ALGEBRAIC METHODS FOR QUANTUM MECHANICS.

by

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SUMMARY.

I have used the term 'algebraic methods' in a strict
sense and have only considered algebras and their
(algebraically) irreducible representations on vector spaces
which are not endowed with a topology. My aim has been to
decide whether such methods are sufficient for a mathematical
description of quantum mechanics.

Algebraic methods have two fundamental limitations.
(a) In an irreducible representation of an algebra on an
infinite dimensional vector space over the complex field, it
is not necessarily true that the only operators which commute
with all the operators of the representation are multiples of
the identity.
(b) There is no analogue of the 'rotation to principal axes'
theorem used in Hilbert spaces.
These limitations dictated my plan of attack, which consists
of the following stages.
(i) Investigate a mathematical theory of quantum mechanics in
which each quantum system is characterised by its algebra of
observables.
(ii) Investigate eigenvalue problems which can be posed and
solved algebraically.

The first stage occupies only the first two chapters.
(1) Chapter 1 contains the arguments, based upon
physics, which support the postulate that an associative algebra
with involution, denoted A^*, should be associated with each
quantum system.
(2) In chapter 2 I have shown that the algebra A^* is
all that must be known. All questions concerning states of

the system and the expectation values of its observables can be formulated in terms of A^* itself.

The second stage, concerned with eigenvalue problems posed in terms of a pair of Lie algebras $K < L$ and various ramifications, occupies the rest of the thesis.

(3) Diagonal operators are rational functions of the invariants of K. If X is an irreducible L-module which is completely reducible into K-modules, then a diagonal operator on X has a diagonal matrix and its spectrum is determined by the decomposition of X. I have employed the characteristic identities for the classical Lie algebras, derived by Bracken and Green (1971) and Green (1971), to present a systematic study of diagonal operators.

(4) Codiagonal operators are polynomials in the elements of L with coefficients which are diagonal operators. When $L = sl(2,C)$, codiagonal operators on X have codiagonal matrices. I have developed a spectral theory for codiagonal operators in this case and have indicated possible extensions to other cases. The algebraic theory shows that the set of Banach space completions of X can be divided into a finite number of classes such that the spectrum of a codiagonal operator is constant within each class.

(5) Green and Triffet (1969) developed a practical method for calculating the spectra of perturbed forms of the special function operators. I have shown that the proper mathematical setting for their work is algebraic.

(6) Chapter 6 contains three examples. The first two exhibit the hypergeometric and Fuchsian differential operators as diagonal and codiagonal operators respectively, so the spectral theory of chapters 3, 4 and 5 can be applied to these

operators. The last example is concerned with the decomposition of an irreducible module for the Poincaré Lie algebra into irreducible modules for the Lorentz Lie algebra. It highlights the differences between the algebraic and analytic approaches to this problem.

(7) Chapter 7 contains a novel application of algebraic methods. To find the <u>analytic</u> continuation of the solutions of an ordinary differential equation with holomorphic coefficients, it is only necessary to solve an <u>algebraic</u> eigenvalue problem.

(8) The final chapter is concerned with Wick's equation and provides an example in relativistic quantum mechanics where it is both burdensome and unnecessary to treat relative coordinates and momenta as observables. I have shown that this is so by proving that the Wick rotation can be justified rigorously for this model.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any other university and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference is made in the text.

D.M. O'Brien.

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CHAPTER 1. INTRODUCTION.

Quantum mechanics and the theory of linear operators on Hilbert spaces are so interwoven that it is difficult to imagine one without the other. Nevertheless, historically their fusion is almost an accident. At the time when physicists were struggling towards quantum mechanics, Hilbert, Hellinger and others happened to be developing the theory of linear integral equations and hermitian forms in an infinite number of variables. Born knew of their work; consequently, he included a chapter on the eigenvalues of hermitian forms in the definitive paper on quantum mechanics, published in collaboration with Heisenberg and Jordan (1926). Von Neumann (1932) developed this material into a mathematically rigorous framework for quantum mechanics, but cast the algebraic methods of matrix mechanics into disrepute by showing (1929) that infinite matrices could not adequately represent unbounded operators on a Hilbert space.

Recent years have seen a revival of algebraic ideas for quantum mechanics and the publication of many systematic studies of algebraic methods, prompted in each case by the successful application of kinematical and dynamical symmetry groups to quantum mechanics. Notable amongst these publications are the following:

(a) the papers by Biedenharn (1971) and his collaborators on the construction of the irreducible representations of the semi-simple Lie groups;

(b) the papers of Moshinsky (1968) and collaborators on the application of algebraic methods to many body problems;

(c) Miller's (1968) text which intertwines the theory of Lie

algebras with that of the special functions and his more recent attempts with Kalnins (1975) to classify separable partial differential equations algebraically.

More important than the symmetry groups have been their Lie algebras; in fact, only for the symmetries of space and time does the group appear to have any physical significance. Lie algebras, commutation rules and representations are nowadays indispensable tools of the mathematical physicist. However, like the matrix methods of Born, Heisenberg and Jordan, many of the algebraic techniques are regarded with suspicion and often are only considered to be formally correct. In order to claim that work involving Lie algebras and representations is rigorous and relevant to quantum mechanics, an author must ensure that the operators of the representation are properly defined on a Hilbert space and that the representation of the Lie algebra can be integrated to a unitary representation of the simply connected Lie group. Only rarely do mathematical physicists perform these steps to the satisfaction of a functional analyst. Also common are papers in which the authors proceed with formal calculations, even though their results clearly demand a framework more general than the Hilbert space theory. Here I could mention the paper by Chakrabarti, Levy-Nahas and Seneor (1968) on the decomposition of representations of the Poincaré group into representations of the Lorentz group; they allow 'self-adjoint' operators to have complex eigenvalues. I think there are two reasons why mathematical physicists have confidence in such calculations and show so little regard for the fine points of analysis.

(1) The results obtained by formal calculations are 'obviously'

correct. Furthermore, experience has shown that all the 'useful' formulae can be obtained in this way.

(2) It is commonly believed that any dubious step of a formal calculation can be justified by recourse to the general results on linear operators on Hilbert spaces.

In short, there is a general confidence that formal techniques are correct, that they can be precisely formulated within a Hilbert space framework, and that therein lies their justification.

I hold a different view, based upon experience with the second of the above points. I believe that the algebraic structure of quantum mechanics should stand alone, rigorously, and that the topological structure, if indeed one is needed, should be added in a manner compatible with the algebraic structure. Thus, I would reverse the usual order which begins with the presentation of a Hilbert space of states and ends with the derivation of a certain algebra of self-adjoint operators. Similarly, the algebraic methods employed in quantum mechanics should not depend for their justification upon von Neumann's theory. I hope to show that this view is tenable.

However, there is an important question which must be answered immediately. Why should quantum mechanics rest upon an algebraic basis? This question is especially important because von Neumann's formulation of non-relativistic quantum mechanics is so successful. I can offer four reasons, the first of which is subjective.

(1) Algebraic methods are both elegant and constructive. Furthermore, the confidence that formal calculations can be justified within von Neumann's framework is naive

because, even when a justification can be found, the simplicity
of the algebraic calculations, their principal asset, is
destroyed in the process.

(2) Von Neumann postulated that the pure states of
a quantum system should comprise a separable Hilbert space and
that the observables should be the self-adjoint operators on
the space. Any two separable Hilbert spaces are unitarily
equivalent and so too are the self-adjoint operators defined
upon them. Thus, the states of any quantum system whatsoever
could be identified with the square-integrable functions on the
real line. Alternatively, the wave function of a hydrogen atom
could be expanded in a series of wave functions for the
plutonium nucleus! Von Neumann's postulates are too general.

(3) I will give another argument with the same
conclusion. Although a separable Hilbert space has an
uncountable (Hamel) basis, it is the closure of an inner product
space of countable dimension. Thus, the set of limit points,
far from being a 'small' set, adds an uncountable number of
dimensions. The important question is whether or not these
extra dimensions are needed for physics. Equivalently, is there
an experiment which can distinguish between the states

$$\psi = \sum_{n=0}^{\infty} c_n \varphi_n \quad \text{and} \quad \psi_m = \sum_{n=0}^{m} c_n \varphi_n \, ,$$

where m is a large integer, $\{\varphi_n\}$ is a complete orthonormal set
and the series converges? Of course the answer is no. However,
in the mathematical theory the limit points are needed to ensure
that self-adjoint operators can be reduced to diagonal form. I
will later argue that this is so because we adhere to a primitive
form of the correspondence principle and protest ignorance of
the true algebra of observables.

(4) Despite the generality of von Neumann's postulates, they are often too rigid. An example will help illustrate this point. Imagine a system of two particles of equal mass whose momenta are p_1 and p_2. When the particles are widely separated, p_1 and p_2 are observables and so must be self-adjoint operators. Consequently,

$$P = p_1 + p_2 \text{ and } p = p_1 - p_2$$

are also self-adjoint. There is little doubt that P should have real expectation values because P generates translations between spatially separated observers. However, p is effectively unobservable and there are many occasions in relativistic quantum mechanics in which it is more natural to allow the relative energy to have imaginary values. The solution of Wick's (1954) equation and Euclidean field theory are just two instances. Another example which demonstrates the inflexibility of von Neumann's axioms occurs in the theory of Regge poles. When the angular momentum l assumes complex values, the corresponding representations of SO(3) are not unitary and the original Hilbert space is no longer appropriate.

It seems to me that von Neumann's postulate that the states should comprise a Hilbert space serves just one purpose: the space of states is so large that it is always possible to find an orthonormal basis in which a given self-adjoint operator is diagonal. In bound state problems this operator would be the Hamiltonian H; in scattering problems it would be the scattering kernel, $G_o V$ in common notation. Once H has been reduced to a diagonal matrix, a small number of operators can be identified, namely, operators which commute with H and which account for the degeneracy of its eigenvalues, and operators which map vectors

from one degenerate multiplet to another. These are the operators which seem important for physics, and all the observables can be constructed from them. The rest of the self-adjoint operators are irrelevant for the physical system under discussion. For example, the magnetic moment operator of the plutonium nucleus and the Hamiltonian operator for hydrogen act upon the same Hilbert space! Similar remarks apply to scattering processes. Von Neumann had to choose such a large class of 'observables' solely to ensure that the true observables could be constructed within this class. Thus, the Hilbert space and the self-adjoint operators defined upon it provide a 'universe' within which the true states and observables must be found.

These arguments indicate the alternative to von Neumann's theory and the principal difficulty that will be encountered in its formulation.

The alternative is to postulate for each quantum system its true algebra of observables. This is not such a radical suggestion and is certainly no more arbitrary than the postulate which assigns a potential function or Lagrangian to each system. In fact, from the point of view of the experimenter this postulate could be decidely advantageous. In his discussion of spectrum generating algebras, Dothan (1970) remarked that the constants which determine the structure of such an algebra are directly related to the observable spectrum of states, and so would be an excellent choice for the set of parameters to be determined from experiments. In contrast, the potential is empirically a poorly defined quantity, for it is well known that the results of calculations with nuclear

models are insensitive to rather large variations in the potential.

This alternative presents a serious difficulty. The true algebra of observables, denoted A, is finitely generated and so its dimension is countably infinite. If X is a vector space which carries an (algebraically) irreducible representation of A, then the dimension of X will also be countable. Thus, if t is an observable, but is not represented by a diagonal matrix on X, then t might not have any eigenvectors in X. In particular, it will not always be possible to find a new basis for X which diagonalises the matrix of t, even when this matrix is hermitian. Hence, if I accept the alternative to von Neumann's theory, then I must also develop a theory of eigenvalue problems on X which is algebraic and does not depend upon a topology defined on X.

My aim is to investigate precisely these problems, the algebraic framework of quantum mechanics and algebraic eigenvalue problems, and the plan of my attack is roughly as follows.

(1) The rest of this introduction will be devoted to identifying a suitable candidate for the algebra A of observables in quantum mechanics, both relativistic and non-relativistic.

(2) In chapter 2 I will assume that the algebra A has been postulated and I will investigate the relation between states and representations of A. My aim will be to associate a left ideal in A with each state, so the study of the states of A can be reduced to a study of the structure of A itself. For this work I have borrowed the Gel'fand-Naimark-Segal construction from the theory of C^*-algebras, with one important

exception, I have not endowed A with a topology.

(3) Chapters 3, 4, 5 and 6 are concerned with eigenvalue problems which can be posed and solved algebraically. I have considered two types, _diagonal_ and _codiagonal_ operators, each defined in terms of a Lie algebra L and a subalgebra K. For diagonal operators the problem of spectral analysis is essentially that of decomposing an irreducible representation of L into irreducible representations of K. For codiagonal operators the situation is far more complicated.

(4) Chapter 7 contains a novel application of algebraic methods. I have argued that it is only necessary to solve an _algebraic_ eigenvalue problem in order to find the _analytic_ continuation of a solution of an ordinary differential equation with holomorphic coefficients. This result is true quite generally, but it is only for the special functions of mathematical physics that all the steps can be completed. Accordingly, I have restricted my discussion to the special functions and, in particular, to the hypergeometric function.

(5) Chapter 8, although the last in the thesis, is the source of several ideas. It is concerned with Wick's equation and proves that the Wick rotation is correct, so it is of interest for its own sake. The proof employs the monodromy group of Heun's equation and it was this proof which prompted the ideas of chapter 7. In addition, Wick's model is an instructive example where it is both burdensome and unnecessary to treat relative four-momenta as self-adjoint operators. This work has already been published (D.M. O'Brien (1975)).

Which algebras are appropriate for quantum mechanics? In the last fifteen years dozens of suggestions have been made:

C^*-algebras, the Lie algebras of the Poincaré group, $SO(4,1)$, $SO(4,2)$, $SU(3)$, $U(6) \times U(6)$, $\cdots$. I could not hope to survey all these possibilities; indeed it would be irrelevant if I did, for I must seek a general class of algebras suitable for quantum mechanics and not the algebras needed for particular problems. Consequently I will divide all the suggestions into three broad categories and extract from each the points that I think are essential.

(1) Von Neumann, Jordan and Wigner (1934) suggested that the observables of a quantum system should comprise a real Jordan algebra, as indeed do the self-adjoint operators on a Hilbert space, and that the task of quantum mechanics should be to find its representations. In this pioneering work, they only considered finite dimensional algebras of observables, so questions of topology were irrelevant. The modern version of their work, promoted by Haag and Kastler (1964), lifts this restriction by demanding that the observables be the self-adjoint elements of a C^*-algebra, a topological algebra of uncountably infinite dimension.

From the point of view of an algebraist, C^*-algebras aggravate the difficulties of quantum mechanics; the observables are still too numerous and all the useful observables, such as the Hamiltonian and momentum, are excluded because they are unbounded. However, there is one very beautiful idea in the theory of C^*-algebras that I will borrow and develop algebraically in the next chapter. This idea is that every state of the quantum mechanical system determines a representation of A via a canonical prescription, and conversely. Thus, greater flexibility is possible. For example, scattering states and

bound states can determine entirely different representations of A.

(2) The second suggestion is very elegant, but dogged with difficulties. It is the suggestion that the symmetry of the space-time manifold may be coupled with an 'internal' symmetry of the dynamics in such a way that the mass (or energy) spectrum is split and accords with experiment. Great hope was held for this theory in the mid-sixties when it was thought that the Poincaré group and SU(3) could be combined into a single Lie group, whose multiplets would accommodate the elementary particles. However, McGlinn (1964) and O'Raifeartaigh (1965) precipitated a flood of 'no-go' theorems which forbade anything but trivial couplings of these groups. Hegerfeldt and Hennig (1968) have reviewed this whole field and have concluded that, although the 'no-go' theorems do not absolutely forbid an explanation of mass splitting with only finite dimensional Lie groups, they do make it seem improbable.

(3) Lastly, there is the proposal by Barut and Kleinert (1967) that the observables should generate a 'dynamical group', or the equivalent proposal by Dothan, Gell-Mann and Ne'eman (1965) that the observables should lie in a 'spectrum generating algebra'. The idea here is that the states of a quantum system at rest should span an irreducible representation of some non-compact Lie group. The states for a moving system are to be obtained by boosts.

The first proposal is too general, but the second and third are too rigid, for they assert that finite dimensional Lie algebras alone will suffice for a description of quantum mechanics. The 'no-go' theorems show that this cannot be so. However, it is clear from the original papers that the

authors knew the limitations of Lie algebras. For example, Dothan (1970) found that unless he generalised the definition of structure constants, that is to say, unless he abandoned the Lie algebra for a more general structure, the only system with a finite dimensional spectrum generating algebra was a set of harmonic oscillators.

There are three features common to all theories of this type.

(1) A fundamental role is played by the Lie group $\underline{G}_R$ of transformations between the coordinate frames of equivalent observers. $\underline{G}_R$ depends upon the model assumed for the empty universe, and for the Newtonian, Minkowski and de Sitter models would be the Galilei group, the Poincaré group and $SO(4,1)$, respectively. In each case $\underline{G}_R$ is a real, linear Lie group. To each element in the real Lie algebra G_R of $\underline{G}_R$ there is an analogous variable in the classical theory. Furthermore, apart from the electric charge, the elements of G_R and combinations of them seem to be the only observables to which the correspondence principle must apply. I will return to this point later.

(2) For a system consisting of just one free elementary particle, G_R and its universal enveloping algebra, denoted $U(G_R)$, are sufficient. To describe interacting or composite systems it is necessary to introduce a further set of variables, which I will denote K_R, and its real, associative enveloping algebra $U(K_R)$. I will assume that K_R is a finite set simply because that is the case in many examples, but it is conceivable that K_R could be countably infinite. Typical elements of K_R might be the variables corresponding to discrete

transformations, such as parity, or the generators of SU(3).

(3) $U(G_R)$ and $U(K_R)$ must be coupled by identities, that is, $U(G_R)$ and $U(K_R)$ must be embedded in a larger real, associative algebra A_R such that certain specified identities are satisfied. These identities could take any form, but probably would be of Lie type

$$[g_i, k_m] = \alpha_{im}{}^j g_j + \alpha_{im}{}^n k_n \ , \quad \alpha_{im}{}^l \in R, \ g_i \in G_R, \ k_m \in K_R \ ,$$

of generalised Lie type in which the structure constants would be polynomials in the invariants of $U(G_R)$ and $U(K_R)$, or would involve anticommutators instead of commutators.

Henceforth A will denote the complexification of A_R,

$$A = A_R + iA_R \ ,$$

where the sum is a direct sum of real vector spaces and the product in A is to be defined in the obvious way.

Flato and Sternheimer (1967) have pointed out that any mass formula can be obtained if the enveloping algebras of the Lie algebras of the Poincaré group and SU(3) are coupled by suitably chosen identities. They concluded that "infinite structures" are too arbitrary to be of any use. Their criticism is justified if K_R and the identities which couple $U(G_R)$ and $U(K_R)$ are postulated solely to obtain the correct mass formula and without the support of physical reasons. However, I have in mind a different approach, based upon an idea that can be found in the thesis by Dr. A. J. Bracken (1970) and also in the work of Takabayasi (1968). With each elementary component of the quantum system must be associated a list of dynamical variables. Furthermore, certain algebraic identities must be valid within the associative algebra D generated by the set of all dynamical variables. Both the dynamical variables and the

identities are characteristic of the system being studied and must be postulated for that system. For example, Bracken investigated a model in which the baryons were constructed from 'three spin $\frac{1}{2}$ objects', each of which had as its dynamical variables a pair of four-vector operators

$$a_\lambda{}^{(k)} \quad , \quad b_\lambda{}^{(k)}$$

and a set of Dirac matrices

$$\gamma_\lambda{}^{(k)} \quad , \quad \begin{cases} 1 \leqq k \leqq 3 \\ 0 \leqq \lambda \leqq 3 \end{cases} .$$

The identities satisfied were

$$[a_\lambda{}^{(k)}, b_\mu{}^{(l)}] = i\delta^{kl} g_{\lambda\mu} ,$$

$$\{\gamma_\lambda{}^{(k)}, \gamma_\mu{}^{(l)}\} = 2g_{\lambda\mu} \text{ for } k = 1 ,$$

and all other commutators were zero. The algebras $U(G_R)$ and $U(K_R)$ must be exhibited as subalgebras of D and the identities which couple $U(G_R)$ and $U(K_R)$ are precisely those inherited from D. I hasten to add that the dynamical variables need not themselves be observable, only certain elements of $U(G_R)$ and $U(K_R)$ are observable.

One property in particular seems essential for A_R; it must be possible to label the basis vectors in any irreducible representation of A_R by the eigenvalues of a _finite_ number of elements of A_R.

The points raised under (1) above require further comment, and I should like to begin with an example.

In the Hamiltonian formulation of non-relativistic quantum mechanics, the interaction of two particles in a region free of external force fields is governed by the Hamiltonian

$$H = \frac{p_1{}^2}{2m_1} + \frac{p_2{}^2}{2m_2} + V(\underline{q}_1 - \underline{q}_2, \underline{j}).$$

Define

$$\underline{P} = \underline{p}_1 + \underline{p}_2 \qquad , \quad \underline{p} = (m_2\underline{p}_1 - m_1\underline{p}_2)/(m_1 + m_2),$$

$$\underline{Q} = (m_1\underline{q}_1 + m_2\underline{q}_2)/(m_1 + m_2), \quad \underline{q} = \underline{q}_1 - \underline{q}_2 \ ,$$

$$\underline{L} = \underline{Q} \times \underline{P} \qquad , \quad \underline{l} = \underline{q} \times \underline{p} \ ,$$

$$\underline{J} = \underline{L} + \underline{j} \qquad , \quad \underline{j} = \underline{l} + \underline{s}_1 + \underline{s}_2 \ ,$$

$$M = m_1 + m_2 \qquad , \quad m = m_1 m_2/(m_1 + m_2) \ .$$

In these formulae, $\underline{p}_i$, $\underline{q}_i$, m_i, $\underline{s}_i$ respectively denote the momentum, coordinate, mass and spin of the ith particle. In the usual way,

$$H = \frac{P^2}{2M} + \frac{p^2}{2m} + V(\underline{q}, \underline{j}).$$

The operators H, $\underline{P}$, $\underline{Q}$, $\underline{J}$ and M satisfy the commutation relations of the Lie algebra of a central extension of the Galilei group (Levy-Leblond (1963)) and generate the transformations between the coordinate frames of different inertial observers. The invariants of this Lie algebra are

$$h = H - \frac{P^2}{2M} = \frac{p^2}{2m} + V(\underline{q}, \underline{j}),$$

$$\underline{j}^2 = (\underline{J} - \underline{Q} \times \underline{P})^2,$$

and
$$M = m_1 + m_2 \ .$$

Thus, an experiment which determines the energy and angular momentum of a bound state of the two particles in fact determines the irreducible representation of the Galilei algebra spanned by the observables which describe the motion of the

centre of mass. Alternatively, the energy and angular momentum of the bound state can be expressed entirely in terms of the variables describing the motion of the centre of mass:

$$h = H - \frac{P^2}{2M} \, ,$$

$$\underline{j} = \underline{J} - \underline{Q} \times \underline{P} \, .$$

It follows that H, $\underline{P}$, $\underline{Q}$, $\underline{J}$ and M are always observables.

On the other hand, there is no need to maintain that the relative momentum $\underline{p}$ and the relative coordinate $\underline{q}$ are observables when the particles are interacting, because these variables cannot be measured. I base this assertion on the following argument. Any measurement takes a finite time to perform. Furthermore, the test probes of the apparatus occupy a finite volume of space. These facts are accounted for in classical physics by assuming that the outcome of a measurement is the average value of the measured quantity over the space-time duration of the experiment. Virtually by definition, the measured quantities in classical physics are constants of the motion during the observation. Consider now the relative momentum $\underline{p}$. $\underline{p}$ does not commute with the Hamiltonian H because of the presence of the potential. Hence

$$\dot{\underline{p}} = i[H,\underline{p}] \neq 0.$$

How can a momentum which is varying with time, very rapidly according to a semi-classical model, be measured? The duration of the observation would have to be so short that the energy required would disintegrate the system. I think that $\underline{p}$ cannot be measured except when V = 0 and the particles are free. Similarly, $\dot{\underline{q}} \neq 0$ and $\underline{q}$ is neither an observable nor a measurable quantity. However, the average of $\underline{p}$ or $\underline{q}$ over a

long period of time is measurable because it is a constant of the motion. The electron clouds described by chemists correspond to this time average.

This observation is important. It shows that the only observables in this example (apart from electric charge) which have analogues in the classical theory are those constructed from H, $\underline{P}$, $\underline{J}$, Q and M. Hence, it is only to these observables that the correspondence principle need be applied.

How is the correspondence principle to be incorporated in the theory? Professor Green suggested that it should be done as follows. Suppose that $g_1, \cdots, g_n$ is a basis for G_R. Each of these quantities has a classical counterpart and is observable. With the product $g_i g_j$ in the classical theory should be associated the observable

$$\tfrac{1}{2}(g_i g_j + g_j g_i)$$

in the quantum theory, with $g_i g_j g_k$ should be associated

$$\tfrac{1}{6}(g_i g_j g_k + g_i g_k g_j + g_j g_i g_k + g_j g_k g_i + g_k g_i g_j + g_k g_j g_i).$$

The general rule is that

$$g_{i_1} g_{i_2} \cdots g_{i_m} \longrightarrow \frac{1}{m!} \sum_{\pi} g_{i_{\pi(1)}} g_{i_{\pi(2)}} \cdots g_{i_{\pi(m)}},$$

where the sum is to be taken over all permutations of $\{1, 2, \cdots, m\}$. The reason behind this proposal is that the canonical equations of motion,

$$[q_j, H] = i\dot{q}_j \quad, \quad [p_j, H] = i\frac{\partial L}{\partial q_j},$$

where

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

and
$$H = -L + \sum_{j} \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j \; ,$$

can be proved in a Lagrangian formulation of quantum mechanics if the Lagrangian $L(q_j, \dot{q}_j)$ is written in symmetrised form, but not otherwise. The observables are the completely symmetric polynomial functions of $g_1, \cdots, g_n$.

There is a further condition to be imposed upon observables, namely, their expectation values must be real (Dyson(1962)), so it is necessary to isolate those representations of A_R in which this condition is satisfied. One class of such representations, which will prove to be sufficient, consists of those which provide a representation of A and an involution * defined on A, with respect to which the observables are self-adjoint. The origin of * is explained in the next few paragraphs.

G_R is a real Lie algebra. Construct its complexification

$$G = G_R + iG_R \; ,$$

and define

$$\alpha : G \longrightarrow G$$
$$g+ig' \longmapsto g-ig' \; , \quad g,g' \; \epsilon \; G_R.$$

α is an automorphism of G and can be uniquely extended to an automorphism of the enveloping algebra of G, denoted U(G). (Varadarajan (1974a)). Note that $\alpha^2 = 1$. The _principal anti-automorphism_ of G is the map

$$\beta : G \longrightarrow G$$
$$g \longmapsto -g \; .$$

It also can be uniquely extended to U(G) and satisfies $\beta^2 = 1$.

(Varadarajan (1974b)). α and β commute and their composition is an involution on $U(G)$, that is, an involutorial anti-automorphism of $U(G)$. Define

$$* : U(G) \longrightarrow U(G)$$
$$g \longmapsto g^* = (\alpha.\beta)(g).$$

With just one further assumption, based on physical principles, $*$ is unique. To see this, suppose the opposite, that γ is another involution on $U(G)$. Then

$$\bar{\alpha} = \beta.\gamma$$

is an automorphism of G, and $\bar{\alpha}^2 = 1$ because β and γ commute. The real forms of G are classified by the involutive automorphisms of G. Thus, the assertion of physics that the particular real form G_R generates the transformations between equivalent observers forces $\alpha = \bar{\alpha}$ and so

$$\gamma(g) = g^*$$

for all g in $U(G)$.

U(G) is only a subalgebra of the complex associative algebra

$$A = A_R + iA_R .$$

However, I will assume that $*$ can be extended to A. I will also extend Professor Green's assertion and consider the observables of the quantum system characterised by A to be the symmetrised polynomials in the self-adjoint generators of A. There is a freedom in this extension of $*$ to A, but it is precisely the type of freedom I have been seeking. For example, certain elements of $U(K)$ will be self-adjoint in some circumstances but skew-adjoint in others.

Note also that α can be assigned a physical meaning. In relativistic quantum mechanics α provides a representation

of charge conjugation, but in non-relativistic quantum mechanics α represents particle creation and annihilation.

All these considerations are summarised in the following postulate.

Postulate.

With a given quantum system can be associated a finitely generated, complex, associative algebra A. A contains a subalgebra isomorphic to U(G). The involution * on U(G) can be extended to A and the observables are the symmetrised polynomials in the self-adjoint generators of A.   ///

It is from this point that the next chapter continues, but before I conclude this introduction I would like to speculate upon the importance of the idea of continuity for quantum mechanics.

I have not assumed that the algebra of observables is endowed with a topology, and possibly that immediately dooms my work to failure. Nevertheless, to my knowledge, nobody has ever inquired how much (or how little) of quantum mechanics depends in an essential way upon the topological assumptions. That is the task I have set myself. The point at which the algebraic structures become unacceptable to the reader is then the point at which a notion of continuity becomes essential.

Suppose that I could establish a theory of quantum mechanics without ever imposing a topology on the algebra of observables. Would the theory so constructed have any relevance to physics? Equivalently, how important is the notion of continuity for quantum physics? There seem to be four quite unrelated uses for continuity.

(1.) Quantum theory predicts the expectation values of observables. It is unlikely that the equations which determine the expectation values will be able to be solved exactly, so approximate values must be found with a computer. Thus, a metric topology will be needed to provide an estimate of the errors in the calculated expectation values. However, I do not think that the topology needed for this purpose should arise from quantum theory itself. For example, Schrödinger's equation for a wave function is a differential equation with associated boundary conditions, dictated by the requirement of square-integrability. Its solutions can be approximated with functions from a variety of topological function spaces. Hilbert space may be very convenient, but it is not the only candidate.

(2) The second application of the notion of continuity is more fundamental. Any experiment is limited in accuracy. Thus, an experiment to discover the state ϕ of a system will merely isolate a 'small' set of states, which hopefully contains ϕ. The set of states so determined could be interpreted as an open neighbourhood of ϕ. Thus, it could be argued that subjecting a quantum system to a number of observations automatically imposes a topology on the set of states. Of course, to obtain a predictive theory it would be necessary to impose mathematically a topology on the set of states, and later to argue that the 'open' sets determined by experiments would be open in this topology. (Haag and Kastler (1964)).

There is another way to incorporate the limited resolution of an experiment in the theory.

An observable r is a property of a <u>microscopic</u>

quantum system. Its expectation value <r> represents a measurable property of the <u>macroscopic</u> system comprised by the measuring apparatus in interaction with the quantum system. More precisely, <r> is the average, over an ensemble of identical experiments, of this property of the combined system. The mapping which takes each observable r into its expectation value <r> <u>is</u> the state of the quantum system. Thus, the interaction between the microscopic quantum system and the macroscopic measuring apparatus determines the state of the quantum system. Prior to measurement, the state is indeterminate.

In the mathematical theory, the calculation of the expectation value usually amounts to forming a trace,

$$<r> = \text{trace } (\rho r),$$

where ρ is called the density matrix. Hence, the <u>linear</u> operator ρ represents the complex interaction between the quantum system and the measuring apparatus. I claim that the limited resolution of the observation will be represented in the mathematical theory if ρ has a finite dimensional range. There is also a technical advantage to be gained from such a choice: trace (ρr) is always well defined because the endomorphisms with finite range form a two-sided ideal in the ring of all endomorphisms of the vector space.

(3) I believe that the notion of causality is intimately related to the continuity of scattering amplitudes as functions of the space-time or momentum variables. For example, if a macroscopic particle were to disappear from one point and simultaneously reappear at another, the process would seem unreal, for it would be both discontinuous and

acausal. Iagolnitzer and Stapp (1969) have investigated this matter within the framework of analytic S-matrix theory. By a series of ingenious arguments, they concluded that a natural definition of <u>macroscopic</u> causality applied to <u>microscopic</u> particles implied analyticity of the scattering amplitudes in the physical region. Thus, it appears that continuity, and hence a topology of some sort, might be necessary to preserve the causal relation between events.

However, if the Minkowski structure of space-time fails at distances of about 10^{-15} centimetres or less, then there is no reason to insist that microscopic particles should satisfy any form of causality. Only macroscopic particles need satisfy the principle of causality formulated by Iagolnitzer and Stapp, and for these the scattering amplitudes must be analytic.

(4) One characteristic of a good physicist, experimental or theoretical, is that he is able to quickly reduce a complicated situation to one which is simple, by ignoring forces whose effects are slight. This skill is based upon the assumption that small forces produce small effects, and hence upon an intuitive notion of continuity.

The reply to this objection is that quantum mechanics is essentially discontinuous and an intuitive understanding of the subject is an impossibility, because two identical quantum systems in the same prepared initial state can evolve to different final states.

I am going to adopt a rather Quixotic attitude and choose to ignore the myriad of pointers which show that continuity and analytical methods are extremely useful, if not essential, in the development of quantum mechanics. I will

develop, without any use of continuity, a mathematical
structure of observables and states that seems acceptable for
the purposes of quantum theory.

Note

Most chapters have an appendix in which are gathered
most of the theorems and proofs. Hopefully, this has made
the text more readable.

CHAPTER 2. MATHEMATICAL STRUCTURE OF QUANTUM MECHANICS.

In the last chapter I postulated that the observables of a quantum system could be identified with the symmetric tensors in an algebra A. Furthermore, an involution * could be defined on A and with respect to * the observables were self-adjoint. I want to continue the investigation of the mathematical structure of quantum mechanics based upon this postulate with the aim of formulating all questions concerning states and expectation values in terms of the algebra of observables.

However, before I start, I must establish two conventions concerning terminology. If A is an associative algebra, with identity element, over a field F whose characteristic is zero, I will simply refer to A as an algebra over F; that A is associative and has an identity element will be implicit. I will distinguish non-associative algebras with an adjective, such as Lie or Jordan, unless the meaning is clear from the text. Secondly, if A is an algebra (associative, Lie, Jordan, · · ·) over a field F, the statement that 'X is an A-module' will always imply that X is a vector space over the same field F.

It is also well to enumerate the essential elements in any theory of quantum mechanics.

Postulate 1.   The mathematical description of any quantum system must contain the following objects:

(1) a set B of observable quantities;

(2) a set S of states of the system;

(3) a mapping which assigns to each state ϕ in S and each observable b in B a real number $<\phi, b>$, which is to be interpreted as the expectation value of b when the system is

in the state ϕ;

(4) a rule which relates the observables and states assigned by different observers.

I have borrowed this postulate from Emch (1972). However, he does not include (4). The rule mentioned in (4) contains all the dynamical information about the quantum system, for it relates the expectation values measured by observers in the future to those measured by an observer in the present.

This postulate defines the task; B, S and the mappings in (3) and (4) must be identified. In the following sections I will examine in turn each of these constituents of quantum theory.

Observables.

The steps leading to the postulate of the last chapter were these.

(1) Take the real Lie algebra G_R of the symmetry group $\underline{G}_R$ of the empty universe.

(2) Construct the complexification of G_R:

$$G = G_R + iG_R.$$

(3) Equip G with an involution *:

$$* : G \longrightarrow G$$

$$g+ig' \longmapsto -g+ig' \quad , \quad g,g' \in G_R.$$

Note that the elements of G which are self-adjoint with respect to this involution have the form

$$ig \quad , \quad g \in G_R.$$

This is the origin of the mysterious i which lurks in quantum mechanical formulae.

(4) Embed U(G) in a complex algebra A, which is finitely
generated, and extend the involution * from U(G) to A. This
extension need not be unique; any arbitrariness is advantageous
because it increases the flexibility of the theory. The
algebra A and the embedding of U(G) must be postulated for
each quantum system.

(5) Select a set of self-adjoint generators of A which includes
a basis for G_R. Denote this set

$$\{b_1, b_2, \cdots, b_n\}$$

and construct the _real_ subspace B of A spanned by the
symmetrised elements

$$\frac{1}{m!} \sum_{\pi} b_{i_{\pi(1)}} b_{i_{\pi(2)}} \cdots b_{i_{\pi(m)}} \quad , \quad m = 1, 2, \cdots,$$

where the summation is over all permutations of $\{1, 2, \cdots, m\}$.

Postulate 2.   The observables of the quantum system
are the elements of B.

Note that every irreducible B-module can be
embedded uniquely in an irreducible A-module, and that every
irreducible A-module remains irreducible when considered as a
B-module. This is so because the set $\{b_1, b_2, \cdots b_n\}$
generates A. Consequently, now that I have identified B, I
can focus my attention on A.

It is worth emphasising that the composition of
the involution * with β, the principal anti-automorphism of
U(G), has a physical interpretation. It provides a
representation of charge conjugation in relativistic quantum
mechanics and particle creation and annihilation in non-
relativistic quantum mechanics. In view of the CPT theorem,
*β also represents the product of the parity and time-reversal

operations.

. Further requirements on A are necessary, but these are best introduced through a discussion of the set of states.

I must pause to introduce a notational device. Whenever the existence of the involution * on A is important in an argument, I will denote the algebra by A^*; otherwise, I will write just A.

Definition. X^* is an A^*-module if each of the following conditions holds.

(1) X^* is an A-module.

(2) X^* is an inner product space.

(3) For all vectors x and x′ in X^* and r in A,

$$(r^*x,x′) = (x,rx′),$$

where (,) is the inner product on X^*.   ///

I will extend my notational device to A and A^*-modules. Thus, if X^* is an A^*-module and I choose to neglect the inner product on X^*, I will write simply X.

Definition. X^* and Y^* are equivalent A^*-modules, written $X^* \simeq Y^*$, if:

(1) X and Y are equivalent A-modules, also written $X \simeq Y$;

(2) the map $\sigma : X \longrightarrow Y$ which establishes the above equivalence satisfies

$$(\sigma x,\sigma x′) = (x,x′) , \quad \text{for all x and x′ in } X^*,$$

where the inner product on the left is in Y^* and that on the right is in X^*.   ///

States.

The process of observation removes the indeterminacy from the state of the quantum system and, by repetition in an experimental ensemble, assigns to each observable its

expectation value. Therefore, it is appropriate that each state ϕ in S should be represented as a mapping of B into the real numbers, in which the image of each observable is its expectation value in the state ϕ:

$$\phi : B \longrightarrow R$$
$$b \longmapsto <\phi,b>.$$

Whatever other properties these real-valued functions on B might have, it is reasonable to expect that they fulfil the following requirements.

(1) The set of all states should 'separate the points' of B. Thus, if a and b are two observables which have identical expectation values in every state ϕ,

$$<\phi,a> = <\phi,b> \quad \text{for all } \phi \text{ in S,}$$

then it should follow that a = b.

(2) Each state ϕ should be a real-valued <u>linear</u> function on B, so that

$$<\phi,\lambda b> = \lambda<\phi,b> , \quad \text{for all real } \lambda,$$

and $\qquad <\phi,a+b> = <\phi,a> + <\phi,b> .$

(It is remarkable that linear functions suffice, because each state represents the complicated interaction between the microscopic quantum system and the macroscopic measuring apparatus.)

(3) The identity of B corresponds roughly to the proposition that the quantum system exists. Thus,

$$<\phi,1> = 1 , \quad \text{for all } \phi \text{ in S,}$$

seems reasonable. Certainly it is harmless.

(4) Finally, if b is any observable, the expectation value of b^2 should be positive:

$$<\phi,b^2> \geqq 0 \quad \text{for all b in B.}$$

The set of all real-valued linear functions on B

separates the points of B. The problem is whether the subset of linear functions which satisfy conditions (3) and (4) also has this property. In general it will not and further postulates are necessary.

Let $\tilde{S}$ denote the set of complex-valued linear functions on A which satisfy the following conditions:

(a) $$\langle\phi,1\rangle = 1;$$

(b) $$\langle\phi,a^*\rangle = \overline{\langle\phi,a\rangle} ;$$

(c) $$\langle\phi,a^*a\rangle \geq 0.$$

It is easy to see that every such function, when restricted to B, has the properties (2), (3) and (4) required of a state. Later I will identify the set S of states with a subset of $\tilde{S}$, but for the moment I will reserve that decision.

If A is not a semisimple algebra, so that its (Jacobson) radical is a proper ideal of A, even $\tilde{S}$ itself would not be large enough to separate the points of A. However, the radical of A contains all those elements which are represented by zero in every irreducible representation of A. Such elements are effectively unobservable and so have no relevance for physics. The natural remedy for this difficulty is to insist, as a postulate, that A should be semisimple.

Unfortunately that assumption is not quite strong enough, because it only ensures that for any r in A there is at least one irreducible representation of A in which the representative of r is non-zero. I need a similar property for irreducible representations of A*, and so I must proceed as follows.

Definition.   The *-radical of A* is the ideal

$$Q^* = \bigcap_{X^*} \{a \mid a \in A, \quad aX = 0\} ,$$

where the intersection is taken over all irreducible A^*-modules. If $Q^* = 0$, A^* is *-semisimple.　　　///

Q^* contains all those elements of A that are represented by zero in every irreducible A^*-module.

Postulate 3.　　　　　The algebra A^* must be *-semisimple.

　　It is obvious that a *-semisimple algebra is semisimple. Jacobson (1956) gives sufficient conditions for the converse to be true; A should be a primitive ring with non-zero socle. However, these conditions are too restrictive and will not always be satisfied in practice.

　　Postulate 3 disposes of the unobservable observables and ensures that $\tilde{S}$ separates the points of A. This is proved in detail in theorem 1 of the appendix, but the outline of the proof is quite elementary. Because A^* is a *-semisimple algebra, for any non-zero element r in A^* it is possible to find an A^*-module X^* and a vector x in X^* such that

$$(x, rx) \neq 0.$$

The formula

$$\langle \phi, a \rangle = (x, ax)/(x, x), \quad \text{for all a in A,}$$

defines a linear function in $\tilde{S}$ which satisfies

$$\langle \phi, r \rangle \neq 0 ,$$

so $\tilde{S}$ separates the points of A^*.

　　Postulate 3 is redundant when A is the universal enveloping algebra of a complex Lie algebra L,

$$L = L_R + iL_R ,$$

and the involution on A is defined by

$$* : L \longrightarrow L$$

$$l + il' \longmapsto -l + il' \quad \text{where } l, l' \in L_R.$$

Harish-Chandra (1950) has shown that the finite dimensional representations of L separate the points of U(L), so that in this case A and B are certainly semisimple. However, this does not imply that the finite dimensional representations of L^* have the same property. In fact, unless L_R is a compact real Lie algebra, L^* does not have any finite dimensional representations. Nevertheless, the set of all L^*-modules, including the infinite dimensional ones, does separate the points of U(L). This follows from the Gel'fand-Raikov (1943) theorem for unitary irreducible representations of locally compact groups. Unfortunately I do not have an algebraic proof of this result.

Every irreducible representation of an algebra A determines by a canonical prescription a maximal left ideal of A, and conversely. I want to develop this result in the next section, and later to use it to formulate questions concerning the states and expectation values of the quantum system in terms of the left ideals of its algebra of observables.

Representations and the Calculus of Ideals.

In this section A can denote any algebra over a field F of characteristic zero.

An A-module X is cyclic if it contains a vector x such that

$$X = Ax.$$

The vector x is called a cyclic vector.

If M is any (proper) left ideal in A, then a well defined cyclic A-module can be associated with M as follows. Construct the factor vector space

$$X_M = A/M$$

and equip X_M with the product

$$a.(r + M) = ar + M \quad \text{for all } a, r \in A.$$

With this product, X_M is a (left) A-module and is irreducible if and only if M is a maximal left ideal. X_M is the A-module <u>canonically associated</u> with M.

Now suppose that X is any cyclic A-module with cyclic vector x. Define

$$M = \{m \in A \mid mx = 0\}.$$

M is a left ideal in A and from it may be constructed the A-module

$$X_M = A/M .$$

X and X_M are equivalent A-modules.

These results are well known and can be found, for example, in the text by Varadarajan (1974c). In contrast, the results which follow are, to my knowledge, quite new.

It is important to know when two left ideals M and N lead via the canonical construction to equivalent A-modules. I have found that an equivalence relation $\simeq$ can be defined on the set of left ideals of A, without reference to the associated A-modules, and in order that X_M and X_N should be equivalent A-modules it is both necessary and sufficient that $M \simeq N$.

Define a relation $\simeq$ on the set of left ideals of A as follows. M is in relation to N, written $M \simeq N$, if elements u and v can be found in A such that the following conditions are satisfied:

$$Mu \leqq N , \qquad uv = 1 \pmod{M} ,$$
$$Nv \leqq M , \qquad vu = 1 \pmod{N} .$$

Lemma 2 of the appendix shows that $\simeq$ is an equivalence

relation on the set of left ideals of A. Theorem 3 establishes
that $X_M \cong X_N$ if and only if $M \cong N$.

It is a difficult task to find all the inequivalent,
irreducible, cyclic A-modules. It is also difficult to find
the set of all maximal left ideals of A and to divide this
set into equivalence classes with respect to $\cong$. However, in
some instances the second task is easier than the first. The
import of the foregoing results is that the two tasks are
equivalent.

If M and N are distinct maximal left ideals of A,
then

$$P = M \cap N$$

is also a left ideal of A, but is not maximal. What is the
relation between the A-modules X_M, X_N and X_P? The answer is
very simple and satisfying:

$$X_P \cong X_M \oplus X_N.$$

Theorem 5 contains an even stronger result, also proved in the
appendix.

<u>Theorem 5.</u>   Suppose that $\{M_1, M_2, \cdots \}$ is a countable
family of distinct maximal left ideals of A. Set

$$M^0 = \bigcap_i M_i$$

and construct the A-modules

$$X^0 = A/M^0 \text{ and } X_i = A/M_i.$$

Then $X^0 \cong \bigoplus_i X_i.$ ///

<u>Gel'fand-Naimark-Segal (GNS) Construction.</u>

I want to be able to identify pure states with
certain maximal left ideals of A. The key to this

identification is the GNS construction which associates with each ϕ in $\tilde{S}$ a certain left ideal of A.

For each ϕ in $\tilde{S}$, define

$$M_\phi = \{m \in A \mid <\phi, m^* m> = 0\}.$$

M_ϕ is a left ideal in A. The factor space

$$X_\phi = A/M_\phi$$

becomes a cyclic A-module with the product

$$a \cdot (r + M_\phi) = ar + M_\phi, \quad \text{for all a and r in A.}$$

The function ϕ can be used to define an inner product on X_ϕ as follows:

$$(b + M_\phi, a + M_\phi) = <\phi, b^* a> \quad \text{for all } a, b \in A.$$

Thus, $$<\phi, a> = (1 + M_\phi, a + M_\phi).$$

With this inner product, X_ϕ^* is a cyclic A^*-module. X_ϕ^* is canonically associated with ϕ. Finally, X_ϕ^* is an irreducible A^*-module if and only if M_ϕ is a maximal left ideal of A.

All these assertions are proved in the appendix.

Let $\tilde{R}$ denote the collection of cyclic A^*-modules. Each ϕ in $\tilde{S}$ determines a cyclic A^*-module X_ϕ^* in $\tilde{R}$. Conversely, each X^* in $\tilde{R}$ determines a positive linear function ϕ in $\tilde{S}$ by

$$<\phi, a> = (x, ax)/(x, x) \, , \quad a \in A,$$

where x is the cyclic vector of X^*. I want to show that $\tilde{R}$ and $\tilde{S}$ can be identified, equivalently, that I can define a map

$$\mu : \tilde{S} \longrightarrow \tilde{R}$$
$$\phi \longrightarrow X_\phi^*$$

which is bijective. However, firstly I must check that μ is well defined. This cannot be so unless the chain

$$X^* \longrightarrow \phi \longrightarrow X_\phi^*$$

closes upon itself. Thus, the function ϕ, defined in terms of

the inner product on X^*, must lead via the canonical
construction to the same A^*-module. This is not the case.
However, theorem 11 of the appendix shows that X^* and X_ϕ^*
are equivalent A^*-modules, so I can proceed as follows.
Firstly, I can divide $\tilde{R}$ into disjoint classes of equivalent
A^*-modules. Secondly, I can define an equivalence relation
$\simeq$ on $\tilde{S}$:

$$\phi \simeq \psi \text{ if and only if } X_\phi^* \simeq X_\psi^*.$$

The map μ is then well defined and bijective, provided I
identify equivalent elements in $\tilde{R}$ and $\tilde{S}$. In loose terms,
the existence of the bijection μ allows me to identify $\tilde{R}$ and
$\tilde{S}$.

When do two linear functions ϕ and ψ in $\tilde{S}$ lead
via the canonical construction to equivalent A^*-modules?
With the help of the results of the last section on the
connection between left ideals of A and A-modules, I have
been able to establish the following beautiful result.

Definition.   Let

$$S_o = \{\phi \in \tilde{S} \mid M_\phi \text{ is a maximal ideal in A}\}.$$

Theorem 15.   Suppose ϕ and ψ are elements of S_o.
X_ϕ^* and X_ψ^* are equivalent A^*-modules if and only if there
exist elements u and v in A such that

$$\langle \psi, a \rangle = \langle \phi, u^* a u \rangle,$$

$$\langle \phi, a \rangle = \langle \psi, v^* a v \rangle, \quad \text{for all } a \in A. \quad ///$$

This theorem provides an alternative definition of the
equivalence relation $\simeq$ on the set S_o.

The situation is as follows. Suppose that M_ϕ and M_ψ are maximal left ideals of A. I have shown that

$$
\begin{array}{ccc}
X_\phi{}^* \simeq X_\psi{}^* & \Longleftrightarrow & \phi \simeq \psi \\
\Downarrow & & \Downarrow \\
X_\phi \simeq X_\psi & \Longleftrightarrow & M_\phi \simeq M_\psi
\end{array}
$$

I would like to complete the diagram and show that

$$ X_\phi \simeq X_\psi \implies X_\phi{}^* \simeq X_\psi{}^* , $$

or equivalently that

$$ M_\phi \simeq M_\psi \implies \phi \simeq \psi . $$

For an arbitrary complex algebra with involution, this result is not true. It fails because the commuting algebra of an irreducible A-module X can contain more than just the scalar multiples of the identity endomorphism of X. In other words, it fails because Schur's lemma does not hold in the strong form familiar from the theory of finite dimensional A-modules. This can be seen in the proof of theorem 17. However, for the algebras needed in quantum mechanics, it is possible to prove an even stronger result, that

$$ M_\phi = M_\chi \implies \phi = \chi , $$

with just one further assumption concerning A that is extremely weak and invariably satisfied in practice. The argument proceeds as follows.

As A-modules, X_ϕ and X_χ are identical because $M_\phi = M_\chi$, but as A^*-modules they possibly will not be so because the inner product on $X_\phi{}^*$ is defined by ϕ whereas the inner product on $X_\chi{}^*$ is defined by χ. The Lie algebra G_R has a compact, real subalgebra, namely,

$$K_R = so(3,R),$$

and so

$$U(K) < U(G) < A ,$$

where
$$K = K_R + iK_R$$

denotes the complexification of K_R. Every irreducible K^*-module is finite dimensional. X_ϕ^* can be regarded as a reducible K^*-module, and as such is completely reducible (lemma 16), so

$$X_\phi^* = \underset{j}{\oplus} X_j^* ,$$

where each X_j^* is an irreducible K^*-module. X_χ^* is also completely reducible,

$$X_\chi^* = \underset{l}{\oplus} X_l^* .$$

Since $X_\phi = X_\chi$, the two decompositions into K-modules can be placed in correspondence. Furthermore, there is essentially only one way in which an inner product can be defined on a finite dimensional K-module in order to obtain a K^*-module, so it should not be surprising that $\phi = \chi$ follows.

There is an obvious flaw in this argument. Because X_ϕ^* is completely reducible as a K^*-module, every submodule has a complementary submodule. However, there is no guarantee that X_ϕ^* contains any irreducible K^*-modules. Alternatively, X_ϕ^* might not satisfy the descending chain condition on its K^*-submodules. This difficulty can be overcome in two ways. Firstly, it is possible to demand that A should contain a finite set of elements which separate the basis of every irreducible A^*-module; such a set is usually called a set of labelling operators. This requirement is very reasonable and is always

satisfied in practice. However, I will follow the second path, because it is slightly more general, and insist that X_ϕ^* can be decomposed into irreducible K^*-modules. This restriction is only apparent, because in practice X_ϕ^* would be constructed by glueing together irreducible K^*-modules.

<u>Postulate 4.</u> X_ϕ^*, $\phi \in \tilde{S}$, when considered as a K^*-module, is a countable direct sum of irreducible K^*-modules, each of which is necessarily finite dimensional.

It is possible that this postulate is redundant; I hope that this is the case. With the help of postulate 4 the announced result is proved in theorem 17.

The chain is complete.

$$X_\phi^* \simeq X_\psi^* \iff \phi \simeq \psi$$
$$\Updownarrow \qquad\qquad \Updownarrow$$
$$X_\phi \simeq X_\psi \iff M_\phi \simeq M_\psi$$

This result is of fundamental importance because it shows that the left ideal M_ϕ uniquely determines ϕ and the expectation values of the observables when the quantum system is in the state ϕ. Thus, questions concerning states and expectation values can be interpreted as questions about the structure of the algebra of observables.

There is one extra link which would round out the theory, although for practical purposes this link is not essential. Given an ideal M and a list of its properties, it would be an advantage to be able to decide whether or not A/M could carry a *-representation of A. Equivalently, what is the condition on the left ideals of A which distinguishes those of the form M_ϕ, $\phi \in \tilde{S}$? This is an unsolved problem, worthy of further research. I refer to Warner (1972).

At last I am in a position to identify the states of the quantum system. Let S denote the convex hull of S_o; $\phi \in S$ if and only if

$$\phi = \sum_{i=1}^{n} \lambda_i \phi_i \, ,$$

where $\phi_1, \phi_2, \cdots, \phi_n$ are elements of S_o and the coefficients satisfy

$$\lambda_i \geqq 0, \quad \sum_i \lambda_i = 1.$$

S is a subset of $\tilde{S}$ because $\tilde{S}$ is also convex.

<u>Postulate 5.</u> The <u>states</u> of the quantum system are the points of S. The <u>pure states</u> are the points of S_o.

The set of states has a number of important properties, listed below but proved in theorem 18 in the appendix.

(1) S separates the points of A.

(2) The points of S_o are the extreme points of S.

(3) If $\phi_1, \cdots, \phi_n$ are distinct elements of S_o and

$$\phi_o = \sum_i \lambda_i \phi_i, \quad \lambda_i \geqq 0, \quad \sum_i \lambda_i = 1,$$

then
$$X_{\phi_o}^* \simeq \bigoplus_i X_{\phi_i}^* \, ,$$

where the sum is a direct sum of A^*-modules.

In the usual version of quantum mechanics, an irreducible representation of the algebra of observables is always (unwittingly) chosen. Each observable is represented by a self-adjoint operator on an inner-product space X. Each state ϕ is represented by a vector x_ϕ in X. The expectation

value of observable r in state ϕ is calculated from

$$<\phi,r> = (x_\phi, rx_\phi).$$

Every vector of X is cyclic, because X is irreducible, so the vector x_ψ which represents a state ψ is the image of x_ϕ under some element u in the algebra of observables:

$$x_\psi = ux_\phi .$$

Thus,
$$<\psi,r> = (x_\psi, rx_\psi)$$
$$= (x_\phi, u^* rux_\phi)$$
$$= <\phi, u^* ru> ,$$

and so $\psi \cong \phi$. It follows that the states employed in the original version of quantum mechanics comprise a single equivalence class with respect to $\cong$ in S_o.

The comparison in the last paragraph is not quite correct, because conventional quantum mechanics uses a Hilbert space and not merely an inner-product space. The inaccuracy is unavoidable because I have not endowed either A or S with a topology.

Each impure state ϕ is a generalisation of a density matrix. To see this, suppose that

$$\phi = \sum_i \lambda_i \phi_i , \quad \lambda_i \geqq 0, \quad \sum_i \lambda_i = 1,$$

where each ϕ_i is a pure state. Now

$$<\phi_i, a> = (x_i, ax_i),$$

where x_i is the cyclic vector $(1+M_{\phi_i})$ in $X_{\phi_i}^*$. Thus, if ρ_i denotes the orthogonal projection onto x_i and if π_i denotes the representation afforded by $X_{\phi_i}^*$, then

$$\langle\phi_i, a\rangle = \text{trace } (\rho_i \pi_i(a)).$$

ϕ is then represented by a density matrix ρ on X_ϕ,

$$\rho = \sum_i \lambda_i \rho_i$$

and $\qquad\qquad \langle\phi, a\rangle = \text{trace } (\rho \pi(a)).$

Note that ρ is the weighted sum of a finite number of projections and therefore has a finite dimensional range.

Alternative Suggestions.

The formation of X_M from the ideal M is a well defined constructive procedure. If I could identify directly those maximal ideals of A which lead via the canonical construction to A^*-modules, the whole procedure would be constructive. Unfortunately I cannot, so I must consider some alternatives.

Firstly, there is the direct approach. Families of irreducible A-modules can be constructed when A is the enveloping algebra of a Lie algebra or a simple generalisation of such an algebra. Essential to this construction is a finite set of elements of A which separate the basis in every irreducible A-module, that is, a set of labelling operators. The existence of such a set for representations of a semisimple compact Lie algebra has been investigated by Biedenharn (1963) and by Gruber and O'Raifeartaigh (1964), but I am not sure that their results hold for infinite dimensional, algebraically irreducible representations of A. From the family of representations constructed, those which also provide a representation of the involution on A can be selected by inspection.

Another interesting suggestion, made by Professor Green, is that the emphasis should be shifted from maximal to minimal left ideals of A. Suppose that the algebra A is isomorphic to the complete ring of endomorphisms of a vector space X. Let ρ_{ij} denote the matrix whose only non-zero element is equal to one and occurs in the ith row and j^{th} column.

Define
$$\text{trace } \rho_{ij} = \delta_{ij} .$$

Let
$$M_{ij} = A\rho_{ij}A$$

be the two-sided ideal in A generated by ρ_{ij}. M_{ij} is a minimal ideal. (Jacobson(1951c)). Note that

$$\rho_{ij}\rho_{kl} = \delta_{jk}\rho_{il} .$$

An arbitrary state can be represented by the density matrix

$$\rho = \sum_{i,j} p_{ij}\rho_{ij} \quad , \quad p_{ij} \in C,$$

where
$$p_{ii} \geqq 0 \quad , \quad \sum_{i} p_{ii} = 1,$$

and the sum is finite. The expectation value of an observable r in this state is

$$\langle r \rangle = \text{trace } (\rho r).$$

The trace is always well defined because ρ has only a finite number of non-zero elements in any row or column. The number p_{ii} is the probability that the system will be found in the pure state ρ_{ii}.

The snag in this argument is the supposition that A has minimal ideals. If A is the enveloping algebra of a Lie algebra, then, according to Jacobson (1956a), A need not be primitive nor need it have minimal left ideals, and these are

precisely the conditions that must be satisfied if A is to be isomorphic to a dense ring of transformations. Either the assumption that A is the enveloping algebra of a Lie algebra must be relaxed or maximal ideals must be used.

Relation between Observers.

An observer perceives space and time as a continuum in which all macroscopic objects are embedded. With clocks and laser beams he is able to chart any small portion of the universe. From his atlas, consisting of all his charts, he can decide upon the geometry of the space-time continuum.

The coordinates established by an observer only label macroscopic objects, such as other observers and experimental apparatus. It has long been accepted as a principle of quantum mechanics that a quantum particle cannot be assigned a definite position. More precisely, a quantum particle only has a determinate position when the particle interacts with a detector, and even then the 'coordinates' of the particle are really those of the detector. In short, coordinates only have meaning in quantum theory when they are used to label observers and their measuring apparatus.

There seem to be only three reasonable models for the _empty_ universe. These are the Newtonian, Minkowski and de Sitter models. With each is associated a symmetry group G_R; respectively, the Galilean group, the Poincaré group and $SO(4,1)$. (It is interesting to note that only the Lie algebra of a central extension of the Galilean group contains coordinates and momenta, elements which satisfy

$$[p_i, p_j] = 0 \ , \ [p_i, q_j] = \delta_{ij} \ , \ [q_i, q_j] = 0.)$$

The principle of relativity asserts that two observers are

equivalent, in the sense that they should formulate the same laws of dynamics, if their coordinate axes are related one to another by a transformation in $\underline{G}_R$. A more precise formulation is as follows.
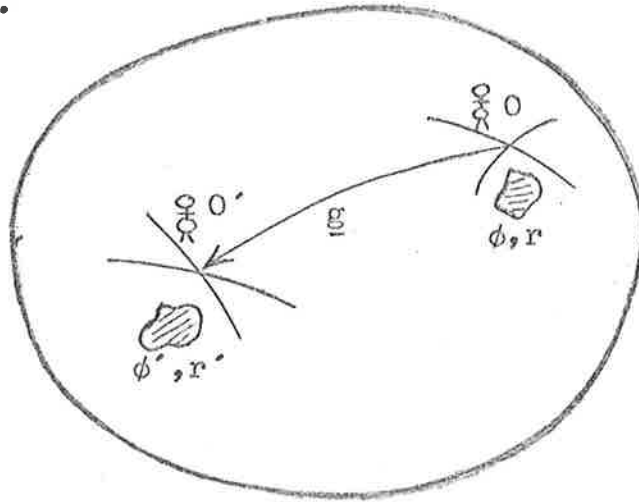
Suppose that O´ is an observer whose coordinate axes are obtained from those of observer O by a transformation $\underline{g}$ in $\underline{G}_R$. Both O and O´ conduct in their laboratories experiments on identical quantum systems. If O assigns the state ϕ to the system and measures an observable r, whereas O´ assigns $\phi´$ and measures r´, then

$$<\phi,r> \ = \ <\phi´,r´>$$

must hold if O and O´ are to be equivalent observers.

The mathematical formulation of the principle of relativity is trivial. Each element of $\underline{G}_R$ induces a transformation on the algebra of observables and, simultaneously, a contravariant transformation on its dual space, so that the expectation values are maintained. For example, in the usual formulation of quantum mechanics, the transformation $\underline{g}$ between the coordinate frames of O and O´ induces

$$r \longmapsto r´ = u_{\underline{g}} r u_{\underline{g}}^* ,$$

$$\Phi \longmapsto \Phi´ = u_{\underline{g}} \Phi ,$$

where $u_{\underline{g}}$ is unitary, so it follows trivially that

$$(\Phi',r'\Phi') = (\Phi,r\Phi).$$

If O is able to measure a certain observable r of the quantum system, then so should O'. If this were not so, then O and O' could never be equivalent observers. Thus, all equivalent observers must use the same algebra A^* of observables to describe a given quantum system. However, there is no need to demand that O and O' should use the same basis for A^*. All that is necessary is that the transformation relating their bases should be either an automorphism or anti-automorphism of A^*. Of these possibilities I will only consider the first. Let

$$\sigma_{\underline{g}} : A \longrightarrow A$$

denote the automorphism of A corresponding to $\underline{g}$.

The contravariant transformation of the states is constructed as follows. A is a vector space over C and the states of A form a subspace of its dual space A'. The transpose of $\sigma_{\underline{g}}$ is the map

$$\sigma'_{\underline{g}} : A' \longrightarrow A'$$
$$\psi \longmapsto \sigma'_{\underline{g}}\psi : A \longrightarrow C$$
$$a \longmapsto \langle\psi,\sigma_{\underline{g}}a\rangle.$$

Thus, $\qquad \langle\sigma'_{\underline{g}}\psi,a\rangle = \langle\psi,\sigma_{\underline{g}}a\rangle$ for all $a \in A$.

Lemma 19 establishes that $\sigma'_{\underline{g}}$ is an automorphism of the vector space S of states. Its inverse, $(\sigma'_{\underline{g}})^{-1}$, is the contravariant transformation of the set of states.

<u>Postulate 6.</u> $\qquad$ Aut (A^*), the group of automorphisms of A^*, must carry a representation of $\underline{G}_R$:

$$\sigma : \underline{G}_R \longrightarrow \text{Aut}(A^*)$$
$$\underline{g} \longmapsto \sigma_{\underline{g}} .$$

If an observer O assigns ϕ and r to a quantum system in his laboratory, then an observer O′, whose coordinate axes are obtained from those of O by a transformation g in $\underline{G}_R$, will assign

$$\phi' = (\sigma'_{\underline{g}})^{-1}\phi \quad \text{and} \quad r' = \sigma_{\underline{g}}r$$

to an identical system in his laboratory. It follows trivially that

$$\langle\phi',r'\rangle = \langle(\sigma'_{\underline{g}})^{-1}\phi,\sigma_{\underline{g}}r\rangle$$

$$= \langle\phi,r\rangle ,$$

so the principle of relativity is satisfied. ///

From where does the representation σ come? If $g \in G_R$, then ad g is an inner derivation of G_R and $\exp(\text{ad } g)$ is an automorphism of G_R. Define

$$\underline{g} = \exp g$$

and

$$\sigma_{\underline{g}} : G_R \longrightarrow G_R$$

$$g' \longmapsto \exp(\text{ad } g)g'.$$

$\sigma_{\underline{g}}$ can be trivially extended to $U(G_R)$ and thence to $U(G)$. In addition, $\sigma_{\underline{g}}$ is a *-automorphism of $U(G)$ because

$$\sigma_{\underline{g}}((g_1 + ig_2)^*) = \sigma_{\underline{g}}(-g_1 + ig_2)$$

$$= -\sigma_{\underline{g}}(g_1) + i\sigma_{\underline{g}}(g_2)$$

$$= (\sigma_{\underline{g}}(g_1+ig_2))^*, \text{ for all } g_1,g_2 \in G_R.$$

The map

$$\sigma : \underline{g} = \exp g \longmapsto \sigma_{\underline{g}} = \exp(\text{ad } g)$$

provides a representation of the connected component of $\underline{G}_R$ on the group of automorphisms of $U(G)$.

<u>Postulate 7.</u>   The representation σ of $\underline{G}_R$ on $\text{Aut}(A^*)$ must be an extension of the representation defined above.

The specification of the extension of $\sigma_{\underline{g}}$ from $U(G)$ to A will be an integral part of the specification of the algebra A appropriate to the quantum system. Discrete symmetries must be treated individually.

This formulation of the principle of relativity avoids the question as to whether or not the irreducible $U(G)$-modules which occur in the decomposition of an irreducible A^*-module can be integrated to provide representations of $\underline{G}_R$. This is an important technical point, because questions of integrability cannot be decided within an algebraic theory. On the other hand, $\exp(\text{ad } g)$ is well defined as an automorphism of $U(G)$, because $(\text{ad } g)$ is a finite dimensional matrix, and the problem of extending $\exp(\text{ad } g)$ from $U(G)$ to A^* can be posed algebraically.

Once the representation σ has been specified, then the temporal and spatial development of the system is known. For example, if $\underline{g}$ represents a translation into the future, and if the dynamics are invariant under reversal of the sense of time, then the expectation value of an observable r at a later time will be

$$<\phi, \sigma_{\underline{g}} r>.$$

Summary.

I have suggested the following mathematical structure for quantum mechanics. With any quantum system must be associated the following objects:

(1) a complex algebra A which extends $U(G)$,

$$U(G) < A \; ;$$

(2) an involution * on A which extends the natural involution

on $U(G)$;

(3) a representation σ of $\underline{G}_R$ on $\text{Aut}(A^*)$ which extends the natural representation on $U(G)$.

Each of these extensions is specific to the quantum system and must be postulated for that system. Every maximal ideal M in A determines by a canonical prescription an irreducible A-module X_M. There is at most one way in which an inner product can be defined on X_M so that $X_M{}^*$ is an A^*-module. When it is possible to do so, then a positive linear function can be defined on A by

$$<\phi,a> = (x,ax)/(x,x) \ , \quad a \ \epsilon \ A,$$

where $\qquad\qquad x = 1 + M$

is the canonically constructed cyclic vector of X_M. ϕ is a pure state if M is a maximal left ideal; ϕ is an impure state if M is the intersection of a finite number of maximal ideals. If $b \ \epsilon \ B < A$ is an observable element of A, then $<\phi,b>$ is its expectation value when the system is in the state ϕ.

In short, all the problems of quantum mechanics can be regarded as problems concerning the structure of the algebra of observables.

APPENDIX 2.

Theorem 1.   If A^* is a *-semisimple algebra, then the linear functions in $\tilde{S}$ separate the points of A^*.

Proof   Suppose a is any non-zero element of A^*. Because A^* is *-semisimple, there is an A^*-module X^* with

$$aX \neq 0 .$$

There is a vector x in X for which

$$(x,ax) \neq 0 .$$

If this were not so, a contradiction could be obtained as follows. The polarisation identity shows that

$$\begin{aligned} 4(x,ay) = \quad &(x + y,a(x + y)) \\ &- (x - y,a(x - y)) \\ &-i(x +iy,a(x +iy)) \\ &+i(x -iy,a(x -iy)) , \end{aligned}$$

so $(x,ay) = 0$ for all x and y in X.

In particular,

$$(ay,ay) = 0 \quad \text{for all y in X.}$$

Thus, $ay = 0$,

because the inner product is non-degenerate, and

$$a = 0 ,$$

contrary to assumption. Define

$$\phi : A \longrightarrow C$$
$$r \longmapsto (x,rx)/(x,x).$$

Then, $\langle\phi,1\rangle = 1$,

$$\langle\phi,r^*\rangle = \overline{\langle\phi,r\rangle} ,$$

$$\langle\phi,r^*r\rangle \geqq 0 , \quad \text{for all r in A.}$$

Thus, ϕ is a linear function in $\tilde{S}$ which satisfies

$$\langle\phi,a\rangle \neq 0 .$$

This proves that $\tilde{S}$ separates the points of A^*.   ///

<u>Lemma 2.</u>   Write $M \simeq N$ if M and N are proper left ideals of A and elements u and v can be found in A with the following properties:

$$Mu \leqq N \, , \quad uv = 1 (\text{mod } M),$$

$$Nv \leqq M \, , \quad vu = 1 (\text{mod } N).$$

$\simeq$ is an equivalence relation on the set of left ideals of A.

<u>Proof</u>   (1) Choose $u = v = 1$.   Then $M \simeq M$ and so $\simeq$ is reflexive.

(2) Suppose $M \simeq N$. Set

$$u' = v \, , \quad v' = u \, .$$

Then

$$Nu' \leqq M \, , \quad u'v' = 1 (\text{mod } N) \, ,$$

$$Mv' \leqq N \, , \quad v'u' = 1 (\text{mod } M) \, .$$

Hence $N \simeq M$ and the relation $\simeq$ is symmetric.

(3) Suppose $M \simeq N$ and $N \simeq P$ with

$$Mu \leqq N \, , \quad uv = 1 (\text{mod } M) \, ,$$

$$Nv \leqq M \, , \quad vu = 1 (\text{mod } N) \, ,$$

$$Nr \leqq P \, , \quad rs = 1 (\text{mod } N) \, ,$$

$$Ps \leqq N \, , \quad sr = 1 (\text{mod } P) \, .$$

Then
$$Mur \leqq P \, ,$$

$$Psv \leqq M \, .$$

Also
$$(ur)(sv) = u(1(\text{mod } N))v$$

$$= 1 (\text{mod } M)$$

$$(sv)(ur) = s(1(\text{mod } N))r$$

$$= 1 (\text{mod } P) \, .$$

Thus, $M \simeq P$ and the relation is transitive.   ///

<u>Theorem 3.</u>   Suppose that M and N are proper left ideals of A. Construct

$$X_M = A/M \text{ and } X_N = A/N.$$

In order that

$$X_M \cong X_N ,$$

it is both necessary and sufficient that

$$M \cong N .$$

Proof of sufficiency.

To prove the sufficiency of this condition I must construct an isomorphism of the vector spaces X_M and X_N which also intertwines the module operations. Since $M \cong N$, there exist elements u and v in A such that

$$Mu \leqq N , \quad uv = 1 \,(\text{mod } M) ,$$
$$Nv \leqq M , \quad vu = 1 \,(\text{mod } N) .$$

Define
$$\sigma : X_M \longrightarrow X_N$$

$$a+M \longmapsto au+N .$$

σ is well defined because

$$a+m+M \longmapsto au+mu+N = au+N$$

for all m in M. It is here that the property $Mu \leqq N$ is used. Similarly, define

$$\tau : X_N \longrightarrow X_M$$

$$a+N \longmapsto av+M .$$

τ is also well defined because $Nv \leqq M$.

Now
$$\sigma.\tau(a+N) = \sigma(av+M)$$
$$= avu+N$$
$$= a+N \quad \text{for all a in A,}$$

because
$$vu = 1\,(\text{mod } N).$$

Similarly,

$$\tau.\sigma(a+M) = \tau(au+N)$$
$$= auv+M$$
$$= a+M \quad \text{for all a in A,}$$

because $\qquad uv = 1 \pmod{M}.$

Thus $\qquad\qquad \tau.\sigma = 1 = \sigma.\tau .$

σ is an isomorphism of the vector spaces X_M and X_N. All that remains to be shown is that σ intertwines the module operations. For all a and b in A,

$$(\sigma a)(b+M) = \sigma(ab+M)$$
$$= abu+N$$
$$= a(bu+N)$$
$$= (a\sigma)(b+M) .$$

Hence, X_M and X_N are equivalent A-modules.

<u>Proof of necessity.</u> $\qquad$ Let σ denote the isomorphism of X_M onto X_N:

$$\sigma : X_M \longrightarrow X_N.$$

(1) Because $(1+M)$ is a non-zero vector of X_M and σ is injective,

$$\sigma(1+M) \neq N.$$

Thus, there exists an element $u \notin N$ such that

$$\sigma(1+M) = u+N.$$

Of course u is not unique since $u+n$, $n \in N$, would still identify the coset $u+N$. Now

$$\sigma(a+M) = \sigma a(1+M)$$
$$= a\sigma(1+M)$$
$$= a(u+N)$$
$$= au+N .$$

The vital, second step in the above chain uses the fact that σ intertwines the module operations on X_M and X_N.

(2) Because σ is injective, if

$$\sigma(a+M) = au+N = N,$$

that is, if $au \in N$, then $a \in M$. Conversely, if $a \in M$, then $au \in N$, so $\qquad\qquad Mu \leq N.$

(3) Because σ is surjective, there must exist a coset $(v+M)$ in X_M such that

$$\sigma(v+M) = vu+N = 1+N.$$

Since $(1+N)$ is a non-zero vector of X_N, $v \notin M$. Thus,

$$vu = 1 \pmod{N}.$$

(4) Define $\qquad \tau : X_N \longrightarrow X_M$

$$a+N \longmapsto av+M.$$

τ will only be well defined provided $nv \in M$ for all $n \in N$. Otherwise two different representations of the same coset $(a+N)$ would lead to different images. Now,

$$\sigma(av+M) = avu+N$$

$$= a+N .$$

In particular, $a \in N$ if and only if $av \in M$ because σ is injective. Thus,

$$Nv \leqq M$$

and τ is well defined.

(5) $$\sigma \cdot \tau(a+N) = \sigma(av+M)$$

$$= avu+N$$

$$= a+N$$

Thus, $\qquad\qquad \sigma \cdot \tau = 1.$

Since σ is known to be bijective,

$$\tau = \sigma^{-1}.$$

$$(\tau \cdot \sigma)(a+M) = \tau(au+N)$$

$$= auv+M .$$

Choose $a = 1$.

$$(\tau \cdot \sigma)(1+M) = uv+M$$

$$= 1+M ,$$

so $\qquad\qquad uv = 1 \pmod{M} .$

(6) I have found u and v in A such that

$$Mu \leqq N \quad , \quad uv = 1 \pmod{M} \ ,$$

$$Nv \leqq M \quad , \quad vu = 1 \pmod{N} \ .$$

This shows that $M \cong N$.   ///

Lemma 4.   Suppose that M_1 is a maximal left ideal of A

and that M' is any other left ideal, subject only to the

condition $M' \nleq M_1$. Set

$$M^0 = M' \cap M_1 \ ,$$

$$X^0 = A/M^0 \ ,$$

$$X' = A/M' \ ,$$

$$X_1 = A/M_1 \ .$$

Then $X^0 \cong X_1 \oplus X' \ ,$

the sum being a direct sum of A-modules.

Proof   Let $N = M_1 + M'$.

Because $M' \nleq M_1$, there is an element m'_1 in M' which does not

belong to M_1. Consequently, M_1 is properly contained in the

ideal N:

$$M_1 < N < A.$$

Since M_1 is a maximal ideal, it follows that $N = A$. Hence,

every element a in A can be expressed

$$a = m_1 + m' \quad ; \quad m_1 \ \epsilon \ M_1, \ m' \ \epsilon \ M' \ ,$$

though this decomposition will not be unique. Thus,

$$(a + M^0) = (m_1 + M^0) + (m' + M^0).$$

Set $Y_1 = M_1/M^0$ and $Y' = M'/M^0$.

Clearly, $X^0 = Y_1 + Y'.$

I want to demonstrate that this sum is direct. Suppose that

$$a + M^0 \ \epsilon \ Y_1 \cap Y' \ .$$

Then $\qquad a + M^0 = m_1 + M^0$, $m_1 \in M_1$,

and $\qquad a + M^0 = m' + M^0$, $m' \in M'$.

Subtract the second equation from the first. This shows that

$$m_1 - m' \in M^0 .$$

But $\qquad\qquad M^0 < M'$

so $\qquad\qquad m_1 - m' \in M'$,

which implies that $\qquad m_1 \in M'$.

Thus, $\qquad\qquad m_1 \in M_1 \cap M' = M^0$

and $\qquad a + M^0 = m_1 + M^0 = M^0 .$

Therefore, $\qquad Y_1 \cap Y' = 0 .$

Y_1 and Y' are A-submodules of X^0. This is easy to see because M_1 and M' are left ideals of A:

$$a(m_1 + M^0) = am_1 + M^0 < Y_1 ,$$
$$a(m' + M^0) = am' + M^0 < Y' .$$

Hence, $\qquad\qquad X^0 = Y_1 \oplus Y' ,$

where the direct sum is not merely of the vector spaces Y_1 and Y' but also of the whole module structures.

Now $\qquad\qquad X^0/Y_1 \simeq Y'$ and $X^0/Y' \simeq Y_1 .$

But

$$X^0/Y_1 = (A/M^0)/(M_1/M^0) \simeq A/M_1 = X_1$$

and

$$X^0/Y' = (A/M^0)/(M'/M^0) \simeq A/M' = X' .$$

Thus $\qquad\qquad Y' \simeq X_1$ and $Y_1 \simeq X'$

This establishes the result, that

$$X^0 \simeq X_1 \oplus X' .$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ///

Theorem 5.

Suppose that $\{M_1, M_2, \cdots\}$ is a countable family of distinct maximal left ideals of A. Set

$$M^0 = \bigcap_{i \geq 1} M_i .$$

Construct the A-modules $X^0 = A/M^0$ and $X_i = A/M_i$. Then

$$X^0 = \bigoplus_{i \geq 1} X_i .$$

Proof.   Let the index i take the values 0, 1, 2, $\cdots$.
Set
$$M^i = \bigcap_{j > i} M_j .$$

Then
$$M^i = M^{i+1} \bigcap M_{i+1} .$$

Construct also the A-modules

$$X^i = A/M^i .$$

Since M_{i+1} is a maximal left ideal of A, the previous lemma can be applied to decompose X^i:

$$X^i \simeq X_{i+1} \oplus X^{i+1} .$$

By induction,

$$X^0 \simeq \bigoplus_{i \geq 1} X_i .$$ ///

Lemma 6.   If X^* is an irreducible A^*-module, then X is an irreducible A-module.

Proof (Trivial).   Suppose that X had a proper A-submodule Y. Then

$$aY \leq Y \quad \text{for all a in A.}$$

Let Y^* denote Y equipped with the inner product inherited from X^*. Choose any two vectors y and y' from Y^*. Because y and y' are also vectors of X^* and X^* is an A^*-module,

$$(y, ay') = (a^* y, y').$$

But this is precisely the condition to be satisfied if Y^* is to be an A^*-submodule of X^*, so Y must be trivial.   ///

Lemma 7 (Schwartz).   I have only included this lemma in order to stress that it is true in inner product spaces as well as Hilbert spaces.

If $\phi \in \tilde{S}$, then for all a and b in A^*

$$|<\phi, a^*b>|^2 \leq <\phi, a^*a><\phi, b^*b> \, .$$

Proof

$<\phi, (a-\lambda b)^*(a-\lambda b)> = <\phi, a^*a>-\bar{\lambda}<\phi, b^*a>-\lambda<\phi, a^*b>+\bar{\lambda}\lambda<\phi, b^*b> \geq 0.$

If $<\phi, b^*b> \neq 0,$

set $\lambda = <\phi, b^*a>/<\phi, b^*b>.$

Then $<\phi, a^*a> \geq |<\phi, a^*b>|^2/<\phi, b^*b>$

as required. However, if $<\phi, b^*b> = 0,$

then a contradiction is obtained for a suitable value of λ unless $<\phi, a^*b> = 0.$ In either case,

$$<\phi, a^*a><\phi, b^*b> \geq |<\phi, a^*b>|^2.$$ ///

Lemma 8.   Suppose that

$$M_\phi = \{m \in A^* \mid <\phi, m^*m> = 0\}.$$

M_ϕ has an alternative presentation,

$$M_\phi = \{m \in A^* \mid <\phi, a^*m> = 0 \text{ for all a in } A^*\}.$$

M_ϕ is a left ideal.

Proof   If $<\phi, a^*m> = 0$ for all a in A^* then certainly $<\phi, m^*m> = 0.$ Since

$$|<\phi, a^*m>|^2 \leq <\phi, a^*a><\phi, m^*m>,$$

if $<\phi, m^*m> = 0$, then $<\phi, a^*m> = 0$ for all a in $A^*.$

Thus, the two presentations of M_ϕ are equivalent. From the second presentation it is clear that M_ϕ is a subspace of A^*. Choose any m in M_ϕ and consider rm, $r \epsilon A$.

$$<\phi, a^*(rm)> = <\phi, (r^*a)^*m> = 0, \text{ for all a in A.}$$

Thus, $rm \epsilon M_\phi$ and M_ϕ is a left ideal. ///

Lemma 9. X_ϕ^* is a cyclic A^*-module for each ϕ in $\widetilde{S}$.

Proof.   Since M_ϕ is a left ideal of A, $X_\phi = A/M_\phi$ is a cyclic A-module. If I can show that the inner product on X_ϕ satisfies

$$((r+M_\phi), a(s+M_\phi)) = (a^*(r+M_\phi), (s+M_\phi))$$

for all a, r and s in A^*, then X_ϕ^* will be an A^*-module. Now

$$((r+M_\phi), a(s+M_\phi)) = (r+M_\phi, as+M_\phi)$$
$$= <\phi, r^*as>$$
$$= <\phi, (a^*r)^*s>$$
$$= (a^*r+M_\phi, s+M_\phi)$$
$$= (a^*(r+M_\phi), (s+M_\phi)).$$

Thus, X_ϕ^* is an A^*-module. ///

Lemma 10. X_ϕ^* is an irreducible A^*-module if and only if X_ϕ is an irreducible A-module. Equivalently, X_ϕ^* is irreducible if and only if M_ϕ is a maximal left ideal of A.

Proof   From lemma 6 it follows that X_ϕ is an irreducible A-module if X_ϕ^* is an irreducible A^*-module. The converse is trivial. X_ϕ can be equipped with the inner product derived

from ϕ, and if X_ϕ is irreducible so too is the space X_ϕ with its inner product. Finally, X_ϕ is irreducible if and only if M_ϕ is a maximal left ideal of A.   ///

<u>Theorem 11.</u>   Suppose X^* and Y^* are cyclic A modules with cyclic vectors x and y.   If

$$(x,ax) = (y,ay) \quad \text{for all a in A,}$$

then $X^* \cong Y^*$.

<u>Proof</u>   Define $\sigma : X \longrightarrow Y$

$$rx \longmapsto ry, \text{ for all r in A.}$$

Because y is a cyclic vector, σ is surjective. σ is also injective for the following reason. If $rx \neq 0$ but $ry = 0$, then

$$0 \neq (rx,rx) = (x,r^*rx)$$

$$= (y,r^*ry)$$

$$= (ry,ry)$$

$$= 0 ,$$

a contradiction. σ also intertwines the module operations in X and Y:

$$(\sigma a)(rx) = \sigma(arx)$$
$$= ary$$
$$= a(ry)$$
$$= (a\sigma)(rx) .$$

Finally, for all r and s in A,

$$(ry,sy) = (r\sigma x, s\sigma y)$$
$$= (\sigma rx, \sigma sy) .$$

By assumption

$$(y,r^*sy) = (x,r^*sx)$$

so $$(rx,sx) = (\sigma rx, \sigma sx) .$$

Thus, X^* and Y^* are equivalent A^*-modules. ///

Lemma 12. Suppose that ϕ is a linear function in $\tilde{S}$ for which M_ϕ is a maximal left ideal of A. Define ψ by

$$\langle \psi, a \rangle = \langle \phi, u^* a u \rangle \quad \text{for all } a \epsilon A,$$

where $\quad u \notin M_\phi$

and u is normalised so that

$$\langle \phi, u^* u \rangle = 1.$$

(1) ψ is also a linear function in $\tilde{S}$.

(2) There exists an element v in A such that

$$\langle \phi, a \rangle = \langle \psi, v^* a v \rangle \quad \text{for all } a \epsilon A,$$

where $\quad v \notin M_\psi$

and v is normalised so that

$$\langle \psi, v^* v \rangle = 1.$$

(3) The left ideals M_ϕ and M_ψ are equivalent,

$$M_\psi \cong M_\phi ,$$

with $\quad M_\psi u \leqq M_\phi , \quad uv = 1 (\text{mod } M_\psi),$

$$M_\phi v \leqq M_\psi , \quad vu = 1 (\text{mod } M_\phi).$$

(4) X_ϕ^* and X_ψ^* are equivalent, irreducible A^*-modules.

Proof (1) This step is trivial.

$$\langle \psi, 1 \rangle = \langle \phi, u^* u \rangle = 1,$$

$$\langle \psi, a^* \rangle = \langle \phi, u^* a^* u \rangle$$

$$= \overline{\langle \phi, u^* a u \rangle}$$

$$= \overline{\langle \psi, a \rangle},$$

$$\langle \psi, a^* a \rangle = \langle \phi, (au)^* (au) \rangle \geqq 0.$$

(2) Since

$$\langle \psi, a^* a \rangle = \langle \phi, (au)^*(au) \rangle \quad , \quad \text{for all } a\epsilon A,$$

it follows that

$$a\epsilon M_\psi \text{ if and only if } au\epsilon M_\phi.$$

In particular,

$$M_\psi u \leqq M_\phi.$$

Because M_ϕ is a maximal left ideal of A, X_ϕ is an irreducible A-module, and so every vector of X_ϕ is cyclic. Thus, there is an element v in A such that

$$v(u + M_\phi) = 1 + M_\phi.$$

Hence, $$vu = 1 (\text{mod } M_\phi).$$

$$\langle \psi, v^* a v \rangle = \langle \phi, u^* v^* a v u \rangle$$

Suppose that $$vu = 1 + m, \quad m \epsilon M_\phi.$$

Then $$\langle \psi, v^* a v \rangle = \langle \phi, (1+m^*) a (1+m) \rangle$$

$$= \langle \phi, a + m^* a \rangle$$

$$= \langle \phi, a \rangle + \overline{\langle \phi, a^* m \rangle}$$

$$= \langle \phi, a \rangle.$$

Also $v \notin M_\psi$ because $vu \notin M_\phi.$

Finally, $$\langle \psi, v^* v \rangle = \langle \phi, 1 \rangle = 1.$$

(3) I have already established that

$$M_\psi u \leqq M_\phi \quad , \quad vu = 1 (\text{mod } M_\phi),$$

so I must now show that

$$M_\phi v \leqq M_\psi \quad , \quad uv = 1 (\text{mod } M_\psi).$$

Since

$$\langle \phi, a^* a \rangle = \langle \psi, (av)^*(av) \rangle \quad , \quad \text{for all } a\epsilon A,$$

it follows that

$$a\epsilon M_\phi \text{ if and only if } av\epsilon M_\psi.$$

In particular,

$$M_\phi v \leqq M_\psi.$$

$$\langle\psi, (uv-1)^*(uv-1)\rangle = \langle\phi, u^*(uv-1)^*(uv-1)u\rangle.$$

Now
$$(uv - 1)u = uvu - u$$
$$= 0 \pmod{M_\phi}.$$

Hence, $\quad \langle\psi, (uv-1)^*(uv-1)\rangle = 0$

and so
$$uv = 1 \pmod{M_\psi}.$$

The conditions established show that $M_\psi \cong M_\phi$.

(4) Because $M_\psi \cong M_\phi$, theorem 3 shows that X_ψ and X_ϕ are equivalent A-modules. Furthermore, the isomorphism which intertwines X_ψ and X_ϕ is

$$\sigma : \quad X_\psi \longrightarrow X_\phi$$

$$a+M_\psi \longmapsto au+M_\phi.$$

I will now show that

$$(\sigma(r+M_\psi), \sigma(s+M_\psi)) = (r+M_\psi, s+M_\psi)$$

for all r and s in A, where the inner product on the left is in X_ϕ but on the right is in X_ψ.

$$(\sigma(r+M_\psi), \sigma(s+M_\psi)) = (ru+M_\phi, su+M_\phi)$$

$$= \langle\phi, u^* r^* su\rangle$$

$$= \langle\psi, r^* s\rangle$$

$$= (r+M_\psi, s+M_\psi).$$

Thus, X_ψ^* and X_ϕ^* are equivalent A^*-modules. Because M_ϕ is a maximal left ideal, X_ϕ^* is irreducible. X_ψ^* obviously is also irreducible. ///

<u>Definition</u>.   Let

$$S_o = \{\phi \in \tilde{\tilde{S}} \mid M_\phi \text{ is a maximal ideal in A}\}.$$

Define a relation $\simeq$ on S_o as follows. ψ is in relation to ϕ, written $\psi \simeq \phi$, if

$$\langle \psi, a \rangle = \langle \phi, u^*au \rangle , \quad \text{for all } a \in A,$$

where
$$u \notin M_\phi$$

and u is normalised so that

$$\langle \phi, u^*u \rangle = 1.$$

<u>Lemma 13</u>. $\simeq$ is an equivalence relation on S_o.

<u>Proof</u>   (1) $\simeq$ is reflexive because $\phi \simeq \phi$ with $u = 1$.

(2) Suppose $\psi \simeq \phi$. In the last lemma it was shown that an element v could be found with the following properties:

$$\langle \phi, a \rangle = \langle \psi, v^*av \rangle \quad \text{for all } a \in A,$$

$$v \notin M_\psi ,$$

$$\langle \psi, v^*v \rangle = 1 .$$

Thus, $\phi \simeq \psi$

and so $\simeq$ is symmetric.

(3) If $\psi \simeq \phi$ and $\phi \simeq \chi$, with

$$\langle \psi, a \rangle = \langle \phi, u^*au \rangle$$

and
$$\langle \phi, a \rangle = \langle \chi, r^*ar \rangle ,$$

then
$$\langle \psi, a \rangle = \langle \chi, (ur)^*a(ur) \rangle \quad \text{for all } a \in A,$$

where
$$ur \notin M\chi$$

and
$$\langle \chi, (ur)^*(ur) \rangle = \langle \psi, 1 \rangle = 1 .$$

Hence, $\simeq$ is transitive.   ///

<u>Lemma 14</u>.   If $X_\phi^* \simeq X_\psi^*$, then $\phi \simeq \psi$.

<u>Proof</u>   Because $X_\phi{}^* \cong X_\psi{}^*$, there exists an isomorphism

$$\sigma : X_\psi \longrightarrow X_\phi$$

which intertwines the A-modules X_ψ and X_ϕ and which also satisfies

$$(\sigma(r+M_\psi), \sigma(s+M_\psi)) = (r+M_\psi, s+M_\psi)$$

for all r and s in A. Define u and v, respectively modulo M_ϕ and M_ψ, by

$$\sigma(1+M_\psi) = u+M_\phi$$

$$\sigma^{-1}(1+M_\phi) = v+M_\psi.$$

Because σ intertwines X_ψ and X_ϕ,

$$\begin{aligned}
\sigma(a+M_\psi) &= (\sigma a)(1+M_\psi) \\
&= (a\sigma)(1+M_\psi) \\
&= a(u+M_\phi) \\
&= au+M_\phi, \quad \text{for all } a\epsilon A.
\end{aligned}$$

Similarly,

$$\sigma^{-1}(a+M_\phi) = av+M_\psi, \quad \text{for all } a\epsilon A.$$

Now

$$\begin{aligned}
<\psi,a> &= ((1+M_\psi), (a+M_\psi)) \\
&= (\sigma(1+M_\psi), \sigma(a+M_\psi)) \\
&= ((u+M_\phi), (au+M_\phi)) \\
&= <\phi, u^*au>.
\end{aligned}$$

In identical fashion,

$$<\phi,a> = <\psi, v^*av>. \qquad ///$$

Note that the irreducibility of $X_\phi{}^*$ and $X_\psi{}^*$ was not needed in the proof of this lemma.

<u>Theorem 15.</u>   Suppose ϕ and ψ are linear

functions in S_o. Then

$$X_\phi{}^* \simeq X_\psi{}^* \Longleftrightarrow \phi \simeq \psi$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$X_\phi \simeq X_\psi \Longleftrightarrow M_\phi \simeq M_\psi.$$

<u>Proof.</u>   The theorem is a trivial consequence of theorem 3 and lemmas 12 and 14.   ///

<u>Lemma 16.</u>   Suppose that X^* is an A^*-module. X^* is completely reducible.

<u>Proof.</u>   Suppose that Y^* is a proper A^*-submodule of X^*. Define

$$Y^\perp = \{x \in X^* \mid (x,y) = 0 \text{ for all } y \in Y\}.$$

Choose any $x \in Y^\perp$. Since $ay \in Y$ for all $y \in Y$,

$$0 = (x,ay) = (a^*x,y).$$

Thus, $a^*x \in Y^\perp$ for all $a \in A$. Because $A^* = A$, it follows that

$$AY^\perp \leq Y^\perp.$$

Since $$X^* = Y \oplus Y^\perp,$$

X^* is completely reducible.   ///

<u>Theorem 17.</u>   Suppose that M_ϕ and M_ψ are maximal left ideals of A. Then

$$M_\phi \simeq M_\psi \Longrightarrow \phi \simeq \psi.$$

A stronger result is also true,

$$M_\phi = M_\chi \Longrightarrow \phi = \chi.$$

There are shorter proofs of this theorem than the one given below, but I chose this one because it demonstrates the difficulties that arise because Schur's lemma does not imply that the commuting algebra of an irreducible A-module consists only of scalars.

<u>Proof.</u>   Because $M_\phi \simeq M_\psi$, there exist elements u and v in A such that

$$M_\phi v \leqq M_\psi \quad , \quad vu = 1 \pmod{M_\phi},$$

$$M_\psi u \leqq M_\phi \quad , \quad uv = 1 \pmod{M_\psi}.$$

Define $\quad \langle \chi, a \rangle = \langle \psi, v^* a v \rangle / \langle \psi, v^* v \rangle \quad$ for all $a \in A$.

Thus, $a \in M_\chi$ if and only if $av \in M_\psi$.

Now, if $av \in M_\psi$, then $avu \in M_\phi$ and so $a \in M_\phi$. Thus, $av \in M_\psi$ if and only if $a \in M_\phi$.

Consequently,

$$M_\phi = M_\chi.$$

By assumption M_ψ is a maximal left ideal, so

$$\psi \cong \chi .$$

Thus, I need only investigate whether

$$M_\phi = M_\chi \implies \phi \cong \chi,$$

or the stronger result,

$$M_\phi = M_\chi \implies \phi = \chi.$$

Set $\qquad\qquad M = M_\phi = M_\chi$

and $\qquad\qquad X = X_\phi = X_\chi.$

On the module X there are defined two inner products,

$$(a + M, b + M)_\phi = \langle \phi, a^* b \rangle$$

and $\qquad\qquad (a + M, b + M)_\chi = \langle \chi, a^* b \rangle,$

and hermitian conjugation with respect to each of these provides a representation of the involution on A. Since A has a countable basis, so too has X. Choose a basis

$$\{x_1 = 1 + M, x_2, x_3, \cdots \}$$

for X which is orthonormal with respect to the first inner product;

$$(x_i, x_j)_\phi = \delta_{ij}.$$

Choose any r in A. The matrices of r and r^* on X are fixed by

the basis.

$$rx_j = \sum_i r_{ij} x_i \ ,$$

$$r^* x_j = \sum_i r_{ij}^* x_i \ .$$

Now
$$(r^* x_j, x_k)_\phi = \sum_i (r_{ij}^* x_i, x_k)_\phi$$

$$= \overline{r}_{kj}^* \ .$$

Similarly,
$$(x_j, r x_k)_\phi = \sum_i (x_j, r_{ik} x_i)_\phi$$

$$= r_{jk} \ .$$

Thus, the matrix of r^* is the hermitian conjugate of the matrix of r, just as would have been expected. Since the matrices of r and r^* are both column-finite, this proves that they are row-finite as well.

The basis vectors will not be orthonormal with respect to the second inner product. In fact,

$$(x_i, x_j)_\chi = g_{ij},$$

where the array (g_{ij}) satisfies

$$\overline{g_{ij}} = g_{ji},$$

but in general need be neither row nor column-finite. A calculation similar to the one above shows that

$$\sum_i \overline{r}_{ij}^* g_{ik} = \sum_i g_{ji} r_{ik} \ .$$

The matrix of r is fixed by the basis and does not depend upon

the inner product defined on X.

Thus,
$$\bar{r}^{*}_{ij} = r_{ji} \, ,$$

as before. Hence,

$$\sum_{i} r_{ji} g_{ik} = \sum_{i} g_{ji} r_{ik} \, .$$

The summations here are well defined because the matrix of r is both row and column-finite.

The matrix (g_{ij}) commutes with all the matrices of the representation. $\underline{If}$ it follows that (g_{ij}) is diagonal,

$$g_{ij} = g_{i} \delta_{ij} \, ,$$

in particular a multiple of the identity, then

$$g_{1} = 1$$

must hold because

$$(x_{1}, x_{1})_{X} = \langle \chi, 1 \rangle = 1 \, .$$

It would then follow that

$$(x_{1}, rx_{1})_{\phi} = (x_{1}, rx_{1})_{X} \quad \text{for all } r \in A,$$

and so

$$\langle \phi, r \rangle = \langle \chi, r \rangle$$

as required.

If A contains a finite set of elements whose matrices separate the basis of X, then it is trivial to show that (g_{ij}) must be diagonal. However, since I know that

$$U(K) < U(G) < A,$$

where K is the complexification of $K_{R} = so(3,R) < G_{R}$, I will proceed as follows.

According to postulate 4, X^{*} can be decomposed into irreducible K^{*}-modules, each of finite dimension;

$$X^{*} = \bigoplus_{j} X_{j}^{*} \, .$$

Furthermore, the irreducible K^*-modules occuring in this decomposition are orthogonal with respect to both inner products on X. Thus, if

$$\{y_1, y_2, \cdots \}$$

is a basis adapted to the decomposition of X^*, with

$$(y_i, y_j)_\phi = \delta_{ij} ,$$
$$(y_i, y_j)_\chi = g_{ij} ,$$

then the matrix $g = (g_{ij})$ can be decomposed similarly,

$$g = \bigoplus_j g_j ,$$

where each g_j is a finite dimensional matrix. Since g_j is an endomorphism of X_j^* and commutes with $U(K)$ on X_j^*, it follows that g_j must be a multiple of the identity endomorphism of X_j^*.

This establishes that g is a diagonal matrix relative to the basis $\{y_1, y_2, \cdots \}$, but g need not be so in the basis $\{x_1, x_2, \cdots \}$. However, for every complex λ, $(g - \lambda)$ lies in the centraliser of the irreducible A^*-module X^*. Schur's lemma states that the centraliser is a division ring, so $(g - \lambda)$ must either be invertible or be zero. Because g has an eigenvector in X, y_1 for example, the first possibility must be discarded and so $g = \lambda$. The condition that

$$<\phi, 1> = <\chi, 1> = 1$$

forces g to be the identity matrix, so $\phi = \chi$ as required. ///

Theorem 18. (1) The set S_o of pure states separates the points of A.

(2) The points of S_o are the extreme points of S.

(3) Suppose $\phi_1, \phi_2, \cdots, \phi_n$ are distinct pure states. If

$$\phi_0 = \sum_i \lambda_i \phi_i \ ,$$

where $\lambda_i \geqq 0$ and
$$\sum_i \lambda_i = 1 \ ,$$

then
$$X_{\phi_0}^* \simeq \bigoplus_i X_{\phi_i}^* \ .$$

<u>Proof.</u>   (1) The proof is identical with the proof of theorem 1 because the irreducible A^*-modules suffice to separate the points of A.

(2) Choose any $\phi_0 \in S_0$. I must show that, if

$$\phi_0 = \sum_{i=1}^n \lambda_i \phi_i$$

with

$$\phi_i \in S_0 \ , \ \lambda_i \geqq 0, \ \sum_{i=1}^n \lambda_i = 1,$$

then
$$\phi_0 = \phi_1 = \cdots = \phi_n .$$

Because
$$\langle \phi_0, a^* a \rangle = \sum_{i=1}^n \lambda_i \langle \phi_i, a^* a \rangle$$

and both λ_i and $\langle \phi_i, a^* a \rangle$ are positive for each i,
$$M_{\phi_0} = \bigcap_i M_{\phi_i} .$$

Thus,
$$M_{\phi_0} \leqq M_{\phi_i} .$$

Since $\phi_0 \in S_0$, M_{ϕ_0} is a maximal left ideal. Therefore,
$$M_{\phi_0} = M_{\phi_i} .$$

By theorem 17,
$$\phi_0 = \phi_i \ , \quad i = 1, 2, \cdots ,n.$$

(3) Because the pure states $\phi_1, \phi_2, \cdots, \phi_n$ are distinct, it follows from theorem 17 that $M_{\phi_1}, M_{\phi_2}, \cdots, M_{\phi_n}$ are also distinct. Now

$$M_{\phi_o} = \bigcap_i M_{\phi_i}.$$

According to theorem 5,

$$X_{\phi_o} \cong \bigoplus_i X_{\phi_i},$$

where the sum is a direct sum of A-modules.

$$(r+M_{\phi_o}, s+M_{\phi_o}) = \langle \phi_o, r^* s \rangle$$

$$= \sum_{i=1}^n \lambda_i \langle \phi_i, r^* s \rangle$$

$$= \sum_{i=1}^n \lambda_i (r+M_{\phi_i}, s+M_{\phi_i}).$$

Thus,

$$X_{\phi_o}^* \cong \bigoplus_i X_{\phi_i}^*,$$

where the sum is now a direct sum of A^*-modules.   ///

Lemma 19.   If σ_g is an automorphism of A^*, then σ_g' is an automorphism of the vector space S.

Proof.   The proof can be broken into four simple stages.

(1) σ_g' is an automorphism of A'. Suppose that $\sigma_g' \psi = 0$. Then

$$\langle \sigma_g' \psi, a \rangle = \langle \psi, \sigma_g a \rangle = 0, \quad \text{for all } a \in A,$$

which implies $\psi = 0$ because σ_g is an automorphism of A. Hence, σ_g' is injective. Now suppose ϕ is an arbitrary linear function on A. Define ψ by

$$\langle \psi, a \rangle = \langle \phi, \sigma_g^{-1} a \rangle, \quad \text{for all } a \in A.$$

Thus, $\langle \psi, \sigma_g b \rangle = \langle \phi, b \rangle$

where $a = \sigma_g b.$

Therefore $$\phi = \sigma'_{\underline{g}}\psi \ .$$

This shows that $\sigma'_{\underline{g}}$ is surjective.

(2) It is trivial to verify that

$$\sigma'_{\underline{g}}\tilde{S} \leqq \tilde{S},$$

because $\sigma_{\underline{g}}$ is an automorphism of A^*, so I need only show that

$$\sigma'_{\underline{g}}\tilde{S} = \tilde{S}.$$

Choose any $\phi \ \epsilon \ \tilde{S}$ and define

$$<\psi,a> \ = \ <\phi,\sigma_{\underline{g}}^{-1}a> \quad \text{for all } a \ \epsilon \ A.$$

ψ also lies in $\tilde{S}$. However, as in part (1),

$$\phi = \sigma'_{\underline{g}}\psi \ ,$$

so $\sigma'_{\underline{g}}$ is a surjective mapping of $\tilde{S}$.

(3) Because $\sigma_{\underline{g}}$ is an automorphism of A, the image and preimage under $\sigma_{\underline{g}}$ of every maximal ideal of A are also maximal. Thus, if M_ϕ is maximal, then so too is M_ψ, where

$$\psi = \sigma'_{\underline{g}}\phi \ .$$

Consequently, $\sigma'_{\underline{g}}$ is an automorphism of S_o.

(4) The extension to S, the convex hull of S_o, is now trivial. ///

CHAPTER 3.   DIAGONAL OPERATORS.

'Spectrum generating algebras' (SGA) were intro-
duced by Dothan, Gell-Mann and Ne´eman (1965) in an attempt to
provide an algebraic description of quantum systems.   The
idea has proved useful and the literature on SGA is voluminous.
Every paper on the subject purports to be algebraic, but in
fact contains a mixture of algebraic and analytic concepts
which cannot easily be disentangled.   Because the framework
for quantum mechanics suggested in chapter 2 accommodates
spectrum generating algebras quite naturally, in this chapter
I will try to unravel the tangle.   Thus, I will try to give
a precise account of the purely algebraic structures present
in the theory of spectrum generating algebras.

Suppose that t is a linear operator defined on an
infinite dimensional vector space X.   To find the eigenvalues
of t is generally a difficult task.   If it is known that t
commutes with a Lie algebra K,

$$[t,K] = 0 ,$$

the task is somewhat simplified, because the eigenvectors of
t divide into degenerate multiplets, each of which carries
an irreducible representation of K.   If all these multiplets
span a single irreducible representation of a larger Lie
algebra L, which contains K as a subalgebra, and if

$$[t,L] \leq L ,$$

then the calculation of the spectrum of t is trivial.   L is
called a spectrum generating algebra for t; K is the symmetry
algebra of t.   To find K and L and to express t in terms of
the elements of the enveloping algebra of L are the aims of
the theory of spectrum generating algebras.

The operator t is usually the Hamiltonian, and so is

a function of several coordinates and conjugate momenta. The
determination of K and L is then a difficult matter. Professor
Green has suggested that the coordinates and momenta which
appear in the Hamiltonian have dubious physical significance,
whereas the algebras K and L are directly related to the
observable spectrum of states. Therefore, he proposed that
the operator t should be defined directly in terms of the
algebras K and L. I wholeheartedly agree with this approach,
and the diagonal and codiagonal operators of this and following
chapters represent the logical development of this idea.

Diagonal operators are extremely simple, as the name
implies. In loose terms, t is a diagonal operator if t is a
function of the invariants of the Lie algebra K. When t acts
upon a space X, which carries an irreducible representation
of L, t will be represented by a diagonal matrix; hence the
name. If X decomposes into irreducible representations of K,
then the spectrum of t is constant on each such representation.
To find the whole spectrum of t, it is only necessary to find
the multiplicity with which any representation of K occurs in
the decomposition of X. Finally, if t is a diagonal operator
defined in terms of the algebras K and L, then K is the
symmetry algebra for t and L is a spectrum generating algebra.

Although these ideas are quite elementary and familiar
to most mathematical physicists, their logical development
within a strictly algebraic framework is not trivial.

The codiagonal operators treated in later chapters
are far more complicated than the elementary diagonal operators.

The Lie Algebras K and L.

What properties should be required of K and L in general? From a survey of the literature on spectrum generating algebras, two points emerge.

(1) The algebras which arise are subalgebras of the reductive, but not semi-simple, Lie algebra gl(n,C).

(2) Loosely speaking, L decomposes into the subalgebra K and a subspace whose elements shift the eigenvalues of the invariants of K.

If the algebra K is to account fully for the degeneracy in the spectrum of t, then a third requirement can be added to the list.

(3) Any irreducible representation of L, which is completely reducible when restricted to K, must not contain in its decomposition any irreducible representation of K more than once.

To fulfil these requirements, I will choose K and L in the following way.

Let a^i_j denote the n × n matrix whose only non-zero element is equal to one and occurs at the intersection of the ith row and j^{th} column. The set

$$\{a^i_j, \ 1 \leqq i,j \leqq n\}$$

is a basis for the Lie algebra gl(n,C). The commutation rules are

$$[a^i_j, a^k_l] = \delta^k_j a^i_l - \delta^i_l a^k_j.$$

All the classical Lie algebras are subalgebras of gl(n,C). To fix upon notation, I want to exhibit their bases explicitly in terms of the basis for gl(n,C).

(a) sl(n,C).   The elements

$$b^i_j = a^i_j - \delta^i_j \sum_k a^k_k / n, \quad 1 \leq i,j \leq n,$$

span sl(n,C). To obtain a basis, one of the elements b^k_k must

be omitted.

(b) <u>so(n,C)</u>.   Define

$$r_{ij} = a^i_{\ j} - a^j_{\ i}, \quad 1 \leq i,j \leq n.$$

It is a trivial matter to verify that

$$[r_{ij}, r_{kl}] = \delta_{jk} r_{il} - \delta_{ik} r_{jl} + \delta_{lj} r_{ki} - \delta_{li} r_{kj}.$$

The set

$$\{r_{ij}, \quad 1 \leq i < j \leq n\}$$

is a basis for so(n,C).

(c) <u>sp(n,C)</u>.   When n is even, define the n × n matrix

$$(g_{ij}) = \begin{bmatrix} 0 & +1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & +1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & 0 & +1 \\ & & & & & & -1 & 0 \end{bmatrix}$$

and the following quantities

$$s_{ij} = g_{ik} a^k_{\ j} + g_{jk} a^k_{\ i},$$

where summation over k is understood. Then

$$[s_{ij}, s_{kl}] = -g_{jk} s_{il} - g_{ik} s_{jl} + g_{lj} s_{ki} + g_{li} s_{kj}$$

and the set

$$\{s_{ij}, \quad 1 \leq i \leq j \leq n\}$$

is a basis for sp(n,C).

sp(n,C) is a subalgebra of sp(n+2,C), whose basis is the set

$$\{s_{ij}, \quad 1 \leq i \leq j \leq n+2\}.$$

There is a Lie algebra of (n+1) × (n+1) matrices which lies between sp(n,C) and sp(n+2,C). Its basis is the set

$$\{s_{ij}, \quad 1 \leq i \leq j \leq n+1\}.$$

This Lie algebra is not semi-simple, since $s^{n+1}_{\ n+1}$ commutes with all the other basis elements, nor is it derived from a symplectic bilinear form, but I will still denote it sp(n+1,C). Thus

$$sp(n,C) < sp(n+1,C) < sp(n+2,C).$$

For each positive integer n, the algebras $gl(n,C)$, $sl(n,C)$, $so(n,C)$, $sp(n,C)$ can be defined. The basis elements of each of these algebras can be assembled into an n × n matrix, which I shall simply call <u>the matrix of the Lie algebra</u>.

Thus $\qquad a = (a^i{}_j)$ for $gl(n,C)$,

$\qquad\qquad r = (r_{ij})$ for $so(n,C)$,

$\qquad\qquad s = (s_{ij})$ for $sp(n,C)$,

and $\qquad b = (b^i{}_j) = a - (\text{trace } a)/n$ for $sl(n,C)$.

The real forms of $so(n,C)$ and $sp(n,C)$ can be constructed in a similar way by the introduction of a real metric h_{ij} on $gl(n,R)$. If h_{ij} is symmetric, then the elements

$$h_{ik}a^k{}_j - h_{jk}a^k{}_i$$

span a real form of $so(n,C)$, but, if n is even and h_{ij} is skew-symmetric, then

$$h_{ik}a^k{}_j + h_{jk}a^k{}_i$$

span a real form of $sp(n,C)$. However, over the complex field, h_{ij} can be reduced to either δ_{ij} or g_{ij}, so I have not forsaken generality by assuming these forms.

I will choose L from the list

$$gl(n+1,C), \quad sl(n+1,C), \quad so(n+1,C), \quad sp(n+1,C),$$

and require K to be the subalgebra of L whose matrix is obtained from the matrix of L by deletion of the last row and column. I shall refer to this choice of L and K as the <u>standard choice</u>. This description of L and K is not as cumbersome as it might at first appear. Matrices over the enveloping algebras of K and L are essential in the solution of eigenvalue problems concerned with diagonal operators. It is then quite natural to think of the basis of the Lie algebra arrayed in a matrix.

The reason why I have chosen K and L in this manner is that condition (3) is satisfied. That is the content of the following theorem.

Theorem 1.   Make the standard choice for L and K. Suppose that X is an irreducible L-module and that X, considered as a K-module, is completely reducible. Then any irreducible K-module can occur at most once in the decomposition of X.   ///

The proof of this theorem will emerge as a subsidiary result from this chapter, whose main purpose is to survey the techniques of spectral analysis and their applications to diagonal operators.

There is a technical advantage in the assumption that K and L should be reductive Lie algebras over the complex field; for such Lie algebras, the theorems on weights, so important in the construction of finite dimensional modules, can be extended to infinite dimensional modules. For completeness, I have listed these theorems below.

Suppose that L is a reductive Lie algebra over C and that X is an L-module. Let H denote a Cartan subalgebra of L, and let L_+ denote the subspace of L spanned by root vectors corresponding to positive roots. A linear function on H,

$$\rho : \quad H \longrightarrow C,$$

is called a weight of X if there is a non-zero vector x in X such that

$$hx = \rho(h)x \quad , \quad \text{for all h in H.}$$

x is called the weight vector with weight ρ. If, in addition,

$$L_+ x = 0 ,$$

x is called a vector of maximum weight ρ.

Theorem (Harish-Chandra (1951)). Let X be an irreducible L-module with a vector x_0 of maximum weight ρ. Then :

(1) the vector x_0 is unique to within a normalisation factor;

(2) the weights of X are of the form

$$\rho - \sum_{\alpha'} n_\alpha \, \alpha \, ,$$

where each n_α is a non-negative integer and the summation is over the simple roots of L;

(3) every weight appears with a finite multiplicity, ρ with multiplicity equal to one;

(4) $X = \underset{\sigma}{+} X_\sigma$, where σ is a weight of X and

$\quad\quad X_\sigma = \{x \in X \mid hx = \sigma(h)x$ for all h in H $\}$.　　　///

Theorem (Harish-Chandra (1951)). Let $\rho : H \longrightarrow C$ be any linear function on H. There exists an irreducible L-module with maximum weight ρ. Two irreducible L-modules X_1 and X_2, with maximum weights ρ_1 and ρ_2, are equivalent if and only if

$\rho_1 = \rho_2$.　　　///

　　　From this result it is clear that, if an infinite dimensional, irreducible L-module has a vector of maximum weight, then the module is uniquely labelled by that weight.

Theorem (E. Cartan, Harish-Chandra).

(1) Every finite dimensional, irreducible L-module has a maximum weight which is both dominant and integral.

(2) Every irreducible L-module with a maximum weight that is dominant and integral is finite dimensional.　　　///

　　　The preference shown for maximum weights is not essential. Similar theorems show that an irreducible module with a vector of minimum weight is uniquely labelled by that weight. It is sometimes even possible to separate the Cartan subalgebra H into two parts and to find modules labelled by a weight which is a maximum for one part of H and a minimum for the other.

Not all the Lie algebras which arise in physics are over the complex field. For example, the Lie algebras of the Lorentz and Poincaré groups are real Lie algebras. Consequently, my assumption that K and L should represent complex Lie algebras probably seems too strong. However, the limitation introduced by this assumption is only apparent. This follows from the theorem presented below and, ultimately, from the definition of (algebraic) irreducibility.

<u>Theorem 5.</u>   Suppose that L is a reductive Lie algebra over the complex field and that L_R is a real form of L.

(1) If X_R is an irreducible L_R-module, then

$$X = X_R + iX_R$$

is an irreducible L-module. The sum indicated is a direct sum of real vector spaces and the module product on X is defined by

$$(a + ib).(x + iy) = (ax - by) + i(bx + ay),$$

for all a and b in L_R and all x and y in X_R.

(2) If X is an irreducible L-module, then X can be decomposed

$$X = X_R + iX_R ,$$

where X_R is an irreducible L_R-module. Furthermore, X_R is unique to within equivalence of L_R-modules.

(3) Suppose X_R and X_R' are two irreducible L_R-modules. Construct the L-modules

$$X = X_R + iX_R \quad \text{and} \quad X' = X_R' + iX_R'.$$

Then X and X´ are equivalent L-modules if and only if X_R and X_R' are equivalent L_R-modules.   ///

Theorem 5 summarises lemmas 2,3 and 4 of the appendix.

This result is well known for finite dimensional L and L_R-modules. The point I want to stress is that it remains true for infinite dimensional L and L_R-modules, provided these modules are <u>algebraically</u> irreducible. For

representations of a Lie algebra on a Banach space, several notions of irreducibility are possible. If one of these is substituted for algebraic irreducibility, theorem 5 will no longer be true generally, unless further conditions are placed upon the representations involved.

Although theorem 5 can be proved quite easily, it has some important consequences. For example, when theorems 1 and 5 are combined, the following result is obtained.

Theorem 6.   Suppose that:

(1) L and K have been chosen in the standard way;

(2) L_R is any real form of L;

(3) K_R is a real form of K such that $K_R < L_R$;

(4) X_R is an irreducible L_R-module, which, when considered as a K_R-module, is completely reducible.

Then any irreducible K_R-module can occur at most once in the decomposition of X_R.   ///

A further example of the importance of theorem 5 will be given in chapter 6. The example is concerned with the construction of irreducible representations of the Poincaré Lie algebra.

K-finite L-modules.

Suppose that L is a reductive Lie algebra over C and that K is a subalgebra of L.

An irreducible L-module X is K-finite if the following conditions hold.

(1) The module X is completely reducible into finite dimensional, irreducible K-modules.

(2) Each irreducible K-module in this decomposition occurs only a finite number of times.

(3) X has a vector of maximum weight.

K-finite L-modules are not the only possible L-modules. For example, an L-module need not have a vector of maximum weight nor need it be decomposable, and, even if these conditions were satisfied, infinite dimensional K-modules could occur in the decomposition. However, they are appropriate in a discussion of algebraic eigenvalue problems, because the spectrum of any diagonal operator on such a module is bounded either above or below and the multiplicity of each eigenvalue is finite.

Bouwer (1968) has investigated irreducible modules for simple Lie algebras that have neither a maximum nor a minimum weight. Instead he assumed the existence of a weight ρ with the following properties;

(1) the multiplicity of ρ was one;

(2) for a subset of the simple roots,

$$\rho + m\alpha$$

was a weight for every integer m, positive or negative;

(3) for the remaining simple roots,

$$\rho + \alpha$$

was never a weight.

Bouwer found that, with certain other minor assumptions, ρ labelled the irreducible module. Similarly, the techniques devised by Green can be extended to such modules. However, for the applications I have in mind, modules with a highest weight will suffice.

Schur's lemma can be sharpened for an irreducible L-module X which is K-finite, a result with important consequences. Let π denote the representation afforded by X and suppose that t is an endomorphism of X which commutes with $\pi(L)$,

$$[t, \pi(L)] = 0.$$

Schur's lemma states that the set of endomorphisms with this

property is a division ring. The lemma does not state that t must be a scalar multiple of the identity; that result is only true generally when X is finite dimensional and complex or when X is a complex Banach space and the representation is topologically irreducible. Nevertheless, because X is K-finite, t must be a scalar multiple of the identity on X. The proof of this assertion is quite simple. It is obvious that t commutes with $\pi(K)$,

$$[t, \pi(K)] = 0.$$

This identity must hold on any irreducible K-submodule Y of X. Since Y is finite dimensional and complex, Schur's lemma implies that t is a scalar multiple of the identity on Y,

$$ty = \lambda y , \quad \text{for all y in Y.}$$

Now $(t - \lambda)$ commutes with $\pi(L)$ on X, and so must either be non-singular or zero, because the set of such endomorphisms of X is a division ring. The first possibility cannot occur because $(t - \lambda)$ vanishes on Y. Hence,

$$tx = \lambda x , \quad \text{for all x in X.}$$

Diagonal Operators and the Algebraic Eigenvalue Problem.

Suppose that L and K have been chosen in the standard way. Let $U(L)$ and $U(K)$ denote the universal enveloping algebras of L and K.

The matrix a of K satisfies a polynomial identity of degree n,

$$a^n + c_1 a^{n-1} + \cdots + c_n = 0 ,$$

in which the coefficients are elements of $Z(K)$, the centre of $U(K)$. The existence of such an identity was proved by Lehrer-Ilamed (1956); the coefficients in the identities for the classical Lie algebras were derived by Bracken and Green (1971)

and Green (1971). It is important to stress that this identity holds within the algebra of matrices over $U(K)$ and is not peculiar to certain representations of K. Equivalently, the identity is true in every representation of K.

Weyl (1939) and Harish-Chandra (1951) have shown that $Z(K)$ is a finitely generated subalgebra of $U(K)$. One set of generators for $Z(K)$ is

$$\{s_i = \text{trace } (a^i), \quad 1 \leq i \leq n\},$$

though not all of its elements need be independent. The traces of higher powers of a can always be expressed in terms of this set as a consequence of the polynomial identity. Thus,

$$Z(K) = C[s_1, s_2, \cdots, s_n],$$

where I have used the common notation for a polynomial ring over C, which can be found, for example, in the text by Jacobson (1951). Because $Z(K)$ is an integral domain, it can be embedded in its field of quotients. This field has a minimal extension field E in which the polynomial

$$x^n + c_1 x^{n-1} + \cdots + c_n$$

splits into linear factors,

$$\prod_{i=1}^{n} (x - l_i), \quad l_i \in E.$$

I will call E the <u>splitting field for K</u>. Because c_i involves only $s_1, s_2, \cdots, s_i$ and s_i appears linearly, s_i can be expressed as a polynomial in $l_1, l_2, \cdots, l_n$. Hence, every point of E is a rational function of the roots $l_1, l_2, \cdots, l_n$, the coefficients being chosen from the complex field C;

$$E = C(l_1, l_2, \cdots, l_n).$$

Of course, not all of the roots need be linearly independent over C.

End(Y)   U(K)   E
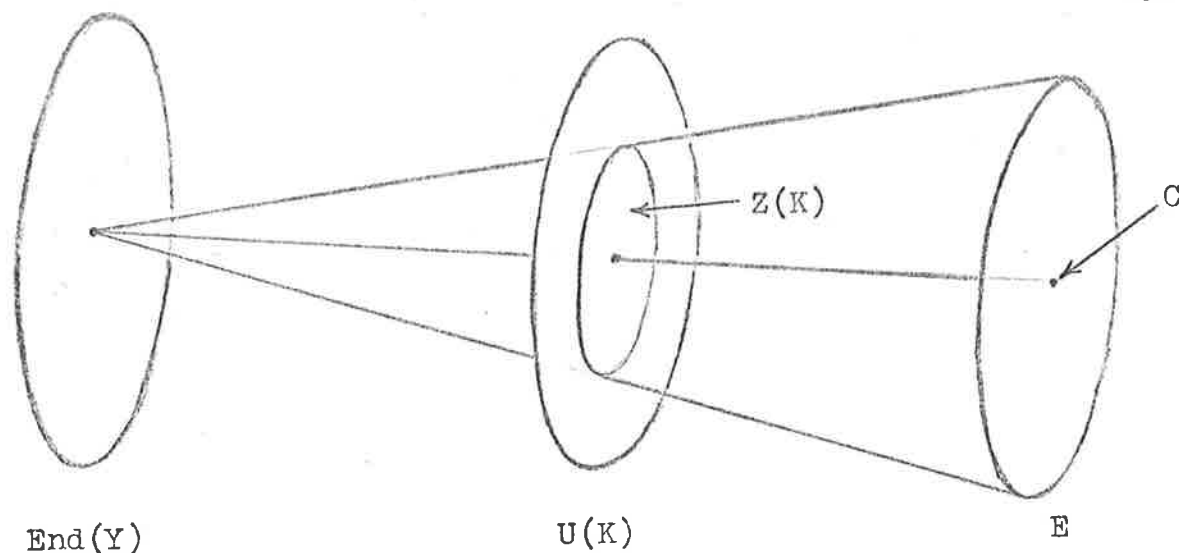
Let X denote a K-finite L-module and suppose that Y
is an irreducible K-submodule of X. As a vector space, Y is
finite dimensional and complex, so Schur's lemma implies that
every element of Z(K) reduces to a multiple of the identity on
Y. In particular,

$$c_i y = \gamma_i y , \quad \gamma_i \in C ,$$

for all y in Y. Let $\lambda_1, \cdots, \lambda_n$ denote the roots in C of the
equation

$$x^n + \gamma_1 x^{n-1} + \cdots + \gamma_{n-1} x + \gamma_n = 0.$$

There is a natural action of E on Y. If $f(1_1, \cdots, 1_n)$
is a rational function of $1_1, \cdots, 1_n$ in E, define

$$f(1_1, \cdots, 1_n)y = f(\lambda_1, \cdots, \lambda_n)y ,$$

for all y in Y. Since

$$X = \bigoplus_\nu Y_\nu ,$$

where each Y_ν is a finite-dimensional, irreducible K-module,
the action of E can be extended to X by the simple definition

$$eX = \bigoplus_\nu eY_\nu , \quad e \in E.$$

Green and Bracken have shown that $\lambda_1, \cdots, \lambda_n$ determine
the components of the highest weight of Y, and conversely. It
also seems clear from their work that infinite dimensional,
irreducible K-modules can be labelled in a similar way and that

the matrix a, its polynomial identity and its 'eigenvalues' $l_1, \cdots, l_n$ will be essential to the study of K and its representations. Consequently, it is necessary to extend U(K) so that it contains E. More precisely, I must extend the field over which U(K) is defined from C to E. This step is not trivial because E contains certain elements of U(K). Nevertheless, in the appendix I have shown how to construct an algebra $\overline{U(K)}$ over E which contains an isomorphic copy of U(K) and which observes all the necessary identities. It is with this algebra that I shall henceforth work.

The polynomial identity can be factorised in $\overline{U(K)}$:

$$\prod_{i=1}^{n} (a - l_i) = 0.$$

For all the classical Lie algebras, the roots $l_1, \cdots, l_n$ are distinct elements of E, though they are not necessarily linearly independent over C. The matrices

$$p_i = \prod_{j=i}^{n} (a - l_j) \, / \, (l_i - l_j)$$

are mutually annihilating projection matrices which provide a resolution of the identity, since

$$p_i p_j = \delta_{ij} p_i \, ,$$
$$1 = \sum_i p_i \, ,$$

and

$$a = \sum_i l_i p_i \, .$$

It is not difficult to verify these assertions.

Similar considerations apply to L as well as K.

<u>Definition.</u>   Suppose that X is a K-finite L-module. A
<u>diagonal operator on X</u> is an element of E acting on X.

Note that the labels which specify the irreducible L-
module X play the role of boundary conditions for a diagonal
operator on X.

Note also the restrictions placed upon the eigenvalues
of the labelling operators in the application to quantum physics.
An inner product must be defined on X and one of the real forms
of L must be represented by skew-symmetric operators on X.
When this real form L_R is the compact real form of L, all the
labels will be real and quantised. When L_R has one non-compact
basis element, one label may be purely imaginary and unquantised.
When L_R has two non-compact basis elements, two labels may be
complex conjugates of one another and may be unquantised, and
so on.

Since every element of E is a rational function, with
complex coefficients, of $l_1, \cdots, l_n$, the spectrum of any
diagonal operator can be found once the spectra of $l_1, \cdots, l_n$
are known. Furthermore, on any irreducible K-submodule of X,
the eigenvalues $\lambda_1, \cdots, \lambda_n$ of $l_1, \cdots, l_n$ are simply related
to the components of the highest weight of the submodule. In
fact, once it is known which K-submodules occur in the decompos-
ition of X, and with what multiplicities they occur, the spectra
of $l_1, \cdots, l_n$ are effectively known.

Thus, the study of diagonal operators can be resolved
into the following steps.

(1) Classify and construct all finite dimensional K-modules.
This is a relatively simple problem and its solution is well
known.

(2) Classify those infinite-dimensional, irreducible L-modules which have a vector of highest weight. This is an easy task because the highest weight itself provides a unique labelling of the module.

(3) Decide which irreducible L-modules are K-finite. In contrast this problem is difficult. However, it can be resolved with the help of the characteristic identities derived by Professor Green and Dr. A. J. Bracken.

(4) For a given K-finite L-module X, determine which irreducible K-modules occur in the decomposition of X and also the respective multiplicities.

(5) Finally, from the results obtained in (4), deduce the spectra of $l_1, \cdots, l_n$ on X and hence the spectrum of any given diagonal operator on X.

For any pair K and L, chosen in the standard way, all these problems can be solved. However, although the method of proof is similar for all pairs K and L, it is not so similar that all cases can be treated simultaneously. This difficulty is akin to the one encountered in the theory of the special functions, where, despite the similarities between the special functions, they must be treated separately. Consequently, I will only show how to solve the problems for one case, namely,

$$K = gl(n,C) < gl(n+1,C) = L.$$

Construction of gl(n,C)-finite gl(n+1,C)-modules.

In this section, the following notation will be in force.

(1) X will denote a gl(n+1,C)-module that is gl(n,C)-finite.

(2) $m_1, \cdots, m_{n+1}$ will denote the roots of the characteristic identity for gl(n+1,C). Because X is gl(n,C)-finite, each m_i reduces to a multiple of the identity on X:

$$m_i x = \mu_i x \ , \quad \mu_i \ \epsilon \ C \ , \quad \text{for all x in X.}$$

(3) Similarly, $1_1, \cdots, 1_n$ will denote the roots of the characteristic identity for $gl(n,C)$. Thus, on every irreducible $gl(n,C)$-submodule Y,

$$1_i y = \lambda_i y \ , \quad \lambda_i \ \epsilon \ C \ ,$$

for all y in Y. Of course the eigenvalue λ_i will depend upon the submodule Y.

(4) For brevity I will use the notation

$$1 = (1_1, \cdots, 1_n) \ , \quad m = (m_1, \cdots, m_{n+1}) \ ,$$
$$\lambda = (\lambda_1, \cdots, \lambda_n) \ , \quad \mu = (\mu_1, \cdots, \mu_{n+1}) \ .$$

(5) x_o will denote the vector of highest weight in X. It is not difficult to verify that this weight is

$$(\mu_1 - n \ , \ \mu_2 - n + 1 \ , \ \cdots \ , \mu_{n+1}) \ .$$

Thus, $\quad a^i_{\ i} x_o = (\mu_i + i - n - 1) x_o \ , \quad 1 \leqq i \leqq n+1,$

and $\quad a^i_{\ j} x_o = 0 \ , \quad 1 \leqq i < j \leqq n+1 \ .$

(6) Similarly, if Y is an irreducible $gl(n,C)$-submodule of X and

$$1_i y = \lambda_i y \ , \quad 1 \leqq i \leqq n \ ,$$

for all y in Y, then the highest weight of Y is

$$(\lambda_1 - n + 1, \ \lambda_2 - n + 2, \ \cdots \ , \lambda_n)$$

and the vector y_o with this weight satisfies

$$a^i_{\ i} y_o = (\lambda_i + i - n) y_o \ , \quad 1 \leqq i \leqq n \ ,$$

and $\quad a^i_{\ j} y_o = 0 \ , \quad 1 \leqq i < j \leqq n \ .$

(7) Finally, the spectral projections for the matrix a of $gl(n,C)$ will be denoted by

$$p_i = \prod_{j=i} (a - 1_j) \ / \ (1_i - 1_j) \ , \quad 1 \leqq i \leqq n \ .$$

Roman indices will lie in the range $1,2,\cdots,n$. When an index is repeated, summation over that index will be understood.

Define

$$b^{ik} = a^{j}{}_{n+1}(p_k)^{i}{}_{j} \text{ and } b_{ik} = a^{n+1}{}_{j}(p_k)^{j}{}_{i} \; .$$

It is easy to verify that

$$a^{j}{}_{j}b^{ik} = b^{ik}(a^{j}{}_{j} + 1)$$

and

$$a^{j}{}_{j}b_{ik} = b_{ik}(a^{j}{}_{j} - 1) \; .$$

Since $\qquad a^{j}{}_{j} = 1_1 + 1_2 + \cdots + 1_n - \tfrac{1}{2}n(n-1) \; ,$

the above formulae indicate that b^{ik} and b_{ik} satisfy

$$1_j b^{ik} = b^{ik}(1_j + \delta_{jk}) \; ,$$
$$1_j b_{ik} = b_{ik}(1_j - \delta_{jk}) \; .$$

This statement is true, but its proof is difficult, so I refer

the matter to the paper by Green (1971).

From the very definition of x_o as the vector of highest

weight in X, it follows that

$$a^{j}{}_{n+1}x_o = 0.$$

It is then quite easy to show that

$$b^{ik}x_o = 0.$$

Define $\qquad\qquad b^{k} = a^{n+1}{}_{i}b^{ik}.$

Obviously, $\qquad\qquad b^{k}x_o = 0 \; .$

b^{k} is a 'mixed invariant', for, although it commutes

with all elements of gl(n,C), it cannot be expressed as a rational

function of 1. However, by an exceedingly tedious calculation,

which will be illustrated later by an example, b^{k} can be expressed

as a rational function of 1 and m. If λ and μ are respectively

the eigenvalues of 1 and m on the vector x_o of highest weight,

then the conditions

$$b^{k}x_o = 0$$

yield n polynomial equations between λ and μ,

$$\varphi^{k}(\lambda,\mu) = 0 \; .$$

Because X is a gl(n,C)-finite gl(n+1,C)-module, the vector x_o of highest weight in X must lie in some finite-dimensional, irreducible gl(n,C)-submodule of X. Furthemore, this submodule will be labelled by the eigenvalue λ of l on x_o. In general, the finite dimensional, irreducible gl(n,C)-modules are labelled by integers $\lambda_1, \cdot \cdot \cdot, \lambda_n$ which satisfy

$$\lambda_1 > \lambda_2 > \cdot \cdot \cdot > \lambda_n \ .$$

Thus, the assumption that X is gl(n,C)-finite places the following restraints on λ and μ:

(1) $\lambda_1, \cdot \cdot \cdot, \lambda_n$ must be integers;

(2) $\lambda_1 > \lambda_2 > \cdot \cdot \cdot > \lambda_n$;

(3) λ and μ must satisfy the polynomial equations

$$\varphi^k(\lambda,\mu) = 0 \ , \quad 1 \leqq k \leqq n \ .$$

An alternative, and more profitable, interpretation is that the irreducible gl(n+1,C)-module, labelled by μ, can only be gl(n,C)-finite when the polynomial equations in (3) have a solution λ which satisfies conditions (1) and (2).

In a similar manner, conditions can be formulated under which X has a vector z_o of minimum weight. From the definition of such a vector,

$$a^{n+1}{}_j z_o = 0.$$

It is not difficult to show that

$$b_{ik} z_o = 0.$$

Define
$$b_k = a^l{}_{n+1} b_{ik}.$$

Obviously,
$$b_k z_o = 0.$$

b_k is a 'mixed invariant' and can be expressed as a rational function of l and m. If λ' is the eigenvalue of l on the vector z_o of lowest weight, then the conditions

$$b_k z_o = 0$$

yield n polynomial equations between λ' and μ,

$$\varphi_k(\lambda',\mu) = 0 \ .$$

If X is irreducible, then each of the differences

$$\nu_i = \lambda_i - \lambda_i'$$

must be an integer.

Again there is a more profitable interpretation of these results. Suppose that μ is given and that λ is a solution of the equations

$$\varphi^k(\lambda,\mu) = 0 \ , \ 1 \leqq k \leqq n.$$

Construct the lattice in R^n whose points have the form

$$\nu = (\nu_1,\nu_2, \cdot \cdot \cdot ,\nu_n) \ ,$$

where each component of ν is an integer. Two points ν and ν' are <u>adjacent</u> in this lattice if $\nu - \nu'$ has only one non-zero component and this component is equal to ± 1. A <u>path</u> from the origin to ν is a finite sequence

$$\{0 = \nu^{(1)},\nu^{(2)}, \cdot \cdot \cdot ,\nu^{(m)} = \nu\}$$

in which each point is adjacent to the next. The point ν is <u>connected to the origin</u> if there is a path from the origin to ν such that

$$\varphi_k(\lambda - \nu^{(i)},\mu) \neq 0, \quad \left\{ \begin{array}{l} 1 \leqq i < m \\ 1 \leqq k \leqq n \end{array} \right. .$$

Let D denote the subset of the lattice connected to the origin and let $X_{\lambda-\nu}$, $\nu \in D$, denote the irreducible gl(n,C)-module labelled by $(\lambda - \nu)$. Define

$$X = \bigoplus_{\nu \in D} X_{\lambda-\nu} \ .$$

Suppose for the moment that it is possible to convert X into an

irreducible $gl(n+1,C)$-module. It is clear from the construction of X that X is completely reducible into irreducible $gl(n,C)$-modules and that each occurs only once. When the numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ are integers and satisfy

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n \, ,$$

X is $gl(n,C)$-finite.

The only surviving problem is to show that the reducible $gl(n,C)$-module X can be converted into an irreducible $gl(n+1,C)$-module. Gel'fand and Cetlin (1950) found a solution for finite dimensional $gl(n+1,C)$-modules; only trivial modifications are required for the infinite dimensional case. For all x in X, define

$$a^{n+1}_{n+1} x = (m_1 + m_2 + \cdots + m_{n+1} - 1_1 - 1_2 - \cdots - 1_n - n)x.$$

The difficulty is to define the action of a^{n+1}_i and a^i_{n+1} on X in a way consistent with the commutation relations. Since

$$a^{n+1}_i = [a^{n+1}_n, a^n_i]$$

and

$$a^i_{n+1} = [a^i_n, a^n_{n+1}] \, ,$$

once a^{n+1}_n and a^n_{n+1} have been defined, a^{n+1}_i and a^i_{n+1} can be defined by these formulae.

The basis vectors of $X_{\lambda - \nu}$ can be represented by the tableaux of integers

$$\left| \begin{array}{ccccc} \lambda_1 - \nu_1 & \lambda_2 - \nu_2 & \cdots\cdots & \lambda_{n-1} - \nu_{n-1} & \lambda_n - \nu_n \\ & \lambda_{1\,n-1} & \cdots\cdots & \lambda_{n-1\,n-1} & \\ & & \cdots\cdots & & \\ & & \lambda_{1\,1} & & \end{array} \right\rangle$$
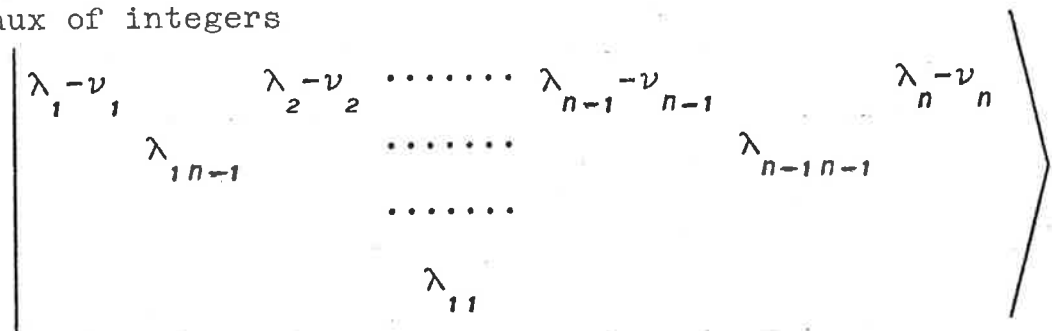
where $\quad \lambda_1 - \nu_1 > \lambda_{1\,n-1} \geqq \lambda_2 - \nu_2 > \cdots\cdots \geqq \lambda_{n-1} - \nu_{n-1} > \lambda_{n-1\,n-1} \geqq \lambda_n - \nu_n \, ,$

$$\cdots\cdots$$

$$\lambda_{1\,2} > \lambda_{1\,1} \geqq \lambda_{2\,2}.$$

The entries of the mth row from the bottom are the eigenvalues of the roots of the characteristic identity of $gl(m,C)$ on the basis vector. Thus, this is essentially the labelling scheme devised by Gel'fand and Cetlin. Now,

$$a^n_{n+1} = a^j_{n+1} \delta^n_j$$

$$= a^j_{n+1} \sum_k (p_k)^n_j$$

$$= \sum_k b^{nk}.$$

Similarly, $\qquad a^{n+1}_n = \sum_k b_{nk}.$

b^{nk} raises the eigenvalue of l_k by $+1$, but leaves unchanged the eigenvalues of l_j, $j \neq k$. Furthermore, b^{nk} commutes with the labelling operators for the subalgebras

$$gl(1,C) < gl(2,C) < \cdots < gl(n-1,C).$$

Thus,

$$b^{nk} \left| \begin{array}{ccc} \lambda_1 - \nu_1 & \cdots \cdots & \lambda_n - \nu_n \\ & \cdots \cdots & \\ & \lambda_{11} & \end{array} \right\rangle = \alpha_k \left[\begin{array}{ccc} \lambda_1 - \nu_1 & \cdots \cdots & \lambda_n - \nu_n \\ & \cdots \cdots & \\ & \lambda_{11} & \end{array} \right] \left| \begin{array}{ccc} \lambda_1 - \nu_1 & \cdots \lambda_k - \nu_k + 1 \cdots & \lambda_n - \nu_n \\ & \cdots \cdots \cdots & \\ & \lambda_{11} & \end{array} \right\rangle.$$

Similarly,

$$b_{nk} \left| \begin{array}{ccc} \lambda_1 - \nu_1 & \cdots \cdots & \lambda_n - \nu_n \\ & \cdots \cdots & \\ & \lambda_{11} & \end{array} \right\rangle = \beta_k \left[\begin{array}{ccc} \lambda_1 - \nu_1 & \cdots \cdots & \lambda_n - \nu_n \\ & \cdots \cdots & \\ & \lambda_{11} & \end{array} \right] \left| \begin{array}{ccc} \lambda_1 - \nu_1 & \cdots \lambda_k - \nu_k - 1 \cdots & \lambda_n - \nu_n \\ & \cdots \cdots \cdots & \\ & \lambda_{11} & \end{array} \right\rangle.$$

The coefficients α_k and β_k must be determined as functions of

$$\left[\begin{array}{ccc} \lambda_1 - \nu_1 & \cdots \cdots & \lambda_n - \nu_n \\ & \cdots \cdots & \\ & \lambda_{11} & \end{array} \right]$$

subject to the following conditions.

(1) α_k and β_k must vanish whenever the vector on the right is without meaning. For example, if

$$\left|\begin{array}{ccc} \lambda_1 - \nu_1 & \cdots\cdots & \lambda_n - \nu_n \\ & \cdots\cdots & \\ & \lambda_{11} & \end{array}\right\rangle$$

is the vector of highest weight in X, then

$$\alpha_k \left[\begin{array}{ccc} \lambda_1 - \nu_1 & \cdots\cdots & \lambda_n - \nu_n \\ & \cdots\cdots & \\ & \lambda_{11} & \end{array}\right] = 0.$$

(2) a^n_{n+1} and a^{n+1}_n must satisfy on X

$$[a^{n+1}_n , a^n_{n+1}] = a^{n+1}_{n+1} - a^n_n .$$

These constraints lead to a set of functional equations for α_k and β_k which have many solutions in general. However, the $gl(n+1,C)$-modules corresponding to the different solutions are equivalent. The solutions found by Gel'fand and Cetlin in the finite dimensional case can be applied equally well here, so the task of constructing the $gl(n+1,C)$-modules is completed.

Example 1. $gl(1,C) < sl(2,C)$.

The construction of irreducible $sl(2,C)$-modules which are $gl(1,C)$-finite is a trivial matter. Nevertheless this example is particularly important, because the theory of the simple spectrum rests upon it.

The matrix of $sl(2,C)$ is

$$b = a - \tfrac{1}{2}\operatorname{trace}(a)$$

$$= \left[\begin{array}{cc} \tfrac{1}{2}(a^1_1 - a^2_2) & a^1_2 \\ a^2_1 & -\tfrac{1}{2}(a^1_1 - a^2_2) \end{array}\right]$$

and satisfies the identity

$$b^2 - b - s = 0 ,$$

where

$$s = \tfrac{1}{4}(2s_2 - s_1{}^2)$$

and

$$s_i = \text{trace } a^i , \quad i = 1,2.$$

The centre of the enveloping algebra of $sl(2,C)$ is generated by s and hence is the polynomial algebra $C[s]$. The extension field E in which the characteristic identity splits is constructed as follows. Firstly, embed $C[s]$ in its field of quotients $F = C(s)$. Secondly, construct the polynomial algebra $F[x]$ in one indeterminate x and the principal ideal J in $F[x]$ generated by

$$x^2 - x - s .$$

Finally, define

$$E = F[x] \, / \, J .$$

For the proof that E is a field I refer to Jacobson (1951a).

Set

$$l = x + J .$$

Then

$$l^2 - l - s = 0$$

and

$$(b - l)(b + l - 1) = 0 .$$

Hence the characteristic identity can be factorised in E and the general techniques outlined in the text can be applied.

However, for this simple example there is a more direct approach. Define

$$h = \tfrac{1}{2}(a^1{}_1 - a^2{}_2) ,$$

$$e_+ = a^1{}_2 ,$$

$$e_- = a^2{}_1 .$$

The set $\{h, e_+, e_-\}$ is a basis for $sl(2,C)$ and the Lie products are

$$[h, e_\pm] = \pm e_\pm , \quad [e_+, e_-] = 2h .$$

Furthermore, $\quad s = h^2 + \tfrac{1}{2}(e_+ e_- + e_- e_+) ,$

so s is just the well known Casimir element for $sl(2,C)$. The

gl(1,C) subalgebra of sl(2,C) is one dimensional and has for its basis {h}.

Every gl(1,C)-finite sl(2,C)-module X is isomorphic to the space S of terminating sequences. h, e_+ and e_- then have the matrix representations

$$
h = \begin{bmatrix} \rho & & & & \\ & \rho-1 & & & \\ & & \rho-2 & & \\ & & & \rho-3 & \\ & & & & \ddots \\ & & & & & \ddots \end{bmatrix}, \quad
e_+ = \begin{bmatrix} 0 & \alpha_1 & & & \\ & 0 & \alpha_2 & & \\ & & 0 & \alpha_3 & \\ & & & 0 & \alpha_4 \\ & & & & \ddots & \ddots \\ & & & & & \ddots \end{bmatrix}, \quad
e_- = \begin{bmatrix} 0 & & & & \\ \beta_1 & 0 & & & \\ & \beta_2 & 0 & & \\ & & \beta_3 & 0 & \\ & & & \ddots & \ddots \\ & & & & & \ddots \end{bmatrix},
$$

where ρ, α_k, $\beta_k \ \epsilon$ C. The commutation relations

$$[h, e_\pm] = \pm\, e_\pm$$

are satisfied no matter what values are assigned to α_k and β_k. However, the third relation is only satisfied if

$$\alpha_k \beta_k = k(2\rho + 1 - k)\,, \quad k \geqq 1.$$

The vector
$$
\underline{\xi} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix}
$$

is the vector of highest weight in S. The corresponding weight, a linear functional on H = {h}, is

$$\rho : H \longrightarrow C$$

$$h \longmapsto \rho.$$

The module X has a vector z of lowest weight if

$$e_- z = 0\,.$$

This can only be so if β_k vanishes for some k, in which case $2\rho + 1 = k$ and the representation is k-dimensional. These are of course the familiar finite dimensional sl(2,C)-modules. When $(2\rho + 1)$ is not a positive integer, $\alpha_k \beta_k$ never vanishes and X is irreducible.

The theory developed by Harish-Chandra shows that the highest weight ρ uniquely labels the sl(2,C)-module. However, in this example it is so simple that it is worthwhile to verify this fact. Suppose that

$$
h = \begin{bmatrix} \rho & & & & \\ & \rho-1 & & & \\ & & \rho-2 & & \\ & & & \rho-3 & \\ & & & & \ddots \\ & & & & & \ddots \end{bmatrix} = h' \, ,
$$

$$
e_+ = \begin{bmatrix} 0 & \alpha_1 & & & \\ & 0 & \alpha_2 & & \\ & & 0 & \alpha_3 & \\ & & & 0 & \alpha_4 \\ & & & & \ddots & \ddots \\ & & & & & \ddots \end{bmatrix} , \quad e'_+ = \begin{bmatrix} 0 & \alpha'_1 & & & \\ & 0 & \alpha'_2 & & \\ & & 0 & \alpha'_3 & \\ & & & 0 & \alpha'_4 \\ & & & & \ddots & \ddots \\ & & & & & \ddots \end{bmatrix} ,
$$

$$
e_- = \begin{bmatrix} 0 & & & \\ \beta_1 & 0 & & \\ & \beta_2 & 0 & \\ & & \beta_3 & 0 \\ & & & \ddots & \ddots \\ & & & & \ddots \end{bmatrix} , \quad e'_- = \begin{bmatrix} 0 & & & \\ \beta'_1 & 0 & & \\ & \beta'_2 & 0 & \\ & & \beta'_3 & 0 \\ & & & \ddots & \ddots \\ & & & & \ddots \end{bmatrix} ,
$$

$$
\alpha_k \beta_k = k(2\rho + 1 - k) = \alpha'_k \beta'_k
$$

are two irreducible representations of sl(2,C) with the same maximum weight. Set

$$
c = \begin{bmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \gamma_3 & \\ & & & \ddots \\ & & & & \ddots \end{bmatrix} , \quad \gamma_1 = 1 \, ,
$$

and require $c e_- = e'_- c$.

Thus, $\beta_k \gamma_{k+1} = \beta'_k \gamma_k$.

Since β_k and β'_k never vanish because the representations are irreducible,

$$
\gamma_{k+1} / \gamma_k = \beta'_k / \beta_k .
$$

Now

$$ce_+ = \begin{bmatrix} 0 & \alpha_1\gamma_1 \\ & 0 & \alpha_2\gamma_2 \\ & & 0 & \alpha_3\gamma_3 \\ & & & \ddots & \ddots \\ & & & & \ddots & \ddots \\ & & & & & \ddots \end{bmatrix} \quad \text{and} \quad e'_+c = \begin{bmatrix} 0 & \alpha'_1\gamma_2 \\ & 0 & \alpha'_2\gamma_3 \\ & & 0 & \alpha'_3\gamma_4 \\ & & & \ddots & \ddots \\ & & & & \ddots \end{bmatrix}.$$

Because

$$\alpha_k\beta_k = \alpha'_k\beta'_k \;,$$

$$\alpha_k\gamma_k = \alpha_k(\gamma_{k+1}\beta_k / \beta'_k)$$

$$= \alpha'_k\gamma_{k+1} \;,$$

and so

$$ce_+ = e'_+c \;.$$

It is trivial that $ch = hc$. Thus, the representations are equivalent.

Example 2. $gl(2,C) \leqslant gl(3,C)$.

The purpose of this example is to illustrate the steps in the general programme for the construction of $gl(n+1,C)$-modules that are $gl(n,C)$-finite. Roman indices will take the values 1 and 2, Greek indices the values 1,2 and 3. When an index is repeated, summation over the index will be understood.

(1) Generators of the centre of the enveloping algebra.

$gl(2,C)$

$$s_1 = a^i{}_i$$

$$s_2 = a^i{}_j a^j{}_i$$

$$s_3 = a^i{}_j a^j{}_k a^k{}_i$$

$$= s_2 + \tfrac{1}{2}s_1(3s_2 - s_1 - s_1^2)$$

$gl(3,C)$

$$t_1 = a^\mu{}_\mu$$

$$t_2 = a^\mu{}_\nu a^\nu{}_\mu$$

$$t_3 = a^\mu{}_\nu a^\nu{}_\sigma a^\sigma{}_\mu$$

These generators may be expressed as symmetric functions of the roots of the characteristic identity.

$\underline{\text{gl}(2,C)}$ $\qquad$ $\underline{\text{gl}(3,C)}$

$$s_1 = l_1 + l_2 - 1$$

$$t_1 = m_1 + m_2 + m_3 - 3$$

$$s_2 = l_1(l_1 - 1) + l_2(l_2 - 1)$$

$$t_2 = m_1(m_1 - 2) + m_2(m_2 - 2) + m_3(m_3 - 2) + 1$$

$$t_3 = m_1(m_1 - 1)^2 + m_2(m_2 - 1)^2 + m_3(m_3 - 1)^2$$
$$- m_1 m_2 - m_2 m_3 - m_3 m_1$$

(2) Projection operators.

$$(p_1)^i{}_j = (a^i{}_j - l_2 \delta^i{}_j)/(l_1 - l_2)$$

$$(p_2)^i{}_j = (a^i{}_j - l_1 \delta^i{}_j)/(l_2 - l_1)$$

(3) Raising and lowering operators.

$$b^{ik} = a^j{}_3 (p_k)^i{}_j, \quad b^k = a^3{}_i b^{ik} \quad,$$

$$b_{ik} = a^3{}_j (p_k)^j{}_i, \quad b_k = a^i{}_3 b_{ik} \quad.$$

(4) Mixed invariants.

b^k and b_k commute with the basis of gl(2,C). By a straight-forward, but exceedingly tedious calculation, they can be expressed in terms of l_1, l_2, s_1, s_2 and t_1, t_2, t_3.

For example,

$$6b^1 = \{2t_3 - 3s_2(1-l_2) - 3t_2(1+l_2) + (t_1-s_1)^3 + 3l_2(t_1-s_1)^2 - $$
$$3(t_1-s_1)(t_2-s_2) + t_1^2 + 3s_1(t_1-s_1) + (3s_1-2t_1)(1+3l_2) + $$
$$s_1^3 - 3s_1 s_2\}/(l_1-l_2) \ .$$

These expressions are very complicated, yet when written in terms of l and m they become remarkably simple.

$$b^k = (-)^{k+1} \varphi^k(l,m)/(l_1 - l_2)$$

$$b_k = (-)^{k+1} \varphi_k(l,m)/(l_1 - l_2)$$

$$\varphi'(l,m)=(m_1-l_1-1)(m_2-l_1-1)(m_3-l_1)+(m_1-l_1-1)(m_3-l_2-1) +$$
$$(m_2-l_2-1)(m_3-l_2-1) \ ,$$

$$\varphi^2(l,m)=(m_1-l_2-1)(m_2-l_2-1)(m_3-l_2)+(m_2-l_2-1)(m_3-l_1-1) +$$
$$(m_1-l_1-1)(m_3-l_1-1) \ ,$$

$$\varphi_1(l,m)=(m_1-l_1)(m_2-l_1)(m_3-l_1) \ ,$$

$$\varphi_2(l,m)=(m_1-l_2)(m_2-l_2)(m_3-l_2) \ .$$

(5) Suppose that the $gl(3,C)$-module X has a vector x_0 of highest weight and that λ and μ are the eigenvalues of l and m on this vector. Then λ and μ must satisfy

$$\varphi^k(\lambda,\mu) = 0 \ , \quad k = 1,2.$$

Similarly, if X has a vector z_0 of lowest weight and λ' is the eigenvalue of l on z_0, then λ' must satisfy

$$\varphi_k(\lambda',\mu) = 0 \ , \quad k = 1,2.$$

(6) <u>Construction of $gl(2,C)$-finite modules.</u>

One solution of the equations

$$\varphi^k(\lambda,\mu) = 0 \ , \quad k = 1,2,$$

is
$$\lambda_1 = \mu_1 - 1 \ , \quad \lambda_2 = \mu_2 - 1 \ .$$

I will construct the irreducible $gl(3,C)$-module in which the labels of the irreducible $gl(2,C)$-submodules are connected to (λ_1,λ_2) in the sense defined in the text. This module will only be $gl(2,C)$-finite if λ_1 and λ_2 are integers which satisfy

$$\lambda_1 > \lambda_2 \ ,$$

or, equivalently, if μ_1 and μ_2 are integers which satisfy

$$\mu_1 > \mu_2 \ .$$

The Array D.

$(\mu_1 - \mu_2 - 1, 0)$
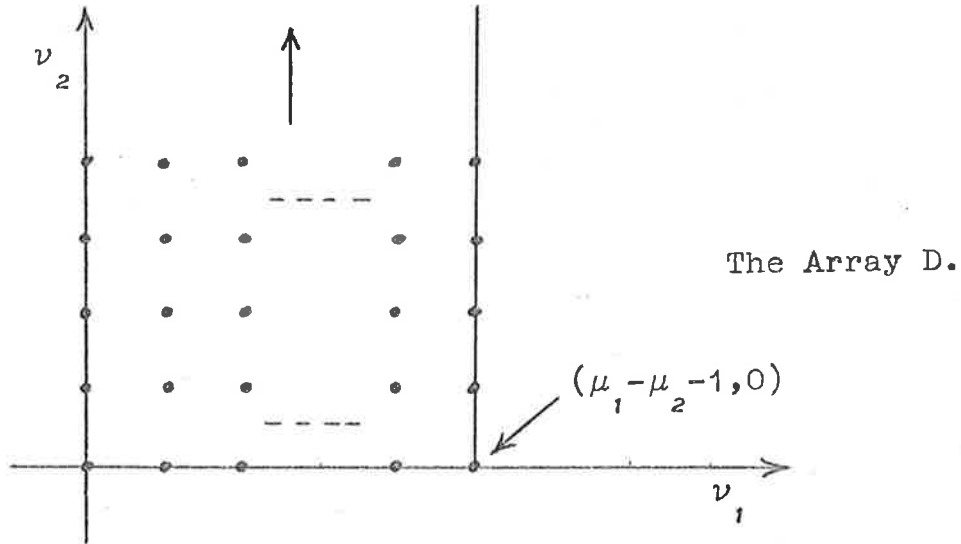
The points of the array D in the (ν_1, ν_2) plane are connected to the origin. Let $X_{\lambda - \nu}$ denote the finite dimensional, irreducible $gl(2,C)$-module with highest weight

$$(\lambda_1 - \nu_1 - 1 , \lambda_2 - \nu_2) ,$$

and construct

$$X = \bigoplus_{\nu \in D} X_{\lambda - \nu} .$$

Define

$$a^3_3 x = (m_1 + m_2 + m_3 - 1_1 - 1_2 - 2)x ,$$

for all x in X. A basis for X consists of the vectors

$$\left| \begin{matrix} \lambda_1 - \nu_1 & \lambda_2 - \nu_2 \\ & \kappa & \end{matrix} \right\rangle \quad , \quad \nu \in D, \ \lambda_1 - \nu_1 > \kappa \geq \lambda_2 - \nu_2 ,$$

where κ is the eigenvalue of a^1_1 on the vector. Define

$$a^3_2 \left| \begin{matrix} \lambda_1 - \nu_1 & \lambda_2 - \nu_2 \\ & \kappa & \end{matrix} \right\rangle$$

$$= \left[(\lambda_1 - \nu_1 - \mu_2)(\lambda_1 - \nu_1 - \mu_3)(\lambda_1 - \nu_1 - \kappa - 1) \left| \begin{matrix} \lambda_1 - \nu_1 - 1 & \lambda_2 - \nu_2 \\ & \kappa & \end{matrix} \right\rangle \right.$$

$$\left. + (\lambda_2 - \nu_2 - \mu_3) \left| \begin{matrix} \lambda_1 - \nu_1 & \lambda_2 - \nu_2 - 1 \\ & \kappa & \end{matrix} \right\rangle \right] / (\lambda_1 - \nu_1 - \lambda_2 + \nu_2)$$

and

$$a^2_3 \left| \begin{matrix} \lambda_1 - \nu_1 & \lambda_2 - \nu_2 \\ & \kappa \end{matrix} \right\rangle$$

$$= \left[(\mu_1 - \lambda_2 + \nu_2 - 1)(\mu_2 - \lambda_2 + \nu_2 - 1)(\kappa - \lambda_2 + \nu_2) \left| \begin{matrix} \lambda_1 - \nu_1 & \lambda_2 - \nu_2 + 1 \\ & \kappa \end{matrix} \right\rangle \right.$$

$$\left. + (\mu_1 - \lambda_1 + \nu_1 - 1) \left| \begin{matrix} \lambda_1 - \nu_1 + 1 & \lambda_2 - \nu_2 \\ & \kappa \end{matrix} \right\rangle \right] / (\lambda_1 - \nu_1 - \lambda_2 + \nu_2).$$

With these definitions, X is an irreducible $gl(3,C)$-module that is $gl(2,C)$-finite. Its highest weight is $(\mu_1 - 2, \mu_2 - 1, \mu_3)$. Note that when μ_3 is an integer and

$$\mu_1 > \mu_2 > \mu_3 ,$$

X is one of the familiar finite dimensional $gl(3,C)$-modules.

APPENDIX 3.

In lemmas 2, 3 and 4 which follow, the following notation will be in force. L will denote a reductive Lie algebra over the complex field and L_R will be a real form of L. Thus,
$$L = L_R + iL_R ,$$
where the sum is a direct sum of vector spaces over R. $U(L)$ and $U(L_R)$ will respectively denote the universal enveloping algebras of L and L_R. A similar decomposition obviously holds for $U(L)$:
$$U(L) = U(L_R) + iU(L_R).$$

Suppose X_R is an L_R-module. Construct
$$X = X_R + iX_R$$
and define a product, denoted by juxtaposition, such that $L \times X \longrightarrow X$ by
$$(a + ib)(x + iy) = (ax - by) + i(bx + ay),$$
where $a,b \in L_R$ and $x,y \in X_R$. With this product, X is an L-module.

Lemma 2. Suppose $X = X_R + iX_R$. X is an irreducible L-module if and only if X_R is an irreducible L_R-module.

Proof (Trivial) (1) Suppose that Y_R is a proper L_R-submodule of X_R. Then $(Y_R + iY_R)$ is a proper L-submodule of X.

(2) Suppose that Y is a proper L-submodule of X. Because X can be decomposed into real and imaginary parts, so too can Y. Thus,
$$Y = Y_R + iY_R.$$
Y_R is a proper L_R-submodule of X_R.   ///

Lemma 3. Suppose that X_R and X_R' are two irreducible L_R-modules. Construct the L-modules
$$X = X_R + iX_R \text{ and } X' = X_R' + iX_R'.$$

Then X and X′ are equivalent L-modules if and only if X_R and $X_R′$ are equivalent L_R-modules.

Proof   (1) If X_R and $X_R′$ are equivalent L_R-modules, there is an isomorphism σ_R,

$$\sigma_R : \quad X_R \longrightarrow X_R′,$$

which intertwines the module operations on X_R and $X_R′$.

Define $\sigma : \quad X \longrightarrow X′$

$$x+iy \longmapsto \sigma_R x + i\sigma_R y \text{ , where } x, y \, \epsilon \, X_R.$$

σ is clearly an isomorphism of the vector spaces X and X′. Furthermore,

$$((a + ib)\sigma)(x + iy) = (a + ib)(\sigma_R x + i\sigma_R y)$$
$$= a(\sigma_R x) - b(\sigma_R y) + i(b(\sigma_R x) + a(\sigma_R y))$$
$$= \sigma_R(ax - by) + i\sigma_R(bx + ay)$$
$$= \sigma((a + ib)(x + iy))$$
$$= (\sigma(a + ib))(x + iy).$$

Thus, X and X′ are equivalent L-modules.

          (2) If X and X′ are equivalent L-modules, there is an isomorphism σ,

$$\sigma : \quad X \longrightarrow X′,$$

which intertwines the module operations on X and X′. Choose any vector x in X_R. X_R can be considered to be an irreducible $U(L_R)$-module. Every vector of X_R, in particular x, is cyclic. Thus,

$$X_R = U(L_R)x.$$

          Let $y = \sigma x$. The vector y can be decomposed

$$y = x′ + ix′′, \text{ where } x′, x′′ \, \epsilon \, X_R′.$$

Since x is non-zero and σ is injective, y is also non-zero, and hence one of x′ and x′′ must be non-zero. I will suppose x′ to be non-zero. A similar proof can be devised in the other case. $X_R′$ is an irreducible $U(L_R)$-module, so x′ is a cyclic vector.

Thus,

$$X_R' = U(L_R)x'$$

Define the map

$$\tau_R: \quad X_R' \longrightarrow X_R$$

$$ax' \longmapsto ax, \text{ where } a \in U(L_R).$$

(i) τ_R is surjective because x is a cyclic vector for X_R.

(ii) τ_R is injective. To show this, suppose ax = 0. Then $\sigma(ax) = 0$. Because σ intertwines the module operations on X and X', $a(\sigma x) = 0$. Thus,

$$ay = ax' + iax'' = 0.$$

Because X_R' and iX_R' are linearly independent over the real field, $ax' = ax'' = 0$. Thus, τ_R is injective.

(iii) τ_R intertwines the module operations on X_R and X_R'.

$$(\tau_R b)(ax') = \tau_R(bax')$$

$$= bax$$

$$= b(\tau_R(ax'))$$

$$= (b\tau_R)(ax').$$

Thus, X_R and X_R' are equivalent L_R-modules. ///

<u>Lemma 4.</u>   If X is an irreducible L-module, then X can be decomposed

$$X = X_R + iX_R,$$

where X_R is an irreducible L_R-module. Furthermore, X_R is unique to within equivalence of L_R-modules.

<u>Proof</u>   Consider X as an irreducible U(L)-module. Every vector x in X is cyclic, so

$$X = U(L)x.$$

Define $\qquad\qquad X_R = U(L_R)x.$

It is clear that X_R is an L_R-module. Furthermore, since

$$U(L) = U(L_R) + iU(L_R),$$

it follows that

$$X = X_R + iX_R.$$

The irreducibility of X_R follows from lemma 2.

If $X = X_R' + iX_R'$ is another decomposition of X, then X_R and X_R' must be equivalent. This is a consequence of lemma 3.   ///

Construction of the Algebra $\overline{U(K)}$.

Both E and $U(K)$ are infinite dimensional algebras over C. Construct the vector space

$$T = E \otimes U(K).$$

With the product

$$\left(\sum_i e_i \otimes u_i\right)\left(\sum_j e_j' \otimes u_j'\right) = \sum_{ij} e_i e_j' \otimes u_i u_j',$$

where e_i, $e_j' \in E$ and u_i, $u_j' \in U(K)$, T is an associative algebra over C. T can be considered to be an algebra over E by defining

$$e\left(\sum_i e_i \otimes u_i\right) = \sum_i ee_i \otimes u_i.$$

However, there are identities which are not satisfied in T. Since

$$Z(K) = E \cap U(K),$$

every element of $Z(K)$ has two presentations, one as an element of E and the other as an element of $U(K)$, so it is important to couple E and $U(K)$ so that the presentations are equal. This may be done as follows. Let

$$s_i(1) = s_i(1_1, 1_2, \cdots, 1_n)$$

be the symmetric polynomial which expresses

$$s_i = \text{trace } a^i \in Z(K), \quad 1 \leqq i \leqq n,$$

in terms of $1_1, 1_2, \cdots, 1_n$ in E. Construct the two-sided ideal

J generated by

$$1 \otimes s_i - s_i(1) \otimes 1 \quad , \quad 1 \leq i \leq n.$$

Since these elements lie in the centre of T,

$$J = \underset{i}{+} \, (1 \otimes s_i - s_i(1) \otimes 1)T.$$

Define the factor algebra

$$\overline{U(K)} = T/J.$$

Finally, let φ denote the natural embedding of U(K) in T,

$$\varphi \, : \, U(K) \longrightarrow T$$

$$u \longmapsto 1 \otimes u,$$

and π denote the canonical projection of T onto $\overline{U(K)}$,

$$\pi \, : \, T \longrightarrow \overline{U(K)}$$

$$t \longmapsto t + J \, .$$

Lemma 5.   The map

$$\psi = \pi \cdot \varphi \, : \, U(K) \longrightarrow \overline{U(K)}$$

$$u \longmapsto 1 \otimes u + J$$

is a monomorphism.

Proof   It is obvious that ψ is a morphism of algebras, so all that must be shown is that ψ is injective. Suppose the contrary, that there exists an element u in U(K) such that

$$\psi(u) = 0,$$

that is, $1 \otimes u \, \epsilon \, J.$

Then $$1 \otimes u = \sum_i (1 \otimes s_i - s_i(1) \otimes 1)t_i$$

$$= \sum_i (1 \otimes s_i - s_i(1) \otimes 1)\Big(\sum_j e_{ij} \otimes u_j\Big),$$

where the elements e_{ij} lie in E and $\{u_1, u_2, \cdots \}$ is the basis of standard monomials in U(K). (Jacobson(1962)).

$$1 \otimes u = \sum_{ij} (e_{ij} \otimes s_i u_j - e_{ij} s_i (1) \otimes u_j).$$

Since s_i is a polynomial in the elements of K, $s_i u_j$ and u_j are linearly independent elements of U(K). Hence, the terms of the sum can never be collected into the form $1 \otimes u$. This shows that $1 \otimes u$ cannot lie in J, and so ψ must be injective.

///

The import of this lemma is that U(K) can be identified with its image in $\overline{U(K)}$. I need a similar result which allows E to be identified with its image in $\overline{U(K)}$. Let σ denote the map

$$\sigma : E \longrightarrow T$$
$$e \longmapsto e \otimes 1.$$

Lemma 6. The map

$$\rho = \pi \cdot \sigma : E \longrightarrow \overline{U(K)}$$
$$e \longmapsto e \otimes 1 + J$$

is a monomorphism.

Proof Again it is obvious that ρ is a morphism of algebras, so I need only prove that ρ is injective. The kernel of ρ is an ideal in E. Because E is a field, its only ideals are trivial. Thus,

$$\ker \rho = 0 \text{ or } E.$$

If I can show that $1 \otimes 1$ does not lie in J, so that

$$\rho(1) = 1 \otimes 1 + J \neq 0,$$

then ρ must be injective. This is trivial because $1 \otimes 1$ is one of the elements $1 \otimes u$, $u \in U(K)$, and the last lemma showed that none of these ever lies in J.

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CHAPTER 4. CODIAGONAL OPERATORS.

Algebraic methods were all that I needed to find the spectrum of eigenvalues of a diagonal operator s on an L-module X because I knew from the very definition of X that X contained a basis consisting of eigenvectors of s. In contrast, if t is an arbitrary linear operator on X, then X need not contain any eigenvectors of t, but there will possibly be extensions of X which do. One such extension is the completion of X with respect to the metric topology defined on X by a norm. Indeed, von Neumann's postulate that the states of a quantum system should comprise a separable Hilbert space simply ensures that the axes of the space can be rotated so that any given self-adjoint operator is diagonal. However, the extension from X to its completion with respect to a norm is not the only possibility, and in this chapter I want to show that for a certain class of operators on X, again connected with the Lie algebras L and K, there is a more elegant procedure that is essentially algebraic.

I want to freely manipulate infinite matrices, but since mistrust of such matrices lingers from the work of von Neumann (1929), I must firstly dispell this taboo. Consequently, there is an introductory section which deals briefly with the algebra of column-finite matrices and its subalgebra of row and column-finite matrices. I should remark that I came to the conclusion that these matrices were appropriate for quantum mechanics in my own round-about way. I later found that Jacobson (1951b) had given a beautiful account of infinite dimensional vector spaces and the algebra of column-finite matrices. His work should be a prerequisite for any course on topological spaces for it establishes the limits of the algebraic theory and thereby defines the role to be played by a norm.

Infinite Matrices.

Suppose that X is a vector space over a field F of characteristic zero, and that the (Hamel) basis of X, denoted

$$\{x_1, x_2, \cdots \},$$

is countably infinite. Every vector of X may be represented as a linear combination of a finite number of basis vectors:

$$x = \sum_k \xi_k x_k, \quad \xi_k \in F,$$

where the number of non-zero coefficients is finite. The map which associates with x the sequence

$$\underline{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

is an isomorphism of X onto the space S comprised by all finite sequences of elements of F.

If $t : X \longrightarrow X$ is an endomorphism of X, then

$$t x_l = \sum_k \tau_{kl} x_k, \quad \tau_{kl} \in F,$$

where the number of summands on the right is necessarily finite. Thus,

$$t \sum_l \xi_l x_l = \sum_{kl} \xi_l \tau_{kl} x_k,$$

where once again the number of summands in each sum is finite. The matrix of coefficients (τ_{kl}) is called the matrix of t relative to the basis $\{x_1, x_2, \cdots \}$ of X. It is uniquely determined by t. (τ_{kl}) is a column-finite matrix, that is to say, the number of non-zero entries in any column of (τ_{kl}) is finite. Conversely, every column-finite matrix over F

determines an endomorphism t of X according to the formula above.

The set of all column-finite matrices over F, with the operations of matrix addition and matrix multiplication, forms an associative algebra over F. The only point worth noting in the proof of this assertion is that the product of two column-finite matrices, (σ_{kl}) and (τ_{kl}) is always well-defined and again column-finite, because the number of non-zero terms in the sum

$$\sum_{j} \sigma_{kj} \tau_{jl}$$

is finite. It is a simple matter to verify that, if s and t are two endomorphisms of X whose matrices are (σ_{kl}) and (τ_{kl}), then the matrices of s+t and st are

$$(\sigma_{kl} + \tau_{kl}) \text{ and } \left(\sum_{j} \sigma_{kj} \tau_{jl} \right)$$

respectively. Thus, the algebra of endomorphisms of X is isomorphic with the algebra of column-finite matrices over F.

For comparison, suppose that X now represents a separable Hilbert space. The Hamel basis of X is now uncountably infinite, and the orthonormal basis, although countable, only spans (algebraically) a dense subspace of X. The infinite matrix, relative to the orthonormal basis, of an unbounded operator t on X does not provide a representation of t unless the domain of t is also specified. For example, all the self-adjoint extensions of a symmetric operator t will have the same matrix elements with respect to an orthonormal basis chosen from the domain of t. The representation of unbounded operators by infinite matrices is fraught with difficulties which cannot

occur in the algebraic theory.

The infinite matrices which arise in many problems of quantum mechanics and spectral analysis are not only column-finite but are also <u>row-finite</u>, that is to say, the number of non-zero entries in any row is finite. This fact is ignored in texts which attempt to establish a rigorous theory of quantum mechanics on Hilbert spaces. Nevertheless, I believe that the justification of many of the formal techniques used in quantum mechanics can be found in this fact.

The row and column-finite matrices form a subalgebra of the algebra of column-finite matrices. The simplest examples of matrices of this type are the codiagonal matrices
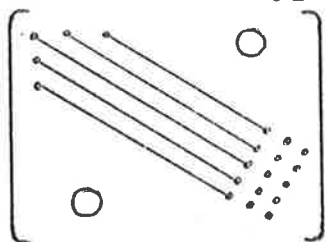
and the block diagonal matrices
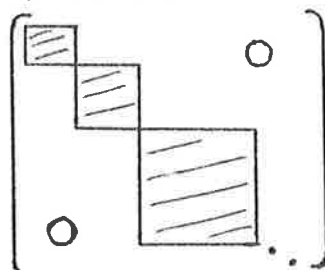
It is not difficult to see that the most general matrix has a swathe of entries along the diagonal
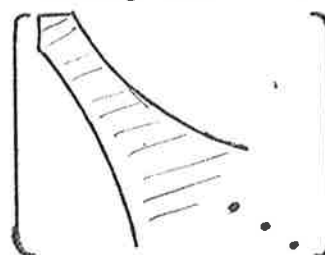
and so is a type of generalised band matrix. For this reason, and also for brevity, I will call an infinite dimensional, row and column-finite matrix a <u>band matrix.</u>

Each band matrix is an endomorphism of S, the space

of terminating sequences of elements of the field F. S can be embedded in a much larger universal space $\overline{S}$, the algebraic completion of S, comprised by all sequences of elements of F. The important property of a band matrix is that it can be uniquely extended from an endomorphism of S to an endomorphism of $\overline{S}$. Suppose that (τ_{kl}) is a band matrix and $\underline{\xi}$ is a vector in S. Set

$$\xi'_k = \sum_l \tau_{kl}\xi_l .$$

The expression for ξ'_k has a well-defined meaning even when the column vector $\underline{\xi}$ has an infinite number of non-zero components. This is so because (τ_{kl}) is row-finite and hence the number of non-zero terms in the summation on the right is always finite. Thus, (τ_{kl}) can be unambiguously extended to an endomorphism of $\overline{S}$. The extension is clearly unique. The isomorphism which maps S onto X can also be extended in a natural way to an isomorphism of $\overline{S}$ onto $\overline{X}$. Here $\overline{X}$, the algebraic completion of X, is the space of all formal series of elements of X,

$$\sum_k \xi_k x_k ,$$

where the sequence $\underline{\xi}$ need no longer terminate. The endomorphism t of X which corresponds to (τ_{kl}) has been extended to an endomorphism $\overline{t}$ of $\overline{X}$. The name 'formal series' suggests that the elements of $\overline{X}$ have dubious mathematical value. This is not so, for $\overline{X}$ is merely an isomorphic copy of $\overline{S}$.

The need for such extensions will be clear in later sections, but for the moment may be illustrated by a simple example. Suppose (τ_{kl}) is a tridiagonal matrix on S,

$$(\tau_{k\,l}) = \begin{bmatrix} \alpha_1 & \beta_1 & & & & \\ \gamma_1 & \alpha_2 & \beta_2 & & & \\ & \gamma_2 & \alpha_3 & \beta_3 & & \\ & & \gamma_3 & \cdot_3 & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot & \cdot \end{bmatrix},$$

in which

$$\beta_k \gamma_k \neq 0 \quad \text{for} \quad k = 1,2,3, \cdots .$$

It is easy to see that the only terminating sequence which satisfies the eigen-equation

$$\sum_{l} \tau_{k\,l} \xi_l = \lambda\, \xi_k \quad , \quad k = 1,2,3, \cdots ,$$

is the trivial sequence, whose elements are all equal to zero. Whereas S does not contain any solutions of the eigen-equation, $\overline{S}$ contains them all. To select particular eigenvectors, I will later define, in an algebraic manner, boundary conditions on $\overline{S}$. Note that if S were embedded in a Banach space, the condition that each eigenvector should have a finite norm would serve the same purpose.

Codiagonal Operators.

Throughout the rest of this chapter, I will use the following notation.

(1) L will denote one of the classical Lie algebras

$$gl(n+1,C) \ , \ sl(n+1,C) \ , \ so(n+1,C) \ , \ sp(n+1,C) \ ,$$

and K will denote the largest subalgebra of L chosen from the list

$$gl(n,C) \ , \ sl(n,C) \ , \ so(n,C) \ , \ sp(n,C) \ .$$

This is the choice for L and K that I called the standard choice in the last chapter. (sp(n,C) for odd n was also

defined there.)

(2) X will be a K-finite L-module of infinite dimension.

(3) Finally, the splitting field of K will be denoted E.

A <u>codiagonal operator</u> on X is a polynomial in the elements of L with coefficients which are diagonal operators on X. Because X is K-finite, every diagonal operator on X has a diagonal matrix. Furthermore, the elements of L have band matrices which must be obtained during the construction of X. It is an elementary step to combine these matrices to find the (band) matrix of any codiagonal operator on X. In particular, when $L = sl(2,C)$, a codiagonal operator is represented by a codiagonal matrix; hence the name.

The aim of this chapter is to investigate the spectrum of a codiagonal operator t on X. In advance I should point out that the spectrum of t on X is usually empty, on $\overline{X}$, the algebraic completion of X, the spectrum is usually the whole complex plane, and on intermediate extensions of X the spectrum lies somewhere between these extremes. The investigation of the spectrum of t more properly consists of the following problems.

(1) Because the matrix of t is a band matrix, the domain of t can be extended from X to $\overline{X}$, and hence to any subspace X´ such that

$$X < X´ < \overline{X}.$$

Let t´ denote the extension of t to X´. If X´ is invariant under t´, then it is quite in order to ask for the spectrum of t´. In fact this is the first problem in the study of the spectrum of t: for every subspace X´, invariant under t´, determine the qualitative character of the spectrum of t´.

I found the concept of a <u>spectral chain for t</u> very

useful in the case $L = sl(2,C)$, and I suspect that its utility is not limited to this case. Suppose that

$$\{X_\gamma , \gamma \geqq 0\}$$

is a family of L-submodules of $\overline{X}$ for which

$$X < X_\beta < X_\gamma < \overline{X} , \quad \beta < \gamma.$$

A finite chain of subspaces

$$X_{\gamma_1} < X_{\gamma_2} < \cdots < X_{\gamma_b} ,$$

where $0 = \gamma_0 \leqq \gamma_1 < \gamma_2 < \cdots < \gamma_b \leqq \gamma_{b+1} = \infty$,

is a spectral chain for t if t has the same spectrum on X_{γ_a} and X_γ for all γ in the range

$$\gamma_a \leqq \gamma < \gamma_{a+1} , \quad 0 \leqq a \leqq b .$$

In loose terms, the spectrum of t is a step function on a spectral chain; it is constant until X_γ reaches a critical size and then it jumps to a new level.

I can rephrase the first problem concerning the spectrum of t. Find a spectral chain for t and determine the qualitative nature of the spectrum of t on each component of the chain.

(2) The second problem is more specific. Given a subspace X', invariant under t' and satisfying

$$X < X' < \overline{X} ,$$

how can the spectrum of t' be calculated in practice?

These are difficult problems, and only in the case of the simple spectrum, where $L = sl(2,C)$, am I able to give a complete solution. Accordingly, I will treat this case in detail and then indicate how the scope of the ideas may be extended.

Simple Spectrum. $L = sl(2,C)$.

In this section, X will denote an irreducible $sl(2,C)$-module of infinite dimension which has a vector x_1 of highest weight ρ. A basis for X consists of the vectors defined recursively by

$$x_{k+1} = e_- x_k \,/\, \beta_k \,, \quad k = 1,2, \cdots ,$$

where $\{\beta_k\}$ is an arbitrary sequence of non-zero complex numbers. Then

$$hx_k = (\rho + 1 - k)x_k \,,$$

$$e_- x_k = \beta_k x_{k+1} \,,$$

$$e_+ x_{k+1} = \alpha_k x_k \,,$$

where

$$\alpha_k \beta_k = k(2\rho + 1 - k) \neq 0 \,.$$

The matrices of h, e_+ and e_- are

$$
h = \begin{bmatrix} \rho & & & & \\ & \rho-1 & & & \\ & & \rho-2 & & \\ & & & \rho-3 & \\ & & & & \ddots \end{bmatrix}, \quad
e_+ = \begin{bmatrix} 0 & \alpha_1 & & & \\ & 0 & \alpha_2 & & \\ & & 0 & \alpha_3 & \\ & & & 0 & \alpha_4 \\ & & & & \ddots \end{bmatrix}, \quad
e_- = \begin{bmatrix} 0 & & & & \\ \beta_1 & 0 & & & \\ & \beta_2 & 0 & & \\ & & \beta_3 & 0 & \\ & & & \ddots & \ddots \end{bmatrix}.
$$

X can obviously be identified with the space S of sequences of complex numbers which have but a finite number of non-zero components.

It is convenient to use unit step operators

$$
d_+ = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ & & & & \ddots \end{bmatrix} \quad \text{and} \quad d_- = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & \ddots & \\ & & & \ddots & \ddots \end{bmatrix}
$$

in place of e_+ and e_- on S. Thus,

$$e_+ = u_+ d_+ \quad \text{and} \quad e_- = d_- u_- \,,$$

where $u_+ = \begin{bmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \ddots \end{bmatrix}$ and $u_- = \begin{bmatrix} \beta_1 & & & \\ & \beta_2 & & \\ & & \beta_3 & \\ & & & \ddots \end{bmatrix}$.

Both u_+ and u_- can be considered to be functions of h since

$$u_+ u_- = (\rho + h)(\rho + 1 - h) .$$

The obvious choices for u_+ and u_- are

$$u_+ = \rho + h \text{ and } u_- = \rho + 1 - h ,$$

but these are not the only possibilites. In general I will allow $u_\pm$ to be rational functions of h. The reason for this choice is that such functions comprise the splitting field E for gl(1,C), that is,

$$E = C(h) .$$

The most general codiagonal operator on S is a polynomial in e_+, e_- and h with coefficients in E. Equivalently, it may be taken as a polynomial in d_+, d_- and h with similar coefficients. Such a polynomial can always be brought to the form

$$t = \sum_{i=-m}^{n} d_i t_i(h) ,$$

where $d_i = \begin{cases} d_+^{\,i} , & \text{for } i \geq 0 , \\ d_-^{\,-i} , & \text{for } i < 0 , \end{cases}$

and $t_i(h) \in E$.

If $t_i(h) = \begin{bmatrix} \tau_{i1} & & & \\ & \tau_{i2} & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix}$, $-m \leq i \leq n$,

then

$$
t = \begin{bmatrix}
\tau_{01} & \tau_{12} & \tau_{23} & \cdot & \cdot & \cdot & \tau_{nn+1} & & \text{O} \\
\tau_{-11} & \tau_{02} & \tau_{13} & \tau_{24} & \cdot & \cdot & & \\
\tau_{-21} & \tau_{-12} & \tau_{03} & \tau_{14} & \tau_{25} & \cdot & & \\
\cdot & \tau_{-22} & \tau_{-13} & \cdot & \cdot & \cdot & & \\
\cdot & & \tau_{-23} & \cdot & \cdot & \cdot & & \\
\cdot & & & & & & & \\
\tau_{-m1} & & & & & & & \\
& \text{O} & & & & & &
\end{bmatrix} ,
$$

so t is a codiagonal matrix with n diagonals above and m below the main diagonal.

When m = O, the matrix of t is upper triangular and the diagonal elements of t are the eigenvalues of t on X. This case is trivial, so I will exclude it and always assume that $m \geqq 1$.

Suppose that $\underline{\xi}$ is an eigenvector of t in $\overline{S}$:

$$
t\underline{\xi} = \lambda\underline{\xi} \ , \quad \lambda \in C, \ \underline{\xi} \in \overline{S}.
$$

The components of $\underline{\xi}$ must satisfy the following equations.

$$
\sum_{i=1-k}^{n} \tau_{i\,k+i}\xi_{k+i} = \lambda\xi_{k} \ , \quad 1 \leqq k \leqq m \ ,
$$

$$
\sum_{i=-m}^{n} \tau_{i\,k+i}\xi_{k+i} = \lambda\xi_{k} \ , \quad m < k.
$$

The equations for k > m constitute a recurrence relation for the components of $\underline{\xi}$, which I will call the recurrence relation associated with t. Its solutions can be classified by the value of

$$
\lim_{k} \sup \ |\xi_{k}|^{1/k} \ ,
$$

according to theorems developed by Poincaré and Perron, and on the basis of this classification a spectral chain for t can be

constructed. 'Lim sup' properly belongs to the realm of analysis, but its only use here will be in the definition of a chain of subspaces of $\overline{S}$.

The equations for $1 \leq k \leq m$ follow from the requirement that S should have a vector of maximum weight.

I have assumed that the coefficients $t_i(h)$ are rational functions of h. Consequently,

$$t_i(h) = p_i(h) + r_i(h),$$

where $p_i(h)$ is a polynomial and $r_i(h)$ is a rational function, whose numerator is of lower degree in h than its denominator. Let

$$1 = \max_{-m \leq i \leq n} \text{degree } p_i(h).$$

I want to exclude the case in which 1 is equal to zero, and for doing so I have three reasons.

(1) All the codiagonal operators which occur in quantum mechanics satisfy the condition $1 > 0$.

(2) When $1 = 0$, the boundary conditions which determine each eigenvalue of t depend upon that eigenvalue, so the eigenvalue problem is no longer linear.

(3) For most purposes, t may be replaced by

$$t' = q(h)t \;,$$

where q(h) is a polynomial in h. If the degree of q is sufficiently high, t' will satisfy the condition $1 > 0$.

Define ζ_i to be the coefficient of h^1 in $p_i(h)$. The characteristic polynomial of the recurrence relation associated with t is the polynomial

$$\sum_{i=-m}^{n} \zeta_i \gamma^i \;, \quad \gamma \in C.$$

Its importance is clear from the following theorem, which is a

minor extension of a result first proved by Perron (1921).

<u>Theorem</u> (Perron).Let $\gamma_1, \cdots, \gamma_b$ be the distinct moduli of the roots of the equation

$$\sum_{i=-m}^{n} \zeta_i \gamma^i = 0,$$

ordered so that

$$0 \leqq \gamma_1 < \gamma_2 < \cdots < \gamma_{b-1} < \gamma_b \leqq \infty .$$

The possibility $\gamma_b = \infty$ only arises when $\zeta_n = 0$. Let m_a be the number of roots whose moduli are all equal to γ_a. Thus,

$$m_1 + m_2 + \cdots + m_b = m + n .$$

Provided that

$$\tau_{-mk} \neq 0 , \quad k \geqq 1,$$

the recurrence relation has a fundamental system of solutions which fall into b classes, such that, for solutions of the ath class and their linear combinations,

$$\lim_{k \to \infty} \sup |\xi_k|^{1/k} = \gamma_a . \qquad ///$$

For any assigned value of λ, there are (n+m) linearly independent solutions of the recurrence relation associated with t, but there will in general be only n independent solutions of the eigen-equations of t. To see this, simply assign arbitrary values to $\xi_1, \cdots, \xi_n$ and compute $\xi_{n+1}, \xi_{n+2}, \cdots$ from the equations recursively. Not every solution of the recurrence relation will lead to a solution of the eigen-equations of t, but the same classification scheme may be used for both.

With the help of Perron's theorem, I can now construct a spectral chain for t.

For each positive real number β, define the following subspace of $\overline{S}$:

$$S^\beta = \{\underline{\xi} \ \epsilon \ \overline{S} \ | \ \lim_{k \to \infty} \xi_k/\beta^k = 0\}, \ \beta > 0.$$

Now define

$$S_\gamma = \bigcap_{\beta > \gamma} S^\beta \ , \quad \gamma \geqq 0 \ ,$$

and

$$S_a = \bigcap_{\beta > \gamma_a} S^\beta \ , \quad 1 \leqq a \leqq b \ ,$$

with the understanding that $S_b = \overline{S}$ if $\gamma_b = \infty$. Since

$$0 \leqq \gamma_1 < \gamma_2 < \cdots < \gamma_b \leqq \infty \ ,$$

it is clear that

$$S < S_1 < S_2 < \cdots < S_b \leqq \overline{S}.$$

Finally, let t_γ and t_a, $1 \leqq a \leqq b$, denote the extensions of t from S to S_γ and S_a, respectively. The first two lemmas of the appendix show that each space S_γ is invariant under t_γ and that a vector $\underline{\xi}$ lies in S_γ if and only if

$$\limsup_{k} |\xi_k|^{1/k} \leqq \gamma \ .$$

I assert that the chain of submodules given above is a spectral chain for t, that is, the spectrum of t_γ on S_γ is constant on each of the intervals

$$0 \leqq \gamma < \gamma_1 \ ,$$

$$\gamma_a \leqq \gamma < \gamma_{a+1} \ , \quad 1 \leqq a < b \ ,$$

$$\gamma_b \leqq \gamma \ .$$

Note that I have assumed that $\gamma_1 > 0$ and $\gamma_b < \infty$. When these conditions are not satisfied only trivial modifications are required. The proof of this assertion is very simple.

(1) If $\qquad\qquad\qquad 0 \leqq \gamma < \gamma_1$,

the space S_γ cannot contain any solutions of the recurrence relation associated with t. Consequently, if t and t_γ have any eigenvectors they must lie in S and the spectra of t and t_γ

must be identical because the restriction of t_γ to S is precisely t.

(2) If
$$\gamma_a \leqq \gamma < \gamma_{a+1} , \quad 1 \leqq a < b ,$$
S_γ can only contain those solutions of the recurrence relation which already lie in S_a. Thus, t_a and t_γ have the same spectra.

(3) Finally, if $\quad \gamma_b \leqq \gamma$,
then S_γ contains every solution of the recurrence relation, and so the spectra of t_b and t_γ are identical.

The spectrum of t on S in general will be empty, for rarely will it be possible to satisfy the recurrence relation associated with t with a sequence which terminates.

The space S_1 contains m_1 linearly independent solutions of the recurrence relation, denoted
$$\underline{\xi}_j^{(1)} , \quad 1 \leqq j \leqq m_1 .$$

Thus, the general solution of the recurrence relation in S_1 is
$$\underline{\xi}^{(1)} = \sum_{j=1}^{m_1} \eta_j^{(1)} \underline{\xi}_j^{(1)} , \quad \eta_j^{(1)} \epsilon C .$$

Now, if $\underline{\xi}^{(1)}$ is to be a solution of the eigen-equations of t, the constants $\eta_j^{(1)}$ must be chosen so that the 'boundary conditions'
$$\sum_{i=1-k}^{n} \tau_{i k+i} \xi_{k+i}^{(1)} = \lambda \xi_k^{(1)}, \quad 1 \leqq k \leqq m ,$$
are also satisfied. Because these equations are homogeneous, there are $(m_1 - 1)$ constants to be chosen so that m linear equations are satisfied.

(1) When
$$m_1 - 1 = m + j , \quad j \geqq 0 ,$$
the eigen-equations of t will have a solution which contains j arbitrary parameters for every value of λ. Thus, the spectrum of t_1 will be the whole complex plane and every point will be

j-fold degenerate.

(2) When $m_1 - 1 < m$, the situation is more interesting. Although the boundary conditions to be satisfied outnumber the parameters to be varied, for certain values of λ there may still be solutions in S_1. Some idea of how these eigenvalues would be distributed throughout the complex plane can be obtained as follows. Let $g_i(\lambda)$ denote the determinant of the infinite matrix obtained from $(t - \lambda)$ by deletion of the ith row and the first column. Infinite determinants are usually not well defined, but, by the insertion of factors to ensure convergence, $g_i(\lambda)$ can be made into an entire function of λ. The eigenvalues of t will then be the zeros of the entire function

$$\sum_{i=0}^{m} (-)^i \tau_{-i,1} g_{i+1}(\lambda) \ ,$$

and consequently will comprise at most a countable subset of the complex plane. Thus, in this second case, the spectrum of t_1 will consist of a discrete set of eigenvalues.

The spectrum of t_a, $2 \leqq a \leqq b$, can be treated in a similar manner. There are m_a solutions of the recurrence relation which lie in S_a but not in S_{a-1}. These are denoted

$$\underline{\xi}_j^{(a)} \ , \quad 1 \leqq j \leqq m_a \ .$$

The most general solution of the recurrence relation in S_a is

$$\underline{\xi}^{(a)} = \sum_{c=1}^{a} \sum_{j=1}^{m_c} \eta_j^{(c)} \ \underline{\xi}_j^{(c)} \ .$$

There are m conditions to be satisfied if $\underline{\xi}^{(a)}$ is to be not only a solution of the recurrence relation but also a solution of the whole system of eigen-equations for t. However, there are now

$$m_1 + m_2 + \cdot \ \cdot \ \cdot + m_a - 1$$

constants to vary in order to find such a solution. The spectrum of t_a clearly includes the spectrum of t_{a-1}. So long as

$$m_1 + m_2 + \cdots + m_a - 1 < m,$$

the spectrum of t_a will be discrete, but once

$$m_1 + m_2 + \cdots + m_a - 1 = m + j, \quad j \geqq 0,$$

the spectrum of t_a will cover the whole complex plane and will depend upon j arbitrary parameters.

Perron's theorem is only applicable if the numbers τ_{-mk}, $k \geqq 1$, are all non-zero. When this is not the case, the analysis presented above is inadequate. However, when $m = 1$, the most common situation, there is a simple remedy for this difficulty. Let j denote the largest integer for which $\tau_{-1j} = 0$. Such an integer can always be found because $t_{-1}(h)$ is a rational function of h. The matrix for t is decomposable:

$$t = \begin{bmatrix} \tau_{01} & \tau_{12} & \cdot & \cdot & \cdot & \cdot & \tau_{nn+1} \\ \tau_{-11} & \tau_{02} & & & & & \\ & \tau_{-12} & & & & & \\ & & \tau_{-1j-1} & \tau_{0j} & \tau_{1j+1} & & \\ & & & 0 & \tau_{0j+1} & & \\ & & & & \tau_{-1j+1} & & \end{bmatrix}.$$

The $j \times j$ block of t can be reduced to its Jordan canonical form in the usual way. Thus, t has j eigenvalues on S, though some of these eigenvalues may be degenerate. A recurrence relation is associated with the lower block of t, Perron's theorem is applicable, and subspaces

$$S < S_1 < S_2 < \cdots < S_b \leqq \bar{S}$$

can be defined as before. However, the difference in this case is that the spectrum of t_a on S_a consists of the whole complex plane, even when $a = 1$. This is easy to see. Choose any λ in

C and construct a solution of the recurrence relation in S_a.
This fixes the components $\xi_{j+1}, \xi_{j+2}, \cdots$ of $\underline{\xi}$. The remaining
components must be chosen to satisfy the following equations.

$$
\begin{bmatrix}
\tau_{01}-\lambda & \tau_{12} & \cdot & \cdot & \cdot & \tau_{j-1\,j} \\
\tau_{-11} & \tau_{02}-\lambda & & & & \tau_{j-2\,j} \\
 & & & & & \\
O & & & & & \\
 & & & & \tau_{-1\,j-1} & \tau_{0j}-\lambda
\end{bmatrix}
\begin{bmatrix}
\xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \cdot \\ \xi_j
\end{bmatrix}
=
$$

$$
-
\begin{bmatrix}
\tau_{j\,j+1} & \cdot & \cdot & \cdot & \tau_{n\,n+1} & O \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
\tau_{1\,j+1} & & & & \tau_{n-j+1\,n+1} & \cdot \cdot
\end{bmatrix}
\begin{bmatrix}
\xi_{j+1} \\ \cdot \\ \cdot \\ \cdot \\ \xi_{j+n}
\end{bmatrix}
\cdot
$$

Provided these equations are linearly independent, as generally
will be the case, there will be a unique solution for $\xi_1, \cdots$,
ξ_j. The vector $\underline{\xi}$ is an eigenvector of t_a with the prescribed
eigenvalue λ. Thus, the spectrum of t_a consists of the whole
complex plane.

I have only discussed the conventional eigenvalue
problem

$$tx = \lambda x \; .$$

Problems of the type

$$tx = \lambda sx \; ,$$

where both t and s are codiagonal operators on X, can be treated
similarly.

Norm Completion of X.

Each of the subspaces $X_1, \cdots, X_b$ has been defined

as the intersection of the kernels of a family of boundary functionals on $\overline{X}$. There are other ways by which particular subspaces of $\overline{X}$ may be isolated, and by far the most important employs a norm defined on X. This section examines the spectrum of t on the completion of X with respect to such a norm.

Suppose that $|\cdot|$ is a norm defined on X:

$$|\cdot| : X \longrightarrow R^+$$
$$x \longmapsto |x| .$$

I will require only one special property of this norm, that the following limit should exist and be different from zero,

$$\lim_{k \to \infty} |x_{k+1}| \, / \, |x_k| = 1/\gamma \neq 0,$$

where, as usual, $\{x_k\}$ denotes the Hamel basis of X. This is a weak restriction, since in most examples $|x_k| = 1$ for all k. Let $\widehat{X}$ denote the completion of X with respect to the metric topology defined on X by the norm.

The matrix of t is a band matrix, so the domain of t can certainly be extended to all of $\widehat{X}$. However, in general t will not leave $\widehat{X}$ invariant. In fact, only when t is continuous with respect to the topology on $\widehat{X}$ can t be extended to an endomorphism of $\widehat{X}$. There will be a maximal extension $\widehat{t}$ of t whose domain Y, necessarily dense in $\widehat{X}$, satisfies

$$X \leqq Y < \widehat{X}$$

and is specified by

$$Y = \{x \in \widehat{X} \mid tx \in \widehat{X}\}.$$

I will seek eigenvectors of $\widehat{t}$ in $\widehat{X}$ as absolutely convergent series

$$x = \sum_{k=1}^{\infty} \xi_k x_k .$$

To decide the question of convergence of such series, I will

use the kth root test; the series converges absolutely if

$$\lim_{k} \sup |\xi_k x_k|^{1/k} = \theta < 1$$

and diverges if $\theta > 1$. I will ignore the case in which $\theta = 1$ and the test fails. Since

$$\lim_{k} \inf |x_{k+1}| / |x_k| \leqq \lim_{k} \inf |x_k|^{1/k}$$

$$\leqq \lim_{k} \sup |x_k|^{1/k}$$

$$\leqq \lim_{k} \sup |x_{k+1}| / |x_k|$$

and, by assumption,

$$\lim_{k \to \infty} |x_{k+1}| / |x_k| = 1/\gamma \; ,$$

the series will converge absolutely if

$$\lim_{k} \sup |\xi_k|^{1/k} < \gamma \; .$$

The numbers

$$0 \leqq \gamma_1 < \gamma_2 < \cdot \cdot \cdot < \gamma_b \leqq \infty$$

partition the positive real axis.

(1) If $0 < \gamma < \gamma_1$, then none of the solutions of the recurrence relation leads to an absolutely convergent series. Consequently, neither t nor $\hat{t}$ has any eigenvectors, other than those in X.

(2) Suppose that

$$\gamma_a \leqq \gamma < \gamma_{a+1} \; , \quad 1 \leqq a < b \; .$$

When $\gamma = \gamma_a$ the kth root test fails, so I will exclude this case and suppose that the strict inequalities hold. Every eigenvector of t_a in X_a has coefficients which satisfy

$$\lim_{k} \sup |\xi_k|^{1/k} \leqq \gamma_a \; ,$$

and so the series corresponding to these solutions are absolutely convergent with respect to the norm $|\cdot|$. Conversely, if

$$x = \sum_{k=1}^{\infty} \xi_k x_k$$

is an absolutely convergent series in X which corresponds to an eigenvector of $\hat{t}$, then

$$\limsup_k |\xi_k|^{1/k} \leq \gamma < \gamma_{a+1} \; ,$$

so x is also an eigenvector of t_a. Thus, t_a and $\hat{t}$ have identical spectra.

(3) Finally, if $\gamma_b < \gamma < \infty$, every solution of the recurrence relation leads to an absolutely convergent series, so once again t_b and $\hat{t}$ have identical spectra.

This result has two important implications. Firstly, nothing of the spectrum is lost if the algebraic extension t_a is employed instead of the topological extension $\hat{t}$. t_a has for its domain the whole of X_a, whereas the domain of $\hat{t}$ is only a dense subspace of $\hat{X}$, so t_a is definitely the easier operator to handle. For the second consequence, suppose that $|\cdot|'$ is any other norm on X for which

$$\lim_{k \to \infty} |x_{k+1}|' \, / \, |x_k|' = 1/\gamma' \neq 0 \; ,$$

that $\hat{X}'$ is the completion of X with respect to the norm $|\cdot|'$, and that $\hat{t}'$ is the maximal extension of t on $\hat{X}'$. If γ and γ' lie in the same range, then the eigenvalue spectra of $\hat{t}$ and $\hat{t}'$ are identical. This result is not so obvious if the algebraic formulation of the problem is not developed before the analytical version because this rescaling of the basis vectors changes the boundaries of $\hat{X}$.

Algebraic and Topological Equivalence.

If this theory of codiagonal operators is to be consistent and of any practical use, the spectrum of a codiagonal operator should not depend upon the basis chosen for X. This requirement can be formulated as follows. Suppose that u is any automorphism of the <u>vector space</u> X. For any codiagonal operator t on X, construct

$$s = utu^{-1}.$$

If t has an eigenvector in X,

$$tx = \lambda x , \quad x \in X, \lambda \in C,$$

then ux is an eigenvector of s with the same eigenvalue, since

$$s(ux) = utu^{-1}ux = \lambda(ux).$$

Similarly, if s has an eigenvector in X, then so too has t. There is no difficulty in this argument because u is an automorphism of X, so ux and $u^{-1}x$ are well defined for all x in X. Now suppose that X′ is an L-module which lies between X and $\overline{X}$,

$$X < X' < \overline{X} ,$$

and let s′, t′, u′ be the extensions to X′ of s, t and u. These extensions exist because the matrices of s, t and u are band matrices. Two problems arise.

(1) There is no guarantee that X′ should be invariant under u′, even though X is invariant under u. For example, suppose that L = sl(2,C) and S′ is the subspace of $\overline{S}$ comprised by sequences which satisfy

$$\lim_{k \to \infty} \xi_k \, / \, \beta^k = 0 , \quad \beta > 0.$$

If
$$u = \begin{bmatrix} \gamma & & & \\ & \gamma^2 & & \\ & & \gamma^3 & \\ & & & \ddots \end{bmatrix} , \quad \text{where } \gamma > \beta ,$$

then

$$u\underline{\xi} = (\gamma^k \xi_k)$$

and it is clear that

$$\lim_{k \longrightarrow \infty} \xi_k (\gamma/\beta)^k$$

need not vanish.

(2) Even if u′ does leave X′ invariant, the extension u′ need not be non-singular. Once again an example will make this point clear. Suppose L, X and X′ are as in the last example, but that

$$u = \begin{bmatrix} \beta & -2 & & & \\ & \beta & -2 & & \\ & & \beta & -2 & \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & \ddots \\ & & & & & \ddots & \ddots \end{bmatrix} \quad .$$

It is easy to verify that u is non-singular on X. However, the sequence with

$$\xi_k = (\beta/2)^k$$

certainly lies in X′ and

$$u'\underline{\xi} = 0.$$

It appears that there are too many automorphisms of X and that it is too strong to insist that s′ and t′ should always have identical spectra. I will moderate the demand as follows. Define a K-automorphism of X to be an automorphism of X which lies in the enveloping algebra of K, considered as an algebra over E. This is the algebra I called $\overline{U(K)}$ in the last chapter. It is clear that

$$u'X' \leqq X'$$

for any K-automorphism u. Furthermore, because X can be completely

decomposed into irreducible K-modules of finite dimension, it seems reasonable that u′ should be non-singular on X′. This is indeed the case and has the immediate consequence that the spectra of s′ and t′ are identical. The proof is very simple, but can be found in the appendix.

For comparison, I want to sketch the procedure that is followed when X is embedded in a topological vector space. Suppose that $|\cdot|$ is a norm on X and that $\hat{X}$ is the completion of X with respect to this norm. Let u denote an automorphism of X. Only when both u and u^{-1} are bounded on X can u be extended to an automorphism $\hat{u}$ of both the algebraic and topological structures of $\hat{X}$. If $\hat{s}$ and $\hat{t}$ are the maximal extensions of s and t, then generally $\hat{s}$ and $\hat{t}$ will only have equal spectra when $\hat{u}$ is an automorphism of $\hat{X}$.

An example will illustrate the difference between the algebraic and topological approaches. Suppose that X is an sl(2,C)-module with highest weight ρ. Define a norm on X by

$$|\cdot| : X \longrightarrow R^+$$

$$\sum_k \xi_k x_k \longmapsto \sum_k |\xi_k| \, ,$$

and let $\hat{X}$ denote the completion of X with respect to this norm. Set

$$u = (\rho + 1 - h) = \begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 3 & & \\ & & & 4 & \\ & & & & \ddots \\ & & & & & \ddots \end{bmatrix}.$$

It is clear that u is a gl(1,C)-automorphism of X. However, u is not bounded on X, so $\hat{u}$ certainly is not a continuous automorphism of $\hat{X}$. Thus, although

$$\underline{\xi} = (\xi_k) = (k^{-2})$$

could be an eigenvector of an operator $\hat{t}$, $\hat{u}\underline{\xi}$ could not be an eigenvector of $\hat{u}\hat{t}\hat{u}^{-1}$ because $u\underline{\xi} \notin \hat{X}$.

Truncation of Codiagonal Matrices.

Through Perron's theorem, I was able to describe qualitatively the spectrum of t on each of the subspaces of the chain

$$S < S_1 < S_2 < \cdots < S_b \lessgtr \bar{\bar{S}}.$$

However, Perron's theorem gave no indication how the eigenvalues of t_a on S_a could be found in practice. A theorem proved by Poincaré, and its generalisation by Perron, show how to solve this problem in most cases, but not all. The idea is very simple. $\underline{\xi}$ satisfies the recurrence relation

$$\sum_{i=-m}^{n} \tau_{i\,k+i} \xi_{k+i} = \lambda \xi_k \ , \quad k \geq m \ ,$$

where

$$\lim_{k \to \infty} \tau_{i\,k}/k^l = (-)^l \zeta_i.$$

If

$$\lim_{k \to \infty} \xi_{k+1}/\xi_k = \gamma,$$

then γ must satisfy the equation

$$\sum_{i=-m}^{n} \zeta_i \gamma^i = 0,$$

that is, γ must be a root of the characteristic equation of the recurrence relation. The numbers $\gamma_1, \cdots, \gamma_b$, introduced earlier, were the distinct moduli of the roots of this equation. Thus,

$$|\gamma| = \gamma_a \ ,$$

for some value of a, and

$$\lim_{k \to \infty} |\xi_{k+1}/\xi_k| = \gamma_a.$$

Therefore, the solution $\underline{\xi}$ is a vector in the subspace S_a. An approximate solution of the eigen-equations can be found as follows. Choose a large positive integer j. Define

$$\xi'_k = \xi_k , \quad j \ge k \ge 1$$

$$\xi'_{j+k} = \gamma^k \xi_j , \quad k > 0.$$

Because

$$\xi_{k+1}/\xi_k \longrightarrow \gamma,$$

$\underline{\xi}'$ should be a good approximation to $\underline{\xi}$. When $\underline{\xi}'$ is substituted for $\underline{\xi}$, the infinite set of eigen-equations collapses to a j × j matrix eigenvalue problem. The eigenvalues of this finite matrix yield approximations to the first j eigenvalues of t_a.

The difficulty with this scheme is that it hinges upon the assumption that the limit

$$\lim_{k \to \infty} \xi_{k+1}/\xi_k$$

exists for every solution of the recurrence relation. Poincaré (1885) has given a set of conditions which are sufficient for this to be true. Perron (1910) has shown by numerous counter examples that it is very difficult to devise better conditions than those of Poincaré.

Poincaré's Theorem. Suppose that $\beta_1, \beta_2, \cdots, \beta_{m+n}$ are the roots of the characteristic equation

$$\sum_{i=-m}^{n} \zeta_i \beta^i = 0$$

and that

$$|\beta_1| < |\beta_2| < \cdots < |\beta_{m+n}| .$$

If $\underline{\xi}$ is any solution of the recurrence relation

$$\sum_{i=-m}^{n} \tau_{i\,k+i}\,\xi_{k+i} = \lambda \xi_k \;,$$

then the limit

$$\lim_{k \to \infty} \xi_{k+1}/\xi_k$$

exists and is equal to one of the numbers $\beta_1, \beta_2, \cdots, \beta_{m+n}$.
<u>Perron's modification</u>. Suppose that the coefficient τ_{-mk},
$k \geqq 1$, never vanishes and that the other conditions of Poincare's
theorem are fulfilled. Then the recurrence relation possesses
$m + n$ fundamental solutions $\underline{\xi}^{(i)}$ which satisfy

$$\lim_{k \to \infty} \xi^{(i)}_{k+1}/\xi^{(i)}_k = \beta_i \;, \quad 1 \leqq i \leqq m+n.$$

I found a minor extension of this result for the
particular case $m + n = 3$. It is included in the appendix, but
I will not bother with it here. Instead I will assume that the
moduli of the roots of the characteristic equation are distinct
and that τ_{-mk} never vanishes.

Suppose that $\underline{\xi}$ is an eigenvector of t_a but not of
t_{a-1}. Then

$$\underline{\xi} = \sum_{c=1}^{a} \eta^{(c)} \underline{\xi}^{(c)} \;, \quad \eta^{(a)} \neq 0.$$

Now, $\displaystyle \lim_{k \to \infty} \xi_{k+1}/\xi_k = \lim_{k \to \infty} \left(\sum_{c=1}^{a} \eta^{(c)} \xi^{(c)}_{k+1}/\xi^{(a)}_k \right) \left(\sum_{c=1}^{a} \eta^{(c)} \xi^{(c)}_k/\xi^{(a)}_k \right)^{-1}$

$$= \beta_a \;,$$

since $\displaystyle \lim_{k \to \infty} \xi^{(c)}_k/\xi^{(a)}_k = 0$ if $c < a$.

I explained previously that the spectrum of t_a is discrete if

$$m_1 + m_2 + \cdots + m_a - 1 < m.$$

In this case,

$$m_i = 1 , \qquad 1 \leqq i \leqq m + n ,$$

so the spectra of $t_1, \cdots, t_m$ are discrete, but every complex number is an eigenvalue of t_{m+i} with multiplicity i if $1 \leqq i \leqq n$. The infinite set of equations for the eigenvalues of t_a, $1 \leqq a \leqq m$, can be reduced to a finite set, suitable for calculation, by the approximation

$$\xi_{j+k} = \beta_a^k \xi_j , \qquad k \geqq 1.$$

The eigenvalue problem for each t_a was defined without recourse to topology. The practical solution of the eigenvalue problem, however, _must_ use some notion of approximation, and hence problems of convergence arise. Numerical analysis is inescapable in the construction of practical solutions. In the theory of quantum mechanics, I believe it should be possible to imitate this arrangement. The basic equations should rest upon algebraic foundations; a norm and a notion of convergence should only be added when numbers are required from the computer.

General Codiagonal Operators.

The definition of a codiagonal operator is quite general. It applies for any standard choice of L and K and any K-finite L-module X. What is peculiar to the case $L = sl(2,C)$ is that the spectrum of a codiagonal operator can be characterised so completely and calculated so easily. The increase in complexity incurred in passing to more general codiagonal operators is of the same order as that in passing from ordinary to partial differential equations. Consequently, I cannot give

an adequate account of the spectrum of a general codiagonal operator.   Instead I will sketch the aims of the theory and indicate where the difficulties arise.

The plan of attack is this.

(1) Given a codiagonal operator t on X, find the matrix of t.

(2) Extend the domain of definition of t from X to $\bar{X}$.

(3) Suppose t′ denotes the extension of t to a subspace X′, where

$$X < X′ < \bar{X}.$$

Identify those subspaces X′ which are invariant under t′ and deduce the character of the spectrum of t′.   This perhaps could involve the construction of a spectral chain for t.

(4) Devise a practical method by which the spectrum of t′ can be calculated.   Equivalently, given a boundary condition on $\bar{S}$, truncate the matrix of t so that the boundary condition is satisfied.

The first stage of this plan is quite elementary.   The matrices of all the elements of L and E are known, and t is simply a polynomial in the elements of L with coefficients chosen from E.

The second stage is also trivial and can be carried out no matter what choice is made for L.   The matrix of t is always a band matrix and this ensures that t can be extended from X to $\bar{X}$.

At the third stage the difficulties begin.   It was Perron's theorem on linear difference equations which enabled the construction of a spectral chain for t in the case $L = sl(2,C)$. In the general case, the eigen-equation for t leads to <u>partial</u> difference equations, for which there is no analogue of Perron's theorem.   The theorem that is needed would have to classify the solutions of a linear partial difference equation according to

the asymptotic behaviour of the solutions. Little seems to be
known about this matter.

The difficulty in the last stage is similar. It was
Poincaré's theorem on difference equations which suggested the
method by which a codiagonal matrix could be truncated in accord-
ance with the boundary conditions imposed on $\overline{S}$. Again, there is
not a generalisation of Poincaré's theorem for partial difference
equations.

I think these difficulties could be overcome for any
codiagonal operator whose matrix is of the type I call a
codiagonal block matrix. Such a matrix is a natural generalis-
ation of a codiagonal matrix and arises as follows. Suppose that
the K-submodules of X are ordered, perhaps according to dimension.
Corresponding to this decomposition of X, a codiagonal block
matrix t would have n diagonals of blocks above the main diagonal
of blocks and m below. Thus, the matrix shown would be a
tridiagonal block matrix.
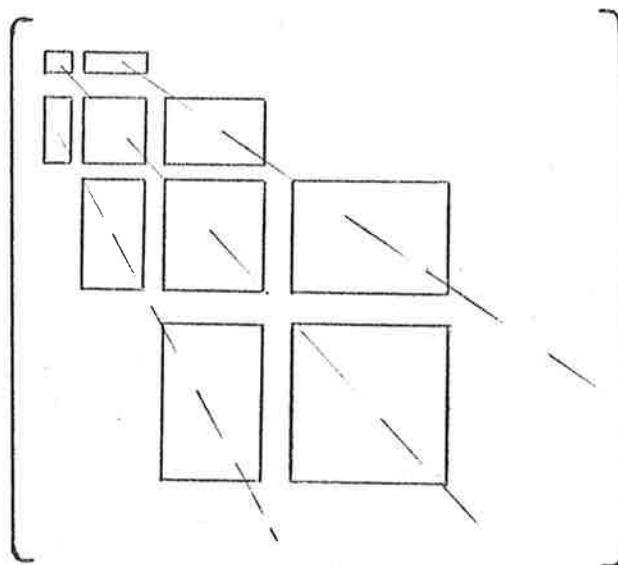
t would map the kth block of X into the linear span of the blocks
between the $(k - m)^{th}$ and the $(k + n)^{th}$. Codiagonal matrices of
course would be a special case in which all the blocks were one
dimensional.

APPENDIX 4.

Lemma 1. For any $\gamma \geqq 0$, S_γ is invariant under the action of L and E.

Proof   To establish the result, I need only show that S_γ is invariant under $f(h)$, e_+ and e_-, where $f(h)$ is a _rational_ function of h.

Suppose $\underline{\xi} \in S_\gamma$. Then,

$$\lim_{k \to \infty} \xi_k/\beta^k = 0 \text{ for all } \beta > \gamma.$$

If (ξ_k) had a subsequence such that

$$\lim_{k_i \to \infty} |\xi_{k_i}|^{1/k_i} \geqq \beta,$$

then

$$\lim_{k \to \infty} \xi_k/\beta^k$$

could not vanish. Thus, for sufficiently large k,

$$|\xi_k|^{1/k}/\beta \leqq \eta < 1.$$

(1) Let $f(h)$ denote a rational function of h.

$$(f(h)\underline{\xi})_k = f(\rho + 1 - k)\xi_k, \quad k \geqq 1.$$

Now, for large values of k,

$$|f(\rho + 1 - k)\xi_k|/\beta^k \leqq |f(\rho + 1 - k)|\eta^k.$$

Since $\eta < 1$,

$$\lim_{k \to \infty} (f(\rho + 1 - k)\xi_k)/\beta^k = 0.$$

Hence, for all $\beta > \gamma$,

$$f(h)\underline{\xi} \in S^\beta,$$

from which it follows that

$$f(h)\underline{\xi} \in S_\gamma.$$

(2) $(e_+\underline{\xi})_k = \alpha_k \xi_{k+1}$

and

$$(e_- \underline{\xi})_{k+1} = \beta_k \xi_k \quad , \qquad k \geq 1.$$

Both α_k and β_k are rational functions of k, so a proof similar to the one above can be used to show that

$$e_+ \underline{\xi} \, \epsilon \, S_\gamma$$

and

$$e_- \underline{\xi} \, \epsilon \, S_\gamma \; . \qquad\qquad ///$$

<u>Lemma 2.</u> $\underline{\xi} \, \epsilon \, S_\gamma$ if and only if

$$\limsup_k \, |\xi_k|^{1/k} \leq \gamma.$$

<u>Proof</u>   (1) Suppose that

$$\limsup_k \, |\xi_k|^{1/k} = \theta \leq \gamma.$$

Then, for all $\beta > \gamma$,

$$\limsup_k \, |\xi_k/\beta^k|^{1/k} = \theta/\beta < 1.$$

Thus, for large values of k,

$$|\xi_k/\beta^k|^{1/k} \leq \theta/\beta < 1 \; ,$$

and so

$$|\xi_k/\beta^k| \leq (\theta/\beta)^k.$$

Since $(\theta/\beta)^k \longrightarrow 0$ as $k \longrightarrow \infty$,

$$\lim_{k \longrightarrow \infty} \xi_k/\beta^k = 0 \; ,$$

so $\underline{\xi} \, \epsilon \, S_\gamma$.

(2) Suppose that

$$\limsup_k \, |\xi_k|^{1/k} = \theta > \gamma \; .$$

There exists a subsequence (ξ_{k_i}) such that

$$\lim_{i \to \infty} |\xi_{k_i}|^{1/k_i} = \theta \quad .$$

If β is chosen so that $\theta > \beta > \gamma$, then

$$\lim_{i \to \infty} |\xi_{k_i}/\beta^{k_i}|^{1/k_i} = \theta/\beta > 1.$$

Thus, for large values of i,

$$|\xi_{k_i}/\beta^{k_i}|^{1/k_i} \geq \eta \, ,$$

where $\theta/\beta \geq \eta > 1$.

Hence, $\qquad |\xi_{k_i}/\beta^{k_i}| \geq \eta^{k_i} \longrightarrow \infty$.

The sequence (ξ_k/β^k) cannot approach zero because it has a subsequence which diverges to infinity. Hence

$$\underline{\xi} \notin S^\beta$$

and so

$$\underline{\xi} \notin S_\gamma \, . \qquad\qquad ///$$

Lemma 3. Suppose that:

(1) u is a K-automorphism of X;

(2) X′ is an L-submodule of $\overline{X}$ with $X < X' < \overline{X}$;

(3) t and $s = utu^{-1}$ are codiagonal operators on X;

(4) t′ and s′ denote the extensions of t and s to X′.

Then t′ and s′ have the same spectrum of eigenvalues.

Proof   Because X′ is an L-submodule of $\overline{X}$ and $u \in \overline{U(K)}$,

$$uX' \leq X' \, .$$

X is K-finite, so

$$X = \bigoplus_k X_k \, ,$$

where each X_k is an irreducible K-submodule of X of finite dimension. Since $u \in \overline{U(K)}$,

$$u = \bigoplus_k u_k \, ,$$

where u_k is an automorphism of X_k. Because the matrix of u is block diagonal, the extension u´ of u to X´ is non-singular, that is, u´ is an automorphism of X´. Thus, s´ and t´ have the same eigenvalues.　　　　　　///

Recurrence Relations.

The codiagonal matrices which occur most frequently in practical problems are tridiagonal. The recurrence relation associated with such a matrix has the form

$$\tau_{-1\,k-1}\xi_{k-1} + \tau_{0\,k}\xi_{k} + \tau_{1\,k}\xi_{k+1} = \lambda\xi_{k}.$$

and the characteristic equation is a quadratic,

$$\zeta_{1}\gamma + \zeta_{0} + \zeta_{-1}/\gamma = 0.$$

Perron and Poincaré assumed that $\zeta_{1} \neq 0$, so both roots of the quadratic were finite. Unfortunately, this condition is not always satisfied in practice, so in the lemmas below I have shown how this restriction may be removed.

Lemma 4.　　　　Suppose that (α_k) and (β_k) are two sequences, neither of which converges to zero, but for which

$$\lim_{k \to \infty} \alpha_k \beta_k = 0.$$

Then there exist two functions m and n on the set N of natural numbers,

m : N ⟶ N 　　　　　　　　　　　，　　　　　　n : N ⟶ N

　　k ⟼ m_k 　　　　　　　　，　　　　　　　k ⟼ n_k,

with the following properties:

(1) $m(N) \cap n(N) = \phi$, 　$m(N) \cup n(N) = N$;

(2) the subsequences (α_{m_k}), (β_{n_k}) both converge to zero.

<u>Proof</u>   (1) Suppose that (α_k) does not have a subsequence which converges to zero. Then

$$\liminf_k |\alpha_k| = \gamma > 0,$$

and so, for all but a finite number of values of k,

$$|\alpha_k \beta_k| \geqq \gamma |\beta_k| \;.$$

Thus, (β_k) must converge to zero, contrary to assumption.

(2) Extract from (α_k) the subsequence (α_{m_k}) which converges to zero in such a way that the remaining sequence does not have zero amongst its set of limit points. Let n be the complementary function of m, that is n_k covers only those integers missed by m_k. (α_{n_k}) does not have a subsequence which converges to zero. Thus,

$$\liminf_k |\alpha_{n_k}| = \delta > 0,$$

and so $$\lim_k |\beta_{n_k}| = 0 \quad .$$

It follows from the very definition of n that

$$m(N) \cap n(N) = \phi,$$
$$m(N) \cup n(N) = N. \qquad\qquad ///$$

<u>Lemma 5.</u>   Suppose that (γ_k) and (δ_k) are two sequences which satisfy

$$\lim_{k \to \infty} \gamma_{k+1}(\gamma_k - \delta_k) = 0,$$

and $$\lim_{k \to \infty} \delta_k = \delta \neq 0.$$

Then $\lim_{k \to \infty} \gamma_k$ exists and is equal to either δ or 0.

<u>Proof</u>   It is clear that, <u>if</u> $\lim_k \gamma_k = \gamma$ exists, then

$$\gamma(\gamma - \delta) = 0 \;,$$

so $\hspace{3cm} \gamma = \delta$ or 0.

All I must do is establish that the limit does exist. Suppose the contrary, that $\lim \gamma_k$ does not exist. Define

$$\alpha_k = \gamma_{k+1}, \quad \beta_k = \gamma_k - \delta_k.$$

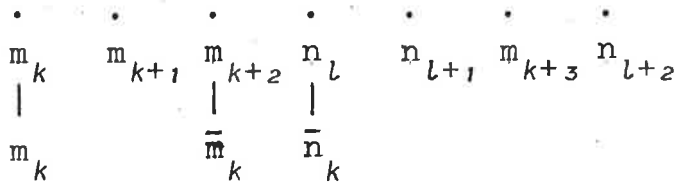Neither (α_k) nor (β_k) is a convergent sequence, but

$$\lim \alpha_k \beta_k = 0.$$

However, according to lemma 4, there are subsequences (α_{m_k}) and (β_{n_k}) which converge to zero. Also,

$$m(N) \cap n(N) = \phi \text{ and } m(N) \cup n(N) = N.$$

Define a subsequence of (m_k) as folows. For each $m_k \in (m_k)$, let $\bar{n}_k$ be the smallest element of (n_k) which exceeds m_k. Such an element must exist because both of the sequences (m_k) and (n_k) diverge to infinity. Define

$$\bar{m}_k = \bar{n}_k - 1 \in (m_k).$$

$$
\begin{array}{cccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
m_k & m_{k+1} & m_{k+2} & n_l & n_{l+1} & m_{k+3} & n_{l+2} & & & \\
| & & | & | & & & & & & \\
m_k & & \bar{m}_k & \bar{n}_k & & & & & &
\end{array}
$$

Note that $\hspace{3cm} \bar{m}_k \geq m_k.$

This implies that the sequence $(\bar{m}_k)$ contains an infinite number of distinct integers, for otherwise, (m_k) would have to be bounded. Now,

$$\alpha_{\bar{m}_k} \longrightarrow 0,$$

so $\hspace{3cm} \gamma_{\bar{m}_k+1} = \gamma_{\bar{n}_k} \longrightarrow 0.$

However, $\hspace{3cm} \beta_{\bar{n}_k} \longrightarrow 0,$

so $\hspace{3cm} \gamma_{\bar{n}_k} - \delta_{\bar{n}_k} \longrightarrow 0.$

The sequence (δ_k) is convergent, so

$$\delta_{\bar{n}_k} \longrightarrow \delta \neq 0.$$

Thus,
$$\gamma_{\bar{n}_k} \longrightarrow \delta \neq 0.$$

I have found a convergent sequence $(\gamma_{\bar{n}_k})$ with two limits, an

impossibility. The only alternative is that $\lim_{k} \gamma_k$ exists. ///

Now consider the difference equation of second order

$$\tau_{-1\,k-1}\xi_{k-1} + \tau_{ok}\xi_k + \tau_{1\,k+1}\xi_{k+1} = \lambda\xi_k \ .$$

Define
$$\zeta_{ik} = (-)^l (\tau_{ik} - \delta_{io}\lambda)/k^l$$

and
$$\zeta_i = \lim_{k \to \infty} \zeta_{ik}.$$

In these formulae, l is the least power of k for which one of

the limits ζ_i is non-zero. The characteristic equation is

$$\zeta_1\gamma + \zeta_o + \zeta_{-1}/\gamma = 0.$$

When $\zeta_1 = 0$, there is only one (finite) solution,

$$\gamma = -\zeta_1/\zeta_o,$$

which I will assume to be non-zero, for otherwise the problem

is trivial. I want to show that even in this degenerate case,

Poincaré's theorem is still applicable.

Theorem 6. Suppose that (ξ_k) is any solution of the

recurrence relation above. Then

$$\lim_{k \to \infty} \xi_{k+1}/\xi_k$$

exists and is equal to either γ or $+\infty$, the two roots of the

characteristic equation.

Proof
$$\zeta_{-1\,k-1}\xi_{k-1} + \zeta_{ok}\xi_k + \zeta_{1\,k+1}\xi_{k+1} = 0.$$

Then
$$\frac{\xi_{k-1}}{\xi_k}\frac{\xi_k}{\xi_{k+1}} + \frac{\zeta_{ok}}{\zeta_{-1\,k-1}}\frac{\xi_k}{\xi_{k+1}} + \frac{\zeta_{1\,k+1}}{\zeta_{-1\,k-1}} = 0.$$

Set $\qquad \gamma_k = \xi_{k-1}/\xi_k \; , \quad \delta_k = -\zeta_{ok}/\zeta_{-1\,k-1} \; .$

Now $\qquad \zeta_{1\,k+1}/\zeta_{-1\,k-1} \longrightarrow \zeta_1/\zeta_{-1} = 0$

and $\qquad -\zeta_{ok}/\zeta_{-1\,k-1} \longrightarrow -\zeta_o/\zeta_{-1} = 1/\gamma \neq 0.$

Thus, $\qquad\qquad\qquad\qquad \gamma_{k+1}(\gamma_k - \delta_k) \longrightarrow 0$

and $\qquad\qquad\qquad\qquad\qquad\qquad \delta_k \longrightarrow 1/\gamma.$

The last lemma shows that $\lim\limits_k \gamma_k$ exists and is equal to either $1/\gamma$ or 0. Hence,

$$\lim_{k \to \infty} \frac{\xi_{k+1}}{\xi_k} = \gamma \text{ or } \infty .$$

///

CHAPTER 5. CODIAGONAL PERTURBATIONS.

In the preceding chapters I assumed that the algebras
L and K were shown from the outset. Eigenvalue problems
which arise in practice are rarely so simple, and often
considerable ingenuity is required to uncover the algebras
L and K. Green and Triffet (1969) devised a technique for
this purpose which works for a number of operators closely
related to the special function operators. Their work has
one disadvantage; they assumed that the operator s under
investigation was a self-adjoint operator on a Hilbert space,
and at several points in their paper the fact seemed crucial.
My aim in this chapter is to show that this is not so, and
that the ideas of Green and Triffet can be developed
rigorously within an algebraic framework without any refer-
ence to Hilbert spaces.

The boson calculus was developed because the formalism
of annihilation and creation operators provided such an
elegant solution of the harmonic oscillator problem. It is
based upon a collection of operators $p_1, p_2, \ldots, p_m$, $q_1, q_2,$
$\ldots, q_m$ which satisfy

$$[p_i, p_j] = 0, \quad [p_i, q_j] = \delta_{ij}, \quad [q_i, q_j] = 0,$$

and so provides a general framework for the study of any
system with just m degrees of freedom. Most linear operators
which arise in non-relativistic quantum mechanics can be
expressed in terms of bosons. For example, any linear
partial differential equation in m variables is a function of
the m conjugate pairs $z_i, \frac{\partial}{\partial z_i}$. Lohe and Hurst (1971) have
shown that presentations of all of the classical Lie algebras

can be found amongst the class of operators constructed from bosons and modified bosons. I want to consider the opposite problem. Given an operator s, constructed from bosons, is there a Lie algebra L which contains s in its enveloping algebra? Of course, in a theory which stressed algebraic considerations, as perhaps quantum mechanics should, this problem of discovering L and K would never arise, for those algebras would be part of the data in the theory.

Let B denote the Lie algebra over C with basis

$$\{1, p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_m\},$$

in which the only non-zero Lie products are

$$[p_i, q_j] = \delta_{ij}.$$

The universal enveloping algebra of B, denoted U(B), is the <u>boson algebra</u>. Unfortunately, very few elements of U(B) hold much interest for physics; I can only think of the oscillator Hamiltonian

$$\sum_i p_i^2 + \omega_i^2 q_i^2, \quad \omega_i \in C.$$

There is clearly a need for a larger algebra based on B.

In the appendix I have shown how to construct a suitable algebra. It is denoted D and consists of all polynomials in p of the form

$$\sum_k a_k p^k, \quad a_k \in C(q),$$

where I have used the common notation

$$p = (p_1, p_2, \ldots, p_m),$$

$$q = (q_1, q_2, \ldots, q_m),$$

$$k = (k_1, k_2, \ldots, k_m),$$

and $\qquad p^k = p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}.$

The coefficients a_k are chosen from the field of rational functions of q. This field is denoted C(q). D looks and behaves like the algebra of all (formal) partial differential operators with coefficients which are rational functions of q. However, I want to stress that D is constructed algebraically from U(B) and that D is independent of any particular representation of p and q.

Suppose that s is a fixed element of D with degree d in p:

$$s = \sum_{|k| \leqslant d} a_k p^k, \qquad a_k \in C(q),$$

where $\qquad |k| = k_1 + k_2 + \ldots + k_m.$

Green and Triffet observed that, under favourable circumstances, it was possible to find elements $r_1, r_2, \ldots, r_n$ in D such that

$$[s, r_j] = \sum_{i=1}^{j} r_i m_{ij} + r_{j+1}, \qquad 1 \leqslant j < n,$$

$$[s, r_n] = \sum_{i=1}^{n} r_i m_{in},$$

where the coefficients m_{ij} were polynomials in s over C. In practice, s was of the second degree in p and $r_1, r_2, \ldots, r_n$ were of the first, so the coefficients m_{ij} were at most linear in s. The first steps of the plan laid

by Green and Triffet to determine the eigenvalues of s (in some representation) were these.

(1) Find the eigenvalues $l_1, l_2, \ldots, l_n$, assumed distinct, and the corresponding eigenvectors $v_1, \ldots, v_n$ of the matrix $m = (m_{ij})$;

$$mv_i = l_i v_i, \quad 1 \leqslant i \leqslant n.$$

Both the eigenvalues and eigenvectors might depend upon s.

(2) Construct

$$w_i = \sum_j r_j v_{ji}, \quad 1 \leqslant i \leqslant n,$$

where v_{ji} is the jth component of v_i. It is then easy to verify that

$$sw_i = w_i(s + l_i), \quad 1 \leqslant i \leqslant n,$$

so that w_i is a shift operator for s.

Green and Triffet stated that s was a self-adjoint operator on a Hilbert space. However, they manipulated s as freely as a finite matrix, and in extracting the roots of the characteristic equation of m formed various algebraic functions of s. I attempted to give a precise account of their techniques, but in so doing encountered several problems. I took p,q and s to be differential operators on the Hilbert space $L^2(I, \mu)$, where I was an open region of R^m and μ a measure on I. The problems were these.

(1) Green and Triffet regarded the self-adjoint s as a function of the skew-adjoint p and self-adjoint q. However, this is not always possible. For example, suppose that the Hilbert space is the space of Lebesgue square-integrable functions on $(0, \infty)$ and that s is a self-adjoint extension of

the symmetric operator s' defined by

$$s' : C_0^\infty(0,\infty) \longrightarrow C_0^\infty(0,\infty)$$

$$f(x) \longmapsto \left[-\frac{d^2}{dx^2} + x^2 + \frac{j(j+1)}{x^2}\right] f(x),$$

where $C_0^\infty(0,\infty)$ is the space of infinitely differentiable functions which vanish outside a compact subset of $(0,\infty)$. The operator

$$p' : C_0^\infty(0,\infty) \longrightarrow C_0^\infty(0,\infty)$$

$$f(x) \longmapsto \frac{d}{dx}f(x)$$

in skew-symmetric, but it is well known that it does not have any skew-adjoint extensions (Dunford and Schwartz (1963)). Thus, s is self-adjoint, but p can never be skew-adjoint.

(2) The only escape from this dilemma is to require merely that p should be skew-symmetric and s symmetric. However, this also results in an impasse. Consider the example above once again. The differential equation

$$\left[-\frac{d^2}{dx^2} + x^2 + \frac{j(j+1)}{x^2}\right] f(x) = \pm i f(x)$$

has two solutions which near the origin behave like x^{j+1} and x^{-j}. If

$$-3/2 < j < 1/2,$$

both of these solutions are square-integrable, the deficiency spaces of s' are non-trivial, and s' has an uncountable number of self-adjoint extensions. Thus, the differential equation has an eigenvector in the Hilbert space with any prescribed complex eigenvalue. This is not the type of operator Green and Triffet wanted to consider; they required

that the eigenvalues of s should form an increasing sequence.
(Note that the case j=0 (S-waves) of importance to physics
is contained within the above range of j.)

(3) It appears that s cannot be considered to be a
function of p and q; the expression for s in terms of p and
q merely defines the action of s on elements of its domain,
but the domain must be specified independently. A symmetric
operator may have an uncountable number of self-adjoint
extensions, and the selection of just one of these is an
unwanted freedom in the theory.

(4) Construct a spectral representation of s. This was
only a problem because Green and Triffet occasionally applied
their technique to operators which were not self-adjoint.

Despite these difficulties, I did manage to present a
rigorous account in Hilbert space of the main points of the
work of Green and Triffet, but the complexity of my results
seemed contrary to the refreshing simplicity of their approach.
I felt that somewhere I had missed the whole point. Indeed I
had. My 'error' was to regard p,q and s as operators on a
Hilbert space; once I discarded the topological structure of
the space, most of the irrelevant complexities vanished with
it. I later realised that I could discard the space as well,
and that most of the results of Green and Triffet were not
dependent upon the representation of p and q as operators.
It is this truly algebraic version that I will present here.

The ring of polynomials in s with complex coefficients,
denoted C[s], is an integral domain. It can be embedded in
its field of quotients C(s). The matrix m is a matrix over
this field and its characteristic polynomial can be calculated
in the usual way:

$$\det(m-1) = (-)^n [1^n + c_1 1^{n-1} + \ldots + c_{n-1} 1 + c_n], \quad c_i \in C[s].$$

In general, the polynomial will not split into linear factors over C(s), but there is an <u>algebraic</u> extension field E of C(s) in which it does split:

$$\det(m-1) = (-)^n \prod_{i=1}^{n} (1-1_i), \quad 1_i \in E.$$

It is not difficult to construct E; for the details I refer to Jacobson (1951a). In the examples treated by Green and Triffet, one of the coefficients c_i involves s linearly; hence, s can be expressed as a symmetric polynomial in the roots $1_1, 1_2, \ldots, 1_n$ and so

$$E = C(1_1, 1_2, \ldots, 1_n).$$

Thus, E is a <u>transcendental</u> extension field of C.

A   C(s) $E = C(1_1, 1_2, \cdots, 1_n)$
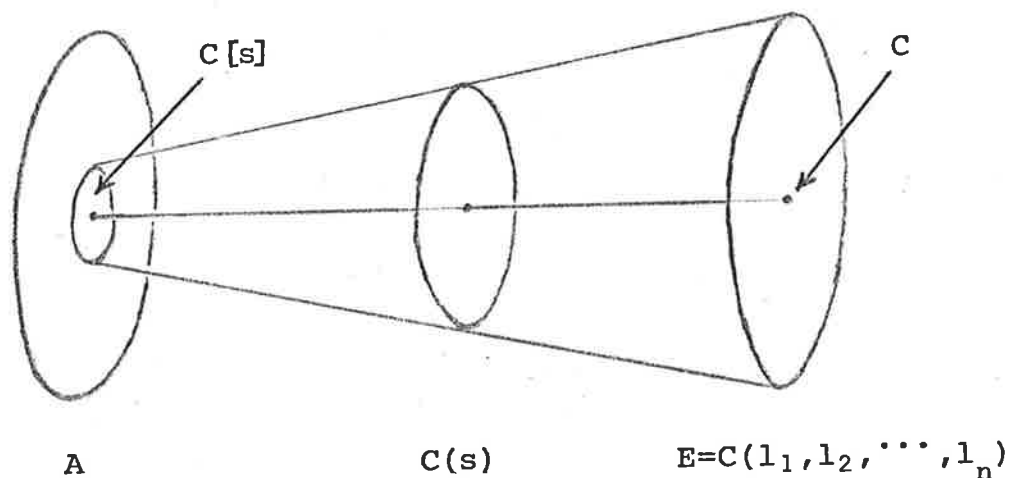
Let A denote the subalgebra of D generated by the set

$$\{1, r_1, r_2, \ldots, r_n, s\}.$$

Since s maps A into itself by commutation, I will focus

attention on A and neglect the rest of D. If a is any element of A and b any element of C[s], then the products ab and ba are well defined; they are simply the products in A. I want to extend this product so that ab and ba are well defined even when b is chosen from E. To do this I must construct an algebra $\overline{A}$ which contains isomorphic copies of A and E which intersect in the copy of C[s]. This construction is almost identical with the construction of $\overline{U(K)}$ in chapter 3, so I will not repeat the details. Although the structures required seem complicated, it is comforting to know that formal calculations according to the elementary rules of algebra always give the correct answers. Many familiar structures are actually no less complicated. For example, who bothers to think that ½ represents an equivalence class in the set of ordered pairs of integers? It is also worth stressing that I have not yet committed myself to a representation of A or E by operators, but, once that step is taken, every product will become an operator product.

I will assume that the roots $l_1,\ldots,l_n$ are distinct elements of E. Define the projection matrices

$$k_i = \prod_{j \neq i} (m-l_j)/(l_i-l_j), \qquad 1 \leq i \leq n.$$

It is easy to verify that

$$k_i k_j = \delta_{ij} k_i,$$

$$\sum_{i=1}^{n} k_i = 1,$$

and

$$\sum_{i=1}^{n} k_i l_i = m.$$

Let $r = (r_1, r_2, \ldots, r_n)$ and construct the row vectors

$$u_i = rk_i$$

Then
$$[s, u_i] = srk_i - rk_i s$$

$$= (sr - rs)k_i$$

$$= rmk_i$$

$$= u_i l_i.$$

Thus,
$$su_i = u_i(s + l_i), \quad 1 \leq i \leq n.$$

These are the commutation relations sought by Green and Triffet. They are identities in the algebra $\overline{A}$ and hold in every representation of $\overline{A}$. In particular, their validity is not restricted to representations of $\overline{A}$ on Hilbert spaces.

In only very few cases is it possible to find $r_1, \ldots r_n$. Green and Triffet found that, when there is only one coordinate q and one momentum p, s must correspond to one of the special function operators, n must equal three, and r_3 reduces to a multiple of the identity in D. I want to consider this case in some detail.

Suppose that

$$s = a_2 p^2 + a_1 p + a_0, \quad a_i \in C(q).$$

If r_1 is a rational function of q, then $[s, r_1]$ is a linear function of p with coefficients which are rational functions of q. $[s, r_1]$ can be written

$$[s, r_1] = \alpha r_1 + r_2$$

where α is a complex number and r_2 is the 'remainder'. Similarly, $[s, r_2]$ is a quadratic polynomial in p, so

$$[s,r_2] = r_1(\beta s+\gamma) + \delta r_2 + r_3,$$

where β,γ,δ are complex numbers and r_3 is the remainder. I will assume that

$$[s,r_3] = 0.$$

Thus,
$$[s,r] = rm,$$

where
$$r = (r_1,r_2,r_3)$$

and
$$m = (m_{ij}) = \begin{pmatrix} \alpha & \beta s+\gamma & 0 \\ 1 & \delta & 0 \\ 0 & 1 & 0 \end{pmatrix} .$$

It is interesting tó note that s can only be factorised exactly in the sense of Infeld and Hull (1951) when r_1, r_2 and r_3 can be found with these properties. The only proof of this fact consists of a case by case verification. A general proof would be of great value, for it would lead to a more precise definition of the class of operators for which the factorisation method works. I failed to find such a proof.

The characteristic equation of the matrix m is

$$1[1^2-(\alpha+\delta)1+\alpha\delta-\beta s-\gamma] = 0.$$

When $\beta=0$, m is simply a matrix over C and the splitting field of its characteristic polynomial is C itself. This case is trivial, so I will exclude it. Let E denote the algebraic extension field of $C(s)$ in which the characteristic polynomial splits into linear factors. Thus,

$$1(1-1_1)(1-1_2) = 0, \quad \text{where} \quad 1_1,1_2,1_3 = 0\in E.$$

Set
$$\mu = (\alpha+\delta)/2, \quad \nu = (\alpha-\delta)/2,$$

and define

$$h = \tfrac{1}{4}(l_1-l_2)/\mu.$$

Since
$$l_1+l_2 = 2\mu,$$

$$l_1 = \mu(1+2h) \quad \text{and} \quad l_2 = \mu(1-2h).$$

Furthermore,

$$s = (4\mu^2h^2-\nu^2-\gamma)/\beta \;,$$

so

$$m = \begin{pmatrix} \mu+\nu & (2\mu h+\nu)(2\mu h-\nu) & 0 \\ 1 & \mu-\nu & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The projection matrices are defined by

$$k_i = \prod_{j \neq i} (m-l_j)/(l_i-l_j);$$

their calculation in this case is trivial.

$$k_1 = \frac{\begin{pmatrix} (\mu+2\mu h)(\nu+2\mu h) & -(\mu+2\mu h)(\nu+2\mu h)(\nu-2\mu h) & 0 \\ \mu+2\mu h & -(\mu+2\mu h)(\nu-2\mu h) & 0 \\ 1 & -(\nu-2\mu h) & 0 \end{pmatrix}}{4\mu^2h(1+2h)} \;,$$

$$k_2(h) = k_1(-h),$$

$$k_3(h) = \frac{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & \mu+\nu & \mu^2(4h^2-1) \end{pmatrix}}{\mu^2(4h^2-1)}.$$

Define

$$r_\pm = r_1(\mu\pm2\mu h)(\nu\pm2\mu h) + r_2(\mu\pm2\mu h) + r_3.$$

It is easy to verify that

$$sr_{\pm} = r_{\pm}(s+\mu\pm2\mu h).$$

Thus,

$$h^2r_{\pm} = r_{\pm}[(h \pm \tfrac{\beta}{4\mu})^2 + \tfrac{\beta}{4\mu}(1 - \tfrac{\beta}{4\mu})].$$

I will assume that $\beta = 4\mu$; this condition is satisfied in all the examples discussed by Green and Triffet. Consequently,

$$h^2r_{\pm} = r_{\pm}(h\pm1)^2.$$

Finally, set

$$b_{\pm} = \tfrac{1}{2}(r_{\pm} + h^{-1}r_{\pm}(h\pm1));$$

then

$$hb_{\pm} = b_{\pm}(h\pm1).$$

By a direct but tedious calculation, it is possible to show that b_-b_+ is a polynomial in h with complex coefficients:

$$b_-b_+ = f(h).$$

The product b_+b_- in the reverse order can then be calculated quite easily as follows.

$$b_-b_+b_- = f(h)b_-$$
$$= b_-f(k-1).$$

Thus, $$b_-(b_+b_- - f(h-1)) = 0.$$

Since $\bar{A}$ is free of divisors of zero,

$$b_+b_- = f(h-1).$$

I have constructed an algebra $\overline{A}$ with the following properties.

(1) $\overline{A}$ contains the subalgebra of D generated by the set $\{1, r_1, r_2, r_3, s\}$.

(2) $\overline{A}$ also contains the field $C(h)$ and

$$s = \mu h^2 - \tfrac{1}{4}(\nu^2 + \gamma)/\mu.$$

(3) There are elements $b_{\pm}$ of $\overline{A}$ which satisfy

$$hb_{\pm} = b_{\pm}(h \pm 1)$$

and hence

$$sb_{\pm} = b_{\pm}(s + \mu \pm 2\mu h).$$

Also $b_- b_+ = f(h)$ and $b_+ b_- = f(h-1)$.

I want to find representations of $\overline{A}$. This is a surprisingly easy task, for I will show that every representation of $sl(2,C)$ can be extended to a representation of $\overline{A}$.

Define

$$e_+ = c_+(h)b_+ \quad \text{and} \quad e_- = b_- c_-(h),$$

where $c_{\pm}(h)$ are rational functions of h which are to be chosen so that

$$[e_+, e_-] = 2h.$$

Now, $$e_+ e_- = g(h)$$

and $$e_- e_+ = g(h+1),$$

where $$g(h) = f(h-1)c_+(h)c_-(h).$$

The commutation relation will be satisfied provided $g(h)$ is

a solution of the following functional equation:

$$g(h) - g(h+1) = 2h.$$

One solution is

$$g(h) = (\rho+h)(\rho+1-h),$$

where ρ is an arbitrarily chosen complex number. In the
appendix I have shown that this is the only solution that is
a rational function of h. I will now allow $c_\pm(h)$ to be any
rational functions of h which fulfil the condition

$$c_+(h)c_-(h) = (\rho+h)(\rho+1-h)/f(h-1).$$

The elements $h, e_\pm$ satisfy

$$[h, e_\pm] = \pm e_\pm, \quad [e_+, e_-] = 2h,$$

and so form a basis for the Lie algebra $sl(2,C)$. The
elements $1, r_1, r_2, r_3, h$, and hence all of $\bar{A}$, can be expressed
in terms of $h, e_\pm$. Consequently, to find the representations
of $\bar{A}$, I need only find representations of $sl(2,C)$ in which
the following identity holds:

$$e_+ e_- = (\rho+h)(\rho+1-h).$$

Before discussing the representations of $\bar{A}$, I want to
remark on the representations of $C(h)$. The field $C(h)$
consists of rational functions of h with complex coefficients.
Every function $a(h)$ in $C(h)$ is invertible in the sense
that there is another function $a^{-1}(h)$ which satisfies

$$a(h)a^{-1}(h) = 1 = a^{-1}(h)a(h).$$

If X is an sl(2,C)-module with a vector of highest weight ρ, then the matrix of h on X is

$$\begin{bmatrix} \rho & & & \\ & \rho-1 & & \\ & & \rho-2 & \\ & & & \ddots \end{bmatrix}$$

It is quite possible that the matrix of a function a(h) will have a <u>finite</u> number of zero eigenvalues. Thus, although a(h) is invertible in C(h), its matrix representation would not be an automorphism of X. I will admit this possibility. Note that a similar situation occurs in the field of rational functions of a complex variable; every such function is invertible within the field, but may have a finite number of poles in the complex plane.

Suppose that X is an irreducible sl(2,C)-module of infinite dimension.

(1) If X has neither a vector of highest nor lowest weight, then the eigenvalues of h form a doubly infinite sequence. The eigenvalues of s can be deduced immediately from those of h. This case is not very interesting.

(2) Suppose instead that X has a vector x_1 of highest weight ρ;

$$hx_1 = \rho x_1,$$
$$e_+x_1 = 0.$$

The matrices of $h, e_\pm$ are

$$h = \begin{bmatrix} \rho & & & \\ & \rho-1 & & \\ & & \rho-2 & \\ & & & \ddots \end{bmatrix}, \quad e_+ = \begin{bmatrix} 0 & \alpha_1 & & \\ & 0 & \alpha_2 & \\ & & 0 & \alpha_3 \\ & & & \ddots \end{bmatrix}, \quad e_- = \begin{bmatrix} 0 & & & \\ \beta_1 & 0 & & \\ & \beta_2 & 0 & \\ & & \beta_3 & \ddots \end{bmatrix},$$

where $\alpha_k \beta_k = k(2\rho+1-k)$, $k = 1,2,\dots$.

Thus, $$e_+ e_- = (\rho+h)(\rho+1-h),$$

as required. The eigenvalues of s on X are

$$\mu(\rho-k+1)^2 - \tfrac{1}{4}(\nu^2+\gamma)/\mu, \quad k = 1,2,\dots .$$

The spectrum of h , and hence of s, depends upon the label ρ of the representation. When s represents a differential operator, it is easy to see that changing the label of the irreducible representation amounts to changing the boundary conditions applied to s. This is a general feature of the algebraic theory of eigenvalue problems: representation labels play the same role for diagonal operators as boundary conditions do for differential operators. Despite these general remarks, certain values of ρ seem to be distinguished, not algebraically but analytically. They are the roots of the equation

$$f(\rho) = 0.$$

When ρ assumes one of these exceptional values and s is represented by a differential operator connected with one of the special functions, the eigenfunctions of s are precisely those branches of the special function which arise most naturally in the classical theory and do not involve linear combinations of those branches, as might be expected.

(3) The case in which X has a vector of lowest weight is very similar to the case above.

(4) X cannot have a vector of highest weight and a vector of lowest weight because X is irreducible and of infinite dimension.

I have discussed only the first two stages in the plan laid by Green and Triffet. Fortunately, the discussion of the remaining two will not be so long. Their aims were these.

(3) Find the matrices of the shift operators $b_\pm$, relative to a basis of eigenvectors of h.

(4) Construct the matrix of any 'codiagonal perturbation' of s:

$$t = \sum_{i=1}^{n} b_+^i t_i(h) + t_o(h) + \sum_{i=1}^{m} t_{-i}(h) b_-^i$$

Truncate the matrix of t after j rows and columns. The eigenvalues of the jxj matrix will approximate, in some sense, the first j eigenvalues of t.

Green and Triffet assumed that s, r_1, r_2, r_3 were self-adjoint operators on a Hilbert space, but in stages (1) and (2) made little use of this fact. However, in the final stages they appeared to use the self-adjointness in a fundamental way. Suppose that X is an irreducible sl(2,C)-module with a vector of highest weight ρ. The matrices of $h, b_\pm$ on X are

$$h = \begin{bmatrix} \rho & & & \\ & \rho-1 & & \\ & & \rho-2 & \\ & & & \cdot \\ & & & & \cdot \end{bmatrix}, \quad b_+ = \begin{bmatrix} 0 & \alpha_1 & & \\ & 0 & \alpha_2 & \\ & & 0 & \alpha_3 \\ & & & \cdot & \cdot \\ & & & & \cdot \end{bmatrix}, \quad b_- = \begin{bmatrix} 0 & & & \\ \beta_1 & 0 & & \\ & \beta_2 & 0 & \\ & & \beta_3 & \cdot \\ & & & \cdot & \cdot \end{bmatrix},$$

where $\alpha_k \beta_k = f(\rho-k)$, $k = 1,2,3,\ldots$,

and f is the polynomial computed earlier. To fix α_k and β_k separately, and not just fix the product, Green and Triffet found a relation between b_- and the adjoint of b_+ which provided a second equation involving α_k and β_k. They obtained

this relation by using the self-adjointness of s. With the matrices of h and $b_{\pm}$, they were able to find the matrix of any codiagonal perturbation t, which they truncated and diagonalised to find approximations to the first few eigenvalues of t.

The 'codiagonal perturbations' are codiagonal operators on X. In the last chapter I showed that the spectrum of t on $\hat{X}$, the completion of X with respect to a norm, is the same as the spectrum of one of the extensions t_a on X_a. Furthermore, the spectra of t_a and $ut_a u^{-1}$ are identical of u is a gl(1,C)-automorphism of X. Every such automorphism is a rational function of h, and so has a diagonal matrix. However, a diagonal matrix is all that is needed to change, by similarity transformation, any given matrix representation of $b_{\pm}$ into any other. Thus, Green and Triffet had no need to single out a particular representation of $b_{\pm}$; all equivalent representations lead to the same spectrum for t. Consequently, the Hilbert space and the self-adjointness properties were irrelevant.

APPENDIX 5.

Construction of the Algebra D.

The ring of polynomials in q with complex coefficients, denoted $C[q]$, is an integral domain. Every element of $U(B)$ can be written in the form

$$\sum_k a_k p^k, \qquad a_k \in C[q].$$

Construct the set of all ordered pairs

$$\left(\sum_k a_k p^k, a\right), \qquad a \neq 0$$

in which the first element lies in $U(B)$ and the second in $C[q]$, and define two ordered pairs to be in the relation $\sim$, written

$$\left(\sum_k a_k p^k, a\right) \sim \left(\sum_k b_k p^k, b\right),$$

if $\qquad \sum_k b a_k p^k = \sum_k a b_k p^k.$

$\sim$ is an equivalence relation. The only facts needed to prove this are that $C[q]$ is commutative and that $U(B)$ is free of zero divisors. Partition the set of all ordered pairs into disjoint equivalence classes with respect to $\sim$. Let D denote the set of equivalence classes and $\sum_k (a^{-1}a_k)p^k$ denote the class which contains $\left(\sum_k a_k p^k, a\right)$, $a \neq 0$. The sum and product on D can now be defined in the obvious way and it is a trivial matter to verify that they are single valued compositions in D.

The map

$$\text{ad } p_i : C[q] \longrightarrow C[q]$$

$$a \longmapsto [p_i, a]$$

is a derivation of $C[q]$ and can be extended uniquely to a derivation of $C(q)$. To see this is quite easy. Because $1 \in C[q]$,

$$[p_i, 1] = 0.$$

If ad p_i is to be a derivation of $C(q)$, then

$$[p_i, aa^{-1}] = a[p_i, a^{-1}] + [p_i, a] a^{-1}$$

$$= [p_i, 1]$$

$$= 0$$

Thus,
$$[p_i, a^{-1}] = -a^{-1} [p_i, a] a^{-1}$$

and the extension of ad p_i to $C(q)$, and hence to all of D, is uniquely determined.

Solution of the Functional Equation.

Suppose that $g_1(h)$ and $g_2(h)$ both satisfy the same functional equation;

$$g_1(h) - g_1(h+1) = 2h,$$

$$g_2(h) - g_2(h+1) = 2h.$$

Set
$$d(h) = g_1(h) - g_2(h).$$

Then
$$d(h) - d(h+1) = 0.$$

If I require that g_1 and g_2 should be rational functions of h, then the only solution for d is

$$d = \rho(\rho+1), \quad \rho \in C,$$

and so any two rational solutions of the functional equation can only differ by a constant. One solution is

$$g_1(h) = h(1-h).$$

Hence, the most general rational solution is

$$g(h) = (\rho+h)(\rho+1-h), \quad \rho \in C.$$

The condition that g should be rational can be weakened. For example, it is sufficient to require that g should be entire. However, for my purposes rational functions suffice.

CHAPTER 6.   EXAMPLES

This chapter contains two examples. The first is
concerned with sl(2,C)-modules and exhibits the hyper-
geometric and Fuchsian operators as diagonal and codiagonal
operators, respectively. The second example shows how
irreducible representations of the Poincaré Lie algebra can
be constructed from irreducible representations of the Lorentz
Lie algebra.

In addition to these examples, I have included a short
list of common presentations of a pair of conjugate operators
p and q on a vector space X. No doubt this material is
familiar to all, but for me it was an important step to real-
ise that p and q could be defined <u>everywhere</u> on X provided X
was not a Banach space.

Presentations of p and q

Suppose that p and q are endomorphisms of a vector
space X over the complex field C, and that

$$[p,q] = 1.$$

X cannot be a Banach space, because the only endomorphisms of
a Banach space are necessarily bounded and it is well known
that the commutation relation cannot be satisfied by bounded
operators. (Putnam (1967)). In particular, X cannot be a
Hilbert space. There are several easily constructed presenta-
tions of the operators p and q listed below, and throughout
the thesis the symbols p and q may be taken to stand for
any of these. Note that the space X must either be infinite
dimensional or trivial, for, if it were of finite dimension,

$$0 = \text{trace } [p,q] = \dim X.$$

(1) Suppose X is the vector space of functions holomorphic in some open region D of the complex plane. Define p and q as follows:

$$p : X \longrightarrow X$$
$$f(z) \longmapsto \frac{df}{dz}(z),$$

$$q : X \longrightarrow X$$
$$f(z) \longmapsto zf(z).$$

Note that $(\lambda - q)$ is invertible whenever λ is a point in the complement of D.

(2) This second presentation of the conjugate operators p and q is very useful for differential operators, such as the hypergeometric operator, whose eigenfunctions have regular singularities. p and q are once again the operations of differentiation with respect to z and multiplication by z, but the functions on which they act are defined over 'double loops'. (Double loops date to Pochhammer (1889) who used them in the construction of integral representations of hypergeometric functions.)
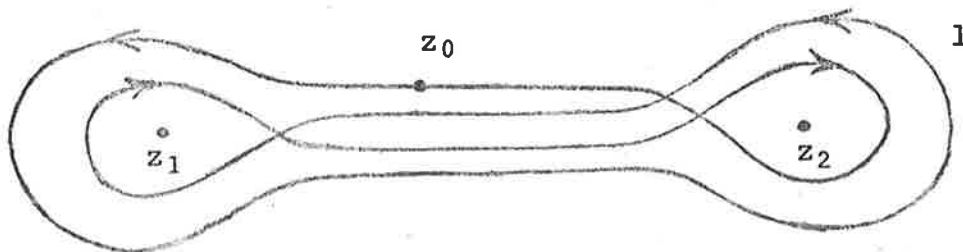
The double loop contour l around points z_1 and z_2 is a smooth contour which commences from a point z_0, distinct from z_1 and z_2, loops z_1 and z_2, loops z_1 and z_2 again, but in the opposite sense, and finally returns to z_0. Let X denote the vector space of functions holomorphic on the contour l and unaffected by analytic continuation around l. X naturally includes all functions holomorphic in a convex region containing the double loop, but X also includes certain functions with branch points at z_1 and z_2. For want of a better name, I have termed the space X the space of <u>periodic functions on the double loop</u>. The operator $(\lambda-q)$ is invertible so long as λ does not lie on l.

(3) Suppose X to be the vector space of all sequences which terminate after a finite number of terms. Define p and q to be the infinite matrices

$$
p = \begin{bmatrix} 0 & \alpha_1 & & & \\ & 0 & \alpha_2 & & \\ & & 0 & \alpha_3 & \\ & & & \cdot & \cdot \\ & & & & \cdot \end{bmatrix}, \quad
q = \begin{bmatrix} 0 & & & & \\ \beta_1 & 0 & & & \\ & \beta_2 & 0 & & \\ & & \beta_3 & \cdot & \\ & & & \cdot & \cdot \end{bmatrix},
$$

where the numbers α_k and β_k are constrained by

$$\alpha_k \beta_k = k, \quad k = 1,2,3,\ldots .$$

(4) Yet another example exhibits p and q as difference operators. Let X be the vector space of entire functions of the complex variable z. Define

$$p : X \longrightarrow X$$
$$f(z) \longmapsto f(z+1),$$

$$q : \quad X \longrightarrow X$$

$$f(z) \longmapsto zf(z-1).$$

Example 1.

The Hypergeometric Operator

I will use the general form of the hypergeometric equation, derived initially by Papperitz (1885):

$$\left\{ (z-z_1)(z-z_2)(z-z_3)\left[\frac{d^2}{dz^2} + \sum_{i=1}^{3} \frac{1-\alpha_i-\beta_i}{z-z_i} \frac{d}{dz} \right] \right.$$

$$+ \frac{\alpha_1\beta_1(z_1-z_2)(z_1-z_3)}{z-z_1} + \frac{\alpha_2\beta_2(z_2-z_3)(z_2-z_1)}{z-z_2} + \left. \frac{\alpha_3\beta_3(z_3-z_1)(z_3-z_2)}{z-z_3} \right\} .$$

$$\cdot g(z) = 0$$

with

$$\sum_{i=1}^{3} (\alpha_i + \beta_i) = 1.$$

The singular points, which are assumed to be distinct, lie at z_1, z_2, z_3. The exponents adopted by the two linearly independent solutions near any one of the singular points, say z_i, are α_i and β_i. The variables η_i and θ_i will denote the sum and difference of the exponents,

$$\eta_i = \alpha_i + \beta_i, \quad \theta_i = \alpha_i - \beta_i$$

Finally, $$\eta = (\eta_1, \eta_2, \eta_3),$$

$$\theta = (\theta_1, \theta_2, \theta_3).$$

Let P denote the space of periodic functions on the double loop around z_1 and z_2, defined in the previous

section. The hypergeometric operator $s(\theta)$, a linear map from P to P which depends upon θ, is defined as follows:

$$s(\theta) \; : \; P \longrightarrow P$$

$$g(z) \longmapsto \sigma g(z),$$

where

$$\sigma = \frac{(z-z_3)^2(z-z_1)(z-z_2)}{(z_3-z_1)(z_3-z_2)} \left[\frac{d^2}{dz^2} + \sum_{i=1}^{3} \frac{1-\eta_i}{z-z_i} \frac{d}{dz} \right]$$

$$- \tfrac{1}{4}(\eta_1^2-\theta_1^2)\frac{(z-z_3)(z_1-z_2)}{(z-z_1)(z_3-z_2)} - \tfrac{1}{4}(\eta_2^2-\theta_2^2)\frac{(z-z_3)(z_2-z_1)}{(z-z_2)(z_3-z_1)} + \tfrac{1}{4}(\eta_3^2-\theta_3^2).$$

The parameter η will be supposed given and fixed, but θ will be allowed to vary freely over C^3. When θ_1 and θ_2 are fixed, the eigenvalues of $s(\theta)$ are the values of $\theta_3^2/4$ for which the kernel of $s(\theta)$, denoted ker $s(\theta)$, is non-trivial. Occasionally I will use a simple extension of this idea and consider a (generalised) eigenvalue of $s(\theta)$ to be any value of θ in C^3 for which ker $s(\theta)$ is non-trivial. The generalised eigenvalues will be seen to lie on surfaces in C^3, eigensurfaces. The conventional eigenvalues, corresponding to fixed θ_1 and θ_2, lie at the points of intersection of the eigensurfaces with a line through $(\theta_1,\theta_2,0)$ parallel to the θ_3 axis.

Although I have defined $s(\theta)$ as a differential operator, $s(\theta)$ yields to the algebraic methods developed in the last chapter. Consequently, for the moment I will ignore the space P and deduce the properties of $s(\theta)$ which depend only on the commutation relation

$$[d/dz,z] = 1.$$

For brevity I will write s for $s(\theta)$ unless the dependence of s upon θ is particularly important.

Define

$$r_1 = 2\frac{(z-z_2)(z_3-z_1)}{(z-z_3)(z_2-z_1)} - 1,$$

$$r_2 = 4\frac{(z-z_1)(z-z_2)}{(z_1-z_2)}\frac{d}{dz} + (\eta_2-\eta_1),$$

$$r_3 = \theta_1{}^2-\theta_2{}^2.$$

By straight-forward calculation it may be verified that

$$[s,r_1] = (1+\eta_3)r_1 + r_2\,,$$

$$[s,r_2] = r_1(4s-\eta_3{}^2+\theta_3{}^2) + (1-\eta_3)r_2 + r_3,$$

$$[s,r_3] = 0.$$

Thus, $$[s,r] = rm,$$

where $$r = (r_1,r_2,r_3)$$

and $$m = \begin{pmatrix} 1+\eta_3 & 4s+\theta_3{}^2-\eta_3{}^2 & 0 \\ 1 & 1-\eta_3 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In the extension field E, s is given by

$$s = h^2 - \theta_3{}^2/4,$$

and the eigenvalues of m are

$$1 \pm 2h, \quad 0.$$

If $$r_{\pm} = r_1(1\pm2h)(\eta_3\pm2h) + r_2(1\pm2h) + r_3,$$

then $$h^2r_{\pm} = r_{\pm}(h\pm1)^2.$$

Thus, $hb_\pm = b_\pm(h \pm 1)$,

where $b_\pm = \frac{1}{2}(r_\pm + h^{-1}r_\pm(h \pm 1))$.

Tedious calculation shows that

$$b_- b_+ = f(h),$$

where

$$f(h) = (2h+1-\theta_1-\theta_2)(2h+1-\theta_1+\theta_2)(2h+1+\theta_1-\theta_2)(2h+1+\theta_1+\theta_2).$$

Finally, set $\varepsilon = (\varepsilon_1, \varepsilon_2)$,   where $\varepsilon_i = \pm 1$, $i = 1,2$.

With this notation,

$$f(h) = \prod_\varepsilon (2h+1+\varepsilon_1\theta_1+\varepsilon_2\theta_2).$$

The quantities $h, r_\pm$, and so on, are all elements of the algebra $\overline{A}$ constructed in the last chapter. I could proceed as I did there and find representations of $\overline{A}$. However, in this case, s is a differential operator on P, and the only solutions of the differential equation

$$\sigma g(z) = 0$$

which are of importance are those which lie in P. Thus, solutions of the eigen-equation of s must satisfy a boundary condition, namely, the condition for admission to P. The parameters which label the irreducible representations of $\overline{A}$ play in the algebraic theory the same role as the boundary conditions imposed on the solutions of the differential equation. Thus, I must determine those representations of $\overline{A}$ in which the boundary conditions dictated by P are satisfied.

Near z_1 the hypergeometric equation has two linearly

independent solutions $f_{1\alpha}$, $f_{1\beta}$ which behave like

$$f_{1\alpha} \sim (z-z_1)^{\alpha_1}, \quad f_{1\beta} \sim (z-z_1)^{\beta_1},$$

provided $(\alpha_1-\beta_1)$ is not an integer. Similarly, near z_2, there are two solutions $f_{2\alpha}$, $f_{2\beta}$ with

$$f_{2\alpha} \sim (z-z_2)^{\alpha_2}, \quad f_{2\beta} \sim (z-z_2)^{\beta_2},$$

provided $(\alpha_2-\beta_2)$ is not an integer. The two sets of solutions are linearly related by analytic continuation:

$$\begin{pmatrix} f_{1\alpha} \\ f_{1\beta} \end{pmatrix} = \begin{pmatrix} \omega_{+-} & \omega_{++} \\ \omega_{--} & \omega_{-+} \end{pmatrix} \begin{pmatrix} f_{2\alpha} \\ f_{2\beta} \end{pmatrix}$$

The entries of the matrix depend only upon θ. An arbitrary solution of the differential equation can be represented in terms of either set of solutions,

$$g = \xi_{1\alpha} f_{1\alpha} + \xi_{1\beta} f_{1\beta}$$

$$= \xi_{2\alpha} f_{2\alpha} + \xi_{2\beta} f_{2\beta}$$

It is not difficult to show that g is a periodic function on the double loop about z_1 and z_2 if and only if one of $\xi_{1\alpha}$ and $\xi_{1\beta}$ is zero and one of $\xi_{2\alpha}$ and $\xi_{2\beta}$ is zero. In other words, g must have a definite exponent both at z_1 and at z_2. The eigensurfaces of $s(\theta)$ are those surfaces in C^3 on which one of the entries vanishes in the matrix above. There are four families of eigensurfaces of $s(\theta)$; the family labelled by $\varepsilon = (\varepsilon_1, \varepsilon_2)$, where $\varepsilon_i = \pm 1$, contains all those surfaces in C^3 on which $\omega_\varepsilon(\theta)$ vanishes. The

functions $\omega_\varepsilon(\theta)$ are well known and involve only Γ functions. (Abramowitz and Stegun (1968)). $\omega_\varepsilon(\theta)$ vanishes whenever

$$\varepsilon_1\theta_1 + \varepsilon_2\theta_2 \pm \theta_3 + (2k-1) = 0, \quad k = 1,2,\ldots,$$

so these planes are the eigensurfaces of $s(\theta)$.

The zeros of the polynomial $f(\rho)$ are

$$\rho_\varepsilon = -\tfrac{1}{2}(\varepsilon_1\theta_1 + \varepsilon_2\theta_2 + 1).$$

Let X_ε denote the irreducible $sl(2,C)$ - module with highest weight ρ_ε. I will assume that $2\rho_\varepsilon + 1$ is not a nonnegative integer, so X_ε is infinite dimensional. The eigenvalues of

$$2h \mp \theta_3$$

on X_ε are

$$-(\varepsilon_1\theta_1 + \varepsilon_2\theta_2 \pm \theta_3 + (2k-1)), \quad k = 1,2,\ldots.$$

Since

$$s = \tfrac{1}{4}(2h+\theta_3)(2h-\theta_3),$$

the eigensurfaces of $s(\theta)$ on X_ε are

$$\varepsilon_1\theta_1 + \varepsilon_2\theta_2 \pm \theta_3 + (2k-1) = 0, \quad k = 1,2,\ldots.$$

It is clear that I can give two equivalent descriptions of $s(\theta)$ and its spectrum. Firstly, I can regard $s(\theta)$ as a differential operator, and insist that the only acceptable solutions of the differential equation

$$\sigma g(z) = 0$$

are those with exponents $\tfrac{1}{2}(\eta_1 + \varepsilon_1\theta_1)$ and $\tfrac{1}{2}(\eta_2 + \varepsilon_2\theta_2)$ at z_1 and

z_2, respectively. Secondly, I can exhibit $s(\theta)$ as an element of the algebra $\overline{A}$ and construct a representation of $\overline{A}$ on the $sl(2,C)$-module X_ϵ with highest weight ρ_ϵ. The algebraic version of the boundary condition imposed by P is that $s(\theta)$ must be represented on the reducible module

$$X = \bigoplus_\epsilon X_\epsilon .$$

The matrices of h and $b_\pm$ on X_ϵ have the familiar form

$$h = \begin{pmatrix} \rho_\epsilon & & & \\ & \rho_\epsilon-1 & & \\ & & \rho_\epsilon-2 & \\ & & & \ddots \end{pmatrix}, \quad b_+ = \begin{pmatrix} 0 & \alpha_1 & & \\ & 0 & \alpha_2 & \\ & & 0 & \alpha_3 \\ & & & \ddots & \ddots \end{pmatrix}, \quad b_- = \begin{pmatrix} 0 & & & \\ \beta_1 & 0 & & \\ & \beta_2 & 0 & \\ & & \beta_3 & \ddots \end{pmatrix}$$

in which the entries α_k and β_k must be so chosen that

$$b_+ b_- = f(h-1) = \prod_{\epsilon'} (2h-1+\epsilon_1'\theta_1+\epsilon_2'\theta_2) .$$

Equivalently,

$$\alpha_k \beta_k = 16k(k+\epsilon_1\theta_1)(k+\epsilon_2\theta_2)(k+\epsilon_1\theta_1+\epsilon_2\theta_2), \quad k = 1,2,\ldots .$$

The choice of α_k and β_k is quite arbitrary provided this condition is satisfied. I will choose

$$\alpha_k = 4(k+\epsilon_1\theta_1)(k+\epsilon_1\theta_1+\epsilon_2\theta_2)$$

and

$$\beta_k = 4k(k+\epsilon_2\theta_2).$$

Thus, $b_+ = u_+ d_+$ and $b_- = d_- u_-$,

where

$$u_+ = (2h-1-\epsilon_1\theta_1-\epsilon_2\theta_2)(2h-1-\epsilon_1\theta_1+\epsilon_2\theta_2) = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \ddots \end{pmatrix},$$

$$u_- = (2h-1+\epsilon_1\theta_1-\epsilon_2\theta_2)(2h-1+\epsilon_1\theta_1+\epsilon_2\theta_2) = \begin{pmatrix} \beta_1 & & & \\ & \beta_2 & & \\ & & \beta_3 & \\ & & & \ddots \end{pmatrix},$$

$$d_+ = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ & & & & \ddots \end{pmatrix},$$

and $d_- = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$

$b_{\pm}$ were defined in terms of r_1, r_2, r_3 and h. It is a simple matter to invert the defining equations to find that

$$r_1 = (b_+(2h+1)^{-1} + b_-(2h-1)^{-1})(4h)^{-1} - (\theta_1{}^2-\theta_2{}^2)(4h^2-1)^{-1}$$

and

$$r_2 = (b_+(2h+1)^{-1}(2h-\eta_3) - b_-(2h-1)^{-1}(2h+\eta_3))(4h)^{-1}$$
$$+ (\theta_1{}^2-\theta_2{}^2)(1+\eta_3)(4h^2-1)^{-1}.$$

If the denominators

$$h, \quad (2h\pm1)$$

are to be well defined automorphisms of X_ε, the numbers $0, \pm\frac{1}{2}$ must not be eigenvalues of h on X_ε. Equivalently,

$$2\rho_\varepsilon + 1$$

must not be a non-negative integer. This condition is satisfied because I assumed X_ε to be infinite dimensional and irreducible. The matrices of r_1 and r_2 are both tri-diagonal on X_ε. In terms of h and $d_\pm$,

$$r_1 = v(2h-1)d_+ + v_0(h) + d_- v(1-2h),$$

$$r_2 = w(2h-1)d_+ + w_0(h) + d_- w(1-2h),$$

where

$$v(a) = \frac{(a-\varepsilon_1\theta_1-\varepsilon_2\theta_2)(a-\varepsilon_1\theta_1+\varepsilon_2\theta_2)}{2a(a-1)},$$

$$v_0(h) = (\theta_2{}^2-\theta_1{}^2)/(4h^2-1),$$

$$w(a) = (a-1-\eta_3)v(a),$$

$$w_0(h) = -(1+\eta_3)v_0(h).$$

Thus, the differential operators

$$r_1 = 2\frac{(z-z_2)(z_3-z_1)}{(z-z_3)(z_2-z_1)} - 1$$

and

$$r_2 = 4\frac{(z-z_1)(z-z_2)}{(z_1-z_2)}\frac{d}{dz} + (\eta_2-\eta_1)$$

have tridiagonal matrices on X_ε.

I will now turn to a class of codiagonal operators on X_ε.

Operators with Regular Singularities

The codiagonal operators I have chosen for this example correspond to ordinary differential equations which have only regular singularities in the complex plane. These equations will appear later in the thesis. Their importance rests upon the following fact: every linear, ordinary differential equation with rational coefficients can be obtained from a member of this class by confluence of singular points. (Ince (1926)). I should also point out that Erdélyi (1944) has given a technique for representing Heun functions in terms of convergent series of hypergeometric functions, and his technique will be seen here to be the analytic version of an algebraic method.

The differential equation is

$$\left[\prod_{i=1}^{m} (z-z_i) \left[\frac{d^2}{dz^2} + \sum_{i=1}^{m} \frac{1-\alpha_i-\beta_i}{z-z_i} \frac{d}{dz} + \sum_{i=1}^{m} \frac{\alpha_i\beta_i}{(z-z_i)^2} \right] \right.$$
$$\left. + a(z) + \lambda b(z) \right] f(z) = 0, \quad m \geq 2.$$

The singular points are $z_1, z_2, \ldots, z_m$ and possibly the point at infinity. The finite singularities are all regular, and the exponents of the solution near any one, say z_i, are α_i and β_i. $a(z)$ and $b(z)$ are polynomials in z. As for the hypergeometric equation, set

$$\eta_i = \alpha_i + \beta_i \quad \text{and} \quad \theta_i = \alpha_i - \beta_i.$$

λ is the eigenvalue to be sought. I will require solutions of the differential equation to have exponents $\frac{1}{2}(\eta_1+\varepsilon_1\theta_1)$ and $\frac{1}{2}(\eta_2+\varepsilon_2\theta_2)$ at z_1 and z_2, respectively. This is

the boundary condition. Finally, I can assume without loss of generality that z_1 and z_2 are both real.

There is an elementary transformation which leads to a considerable simplification in the differential equation. Suppose that

$$f(z) = g(z) \prod_{i=3}^{m} (z-z_i)^{\beta_i}.$$

g satisfies the same equation as f, except that $\beta_3, \beta_4, \ldots, \beta_m$ must be set equal to zero and $a(z)$ must be replaced by another polynomial $c(z)$. The boundary conditions imposed on f and g are identical. It is in this second form that I will use the differential equation.

Let P denote the vector space of periodic functions on the double loop which encircles z_1 and z_2, but neither encloses nor passes through any of the other singular points.

Define
$$s(\theta) : P \to P$$
$$g(z) \mapsto \sigma g(z),$$

where
$$\sigma = \bar{\sigma} + \tfrac{1}{4}(\bar{\eta}_3{}^2 - \bar{\theta}_3{}^2)$$

and
$$\bar{\sigma} = (z-z_1)(z-z_2) \left[\frac{d^2}{dz^2} + \left(\frac{1-\eta_1}{z-z_1} + \frac{1-\eta_2}{z-z_2} \right) \frac{d}{dz} \right]$$
$$+ \tfrac{1}{4}(\eta_1{}^2 - \theta_1{}^2) \frac{z_1-z_2}{z-z_1} + \tfrac{1}{4}(\eta_2{}^2 - \theta_2{}^2) \frac{z_2-z_1}{z-z_2}.$$

$s(\theta)$ is the hypergeometric operator with singularities at z_1, z_2 and ∞. The variable $\bar{\eta}_3$ and $\bar{\theta}_3$ are spurious; $\bar{\eta}_3$ satisfies

$$\eta_1 + \eta_2 + \bar{\eta}_3 = 1,$$

and $\bar{\theta}_3$ is arbitrary; they only appear in order to disappear. The <u>Fuchsian differential operator</u> is defined in terms of s. Define

$$t(\lambda) : \quad P \longrightarrow P$$

$$g(z) \longmapsto \tau g(z)$$

where

$$\tau = \prod_{i=3}^{m} (z-z_i) \left[4\bar{\sigma} + \sum_{i=3}^{m} \frac{1-\theta_i}{z-z_i} (z-z_1)(z-z_2)\frac{d}{dz} \right] + c(z) + \lambda b(z)$$

Both of the differential equations

$$\tau g = 0 \quad \text{and} \quad \sigma g = 0$$

have regular singular points at z_1 and z_2. Furthermore, the exponents at these points are the same for the two equations. Thus, the boundary conditions imposed on solutions of the first equation can also be imposed on solutions of the second. If this is done, the functions in the kernel of $s(\theta)$, denoted

$$\{x_k, \, k = 1,2,\ldots\}$$

span the irreducible $sl(2,C)$-module X_ε. The matrices of the operators

$$s = h^2 - \bar{\theta}_3{}^2/4,$$

$$r_1 = 2(z-z_2)/(z_1-z_2) - 1,$$

and $\quad r_2 = 4\frac{(z-z_1)(z-z_2)}{(z_1-z_2)} \frac{d}{dz} + (\eta_2-\eta_1)$

are known on X_ε; s is diagonal, r_1 and r_2 are tridiagonal. The 'perturbed' operator t is a polynomial in

s, r_1 and r_2, so it is a trivial matter to compute the codiagonal matrix of t.

To construct a spectral chain for t, I must find the roots of the characteristic equation of the recurrence relation associated with t. To do this requires several steps.

(1) s is diagonal and, asymptotically, the kth diagonal element of s is k^2. I will write

$$s \sim \begin{pmatrix} 1 & & & & & & \\ & 4 & & & & & \\ & & 9 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & k^2 & \\ & & & & & & \cdot \end{pmatrix}$$

to indicate that these matrices are asymptotically equal.

(2) r_1 is tridiagonal and

$$r_1 \sim \tfrac{1}{2}v = \tfrac{1}{2}\begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & 1 & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \end{pmatrix}$$

Since $(z-z_i) = \tfrac{1}{2}((z_1-z_2)r_1 + z_1+z_2-2z_i)$, $1 \leqslant i \leqslant m$,

$$z-z_i \sim \tfrac{1}{4}(z_1-z_2)(v+y_i),$$

where $y_i = 2(z_1+z_2-2z_i)/(z_1-z_2)$.

(3) Similarly,

$$r_2 \sim \begin{pmatrix} 0 & -1 & & & & \\ 1 & 0 & -2 & & & \\ & 2 & 0 & -3 & & \\ & & 3 & 0 & . & \\ & & & . & . & . \\ & & & & . & . & . \end{pmatrix}.$$

(4) t has a codiagonal matrix. The characteristic equation of the recurrence relation associated with t has the form

$$\sum_i \zeta_i \, \gamma^i = 0;$$

ζ_i is the asymptotic value of the sequence in the ith diagonal of t. The term

$$4\bar{\sigma} \prod_{i=3}^{m} (z-z_i)$$

dominates the matrix of t and alone determines the characteristic equation because, relative to this term, the other terms are asymptotically equal to zero.

(5) In the first section of the appendix, I have shown that the characteristic equation reduces to the following:

$$\prod_{i=3}^{m} (\gamma + y_i + 1/\gamma) = 0.$$

The roots are the solution of

$$\gamma + y_i + 1/\gamma = 0, \qquad 3 \leqslant i \leqslant m.$$

It is a simple matter to substitute the expression for y_i to find that the $2(m-2)$ roots are

$$\gamma_{i\pm} = 2(2z_i - z_1 - z_2 \pm 2\sqrt{(z_i - z_1)(z_i - z_2)})/(z_1 - z_2).$$

It is interesting that these only depend upon the locations of the singular points.

(6) In order to construct a spectral chain for t, I only need to know the roots of the characteristic equation. However, to calculate the spectrum of t, I must be able to truncate the infinite set of equations to a finite set, and only when the moduli of the roots are all distinct do I know how this can be done. Consequently, in the appendix I have sought conditions on the locations of the singular points which are necessary and sufficient for the moduli of the roots to be distinct. The most useful result is the following. If all the singular points are real, then the moduli of the roots are distinct if and only if z_1 and z_2 are adjacent. I will assume that this is so.

If the degrees of the polynomials $b(z)$ and $c(z)$ do not exceed $(m-2)$, then the order of the recurrence relation is equal to the order of its characteristic equation, and Poincaré's theorem is applicable. When the recurrence relation has a higher order, I believe that a modified form of Poincaré's theorem could be applied, but I have only been able to prove this in very simple circumstances.

Let

$$0 \leqslant \gamma_1 < \gamma_2 < \quad < \gamma_{2(m-2)} < \infty$$

be the moduli of the roots of the characteristic equation. As in chapter 4, let X_a denote the subspace of $\overline{X}_\epsilon$ comprised by sequences which satisfy

$$\lim_{k \to \infty} \xi_k / \beta^k = 0$$

for all $\beta > \gamma_a$, $a = 1, \ldots, 2(m-2)$. Provided $a \leq m-2$, the sprectrum of t_a on X_a is discrete. The matrix of t can be truncated to a finite matrix, in accordance with the boundary condition on X_a, by setting

$$\xi_{k+n} = \beta_a^k \; \xi_n, \quad k \geq 0;$$

in this formula, β_a is the root whose modulus is γ_a and n can be chosen at will. The first n eigenvalues of t_a are the values of λ for which the (truncated) finite matrix equation

$$t(\lambda)x = 0$$

has a non-trivial solution. There are well established methods for solving such equations.

Example 2. Poincaré Modules.

I have included this example as a warning to show that the analytical and algebraic approaches to some problems are not equivalent and that it is not always possible to choose the best of both methods.

I will use the following notation. The Poincaré and Lorentz groups are real Lie groups so I will denote them $\underline{P}_R$ and $\underline{L}_R$. Their real Lie algebras are P_R and L_R. Finally, P and L will denote the complexifications of P_R and L_R.

Each unitary irreducible representation of the Poincaré group $\underline{P}_R$ is labelled by the values of the mass m, the spin s and the sign of the energy. Such a representation is denoted

$$D(m,s,\pm).$$

The decomposition of the representation $D(m,s,+)$ into unitary irreducible representations of the Lorentz group $\underline{L}_R$ was derived by Joos (1962) and takes the form

$$D(m,s,+) = \sum_{k=-s}^{s} \oplus \int_{O}^{\infty} d\rho[\,k,i\rho]$$

where $[\,k,i\rho]$ is the notation introduced by Naimark (1964) for the unitary irreducible representations of $\underline{L}_R$. These are analytical statements; the unitary irreducible representations of $\underline{P}_R$ are classified, one is assumed to be given and this one is dissected.

In the algebraic approach, the order of these steps is

reversed. Irreducible so(3)-modules are known. These can be glued together to form irreducible L_R-modules. (Gel'fand and Ponomarev (1968)) It is also possible to glue together L_R-modules to form an irreducible P_R-module with any preassigned values of the mass and spin. The algebraists' plan is to synthesise rather than analyse P_R-modules.

However, the two programmes are not equivalent, at least in any obvious sense. To see why this is so it is necessary to examine the direct integral decomposition more closely. It will suffice to examine just one term

$$\int_0^\infty d\rho[\,k,i\rho]\,,$$

since the direct summation over k is well understood.

Let H_ρ denote the Hilbert space which carries the unitary irreducible representation of $\underline{L}_R$ labelled $[k,i\rho]$, and construct

$$V = \bigcup_{\rho \geqslant 0} H_\rho.$$

Let F denote the set of all functions

$$f \,:\, [\,0,\infty) \longrightarrow V$$

$$\rho \longmapsto f(\rho) \in H_\rho.$$

Each such function maps ρ into a vector in H_ρ. Define

$$|f|^2 = \int_0^\infty d\rho |f(\rho)|_\rho^2\,,$$

where $|\cdot|_\rho$ denotes the norm on H_ρ, and set

$$H = \{f \in F|\quad |f|^2 < \infty\}.$$

It is the Hilbert space H that carries the direct
integral representation

$$\int_0^\infty d\rho[\,k,i\rho]\,.$$

The operators of this representation on H, denoted $\Pi(g)$
and defined weakly, are

$$\Pi(g) \;=\; \int_0^\infty d\rho\;\Pi_\rho(g)\,,\;\; g\in\underline{L}_R\,,$$

where $\Pi_\rho(g)$ are the operators of the representation of
$\underline{L}_R$ on H_ρ and

$$(\Pi(g)f)(\rho) \;=\; \Pi_\rho(g)f(\rho) \quad \text{for all } f\in H.$$

The point I want to stress is that the elements of
H are functions from $[o,\infty)$ to V, so it is not correct
to consider the representation spaces H_ρ as subspaces of
H. (Only with the help of a rigged Hilbert space and
distributions outside H is it possible to view H_ρ as a
subspace of H.) This conflicts with the algebraic approach
where the L_R-modules are actually embedded in the P_R-module.

There is a serious difficulty here. Suppose values
are prescribed for m and s. The representation
$D(m,s,+)$ of $\underline{P}_R$ can be decomposed into a direct integral
of unitary irreducible representations of $\underline{L}_R$. However, it
is also possible to construct an algebraically irreducible
P_R-module, with the same values for m and s, as the
countable direct sum of algebraically irreducible L_R-modules.
The difficulty is that it is not possible to embed each

L_R-module in a Hilbert space which carries a unitary representation of $\underline{L}_R$. Chakrabarti, Levy-Nahas and Seneor (1968) noticed this difficulty but failed to resolve it, so they only claimed that their results were formally correct. They did not seem to realize that their results were rigorously correct in an algebraic framework and that the representations they had constructed were algebraically irreducible representations of P_R, not unitary topologically irreducible representations of $\underline{P}_R$. MacDowell and Roskies (1972) dismissed the work of Chakrabarti et al. with the observation that the basis vectors they had employed lay outside the domain of the momentum operators, which are necessarily unbounded in a unitary representation of $\underline{P}_R$. This observation is correct but irrelevant, for it does not help to establish the connection between the algebraically irreducible representations of P_R and the topologically irreducible unitary representations of $\underline{P}_R$ with the same mass and spin.

I do not intend to speculate upon this connection because more research is still needed. Instead I will outline the steps in the construction of (algebraically) irreducible P_R-modules, since this illustrates the general methods of chapter 3 and does not seem to be widely known. Since every irreducible P_R-module provides an irreducible P-module and vice versa, I can without loss of generality deal only with the complex Lie algebra P.

P is a ten dimensional Lie algebra over C with basis.

$$\{r_{\lambda\mu} = -r_{\mu\lambda},\ p_\lambda,\ 1 \leqslant \lambda, \mu \leqslant 4\}$$

and Lie products

$$[r_{\lambda\mu}, r_{\nu\rho}] = \delta_{\mu\nu} r_{\lambda\rho} - \delta_{\lambda\nu} r_{\mu\rho} + \delta_{\rho\mu} r_{\nu\lambda} - \delta_{\rho\lambda} r_{\nu\mu},$$

$$[r_{\lambda\mu}, p_\nu] = \delta_{\mu\nu} p_\lambda - \delta_{\lambda\nu} p_\mu,$$

$$[p_\lambda, p_\nu] = 0.$$

The 'spin vector' in $U(P)$ is

$$s_\lambda = \tfrac{1}{2}\epsilon_{\lambda\mu\nu\rho} r_{\mu\nu} p_\rho$$

The centre of $U(P)$ is the subalgebra generated by the set

$$\{1, p_\lambda p_\lambda, s_\lambda s_\lambda\}.$$

The anti-symmetric matrix

$$r = (r_{\lambda\mu}),$$

with entries from $U(L)$, satisfies a fourth order polynomial identity (Bracken and Green (1971)):

$$(r-1)^4 - (1+\tfrac{1}{2}\sigma_2)(r-1)^2 + \tfrac{1}{4}(\sigma_2(1+\tfrac{1}{2}\sigma_2)-\sigma_4) = 0,$$

where $\sigma_i = \text{trace } r^i$.

$U(L)$ can be embedded isomorphically in an algebra $\overline{U(L)}$ in which the polynomial identity can be factorised,

$$(r-1-k_1)(r-1+k_1)(r-1-k_2)(r-1+k_2) = 0,$$

so that $\sigma_2 = 2(k_1{}^2+k_2{}^2-1),$

$$\sigma_4 = 2(k_1{}^2(k_1{}^2-1) + k_2{}^2(k_2{}^2-1)).$$

Each irreducible $\overline{U(L)}$-module provides an irreducible $U(L)$-module, and vice versa. Furthermore, each irreducible $\overline{U(L)}$-module in which k_1 and k_2 are represented by scalar multiples of the identity is uniquely labelled by these values.

The procedure by which irreducible P-modules may be constructed is now very similar to that given in chapter 3 for $gl(n+1,C)$-modules, so I will only list the steps.

(1) Set

$$k_3 = -k_1 \quad \text{and} \quad k_4 = -k_2,$$

so that the polynomial identity can be written

$$\prod_{i=1}^{4}(r-1-k_i) = 0.$$

Define

$$e_i = \prod_{j \neq i}(r-1-k_j)/(k_i-k_j), \quad i=1,2,3,4.$$

It is easy to verify that

$$e_i e_j = \delta_{ij} e_i,$$

$$1 = \sum_i e_i,$$

$$r = \sum_i k_i e_i.$$

(2) Construct the 'eigenvectors' of r,

$$(p_i)_\lambda = p_\mu (e_i)_{\mu\lambda}.$$

These satisfy

$$(p_i)_\lambda (r_{\lambda\mu} - k_i \delta_{\lambda\mu}) = 0,$$

$$k_1 (p_i)_\lambda = (p_i)_\lambda (k_1 - \delta_{i1} + \delta_{i3}),$$

and
$$k_2 (p_i)_\lambda = (p_i)_\lambda (k_2 - \delta_{i2} + \delta_{i4}).$$

(3) Construct the 'mixed invariants'

$$d_i = p_\lambda p_\mu (e_i)_{\mu\lambda} \quad , \quad i = 1,2,3,4.$$

Their calculation is tedious but straight-forward. The result is

$$d_i = (-)^i \tfrac{1}{2} m^2 (k_i + s)(k_i - s - 1)/(k_1{}^2 - k_2{}^2).$$

The constants m and s are the mass and spin; they can be assigned arbitrary values, real or complex, but I will assume that (2s+1) is an integer. Furthermore,

$$p_\lambda p_\lambda x = -m^2 x$$

and
$$s_\lambda s_\lambda x = m^2 s(s+1) x$$

for every vector x in X, the irreducible P-module that will be constructed.

(4) Let X(k,c) denote the irreducible L-module labelled [k,c] in Naimark's notation. Construct the module

$$X = \sum_{\substack{-s \leqslant k \leqslant s \\ -\infty < n < \infty}} X(k, i\rho + n),$$

where the sum is direct and ρ is a fixed constant. The

action of p_λ can be defined on X. For example, the matrix elements for p_λ given by Chakrabarti et al. will suffice. Hence X is a P-module. p_λ maps the L-module $X(k,i\rho+n)$ into the four show in the diagram below, and only when the polynomial d_i vanishes is the corresponding transition impeded.

$$
\begin{array}{ccccc}
 & & X(k,i\rho+n+1) & & \\
 & & \uparrow{\scriptstyle d_4} & & \\
X(k-1,i\rho+n) & \xleftarrow{\ d_1\ } & X(k,i\rho+n) & \xrightarrow{\ d_3\ } & X(k+1,i\rho+n) \\
 & & \downarrow{\scriptstyle d_2} & & \\
 & & X(k,i\rho+n-1) & &
\end{array}
$$

d_1 and d_3 vanish when $k_1 = \pm s$, but, if ρ is real, d_2 and d_4 never vanish. Thus, X is an irreducible P-module.

The modules $X(k,i\rho+n)$, $n \neq 0$, can never be integrated to unitary representations of $\underline{L}_R$. How are these non-unitary representations to be interpreted? Prof. Green suggests that the same set of generators for $\underline{L}_R$ cannot be used in every irreducible representation occurring in the decomposition of X. For example, if ϕ is a scalar wave function, then $p_\lambda \phi$ has 'spin' associated with the index λ and so belongs to a (reducible) representation with generators

$$
r_{\lambda\mu} + s_{\lambda\mu},
$$

where $s_{\lambda\mu}$ accounts for this spin.

The existence of these algebraically irreducible

P_R-modules poses two problems. The unitary irreducible
representation $D(m,s,+)$ and the algebraically constructed
representation have the same values for the mass and spin.
In what sense, if any, are they equivalent? Secondly, if
they are inequivalent, which is needed in quantum physics?

P_R-modules poses two problems. The unitary irreducible

representation $D(m,s,+)$ and the algebraically constructed

representation have the same values for the mass and spin.

APPENDIX 6

Derivation of the Characteristic Equation

The term which dominates the matrix of t is

$$4\,\bar{\sigma}\prod_{i=3}^{m}(z-z_i) \sim 4\bar{\sigma}\prod_{i=3}^{m}\tfrac{1}{4}(z_1-z_2)(v+y_i),$$

where
$$v = \begin{pmatrix} 0 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ & 1 & 0 & 1 & & & \\ & & 1 & \cdot & \cdot & & \\ & & & \cdot & & & \\ & & & & \cdot & \cdot & \\ & & & & & \cdot & \cdot \end{pmatrix}.$$

Since $\bar{\sigma}$ is diagonal, the coefficients in the characteristic equation are determined by the matrix

$$p(v) = \prod_{i=3}^{m}(v+y_i).$$

Since y_i commutes with v, this product can be expanded as an ordinary product to obtain

$$p(v) = \sum_{j=0}^{m-2}\alpha_j v^j.$$

The coefficients here will depend upon $y_3, y_4, \ldots, y_m$; their calculation is trivial. The matrix elements of v^j are computed in the following lemma.

Lemma 1.

$$(v^j)_{kl} = \sum_{i=0}^{j}\delta_{k\ l-j+2i}\binom{j}{i}, \quad l > j.$$

Proof.   When $j=1$, the result is trivial. Assume that it is true for $j \leqslant j'$. Then,

$$(v^{j+1})_{kn} = \sum_l (v^j)_{kl} \, v_{ln}, \qquad n > j+1$$

$$= \sum_l \sum_{i=0}^{j} \delta_{k \ \ l-j+2i} \binom{j}{i} (\delta_{l \ \ n-1} + \delta_{l \ \ n+1})$$

$$= \sum_{i=0}^{j} (\delta_{k \ \ n-j-1+2i} + \delta_{k \ \ n-j+1+2i}) \binom{j}{i}$$

$$= \sum_{i=0}^{j} \delta_{k \ \ n-j-1+2i} \binom{j}{i} + \sum_{i=1}^{j+1} \delta_{k \ \ n-j-1+2i} \binom{j}{i-1}$$

$$= \delta_{k \ \ n-j-1} + \delta_{k \ \ n+j+1} + \sum_{i=1}^{j} \delta_{k \ \ n-j-1+2i} \left(\binom{j}{i} + \binom{j}{i-1} \right)$$

$$= \delta_{k \ \ n-j-1} + \delta_{k \ \ n+j+1} + \sum_{i=1}^{j} \delta_{k \ \ n-j-1+2i} \binom{j+1}{i}$$

$$= \sum_{i=0}^{j+1} \delta_{k \ \ n-j-1+2i} \binom{j+1}{i}$$

Thus, the result is true for all j.   ///

The matrix $p(v)$ can now be found easily.

$$(p(v))_{kl} = \sum_{j=0}^{m-2} \alpha_j \sum_{i=0}^{j} \delta_{k \ \ l-j+2i} \binom{j}{i}, \qquad l > m-2.$$

Define $\zeta_l = (p(v))_{k \ k+l}$, $-(m-2) \leqslant l \leqslant (m-2)$, for some large value of k. (Since the elements on any diagonal of $p(v)$ are constant after the first m or so, ζ_l is independent of k.)

$$\zeta_l = \sum_{j=0}^{m-2} \alpha_j \sum_{i=0}^{j} \delta_{l \ \ j-2i} \binom{j}{i} .$$

The characteristic equation of the difference equation is

$$\sum_{l=-(m-2)}^{m-2} \zeta_l \gamma^l = 0.$$

This equation can be rearranged:

$$\sum_{l=-(m-2)}^{m-2} \zeta_l \gamma^l =$$

$$= \sum_{j=0}^{m-2} \alpha_j \ (\gamma + 1/\gamma)^j$$

$$= \prod_{i=3}^{m} \ (\gamma + y_i + 1/\gamma).$$

Moduli of the Roots.

__Lemma 2.__ The roots $\gamma_{i\pm}$ of the characteristic equation have distinct moduli if and only if z_i does not lie on the line joining z_1 and z_2.

__Proof__
$$\gamma_{i\pm} = \tfrac{1}{2}(-y_i \pm \sqrt{y_i^2 - 4})$$

(1) The roots are equal, and hence have equal moduli, if and only if

$$y_i^2 - 4 = 16(z_i - z_1)(z_i - z_2)/(z_1 - z_2)^2 = 0.$$

This can only be so if $z_i = z_1$ or $z_i = z_2$, so the result is trivial in this case.

(2) Suppose then that

$$\sqrt{y_i^2 - 4} = \alpha + i\beta \neq 0 \quad \text{and} \quad y_i = \gamma + i\delta.$$

Thus,
$$\alpha^2 - \beta^2 = \gamma^2 - \delta^2 - 4,$$

and
$$\alpha\beta = \gamma\delta.$$

Require that the roots have equal moduli:

$$|(\alpha - \gamma) + i(\beta - \delta)| = |(\alpha + \gamma) + i(\beta + \delta)|.$$

Equivalently,
$$\alpha\gamma + \beta\delta = 0.$$

I will show that this can only be so if one of y_i and $\sqrt{y_i^2-4}$ is real and the other is purely imaginary.

If $\alpha \neq o$, multiply the last equation by α:

$$\gamma\alpha^2 + \delta(\alpha\beta) = o$$

Then
$$\gamma(\alpha^2+\delta^2) = o,$$

and so
$$\gamma = o \text{ and } \beta = o.$$

Thus, y_i is imaginary and $\sqrt{y_i^2-4}$ is real. If $\alpha = o$, so that $\sqrt{y_i^2-4}$ is imaginary, then

$$\beta\delta = o,$$

from which it follows that $\delta = 0$ and that y_i is real.

By assumption, z_1 and z_2 are real. The only way in which one of

$$y_i = 2(z_1+z_2- 2z_i)/(z_1-z_2)$$

and

$$\sqrt{y_i^2-4} = 4\sqrt{(z_i-z_1)(z_i-z_2)}/(z_1-z_2)$$

can be real and the other imaginary is if z_i is real and satisfies

$$z_1 \leqslant z_i \leqslant z_2. \qquad\qquad /// $$

I now want to investigate whether it is possible for $\gamma_{i\pm}$ and $\gamma_{j\pm}$, $i \neq j$, to have equal moduli. Note that the $\pm$ signs need not correspond. A necessary and sufficient condition for

$$|\gamma_{i\pm}| = |\gamma_{j\pm}|$$

is that

$$\gamma_{i\pm} = e^{i\phi}\gamma_{j\pm},$$

which can easily be reduced to

$$\cos \phi = y_i y_j \pm \sqrt{(y_i^2-4)(y_j^2-4)}.$$

Equivalently,

$$\cos \phi = \frac{4}{(z_1-z_2)^2} \left((z_1+z_2-2z_i)(z_1+z_2-2z_j) \right.$$

$$\left. \pm \sqrt{(z_1-z_i)(z_2-z_i)(z_1-z_j)(z_2-z_j)} \right).$$

Lemma 3.   Suppose that all the singular points are real. If z_1 and z_2 are adjacent, then the moduli of the roots of the characteristic equation are all distinct.

Proof.   According to lemma 2,

$$|\gamma_{i+}| \neq |\gamma_{i-}|, \quad 3 \leq i \leq m.$$

I need only show that $\cos \phi$, defined above, does not satisfy

$$-1 \leq \cos \phi \leq +1$$

for any choice of z_i and z_j, $i \neq j$.

$$4(y_i^2+y_j^2)-2y_i y_j \cos \phi + \cos^2\phi - 16 = 0.$$

Solve for y_j.

$$4y_j = y_i\cos\phi \pm \sqrt{(\cos^2\phi-16)(y_i^2-4)}$$

Since both y_i and y_j are real,

$$(\cos^2\phi-16)(y_i^2-4) \geq 0.$$

Furthermore, ϕ is a real angle so

$$\cos^2\phi - 16 < 0$$

and $$y_i{}^2 - 4 \leqslant 0.$$

Therefore, $$(z_i - z_1)(z_i - z_2) \leqslant 0.$$

This implies that z_i lies between z_1 and z_2, contrary to assumption. Hence,

$$|\gamma_{i\pm}| \neq |\gamma_{j\pm}| \quad \text{for all } i \neq j. \qquad ///$$

CHAPTER 7. THE SPECTRAL FUNCTION AND ANALYTIC CONTINUATION

Suppose that K is a subalgebra of a Lie algebra L
and that X is an irreducible L-module. The aim of
spectral analysis of X is to answer the following questions.

(1) Is X completely reducible as a K-module?

(2) If so, with what multiplicity does each irreducible
K-module occur in X?

(3) Otherwise, which indecomposable K-modules occur in X.

There are two important circumstances in which X is
always completely reducible. The first arises when L is
a reductive Lie algebra over C and X is finite dimen-
sional. The second arises when X is a K-finite
L-module. In each of these cases, the second problem can
usually be solved quite readily with the techniques given
in earlier chapters. As usual, when L represents
sl(2,C), all the problems are trivial. I will use this
case to introduce the spectral function and its connection
with analytic continuation.

Take a finite dimensional sl(2,C)-module X. The
matrix of the element h of the Cartan subalgebra is
diagonal;

$$
h = \begin{pmatrix} m & & & & & \\ & m-1 & & & & \\ & & m-2 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & & -m \end{pmatrix},
$$

where m is either an integer of half an odd integer. The
function

$$\phi(\lambda) = \det(h-\lambda) = \prod_{i=o}^{2m} (m-i-\lambda)$$

is an entire function of λ which vanishes whenever λ is
an eigenvalue of h. In more sophisticated terms, $\phi(\lambda)$
vanishes whenever λ is the highest weight of a
gl(1,C)-submodule of X. Apart from a scaling factor, $\phi(\lambda)$
is unique.

Now suppose that X is an infinite dimensional,
irreducible sl(2,C)-module with highest weight ρ. The
matrix of h on X is

$$h = \begin{pmatrix} \rho & & & \\ & \rho-1 & & \\ & & \rho-2 & \\ & & & \ddots \end{pmatrix}.$$

Define the <u>spectral function</u> $\phi(\lambda)$ of X to be the entire
function of λ, of lowest possible order and type, whose
zeros are the highest weights of the gl(1,C)-submodules of
X. The exponent of convergence of the eigenvalues of h
is equal to unity. Thus, the spectral function $\phi(\lambda)$ is
the canonical product

$$\phi(\lambda) = \prod_{k=1}^{\infty} (1-\frac{\lambda}{\rho-k+1}) \, e^{\lambda/(\rho-k+1)}$$

(Note. The definitions of order, type and exponent of

convergence are those given by Titchmarsh (1939).) Since
I assumed X to be irreducible, the highest weight ρ can
never be a non-negative integer or half-integer, so
$(\rho-k+1)$ never vanishes and the product is well defined.
This infinite product can be evaluated quite easily:

$$\phi(\lambda) = \exp(\alpha\lambda) \; \Gamma(-\rho)/\Gamma(\lambda-\rho),$$

where
$$\alpha = \frac{1}{\rho} - \gamma - \rho \sum_{k=1}^{\infty} \frac{1}{k(k-\rho)} \;,$$

and γ is Euler's constant. If the assumption that X
should be irreducible is relaxed, then $\phi(\lambda)$ will have a
simple pole when ρ is a non-negative integer.

In this example, $\phi(\lambda)$ is simply the 'determinant' of
$(h-\lambda)$. This definition can be extended. If

$$s = \prod_{i=1}^{m} (h-\lambda_i), \quad m \geqslant 1,$$

then define the <u>determinant</u> of s to be

$$\det s = \prod_{i=1}^{m} \phi(\lambda_i).$$

Thus,
$$\det s = \prod_{i=1}^{m} \det (h-\lambda_i).$$

It is not difficult to generalise the definition of
a spectral function to include the cases where L is
other than sl(2,C). Suppose that L is a reductive Lie
algebra over C and that X is a K-finite L-module. If
the Cartan subalgebra of K has dimension n, then the
highest weight of any irreducible K-submodule of X has

n components. Furthermore, these components uniquely
label that K-module. The spectral functions of X are
n entire functions $\phi_1(\lambda),\ldots,\phi_n(\lambda)$ of the complex
variable $\lambda = (\lambda_1,\ldots,\lambda_n)$, of lowest possible order and
type, with the following property. A point λ in C^n is
a simultaneous zero of $\phi_1(\lambda),\ldots,\phi_n(\lambda)$ of order k if
and only if the irreducible K-module labelled by $(\lambda_1,\ldots,\lambda_n)$
occurs exactly k times in the decomposition of X.

I have only given this definition for completeness;
I will not use it in the sequel.

It is clear that the calculation of the spectral
function for an $sl(2,C)$-module X is a trivial matter.
However, in the rest of this chapter I want to show that it
has a non-trivial consequence, connected with analytic
continuation. The result is this. Suppose that s is a
diagonal operator on X which can also be represented by
an ordinary differential operator with analytic coefficients,
together with boundary conditions imposed at two points in
the complex plane. Generally it is quite easy to construct
solutions of the differential equation in the vicinity of
one of the boundary points, but it is quite difficult to
analytically continue the solutions into the vicinity of
the other point. However, once the spectral function of X
is known, the analytic continuation is trivial. Some
remarks concerning this assertion are needed.

There is a sense in which every operator s, whose
spectrum of eigenvalues is simple, bounded below and count-
able, can be exhibited as a diagonal operator on an

sl(2,C)-module. Because of the assumed properties, the eigenvalues of s can be arranged in a sequence

$$\{\gamma_k, \ k = 1,2,\ldots\}.$$

There is a uniquely determined entire function g of γ which interpolates the points of this sequence. Thus,

$$g(k) = \gamma_k$$

Let X denote the vector space spanned by the eigenvectors $\{x_k, \ k = 1,2,\ldots\}$ of s. On X define

$$h \ : \ X \to X$$
$$x_k \to (\rho - k + 1)x_k, \ \rho \in C,$$
$$e_+ \ : \ X \to X$$
$$x_k \to \alpha_{k-1} x_{k-1},$$
$$e_- \ : \ X \to X$$
$$x_k \to \beta_k x_{k+1},$$

where α_k and β_k are chosen to satisfy

$$\alpha_k \beta_k = k(2\rho + 1 - k).$$

Finally, define the operator $g(\rho - h + 1)$ by

$$g(\rho - h + 1) \ : \ X \to X$$
$$x_k \to g(k)x_k.$$

It is clear that $s = g(\rho - h + 1)$ on X, so that (loosely) s is a diagonal operator. The Lie algebra sl(2,C) matches the qualitative characteristics of the spectrum of s, a contortion $g(\rho - h + 1)$ of the operator h reproduces the

spectrum of s. Almost any differential operator of the second order will have a simple spectrum of eigenvalues, if only they form a discrete set. This is so because the multiplicity of any eigenvalue can at most equal two, the number of linearly independent solutions of the differential equation, and in practice will equal one, because only one solution of the equation will fulfil the boundary conditions. For these operators then, the algebra which matches the qualitative aspects of the spectrum is sl(2,C).

The difficulty, of course, is that the eigenvalues of s are not known in advance, and, even if they were, the function g could not always be obtained in closed form. It turns out in fact that the only ordinary differential operators which can usefully be represented as diagonal operators are the special function operators. Nevertheless, it is a beautiful result that the analytic continuation of a special function can be deduced once the operator is given as a diagonal operator.

There are two procedures which occur in the theory of each of the special functions of mathematical physics. Both deal with the differential equation satisfied by the special function. The first is the construction of a representation of the general solution of the equation in regions which include one singularity of the equation and no other. This involves the construction in each of these regions of a local basis for the vector space of solutions of the equation, and is usually achieved through the development of a set of independent solutions in (asymptotic) series which converge only in that region. The second is

the determination of the <u>connecting matrices</u> which relate
the several local bases of the vector space of solutions.
Once the local bases and the connecting matrices have been
found, the analytic structure of the special function is
known and, for practical purposes, the theory is complete.
The first step is relatively simple, and can be executed
not only for the special functions but also for the solutions
of most linear differential equations whose coefficients
are holomorphic at all but a finite number of points. The
second step is difficult. In fact, the reason why the
special functions are special is precisely that both steps
of the programme can be completed for these functions.

In the next sections I will examine just one of the
special functions, the hypergeometric function. My aim will
be to show that the connecting matrices for this function
are known once the operator is exhibited as a diagonal
operator. I stress that similar results hold for all the
other special functions. The central idea, that the spectral
function determines the connecting matrices, can be extended
to more difficult ordinary differential equations. However,
the results are incomplete because, as I have already
pointed out, it is as difficult to represent an arbitrary
differential operator as a diagonal operator as it is to
determine the analytic structure of its eigensolutions.
Consequently, I have omitted them. All I will say is that
many differential operators can be represented as codiagonal
operators, and so a finite number of their eigenvalues can
be calculated on a computer. From these eigenvalues it is
possible to calculate approximately the monodromy group of

the differential equation.

The hypergeometric operator $s(\theta)$ is the diagonal operator

$$s = h^2 - \tfrac{1}{4}\theta_3{}^2$$

On the $sl(2,C)$-module X_ε, the determinant of s is the function

$$\det s = \det(h-\tfrac{1}{2}\theta_3)\ \det(h+\tfrac{1}{2}\theta_3) = \phi(\varepsilon_1\theta_1,\varepsilon_2\theta_2,\theta_3),$$

where $\phi(\theta_1,\theta_2,\theta_3) = \dfrac{\Gamma(\tfrac{1}{2}(1+\theta_1+\theta_2))^2}{\Gamma(\tfrac{1}{2}(1+\theta_1+\theta_2+\theta_3))\Gamma(\tfrac{1}{2}(1+\theta_1+\theta_2-\theta_3))}$.

I assert that the four functions,

$$\phi(\varepsilon_1\theta_1,\varepsilon_2\theta_2,\theta_3), \quad \varepsilon_i = \pm 1, \quad i = 1,2,$$

are essentially the four elements of the matrix connecting the hypergeometric functions constructed near z_1 to those constructed near z_2. To prove the assertion I must examine the analytical aspects of the problem. For a short time I will consider a differential equation with an arbitrary number of regular singularities in order to decide upon which parameters the connecting matrices may depend. I believe that these results on an old topic are new.

Differential Equations with Regular Singularities

The hypergeometric differential equation has three regular singular points. It is a special case of the following equation:

$$\sum_{i=1}^{m}(1-\alpha_i-\beta_i)\left[\frac{d^2}{dz^2}+\frac{2}{z-z_i}\frac{d}{dz}+\frac{2}{(1-\alpha_i-\beta_i)}\left(\frac{\alpha_i\beta_i}{(z-z_i)^2}-\frac{\gamma_i}{(z-z_i)}\right)\right]f(z) = 0.$$

When the parameters $\alpha_i, \beta_i, \gamma_i$, $1 \leqslant i \leqslant m$, satisfy

$$\sum_i (1-\alpha_i-\beta_i) = 2,$$

$$\sum_i \gamma_i = 0,$$

$$\sum_i \gamma_i z_i = \sum_i \alpha_i \beta_i,$$

$$\sum_i \gamma_i z_i^2 = 2\sum_i \alpha_i \beta_i z_i,$$

the points $z_1, \ldots z_m$ are the only singularities of the equation and all are regular. In place of the variables α_i and β_i, I will often use

$$\eta_i = \alpha_i + \beta_i \quad \text{and} \quad \theta_i = \alpha_i - \beta_i.$$

Near any one of the singular points, say z_i, two solutions with exponents α_i and β_i can be developed in convergent series. Riemann's notation for the hypergeometric function can be extended to denote the general solution of this equation:

$$P \begin{Bmatrix} z_1 & - & - & - & z_m \\ \alpha_1 & - & - & - & \alpha_m \\ \beta_1 & - & - & - & \beta_m \\ \gamma_1 & - & - & - & \gamma_m \end{Bmatrix} z \equiv P \begin{Bmatrix} z_i \\ \gamma_i \\ \beta_i \\ \gamma_i \end{Bmatrix} z \; .$$

Choose a point z_o that is not collinear with any pair of singular points and cut the plane as shown in the diagram.
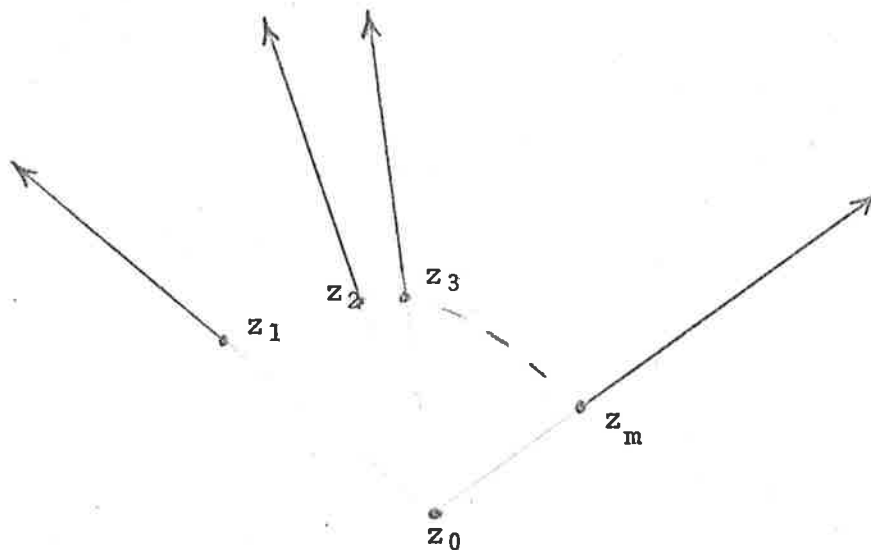
Near the singularity z_i, two solutions with exponents α_i and β_i can be developed in convergent power series. Let $f_{i\alpha}$ and $f_{i\beta}$ denote these solutions:

$$f_{i\alpha} = (z-z_i)^{\alpha_i}(1+\sum_{k=1}^{\infty} a_k(z-z_i)^k),$$

$$f_{i\beta} = (z-z_i)^{\beta_i}(1+\sum_{k=1}^{\infty} b_k(z-z_i)^k).$$

The disadvantage of these series is that whenever θ_i is an integer, other than zero, the coefficients in one of them are undefined. To remedy this, define

$$g_{i\alpha}(z) = f_{i\alpha}(z)/\Gamma(1+\theta_i)$$

and

$$g_{i\beta}(z) = f_{i\beta}(z)/\Gamma(1-\theta_i).$$

These series are well defined for all values of θ_i. In fact they are very well behaved, as can be seen from the following lemma.

<u>Lemma 1</u>. The functions $g_{i\alpha}$ and $g_{i\beta}$ are entire functions of θ, of order at most equal to one, for all values of z, other than z_i, for which they converge uniformly.

A result of this type is well known for any solution $g(z)$ developed in series about a point z_0 which is not a singular point, provided that the 'initial values'

$$g(z_0) \quad \text{and} \quad g'(z_0)$$

are entire functions of θ of order at most equal to one. (Hille (1969)). I have not seen a published proof of this result when $g(z)$ is developed in series about a singular point. The proof is not difficult, but is rather long, so I will omit it. The solutions $g_{i\alpha}$ and $g_{i\beta}$ are linearly dependent if and only if θ_i is an integer. Set

$$\underline{g}_i = \begin{pmatrix} g_{i\alpha} \\ g_{i\beta} \end{pmatrix} :$$

The components of $\underline{g}_i$ may be analytically continued throughout the cut plane. In particular, they may be continued into the domain of definition of $\underline{g}_j$ near z_j. Provided θ_j is not an integer, the components of $\underline{g}_j$ will be linearly independent, so the continuation of $\underline{g}_i$, which I still denote $\underline{g}_i$, can be expressed in terms of $\underline{g}_j$:

$$\underline{g}_i = u_{ij}\underline{g}_j$$

The matrix u_{ij} is independent of z, but could conceivably depend upon the locations of the singular points $z_1,\ldots,z_m$ and the parameters α_k, β_k and γ_k, $1 \leq k \leq m$,

which appear in the differential equation. However, in the appendix I have shown that the only variables upon which u_{ij} may depend are the following:

(1) the exponent differences $\theta_k = \alpha_k - \beta_k$;

(2) the independent cross ratios $\chi_1, \ldots, \chi_n$ of the singular points;

(3) the variables

$$\zeta_k = \{(z_{k+1} - z_k)(z_{k-1} - z_k)(\gamma_k + \tfrac{1}{2}(1 - \alpha_k - \beta_k)\sum_{l \neq k}(1 - \alpha_l - \beta_l)/(z_k - z_l))$$

$$+ \tfrac{1}{4}(1 - \theta_k^2)(z_{k+1} + z_{k-1} - 2z_k)\}/(z_{k+1} - z_{k-1}),$$

where it is understood that $z_{m+1} = z_1$ and $z_0 = z_m$.

Thus, if
$$\theta = (\theta_1, \ldots, \theta_m),$$

$$\zeta = (\zeta_1, \ldots, \zeta_m),$$

and
$$\chi = (\chi_1, \ldots, \chi_n),$$

then
$$u_{ij} = u_{ij}(\zeta, \theta, \chi).$$

The reason why these variables are distinguished is that they are the invariants of the transformations which leave the form of the differential equation unchanged.

The matrix u_{ij} has a symmetry which I have not yet employed. The components of g_i are interchanged by the interchange of α_i and β_i. This requires that the rows of u_{ij} should be interchanged when the sign of θ_i is reversed. Similarly, the columns of u_{ij} must be interchanged when the sign of θ_j is reversed. Thus, if the

entry in the first row and second column of u_{ij} is

$$u(\theta_i, \theta_i, \cdot),$$

where the dot stands for all the other variables upon which u may depend, then

$$u_{ij} = \begin{pmatrix} u(\theta_i, -\theta_j, \cdot) & u(\theta_i, \theta_j, \cdot) \\ u(-\theta_i, -\theta_j, \cdot) & u(-\theta_i, \theta_j, \cdot) \end{pmatrix}.$$

When θ_j is not an integer, the matrix u_{ij} is given by

$$u_{ij} = (\underline{g}_i, \underline{g}_i{'})(\underline{g}_j, \underline{g}_j{'})^{-1}$$

$$= \begin{pmatrix} g_{i\alpha} & g_{i\alpha}' \\ g_{i\beta} & g_{i\beta}' \end{pmatrix} \begin{pmatrix} g_{j\beta}' & -g_{j\alpha}' \\ -g_{j\beta} & g_{j\alpha} \end{pmatrix} (g_{j\alpha}g_{j\beta}' - g_{j\alpha}'g_{j\beta})^{-1},$$

where each function on the right is to be evaluated at some fixed point z_o, distinct from the singular points. The prime denotes differentiation. The entries of the matrices are entire functions of θ, of order at most equal to one, so the entries in the product of the matrices share this property. The denominator, the Wronskian of $g_{j\alpha}$ and $g_{j\beta}$, has simple poles whenever θ_j is an integer. However, if I define

$$v_{ij} = \frac{u_{ij}}{\Gamma(1+\theta_j)\Gamma(-\theta_j)},$$

the poles of the Γ functions cancel with the zeros of the denominator of u_{ij}, so v_{ij} is an entire function of θ and its order also does not exceed one. I have verified

that all these assertions are correct, but because the results are self-evident and the proofs are tedious, I omit the proofs.

The Hypergeometric Equation.

In the case of the hypergeometric equation, the situation is particularly simple. With only three singular points, it is impossible to form any cross ratios. Furthermore, becuase the parameters γ_1, γ_2 and γ_3 must satisfy three linear conditions, they can be eliminated in favour of θ_1, θ_2 and θ_3. Thus, the connecting matrices u_{ij} must be functions of θ alone:

$$u_{ij} = u_{ij}(\theta).$$

The hypergeometric operator $s(\theta)$ is defined in the previous chapter. It acts on P, the vector space of periodic functions on the double loop l about z_1 and z_2. Consequently, eigenfunctions of $s(\theta)$ are solutions of the hypergeometric equation which are unaffected by analytic continuation around l. It is not difficult to show that the condition of periodicity on l is only satisfied by hypergeometric functions which have a definite exponent at both z_1 and z_2. In other words, the (generalised) eigenvalues of $s(\theta)$ are the zeros of the elements of the matrix u_{12}, and the correspondence between these sets is one-to-one. If θ is a zero of

$$u(\epsilon_1\theta_1, \epsilon_2\theta_2, \theta_3), \quad \epsilon_i = \pm 1,$$

then one branch of the hypergeometric function has exponents

$$\tfrac{1}{2}(\eta_1 + \varepsilon_1 \theta_1) \qquad \text{and} \qquad \tfrac{1}{2}(\eta_2 + \varepsilon_2 \theta_2)$$

at z_1 and z_2, respectively, and hence θ must lie on an eigensurface in the family labelled by $\varepsilon = (\varepsilon_1, \varepsilon_2)$. Conversely, if θ is any point of this eigensurface, then the branch of the hypergeometric function with exponent $\tfrac{1}{2}(\eta_1 + \varepsilon_1 \theta_1)$ at z_1 must also have exponent $\tfrac{1}{2}(\eta_2 + \varepsilon_2 \theta_2)$ at z_2, and so $u(\varepsilon_1 \theta_1, \varepsilon_2 \theta_2, \theta_3)$ must vanish. Furthermore, the multiplicities of eigenvalues and zeros are equal. This is easily shown by converting the differential equation to an integral equation, in which case the elements of u_{12} are the Fredholm determinants.

On the $sl(2,C)$-module X_ε, $s(\theta)$ is a diagonal operator and its eigensurfaces, found algebraically in the preceding chapter, are

$$\varepsilon_1 \theta_1 + \varepsilon_2 \theta_2 \pm \theta_3 + (2k-1) = 0, \quad k=1,2,\ldots .$$

Consequently, I can identify the surfaces on which $u(\varepsilon_1 \theta_1, \varepsilon_2 \theta_2, \theta_3)$ vanishes with these planes.

I note in particular one simple eigensolution of $s(\theta)$ on X_ε:

$$f(z) = y^{\alpha_1}(1-y)^{\alpha_2},$$

where

$$y = \frac{(z-z_1)(z_3-z_2)}{(z-z_3)(z_1-z_2)}$$

and

$$1 + \theta_1 + \theta_2 \pm \theta_3 = 0.$$

That this is an eigensolution is only a matter for verification. It corresponds to the trivial solution

$$F(a,b;c;y) = 1$$

of Gauss' hypergeometric equation when one exponent at infinity vanishes. If f is developed in powers of $(z-z_1)$, the leading term is $(z-z_1)^{\alpha_1}$, so

$$g_{1\alpha} = f/\Gamma(1+\theta_1).$$

Similarly,

$$g_{2\alpha} = f/\Gamma(1+\theta_2).$$

Thus, when

$$1 + \theta_1 + \theta_2 \pm \theta_3 = 0,$$

$$g_{1\alpha} = \frac{\Gamma(1+\theta_2)}{\Gamma(1+\theta_1)} g_{2\alpha},$$

so

$$u(\theta_1,-\theta_2,\theta_3) = \frac{\Gamma(1+\theta_2)}{\Gamma(1+\theta_1)}$$

and

$$u(\theta_1,\theta_2,\theta_3) = 0.$$

The determinant of s on X_ϵ, $\epsilon = (1,1)$, is

$$\phi(\theta_1,\theta_2,\theta_3) = \frac{\Gamma(\tfrac{1}{2}(1+\theta_1+\theta_2))^2}{\Gamma(\tfrac{1}{2}(1+\theta_1+\theta_2+\theta_3))\Gamma(\tfrac{1}{2}(1+\theta_1+\theta_2-\theta_3))} \cdot$$

ϕ is an entire function of θ_3, but as a function of $\theta = (\theta_1,\theta_2,\theta_3)$ has double poles whenever

$$\theta_1 + \theta_2 = -(2k-1), \quad k = 1,2,\ldots$$

However, the factor

$$\overline{\phi}(\theta) = [\Gamma(\tfrac{1}{2}(1+\theta_1+\theta_2+\theta_3))\Gamma(\tfrac{1}{2}(1+\theta_1+\theta_2-\theta_3))]^{-1}$$

is an entire function of θ, of order at most one, which vanishes on the eigensurfaces of $s(\theta)$ on X_ϵ. The function

$$\frac{u(\theta_1,\theta_2,\theta_3)}{\Gamma(1+\theta_2)\Gamma(-\theta_2)}$$

is entire and has precisely the same zeros with the same multiplicities, so the ratio of the two is entire and never vanishes. Therefore, it must take the form

$$\exp(r(\theta))$$

where r is entire. Thus,

$$\frac{\overline{\phi}(\theta_1,\theta_2,\theta_3)\Gamma(1+\theta_2)\Gamma(-\theta_2)}{u(\theta_1,\theta_2,\theta_3)} = \exp(r(\theta)).$$

Since the ratio has order one at most, $r(\theta)$ must simply be a linear function of θ:

$$r(\theta) = c+c_1\theta_1+c_2\theta_2+c_3\theta_3.$$

Both $u(\theta)$ and $\overline{\phi}(\theta)$ are unchanged when the sign of θ_3 is reversed, so $c_3 = 0$. The constants c, c_1 and c_2 are uniquely determined by the values of $u(\theta_1,-\theta_2,\theta_3)$ and $\overline{\phi}(\theta_1,-\theta_2,\theta_3)$ when

$$1 + \theta_1 + \theta_2 \pm \theta_3 = 0.$$

In fact, when this is the case,

$$\overline{\phi}(\theta_1,-\theta_2,\theta_3) = \frac{1}{\Gamma(1+\theta_1)\Gamma(-\theta_2)}$$

and

$$\frac{u(\theta_1,-\theta_2,\theta_3)}{\Gamma(1-\theta_2)\Gamma(\theta_2)} = \frac{\Gamma(1+\theta_2)}{\Gamma(1+\theta_1)}\frac{-1}{\Gamma(1+\theta_2)\Gamma(-\theta_2)} = \frac{-1}{\Gamma(1+\theta_1)\Gamma(-\theta_2)}$$

Thus, $c_1 = c_2 = c_3 = 0$ and $\exp(r(\theta)) = -1$,

so $u(\theta_1,\theta_2,\theta_3) = -\dfrac{\Gamma(1+\theta_2)\Gamma(-\theta_2)}{\Gamma(\frac{1}{2}(1+\theta_1+\theta_2+\theta_3))\Gamma(\frac{1}{2}(1+\theta_1+\theta_2-\theta_3))}$

$$= \frac{\Gamma(1-\theta_2)\Gamma(\theta_2)}{\Gamma(\frac{1}{2}(1+\theta_1+\theta_2+\theta_3))\Gamma(\frac{1}{2}(1+\theta_1+\theta_2-\theta_3))}\ .$$

The analytic continuation of $g_{1\alpha}$ into the vicinity of z_2 is

$$g_{1\alpha} = \frac{\Gamma(1+\theta_2)\Gamma(-\theta_2)}{\Gamma(\tfrac{1}{2}(1+\theta_1-\theta_2+\theta_3))\Gamma(\tfrac{1}{2}(1+\theta_1-\theta_2-\theta_3))} g_{2\alpha}$$

$$+ \frac{\Gamma(1-\theta_2)\Gamma(\theta_2)}{\Gamma(\tfrac{1}{2}(1+\theta_1+\theta_2+\theta_3))\Gamma(\tfrac{1}{2}(1+\theta_1+\theta_2-\theta_3))} g_{2\beta}.$$

Hence,

$$f_{1\alpha} = \frac{\Gamma(1+\theta_1)\Gamma(-\theta_2)}{\Gamma(\tfrac{1}{2}(1+\theta_1-\theta_2+\theta_3))\Gamma(\tfrac{1}{2}(1+\theta_1-\theta_2-\theta_3))} f_{1\alpha}$$

$$+ \frac{\Gamma(1+\theta_1)\Gamma(\theta_2)}{\Gamma(\tfrac{1}{2}(1+\theta_1+\theta_2+\theta_3))\Gamma(\tfrac{1}{2}(1+\theta_1+\theta_2-\theta_3))} f_{2\beta},$$

in agreement with the classical result.

From this long calculation I can draw the following interesting conclusion. The classical problem connected with the hypergeometric function was to find the analytic continuation of each of its branches throughout the complex plane. The algebraic problem is to exhibit the hypergeometric operator as a diagonal operator on an $sl(2,C)$-module. The solution of the algebraic problem and the subsequent construction of the spectral function solves the classical problem, apart from some inessential analytical complications.

APPENDIX 7.

Invariant Parameters.

It is well known that there are two transformations which map the equation with regular singularities into another of the same type (Ince (1926)). What I will do is show that there are certain combinations of the parameters of the equation which are unchanged by these transformations.

Lemma 2.   If the P-function is multiplied by a factor of the form

$$\left(\frac{z-z_k}{z-z_1}\right)^{\alpha},$$

another P-function is produced. In fact,

$$\left(\frac{z-z_k}{z-z_1}\right)^{\alpha} P\left(\begin{array}{ccc} z_i & & \\ \alpha_i & z & \\ \beta_i & & \\ \gamma_i & & \end{array}\right) = P\left(\begin{array}{ccc} z_i & & \\ \alpha_i' & z & \\ \beta_i' & & \\ \gamma_i' & & \end{array}\right),$$

where $\alpha_i' = \alpha_i + (\delta_{ik} - \delta_{i1})\alpha$

$\beta_i' = \beta_i + (\delta_{ik} - \delta_{i1})\alpha$

and $\gamma_i' = \gamma_i + \frac{1}{2}(1 - \alpha_i - \beta_i) \sum_{j \neq i} (1 - \alpha_j - \beta_j)/(z_i - z_j)$

$- \frac{1}{2}(1 - \alpha_i' - \beta_i') \sum_{j \neq i} (1 - \alpha_j' - \beta_j')/(z_i - z_j).$

The proof of this result is tedious, uninteresting and omitted. Its importance is this.

Define $\delta_i = \gamma_i + \tfrac{1}{2}(1-\alpha_i-\beta_i)\sum_{j \neq i}(1-\alpha_j-\beta_j)/(z_i-z_j),$

$$\delta = (\delta_1, \delta_2, \ldots, \delta_m),$$

$$\eta = (\eta_1, \eta_2, \ldots, \eta_m),$$

$$\theta = (\theta_1, \theta_2, \ldots, \theta_m).$$

It is clear that δ, θ and z_1, z_2, ..., z_m are independent variables which are unchanged by the transformation. Combinations such as

$$\sum_i \alpha_i \quad \text{and} \quad \sum_i \beta_i$$

are also invariant, but since

$$\sum_i \beta_i = \sum_i (1-\alpha_i) - 2$$

and

$$\sum_i \alpha_i = \tfrac{1}{2}\sum(\alpha_i+\beta_i) + \tfrac{1}{2}\sum_i(\alpha_i-\beta_i)$$

$$= \tfrac{1}{2}(m-2) + \tfrac{1}{2}\sum_i \theta_i,$$

these are dependent upon θ. δ, θ and $z_1, z_2, \ldots, z_m$ is the largest set of independent invariants.

Lemma 3. If
$$z' = \frac{az+b}{cz+d},$$

$$z_i' = \frac{az_i+b}{cz_i+d},$$

and
$$ad - bc \neq 0,$$

then
$$P \begin{pmatrix} z_i \\ \alpha_i \\ \beta_i \\ \delta_i \end{pmatrix} z = P \begin{pmatrix} z_i' \\ \alpha_i \\ \beta_i \\ \delta_i' \end{pmatrix} z',$$

where $\quad \delta_i' = \{\delta_i(cz_i+d) - \tfrac{1}{2}c(1-\theta_i^2)\}/(-cz_i'+a).$

<u>Lemma 4</u>. If z_i, z_j, z_k are any three distinct singular points, then

$$\{(z_j'-z_i')(z_k'-z_i')\delta_i' + \tfrac{1}{4}(1-\theta_i^2)(z_j'+z_k'-2z_i')\}/(z_j'-z_k')$$

$$= \cdot\{(z_j-z_i)(z_k-z_i)\delta_i + \tfrac{1}{4}(1-\theta_i^2)(z_j+z_k-2z_i)\}/(z_j-z_k)$$

To prove these lemmas, like lemma 2, once the answer is known, requires only very tedious straight-forward calculation. Without fore-knowledge of the result required, they are not so easy to establish.

The quantities invariant under a linear fractional transformation of the independent variable are:

(1) η and θ;

(2) the collection $\chi = (\chi_1, \chi_2, \ldots, \chi_n)$ of independent cross ratios of the singular points;

(3) $\zeta_{ijk} = \{(z_j-z_i)(z_k-z_i)\delta_i + \tfrac{1}{4}(1-\theta_i^2)(z_j+z_k-2z_i)\}/(z_j-z_k),$

where for each value of i, the indices j and k must be chosen so that

$$i \neq j \neq k \neq i.$$

ζ_{iln} may be constructed from ζ_{ijk} by means of the relations

$$\zeta_{ijk} = -\zeta_{ikj}$$

and

$$\frac{(z_i-z_1)(z_j-z_k)}{(z_i-z_j)(z_1-z_k)}\left[\zeta_{ijk} + \tfrac{1}{4}(1-\theta_i{}^2)\right] - \tfrac{1}{4}(1-\theta_i{}^2) = \zeta_{ilk},$$

so ζ_{ijk} and ζ_{iln} are not independent quantities. Set

$$\zeta_i = \{(z_{i+1}-z_i)(z_{i-1}-z_i)\delta_i + \tfrac{1}{4}(1-\theta_i{}^2)(z_{i+1}+z_{i-1}-2z_i)\}/$$

$$(z_{i+1}-z_{i-1}),$$

where it is understood that $z_{m+1} = z_1$ and $z_0 = z_m$. Finally, set

$$\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m).$$

The variables $\zeta, \eta, \theta, \chi$ form a complete set of invariants under this second type of transformation. However, only ζ, θ and χ are invariant under transformations of the dependent variable described previously, so these constitute a complete set of invariants when both types of transformations are considered.

Functional Dependence of the Connecting Matrices.

The solutions at z_i and z_j are related by the matrix u_{ij}:

$$g_i = u_{ij}g_j$$

I want to show that u_{ij} can only depend upon ζ, θ and χ.

Firstly, multiply both sides of the equation above by

$$\left(\frac{z-z_k}{z-z_1}\right)^{\alpha}$$

to obtain

$$\left(\frac{z-z_k}{z-z_1}\right)^{\alpha} \underline{g}_i = u_{ij} \left(\frac{z-z_k}{z-z_1}\right)^{\alpha} \underline{g}_j.$$

The components of $\underline{g}_i$ satisfy the differential equation with only regular singular points. The components of

$$\left(\frac{z-z_k}{z-z_1}\right)^{\alpha} \underline{g}_i$$

satisfy a similar equation except that the exponents at z_k are increased by α and those at z_1 are decreased by α. The same matrix u_{ij} relates the branches at z_i and z_j for both of those differential equations. Since $k, 1$ and α are arbitrary, u_{ij} can only depend upon the invariants of this transformation.

Secondly, in the equation

$$\underline{g}_i (z) = u_{ij} \underline{g}_j (z),$$

set

$$z = \frac{dz'-b}{-cz'+a}, \quad ad-bc \neq 0.$$

$$z_i = \frac{dz_i'-b}{-cz_i'+a},$$

and

$$\delta_i = \{\delta_i'(-cz_i'+a)+\tfrac{1}{2}c(1-\theta_i^2)\}/(cz_i+d).$$

$\underline{g}_i(z)$ can be considered as a function of the primed variables and as such satisfies the Fuchsian differential equation with primed parameters. Once again, the same

matrix u_{ij} relates the branches at z_i' to those at z_j'. Hence, u_{ij} may only depend upon the invariants of transformations of this type.

Together these results imply that u_{ij} may only depend upon the exponent differences θ, the independent cross ratios χ of the singular points, and the parameters ζ:

$$u_{ij} = u_{ij}(\zeta,\theta,\chi).$$

CHAPTER 8. IMAGINARY RELATIVE TIME?

I have argued the case for algebraic methods for quantum mechanics and the eigenvalue problems which arise therefrom. In the last chapter I inferred that even analytic continuation could be reduced to algebraic terms. Algebraic methods are obviously powerful, but it might be argued that an algebraic formulation of quantum mechanics is too free, because it lacks the restrictions imposed by a topological structure. In this chapter I want to discuss just one freedom of the algebraic theory, the freedom to regard the relative energy and relative time coordinates of two particles as imaginary quantities. My aim will be to show that, not only is this freedom not unwanted but that it is advantageous and, surprisingly, not necessarily incompatible with the usual formulation of relativistic quantum mechanics.

In chapter 3 I demonstrated that an irreducible representation of a complex Lie algebra L remained irreducible when restricted to one of its real forms L_R; conversely, every irreducible representation of L_R could be uniquely extended to an irreducible representation of L. The complex Lie algebra so(4,C) has two important real forms; the Lorentz Lie algebra so(3,1) and the compact real form so(4,R). Thus, an irreducible representation of so(3,1) can be extended to so(4,C) and then restricted to so(4,R); the resulting representation is irreducible. The process can be reversed.

so(4,C)

so(3,1)   so(4,R)
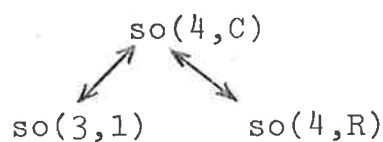
<u>Algebraically</u>, there is little difference between irreducible representations of so(4,R) and so(3,1). <u>Analytically</u> this is not so. Quantum physics demands unitary representations of the Lorentz group SO(3,1) and these cannot be obtained by analytic continuation from the unitary representations of SO(4,R). This is obvious because the unitary, irreducible representations of SO(4,R) are finite dimensional, whereas none of the finite dimensional representations of SO(3,1) is unitary. Thus, it is the requirement of unitarity of the representations that distinguishes between SO(3,1) and SO(4,R).

(I believe that the demand for unitary representations of the Lie group can be relaxed. *-representations of the enveloping algebra of the Lie algebra, equipped with an involution *, should suffice. In fact, this is precisely the structure studied by most physicists. Whether or not these *- representations of the <u>algebra</u> are integrable to unitary representations of the <u>group</u> can be decided by Nelson's theorem. (Nelson (1959)). Whether or not they need to be integrated depends upon the framework adopted for quantum mechanics.)

The relative angular momentum of two spinless particles is

$$l_{\lambda\mu} = i(x_\lambda p_\mu - x_\mu p_\lambda), \quad 0 \le \lambda, \; \mu \le 3,$$

where x_λ and p_λ are respectively the relative coordinates and momenta of the particles. The six independent components of the angular momentum provide a basis for so(3,1). The transformation from so(3,1) to so(4,R) is easy to effect; simply multiply the relative time and energy by i. Thus,

$$k_{lm} = i(x_l p_m - x_m p_l), \quad l \le 1, \ m \le 4,$$

where $\qquad x_4 = ix_0 \qquad$ and $\qquad p_4 = ip_0,$

is a basis for $so(4,R)$. Consequently, the question raised because there is so little difference algebraically between representations of $so(3,1)$ and $so(4,R)$ is whether or not the relative time and energy variables can take imaginary values.

The momentum (or angular momentum) of the centre of mass of a system of particles is measurable; it generates the translation (rotation) from the coordinate frame of the observer to the centre-of-mass frame of the system. Measurable quantities are real, so I think it is inescapable that the _total_ four-momentum should be a Minkowski four-vector. It is not so obvious that _relative_ momenta are observables; in fact, in chapter 1 I argued that they are not. Hence, it is conceivable that relative time and energy could be imaginary or, equivalently, that the relative coordinates and momenta could be Euclidean four-vectors.

How am I to test such a hypothesis? I need a soluble model in relativistic quantum mechanics which can be solved in two ways, the relative momenta being treated as Euclidean four-vectors in one and as Minkowski four-vectors in the other. A suitable model is Wick's model for the interaction of two spinless particles by the exchange of spinless 'photons'. Wick (1954) proposed the model in 1954 and showed that the equations could be reduced to a single ordinary differential equation, if the relative energy of the two particles could assume imaginary values. In 1957 Green (1957)

found that Wick's model could be solved without this assumption. However, Green's solutions appeared to differ from Wick's at low energies, and this discrepancy remained unexplained for fourteen years. In 1971 I proved that the discrepancy is only apparent and that the results found by Green and Wick are equivalent. Thus, in Wick's model the relative momentum can be regarded as a Euclidean four-vector and I suspect that this is a general feature of relativistic quantum mechanics.

Not only does this problem have important implications for physics, it is also interesting mathematically. The work of Green and Wick can be represented as follows. Both of them derived the same ordinary differential equation, a special case of Heun's equation. Wick applied boundary conditions at two points a and b and sought eigenvalues and eigenfunctions of the boundary value problem. Green did likewise, but imposed his boundary conditions at points b and c. The question raised is whether the boundary conditions at a and b determine the same eigenvalues as the boundary conditions at b and c.

Green and Biswas (1972) have recently shown that Wick's equation in the centre-of-mass coordinates can be transformed so that it is invariant under SO(3,1). This 'internal Lorentz group' is a symmetry of the interaction and is independent of the kinematical Lorentz group. The analytic continuation from SO(3,1) to SO(4,R) is essentially the transformation from real to imaginary time made by Wick.

Wick's Equation

The Bethe-Salpeter (1951) equation describes in a

relativistic fashion the quantum mechanics of two particles which are bound one to another by the exchange of a third particle. When all three particles are spinless, the 'ladder approximation' to the Bethe-Salpeter equation is

$$\left[\left(p+\frac{P}{2}\right)^2 - m^2\right]\left[\left(p-\frac{P}{2}\right)^2 - m^2\right]\Psi(p) = \frac{i\lambda}{\pi^2}\int\frac{\Psi(k)d^4k}{(p-k)^2-\mu^2-i\varepsilon} \cdot$$

The symbols in this equation have the following meanings. m and μ are the masses of the bound and the exchanged particles, respectively. P is the total four-momentum of the system and p is the relative four-momentum of the bound particles. $\Psi(p)$ is the wave function for the whole system. Finally, λ is the coupling constant, whose magnitude determines the strength of the interaction. In general this equation is too difficult to solve.

Wick (1954) made two important contributions to the theory. The first was his suggestion that the relative energy p_0 could assume imaginary values and that the contour of integration over k_0 could be rotated from the real to the imaginary axis. This is the 'Wick rotation'. Its validity clearly depends upon the analytic properties of the wave function. Wick's second contribution was to propose a set of approximations for the Bethe-Salpeter equation which rendered this equation soluble, and yet not so simple that all the relativistic features were obliterated. These approximations were that the bound particles should be spinless, that the exchanged particles should be both massless and spinless, and that the bound state could be adequately represented by the ladder Feynman diagram.

It is questionable whether Wick's model has any connection

with physics, for the assumption that the exchanged particles have neither mass nor spin excludes all real particles. Nevertheless, applications have been found. For example, Biswas (1958) used Wick's equation in a composite model for K-mesons; Nakanishi (1969) lists many others. However, the value of Wick's equation lies not in its applications, but rather in the fact that it is the only soluble example of the Bethe-Salpeter equation. It is hoped that the relativistic aspects of its solutions, namely, the appearance of a new quantum number and the possibility of imaginary time and energy, are not peculiar to the model, but are general features of the Bethe-Salpeter equation.

There is a definite advantage to be gained by allowing imaginary values for p_0 and k_0, so that

$$p_4 = ip_0 \quad \text{and} \quad k_4 = ip_0$$

are real. The bilinear product

$$(p-k,p-k) = (p_0-k_0)^2 - (\underline{p}-\underline{k})^2$$

is indefinite when p_0 and k_0 are real, but is negative definite when p_4 and k_4 are real. Wick found that the integral equation was amenable to standard mathematical techniques once p_0 and k_0 were assigned imaginary values. To accomplish this change from the Minkowski to the Euclidean metric on space-time, Wick needed to prove three facts: (1) that the wave function $\Psi(p)$ was analytic in the upper and lower halves of the complex plane of the variable p_0, and could be analytically continued from one region to the other;

(2)   that the contour of integration over k_0 could be rotated from the real to the imaginary axis without encountering any singularities of the integrand;

(3)   that the contributions to the integral from the quarter circles at infinity in the k_0-plane were zero, or, equivalently, that $\Psi(k)$ approached zero as rapidly as k_0^{-2} when k_0 approached infinity along any ray in the first or third quadrants of the k_0-plane.

Wick formulated stability conditions for the bound particles and found that these were sufficient to establish points (1) and (2) concerning the wave function. However, he was forced to assume the validity of (3). Wick reduced the four-dimensional integral equation to an ordinary differential operator. He deduced that, at a given centre-of-mass energy, bound states could only occur if the coupling constant assumed one of a countably infinite set of values, and that these values were the eigenvalues of the differential operator.

Green (1957) discovered that it was not necessary to rotate the contour of integration in order to find solutions of Wick's equation. He replaced the integral equation by a partial differential equation with associated boundary conditions and demonstrated that an ingenious bipolar co-ordinate transformation rendered the partial differential equation separable. Green also found that the permissible values of the coupling constant were the eigenvalues of an ordinary differential operator, which differs from the operator derived by Wick.

To reconcile the two studies, and hence to justify

Wick's assumption (3), it must be shown that the two ordinary differential operators have the same spectrum. The solution of this problem is here resolved into two stages. Firstly, I will reduce the two differential equations to the same form, so that the operators differ only in the boundary conditions. Secondly, I will show that solutions which fulfil one set of boundary conditions also fulfil the other set, and vice versa. I need only consider the case of particles of equal mass, since Cutkosky (1954) has shown that the equations for unequal masses can be reduced to the former case.

I would like to mention some numerical results which I find interesting, even though they are somewhat irrelevant. The eigenvalues of Wick's equation are the eigenvalues of a tridiagonal matrix. The difficulty is that Green and Wick derived different matrices. For large energies it is easy to verify numerically that the two matrices have the same eigenvalues. However, in the non-relativistic limit, the matrices are inordinately large, often 10^5 x 10^5, and the numerical techniques, which involve iteration of recurrence relations, become unreliable. Nevertheless, the two matrices do seem to have the same eigenvalues, though to obtain the same accuracy Wick's matrix must be much larger than Green's. The most interesting point is that for neither matrix does the lowest eigenvalue agree with the value predicted from Balmer's formula, as Wick claimed it should. The discrepancy is of the same order as λ^2, the square of the coupling constant.

Wick's Differential Operator.

It would be pointless if I were to re-derive all of

Wick's work. Instead I will state only his final equation.

$$\left[(1-z^2)\frac{d^2}{dz^2} + 2(n-1)z\frac{d}{dz} - n(n-1) + \frac{\lambda}{1-a+az^2}\right]g(z) = 0 \qquad (1)$$

with boundary conditions

$$g(\pm 1) = 0. \qquad (2)$$

In this equation, the parameter a is given by

$$a = E^2/m^2,$$

λ is the coupling constant, and n is an integral quantum number. E is one half of the total energy in the centre-of-mass frame and is supposed fixed. Thus, for bound states,

$$a < 1.$$

The allowable values of the coupling constant λ are sought as eigenvalues.

Equation (1) is invariant under the transformation

$$z \rightarrow -z.$$

The theory of Sturm-Liouville asserts that the eigenvalues are simple. Thus, the eigensolutions must have definite parity, for otherwise, g(z) and g(-z) would be independent eigensolutions with the same eigenvalue. The boundary conditions (2) may be replaced by

$$g(0) = g(1) = 0$$

for odd solutions of (1), and by

$$\frac{dg}{dz}(0) = g(1) = 0$$

for even solutions. Define

$$x = (1-z^2)^{-1}$$

and
$$f(x) = x^{\frac{1}{2}n}g((1-1/x)^{\frac{1}{2}}).$$

The function $f(x)$ satisfies the following differential equation, a particular case of Heun's equation (Erdélyi (1955)):

$$\left[\frac{d^2}{dx^2} + \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1}\right)\frac{d}{dx} + \frac{\lambda-n^2(x-a)}{4x(x-1)(x-a)}\right]f(x) = 0 \quad . \tag{3}$$

The boundary conditions to be satisfied by an 'odd' solution of (3), that is, a solution (3) derived from an odd solution of (1), are

$$\lim_{x\to\infty} x^{-\frac{1}{2}n}f(x) = 0 \tag{4}$$

and
$$f(1) = 0 \quad . \tag{5}$$

'Even' solutions of equation (3) must also satisfy (4) but, instead of (5), they must satisfy

$$\lim_{x\to 1} (x-1)^{\frac{1}{2}}\frac{df}{dx} = 0 \quad . \tag{6}$$

Green's Differential Operator.

In the centre of mass frame, Wick's equation can be written

$$[(p_0+E)^2-\underline{p}^2-m^2][(p_0-E)^2-\underline{p}^2-m^2]\,\Psi(p) = \frac{i\lambda}{\pi^2}\int\frac{d^4k\Psi(k)}{(p-k)^2-i\epsilon} \quad .$$

Green reduced the integral equation to a differential equation as follows. Set

$$\Phi(p) = \frac{i\lambda}{\pi^2}\int\frac{d^4k\Psi(k)}{(p-k)^2-i\epsilon} \quad .$$

Then $$\Box\Phi(p) = 4\lambda\Psi(p)$$

and so

$$[(p_0+E)^2-\underline{p}^2-m^2] \, [(p_0-E)^2-\underline{p}^2-m^2]\Box\Phi(p) = 4\lambda\Phi(p). \qquad (7)$$

The boundary conditions which supplement the differential equation are that:

(1) $\Phi(p)$ should be bounded near the origin;

(2) $p^2\Phi(p)$ should be bounded for all p.

Green found that (7) was separable if the following coordinate transformations were made. Firstly, represent p in polar coordinates:

$$p_1 = q \sin\theta \cos\phi,$$
$$p_2 = q \sin\theta \sin\phi,$$
$$p_3 = q \cos\theta,$$

so that $$\underline{p}^2 = q^2.$$

Secondly, introduce bipolar coordinates in the (p_0,q) plane:

set $$c^2 = m^2 - E^2 > 0$$

and define

$$p_0 = c \sin\alpha / (\cos\alpha - \cos\beta),$$
$$q = c \sin\beta / (\cos\alpha - \cos\beta).$$

(Note that these are not the usual bipolar coordinates which can be found in the handbook on special functions by Magnus, Oberhettinger and Soni (1966). To obtain their bipolar coordinates, c and α must be replaced by imaginary quantities.) The ranges

$$-\infty < p_0 < \infty, \qquad q \geqslant 0$$

are covered if α and β are restricted to the triangular region

$$-\pi < \alpha < \pi, \qquad |\alpha| < \beta \leqslant \pi .$$

In the new variables, the differential equation for Φ becomes

$$(E \ \sin\alpha - c \ \cos\alpha)(E \ \sin\alpha + c \ \cos\alpha)\left(\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\beta^2} + \frac{1(1+1)}{\sin^2\beta}\right)(q\Phi) = \lambda q\Phi.$$

This equation has separable solutions:

$$q\Phi = u(\alpha)v(\beta)Y_{1m}(\theta,\phi)$$

where $u(\alpha)$ and $v(\beta)$ satisfy

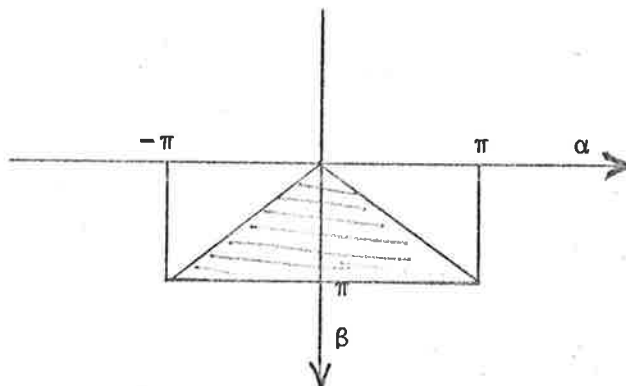$$\left(\frac{d^2}{d\beta^2} + n^2 - \frac{1(1+1)}{\sin^2\beta}\right)v(\beta) = 0 \qquad (8)$$

and

$$\left(\frac{d^2}{d\alpha^2} + n^2 + \frac{\lambda/m^2}{a-\cos^2\alpha}\right)u(\alpha) = 0 \quad . \qquad (9)$$

The boundary conditions to be imposed upon the solutions $u(\alpha)$ and $v(\beta)$ of the separated equation have caused some confusion, so I will examine this point carefully.

The variables α and β are restricted to the triangular region

$$-\pi < \alpha < \pi, \qquad |\alpha| < \beta \leqslant \pi \quad .$$

Consequently, the boundary conditions should be applied on the perimeter of the triangle. Green did not do this. Instead he applied conditions on the perimeter of the rectangle

$$-\pi < \alpha < \pi,$$
$$0 < \beta \leqslant \pi.$$

The conditions were that:

(1) $u(\alpha)$ should be a bounded, periodic function of α with period 2π;

(2) $v(\beta)/\sin \beta$ should be bounded, periodic function of β with period π.

These conditions were derived by substituting the differential equation in the integral equation and then integrating by parts.

There is a difficulty here. A former Ph.D student at Adelaide, L.H.D. Reeves (1962), found that the periodic boundary conditions did not rigorously follow by the technique Green had used. The reason for this is easily understood. When the integral equation is written in terms of the bipolar coordinates, it has the form

$$\Psi(\alpha,\beta) = \lambda \int_0^\pi d\beta' \int_{-\beta'}^{\beta'} d\alpha' k(\alpha,\beta,\alpha',\beta')\Psi(\alpha',\beta'),$$

where the kernel k is an elementary function of α, β, α' and β', whose detailed form is of no importance for this argument. Because the range of integration over α' depends upon β', the integral equation, unlike the differential equation, cannot be separated. Reeves did not assert that Green's boundary conditions on the rectangle were incorrect, but only that they

did not follow from the integral equation by the steps Green outlined. Reeves attempted to find boundary conditions appropriate to the triangular region, but failed. His solution to the dilemma was to 'rotate' α to a purely imaginary quantity $i\bar{\alpha}, \bar{\alpha}$ real. Under this mapping, equivalent in fact to Wick's rotation of the contour of integration, the triangular region could be replaced by a rectangular region

$$-\infty < \bar{\alpha} < \infty, \quad 0 < \beta < \pi.$$

The variables in the integral equation could then be separated easily.

The point Reeves missed was that the periodic boundary conditions used by Green were <u>sufficient</u> to ensure that $\Phi(p)$ should be bounded near the origin and that $p^2 \Phi(p)$ should be bounded for all p. Also, these conditions on Φ were <u>sufficient</u> to ensure that Ψ, given by

$$4\lambda\Psi(p) = \Box\Phi(p),$$

was an acceptable solution of Wick's integral equation. It is not difficult to verify these assertions.

Green's separable solutions of Wick's equation are correct. So too are Wick's, provided it is possible to justify Wick's assumption that the contour of integration over k_0 can be rotated from the real to the imaginary axis. However, the answers obtained by Green and Wick seem to be inequivalent, so the validity of Wick's assumption concerning the asymptotic behaviour of the wave function is in doubt.

Green produced the equation (9), to be satisfied by bounded, periodic functions with period 2π. If

$$x = \cos^2\alpha$$

and
$$f(x) = u(\arccos x^{\frac{1}{2}}),$$

then $f(x)$ also satisfies equation (3). The condition that $u(\alpha)$ should be bounded requires that $u(\alpha)$ should vanish as rapidly as $(\cos^2\alpha-a)$ near $\cos^2\alpha = a$, for otherwise $u(\alpha)$ would contain a factor $\log(\cos^2\alpha-a)$. (Such a factor would confound the definition of periodicity!) For the function $f(x)$, this implies that

$$\lim_{x\to a} f(x)/(x-a) < \infty. \tag{10}$$

The second boundary condition on f also depends upon whether f is derived from an even or an odd function. If $u(\alpha)$ is odd,

$$u(0) = 0$$

and so,
$$f(1) = 0, \tag{11}$$

but if $u(\alpha)$ is even,

$$\frac{du}{d\alpha}(0) = 0$$

and so,
$$\lim_{x\to 1} (x-1)^{\frac{1}{2}} \frac{df}{dx} = 0. \tag{12}$$

The various boundary conditions imposed on solutions of (3) are summarised in the table.

	x = a	x = 1	x = ∞
'Even' solutions	$\lim_{x \to a} f(x)/(x-a) < \infty$	$\lim_{x \to 1}(x-1)^{\frac{1}{2}}\dfrac{df}{dx} = 0$	$\lim_{x \to \infty} f(x)/x^{\frac{1}{2}n} = 0$
'Odd' solutions	"	$f(1) = 0$	"

Wick's conditions.

Green's conditions.

I will show that any solution of (3) which satisfies the
boundary conditions at $x = 1$ and $x = \infty$ has an analytic
continuation which satisfies the boundary condition at $x = a$.
Conversely, if the boundary conditions at $x = a$ and $x = 1$
are satisfied, then the analytic continuation satisfies the
boundary condition at infinity. The work of Erdélyi (1944)
on the representation of Heun functions as convergent series
of hypergeometric functions can be used to solve this problem.
There is, however, a simpler solution which uses only
elementary properties of Heun functions.

Heun Functions.

Heun's equation is the second-order differential
equation with just four singular points, all of which are
regular. In its general form

$$\left[\frac{d^2}{dx^2} + \sum_{i=1}^{3}\left(\frac{1-\alpha_i-\beta_i}{x-x_i}\frac{d}{dx} + \frac{\alpha_i\beta_i}{(x-x_i)^2}\right) - \frac{x(\alpha_4\beta_4 - \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3) - \zeta}{(x-x_1)(x-x_2)(x-x_3)}\right]f(x) = 0,$$

with
$$\sum_{i=1}^{4}\alpha_i + \beta_i = 2,$$

the singularities lie at x_1, x_2, x_3 and $x_4 = \infty$, and the

parameters may be assumed to satisfy

$$\text{Re } (\alpha_i - \beta_i) \geqslant 0 \quad \text{for} \quad i = 1,2,3,4.$$

Solutions of Heun's equation are represented globally by

$$P \begin{bmatrix} x_1 & x_2 & x_3 & x_4 = \infty \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & & x \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & & \end{bmatrix}, \tag{13}$$

a simple extension of the notation invented by Riemann for solutions of the hypergeometric equation. The entries α_i and β_i below a given x_i are the exponents of the two independent branches which can be developed in series near x_i. For certain values of the parameter ζ, two branches of the P-function may become linearly dependent; such exceptional solutions are called Heun functions. In this section it is shown that for certain cases of Heun's equation, one of which is equation (3), there are Heun functions which adopt definite exponents at not just two singular points but instead at all four.

Suppose that it is possible to choose one exponent at each singular point so that the sum of the four chosen is an integer. If the difference of the exponents at any singular point is an integer, suppose in addition that the exponent with the larger real part has been chosen. For clarity of argument, I will assume that exponents $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ have been chosen and that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = m, \tag{14}$$

where m is an integer. Let $f(x)$ denote a Heun function

with exponents α_3 and α_4 at x_3 and x_4 respectively. Finally,
let l denote a simple closed contour which commences at some
point x_0 and encircles in a positive sense the singular
points x_1 and x_2.
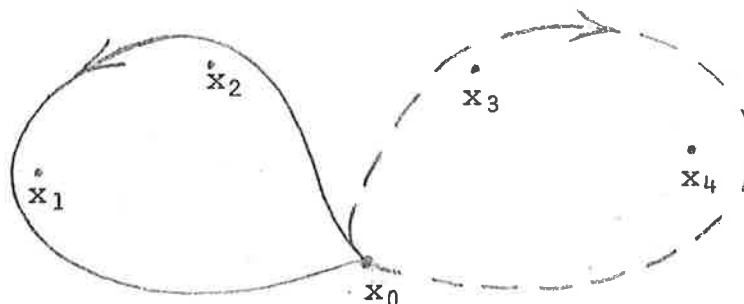
The analytic continuation of $f(x)$ from x_0 around l
produces the same effect as continuation of $f(x)$ around a
similar contour which encircles x_3 and x_4 in the opposite
sense. This is readily seen if the compactified plane is
projected onto the Riemann sphere. The latter continuation
replaces $f(x)$ by

$$\exp(-2\pi i(\alpha_3 + \alpha_4))f(x).$$

By the assumption (14),

$$\exp(-2\pi i(\alpha_3 + \alpha_4)) = \exp(2\pi i(\alpha_1 + \alpha_2)).$$

Thus, continuation of $f(x)$ from x_0 around l maps $f(x)$
into

$$\exp(2\pi i(\alpha_1 + \alpha_2))f(x).$$

A sufficient condition for this to be true is that $f(x)$
should have exponents α_1 and α_2 at x_1 and x_2 respectively.
When the difference of the exponents at one of these points,
say x_2, is an integer, this condition is also necessary.
The details of the proof of this assertion are messy so I

will not give them, but the result can easily be understood.
The branch of the P-function (13) with exponent β_2 at x_2
contains a term with a factor $\log(x-x_2)$ whenever $\alpha_2-\beta_2$ is
a non-negative integer. The assumption that $f(x)$ depends
upon this branch, and hence that $f(x)$ also contains a term
with a logarithmic factor, is impossible to reconcile with
the result that continuation of $f(x)$ around 1 merely
multiplies $f(x)$ by the factor $\exp(2\pi i(\alpha_1+\alpha_2))$. Thus,
when the condition (14) holds and either $(\alpha_1-\beta_1)$ or $(\alpha_2-\beta_2)$
is an integer, the Heun function with exponents α_3 and α_4
at x_3 and x_4 also adopts exponents α_1 and α_2 at x_1 and
x_2.

Application to Wick's Equation.

Equation (3) is a particular case of Heun's equation,
the solutions of which in Riemann's notation are

$$P \begin{bmatrix} 0 & a & 1 & \infty \\ \tfrac{1}{2} & 1 & \tfrac{1}{2} & \tfrac{1}{2}n & x \\ 0 & 0 & 0 & -\tfrac{1}{2}n \end{bmatrix} .$$

The boundary conditions at $x = 1$, to be satisfied respectively
by 'odd' and 'even' solutions of equation (3), are fulfilled
if and only if the 'odd' solutions have exponent $\tfrac{1}{2}$ and the
'even' solutions have exponent 0 at $x = 1$. The condition
at infinity imposed by Wick requires $f(x)$ to have exponent
$\tfrac{1}{2}n$ at infinity. Thus, the 'odd' and 'even' eigensolutions
of Wick's boundary value problem are Heun functions with
exponent $\tfrac{1}{2}n$ at infinity and exponents $\tfrac{1}{2}$ and 0 respectively
at $x = 1$. The condition at $x = a$ imposed by Green requires

f(x) to have exponent 1 at x=a, so the eigensolutions of Green's boundary value problem are also Heun functions.

Suppose f(x) is an 'odd' eigensolution of Wick's boundary value problem. If n is even, then

$$\tfrac{1}{2} + 1 + \tfrac{1}{2} + \tfrac{1}{2}n$$

is an integer and, since the difference of the exponents at x = a is an integer, it follows from the argument of the preceding section that f(x) has exponents

$$\tfrac{1}{2}, \ 1, \ \tfrac{1}{2}, \ \tfrac{1}{2}n$$

at   0, a, 1, ∞, respectively.

Thus, f(x) satisfies the boundary condition at x = a and so is an eigensolution of Green's boundary value problem. This argument can be reversed because the difference of the exponents at infinity is also an integer. Consequently, f(x) is an eigensolution of one operator if and only if it is an eigensolution of the other. The same conclusion applies in the other cases, corresponding to the other possible choices of parity for the eigensolution and n.

Conclusions.

I have shown that the analytic continuation of an eigen-solution of Wick's differential operator satisfies the differential equation and boundary conditions which together compromise Green's differential operator, and vice versa. The spectra of the operators are identical. Consequently, Wick's approach, which employs the contour-rotation, is equivalent to Green's which does not. Hence, the use of the

contour rotation is justified in this model.

The proof of the equivalence rests upon a fortuitous combination of exponents in Heun's equation, which permits the boundary conditions at one pair of singular points to be transferred to equivalent boundary conditions at another pair. It is most certainly not a general feature of Heun functions that they adopt a definite exponent at each of the four singular points. In fact, the argument can be reversed to imply that the contour rotation is only possible because Wick's equation leads to such a special case of Heun's equation. It could be argued that the unusual trick needed to justify the rotation is a reflection of the special nature of Wick's model and that the possibility of the Wick rotation only arises for solutions of Wick's equation. However, an alternative and broader view is that in any fundamental process the internal or relative momenta can be treated as Euclidean four-vectors or, equivalently, that the relative times may be taken to be imaginary. If this view is accepted, as is done for example in Euclidean field theory, then it is not at all surprising that the rotation can be justified.

In algebraic terms, the implication of this result is that non-unitary representations of Lie groups may have a role to play in quantum mechanics.

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