, as does the matrix [E F], and so by Lemma 1 the matrix A has ℓ.d. rows. But in most practical situations, the network of empty links is connected, and so when any row of [E F] is deleted, the remaining rows of E, and of [E F], are ℓ.i. Henceforth it is assumed that E and [E F] represent these ℓ.i. rows. Then the matrix A defined by (49) has ℓ.i. rows. In the non-cyclic case, the matrix E has ℓ.i. rows and it is not necessary to delete a redundant row.

3.2 Integer Property Theorems

In the last section, the constraints of the two-commodity problem were expressed in the form $A\underset{\sim}{y} = \underset{\sim}{b}$, $\underset{\sim}{y} \geq \underset{\sim}{0}$ and the optimal vector $\underset{\sim}{y}$ was required to be integral. It is well-known (ch.7, [7]) that a problem with the constraints $A\underset{\sim}{y} = \underset{\sim}{b}$, $\underset{\sim}{y} \geq \underset{\sim}{0}$ has an optimal solution at one of the extreme points of the convex set

$$\mathcal{Y}(A,\underset{\sim}{b}) = \{\underset{\sim}{y}: A\underset{\sim}{y} = \underset{\sim}{b},\ \underset{\sim}{y} \geq \underset{\sim}{0}\}.$$

Consequently this section investigates conditions on an arbitrary integral matrix A in order that the extreme points of $\mathcal{Y}(A,\underset{\sim}{b})$ are integral.

Definition 1 (Integer Property)

An integral matrix A is said to have the integer property (i.p.) if and only if the extreme points of $\mathcal{Y}(A,\underset{\sim}{b})$ are integral for all $\underset{\sim}{b} \in \mathcal{B}(A)$, where $\mathcal{B}(A) = \{\underset{\sim}{b} : \underset{\sim}{b}$ integral, $A\underset{\sim}{y} = \underset{\sim}{b}$ is soluble$\}$.

As noted by Hu (p.125, [24]), various papers have determined conditions on a matrix in order that an integer property holds. These conditions will be examined shortly.

For the present, assume that the rows of A are ℓ.i. Then Dantzig (ch.7, [7]) has shown that there is a one-to-one correspondence between extreme points and basic feasible solutions of $\mathcal{Y}(A,\underset{\sim}{b})$, so that

$$\{\underset{\sim}{y}:\underset{\sim}{y} \text{ is an extreme point of } \mathcal{Y}(A,\underset{\sim}{b})\}$$
$$= \left\{\underset{\sim}{y}:\underset{\sim}{y} = \begin{bmatrix} \underset{\sim}{y}_B \\ \underset{\sim}{y}_N \end{bmatrix}, \text{ where } \underset{\sim}{y}_B = B^{-1}\underset{\sim}{b} \geq \underset{\sim}{0} \text{ for some basis } B \text{ from } A,\ \underset{\sim}{y}_N = \underset{\sim}{0}\right\}.$$

(If A has r (ℓ.i.) rows, a <u>basis</u> from A is any $r \times r$ nonsingular submatrix of A.)

Using the above result, Veinott and Dantzig [48] have derived necessary and sufficient conditions in order that the matrix A has the i.p. Their result may be stated as follows:

<u>Theorem 1</u> (Veinott and Dantzig)

Let A be an integral matrix which has ℓ.i. rows. The matrix A has the i.p. if and only if

every basis B from A has $\det B = \pm 1$. (52)

Using Theorem 1, the above authors were able to provide a simple proof of an important theorem first proved by Hoffman and Kruskal [19].

<u>Theorem 2</u> (Hoffman and Kruskal)

Let A be an integral matrix (not necessarily with ℓ.i. rows).

The extreme points of the set

$$\mathcal{Y}^*(A,\underset{\sim}{b}) = \{\underset{\sim}{y}:A\underset{\sim}{y} \leq \underset{\sim}{b},\ \underset{\sim}{y} \geq \underset{\sim}{0}\}$$

are integral for all integral $\underset{\sim}{b}$ if and only if

A is totally unimodular,

where

Definition 2 (Totally Unimodular)

A matrix A is said to be totally unimodular if and only if every square submatrix of A has determinant +1, -1 or 0.

Note that the set involved in Theorem 2 is $\mathcal{Y}^*(A,\underset{\sim}{b})$ and not $\mathcal{Y}(A,\underset{\sim}{b})$, and so Theorem 2 does not give necessary and sufficient conditions for A to have the integer property of Definition 1. On the other hand, it is easy to see that the condition that A is totally unimodular is sufficient (but not necessary) for A to have the i.p.

Theorem 3

Let A be an integral matrix (not necessarily with ℓ.i. rows).

For A to have the i.p. it is sufficient (but not necessary) that A is totally unimodular.

Proof of Theorem 3

(i) Let the rank of A be r, and let $[A'\underset{\sim}{b}']$ be r ℓ.i. rows chosen from $[A\ \underset{\sim}{b}]$, where $\underset{\sim}{b} \in \mathcal{B}(A)$. Then $\mathcal{Y}(A',\underset{\sim}{b}') = \mathcal{Y}(A,\underset{\sim}{b})$ and so these sets have the same extreme points. Since A is totally unimodular by hypothesis, so is A' and in particular every basis from A' has determinant ± 1. By Theorem 1, A' has the i.p. i.e. the extreme points of $\mathcal{Y}(A',\underset{\sim}{b}')$ are integral.

Thus the extreme points of $\mathcal{Y}(A,\underset{\sim}{b})$ are integral for arbitrary $\underset{\sim}{b} \in \mathcal{B}(A)$, and so A has the i.p.

(ii) The matrix $A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ has ℓ.i. rows and det A=1, By Theorem 1, A has the i.p., but clearly A is not totally unimodular. The condition that A is totally unimodular is not necessary for A to have the i.p. #

Necessary and sufficient conditions for an integral matrix A with ℓ.d. rows to have the i.p. have been derived. These conditions are shortly stated as Theorem 4, but first several preparatory lemmas are required.

<u>Lemma 2</u>

Let A be an $m \times n$ integral matrix with rank r. Then there exists an $m \times m$ integral matrix U with det U = ±1 such that

$$UA = \begin{bmatrix} A' \\ 0 \end{bmatrix}$$

where the (integral) matrix A' has r rows, which are ℓ.i.

<u>Proof of Lemma 2</u>

A theorem in linear algebra [25] states that any $m \times n$ integral matrix A of rank r can be put in the form

$$A = VDW$$

where V is an $m \times m$ integral matrix with det V = ±1, W is an $n \times n$ integral matrix with det W = ±1, and D is an $m \times n$ non-negative integral diagonal matrix with non-zero diagonal elements $d_{11}, d_{22}, \ldots, d_{rr}$. It follows that

$U = V^{-1}$ is an $m \times m$ integral matrix with $\det U = \pm 1$, and that DW is of the required form

$$\begin{bmatrix} A' \\ 0 \end{bmatrix}. \quad \#$$

<u>Lemma 3</u>

Let A be an $m \times n$ integral matrix with rank r, and let U be any $m \times m$ <u>nonsingular</u> integral matrix such that

$$UA = \begin{bmatrix} A' \\ 0 \end{bmatrix}$$

where the integral matrix A' has r rows, which are ℓ.i.

Let

$$U\mathcal{B}(A) = \{U\underset{\sim}{b} : \underset{\sim}{b} \in \mathcal{B}(A)\}$$

and let

$$\mathcal{B}_0(A') = \{\underset{\sim}{b}_0 : \underset{\sim}{b}_0 = \begin{bmatrix} \underset{\sim}{b}' \\ \underset{\sim}{0} \end{bmatrix}, \underset{\sim}{b}' \in \mathcal{B}(A')\}.$$

Then (i) $U\mathcal{B}(A) \subseteq \mathcal{B}_0(A')$;

(ii) If $\det U = \pm 1$, $U\mathcal{B}(A) = \mathcal{B}_0(A')$.

<u>Proof of Lemma 3</u>

First note that since A' has ℓ.i. rows, $A'\underset{\sim}{y} = \underset{\sim}{b}'$ is soluble for all integral $\underset{\sim}{b}'$, and so

$$\mathcal{B}(A') = \{\underset{\sim}{b}' : \underset{\sim}{b}' \text{ integral}\}.$$

(i) Let $\underset{\sim}{b} \in \mathcal{B}(A)$.

Then for some vector $\underset{\sim}{y}$, $A\underset{\sim}{y} = \underset{\sim}{b}$, and so $UA\underset{\sim}{y} = U\underset{\sim}{b}$

$$\text{viz.} \quad \begin{bmatrix} A' \\ 0 \end{bmatrix} \underset{\sim}{y} = U\underset{\sim}{b}. \tag{53}$$

Since U and $\underset{\sim}{b}$ are integral, $U\underset{\sim}{b}$ is integral, and by (53) is of the form

$$\begin{bmatrix} \underset{\sim}{b}' \\ \underset{\sim}{0} \end{bmatrix}$$

where $\underset{\sim}{b}'$ is integral. Consequently $U\underset{\sim}{b} \in \mathcal{B}_0(A')$.

Thus $U\mathcal{B}(A) \subseteq \mathcal{B}_0(A')$.

(ii) Let $\underset{\sim}{b}' \in \mathcal{B}(A')$.

Then for some vector $\underset{\sim}{y}$, $A'\underset{\sim}{y} = \underset{\sim}{b}'$ and so

$$U^{-1} \begin{bmatrix} A' \\ 0 \end{bmatrix} \underset{\sim}{y} = U^{-1} \begin{bmatrix} \underset{\sim}{b}' \\ \underset{\sim}{0} \end{bmatrix}$$

viz. $A\underset{\sim}{y} = U^{-1}\underset{\sim}{b}_0$, where $\underset{\sim}{b}_0 \in \mathcal{B}_0(A')$.

But if $\det U = \pm 1$, U^{-1} is integral and so $U^{-1}\underset{\sim}{b}_0$ is integral. Consequently $U^{-1}\underset{\sim}{b}_0 = \underset{\sim}{b}$ where $\underset{\sim}{b} \in \mathcal{B}(A)$, or $\underset{\sim}{b}_0 = U\underset{\sim}{b}$. Thus if $\det U = \pm 1$, $\mathcal{B}_0(A') \subseteq U\mathcal{B}(A)$ which together with (i) implies $U\mathcal{B}(A) = \mathcal{B}_0(A')$. #

<u>Lemma 4</u>

Let A, U and A' satisfy the conditions of Lemma 3.

Then (i) For any $\underset{\sim}{b} \in \mathcal{B}(A)$ there exists $\underset{\sim}{b}' \in \mathcal{B}(A')$ such that $\mathcal{Y}(A,\underset{\sim}{b}) = \mathcal{Y}(A',\underset{\sim}{b}')$.

(ii) If $\det U = \pm 1$, for any $\underset{\sim}{b}' \in \mathcal{B}(A')$ there exists $\underset{\sim}{b} \in \mathcal{B}(A)$ such that $\mathcal{Y}(A,\underset{\sim}{b}) = \mathcal{Y}(A',\underset{\sim}{b}')$

Proof of Lemma 4

(i) By Lemma 3, if $\underset{\sim}{b} \in \mathcal{B}(A)$ there exists $\underset{\sim}{b}' \in \mathcal{B}(A')$ such that

$$U\underset{\sim}{b} = \begin{bmatrix} \underset{\sim}{b}' \\ \underset{\sim}{0} \end{bmatrix}.$$

Since U is nonsingular,

$$\begin{aligned} \mathcal{Y}(A,\underset{\sim}{b}) &= \{\underset{\sim}{y} : A\underset{\sim}{y} = \underset{\sim}{b},\ \underset{\sim}{y} \geq \underset{\sim}{0}\} \\ &= \{\underset{\sim}{y} : UA\underset{\sim}{y} = U\underset{\sim}{b},\ \underset{\sim}{y} \geq \underset{\sim}{0}\} \\ &= \{\underset{\sim}{y} : \begin{bmatrix} A' \\ 0 \end{bmatrix}\underset{\sim}{y} = \begin{bmatrix} \underset{\sim}{b}' \\ \underset{\sim}{0} \end{bmatrix},\ \underset{\sim}{y} \geq \underset{\sim}{0}\} \\ &= \{\underset{\sim}{y} : A'\underset{\sim}{y} = \underset{\sim}{b}',\ \underset{\sim}{y} \geq \underset{\sim}{0}\} \\ &= \mathcal{Y}(A',b'). \end{aligned}$$

(ii) The proof is similar to (i). #

Theorem 4

Let A be an $m \times n$ integral matrix with rank r, and let U be any $m \times m$ nonsingular integral matrix such that

$$UA = \begin{bmatrix} A' \\ 0 \end{bmatrix} \qquad (54)$$

where the integral matrix A' has r rows, which are ℓ.i.

Then (i) For A to have the i.p. it is sufficient that A' has the i.p.

(ii) If $\det U = \pm 1$, for A to have the i.p. it is necessary that A' has the i.p.

<u>Proof of Theorem 4</u>

(i) If A' has the i.p., the extreme points of $\mathcal{Y}(A',\underset{\sim}{b}')$ are integral for all $\underset{\sim}{b}' \in \mathcal{B}(A')$. By Lemma 4, for each $\underset{\sim}{b} \in \mathcal{B}(A)$ there exists $\underset{\sim}{b}' \in \mathcal{B}(A')$ such that $\mathcal{Y}(A,\underset{\sim}{b}) = \mathcal{Y}(A',\underset{\sim}{b}')$.

Thus the extreme points of $\mathcal{Y}(A,\underset{\sim}{b})$ are integral for all $\underset{\sim}{b} \in \mathcal{B}(A)$. The matrix A has the i.p.

(ii) If A has the i.p., and $\det U = \pm 1$, a similar proof to (i) shows that A' has the i.p. #

Theorem 4 shows how to convert a matrix A with ℓ.d. rows into an associated matrix A' which has ℓ.i. rows and to which the theorem of Veinott and Dantzig (Theorem 1) may be applied. By Lemma 2, there exists an integral matrix U with $\det U = \pm 1$ such that (54) holds, and then A has the i.p. if and only if every basis from A' has determinant ± 1. For example, the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \tag{55}$$

has ℓ.d. rows. Now let the matrix U be

$$U = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix}$$

so that $\det U = +1$, and note that

$$UA = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

By Theorem 4, A has the i.p. if and only if the matrix

$$A' = [1 \quad 1]$$

has the i.p. Since A' has the i.p. by Theorem 1, A has the i.p.

When (54) holds and A has the i.p., it does not necessarily follow that A' has the i.p. if $\det U \neq \pm 1$. For example if A is given by (55) and

$$U = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$$

so that $\det U = 2$, then

$$UA = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$$

and $A' = [2 \; 2]$ does not have the i.p.

A number of sufficient conditions for a matrix to be totally unimodular (and hence to have the i.p.) have been given by Hoffman and Kruskal in [19]. Only the following theorem proved by Hoffman in the appendix of Heller and Tompkins [18] is given here:

Theorem 5 (Heller and Tompkins)

A matrix A is totally unimodular if the following four conditions are satisfied:

(a) Every column contains at most two non-zero entries;

(b) Every entry is $0, \pm 1$;

The matrix A can be partitioned into two disjoint sets of rows $\mathcal{R}_1$ and $\mathcal{R}_2$ such that:

(c) If a column of A contains two non-zero entries and both have the same sign, then one is in $\mathcal{R}_1$ and one is in $\mathcal{R}_2$;

(d) If a column of A contains two non-zero entries and they are of opposite sign, then both are in $\mathcal{R}_1$ or both in $\mathcal{R}_2$.

Theorem 5 is true even if one of the sets $\mathcal{R}_1, \mathcal{R}_2$ is empty.

In the appendix of Heller and Tompkins [18], Gale has proved an interesting converse result to Theorem 5. His result is that if a matrix A satisfies condition (a), then in order that A is totally unimodular, conditions (b), (c) and (d) are <u>necessary</u>.

In the following sections, some of the above theorems are used to investigate conditions under which the two-commodity matrix A defined by (49) has the i.p.

3.3 Two-Commodity Fractional Examples

The constraints of a one-commodity network problem may be written in the form

$$A\underset{\sim}{y} = \underset{\sim}{b}, \qquad \underset{\sim}{\ell} \leqslant \underset{\sim}{y} \leqslant \underset{\sim}{u}$$

where A is a node-link incidence matrix. Since a node-link incidence matrix satisfies the conditions of Theorem 5 where $\mathcal{R}_1$ is the whole matrix, A is totally unimodular. The well-known integrality property of one-commodity network problems follows [9].

In general, multi-commodity network flow problems have fractional solutions, even though all the link capacities are integral. Ford and Fulkerson have given a three-commodity fractional example (p.17, [9]), and Figure 6 depicts a two-commodity example due to Jewell [30]. The variable v^k represents the amount of flow from origin k to destination k' $(k=1,2)$. All links are given joint upper capacities of 1 i.e. link capacities are integral. The maximum two-commodity flow (v^1+v^2) is $1\frac{1}{2}$ and is achieved by sending $\frac{1}{2}$ unit of commodity 1 along each of the chains $(1,3,5,6,4,1')$ and $(1,3,7,8,4,1')$, and $\frac{1}{2}$ unit of commodity 2 along the chain $(2,5,6,7,8,2')$.

Jewell's example is not in the same mathematical form (49) - (51) as the two-commodity problem formulated in section 3.1, but it is not difficult to modify his example to produce a fractional example of the form

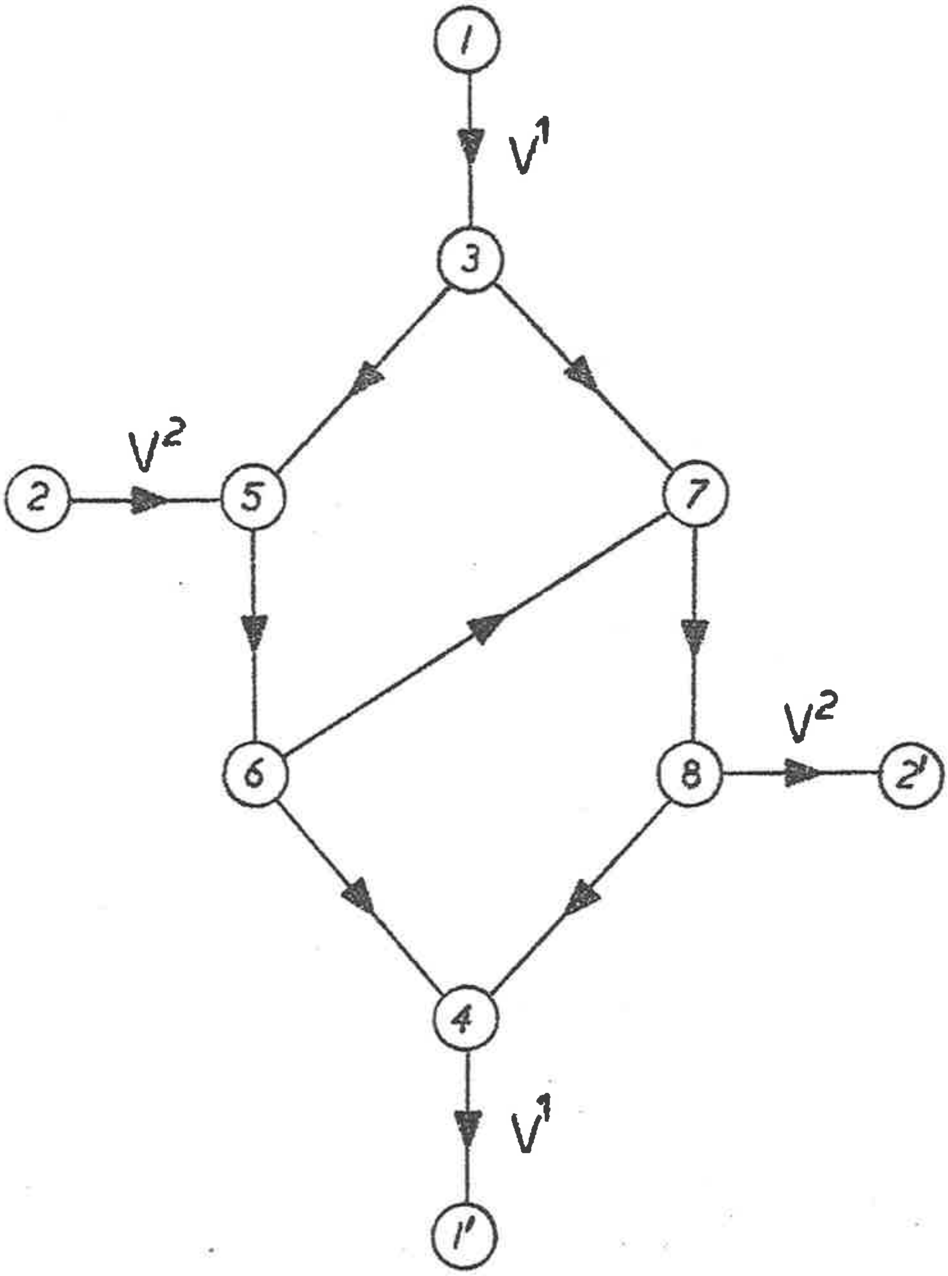

FIGURE 6.

Jewell's two-commodity fractional example.
All links have joint upper capacities of 1.
The maximum two-commodity flow (v^1+v^2) is $1\frac{1}{2}$.

(49) - (51). This example is shown in Figure 7. The lines with double arrows represent full links, while the lines with single arrows represent empty links. The sub-network comprising just the empty links (1,4), (2,3), (2,4) (3,1) is a connected network. The node-link incidence matrix [E F] is:

		empty links				full links		
		(1,4)	(2,3)	(2,4)	(3,1)	(1,2)	(3,1)	(4,3)
nodes	1	1	0	0	-1	1	-1	0
	2	0	1	1	0	-1	0	0
	3	0	-1	0	1	0	1	-1
	4	-1	0	-1	0	0	0	1 .

Choose $a^3_{12} = 2$, $a^2_{31} = a^3_{31} = 1$, $a^3_{43} = 2$ and all other $a^k_{ij} = 0$. When the redundant row 4 is deleted, the two-commodity matrix A defined by (49) and the vector $\underset{\sim}{b}$ defined by (51) are given by:

$$
\begin{array}{c}
\begin{array}{cccccccccccccc}
* & * & * & * & * & * & * & & & & & * & & *
\end{array} \\
A = \left[\begin{array}{cccccccccccccc}
1 & 0 & 0 & -1 & 1 & -1 & 0 & & & & & & & \\
0 & 1 & 1 & 0 & -1 & 0 & 0 & & & & & & & \\
0 & -1 & 0 & 1 & 0 & 1 & -1 & & & & & & & \\
 & & & & & & & 1 & 0 & 0 & -1 & 1 & -1 & 0 \\
 & & & & & & & 0 & 1 & 1 & 0 & -1 & 0 & 0 \\
 & & & & & & & 0 & -1 & 0 & 1 & 0 & 1 & -1 \\
 & & & & 1 & & & & & & & 1 & & \\
 & & & & & 1 & & & & & & & 1 & \\
 & & & & & & 1 & & & & & & & 1
\end{array}\right], \quad \underset{\sim}{b} = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 2 \\ 1 \\ 2 \end{array}\right]
\end{array}
\tag{56}
$$

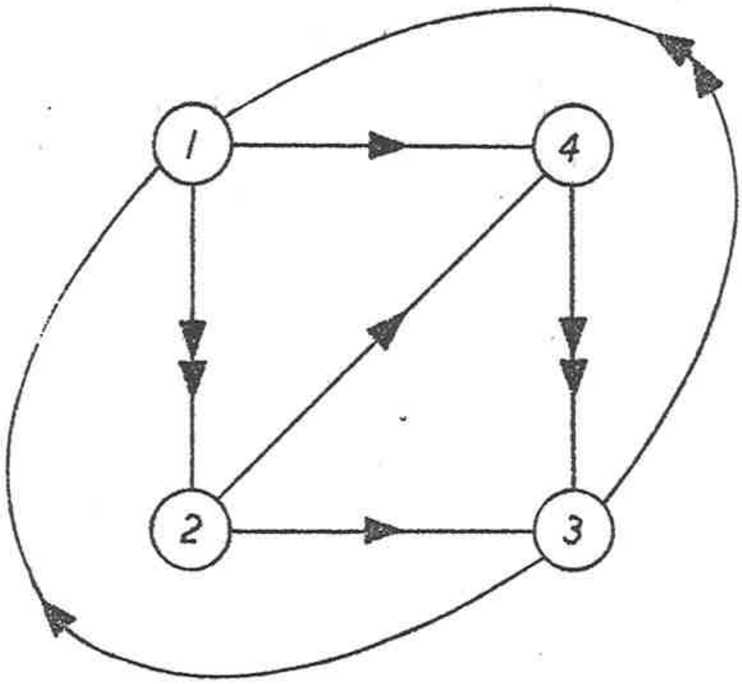

FIGURE 7.

The lines with double arrows represent full links, and the lines with single arrows represent empty links.

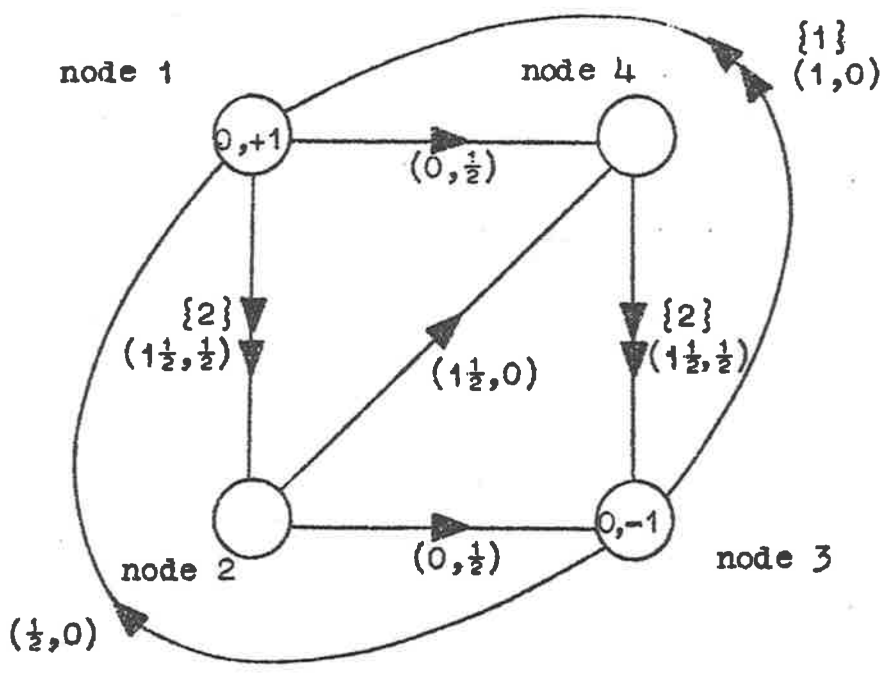

FIGURE 8.

Representation of the two-commodity problem (56) and its solution (57) for the Fig.7 network.

where the elements of A not shown in (56) are all zeros.

The matrix A in (56) does not have the integer property, for when the asterisked columns are chosen to form a basis B it is found that det B = 2, and so A does not satisfy condition (52) of Theorem 1. When $\underset{\sim}{b}$ is given by (56), the basic feasible solution corresponding to the basis B is

$$[0,0,1\tfrac{1}{2},\tfrac{1}{2},1\tfrac{1}{2},1,1\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},0,\tfrac{1}{2}] \qquad (57)$$

where the variables at zero level are non-basic. It is possible to choose the link costs c^k_{ij} and d^k_{ij} such that this basic feasible solution minimizes the two-commodity network flow cost. When the link costs are

$$[1,1,2,1,1,1,1,2,2,3,2,2,2,2] \qquad (58)$$

the solution (57) minimizes the network cost, and the minimum cost is $11\tfrac{1}{2}$. Since the link costs (58) are integral, and the minimum cost is $11\tfrac{1}{2}$, it is clear that any other optimal solution to this problem <u>must be fractional</u>. In fact (57) is the unique minimum cost solution.

Figure 8 represents the problem (56) and its solution (57) on the network. If e^k_{ij} is the flow of commodity k (k=1,2) on the empty link (i,j), this two-commodity flow is represented on the network as the ordered pair of numbers (e^1_{ij},e^2_{ij}) attached to the link (i,j). Similarly the ordered pair of numbers (f^1_{ij},f^2_{ij}) represents the two-commodity flow on the full link (i,j). The flow of each commodity may not be conserved at all nodes of the network. If the number of units of commodity k $(k=1,2)$ created at node i is b_i^k, node i is assigned the ordered pair of numbers b_i^1, b_i^2. In the example (56), commodity 1 flow is conserved at every node. But 1 unit of commodity 2 is created at node 1 (i.e. $b_1^2 = +1$), and 1 unit of commodity 2 is destroyed at node 3 (i.e. $b_3^2 = -1$). Thus in Figure 8, node 1 is assigned the ordered pair of numbers $0,+1$ and node 3 the pair $0,-1$. Nodes 2 and 4 are not assigned pairs of numbers, it being understood that the flow of each commodity is conserved at both these nodes. Finally, if (i,j) is a full link, it is necessary to indicate that the two-commodity flow (f_{ij}^1, f_{ij}^2) must satisfy

$$f_{ij}^1 + f_{ij}^2 = a_{ij}^0.$$

This is achieved by assigning to each full link (i,j) the number $\{a_{ij}^0\}$. Thus in Figure 8, the full links $(1,2)$, $(3,1)$ and $(4,3)$ are assigned the numbers $\{2\}$, $\{1\}$ and $\{2\}$ respectively.

Since a two-commodity problem with the mathematical form (49) - (51) may have a fractional solution, it would be expected that the two-commodity problem (29)-(32) would have a fractional solution in general. However in the following section it is shown that under certain circumstances the solution of (29) - (32) will be integral.

3.4 Steady-State Solutions

The two-commodity problem (29)-(32) is a multi-period problem, since t takes values 1,2,...,T. The steady-state version of (29)-(32) is:

$$\text{Minimize} \quad \sum_k [\sum_{i,j} c^k_{ij} e^k_{ij} + \sum_{i,j} d^k_{ij} f^k_{ij}] \tag{59}$$

subject to

$$\sum_j \{e^k_{ij}+f^k_{ij}\}-\sum_j \{e^k_{ji}+f^k_{ji}\} = \sum_j a^k_{ji}-\sum_j a^k_{ij} \tag{60}$$

$$\sum_k f^k_{ij} = a^s_{ij} \tag{61}$$

$$e^k_{ij} \geq 0, \quad f^k_{ij} \geq 0. \tag{62}$$

Since $e^k_{ii} = 0$ in a steady-state system, the summations in (59) and (60) are taken over $j \neq i$. The costs c^k_{ij} and d^k_{ij} are given by

$$c^k_{ij} = \gamma_{ij} + s_{ij}\alpha^k \tag{63}$$

and

$$d^k_{ij} = t_{ij}\alpha^k, \tag{64}$$

where it is assumed that $\gamma_{ij} \geq 0$, $\alpha^k > 0$ and $t_{ij} > s_{ij} > 0$.

In this section it is shown that for certain network configurations, the two-commodity steady-state problem (59)-(64) has integral optimal solutions. In fact arbitrary costs c^k_{ij} and d^k_{ij} are allowed.

<u>Two Stockpile Steady-State Problem</u>

A very macroscopic model of the Australia-Europe system might involve just two stockpiles, one representing Australia, the other Europe. The resultant two stockpile steady-state problem may be solved fairly readily by hand calculations, and the solution affords insight into some of the factors affecting container movement patterns in the Australia-Europe system. First the two stockpile steady-state problem and its dual are formulated, and then the complete solution is obtained and interpreted with reference to the Australia-Europe system.

When the redundant equation for stockpile 2 is omitted, the two stockpile steady-state problem reads:

$$\text{Minimize} \quad \sum_k [c^k_{12}e^k_{12}+d^k_{12}f^k_{12}+c^k_{21}e^k_{21}+d^k_{21}f^k_{21}] \tag{65}$$

subject to

$$e^k_{12}+f^k_{12}-e^k_{21}-f^k_{21} = a^k_{21}-a^k_{12} \tag{66}$$

$$\sum_k f^k_{12} = a^s_{12} \tag{67}$$

$$\sum_k f^k_{21} = a^s_{21} \tag{68}$$

with all variables required to be non-negative.

The dual linear program, obtained by assigning multipliers π^k, θ_{12}, θ_{21} to (66), (67), (68) respectively, is:

Maximize $[a^3_{12}\theta_{12}+a^3_{21}\theta_{21} + \sum_k(a^k_{21}-a^k_{12})\pi^k]$

subject to

$$\pi^k \leq c^k_{12} \tag{69}$$

$$-\pi^k \leq c^k_{21} \tag{70}$$

$$\pi^k+\theta_{12} \leq d^k_{12} \tag{71}$$

$$-\pi^k+\theta_{21} \leq d^k_{21} \tag{72}$$

with the variables π^k, θ_{12}, θ_{21} unrestricted.

In the following, assume without loss of generality that $\alpha^2 \geq \alpha^1$ and $a^2_{21} - a^2_{12} \geq 0$. In the Australia-Europe system, this corresponds to assuming that commodities 1,2 represent general, insulated containers respectively, and that stockpiles 1,2 represent Europe, Australia respectively

Lemma 5

When $\alpha^2 \geq \alpha^1$, $a^2_{21} - a^2_{12} \geq 0$ and

$$a^1_{21} + a^3_{21} - a^1_{12} - a^3_{12} \geq 0 \tag{73}$$

an optimal solution to (65) - (72) is:

$e^1_{12} = a^1_{21}+a^3_{21}-a^1_{12}-a^3_{12}$ $\qquad$ $e^2_{12} = a^2_{21}-a^2_{12}$

$f^1_{12} = a^3_{12}$ $\qquad$ $f^2_{12} = 0$

$e^1_{21} = 0$ $\qquad$ $e^2_{21} = 0$

$f^1_{21} = a^3_{21}$ $\qquad$ $f^2_{21} = 0$

$\pi^1 = c^1_{12}$ $\qquad$ $\pi^2 = c^2_{12}$

$\theta_{12} = d^1_{12}-c^1_{12}$ $\qquad$ $\theta_{21} = d^1_{21}+c^1_{12}$.

Proof of Lemma 5

Since c^k_{ij} and d^k_{ij} are given by (63) and (64), it is easy to verify that the above relations give primal and dual feasible solutions which have the same objective value. From the Duality Theorem of Linear Programming (ch.6, [7]) it follows that the solutions are optimal. #

Lemma 6

When $\alpha^2 \geq \alpha^1$, $a^2_{21} - a^2_{12} \geq 0$ and

$$a^1_{21} + a^3_{21} - a^1_{12} - a^3_{12} < 0 \qquad (74)$$

and

$$d^2_{12} - c^2_{12} \geq d^1_{12} + c^1_{21} \qquad (75)$$

an optimal solution to (65) - (72) is:

$e^1_{12} = 0$ $\qquad e^2_{12} = a^2_{21}-a^2_{12}$

$f^1_{12} = a^3_{12}$ $\qquad f^2_{12} = 0$

$e^1_{21} = a^1_{12}+a^3_{12}-a^1_{21}-a^3_{21}$ $\qquad e^2_{21} = 0$

$f^1_{21} = a^3_{21}$ $\qquad f^2_{21} = 0$

$\pi^1 = -c^1_{21}$ $\qquad \pi^2 = c^2_{12}$

$\theta_{12} = d^1_{12}+c^1_{21}$ $\qquad \theta_{21} = d^1_{21}-c^1_{21}$.

Lemma 6 may be proved in the same way as Lemma 5.

Lemma 7

When $\alpha^2 \geq \alpha^1$, $a^2_{21} - a^2_{12} \geq 0$ and

$$a^1_{21} + a^3_{21} - a^1_{12} - a^3_{12} < 0$$

and

$$d^2_{12} - c^2_{12} < d^1_{12} + c^1_{21} \qquad (76)$$

an optimal solution to (65) - (68) is:

$e^1_{12} = 0$ $\qquad$ $e^2_{12} = a^2_{21}-a^2_{12}-m$

$f^1_{12} = a^3_{12}-m$ $\qquad$ $f^2_{12} = m$

$e^1_{21} = a^1_{12}+a^3_{12}-a^1_{21}-a^3_{21}-m$ $\qquad$ $e^2_{21} = 0$

$f^1_{21} = a^3_{21}$ $\qquad$ $f^2_{21} = 0$

where $m = \min\{a^3_{12}, a^2_{21}-a^2_{12}, a^1_{12}+a^3_{12}-a^1_{21}-a^3_{21}\}$.

An optimal solution to (69) - (72) is:

<u>$m = a^3_{12}$</u>

$\pi^1 = -c^1_{21}$ $\qquad$ $\pi^2 = c^2_{12}$

$\theta_{12} = d^2_{12}-c^2_{12}$ $\qquad$ $\theta_{21} = d^1_{21}-c^1_{21}$

<u>$m = a^2_{21}-a^2_{12}$</u>

$\pi^1 = -c^1_{21}$ $\qquad$ $\pi^2 = d^2_{12}-d^1_{12}-c^1_{21}$

$\theta_{12} = d^1_{12}+c^1_{21}$ $\qquad$ $\theta_{21} = d^1_{21}-c^1_{21}$

<u>$m = a^1_{12}+a^3_{12}-a^1_{21}-a^3_{21}$</u>

$\pi^1 = d^1_{12}-d^2_{12}+c^2_{12}$ $\qquad$ $\pi^2 = c^2_{12}$

$\theta_{12} = d^2_{12}-c^2_{12}$ $\qquad$ $\theta_{21} = d^1_{21}+d^1_{12}-d^2_{12}+c^2_{12}$.

Lemma 7 may be proved in the same way as Lemma 5.

In the Australia-Europe system (with stockpiles 1,2 representing Europe, Australia respectively), the constraint (73) in Lemma 5 corresponds to the assumption that Australia exports more type 1 and type 3 cargo than she imports. Thus Lemma 5 considers a case of no practical interest in the Australia-Europe system. In Lemmas 6 and 7, the realistic constraint (74) replaces (73).

Essentially only two cases arise:

(i) It is not worthwhile to pack type 3 cargo in insulated containers when (75) holds (Lemma 6);

(ii) It is worthwhile to pack type 3 cargo in insulated containers when (76) holds (Lemma 7).

Note that in view of (63) and (64), the constraint (75) reads

$$t_{12}\alpha^2 - (\gamma_{12}+s_{12}\alpha^2) \geq t_{12}\alpha^1 + (\gamma_{21}+s_{21}\alpha^1)$$

i.e.

$$(t_{12}-s_{12})\alpha^2 \geq \gamma_{12}+\gamma_{21}+(t_{12}+s_{21})\alpha^1. \qquad (77)$$

The relation (77) holds only when the insulated container inventory cost is very expensive relative to the cost of empty movements and the general container inventory cost. On the basis of the data given in chapter 2, it is reasonable to take

$$s_{12} = s_{21} = 4 \text{ (weeks)}$$

$$t_{12} = 6 \text{ (weeks)}$$

$$\gamma_{12} = \gamma_{21} = 40 \text{ (dollars)}$$

and $$\alpha^1 = 5 \text{ (dollars per week)}.$$

Then (77) holds only when

$$\alpha^2 \geq 65 \text{ (dollars per week)}.$$

In fact the insulated container inventory cost is only 10 dollars per week, and so the constraint (76) is realistic for the Australia - Europe system and Lemma 7 is applicable.

On the basis of the approximate figures per voyage given in section 2.5, it is easy to see that in Lemma 7, $m = a^2_{21} - a^2_{12}$. For $a^1_{12} = a^2_{12} = 600$, $a^1_{21} = 800$, $a^2_{21} = 0$, $a^2_{12} = 10$ and $a^2_{21} = 210$, so that

$$\begin{aligned} m &= \min\{600,\ 210-10,\ 600+600-800-0\} \\ &= \min\{600,\ 200,\ 400\} \\ &= 200. \end{aligned}$$

Thus the Lemma 7 solution with $m = a^2_{21} - a^2_{12}$ corresponds to the heuristic solution suggested in section 2.5.

It should be noted that the optimal solution to the two stockpile two-commodity steady-state problem - as given by Lemmas 5,6 and 7 - is always integral. In a later part of this section, it is shown that the two stockpile problem is just a special case from a general class of multi-commodity steady-state problems which have integral optimal solutions.

Three Stockpile Steady-State Problem

In the most general three stockpile steady-state problem, the network is that depicted in Figure 9. For such a problem, it is shown that the two-commodity matrix A defined by (49) is totally unimodular, and consequently the three stockpile steady-state problem has integral optimal solutions.

In Figure 9, the node-link incidence matrices of the subnetworks comprising empty links and full links respect-

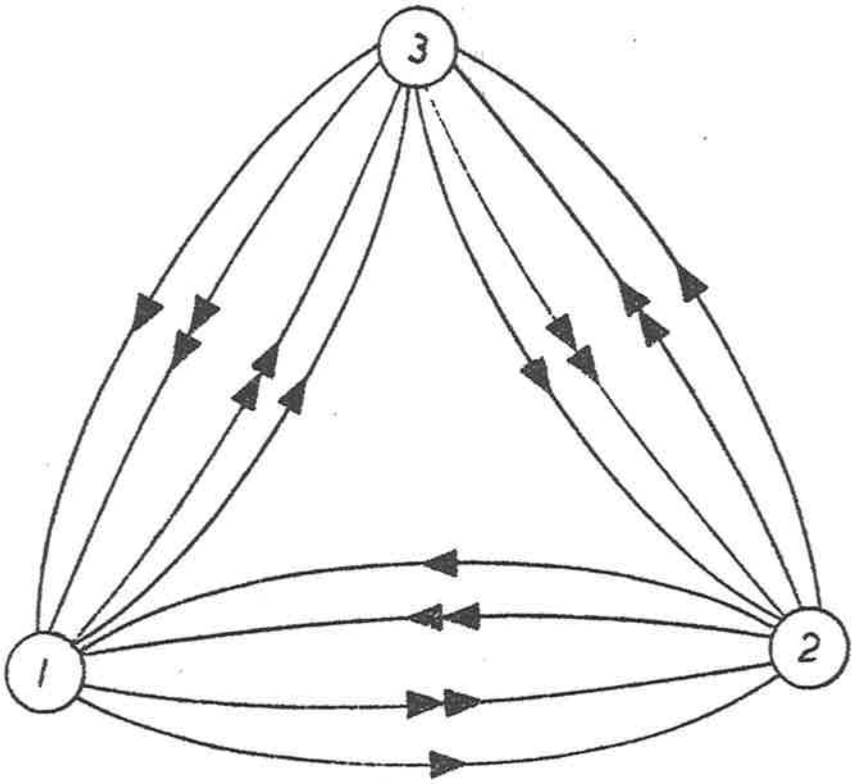

FIGURE 9.

Network for general three stockpile steady-state problem.

ively, are identical and given by

		links					
		(1,3)	(3,1)	(2,3)	(3,2)	(1,2)	(2,1)
	1	1	-1	0	0	1	-1
nodes	2	0	0	1	-1	-1	1
	3	-1	1	-1	1	0	0 .

(It is convenient here not to order the links lexicographically.)

When the redundant row corresponding to node 3 is deleted, the matrices E and F are given by

$$E = F = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}. \tag{78}$$

<u>Theorem 6</u>

When the matrices E,F are given by (78), the matrix A defined by

$$A = \begin{bmatrix} E & F & 0 & 0 \\ 0 & 0 & E & F \\ 0 & I & 0 & I \end{bmatrix}$$

is totally unimodular.

<u>Proof of Theorem 6</u>

The matrix A is totally unimodular if and only if the matrix A′ defined by

$$A' = \begin{bmatrix} E & F & 0 & 0 \\ 0 & 0 & E & F \\ 0 & I' & 0 & I' \end{bmatrix}$$

is totally unimodular, where

$$I' = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(Note that A' is obtained from A by multiplying two rows of A by -1. This does not change the determinants of any square submatrices, except perhaps for sign.) It is now proved by induction that if A_k is any $k \times k$ submatrix of A', then $\det A_k = \pm 1, 0$.

Clearly $\det A_1 = \pm 1, 0$ since every element of A' is $\pm 1, 0$. Now assume that $\det A_{k-1} = \pm 1, 0$ for every $(k-1) \times (k-1)$ submatrix A_{k-1} of A', and let A_k be any $k \times k$ submatrix of A'. If any column (or any row) of A_k is all zeros, $\det A_k = 0$. If any column (or any row) of A_k contains just one non-zero, then expand $\det A_k$ by that column (or row) and obtain $\det A_k = \pm 1 \det A_{k-1}$ where A_{k-1} is the cofactor of the non-zero entry, and has determinant $\pm 1, 0$ by hypothesis. Thus it may be assumed that

every column and every row of A_k contains at least two non-zero entries. (79)

Consequently, the columns of E corresponding to the empty links (1,3), (3,1), (2,3) and (3,2) are not present in A_k, and A_k is a submatrix of

$$
H_1 = \begin{array}{c|cccccccccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\hline
1 & 1 & -1 & 1 & -1 & 0 & 0 & 1 & -1 & & & & & & & & \\
2 & -1 & 1 & 0 & 0 & 1 & -1 & -1 & 1 & & & & & & & & \\
3 & & & & & & & & & 1 & -1 & 1 & -1 & 0 & 0 & 1 & -1 \\
4 & & & & & & & & & -1 & 1 & 0 & 0 & 1 & -1 & -1 & 1 \\
5 & & & -1 & & & & & & & & -1 & & & & & \\
6 & & & & 1 & & & & & & & & 1 & & & & \\
7 & & & & & -1 & & & & & & & & -1 & & & \\
8 & & & & & & 1 & & & & & & & & 1 & & \\
9 & & & & & & & 1 & & & & & & & & 1 & \\
10 & & & & & & & & 1 & & & & & & & & 1
\end{array} .
$$

Note that columns 1-6 and 9-14 of H_1 each contain exactly two non-zero entries of opposite sign, while columns 7,8,15, 16 each contain three non-zero entries.

Since A_k satisfies (79), if column 3 of H_1 is involved in A_k, then so is the corresponding column 11, and similarly for columns 4 and 12, 5 and 13, 6 and 14.

It is now convenient to define

$$\mathcal{R}(A_k) = \{i : A_k \text{ involves row } i \text{ of } H_1\}$$
$$\mathcal{C}(A_k) = \{j : A_k \text{ involves column } j \text{ of } H_1\}$$

and to let $\mathcal{R}'(A_k)$, $\mathcal{C}'(A_k)$ denote the complements of the sets $\mathcal{R}(A_k)$, $\mathcal{C}(A_k)$. Then because of the symmetry of H_1, there are essentially just five cases to consider.

(a) $\mathcal{R}(A_k) \supseteq \{1,2,3,4\}$ i.e. A_k involves (at least) rows 1,2,3,4 of H_1. Then since A_k satisfies (79), the sum of all rows i for which

$$i \in \mathcal{R}(A_k) \quad \text{and} \quad 1 \leqslant i \leqslant 8,$$

is zero, and so $\det A_k = 0$.

(b) $\mathcal{R}(A_k) \supseteq \{1,2,3\}$ and $\mathcal{R}'(A_k) \supseteq \{4\}$.

In this case, A_k does not involve columns 5,6,9,10,13,14 and rows 7,8 of H_1 and so A_k is a submatrix of

$$H_2 = \begin{array}{c|cccccccccc|} & 1 & 2 & 3 & 4 & 7 & 8 & 11 & 12 & 15 & 16 \\ \hline 1 & 1 & -1 & 1 & -1 & 1 & -1 & & & & \\ 2 & -1 & 1 & 0 & 0 & -1 & 1 & & & & \\ 3 & & & & & & & 1 & -1 & 1 & -1 \\ 5 & & & -1 & & & & -1 & & & \\ 6 & & & & 1 & & & & 1 & & \\ 9 & & & & & 1 & & & & 1 & \\ 10 & & & & & & 1 & & & & 1 \\ \hline \end{array} . \qquad (80)$$

Now if $\mathscr{C}(A_k) \supseteq \{3,4,11,12\}$, then since col.3 + col.4 = col.11 + col.12, det A_k = 0. Otherwise suppose that columns 4 and 12 are not involved in A_k. Then when row 1 of A_k is replaced by the sum of all rows i for which

$$i \in \mathcal{R}(A_k) \quad \text{and} \quad i \notin \{3,9,10\},$$

det A_k is unaltered, and the new row 1 has all zeros, except for a -1 in the column 11 position. So det $A_k = \pm 1 \times$ det A_{k-1} and the inductive hypothesis gives det $A_k = \pm 1, 0$.

(c) $\mathcal{R}(A_k) \supseteq \{1,3\}$ and $\mathcal{R}'(A_k) \supseteq \{2,4\}$.

In this case, A_k is a submatrix of

$$H_3 = \begin{array}{c|cccccccc|} & 3 & 4 & 7 & 8 & 11 & 12 & 15 & 16 \\ \hline 1 & 1 & -1 & 1 & -1 & & & & \\ 3 & & & & & 1 & -1 & 1 & -1 \\ 5 & -1 & & & & -1 & & & \\ 6 & & 1 & & & & 1 & & \\ 9 & & & 1 & & & & 1 & \\ 10 & & & & 1 & & & & 1 \\ \hline \end{array} .$$

But the matrix H_3 is totally unimodular, for when row 9 is multiplied by -1, every column contains two non-zeros of opposite sign, and Theorem 5 applies. So in this case too, $\det A_k = \pm 1, 0$.

(d) $\mathcal{R}(A_k) \supseteq \{2,3\}$ and $\mathcal{R}'(A_k) \supseteq \{1,4\}$.

When it is remembered that A_k satisfies (79), consideration of (80) shows that in this case A_k must be precisely the matrix

$$H_4 = \begin{array}{c} \\ 2 \\ 3 \\ 9 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccc} 7 & 8 & 15 & 16 \end{array} \\ \left[\begin{array}{cccc} -1 & 1 & & \\ & & 1 & -1 \\ 1 & & 1 & \\ & 1 & & 1 \end{array}\right] \end{array} .$$

But the sum of the first two columns of H_4 equals the sum of the last two columns, and so $\det H_4 = 0$.

(e) $\mathcal{R}'(A_k) \supseteq \{3,4\}$.

Consideration of (80) shows that A_k must be a 2×2 sub-matrix of the matrix

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 7 & 8 \end{array} \\ \left[\begin{array}{cccc} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{array}\right] \end{array} ,$$

and hence $\det A_k = 0$.

This completes the inductive proof that A', and hence A, is totally unimodular. #

Four Stockpile Steady-State Problem

In section 3.3, a four stockpile problem (Figure 8) was given which has a unique fractional optimal solution. Thus in general, two-commodity steady-state problems involving four or more stockpiles have fractional optimal solutions.

A Special Class of Steady-State Problems

It is now shown that there is a special class of steady-state problems which have integral optimal solutions. For any problem in the class, the subnetwork of full links is assumed to have the following property:

> there is a distinguished node in the network such that every full link either originates or terminates at this node. (81)

If it is assumed that the distinguished node is node 1, and the redundant equation corresponding to node 1 is omitted, any problem in the class may be formulated as follows:

$$\text{Minimize} \quad \sum_k \Big[\sum_{i,j} c^k_{ij} e^k_{ij} + \sum_i \{ d^k_{i1} f^k_{i1} + d^k_{1i} f^k_{1i} \} \Big] \tag{82}$$

subject to

$$f^k_{i1} - f^k_{1i} + \sum_j e^k_{ij} - \sum_j e^k_{ji} = a^k_{1i} - a^k_{i1} \tag{83}$$

$$\sum_k f^k_{i1} = a^3_{i1} \tag{84}$$

$$\sum_k f^k_{1i} = a^3_{1i} \tag{85}$$

with all variables required to be non-negative.

In (82) - (85), i takes values $2,3,\ldots$ and in (82) and (83) the summations are taken over the values $j=1,2,3,\ldots$.

<u>Theorem 7</u>

> The two-commodity problem (82) - (85) is equivalent to a one-commodity problem, and hence has integral optimal solutions.

<u>Proof of Theorem 7</u>

Replace the constraint (84) by the equivalent constraint

$$-\sum_k f^k_{i1} = -a^3_{i1} \tag{86}$$

and sum equations (83), (85) and (86) over i (for $i = 2,3,\ldots$) and over k. This yields

$$\sum_k \sum_i (e^k_{i1}-e^k_{1i}) = \sum_k \sum_i (a^k_{1i}-a^k_{i1}) + \sum_i (a^3_{1i}-a^3_{i1})$$

or

$$\sum_k \sum_i (e^k_{1i}-e^k_{i1}) = \sum_k \sum_i (a^k_{i1}-a^k_{1i}) + \sum_i (a^3_{i1}-a^3_{1i}). \tag{87}$$

The problem (82), (83), (85), (86) and (87) is a one-commodity network flow problem which is equivalent to the two-commodity problem (82) - (85). Each variable e^k_{ij}, f^k_{i1}, f^k_{1i} occurs in just two of the equations (83), (85) - (87) and has coefficient +1 in one equation and -1 in the other. #

Figure 10 depicts a simple three stockpile example for which the subnetwork of full links has property (81), and Figure 11 shows the equivalent one-commodity network flow problem. In Figure 11 it is assumed that

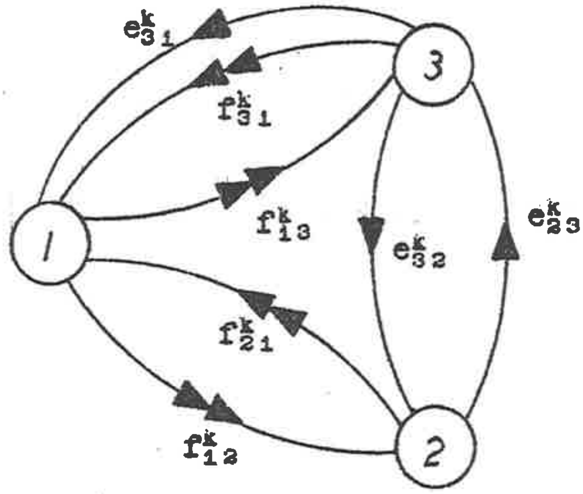

FIGURE 10.

Network for which every full link originates or terminates at node 1.

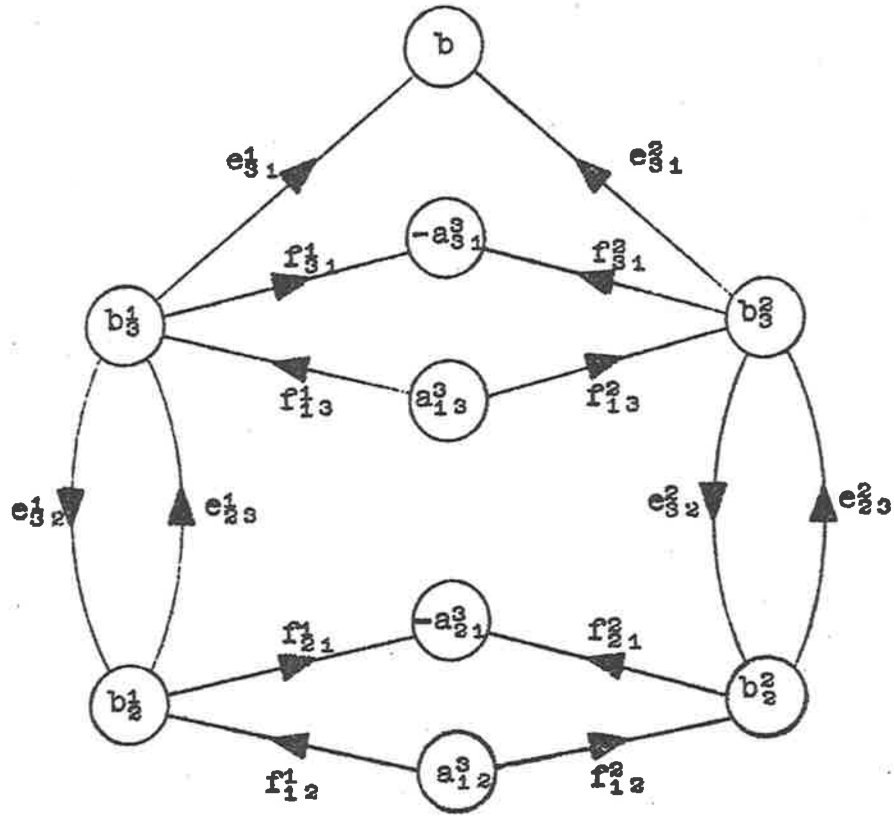

FIGURE 11.

One-commodity network flow problem which is equivalent to two-commodity problem on Fig.10 network.

$$b_i^k = a^k_{1i} - a^k_{i1}$$

$$b = \sum_k \sum_i (a^k_{i1} - a^k_{1i}) + \sum_i (a^3_{i1} - a^3_{1i}),$$

and that the number assigned to each node is the number of units of flow created at that node.

In the problem (82) - (85), k takes values 1,2. Theorem 7 generalizes to the K-commodity problem where k takes values $1,2,\ldots,K$ in (82), (83) and where (84) and (85) are replaced by

$$\Sigma_{k=1}^{K} f^k_{i1} = a^{K+1}_{1i}$$

$$\Sigma_{k=1}^{K} f^k_{1i} = a^{K+1}_{i1}.$$

Note that the two stockpile problem solved in Lemmas 5-7 is a special case of (82) - (85), but the most general form of the three stockpile steady-state problem is not. In fact while it is true that the general two-commodity three stockpile problem has integral optimal solutions (as proved in Theorem 6), the corresponding <u>three</u>-commodity problem does not. Figure 12 shows a three-commodity example which is similar to the example of Ford and Fulkerson (mentioned in section 3.3). Set

$$c^1_{12} = d^1_{23} = d^1_{31} = 1$$

$$c^2_{23} = d^2_{31} = d^2_{12} = 1$$

$$c^3_{31} = d^3_{12} = d^3_{23} = 1$$

and let all other c^k_{ij}, d^k_{ij} be large positive integers. Then the unique minimum cost solution is shown on the

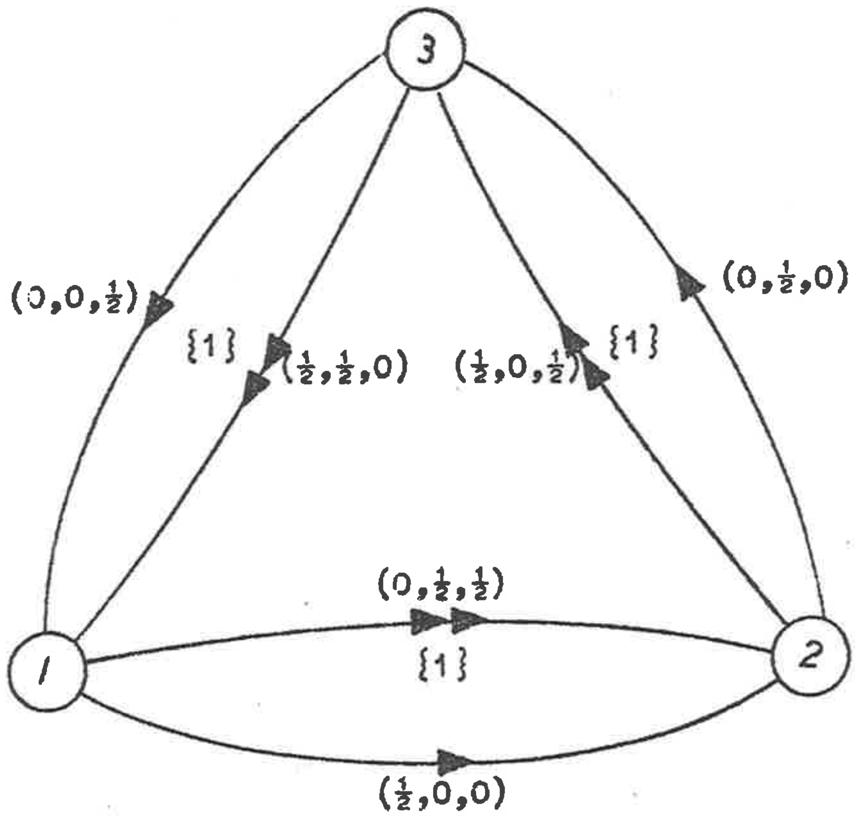

FIGURE 12.

Three stockpile three-commodity fractional example.

Figure 12 network, and the minimum cost is $4\frac{1}{2}$.

In the Australia - Europe container system, there was originally just one European port of call - Tilbury - and there were several Australian ports of call. In that system, all full container movements either originated or terminated at Tilbury and consequently the resultant steady-state container network possessed the property (81). However in the present Australia - Europe system, there are several European ports of call and several Australian ports of call. The property (81) does not apply, and in general the two-commodity steady-state problem has fractional optimal solutions.

3.5 Multi-Period Solutions

In this section it is shown that in general the optimal solution of the two-commodity multi-period problem (29) - (32) is fractional, even when the corresponding steady-state problem has an integral optimal solution. The examples given here are based on realistic travel times and costs for the Australia - Europe system, but involve rather simplified versions of the container network. It has been possible to obtain optimal solutions to these rather simple two-commodity multi-period problems fairly readily using a linear programming package.

Figure 13 depicts a two stockpile multi-period problem and its fractional optimal solution. As in section 3.4, stockpiles 1,2 represent Europe, Australia respectively and it is assumed that

$$\begin{aligned} s_{12} &= s_{21} = 4 \text{ (weeks)} \\ t_{12} &= t_{21} = 6 \text{ (weeks)} \\ \gamma_{12} &= \gamma_{21} = 40 \text{ (dollars)} \\ \alpha^1 &= 5,\ \alpha^2 = 10 \text{ (dollars per week).} \end{aligned} \tag{88}$$

Note that the link costs are integral, and the minimum cost is $332\frac{1}{2}$, so that any alternative optimal solutions are also fractional.

In the Figure 13 example, type 3 cargo is sent from stockpile 1 to stockpile 2, and from stockpile 2 to stockpile 1. In the example shown in Figure 14, there is no

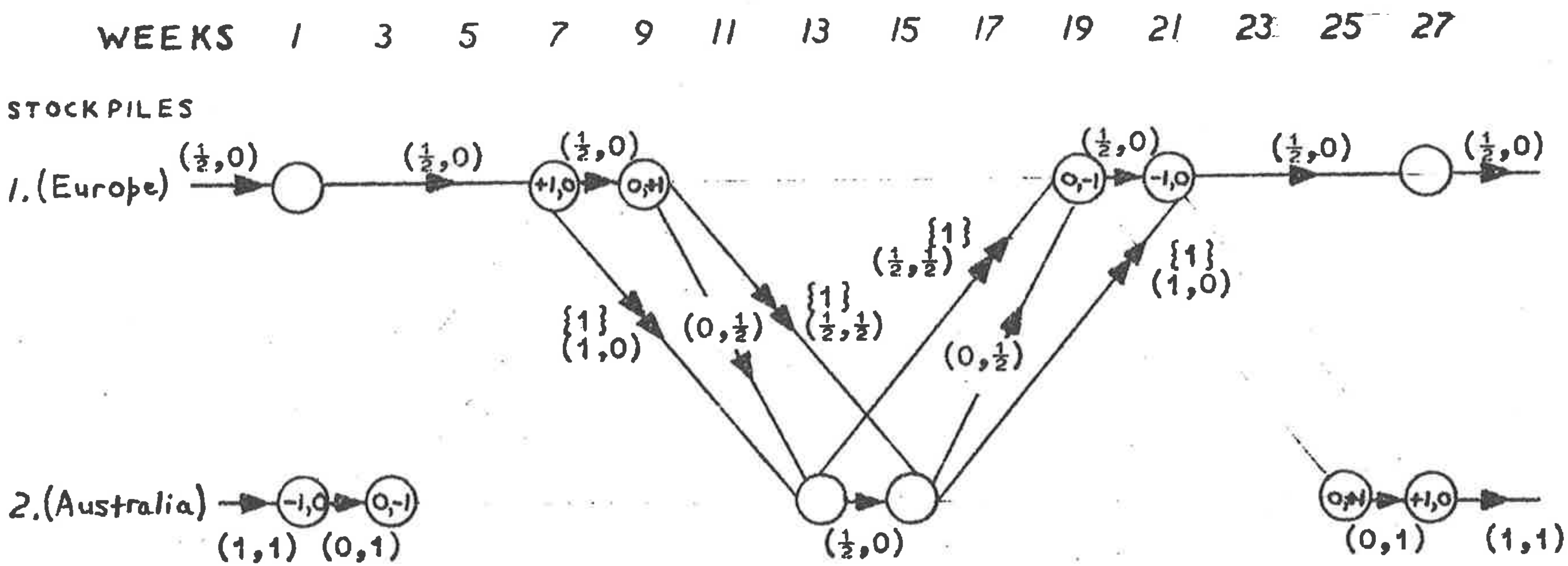

FIGURE 13.

Two stockpile two-commodity multi-period fractional example.
(Only those links on which non-zero flow occurs are shown.)

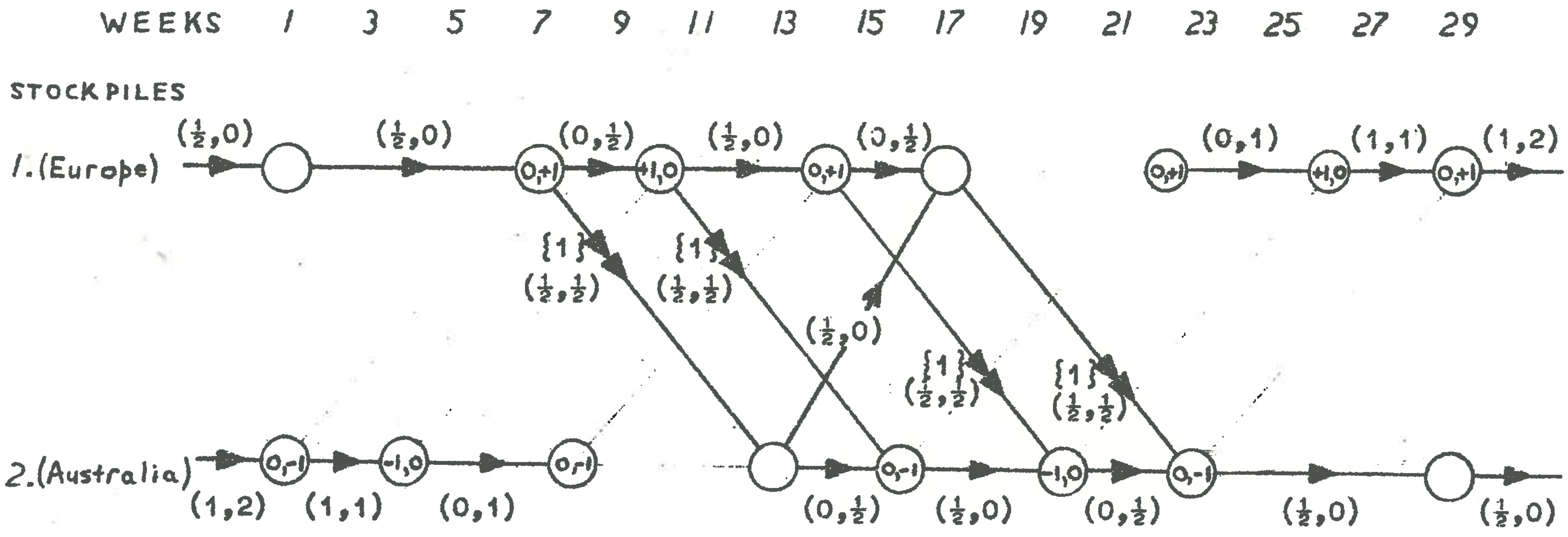

FIGURE 14.

Two stockpile two-commodity multi-period fractional example.
No type 3 cargo from stockpile 2 to stockpile 1.

type 3 cargo from stockpile 2 to stockpile 1. This corresponds to the situation in the Australia - Europe system, where there is virtually no type 3 cargo from Australia to Europe. In Figure 14 the minimum-cost solution is again fractional, and the minimum cost is $517\frac{1}{2}$.

Figure 15 shows an example which is similar to the Figure 14 example, but involves three stockpiles. Stockpiles 1,2,3 represent Tilbury, Sydney and Melbourne respectively. In addition to the travel times and costs given by (88), it is assumed that

$$s_{23} = 1 \text{ (week)}$$
$$t_{13} = 7, \; t_{31} = 6 \text{ (weeks)}$$
$$\gamma_{23} = 40 \text{ (dollars)}.$$

Note that there is no type 3 cargo from either Sydney or Melbourne to Tilbury. The minimum-cost solution is fractional, and the minimum cost is $592\frac{1}{2}$.

Thus where a steady-state problem has integral optimal solutions, the corresponding multi-period problem may have fractional optimal solutions. On the other hand, the examples given in Figures 13, 14, 15 do not involve realistic cargo movements (although travel times and costs <u>are</u> realistic). While the examples show that in general the optimal solution of a multi-period problem is fractional, it may easily happen that for a particular set of cargo movements, the optimal solution is integral.

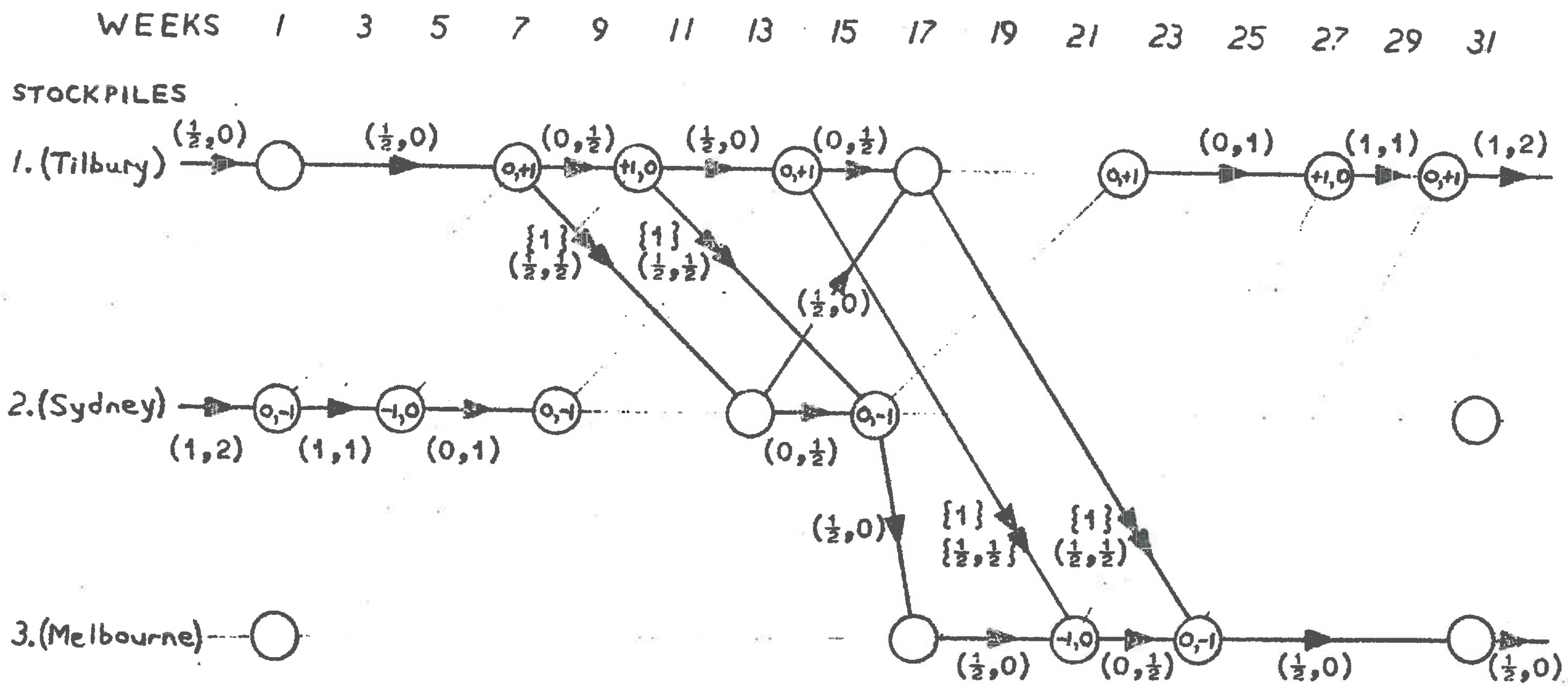

FIGURE 15.

Three stockpile two-commodity multi-period fractional example.
No type 3 cargo from stockpiles 2,3 to stockpile 1.

CHAPTER 4. CONCLUSION

This thesis has formulated and solved a practical problem arising in a container shipping system. The problem of determining container inventory and movements in the Australia-Europe system has been formulated as a two-commodity network flow model, with the variables required to take integral values. The difficulties inherent in solving large two-commodity flow problems have been avoided by using an efficient heuristic procedure which obtains near-optimal solutions. Computer programs have been written to implement the heuristic procedure and used to obtain realistic answers to many practical question about the Australia - Europe container shipping system.

The model has the advantage that it may readily be generalized to consider systems involving more than two container types, although it is not clear that the resultant multi-commodity flow problem could be solved efficientl by heuristic means. While it has been assumed in the model description that the basic unit of time is one week, the formulation allows any time unit. With a basic time unit of a day, the model might prove useful in answering detailed questions of short-term container control.

Perhaps the major limitation of the model is that it is deterministic. In this respect the model is less general than that of Horn [20], who allows probabilistic cargo demands. But for a container shipping system, it is difficult to obtain cargo estimates and almost impossible to obtain meaningful probability density functions representing cargo demands. On the macroscopic scale at which the model operates, it seems preferable to use deterministic cargo demands, and to conduct sensitivity analyses to investigate the effects of cargo variations. As noted in section 2.7, such analyses may be conducted very rapidly using the heuristic solution procedure. In any case, the introduction of probabilistic cargo demands into even a one-commodity model of the Australia - Europe container system cannot be handled in the elegant fashion of Horn, since different assumptions are involved. The principal differences in Horn's paper are the assumption that there is an alternative (expensive) non-container mode which may be used if necessary to carry some cargo (the "cargo" is mail in the system considered by Horn), and the assumption of equal travel times for full and empty containers.

While the heuristic procedure provides a near-optimal solution of the two-commodity problem, one area for future work would be to program an algorithm to obtain the optimal solution. It would be interesting to compare the application of the various approaches suggested in the literature to this problem. Jewell's generalized out-of-kilter approach [29] would seem to be particularly suitable.

Whichever approach was employed, the heuristic procedure could be used to provide a good starting point. In fact it is not clear that intelligent application of a large-scale linear programming package might not provide an efficient means of solution. The two-commodity matrix is very sparse and suited for a package based on the revised simplex algorithm, using the product form of the inverse (ch.9, [7]).

But even if the optimal solution to the two-commodity problem could be obtained reasonably quickly using one of these methods, there would still remain the problem of non-integral solutions. Chapter 3 of the thesis has shown that in general the optimal solution to the two-commodity problem will be non-integral. It does seem possible that under restrictive conditions on cargo imbalances and costs (which might be satisfied in the Australia - Europe system), a class of two-commodity multi-period problems with integral optimal solutions could be established, and this is another direction in which further research could proceed. However for all practical purposes, the problem of solving the two-commodity problem may be considered finished, since the heuristic procedure provides completely realistic and adequate integral solutions.

It has already been noted that the introduction of containerisation into the field of shipping transportation has resulted in considerable system simplification. While it is true that a network flow approach could be used to model the movement of containers by irregular transport modes, the formulation of section 2.4 relies upon the basic periodicity of the various transport modes between stockpiles. It would seem that the basic simplicity of the containerisation system should allow a profitable study of other problems to be undertaken. Thus in conclusion, it is suggested that the introduction of containerisation has opened up a new field for the application of operations research techniques to significant practical problems.

APPENDIX. DEFINITIONS AND RESULTS FROM NETWORK THEORY

A directed network $[N;L]$ consists of a finite set N of unordered elements called nodes, and a set L of ordered pairs of elements of N called links.

A subnetwork of $[N;L]$ is a directed network $[N';L']$ such that $N' \subseteq N$ and $L' \subseteq L$.

If $(n_1, n_2) \in L$, the link (n_1, n_2) is said to originate at the node n_1 and terminate at the node n_2.

A path between nodes n_1 and n_r is a sequence

$$n_1, \ell_1, n_2, \ell_2, \ldots, n_{r-1}, \ell_{r-1}, n_r$$

of distinct nodes $n_i (i = 1,2,\ldots,r)$ and distinct links $\ell_i (i = 1,2,\ldots,r-1)$ such that

$$\ell_i = (n_i, n_{i+1}) \in L$$

$$\text{or} \quad \ell_i = (n_{i+1}, n_i) \in L.$$

A directed network is connected if there is a path between every pair of nodes in the network.

Suppose that a network $[N;L]$ comprises n nodes and ℓ links. The node-link incidence matrix associated with $[N;L]$ is an $n \times \ell$ matrix where the element in the row corresponding to node n_i and in the column corresponding to link ℓ_j is

+1 is ℓ_j originates at n_i

-1 if ℓ_j terminates at n_i

0 otherwise.

Thus each column of a node-link incidence matrix contains just two non-zero entries, which are +1 and -1.

The rows of a node-link incidence matrix are linearly dependent because if the rows are $\underset{\sim}{r}_1, \underset{\sim}{r}_2, \ldots, \underset{\sim}{r}_n$ then

$$\underset{\sim}{r}_1 + \underset{\sim}{r}_2 + \ldots + \underset{\sim}{r}_r = \underset{\sim}{0}.$$

The node-link incidence matrix of a connected network with n nodes has rank (n-1). If any row of such a matrix is deleted, the remaining (n-1) rows are linearly independent

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