DIOPHANTINE INEQUALITIES IN MANY VARIABLES

by

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SUMMARY

The two parts of this thesis deal with two special kinds of systems of Diophantine inequalities involving real homogeneous forms in many variables. In both parts, the basic tool used is a modified version of the Hardy-Littlewood method as developed by Davenport and others (see, for example, Davenport [3], Birch [1] and Cook [2])and, in addition, special methods are needed to deal with the case when this method does not apply.

In Part I, the system considered involves R real diagonal forms of odd degree, and the final result is as follows:

Let

$$F_i(\underline{x}) = \sum_{j=1}^{n} \lambda_{ij} x_j^k, \quad (1 \leqslant i \leqslant R)$$

be R independent forms of degree k, in n variables where $k \geqslant 3$ is a fixed odd integer and the coefficients λ_{ij} are real numbers. Suppose $n \geqslant [9R^3 k^2 \log 3Rk]$. Then, for any $\varepsilon > 0$, the system

$$|F_i(\underline{x})| < \varepsilon, \quad (1 \leqslant i \leqslant R)$$

is solvable non-trivially in integers.

In order to prove the above result, a preliminary result on bounds for solutions of a corresponding system of R diagonal Diophantine equations $F_i = 0$ is needed. The solvability of such a system has been already proved by Davenport and Lewis [4]; by using their method with some refinements and a careful consideration of dependence on the coefficients, the required bounded result is proved. Then a similar result on a system of r equations and $R - r$ inequalities is proved and the main result is obtained by induction on $R - r$. The starting point

for this part of the thesis was an earlier version (covering odd degree only) of the work on the special case $R = 2$ in Pitman [6].

In Part II, the system considered involves a single real form of degree 2 or 3 of the shape

$$F(\underline{x}) = \lambda_1 F_1(\underline{x}) + \ldots + \lambda_R F_R(\underline{x}) ,$$

where $\lambda_1, \ldots, \lambda_R$ are real numbers independent over the rationals and $F_1(\underline{x}), \ldots, F_R(\underline{x})$ are either all integral quadratic forms ~~of~~ or all integral cubic forms. The results of Birch [1] are used to estimate the relevant trigonometric sum, which is the same as that for the system of equations $F_i = 0$, $(1 \leqslant i \leqslant R)$; however, there is considerable difficulty in dealing with the residual case when appropriate estimates are not available, which can occur when the system $F_i = 0$, $(1 \leqslant i \leqslant R)$ is singular.

For degree 2, the residual case is dealt with by a method somewhat similar to that for Part I, and the method in fact yields solutions which are bounded in terms of the coefficients; in particular, the result obtained implies that if the F_i's are all integral quadratic forms such that F as above is indefinite, then for any $\varepsilon > 0$, the inequality $|F(\underline{x})| < \varepsilon$ has a non-trivial integral solution provided that $n \geqslant 2R + 6$. (For $R \leqslant 7$, this gives an improvement for this particular type of form, on the results of Birch, Davenport and Ridout on the general quadratic inequality [5].)

For degree 3, the methods used to deal with the residual case involve extensions of the ideas and results of Davenport [3] on cubic forms. It is shown that if the F_i's are all integral cubic forms then for any $\varepsilon > 0$ the inequality $|F(\underline{x})| < \varepsilon$ has a non-trivial integral solution provided that

$$n \geqslant 4R(R + 3).$$

(Results previously known [7] for the general cubic inequality involve an extremely large number of variables.)

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STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university and, to the best of my knowledge, contains no material previously published or written by another person, except when due reference is made in the thesis.

T. Ponnudurai.

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DEDICATION

I dedicate this thesis to the memory of my teacher and colleague

Professor P. Kanagasabapathy.

GENERAL INTRODUCTION

Progress made so far on Diophantine problems on forms in many variables mainly concerns systems of Diophantine equations

(1) $$F_i(\underline{x}) = 0, \quad (1 \leqslant i \leqslant R),$$

where $F_i(\underline{x})$, $(1 \leqslant i \leqslant R)$ are homogeneous forms of the same degree with integral coefficients; and of Diophantine inequalities

(2) $$|F_i(\underline{x})| < \varepsilon, \quad (1 \leqslant i \leqslant R),$$

where $F_i(\underline{x})$, $(1 \leqslant i \leqslant R)$ are real homogeneous forms of the same degree. By a solution, we mean a non-trivial integral solution (that is an integer solution other than $\underline{0}$). As we shall see later, the two kinds of problems are closely inter-connected, and progress in determining conditions of solvability for either type of problem has often stemmed from detailed results about systems of special kinds. In this thesis, I shall be concerned with two special kinds of systems.

In Part I, I shall study a system of R simultaneous inequalities (2), in the case when the R forms $F_i(\underline{x})$, $(1 \leqslant i \leqslant R)$ are all real diagonal forms of odd degree $k \geqslant 3$, so that in the terminology of Davenport and Lewis [21], the system is of additive type. In this part, I shall also be concerned with results on the corresponding problem of a system of R simultaneous diagonal equations to the extent that such results are relevant to the main problem.

In Part II, I shall study a single inequality $|F(\underline{x})| < \varepsilon$, in the case where $F(\underline{x})$ is of the type

$$F(\underline{x}) = \lambda_1 F_1(\underline{x}) + \ldots + \lambda_R F_R(\underline{x}),$$

where the $F_i(\underline{x})$, $(1 \leq i \leq R)$ are either all integral quadratic forms or all integral cubic forms and λ_i, $(1 \leq i \leq R)$ are real numbers independent over the rationals.

Both parts of this thesis depend on the use of a modified form of the Hardy-Littlewood method and of special methods when this approach fails. Although the details of these special methods used differ considerably in the two parts (because of the nature of the forms), the general spirit is similar.

In Chapter 1 of Part I, I shall give an introduction to diagonal Diophantine equations and inequalities and to the analytic methods used; this chapter contains introductory material relevant to both parts of the thesis and concludes with a more detailed introduction to the results and methods of Part I. In Chapter 1 of Part II, I shall give an introduction to Diophantine equations and inequalities involving general forms (rather than diagonal forms) and shall also give a detailed introduction to Part II, which does not depend on any chapters of Part I other than chapters 1 and 2. Except for this and the common bibliography, both the parts are self-contained.

There are four chapters in Part I and three in Part II and each chapter is divided into sections numbered consecutively throughout the chapter. In each section, the lemmas and definitions are numbered separately in sequence, where the numbering includes the section number and the chapter number in Part I and the section number but not the chapter number in Part II. For example, Lemma 2.3.5 (or Definition 2.3.5) is followed by Lemma 2.3.6 (or Definition 2.3.6) in §3 of Chapter 2 in Part I, and in Part II, Lemma 3.4 (or Definition 3.4) in §3 of any chapter is followed by Lemma 3.5 (or Definition 3.5) in §3 of the same chapter. The displayed formulae and equations which are required for later reference are numbered together in another sequence, in the same manner as that for the lemmas. The numbers in square brackets refer to the bibliography at the end of the thesis.

The notation displayed below is standard for both the parts.

TABLE OF NOTATIONS FOR PART I AND PART II

Expression	Meaning
$\mathbb{R}$	set of real numbers
$\mathbb{R}^n$	set of real n-tuples
$\mathbb{Z}$	set of integers
$b \mid a$	b divides a
$b \nmid a$	b does not divide a
$(a_1, \ldots, a_n)$	greatest common divisor of $a_1, \ldots, a_n$
$[x]$	integer part of x
$\lVert y \rVert$	distance from real y to the nearest integer
$\mu(E)$	measure of a set E
$e(y)$	$e^{2\pi i y}$
$\lvert \underline{x} \rvert$	$\max_{1 \leqslant i \leqslant n} \lvert x_i \rvert$, for $\underline{x} = (x_1, \ldots, x_n)$
$\lvert M \rvert$	$\max_{i,j} \lvert m_{ij} \rvert$, for matrix $M = (m_{ij})$
$\lvert F \rvert$	max|coefficients of $F(\underline{x})$|, for polynomial $F(\underline{x})$
P	large positive integer
$\delta, \tau, \omega, \eta$	small positive numbers
ε	arbitrarily small positive number
$A \ll B$, $A = O(B)$	$\lvert A \rvert \leqslant c\,B$, for some constant c, independent of P and of the appropriate coefficients
$A \gg B$	$B \ll A$, provided A, B and the implied constant c are all positive

PART I

SIMULTANEOUS DIAGONAL INEQUALITIES OF ODD DEGREE

CHAPTER 1

INTRODUCTION TO PART I

1. INTRODUCTION

In this part of the thesis, we consider a system of R ($\geqslant 1$) Diophantine inequalities

$$|F_i(\underline{x})| < \varepsilon\,, \quad (1 \leqslant i \leqslant R), \tag{1.1.1}$$

where the $F_i(\underline{x})$'s are diagonal forms in n variables $\underline{x} = (x_1, \ldots, x_n)$

$$F_i(\underline{x}) = \sum_{j=1}^{n} \lambda_{ij}\, x_j^k \quad , \quad (1 \leqslant i \leqslant R), \tag{1.1.2}$$

λ_{ij} $(1 \leqslant i \leqslant R,\ 1 \leqslant j \leqslant n)$ are all real numbers and the degree k is a fixed *odd* integer $\geqslant 3$. The aim will be to find a condition on n which will ensure that such a system is solvable for arbitrarily small positive ε. In fact we will ultimately show (in Theorem 3) that the condition

$$n \geqslant [9R^3 k^2 \log 3Rk]$$

is sufficient.

As an introduction to this problem, I shall start by giving an account of previous work on diagonal equations and inequalities. In §2, I shall discuss single diagonal equations and in §3, single diagonal inequalities. Then I shall discuss simultaneous equations and inequalities in §4 and §5.

2. SINGLE DIAGONAL EQUATIONS

There are two kinds of results on a single diagonal equation, namely, on the existence of integral solutions and on the existence of (integral) solutions bounded in terms of the coefficients of the equation.

SOLUTIONS

We shall start with quadratic equations. Meyer in 1884, proved [26] the following result:

RESULT 1. If $n \geq 5$ and the non-zero integers $\lambda_1, \ldots, \lambda_n$ are not all of the same sign, then there exists a solution $\underline{x}$ of

$$\lambda_1 x_1^2 + \ldots + \lambda_n x_n^2 = 0.$$

The restriction $n \geq 5$ is the best possible. Extensions of this result to higher degree k, say, diagonal (or additive) equations were done in stages, and, in particular the result 2 stated below was proved by Davenport [17], (in Theorem 7 and Theorem 9).

RESULT 2. Let

$$\lambda_1 x_1^k + \ldots + \lambda_n x_n^k = 0, \tag{1.2.1}$$

where $\lambda_1, \ldots, \lambda_n$ are non-zero integers, not all of the same sign if k is even and

$$n \geq \begin{cases} k^2(2k-1) + 1 & , \text{ if } 1 \leq k \leq 11 \\ k^2(2k-1) + 3(2k \log 3k+1) + 1, & \text{ if } k \geq 12 \end{cases}.$$

Then the equation (1.2.1) has a non-trivial integral solution.

This result was later improved [19] by Davenport and Lewis to $n \geq k^2+1$, for $k \leq 6$ and $k \geq 18$, also to $n = 8$ for the particular case $k = 3$.

SOLUTIONS WITH BOUNDS

I shall always mean, by solutions with bounds or sometimes by "bounded solutions", solutions that are bounded by an explicit function of the coefficients. This sort of result was obtained for diagonal forms, only after Cassels's result [5] in 1955, on bounds for solutions of indefinite quadratic Diophantine equations.

RESULT 3. Let $k \geqslant 2$ be an integer and let n be the integer defined by

(1.2.2)
$$n \begin{cases} = 2^k + 1, & \text{if } 2 \leqslant k \leqslant 11 \\ \geqslant 2k^2(2\log k + \log\log k + 3) + 1, & \text{if } k \geqslant 12. \end{cases}$$

Then for any $\theta > 0$, there exists a constant C_θ, depending only on θ and k, and the following property. If $\lambda_1, \ldots, \lambda_n$ are non-zero integers which are not of the same sign if k is even, then the Diophantine equation (1.2.1) has a solution in non-zero integers such that

(1.2.3)
$$|\lambda_1 x_1^k| + \ldots + |\lambda_n x_n^k| < C_\theta |\lambda_1 \lambda_2 \ldots \lambda_n|^{k\psi+\theta},$$

where

$$\psi = \begin{cases} \tfrac{1}{2}, & \text{if } 2 \leqslant k \leqslant 11 \\ 1, & \text{if } k \geqslant 12. \end{cases}$$

The particular case $k = 2$ was proved by Birch and Davenport [2] and that for $k = 3$ by Pitman and Ridout [34] . The result 3 by Pitman [32] is an extension of these. Recently Schmidt [38] has obtained improved bounds for a very large number of variables:

RESULT 4. Let λ_i, μ_i $(1 \leqslant i \leqslant n)$ be positive integers. Then the equation

$$\sum_{i=1}^{n} \lambda_i x_i^k - \sum_{i=1}^{n} \mu_i y_i^k = 0$$

has a non-trivial, non-negative solution x_i , y_i $(1 \leqslant i \leqslant n)$, for large values of n (as a function of k), such that

$$\max_{1 \leqslant i \leqslant n} (x_i , y_i) < m^{1/k+\epsilon} ,$$

where

$$m = \max_{1 \leqslant i \leqslant n} (\lambda_i , \mu_i) .$$

METHOD.

The basic tool used to prove these results was an adaptation of the Hardy-Littlewood circle method. By this method, the number $N(P)$, say, of integral solutions $\underline{x}$ of (1.2.1) such that $1 \leqslant x_i \leqslant P$, $(1 \leqslant i \leqslant n)$ is

$$N(P) = \int_0^1 \prod_{i=1}^{n} \sum_{x} e(\lambda_i \alpha x_i^k).d\alpha , \tag{1.2.4}$$

where x_i runs over all integers from 1 to P . (For the meaning of $e(y)$, see the table of notations in page 3). The results 1 and 2 were proved by showing, using various techniques, that the integral in (1.2.4) is positive for very large values of P . Whereas to prove the results 3 and 4, a value of P in terms of the coefficients $|\lambda_i|$ was obtained such that the above integral is positive for this value of P . This is difficult and more work is involved in comparison to problems on solutions.

For these types of problems, the solvability of (1.2.1) in all p-adic fields is an essential preliminary, and so one needs to solve the corresponding congruences as well.

3. SINGLE DIAGONAL INEQUALITIES

Analogous results have been obtained for inequalities of the type

(1.3.1) $$|\lambda_1 x_1^k + \ldots + \lambda_n x_n^k| < \varepsilon ,$$

for arbitrarily small $\varepsilon > 0$, where $\lambda_1, \ldots, \lambda_n$ are real numbers.

SOLUTIONS.

Davenport and Heilbronn [18] in 1946 proved the existence of solutions to (1.3.1):

RESULT 5. If $\lambda_1, \ldots, \lambda_n$ are not all of the same sign if k is even, and are not all in rational ratios and $n \geqslant 2^k+1$, then the inequality (1.3.1) is solvable in integers.

Later, the condition on n was improved by Davenport and Roth [23] to $n \geqslant ck \log k$ for $k \geqslant 12$ and by Danicic [11] to $n \geqslant 14$ for $k = 4$. We easily see that for solutions of the above type, it is enough to get a corresponding result for that with $\varepsilon = 1$. That is for

(1.3.2) $$|\lambda_1 x_1^k + \ldots + \lambda_n x_n^k| < 1.$$

SOLUTIONS WITH BOUNDS.

By using result 3 above, the following result has been proved by Birch and Davenport [3] for $k = 2$, Pitman and Ridout [34] for $k = 3$ and Pitman [31] for $k \geqslant 3$.

RESULT 6. Let k be an integer $\geqslant 2$ and let n be the integer defined by (1.2.2). Then for any $\theta > 0$, there exists a constant K_θ , depending only on θ and k such that if $\lambda_1, \ldots, \lambda_n$ are real numbers, $|\lambda_i| \geqslant 1$ for all i , and not all of the same sign if k is even, then (1.3.2) has a non-trivial solution $\underline{x}$ satisfying (1.2.3) with K_θ instead of C_θ.

Similarly using result 4, the following result was proved by Schlickewei [36] in 1978:

RESULT 7. Let $k \geq 2$ be an integer and $\varepsilon > 0$. Suppose $s \geq c_2(k,\varepsilon)$ and $\lambda_1, \dots, \lambda_s$, $\mu_1, \dots, \mu_s$ are real, positive numbers and $M = \max_{1 \leq i \leq s} \{|\lambda_i|, |\mu_i|\}$. Then for any $N \geq 1$,

$$(1.3.3) \qquad |\lambda_1 x_1^k + \dots + \lambda_s x_s^k - \mu_1 y_1^k - \dots - \mu_s y_s^k| < MN^{-k+\epsilon}$$

has a non-trivial solution in non-negative integers $x_1, \dots, x_s$, $y_1, \dots, y_s$ such that

$$\max_{1 \leq i \leq s} \{x_i , y_i\} \leq N .$$

For $N = M^{1/k+\epsilon}$, the inequality (1.3.3) takes the form (1.3.2) and so the result is the existence of small solutions and is an improvement of the result 6, for this particular case.

METHOD.

The main approach for dealing with problems on inequalities is again a modification of the Hardy-Littlewood circle method. The basic idea is due to Davenport and Heilbronn; it was subsequently refined by Davenport [13] , who proved the lemma below, which give an improved kernel K in place of $K(\alpha) = \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2$ which was used by Davenport and Heilbronn [18] .

Lemma 1.3.1. For any given positive integer r , there exists a real valued function $K(\alpha)$, say, of a real variable α such that

$$(1.3.4) \qquad |K(\alpha)| < C(r) \min\{1, \alpha^{-r-1}\} ,$$

for $\alpha > 0$, and $C(r)$ a constant depending only on r , satisfying the following condition: If

(1.3.5) $$f(\eta) = \int_{-\infty}^{\infty} e(\eta\alpha)K(\alpha)d\alpha ,$$

then

$$0 \leqslant f(\eta) \leqslant 1, \text{ for all real } \eta ,$$
$$f(\eta) = 0, \text{ for } |\eta| \geqslant 1,$$

and

$$f(\eta) = 1, \text{ for } |\eta| \leqslant \frac{1}{3} .$$

From this lemma, we easily see that if $N_1(P)$ is the number of solutions $\underline{x}$ of (1.3.1) such that $1 \leqslant x_i \leqslant P$, then

(1.3.6) $$N_1(P) \geqslant \int_{-\infty}^{\infty} S(\alpha)K(\alpha)d\alpha ,$$

where

$$S(\alpha) = \prod_{i=1}^{n} \sum_{x=1}^{P} e(\lambda_i x^k \alpha) = \sum_{x_1,\ldots,x_n} e((\lambda_1 x_1^k + \ldots + \lambda_n x_n^k)\alpha) ,$$

$x_1,\ldots,x_n$ run over all the integers such that $1 \leqslant x_i \leqslant P$, $(1 \leqslant i \leqslant n)$. Therefore for the existence of solutions of (1.3.1), it is enough to prove that there are values of P for which the integral (1.3.6) is positive. And to prove results on "bounded solutions", as in the case of equations, the values of such P's must be found in terms of the absolute values of the coefficients.

Problems associated with inequalities do not have associated "singular series" arising from corresponding congruence problem and thus avoid one complication which have to be faced with equations. However, they involve a different difficulty, in that with inequalities, a situation can arise where adequate estimates are not available and so the above method does not apply. Davenport and Heilbronn [18] avoided this by assuming λ_i/λ_j irrational and choosing P as the square of the denominator of a convergent to the continued fraction of λ_i/λ_j and using properties of continued fractions, but this approach does not yield bounds.

Though equations are a particular case of the corresponding inequalities (since $0 < \varepsilon$ is always true), it is found in many results, especially those involving bounds, that a result on equations (usually a "bounded result") is a vital tool to deal with the corresponding inequalities. As for example, result 3 was an essential preliminary to the proof of result 6.

4. A SYSTEM OF SEVERAL EQUATIONS

We now consider the extensions of the results of §2 to systems of several simultaneous equations, all of additive type, of the same degree.

SOLUTIONS.

RESULT 8. Let

$$\left\{\begin{array}{l} F(\underline{x}) = \lambda_1 x_1^k + \dots + \lambda_n x_n^k , \\ G(\underline{x}) = \mu_1 x_1^k + \dots + \mu_n x_n^k , \end{array}\right. \tag{1.4.1}$$

where λ_i , μ_i $(1 \leqslant i \leqslant n)$ be all integers, k an odd positive integer. Then for $n \geqslant 2^{k+1}+1$, the equations $F(\underline{x}) = 0$, $G(\underline{x}) = 0$ are simultaneously solvable in integers.

This is due to Cook [7] in 1972, and is an improvement of the result [20] by Davenport and Lewis for $k = 3$ in which $n \geqslant 18 = 2^4+2$ was required. Recently, in 1977, this condition was improved to $n \geqslant 16 = 2^4$, by Vaughan [40] . Also it was proved by Cook [6] that the above equations are solvable for $k = 2$, if $n \geqslant 9$ and all non-trivial linear combinations $a\,F + b\,G$ are indefinite and explicitly contain at least 5 variables.

Cook [8] also proved the solvability of a system of 3 additive quadratic equations, namely the following result:

RESULT 9. Let

$$F_i(\underline{x}) = \sum_{j=1}^{n} a_{ij}\, x_j^2, \quad (i = 1,2,3),$$

where the coefficients a_{ij} are all integers. Suppose the forms are such that all real non-zero linear combinations of F_1, F_2, F_3 contain at least 11 variables and there exist non-singular solutions of

$$F_i(\underline{x}) = 0, \quad (1 \leqslant i \leqslant 3),$$

in real and 2-adic fields. Then the above equations have a non-trivial integral solution.

Davenport and Lewis [21], in 1969, considered a system of R additive equations of the same degree k, for any integer $k \geqslant 3$ and proved the following result:

RESULT 10. Let

$$F_i(\underline{x}) = \sum_{j=1}^{n} a_{ij}\, x_j^k, \quad (1 \leqslant i \leqslant R), \quad k \geqslant 3$$

be R diagonal (additive) forms, where the coefficients a_{ij} are integers. Then the system of equations

$$F_i(\underline{x}) = 0, \quad (1 \leqslant i \leqslant R)$$

has a non-trivial integral solution if

(i) k is odd and $n \geqslant [9R^2 k \log 3Rk]$,

or (ii) k is even, the above equations have a real non-singular solution, every set of S independent integral (not all zero) linear combinations of $F_1, \ldots, F_R$ contains at least $[48SRk^3 \log 3Rk^2]$ variables, where $S = 1, \ldots, R$. (Then $n \geqslant [48R^2k^3 \log 3Rk^2]$ for k even.)

For k odd and < 12, the result 8 is an improvement of result 10 with $R = 2$.

SOLUTIONS WITH BOUNDS

Toliver [39] has obtained bounds for solutions of (1.4.1), in his Ph.D. thesis and the result is as follows:

RESULT 11. Let $F(\underline{x})$, $G(\underline{x})$ be as in (1.4.1) and k an odd positive integer and $n = 2^{k+1}+1$. Then there exists a constant $B_1 = B_1(k)$ such that for any $\theta > 0$, the Diophantine equations $F = G = 0$ are solvable with

$$0 < |\underline{x}| \ll \{\prod_{i=1}^{n} \max(1, |\lambda_i|, |\mu_i|)\}^{B_1+\theta} ,$$

where the constant implied by $\ll$ depends only on k and θ.

Explicit values of B_1 were obtained in his result, for all odd k, and, in particular,

$$B_1 = k\left\{\tfrac{1}{2} + \frac{k(2k-1)}{n-2k(2k-1)}\right\}, \quad \text{for odd } k \geqslant 5 .$$

A similar result was obtained by Lloyd [25], for $k = 2$, $n \geqslant 9$.

METHOD.

The method uses an R-dimensional (where R is the number of equations in the system) adaptation of the circle method and thus the integral for $N(P)$ (in (1.2.4)) becomes an R-fold integral over the unit cube (in R-dimensions).

5. A SYSTEM OF INEQUALITIES

We now consider a system of R inequalities (1.1.1), namely,

$$|F_i(\underline{x})| < \varepsilon, \quad (1 \leqslant i \leqslant R),$$

for arbitrarily small $\varepsilon > 0$ and $R \geqslant 2$, where the $F_i(\underline{x})$'s are real diagonal forms of the type (1.1.2), (of degree k).

For the case $k = 2$, $R = 2$, Cook [9] investigated this problem by applying the 2-dimensional extension of the modification of the Hardy-Littlewood method (as described in §3). Unfortunately, he found that the case when the analytic argument might fail cannot be eliminated by the approach of Davenport and Heilbronn [18]. He avoided this difficulty by imposing extra conditions on the forms; in particular, these conditions are satisfied if all the coefficients of the system are algebraic. He proved the following result:

RESULT 12. Let $F_1(\underline{x})$, $F_2(\underline{x})$ be diagonal quadratic forms, (as in (1.1.2)), having real algebraic coefficients, in $9(= 2^3+1)$ variables. Suppose that every non-trivial linear combination $a\,F + b\,G$ is indefinite with at least 5 non-zero coefficients; and that not all the ternary linear forms

$$L_{ijk}(u,v,w) = \begin{vmatrix} u & v & w \\ \lambda_{1i} & \lambda_{1j} & \lambda_{1k} \\ \lambda_{2i} & \lambda_{2j} & \lambda_{2k} \end{vmatrix}$$

associated with F_1, F_2 have coefficients which are linearly dependent over the rationals. Then for any $\varepsilon > 0$, the inequalities

$$|F_1(\underline{x})| < \varepsilon, \quad |F_2(\underline{x})| < \varepsilon$$

are solvable.

In 1975 (in an earlier version of [33], covering odd degree only), Dr. Pitman investigated the case when $R = 2$, k is odd and $n \geqslant 2^{k+1}+1$, by using Cook's approach together with an approximation argument which works when the analytic method fails. She showed that if a suitable bound result holds for pairs of diagonal equations $F_1(\underline{x}) = F_2(\underline{x}) = 0$ of odd degree, in at least $2^{k+1}+1$ variables, then the solvability of $|F_1(\underline{x})| < \varepsilon$, $|F_2(\underline{x})| < \varepsilon$, for any $\varepsilon > 0$, can be proved for pairs of real forms of the same type. The argument involved an intermediate result on bounds for solutions of a "mixed system" of the type $|F_1| = 0$, $|F_2| < 1$, where F_1 is integral and F_2 is real. The result 11 of Toliver provided a bound result of the required kind, for pairs of diagonal equations, and so led to the following result.

RESULT 13. Suppose $F_1(\underline{x})$ and $F_2(\underline{x})$ are real diagonal forms (of the type (1.1.2)), of odd degree $k \geqslant 3$ and that $n \geqslant 2^{k+1}+1$. Then for any $\varepsilon > 0$, the system of inequalities

$$|F_1(\underline{x})| < \varepsilon, \quad |F_2(\underline{x})| < \varepsilon$$

is solvable.

The main problem of this part of the thesis (which was suggested to me by Dr. Pitman) is to obtain a corresponding result for R diagonal inequalities of odd degree, where $R > 2$. The approach used is an extension of the argument for $R = 2$, but there are additional complications, particularly in connection with the so-called "singular series". Attention has been restricted to odd k, because it appeared from the work of Lloyd [25] and Toliver [39] that for even k there would be further serious difficulties, particularly in connection with the so-called "singular integral". However, the work on the case k even and $R = 2$ in Pitman [33] (which was done only very recently, after the completion of the work for this thesis) now suggests that further extension may be possible.

6. PLAN OF PART I

In Chapter 2, I shall give all the preliminary results needed for the rest of the chapters of this part. As in the argument outlined above for $R = 2$, we need a preliminary result on bounds for solutions of the corresponding system of R equations, and this will be done in Chapter 3, which relies heavily on the methods used by Davenport and Lewis [21] to prove result 10. Similarly, we also need an intermediate result of the same kind for a mixed system of r equations $F_i(\underline{x}) = 0$, $(1 \leqslant i \leqslant r)$ and $R - r$ inequalities $|F_i(\underline{x})| < 1$, $(r+1 \leqslant i \leqslant R)$. In Chapter 4, I shall obtain the required intermediate result, by induction on $R - r$, and then use the case $r = 0$ to derive the main result (Theorem 4). In fact it is possible to combine the material in Chapters 3 and 4 in a single argument, but the present division of work is a natural one which enables us to separate off some difficulties which are handled in more detail, in Chapter 3.

Throughout this part of the thesis, the unspecified constants implied by the notations $\gg$, $\ll$, O are independent of the coefficients of the forms; they will ultimately depend only on n, R and k.

CHAPTER 2

PRELIMINARY LEMMAS

1. INTRODUCTION

To simplify the discussion in this part of the thesis, it will be convenient if we first give the preliminary results needed for our proofs. When these results are well known, I shall not give their proofs, but the references where their proofs can be found.

In §2 of this chapter, I shall give the basic lemmas needed for dealing both with a system of diagonal equations and with a "mixed system" of equations and inequalities, all diagonal. In §3, the results that are useful for our theorem on a system of equations are given, and in §4, I shall give all the preliminaries needed on the so-called "singular series" for equations.

We shall use the abbreviation D-L in Part I for [21] of Davenport and Lewis.

2. BASIC LEMMAS

We shall begin this section with the following definition:

Definition 2.2.1. Let P be a (large) positive integer and $B(P)$, the interval $[-P,-1] \cup [1,P]$. Then we define for any real number Λ , and integers a,q such that $q > 0$ and $(a,q) = 1$, the following trigonometric sums and integrals:

$$T(\Lambda) = \sum_{x \in B(P)} e(\Lambda x^k), \tag{2.2.1}$$

$$I(\Lambda) = \int_{B(P)} e(\Lambda\xi^k)\,d\xi, \tag{2.2.2}$$

and

$$S(a,q) = \sum_{x=1}^{q} e\left(\frac{ax^k}{q}\right) . \tag{2.2.3}$$

<u>Lemma 2.2.1</u>. Suppose Λ is any real number and $T(\Lambda)$ as in (2.2.1), $0 < \eta < 1$ and $\varepsilon > 0$. Then *either*

(i) there exist integers a,q such that

$$\begin{cases} \Lambda = \dfrac{a}{q} + \gamma , & (a,q) = 1 \\ 0 < q \leqslant P^{\eta} , & |\gamma| < q^{-1}\ P^{-k+\eta} \end{cases} \tag{2.2.4}$$

or (ii) we have

$$|T(\Lambda)| \ll P^{1-\rho+\varepsilon} , \tag{2.2.5}$$

where (2.2.6) $\rho = \eta/2^{k-1}$.

$$\rho = \begin{cases} \dfrac{\eta}{2^{k-1}} , & \text{for } k \leqslant 11 \\ \dfrac{\eta}{2k^2(2\log k + \log\log k + 3)} , & \text{for } k \geqslant 12 . \end{cases} \tag{2.2.6}$$

Moreover, when the alternative (i) holds, we have

$$|T(\Lambda)| \ll q^{-1/k} \min\{P, P^{-k+1}\ |\gamma|^{-1}\} . \tag{2.2.7}$$

<u>Proof</u>. By Dirichlet's theorem, since Λ is a real number it has a rational approximation $\frac{a}{q}$ such that

$$\Lambda = \frac{a}{q} + \gamma, \quad (a,q) = 1$$

and

$$0 < q \leqslant P^{k-\eta}, \quad |\gamma| < q^{-1}\ P^{-k+\eta} .$$

When $q \leqslant P^{\eta}$, we get the alternative (i) and when $q > P^{\eta}$, we get (2.2.5) by Weyl's inequality (Lemma 1 of [17]) for $k \leqslant 11$, and by Vinogradov's inequality (Lemma 16 of D-L or see Lemma 9 of Davenport and Lewis [19]) for $k \geqslant 12$. The last part of the lemma is Lemma 18 of D-L.

Lemma 2.2.2. Suppose N, Λ are given positive integers and η, Λ are real numbers such that

(2.2.8) $$0 < \eta < 1/4, \quad P > 4N^2k^2\Lambda^4$$

and that there are integers a, q such that

$$\Lambda = \frac{a}{q} + \gamma, \quad (a,q) = 1$$

(2.2.9) $$0 < q \leqslant \Lambda P^{\eta}, \quad |\gamma| < N\Lambda P^{-k+\eta} .$$

Then for $T(\Lambda)$, $I(\gamma)$, $S(a,q)$ as in Definition 2.2.1, we have

(2.2.10) $$T(\Lambda) = q^{-1} S(a,q)I(\gamma) + O(q) ,$$

where the constant implied by O is an absolute constant (which may depend on n).

Proof. Follows steps similar to the proof of Lemma 17 of Davenport [17] and is an application of Van der Corput's Lemma.

Lemma 2.2.3. Suppose that a, q are integers such that

$$\Lambda = \frac{a}{q} + \gamma, \quad (a,q) = 1$$

$$0 < q \leqslant P^{\delta}, \quad |\gamma| < q^{-1} P^{-k+\delta} ,$$

where δ is a small positive constant. Then

(2.2.11) $$T(\Lambda) = q^{-1} S(a,q)I(\gamma) + O(P^{2\delta}),$$

and if $0 < \delta < 1/3$ then

$$T(\Lambda) \ll q^{-1/k} \min(P, |\gamma|^{-1/k}).$$

Proof. This is Lemma 26 of D-L. For its proof, see Lemma 4 of Davenport [17]. The last part follows from (2.2.11) and the next two lemmas.

Lemma 2.2.4. Let a,q be integers such that $(a,q) = 1$ and let $S(a,q)$ be as in (2.2.3). Then

$$S(a,q) \ll q^{1-1/k}, \tag{2.2.12}$$

where the implied constant depends only on k.

Proof. This is Lemma 15 of Davenport [17] and the dependence of the implied constant is checked.

Lemma 2.2.5. Let $I(\beta)$ be as in (2.2.2) for all real β. Then we have

$$|I(\beta)| \ll \min\{P, |\beta|^{-1/k}\}, \tag{2.2.13}$$

where the implied constant depends only on k.

Proof. See page 69 of Davenport [17].

Lemma 2.2.6. Let α be any real number and r be an integer such that $r \geqslant s_0$, where

$$s_0 = [9k^2R\log 3Rk] - 2, \tag{2.2.14}$$

and let $T(\alpha)$ be as in (2.2.1). Then we have

$$\int_0^1 |T(\alpha)|^r \, d\alpha \ll P^{r-k+\varepsilon}. \tag{2.2.15}$$

Proof. We prove this by using Hua's inequality and Vinogradov's inequality. We note that for $k \leqslant 11$, $2^k < s_0 \leqslant r$, where s_0 is as in (2.2.14). Therefore for $k \leqslant 11$, we use the trivial estimate P for $r - 2^k$, $T(\alpha)$'s and then we apply Hua's inequality. We then get

$$\int_0^1 |T(\alpha)|^r \, d\alpha << P^{r-2^k} \int_0^1 |T(\alpha)|^{2^k} \, d\alpha$$

$$<< P^{r-2^k+2^k-k+\varepsilon}$$

which gives (2.2.15) for $k \leqslant 11$.

For $k \geqslant 12$, and even $t \geqslant 2k^2(2 \log k + \log\log k + 2.5) - 4$, by Lemma 7.13 [24] of Hua, we have that

(2.2.16) $$\int_0^1 |T(\alpha)|^t \, d\alpha << P^{t-k}.$$

for the smallest even t as above

We also note that $s_0 > t$ for $k \geqslant 12$ and so $r > t$. Therefore as before by using the trivial estimate for $r - t$, $T(\alpha)$'s and (2.2.16) for the remaining $T(\alpha)$'s, we have

$$\int_0^1 |T(\alpha)|^r \, d\alpha = \int_0^1 |T(\alpha)|^{r-t+t} \, d\alpha$$

$$<< P^{r-t+t-k} << P^{r-k} .$$

Hence we have the result (2.2.15).

Lemma 2.2.7. Let $\alpha_1, \ldots, \alpha_R$ be any real numbers and $\Lambda_1, \ldots, \Lambda_R$ be independent linear forms in $\alpha_1, \ldots, \alpha_R$ such that

(2.2.17) $$\Lambda_j = \sum_{i=1}^{R} \lambda_{ij} \alpha_i , \quad (1 \leqslant j \leqslant R)$$

where (λ_{ij}) is a non-singular real matrix. Let U be the R-dimensional unit cube $[0,1] \times \ldots \times [0,1]$ and $T(\Lambda_j)$, $(1 \leqslant j \leqslant R)$ be as in (2.2.1). Then for $2r \geqslant s_0$, s_0 as in (2.2.14) and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_R)$, we have

(2.2.18) $$\int_U \prod_{j=1}^{R} |T(\Lambda_j)|^{2r} \, d\underline{\alpha} << P^{R(2r-k)} .$$

Proof. This is the R-dimensional analog of Lemma 2.2.6. Our proof follows the lines of proof of the corresponding lemma of Toliver [39].

We write the left hand side of (2.2.18) as

$$\int_U \Big| \prod_{j=1}^{R} T(\Lambda_j)T(-\Lambda_j)\Big|^r \, d\underline{\alpha} = \sum_{\underline{u}_i, \underline{v}_i} \int_U e\Big(\sum_{i=1}^{R} \cdot \sum_{j=1}^{R} \alpha_j \cdot \lambda_{ji} \cdot F(\underline{u}_i, \underline{v}_i)\Big) d\underline{\alpha},$$

by (2.2.1) and (2.2.17), where $F(\underline{u}_i, \underline{v}_i) = \sum_{t=1}^{r} (u_{it}^k - v_{it}^k)$. And this is

$$= \sum_{\underline{u}_i, \underline{v}_i} \int_U \prod_{j=1}^{R} e\Big(\alpha_j \sum \lambda_{ji} \; F(\underline{u}_i, \underline{v}_i)\Big) d\underline{\alpha}$$

$$= \sum_{\underline{u}_i, \underline{v}_i} \prod_{j=1}^{R} \int_0^1 e\Big(\alpha_j \sum \lambda_{ji} \; F(\underline{u}_i, \underline{v}_i)\Big) d\alpha_j \, .$$

This in fact is the number of integral solutions in all variables of the system of R equations

$$\sum_{i=1}^{R} \lambda_{ji} \; F(\underline{u}_i, \underline{v}_i) = 0, \quad (1 \leqslant j \leqslant R),$$

where $\underline{u}_i = (u_{i1}, \ldots, u_{ir})$, $\underline{v}_i = (v_{i1}, \ldots, v_{ir})$, $(1 \leqslant i \leqslant R)$ such that $1 \leqslant |u_{it}|$, $|v_{it}| \leqslant P$, $(1 \leqslant i \leqslant R,\ 1 \leqslant t \leqslant r)$.

Since $\det|(\lambda_{ij})| \neq 0$, the number of solutions of the above system is the same as that of the system

$$F(\underline{u}_i, \underline{v}_i) = 0, \quad (1 \leqslant i \leqslant R),$$

such that $1 \leqslant |\underline{u}_i|$, $|\underline{v}_i| \leqslant P$ and so it is given by

$$\prod_{i=1}^{R} \int_0^1 |T(\Lambda_i)|^{2r} \, d\Lambda_i \ll P^{R(2r-k)} ,$$

by Lemma 2.2.6, since $2r \geqslant s_0$. Hence the result.

In fact, for any $r \geqslant s_0 +1 = s - 1$, we will get the result (2.2.18) with $2r$ replaced by r, where $s = [9k^2 R \log 3kR]$ (in the notation to be used in Chapters 3 and 4).

Lemma 2.2.8. Let $\Lambda_1, \ldots, \Lambda_R$ be independent linear forms in $\alpha_1, \ldots, \alpha_R$ with integral coefficients as in (2.2.17), where $\underline{\alpha}$ is in the unit cube U and let $\Lambda = \max_{1 \leq i,j \leq R} |\lambda_{ij}|$.
Let $T(\Lambda_j)$ $(1 \leqslant j \leqslant R)$ be as in (2.2.1). Then

either (i)

$$\prod_{j=1}^{R} |T(\Lambda_j)| << P^{R-\sigma} \tag{2.2.19}$$

or (ii)

$\alpha_1, \ldots, \alpha_R$ have simultaneous rational approximations $\frac{A_1}{Q}, \ldots, \frac{A_R}{Q}$ satisfying

$$\alpha_j = \frac{A_j}{Q} + \beta_j \ , \quad (Q, A_1, \ldots, A_R) = 1,$$

$$1 \leqslant Q \leqslant \Lambda^R P^{k\sigma} \ , \ |\beta_j| < Q^{-1} \Lambda^{R-1} P^{-k+k\sigma} \tag{2.2.20}$$

for $1 \leqslant j \leqslant R$, where $\sigma = \dfrac{1}{8k^2 \log k}$.

Proof. This is Lemma 19 of D-L, since his $\Delta = \det |\lambda_{ij}| << \Lambda^R$ and his $D = \max|\text{cofactors of } \lambda_{ij}| << \Lambda^{R-1}$, where the constant implied by $<<$ is an absolute constant depending only on R. [Our lemma 2.2.5 is used in place of lemma 18 of DL, but this does not affect the results.]

Lemma 2.2.9. Let $s \geqslant 2$ be an arbitrary positive integer. Suppose that every non-trivial linear combination of the forms $F_1, \ldots, F_R$ contains more than $(s-1)R$ variables with coefficients not all zero. Then the forms are linearly independent and for any specified non-singular $R \times R$ submatrix of the coefficient matrix (λ_{ij}), there exist $s - 1$ others as above.

Proof. See Lemma 12 of D-L with $3H + 2k$ replaced by s.

Lemma 2.2.10. Let n, R be integers such that $n > 2R$ and let $M = (M_{ij})$ be an $R \times n$ matrix having real coefficients whose non-zero coefficients have absolute value at least 1 and having its first R columns and the second R columns, both non-singular. Also let the determinant Δ, say, of the first R columns be such that every non-singular $R \times R$ submatrix of M has $|\text{determinant}| \geqslant |\Delta| \geqslant 1$. Suppose Q is a given integer such that

(2.2.21) $$Q > 6(3nR!)^{2R^2} m^{2R^2} ,$$

where

$$m = \max|M_{ij}| .$$

Then there exists a set $S(Q)$, say, which is a union of boxes in $(n-R)$-dimensional space having the following properties:

(i) For all $\underline{z}=(z_1,\dots,z_R)$ such that

(2.2.22) $$|z_i| < \frac{1}{R!m^{nR}} , \quad (1 \leqslant i \leqslant R)$$

and for all points $(y_{R+1},\dots,y_n)$ in $S(Q)$, we have $1 \leqslant |y_i| \leqslant Q$, $(R+1 \leqslant i \leqslant n)$ and the unique $(y_1,\dots,y_R)$ such that

(2.2.23) $$z_i = \sum_{j=1}^{n} M_{ij} \; y_j , \quad (1 \leqslant i \leqslant R)$$

satisfy $1 \leqslant |y_i| \leqslant Q$, $(1 \leqslant i \leqslant R)$ and

(ii) measure ($n-R$ dimensional) of $S(Q)$ is $\geqslant \dfrac{Q^{n-R}}{m^{(3n)R^2}}$.

The set $S(Q)$ is specified by (2.2.29) below, and is of the form $QU = \{Q\underset{\sim}{x};\ \underset{\sim}{x} \in U\}$, where U is a finite disjoint union of bounded boxes.

<u>Proof</u>. We write the given matrix M as

$$M = [M_1|M_2] ,$$

where M_1 consists of the first R columns of M and M_2 has the remaining $n-R$ columns. Then M_1 is non-singular and so let

(2.2.24) $$M_1^{-1} = \frac{A}{\Delta} ,$$

where $\Delta = \det M_1$ and A is the adjoint of M_1 . Then (2.2.23) is equivalent to

(2.2.25) $$\Delta\begin{pmatrix} y_1 \\ \vdots \\ y_R \end{pmatrix} = - AM_2 \begin{pmatrix} y_{R+1} \\ \vdots \\ y_n \end{pmatrix} + A\underline{z}, \quad \underline{z} = (z_1,\dots,z_R),$$

and $\max|\text{coefficient of } A| \leqslant (R-1)!\; m^{R-1}$.

Therefore for all $\underline{z}$ satisfying (2.2.22), we have

$$(2.2.26) \qquad |A\underline{z}| \leqslant m^{-nR+R-1}, \quad (1 \leqslant i \leqslant R),$$

with the notation as in the table given on page 3 (general introduction).

Let us write, for convenience, that

$$- AM_2 = B = (B_{ij}) .$$

Then we have

$$(2.2.27) \qquad |B| \leqslant R!m^R = \bar{m}_2 , \text{ say.}$$

Also

$|B_{ij}| = |$determinant of the matrix obtained by replacing the i^{th} column of M_1 by the j^{th} column of $M_2|$.

And so by the hypothesis of the lemma,

$$|B_{ij}| = 0 \quad \text{or} \quad |B_{ij}| \geqslant |\Delta| .$$

Thus by the non-singularity assumption, for $1 \leq i \leq R$ there exists a *least* $k = k(i)$, say, in $[1,R]$ such that

$$|B_{ik}| \geqslant |\Delta| .$$

We then have, from (2.2.25),

$$(2.2.28) \qquad \Delta y_i = \sum_{j=R+k}^{n} B_{ij}\, y_j + \sum_{j=1}^{R} a_{ij} z_j , \quad (1 \leqslant i \leqslant R),$$

where $1 \leqslant k = k(i) \leqslant R$, $|B_{i,R+k}| \geqslant |\Delta|$ and $A = (a_{ij})$.

Let $S(Q)$ be the $(n-R)$-dimensional region defined by

$$(2.2.29)\quad \begin{cases} \dfrac{Q}{\{3(n-R)m_2\}^{i+R}} \leqslant |x_i| \leqslant \dfrac{2Q}{\{3(n-R)m_2\}^{i+R}}, & (1 \leqslant i \leqslant R), \\[2ex] \dfrac{Q}{\{3(n-R)m_2\}^{2R+i}} \leqslant |x_i| \leqslant \dfrac{2Q}{\{3(n-R)m_2\}^{2R+i}}, & (R+1 \leqslant i \leqslant n-R). \end{cases}$$

It is then clear that if $\underline{y}^{(R)} = (y_{R+1}, \ldots, y_n)$ is in $S(Q)$, then $1 \leqslant |y_i| \leqslant Q$, $(R+1 \leqslant i \leqslant n)$,

$$(2.2.30)\qquad |B_{i,R+k}\, y_{R+k}| \geqslant \frac{|\Delta|\, Q}{\{3(n-R)m_2\}^{R+k}}\ k = k(i), \qquad (1 \leqslant i \leqslant R)$$

and

$$\left|\sum_{j=R+k+1}^{n} B_{ij}\, y_j + \sum_{j=1}^{R} a_{ij}\, z_j\right| \leqslant \frac{(n-R-k-1)}{3(n-R)} \cdot \frac{2Q}{\{3(n-R)m_2\}^{R+k}} + m^{-nR+R-1}$$

$$\leqslant \frac{2Q}{3\{3(n-R)m_2\}^{R+k}} + \frac{Q}{6\{3nR!m\}^{2R^2}},$$

by (2.2.21). And this is

$$\leqslant \frac{5|\Delta|\, Q}{6\{3(n-R)m_2\}^{R+k}},$$

since $|\Delta| \geqslant 1$. Using this and (2.2.30) in (2.2.28), we have

$$\frac{Q}{6\{3(n-R)m_2\}^{R+k}} \leqslant |y_i| \leqslant \frac{2Q}{\{3(n-R)m_2\}^{R+k-1}} + m^{-nR+R-1}, \qquad (1 \leqslant i \leqslant R)$$

(as $|\Delta| \geqslant 1$). Hence by (2.2.21) since $m_2 = R!m^R$, we have

$$1 \leqslant |y_i| \leqslant Q, \quad (1 \leqslant i \leqslant R).$$

Thus result (i) is proved.

From the Definition (2.2.29) of $S(Q)$, result (ii) follows, since the volume of $S(Q)$ is

$$\gg \frac{Q^{n-R}}{m^{R^3}(m^R)^{1+2+\cdots+R+(n-2R)(2R+i)}} \gg \frac{Q^{n-R}}{m^{(3n)R^2}}.$$

3. LEMMAS FOR EQUATIONS

In this section, I shall give some preliminary results which have applications in the next chapter (for a system of equations), for certain ranges of values of α, called major arcs (defined in the next chapter).

Let $\Lambda_1, \ldots, \Lambda_n$ be integral linear forms in $\alpha_1, \ldots, \alpha_R$

(2.3.1) $$\Lambda_j = \sum_{i=1}^{R} \lambda_{ij} \alpha_i, \quad (1 \leqslant j \leqslant n).$$

For $1 \leqslant i \leqslant R$, $1 \leqslant j \leqslant n$, let Q, A_i, a_j, q_j be integers and β_i, γ_j be real numbers satisfying the relations

(2.3.2)
$$(Q, A_1, \ldots, A_R) = 1, \quad 0 \leqslant A_1, \ldots, A_R < Q, \quad \alpha_i = \frac{A_i}{Q} + \beta_i \quad (1 \leqslant i \leqslant R)$$

(2.3.3)
$$(a_j, q_j) = 1, \quad \frac{a_j}{q_j} = \frac{1}{Q} \sum_{i=1}^{R} \lambda_{ij} A_i, \quad \gamma_j = \sum_{i=1}^{R} \lambda_{ij} \beta_i, \quad (1 \leqslant j \leqslant n).$$

And so $\Lambda_j = \dfrac{a_j}{q_j} + \gamma_j$. For a given fixed (small) $\tau > 0$, let

(2.3.4) $$T = \{\underline{\beta} = (\beta_1, \ldots, \beta_R); \max_{1 \leqslant j \leqslant R} |\beta_j| > P^{-k+\tau}\}.$$

<u>Lemma 2.3.1</u>.* Let $\Lambda_1, \ldots, \Lambda_R$ be independent linear forms in $\alpha_1, \ldots, \alpha_R$ as in (2.3.1) and let T be the R-dimensional region, as in (2.3.4) with notations (2.3.2), (2.3.3) and W be the whole of the R-dimensional space. Suppose $s = [9k^2 \log 3Rk]$. Then

(2.3.5) $$\int_W \prod_{j=1}^{R} \min\{P, P^{-k+1} |\gamma_j|^{-1}\}^s \, d\underline{\beta} << |\Delta|^{-1} P^{(s-k)R}$$

and

(2.3.6) $$\int_T \prod_{j=1}^{R} \min\{P, P^{-k+1} |\gamma_j|^{-1}\}^s \, d\underline{\beta} << |\Delta|^{-s} \Lambda^{(s-1)(R-1)} P^{(s-k)R-\tau(s-}$$

where $\Delta = \det|(\lambda_{ij}) \ (1 \leqslant i,j \leqslant R)|$.

<u>Proof</u>. By (2.3.3) for $1 \leqslant i,j \leqslant R$, we see that the numbers $\gamma_1, \ldots, \gamma_R$ and $\beta_1, \ldots, \beta_R$ are related by the relations

(2.3.7) $$\gamma_j = \sum_{i=1}^{R} \lambda_{ij} \beta_i, \quad (1 \leqslant j \leqslant R).$$

* The estimates (2.3.6) and (2.3.7) are not in fact required in what follows, but estimates required are obtained in a similar manner.

Since $\Lambda_1,\ldots,\Lambda_R$ are independent, $\det|(\lambda_{ij})_{R\times R}| = \Delta \neq 0$ and so we can apply a change of variables in the integrals in question from $\underline{\beta}$ to $\underline{\gamma}$. Also on solving (2.3.7) for $\beta_1,\ldots,\beta_R$ we get

$$\beta_i = \Delta^{-1} \sum_{j=1}^{R} D_{ij}\gamma_j, \quad (1 \leq i \leq R),$$

where D_{ij} are the cofactors of λ_{ij} in (λ_{ij}) $(1 \leq i,j \leq R)$. Also we easily see that the T is contained in the region (in γ space), where

(2.3.8) $$\max_{1\leq j\leq R} |\gamma_j| > (R!)^{-1}\Delta\,\Lambda^{-(R-1)}\,P^{-k+\tau} = C_0\,P^{-k+\tau}, \quad \text{say}.$$

Now after the change of variables both the integrals factorise and the result (2.3.5) follows, since

(2.3.9) $$\int_{-\infty}^{\infty} \min\{P, P^{-k+1}|\gamma|^{-1}\}^s \, d\gamma \ll P^{s-k},$$

and (2.3.6) follows from (2.3.9) and the following (used on one of the R factors)

$$\int_{C_0P^{-k+\tau}} \min\{P, P^{-k+1}|\gamma|^{-1}\}^s \, d\gamma \ll P^{(-k+1)s}(C_0P^{-k+\tau})^{-s+1},$$

$$\ll C_0^{-s+1}\,P^{s-k-\tau(s-1)},$$

since T is contained in the region for which (2.3.8) holds.]

Lemma 2.3.2. Suppose the relations (2.3.1) to (2.3.3) hold for $1 \leq i,j \leq R$ and $\Lambda_1,\ldots,\Lambda_R$ are independent. Let $\omega > 0$ be a given small real number and $s = [9k^2R\log 3Rk]$. Then we have

(2.3.10) $$\sum_{\underline{A}} (q_1 \ldots q_R)^{-s/k} \ll \Lambda^R \left\{\frac{Q}{(Q,\Delta)}\right\}^{-((s/k)-1)+(R-1)\epsilon}$$

and

(2.3.11) $$\sum_{Q>P^\omega} \sum_{\underline{A}} (q_1 \ldots q_R)^{-s/k} \ll \Lambda^R\,\Delta^{\epsilon_0}\,\min\{1, \Delta^{s/k-2-\epsilon_0}\,P^{-\omega(s/k-2-\epsilon_0)}\}$$

where $\underline{A}$ runs over all the integers $A_1,\ldots,A_R$ satisfying (2.3.2), $\Delta = \det|(\lambda_{ij})|$, $\varepsilon > 0$, $\varepsilon_0 = (R-1)\varepsilon > 0$ and $\Lambda = \max\limits_{1\le i,j\le R} |\lambda_{ij}|$.

Proof. We use the ideas of Lemma 23 of D-L. From (2.3.3), we have

$$\frac{a_j}{q_j} = \sum_{i=1}^{R} \lambda_{ij} \frac{A_i}{Q},$$

where $(a_j, q_j) = 1$ and so $q_j | Q$. Let

$$u_j = \left(Q, \sum_{i=1}^{R} \lambda_{ij} A_i\right), \quad (1 \le j \le R) \tag{2.3.12}$$

and

$$d = (u_1, \ldots, u_R).$$

Then $d|Q$,

$$q_j = \frac{Q}{u_j}, \quad (1 \le j \le R) \tag{2.3.13}$$

and

$$\sum_{i=1}^{R} \lambda_{ij} A_i \equiv 0 \pmod{u_j}, \quad (1 \le j \le R).$$

Hence $d \mid \Delta A_i \ (1 \le i \le R)$, ~~$d|(\Delta, A_1,\ldots,A_R)$~~, where $\Delta = \det|(\lambda_{ij})|$. From this and (2.3.2) since $d|Q$, it follows that $d|\Delta$. Therefore for $\underline{A}$ satisfying (2.3.2), using (2.3.13) we have

$$\sum_{\underline{A}} (q_1 \ldots q_R)^{-s/k} = \sum_{\underline{A}} Q^{-R.s/k} . (u_1 \ldots u_R)^{s/k}. \tag{2.3.14}$$

Let

$$C_j = \sum_{i=1}^{R} \lambda_{ij} A_i, \quad (1 \le j \le R). \tag{2.3.15}$$

Then $|C_j| \le R\Lambda Q$, since (2.3.2) holds, where $\Lambda = \max\limits_{1\le i,j\le R} |\lambda_{ij}|$. Also by (2.3.12), C_j is a multiple of u_j. Thus the number of possibilities for C_j is $\ll \frac{\Lambda Q}{u_j}$. For given values of $u_1,\dots,u_R$, the number of possibilities for $C_1,\dots,C_R$ is therefore

$$\ll \frac{\Lambda Q}{u_1} \cdots \frac{\Lambda Q}{u_R} \ll \frac{\Lambda^R Q^R}{u_1 \cdots u_R} .$$

But by (2.3.15), since $\Delta \neq 0$ (as $\Lambda_1,\dots,\Lambda_R$ are independent) for given values of $C_1,\dots,C_R$, there exist unique values of $A_1,\dots,A_R$. And so the sum in the right hand side of (2.3.14) is

$$\ll \sum_{\substack{u_j \mid Q \\ 1 \leq j \leq R}} Q^{-R.s/k+R} \Lambda^R (u_1 \dots u_R)^{(s/k)-1} ,$$

$$\ll \Lambda^R Q^{-R(s/k-1)} \sum_{\substack{u_j \mid Q \\ 1 \leq j \leq R}} (u_1 \dots u_R)^{(s/k)-1} .$$

Now we write

$$u_j = dv_j , \quad (1 \leq j \leq R); \quad (v_1,\dots,v_R) = 1.$$

Then the sum in the last inequality is

$$\ll \sum_{d \mid (Q,\Delta)} d^{R(s/k-1)} \sum_{\substack{v_j \mid \frac{Q}{d} \\ 1 \leq j \leq R}} (v_1 \dots v_R)^{(s/k)-1} ,$$

$$\ll \sum_{d \mid (Q,\Delta)} d^{R(s/k-1)} \sum_{v_1 \dots v_R \mid (\frac{Q}{d})^{R-1}} (v_1 \dots v_R)^{(s/k)-1+\epsilon} ,$$

, by the same argument as in Lemma 23 of D-L. And this is

$$\ll \sum_{d \mid (Q,\Delta)} d^{R(s/k-1)} \left(\frac{Q}{d}\right)^{(R-1)(s/k-1+\epsilon)}$$

$$\ll Q^{(R-1)(s/k-1+\epsilon)} \sum_{d \mid (Q,\Delta)} d^{s/k-1-(R-1)\epsilon}$$

$$<< Q^{(R-1)(s/k-1+\epsilon)} \quad (Q,\Delta)^{s/k-1-(R-1)\epsilon} \; .$$

Hence (2.3.10) follows.

Now using this we prove (2.3.11). The left hand side of (2.3.11) is, by (2.3.10)

$$<< \Lambda^R \sum_{Q>P^\omega} \left\{\frac{Q}{(Q,\Delta)}\right\}^{-(s/k-1)+(R-1)\epsilon}$$

$$<< \Lambda^R \sum_{\delta|\Delta} \sum_{\substack{Q>P^\omega \\ \delta|\Delta,\delta|Q}} \left(\frac{Q}{\delta}\right)^{-(s/k-1)+(R-1)\epsilon}$$

$$<< \Lambda^R \sum_{\delta|\Delta} \max\left\{1, \frac{P^\omega}{\delta}\right\}^{-(s/k-1)+(R-1)\epsilon+1}$$

$$<< \Lambda^R \sum_{\delta|\Delta} \min\left\{1, \; |\Delta|^{s/k-2-(R-1)\epsilon} \; (P^\omega)^{-(s/k-2)+(R-1)\epsilon}\right\}$$

$$<< \Lambda^R \; |\Delta|^{\epsilon_0} \; \min\left\{1, \; |\Delta|^{s/k-2-(R-1)\epsilon} (P^\omega)^{-(s/k-2)+(R-1)\epsilon}\right\},$$

since there are $<< |\Delta|^{\epsilon_0}$ choices for δ , the divisors of Δ , for any fixed $\varepsilon_0 > 0$. Hence our result follows by taking $\varepsilon_0 = (R-1)\varepsilon$.

4. LEMMAS FOR SINGULAR SERIES

In this section, I give some results regarding the singular series of a system of equations. Throughout this section p denotes a prime number.

Definition 2.4.1. Let A_i , $(1 \leqslant i \leqslant R)$, Q be integers such that

$$(Q,A_1,\ldots,A_R) = 1, \quad 1 \leqslant A_1,\ldots,A_R \leqslant Q \; .$$

Then we write

$$(2.4.1) \qquad S_0(\underline{A},Q) = \sum_{x_1=1}^{Q} \cdots \sum_{x_n=1}^{Q} e\left(\frac{A_1F_1 + \ldots + A_RF_R}{Q}\right),$$

where $F_i(\underline{x})$ $(1 \leqslant i \leqslant R)$ are integral forms of the type (1.1.2) and we define the following sum, denoted by $\mathfrak{S}$, as the *singular series*:

$$(2.4.2) \qquad \mathfrak{S} = \sum_{Q=1}^{\infty} \sum_{\underline{A}} Q^{-n} S_0(\underline{A},Q).$$

Then it follows by standard arguments (see, for example, Lemma 7 of Davenport [17]) that

$$(2.4.3) \qquad \mathfrak{S} = \prod_{p} \chi(p)$$

where p runs through all the primes and

$$(2.4.4) \qquad \chi(p) = \sum_{\upsilon=0}^{\infty} \sum_{\underline{A}} (p^{\upsilon})^{-n} S_0(\underline{A},p^{\upsilon}),$$

$\underline{A}$ runs through the residue classes $\bmod(p^{\upsilon})$ with $(p,A_1,\ldots,A_R) = 1$.

<u>Definition 2.4.2</u>. For any prime p, a solution $\underline{x}$ of the congruences

$$(2.4.5) \qquad F_i(\underline{x}) \equiv 0 \pmod{p^s}, \quad (1 \leqslant i \leqslant R)$$

is said to be non-singular $(\bmod\ p^s)$, if the matrix $\frac{1}{h}\left(\frac{\partial F_i}{\partial x_j}\right)$ evaluated at this solution has rank $R \pmod p$; that is if the matrix $(\lambda_{ij}x_j)$ has rank $R \pmod p$.

<u>Lemma 2.4.1</u>. For any prime p, let $M(p^s)$ be the number of solutions of the system of congruences (2.4.5) and $\chi(p)$ as in (2.4.4). Then we have,

$$(2.4.6) \qquad \chi(p) = \lim_{s\to\infty} \frac{M(p^s)}{p^{s(n-R)}}.$$

Proof. This follows by the standard arguments (as in Lemma 8, for $R = 1$, of Davenport [17]).

Lemma 2.4.2. For any prime p, let t be the exact power of p dividing k, that is $k = p^t k_0$, $(k_0, p) = 1$ and let

$$\gamma = \begin{cases} 1 & , \text{ if } t = 0 \\ t+1 & , \text{ if } t > 0 \text{ and } p > 2 \\ t+2 & , \text{ if } t > 0 \text{ and } p = 2 \end{cases}.$$

Also let $N(p^s)$ be the number of non-singular solutions (mod p^s) (see Definition 2.4.2) of the congruences (2.4.5) and $M(p^s)$ as in Lemma 2.4.1. Then

(i) for any r such that $r > \gamma$,

(2.4.7) $$M(p^r) \geqslant p^{(r-\gamma)(n-R)} N(p^{\gamma}),$$

and

(ii) if we can write for $p \nmid k$, after re-ordering the variables if necessary,

$$F_i(\underline{x}) = F_{i0}(x_1, \dots, x_m) + pF_{i1}(x_{m+1}, \dots, x_n), \quad 1 \leqslant i \leqslant R,$$

where each of the variables $x_1, \dots, x_m$ occurs in at least one of $F_{10}, \dots, F_{R0}$ with a coefficient not divisible by p, then

(2.4.8) $$M(p^r) \geqslant p^{(r-1)(n-R)+n-m} N_0(p),$$

where $N_0(p)$ is the number of non-singular solutions of the system

(2.4.9) $$F_{i0} \equiv 0 \pmod{p}, \quad (1 \leqslant i \leqslant R).$$

Proof. The proof of (2.4.7) corresponds to that of Lemma 10, for $R = 1$, of Pitman and Ridout [34]. (It depends on Hensel's lemma.) In (ii), since $p \nmid k$, $\gamma = 1$ and so to prove (2.4.8) it is enough to prove that

$$N(p) = p^{n-m} N_0(p) .$$

Given a non-singular solution of (2.4.9), each choice of x_i (mod p) for i such that $m+1 \leqslant i \leqslant n$ yields a unique non-singular solution (mod p) to (2.4.5) with $s = 1$ and all non-singular solutions (mod p) of the latter can be obtained by this method. Hence there exists a 1 to p^{n-m} correspondence between the solutions of (2.4.9) and of (2.4.5) with $s = 1$ and thus the above equation follows. (This is an idea of Toliver [39].)

<u>Lemma 2.4.3.</u> Let $p \nmid k$ and for i, $(1 \leqslant i \leqslant R)$, let F_i be as in (1.1.2) with integral coefficients and $F_{i\,0}$ as in Lemma 2.4.2. Suppose that any non-trivial integer linear combination of $F_{i\,0}$, (mod p) $(1 \leqslant i \leqslant R)$ has at least q variables explicitly with non-zero coefficients (mod p), $q \geqslant 1$. Then the number of singular solutions (mod p) of (2.4.9) is

(2.4.10) $$\ll p^{m-q-R+1} .$$

<u>Proof</u>. (The proof uses the ideas of Toliver [39].) Without loss of generality, we may take the $F_{i\,0}$, $(1 \leqslant i \leqslant R)$, to involve the first m variables $x_1, \ldots, x_m$. By the hypothesis of this lemma, their coefficient matrix has rank R & so $m \geqslant R$; and so at every singular solution $\underline{X}$, say, there exist $i_1, \ldots, i_R$ subscripts such that

either (i) at least one of $X_{i_1}, \ldots, X_{i_R} \equiv 0$ (mod p)

or (ii) $\Delta(i_1, \ldots, i_R) \equiv 0$ (mod p).

If the above $\underline{X}$ is such that all $X_{i_j} \not\equiv 0$ (mod p), $(1 \leqslant j \leqslant R)$, then the alternative (ii) must hold. And the non-zero variables of such an $\underline{X}$ correspond to a set of columns of rank atmost $R - 1$, and this set of columns is contained in a maximal set of columns of rank $\leqslant R$. This maximal set will necessarily have rank exactly $R - 1$. Suppose there are m_0 columns in this maximal set. Then since (ii) holds, we can find a linear combination of F_{io}'s, having $m - m_0$ columns and so by the hypo--thesis of the lemma

$$m - m_0 \geqslant q .$$

That is

(2.4.11) $$m_0 \leqslant m - q .$$

By changing the ordering of the variables, if necessary, we may take these m_0 to be the first m_0 columns, with the first $R-1$ columns linearly independent.

Now consider the system

$$\sum_{j=1}^{m_0} \lambda_{ij} x_j^k \equiv 0 \pmod p, \quad (1 \leqslant i \leqslant R).$$

By the definition of m_0, the rank of the coefficient matrix of this system is $R-1$; and so the system is equivalent to $R-1$ independent congruences. Then for any choice of $x_R, \ldots, x_{m_0} \pmod p$ there are at most k^{R-1} values of $x_1, \ldots, x_{R-1} \pmod p$ satisfying these $R-1$ independent congruences. And $x_R, \ldots, x_{m_0} \pmod p$ can be chosen in $p^{m_0-(R-1)}$ ways and so the number of solutions (mod p) of the above system of congruences is

$$\leqslant k^{R-1} p^{m_0-(R-1)} .$$

Therefore the number of singular solutions (mod p) of (2.4.9) with non-zero variables corresponding to this maximal set of columns is

$$\begin{aligned} &\leqslant k^{R-1} p^{m_0-(R-1)} \\ &\ll p^{m_0-(R-1)} \\ &\ll p^{m-q-R+1} , \end{aligned}$$

by (2.4.11). ~~This is (2.4.10), where the implied constant depends on k and R only.~~ Since the number of such maximal sets of columns is certainly $\leq 2^m \leq 2^N$ and hence $\ll 1$, this gives (2.4.10).

Lemma 2.4.4. Suppose F_i, F_{i0} $(1 \leqslant i \leqslant R)$ satisfy the hypothesis of Lemma 2.4.3 with

(2.4.12) $$q > Rk,$$

and let $N_0(p)$ be the number of non-singular solutions of (2.4.9) as in Lemma 2.4.2. Then we have,

(2.4.13) $$N_0(p) \gg p^{m_0 - R - \varepsilon_0}, \quad \varepsilon_0 = \frac{q}{k} - R.$$

Proof. By the usual arguments (see Lemma 12 of Pitman [32]), it can be shown that the total number of solutions of (2.4.9) is

(2.4.14) $$= p^{m-R} + p^{-R} \sum_{\underline{t}} \prod_{i=1}^{m} S\Big(\sum_{j=1}^{R} t_j \lambda_{ji}, p\Big),$$

where the summation is over $0 \leqslant t_1, \ldots, t_R \leqslant p-1$ such that $\underline{t} \neq \underline{0}$ and $S(a,p)$ as in Definition 2.2.1. From the definition of $S(a,p)$ and by Lemma 2.2.4, we note that

(2.4.15) $$S(a,p) = \begin{cases} p, & a \equiv 0 \pmod p \\ \ll p^{1-1/k}, & \text{otherwise}. \end{cases}$$

~~Let m_0 be as in Lemma 2.4.3. The the first m_0 columns of the coefficient matrix of (2.4.9) has rank (mod p) at most $R-1$, and so (2.4.11) holds. And (by rearranging these m_0 columns, if necessary)~~

For each $\underline{t} \neq \underline{0}$ as above, the weaker estimate for the i^{th} factor of the second term of (2.4.14) must be used whenever

(2.4.16) $$\sum_{j=1}^{R} t_j \lambda_{ji} \equiv 0 \pmod p.$$

holds, However, by the definition of q, (2.4.16) holds for $\leq m - q$ values of i whatever $\underline{t}$ is; and for each choice of $\underline{t}$, the product in the second term of (2.4.14), by (2.4.15) is therefore

$$\ll p^{m-q} p^{q(1-1/k)}.$$

$$\ll p^{m_0+(1-1/k)(m-m_0)} \ll p^{(1-1/k)m+m_0/k} .$$

And so the second term in (2.4.14) is

$$\ll p^{-R}\{R(p-1)p^{(1-1/k)m+m_0/k} + (p^R - 1 - R(p-1))p^{(1-1/k)m}\} ,$$

$$\ll p^{-R+m} \max\{p^{-q/k+1} , p^{R-m/k}\} ,$$

Since there are (p^R-1) choices of $\underset{\sim}{c} \neq \underset{\sim}{0}$, the second term in (2.4.14) is

$$\ll p^{-R}(p^R-1)\,p^{m-(q/k)}$$

$$\ll p^{-R+m-\epsilon_0} ,$$

where $\epsilon_0 > 0$, since (2.4.12) holds.

Using this, the estimate (2.4.10) for the number of singular solutions and the relation (2.4.14), the number of non-singular solutions of (2.4.9) is

$$\geqslant p^{m-R} - p^{m-R-\epsilon_0} - p^{m-q-R+1}$$

$$\gg p^{m-R}(1 - p^{-\epsilon_0} - p^{-q+1})$$

$$\gg p^{m-R-\epsilon_0} ,$$

since $\epsilon_0 \leq q-1$, and this is the required result.

<u>Lemma 2.4.5</u>. Suppose the hypothesis of Lemma 2.4.3 and (2.4.12) hold. Let $\chi(p)$ be as in (2.4.4). Then for $p > k$, we have

(2.4.17) $$\chi(p) \gg p^{-\epsilon_0}, \quad \epsilon_0 = \frac{q}{k} - R .$$

<u>Proof</u>. Since $p > k$, $p \nmid k$ and so the Lemmas 2.4.2 (ii), 2.4.3, 2.4.4 are applicable. Thus by (2.4.6), (2.4.8) and (2.4.13), the result (2.4.17) follows.

CHAPTER 3

A SYSTEM OF R DIAGONAL EQUATIONS

1. INTRODUCTION

In this chapter, we obtain a result on bounded solutions of a system of equations

$$(3.1.1) \qquad F_i(\underline{x}) = 0, \quad 1 \leqslant i \leqslant R,$$

where

$$(3.1.2) \qquad F_i(\underline{x}) = \sum_{j=1}^{n} \lambda_{ij} x_j^k, \quad (1 \leqslant i \leqslant R),$$

λ_{ij} $(1 \leqslant i \leqslant R, \ 1 \leqslant j \leqslant n)$ are all integers and $k \geqslant 3$ is an odd integer. The work in §2 to §7 will be devoted to proving the following theorem, which is essential as an auxiliary result for our work on inequalities (in the next chapter):

THEOREM 1. There exist constants $A_1(k)$, $A_2(n,R,k)$ such that the following result holds: Let $F_1(\underline{x}),\ldots,F_R(\underline{x})$ be integral forms as in (3.1.2), in

$$(3.1.3) \qquad n \geqslant [9R^3k^2 \log 3Rk]$$

variables and let

$$(3.1.4) \qquad \Lambda = \max_{1 \leqslant i \leqslant R} |F_i| .$$

(For the meaning of $|F_i|$, see the table of notations.) Then the system (3.1.1) has an integral solution $\underline{x}$ satisfying

(3.1.5) $$0 < |\underline{x}| \leqslant A_2 \Lambda^{A_1} .$$

OUTLINE OF THE METHOD

The basic approach to prove Theorem 1 is to use the R-dimensional extension of the Hardy-Littlewood circle method (see §2 of Chapter 1), as used in D-L. To give a brief outline of this method, we need the following definition:

Definition 3.1.1. Let P be a large positive integer to be determined later in this chapter. Let $\mathcal{B}(P)$ be the n-dimensional box given by

(3.1.6) $$\mathcal{B}(P) = \{\underline{x};\ x_i \in B(P),\ (1 \leqslant i \leqslant n)\} ,$$

where, as in Definition 2.2.1,

$$B(P) = [-P,-1] \cup [1,P] .$$

Then let $N(P)$ be the number of integral solutions of the system of equations (3.1.1), inside the box $\mathcal{B}(P)$.

Also let $T(\Lambda)$ be the trigonometric sum

(3.1.7) $$T(\Lambda) = \sum_{\underline{x} \in B(P)} e(\Lambda x^k),$$

as in (2.2.1), for any real Λ. Then

(3.1.8) $$N(P) = \int_U \prod_{j=1}^{n} T(\Lambda_j) d\underline{\alpha} ,$$

where

(3.1.9) $$U = \{\underline{\alpha};\ 0 < \alpha_i < 1,\ (1 \leqslant i \leqslant R)\} ,$$

(3.1.10) $$\Lambda_j = \sum_{i=1}^{R} \lambda_{ij} \alpha_i ,\quad (1 \leqslant j \leqslant n)$$

and $T(\Lambda_j)$'s are as in (3.1.7) for these Λ_j's .

Therefore to prove Theorem 1, it is enough to show that there exist constants A_1, A_2 such that for $P > A_1\Lambda^{A_2}$, the integral on the right hand side of (3.1.8) is greater than 0 and hence $N(P)$ is at least 1. This forms the main idea.

To do this, we divide the unit cube U which is the region of our integration, into two parts called "*major arcs*" and "*minor arcs*" and estimate the contributions of them to the integral in (3.1.8). We apply Hua's and Vinogradov's inequalities in appropriate places. The major arc contribution is large compared to that of the minor arcs and we attempt to find values of P as a function of Λ for which the former contribution dominates the latter. We follow the lines of D-L for the subdivision of U and for the estimations of the contributions, but we must keep track of the dependence on Λ, thus in contrast to the situation in D-L, our implied constants must be independent of the coefficients of the F_i's.

PLAN OF THIS CHAPTER

We shall define a hypothesis (block condition) and a normalisation technique in §2 and shall first prove that it is sufficient to prove Theorem 1 for a normalised system that satisfies the block condition and from there onwards we consider only such a system. In §3, we shall describe the subdivision of U and estimate the contribution of the minor arcs. In §4, we shall discuss the pruning of the major arcs and in §5 and §6 the singular integral and the singular series respectively are discussed. Finally the proof is completed in §7.

2. REDUCTION AND CONGRUENCES

We have seen from the known results given in Chapter 1 that to prove our theorem, we need conditions that ensure the solvability (non-singular), of the equations (3.1.1) in every p-adic field. This needs results on the corresponding congruences, which we shall discuss in this section. Before starting on this subject, we shall first reduce our Theorem 1 by a prescribed hypothesis, called the block condition and by a normalisation technique, which I shall describe now.

HYPOTHESIS H (or "BLOCK CONDITION")

Let $s = [9\, k^2 R \log 3Rk]$. The forms $F_1, \ldots, F_R$ are linearly independent and, if any non-singular $R \times R$ submatrix of the coefficient matrix (λ_{ij}) is specified, then there exist $s - 1$ other non-singular $R \times R$ submatrices which together with the given one, are all disjoint.

Lemma 3.2.1. Suppose Theorem 1 holds for $n \geq [9R^3 k^2 \log 3Rk]$ when the hypothesis H holds for $s = [9k^2 R \log 3Rk]$. Then the theorem holds for $n \geq [9R^3 k^2 \log 3Rk]$ regardless of the hypothesis H.

Proof. We prove this by induction on R. Suppose the lemma holds for $R - 1$ forms and $n \geq [9(R-1)^3 k^2 \log 3(R-1)k]$.

Suppose every non-trivial linear combination of $F_1, \ldots, F_R$ contains more than $(s-1)R$ variables. Then by Lemma 2.2.9, the hypothesis H holds and so by the hypothesis of the present lemma, Theorem 1 holds. Now we suppose that there exists a non-trivial linear combination of $F_1, \ldots, F_R$ having at most $(s-1)R$ variables. Let this be

$$g = \sum_{i=1}^{R} \lambda_i F_i, \quad \lambda_t \neq 0 \text{ for some } t, \quad 1 \leq t \leq R.$$

We note that since $\lambda_t \neq 0$, $\underline{x}$ is a solution of

$$(3.2.1) \qquad \begin{cases} g = 0 \\ F_i = 0, \quad 1 \leqslant i \leqslant R, \quad i \neq t \end{cases}$$

iff it is a solution of $F_i = 0$, $1 \leqslant i \leqslant R$. We give therefore the value zero to all the variables appearing in g with non-zero coefficients. Then the system of $R - 1$ equations

$$(3.2.2) \qquad F_i = 0, \quad 1 \leqslant i \leqslant R, \quad i \neq t$$

has at least $[9R^3k^2 \log 3Rk] - (s-1)R$ variables and so the number of variables in this sytem is

$$\begin{aligned}
&\geqslant 9R^3k^2 \log 3Rk - 1 - ([9k^2R\log 3Rk] - 1)R \\
&\geqslant 9R^3k^2 \log 3Rk - 1 - (9k^2R\log 3Rk - 1)R \\
&\geqslant 9(R^2-R)k^2R\log 3Rk + (R-1) \\
&> 9(R-1)^3k^2 \log 3(R-1)k .
\end{aligned}$$

And so by our assumption, this system (3.2.2) has a solution which is bounded in terms of Λ. Thus the system (3.2.1) also has a solution satisfying the same bound and this solution is also a solution of (3.1.1). Since Λ' (where $\Lambda' = \max_{i \neq t} |F_i|$) is at most equal to Λ (as in (3.1.4)), the above bound is less than a bound in terms of Λ of $F_1, \ldots, F_R$. Hence if the theorem holds for $R - 1$ forms, it also holds for R forms.

We now consider the case $R = 1$, and consider a system consisting of one form. If the hypothesis H does not hold then the above form has more than one coefficient zero. So we get a bounded solution with all the variables corresponding to these zero coefficients as 1 and the rest 0. If the hypothesis H holds then a bounded solution exists by the hypothesis of the present lemma. Hence in any case, Theorem 1 holds for $R = 1$.

Thus the result follows by induction.

NORMALISATION

In this section, I shall discuss a normalisation technique that is needed to establish p-adic solvability which is used later to deal with the singular series. This is based on the normalisation in D-L, but simpler. We assume that $F_1, \ldots, F_R$ are linearly independent (since we shall be assuming the Block Condition). Let

$$(3.2.3) \qquad \Theta(F_1, \ldots, F_R) = \prod_{(j_1, \ldots, j_R)} \det(\lambda_{i j_h}), \quad (1 \leq i, h \leq R),$$

where the product is taken over all subsets of R disjoint $j_1, \ldots, j_R$ such that $1 \leq j_1, \ldots, j_R \leq n$ (and the corresponding determinant is non-zero). Let M be the number of such subsets. Our proofs will use the following lemma of D-L.

Lemma 3.2.2. Let $F_i(\underline{x})$, $(1 \leq i \leq R)$ be given forms as in (3.1.2). Then

(i) if $F_i'(\underline{x}) = F_i(p^{\nu_1}x_1, \ldots, p^{\nu_n}x_n)$, $(1 \leq i \leq R)$ and $\nu = \nu_1 + \ldots + \nu_n$, then

$$(3.2.4) \qquad \Theta(F_1', \ldots, F_R') = p^{kRM\nu/n}\Theta(F_1, \ldots, F_R),$$

and (ii) if $F_i'(\underline{x}) = \sum_{j=1}^{R} d_{ij} F_j(\underline{x})$, $(1 \leq i \leq R)$,

where d_{ij} are integers and let $\det(d_{ij}) = D$, say, $\neq 0$, then

$$(3.2.5) \qquad \Theta(F_1', \ldots, F_R') = D^M\, \Theta(F_1, \ldots, F_R).$$

Proof. See lemma 10 of D-L .

D-L considered "local" normalisation with respect to a particular prime p and was based on operations of the type in the above lemma. Ours, following Lloyd [25], is a "global" normalisation with respect to all primes p simultaneously and will be based on a narrower class of operations, namely the following:

O_1. Divide one or more rows of (λ_{ij}) by p; this is equivalent to pre-multiplying (λ_{ij}) by a $R \times R$ diagonal matrix D_1, say, having its components either p^{-1} or 1, at least one is p^{-1};

O_2. Multiply one or more variables x_i by p; this is equivalent to multiplying one or more columns of (λ_{ij}) by p^k, which is the same as post-multiplying (λ_{ij}) by a $n \times n$ diagonal matrix D_2, say, having components either p^k or 1;

O_3. Replace one of the forms $F_1, \ldots, F_R$, say F_k by F_k + linear combination of others with integer coefficients in $[0,p-1]$; this is equivalent to pre-multiplying (λ_{ij}) by an $R \times R$ integral matrix U, say, having 1 as its diagonal coefficients and 0's elsewhere except in the k-th row whose coefficients are integers in $[0,p-1]$.

From Lemma 3.2.2, it is clear that the only operation of this kind which could take a set of R integral forms to another set of integral forms with lower value of Θ (as in (3.2.3)) are those of the type O_1 such that $p|\Theta$; the other operations, however, can make such reduction possible. ~~We note also that for a particular p, operations of the type O_1 and O_2 commute with those of that of the type O_3.~~

<u>Definition 3.2.1.</u> A *p-operation*: A single operation of the type O_3 for a prime p, followed by at most one operations of the type O_2 and an O_1 for the *same* p such that

(i) the R new forms thus obtained are integral,

(ii) these new forms have a lower value for Θ (lower than that of the old forms).

A system of forms $F_1, \ldots, F_R$ is called *normalised* iff no p-operation as above is possible on this system.

Since each p-operation reduces Θ for some p such that $p|\Theta$, only a finite number of successive ones are possible; and starting from any given system of forms by applying possible p-operations until we reach a situation where for *all* p no further p-operations are possible, we will always arrive at a normalised system.

EFFECT OF NORMALISATION ON INTEGER SOLUTIONS

We shall now show that if a normalised system obtained from the given system (3.1.1), as above, has a "bounded solution" (solution bounded as a function of Λ) then a "bounded solution" exists for the given system also.

Lemma 3.2.3. Let $F_1(\underline{x}), \ldots, F_R(\underline{x})$ be as in (3.1.2), Θ be as in (3.2.3) and p a prime dividing Θ. Suppose $F_1'(\underline{x}), \ldots, F_R'(\underline{x})$ are obtained from $F_1, \ldots, F_R$ by a p-operation O_1 or O_2 and suppose the system

(3.2.6) $$F_i'(\underline{x}) = 0, \quad (1 \leq i \leq R)$$

has a solution $\underline{x}$ satisfying

(3.2.7) $$|\underline{x}| << \Lambda'^{B_1},$$

where $\Lambda' = \max_{1 \leq i \leq R} |F_i'|$. Then there exists a constant $B_2 = B_2(B_1, n, R, k)$ such that the system (3.1.1) has a solution $\underline{y}$, say, satisfying

$$|\underline{y}| << \Lambda^{B_1} p^{B_2}.$$

Proof. We first consider $F_1', \ldots, F_R'$ obtained from $F_1, \ldots, F_R$ by performing O_1 with the first t, say, rows of (λ_{ij}) divided by p (rearranging, if necessary) and let $\underline{x}$ be a solution of (3.2.6) satisfying (3.2.7). Then we easily see that $\underline{y} = \underline{x}$ satisfies (3.1.2) and $\Lambda' \leq \Lambda$, so the result follows with $B_2 = 0$.

see that $\underline{y} = (px_1, \ldots, px_t, x_{t+1}, \ldots, x_n)$ is a solution of (3.1.1) and the result follows in this case, since

$$|\underline{y}| \ll p\,\Lambda'^{B_1} \ll p(p^{-t}\Lambda)^{B_1} \ll \Lambda^{B_1} p^{-tB_1+1}$$

and taking $B_2 = -tB_1 + 1$.

Similarly if $F_1', \ldots, F_R'$ were obtained from $F_1, \ldots, F_R$ by a p-operation O_2, $\Lambda' \ll p^{k}\Lambda$ and $|\underline{y}| < p|\underline{x}|$ and so with $B_2 = B_1(k+1)$.

Lemma 3.2.4. Let $F_1, \ldots, F_R, \Theta$, p be as in Lemma 3.2.3 and let $F_1', \ldots, F_R'$ be forms obtained from $F_1, \ldots, F_R$ by a p-operation O_3. Suppose the system (3.2.6) has a solution $\underline{x}$ satisfying (3.2.7). Then the system (3.1.1) has a solution $\underline{y}$ such that

$$|\underline{y}| \ll p^{B_1}\Lambda^{B_1} \quad .$$

Proof. The p-operation O_3 is equivalent to pre-multiplying the coefficient matrix by an $R \times R$ integral matrix U, which is as described in O_3 and so $\det U = 1$. Thus $\underline{y} = \underline{x}$ is also a solution of the original system (3.1.1). But

$$\Lambda' \leqslant (p-1)R\Lambda \ll p\,\Lambda \quad .$$

Therefore

$$|\underline{y}| \ll (p\,\Lambda)^{B_1} \quad ,$$

which gives the result.

Lemma 3.2.5. Let $F_1', \ldots, F_R'$ be a *normalised* system obtained from a system $F_1, \ldots, F_R$ by a finite number of p-operations for primes p dividing Θ, as in (3.2.3). Suppose for this normalized system, (3.2.6) has a solution $\underline{x}$ satisfying

$$|\underline{x}| \ll (\Lambda')^{B'} .$$

Then the system (3.1.1) has a solution $\underline{y}$ satisfying

$$|\underline{y}| \ll \Lambda^{B},$$

where $B = B(B',k,R,n)$.

Proof. All primes involved in the normalisation operations must divide $\Theta(F_1,\ldots,F_R)$. And so in the process of normalisation at each step of reduction, at worst we would have performed one from each operation O_1, O_2, O_3. There would have been at most $\Omega(\Theta)$ such operations. Therefore by repeated application of Lemmas 3.2.3, 3.2.4, there will be a solution $\underline{y}$ of (3.1.1) such that

$$|\underline{y}| \ll \Theta^{B_2} \Lambda^{B'} , \quad B_2 = B_2(B',k,n,R)$$

$$\ll \Lambda^{MRB_2 + B'} \ll \Lambda^{B} , \text{ say},$$

since Θ has M products, each of which is a determinant of $R \times R$ submatrix of (λ_{ij}) and so is $\ll \Lambda^{R}$.

From this lemma, it follows that to prove our Theorem 1, it is enough to prove the same for a normalised system. Also from Lemma 3.2.1, it follows that it is enough to prove the theorem for a system which satisfies the hypothesis H with $s = [9k^2 R \log 3Rk]$.

Hence it is enough to prove the following theorem:

THEOREM 2. There exist constants $B_1(k)$, $B_2(n,R,k)$ such that the following result holds: Let $F_1(\underline{x}),\ldots,F_R(\underline{x})$ be integral forms as in (3.1.2), in n variables. Suppose that $F_1,\ldots,F_R$ are normalised, n satisfies (3.1.3), and the hypothesis H with $s = [9k^2 R \log 3Rk]$ holds. Then the system (3.1.1) has an integral solution $\underline{x}$ satisfying

$$0 < |\underline{x}| \leqslant B_2 \Lambda^{B_1} ,$$

where Λ is as in (3.1.4).

CONGRUENCES

For the reasons given above, from now on we consider forms $F_1,\dots,F_R$ which are normalised and for which the hypothesis H holds. Thus in this subsection, we shall study the p-adic solvability of normalised forms.

Lemma 3.2.6. Let $F_i(\underline{x})$ be as in (3.1.2) and k odd. Suppose

(3.2.8) $$n_0 = [9R \log 3Rk] ,$$

and let p^τ be the exact power of p dividing k and let

(3.2.9) $$\gamma = \gamma(k,p) = \begin{cases} 1 , & \text{if } \tau = 0 \\ \tau + 1, & \text{if } \tau > 0 \text{ and } p > 2 \\ \tau + 2, & \text{if } \tau > 0 \text{ and } p = 2. \end{cases}$$

Then if $n \geqslant n_0$, then the congruences

(3.2.10) $$F_i(\underline{x}) \equiv 0 \pmod{p^\gamma}, \quad (1 \leqslant i \leqslant R)$$

have a solution with not all of $x_1,\dots,x_n$ divisible by p.

Proof. See Lemma 6 of D-L.

Definition 3.2.2. We call a solution $\underline{x}$ of the system of equations (3.1.1) *non-singular* if the matrix $\frac{1}{h}\left(\frac{\partial F_i}{\partial x_j}\right)$ evaluated at this solution has rank R.

Lemma 3.2.7. Suppose $F_i(\underline{x})$ are as in (3.1.2) and let

(3.2.11) $$F_i(\underline{x}) \equiv \phi_i(\underline{x}) \pmod{p^\gamma}, \quad (1 \leqslant i \leqslant R).$$

Suppose for every j there exists at least one i for which $\lambda_{ij} \not\equiv 0 \pmod p$ (that is no column $\equiv \underline{0} \pmod p$) and any form

$$g_1\phi_1 + \dots + g_R\phi_R$$

such that some $g_i = 1$ and $0 \leqslant g_i \leqslant p-1$, $(1 \leqslant i \leqslant R)$ contains at least n_0 coefficients not divisible by p, where n_0 satisfies (3.2.8). Then the congruences (3.2.10) with γ replaced by s have a non-singular solution $\pmod{p^s}$, for all $s \geqslant \gamma$, (see Definition 2.4.2).

Proof. Let $F = g_1\phi_1 + \ldots + g_R\phi_R$ be any form where the g_i's are not divisible by p, say $p \nmid g_1$. Then the number of variables with coefficients not divisible by p is the same as for the form $h_1\phi_1 + \ldots + h_R\phi_R$, where $h_i \equiv g_1' g_i \pmod p$, $g_1' g_1 \equiv 1 \pmod p$ and $0 \leqslant h_i \leqslant p-1$ for all i. And so by the hypothesis of this lemma, this number is at least n_0. Then by Lemma 7 of D-L, the congruences (3.2.10) have a solution of rank R.

From this solution, by the standard procedure, as in Lemma 9 of D-L, we can obtain a non-singular solution of the congruences (3.2.10) with γ replaced by $s > \gamma$. Thus the proof is completed.

(From the existence of non-singular solution of (3.2.10) for every $s \geqslant \gamma$, by p-adic compactness property, it follows that the equations

$$\phi_i(\underline{x}) = 0, \quad (1 \leqslant i \leqslant R)$$

have a non-singular solution in the p-adic field.)

Lemma 3.2.8. Suppose $F_1(\underline{x}), \ldots, F_R(\underline{x})$ is a system of linearly independent normalised forms of the type (3.1.2). Then for each prime p, they can be written as follows, after re-ordering the variables if necessary,

$$(3.2.12) \qquad F_i(\underline{x}) = F_{i\,0}(x_1, \ldots, x_m) + p\, F_{i\,1}(x_{m+1}, \ldots, x_n), \quad (1 \leqslant i \leqslant R),$$

where $m = m(p)$ and each variable $x_1, \ldots, x_m$ occurs in some $F_{i\,0}$ with coefficient not divisible by p and

$$(3.2.13) \qquad m \geqslant \frac{n}{k}.$$

Also let $q = q(p)$ be the minimal number of variables which occur in any $\sum_{i=1}^{R} g_i F_{i\,0}$ with some $g_i = 1$ and $0 \leqslant g_i \leqslant p-1$ for all i, and with coefficients not divisible by p. Then

(3.2.14) $$q \geqslant \frac{n}{Rk} \quad .$$

Proof. The results (3.2.12) and (3.2.13) correspond to the results (34) and (35) respectively in Lemma 11 of D-L.

So we shall prove (3.2.14). Let $\sum_{i=1}^{R} g_i F_{i\,0}$ be the linear combination as in the lemma,having the minimal number of variables q as defined above and let $g_k = 1$ $(0 \leqslant g_i \leqslant p-1)$. Consider

$$F_i'(\underline{x}) = F_i(\underline{x}), \quad i \neq k$$

$$F_k'(\underline{x}) = \sum_{i=1}^{R} g_i F_i(\underline{x}), \quad g_k = 1, \quad 0 \leqslant g_i \leqslant p-1.$$

Thus $F_1',\ldots,F_R'$ are obtained from $F_1,\ldots,F_R$ by an operation of the type O_3 with $\det U = 1$.

Since q is the number of variables in $\sum_{i=1}^{R} g_i F_{i\,0}$ with coefficients not divisible by p , let these variables be $x_1,\ldots,x_q$, by rearranging if necessary. Now consider

$$F_k'' = p^{-1} F_k'(px_1,\ldots,px_q,x_{q+1},\ldots,x_n)$$
$$F_i'' = F_i'(px_1,\ldots,px_q,x_{q+1},\ldots,x_n), \quad i \neq k .$$

Then since $\sum g_i F_{i\,0}$ has integral coefficients, it is clear from the defining property of F_k' that F_k'' has integral coefficients. And

$$\Theta(F_1'',\ldots,F_R'') = p^{kRMq/n-M}\Theta(F_1,\ldots,F_R),$$

by (3.2.4) and (3.2.5) of Lemma 3.2.2, with $\nu_1 = \ldots = \nu_q = 1$, $\nu_i = 0$ for $q+1 \leqslant i \leqslant n$ and $D = p^{-1}$. The given forms $F_1,\ldots,F_R$ are normalised and so the power of p in Θ cannot be reduced by any p-operations (in the sense of Definition 3.2.1). Thus the power of p in $\Theta(F_1'',\ldots,F_R'')$ cannot be negative and so

$$\frac{kRMq}{n} - M \geq 0,$$

from which (3.2.14) follows. This completes the proof of this lemma.

Lemma 3.2.9. Suppose $F_1, \ldots, F_R$ (as in (3.1.2)) is a normalised system of forms in n variables $x_1, \ldots, x_n$ where

(3.2.15) $$n \geq Rk[9R \log 3Rk] .$$

Then the system of congruences

$$F_i(\underline{x}) \equiv 0 \pmod{p^s}, \quad (1 \leq i \leq R)$$

for $s \geq \gamma$ has a non-singular solution $\pmod{p^s}$ for every prime p.

Proof. The proof follows steps similar to those of Theorem 3 (for odd k) of D-L; the difference is that we obtain a non-singular solution $\pmod{p^s}$. We appeal to our Lemmas 3.2.6 to 3.2.8.

The value of q in Lemma 3.2.8, by the same lemma satisfies $q \geq \frac{n}{Rk} \geq [9R \log 3Rk]$, by (3.2.15). That is $q \geq n_0$, where n_0 is as defined in Lemma 3.2.6. Thus the hypothesis of Lemma 3.2.7 is satisfied and so the result follows by the same lemma.

3. DISSECTION AND MINOR ARCS

In order to estimate the integral (3.1.8), and hence to estimate N(P), the number of solutions of our system of equations (3.1.1), we follow the usual convention of dividing the region of integration U (as in(3.1.9)) into smaller regions called major arcs and minor arcs. Then we estimate the contribution of these arcs to the integral (3.1.8) separately,using the available tools (the results in Chapter 2).

In this section, I shall give this dissection and estimate the contribution of minor arcs to

$$(3.3.1) \qquad \int_U \prod_{j=1}^{n} T(\Lambda_j)\, d\underline{\alpha}$$

where Λ_j's are as in (3.1.10) and $T(\Lambda_j)$ as in (3.1.7).

Major Arcs and Minor Arcs

For any integers $Q, A_1, \ldots, A_R$ satisfying

$$(3.3.2) \qquad 0 \leqslant A_i \leqslant Q, \quad (Q, A_1, \ldots, A_R) = 1, \quad 1 \leqslant Q \leqslant \Lambda^R P^{k\sigma} \quad ,$$

where Λ is as in (3.1.4), we define the *major arcs*, denoted by $M(\underline{A}, Q)$, to consist of all $\underline{\alpha}$ in U (see (3.1.9)) that satisfy

$$(3.3.3) \qquad \alpha_i = \frac{A_i}{Q} + \beta_i\,, \quad |\beta_i| < Q^{-1} \Lambda^{R-1} P^{-k+k\sigma}, \quad (1 \leqslant i \leqslant R)$$

where $\sigma = \dfrac{1}{8k^2 \log k}$. Let M be the union of all $M(\underline{A}, Q)$ and m called the *minor arcs*, be the complement of M in U.

Suppose

$$(3.3.4) \qquad P^{k-2k\sigma} > 2\Lambda^{R-1} \; .$$

Then it is easily checked that two major arcs do not overlap unless they are identical. So from now on, we suppose (3.3.4) holds. Now I shall estimate the contribution of m to our integral.

The contribution of the minor arcs

Lemma 3.3.1. Let m be the set of minor arcs and suppose the hypothesis H holds for the system (3.1.1). Then for $n \geqslant [9R^3k^2 \log 3Rk]$, we have

$$(3.3.5) \qquad \int_m \prod_{j=1}^{n} |T(\Lambda_j)|\, d\underline{\alpha} \ll P^{n-Rk-\sigma} \quad ,$$

where Λ_j, $T(\Lambda_j)$ $(1 \leqslant j \leqslant n)$ are as in (3.1.~~9~~ 10), (3.1.7) respectively and $\sigma = \dfrac{1}{8k^2 \log k}$.

<u>Proof</u>. The forms $F_1, \ldots, F_R$ are independent forms and so the rank of the coefficient matrix (λ_{ij}) is R. Therefore there is an $R \times R$ non-singular submatrix. We specify this submatrix. Then since the hypothesis H holds, there are $s - 1$ more disjoint non-singular $R \times R$ submatrices other than the specified one, where s is as in the hypothesis H.

For $\underline{\alpha}$ in m and for any R independent Λ_j's the alternative (i) of Lemma 2.2.8 holds. And so we use this for the j's corresponding to the above specified non-singular submatrix. Also we use the trivial estimate P for the j's not corresponding to any of the above s submatrices. Then the left hand side of (3.3.5) is

$$<< P^{n-sR} P^{R-\sigma} \int_m \prod_j |T(\Lambda_j)| d\underline{\alpha},$$

where the product is over the j's corresponding to the above $(s-1)$ $R \times R$ submatrices. If $s-1$ is even, then by writing this integrand as a product of $s - 1$ sets of R, $|T(\Lambda_j)|$'s with Λ_j's belonging to the same set, we can apply Lemma 2.2.7, since $s - 1 = s_0 + 1$, and then Hölder's inequality, so the above integral becomes

$$<< P^{R(s-1-k)}.$$

Hence the integral in (3.3.5) is

$$<< P^{n-sR+R-\sigma+R(s-1-k)},$$

$$<< P^{n-Rk-\sigma},$$

which is the required result. If $s-1$ is odd, we can use a similar argument with $s-2$ sets of R $|T(\Lambda_j)|$'s.

4. PRUNING THE MAJOR ARCS

In this section, we first recall the definition of the major arcs $M(\underline{A},Q)$; namely (3.3.3). We shall reduce the range of Q in (3.3.2) to

$$1 \leqslant Q \leqslant P^{\omega} ,$$

where $\omega > 0$ is a given small number (which we shall determine suitably later) and then contract $M(\underline{A},Q)$ to $M'(\underline{A},Q)$, say, for which

(3.4.1) $$|\beta_i| < P^{-k+\tau}, \quad (1 \leqslant i \leqslant R),$$

where $\tau > 0$ is also a given number, the value of which is to be determined later.

We note that this is a contraction for $Q \leqslant P^{\omega}$, if

$$Q^{-1}\Lambda^{R-1}P^{-k+k\sigma} > P^{-k+k\sigma-\omega} > P^{-k+\tau} .$$

Hence, from now on we shall take

(3.4.2) $$\omega + \tau < k\sigma.$$

The approximations $\frac{A_i}{Q}$ to α_i $(1 \leqslant i \leqslant R)$ imply approximations $\frac{a_j}{q_j}$, say, to the linear forms Λ_j $(1 \leqslant j \leqslant n)$, where Λ_j's are as in (3.1.10). These are related by the following:

$$\left.\begin{aligned} &(3.4.4) \qquad \Lambda_j = \sum_{i=1}^{R} \lambda_{ij}\, \alpha_i = \frac{a_j}{q_j} + \gamma_j\,, \\ &\text{where} \\ &(3.4.5) \qquad \frac{a_j}{q_j} = \frac{1}{Q}\sum_{i=1}^{R} \lambda_{ij}\, A_i\,, \quad (a_j\,,q_j) = 1 \\ &\text{and} \\ &(3.4.6) \qquad \gamma_j = \sum_{i=1}^{R} \lambda_{ij}\, \beta_i \end{aligned}\right\} (1 \leqslant j \leqslant n)\ .$$

The contribution of the arcs $M'(\underline{A},Q)$ constitutes the main term of (3.1.8) and that of the arcs $M(\underline{A},Q) - M'(\underline{A},Q)$ forms a part of the error term. In this section, I shall estimate the latter contribution.

Lemma 3.4.1. Let n,Λ be as in (3.1.3), (3.1.4) respectively; and $\delta > 0$ and integer P be such that

$$(3.4.7) \qquad 1 - \delta - k\sigma > 0, \quad 0 < \delta < 1/3$$

and

$$(3.4.8) \qquad P^{1-\delta-k\sigma} > \Lambda^{R} R.$$

Suppose the hypothesis H holds. Then the contribution to the integral in (3.1.8) by $M(\underline{A},Q)$ with $Q > P^{\omega}$ is

$$(3.4.9) \qquad \ll \Lambda^{R(s/R-1)}\, P^{n-kR-\omega(s/k-2-\epsilon_0)}\,,$$

where $s = [9k^2R\log 3Rk]$, $\varepsilon_0 = (R-1)\varepsilon > 0$.

Proof. Since the rank of the coefficient matrix (λ_{ij}) is R and the hypothesis H holds, we have s disjoint sets of R, j's (in $[1,n]$) such that each set of R, Λ_j's for j's corresponding to the same set are independent (one of these s sets is a specified one). Then by taking the trivial estimate P for all the $|T(\Lambda_j)|$'s for the j's not corresponding to any of the above s sets, we can write

$$(3.4.10) \qquad \prod_{j=1}^{n} |T(\Lambda_j)| \leqslant P^{n-sR} \prod_{(j_1,\ldots,j_R)} |T(\Lambda_{j_1}) \ldots T(\Lambda_{j_R})| \ ,$$

where the product on the right hand side is over the disjoint R-tuples of the above mentioned s sets.

Now we consider (3.4.4) to (3.4.6) for $j = j_1,\ldots,j_R$ belonging to any one of the s sets. By (3.4.5), $q_j\,|Q$ and so we have (3.4.4) where

$$q_j \leqslant Q \leqslant \Lambda^R P^{k\sigma} \leqslant P^{1-\delta}$$

by (3.3.2), (3.4.7) and (3.4.8); and so

$$|\gamma_j| \leqslant R\Lambda^R Q^{-1} P^{-k+k\sigma} < q_j^{-1} P^{-k+1-\delta}$$

by (3.4.6), (3.3.3), (3.4.7) and (3.4.8), since $\underline{\alpha}$ is in $M(\underline{A},Q)$. Then the alternative (i) of Lemma 2.2.1 holds, and by the last part of lemma 2.2.3 we have,

$$(3.4.11) \qquad |T(\Lambda_j)| \ll q_j^{-1/k} \min\{P, \ |\gamma_j|^{-\frac{1}{k}}\}, \quad j = j_1,\ldots,j_R \ .$$

Using this and extending the region of integration of (3.1.8) for $M(\underline{A},Q)$ with $Q > P^{\omega}$, to the whole R-space W, say, we get

$$\int \prod_{j=j_1,\ldots,j_R} |T(\Lambda_j)|^s \, d\underline{\alpha}$$

$$\ll \sum_{Q>P^{\omega}} \sum_{\underline{A}} \prod_{j=j_1,\ldots,j_R} |q_j|^{-s/k} \int_W \prod_{j=j_1,\ldots,j_R} \min\{P, \ |\gamma_j|^{-\frac{1}{k}}\}^s \, d\underline{\beta}$$

$$\ll \sum_{Q>P^{\omega}} \sum_{\underline{A}} \prod_{j=j_1,\ldots,j_R} q_j^{-s/k} |\Delta(j_1,\ldots,j_R)|^{-1} P^{(s-k)R} \ ,$$

(where $\Delta(j_1,\ldots,j_R) = \det|(\lambda_{ij})|$, $1 \leqslant i \leqslant R$, $j = j_1,\ldots,j_R$), by arguing as in Lemma

2.3.1,since $\Lambda_{j_1},\ldots,\Lambda_{j_R}$ are independent as the $j_1,\ldots,j_R$ belong to one of the s sets.

Using this last inequality, it follows by Hölder's inequality that the left hand side of (3.4.9) is

$$(3.4.12) \qquad \ll \sum_{Q>P^{\omega}} \sum_{\underline{A}} \prod_{j \in s \text{ sets}} q_j^{-1/k} \cdot P^{(s-k)R+n-sR},$$

since $|\Delta(j_1,\ldots,j_R)| \geqslant 1$ (as it is non-zero and λ_{ij} integers). So now it is enough to estimate

$$\sum_{Q>P^{\omega}} \sum_{\underline{A}} \prod_{j=j_1,\ldots,j_R} q_j^{-s/k},$$

as then we can estimate the required sum by applying Hölders inequality. And the above sum is, by Lemma 2.3.2, (since $\Lambda_{j_1},\ldots,\Lambda_{j_R}$ are independent)

$$\ll \Lambda^{R+R\epsilon_0} \min\{1, \Lambda^{R(s/k-2-\epsilon_0)} P^{-\omega(s/k-2-\epsilon_0)}\},$$

since $|\Delta| \ll \Lambda^R$. Using this and Hölder's inequality in (3.4.10), our result (3.4.9) follows.

<u>Lemma 3.4.2</u>. Suppose n is as in (3.1.3) and the hypothesis H holds. Then the difference between the contributions of $M(\underline{A},Q)$ and $M'(\underline{A},Q)$ with $Q \leqslant P^{\omega}$, to the right hand side of (3.1.8) is

$$(3.4.13) \qquad \ll \Lambda^{-R+1+\frac{4}{k}(2R-1)} \cdot P^{n-Rk-\tau(\frac{s}{k}-1)},$$

where Λ is as in (3.1.4).

<u>Proof</u>. As in the proof of the previous lemma, there exist s disjoint sets of R, Λ_j's such that the Λ_j's in each set are independent. Also reasoning as in the same lemma, since $\underline{\alpha}$ is in $M(\underline{A},Q)$, (3.4.11) holds for Λ_j's with j in any of these s sets.

To prove our result (3.4.13), it is enough to estimate

$$(3.4.14) \qquad \sum_{Q \leqslant P^{\omega}} \sum_{\underline{\Lambda}} \int \prod_{j=1}^{n} |T(\Lambda_j)| d\underline{\beta} ,$$

where the integration is over the region

$$P^{-k+\tau} < \max_{1 \leqslant i \leqslant R} |\beta_i| < Q^{-1} \Lambda^{-1} P^{-k+k\sigma} .$$

In this region, the integral in (3.4.14) is

$$< \int_T \prod_{j=1}^{n} |T(\Lambda_j)| d\underline{\beta} ,$$

where

$$T = \{\underline{\beta};\ \max_{1 \leqslant i \leqslant R} |\beta_i| > P^{-k+\tau}\} .$$

By taking the trivial estimate for $n - sR$, Λ_j's j not corresponding to any of the above s sets and applying (3.4.11) for the remaining, the above integral is

$$< \int_T P^{n-sR} \prod_{(j_1,\ldots,j_R)} \Big(\prod_{j=j_1,\ldots,j_R} q_j^{-1/k} \min\{P, |\gamma_j|^{-\frac{1}{k}}\} \Big) d\underline{\beta}$$

where $j_1,\ldots,j_R$ belong to the same set amongst the s sets and $(j_1,\ldots,j_R)$ runs over the s sets.

Now we can apply the result analogous to (2.3.6) of Lemma 2.3.1, and then by Hölder's inequality we get the above inequality as

$$<< P^{n-sR} \prod_{(j_1,\ldots,j_R)} (q_{j_1} \cdots q_{j_R})^{-1/k} \Lambda^{(\frac{s}{k}-1)(R-1)} P^{(s-k)R-\tau(\frac{s}{k}-1)} ,$$

since $|\Delta| \geqslant 1$ (the Δ occurring in (2.3.6)).

Using this in (3.4.14), the required estimate is

$$<< \Lambda^{(\frac{s}{k}-1)(R-1)} P^{n-Rk-\tau(\frac{s}{k}-1)} \sum_{Q \leqslant P^{\omega}} \sum_{\underline{\Lambda}} \prod_{(j_1,\ldots,j_R)} (q_{j_1} \cdots q_{j_R})^{-1/k}$$

$$<< \Lambda^{(\frac{s}{k}-1)(R-1)} P^{n-Rk-\tau(\frac{s}{k}-1)} \Lambda^{R+(\frac{s}{k}-1)R} ,$$

by applying (2.3.10) (of Lemma 2.3.2) and Hölder's inequality for the inner summation and then allowing Q to go from 1 to ∞.

We note that we are now left only with the contribution of $M'(\underline{A},Q)$ (the rest have been all estimated) and this forms the main term of our integral (3.1.8).

Lemma 3.4.3. For any integers a,q and real number γ, let $S(a,q)$, $I(\gamma)$ be as in (2.2.3), (2.2.2) respectively; n,Λ be as in (3.1.3), (3.1.4) respectively and the hypothesis H holds. Suppose δ, ω, τ, P be such that

(3.4.15) $$\delta > \omega + \tau$$

and

(3.4.16) $$P^{\delta-\omega-\tau} > \Lambda R.$$

Then we have,

(3.4.17) $$\sum_{Q\leq P^{\omega}} \sum_{\underline{A}} \int_{M'(\underline{A},Q)} \prod_{j=1}^{n} T(\Lambda_j)\,d\underline{\alpha} = \mathfrak{S}(P^{\omega})J(P^{-k+\tau}) + E,$$

where $\underline{A} = (A_1,\ldots,A_R)$ is over $0 \leq A_1,\ldots,A_R < Q$, $(A_1,\ldots,A_R,Q) = 1$,

(3.4.18) $$\mathfrak{S}(P^{\omega}) = \sum_{Q\leq P^{\omega}} \sum_{\underline{A}} \prod_{j=1}^{n} q_j^{-1} \; S(a_j,q_j),$$

(3.4.19) $$J(P^{-k+\tau}) = \int_{B} \prod_{j=1}^{n} I(\gamma_j)\,d\underline{\beta},$$

the integration is over the region

(3.4.20) $$B = \{\underline{\beta};\ |\beta_i| \leq P^{-k+\tau},\ (1 \leq i \leq R)\}$$

and

(3.4.21) $$E \ll P^{n-Rk-1+2\delta+(R+1)\omega+\tau R}.$$

<u>Proof</u>. For $\underline{\alpha}$ in $M'(\underline{A},Q)$, $\alpha_i = \dfrac{A_i}{Q} + \beta_i \quad (1 \leqslant i \leqslant R)$ with

$$1 \leqslant Q \leqslant P^{\omega}, \quad |\beta_i| \leqslant P^{-k+\tau} .$$

And the corresponding Λ_j, $(1 \leqslant j \leqslant n)$ are given by (3.4.4) with

$$q_j \leqslant Q \leqslant P^{\omega} < P^{\delta}, \quad |\gamma_j| << \Lambda P^{-k+\tau} << P^{-k+\delta-\omega} << q_j^{-1} P^{-k+\delta}$$

by (3.4.15), (3.4.16); and so by Lemma 2.2.3 we have,

$$\text{(3.4.22)} \qquad T(\Lambda_j) = q_j^{-1} S(a_j, q_j) I(\gamma_j) + O(P^{2\delta}), \quad (1 \leqslant j \leqslant n).$$

In order to approximate $\prod_{j=1}^{n} T(\Lambda_j)$, we have to multiply together the approximations in (3.4.22). The error term can be estimated by multiplying the estimates of any set of main terms of (3.4.22) and of the complementary set of error terms. It is clear that any main term is majorised by P and any error term by $P^{2\delta}$. The error term is thus majorised by taking the product of $n - 1$ main terms and a single error term. ~~Hence the error while estimating~~ $\prod_{j=1}^{n} T(\Lambda_j)$ ~~is~~

$$<< P^{2\delta} \prod_{\substack{j=1 \\ j \neq t}}^{n} T(\Lambda_j)$$

$$<< P^{2\delta} P^{n-1-sR} \prod_{(j_1,\ldots,j_R)} \left(\prod_{j=j_1,\ldots,j_R} q_j^{-1/k} \min\{P, P^{-k+1} |\gamma_j|^{-1}\} \right)$$

~~where~~ $(j_1,\ldots,j_R)$ ~~corresponds to the same set amongst the~~ s ~~sets as in the previous lemma.~~ And so the error in approximating the integral in (3.4.17) is $\ll P^{n-1} . P^{2\delta}$ times the total area of the pruned major arcs $M'(\underline{A},Q)$, and so is

$$<< P^{2\delta+n-1} . P^{(R+1)\omega} . P^{(-k+\tau)R} ;$$

since $|A_i| \leq Q \leq P^{\omega}$ and $|\beta_i| \leq P^{-k+\tau}$ in $M'(\underline{A},Q)$, for $1 \leq i \leq R$, This is obtained by extending the area of integration for this error to the whole space and using the result (2.3.5) of Lemma (2.3.1), and Hölder's inequality.

The total error term is obtained by summing this error over $\underline{A}$ and then over $Q \leqslant P^{\omega}$. This gives (3.4.21). The product of the main terms gives the first term of the right hand side of (3.4.17). Hence the lemma is proved.

5. SINGULAR INTEGRAL

This section is devoted to find a lower bound in terms of Λ and P of the integral $J(P^{-k+\tau})$, as in (3.4.19). We shall do this in the following lemma:

Lemma 3.5.1. Let $J(P^{-k+\tau})$ be as in (3.4.19). Suppose

(3.5.1) $$P^{k} > 6(3nR!)^{2R^2}\Lambda^{2R^2}, \quad 0 < \eta < \tfrac{1}{3}.$$

Then we have ,

(3.5.2) $$J(P^{-k+\tau}) \gg \Lambda^{-R} P^{n-Rk}\{C_0\Lambda^{-(3n)R^2} + O(\Lambda^{\frac{s}{h}(R-1)+1}P^{-\tau(\frac{s}{h}-1)})\},$$

where C_0 is a positive constant depending only on n, R, h, and $s = [9k^2 R \log 3Rk]$. (For the meaning of the notation 'O', see the table of notation in page 3.)

Proof. We first recall B given by (3.4.20), which is the region of integration of $J(P^{-k+\tau})$. Now we extend this integral to the whole of $\underline{\beta}$ space. The remainder space is then

$$\{\underline{\beta};\ \max_{1\leqslant j\leqslant R}|\beta_j| > P^{-k+\tau}\},$$

which is the region T as in (2.3.4). We also have, by Lemma 2.2.3,

$$|I(\gamma_j)| \ll \min\{P, |\gamma_j|^{-1/h}\},$$

and s sets of R independent γ_j's , as in the proof of Lemma 3.4.1. So by taking the trivial estimate P for the remaining $n - sR$, $|I(\gamma_j)|$'s and applying Hölder's inequality and the result analogous to (2.3.6) of Lemma 2.3.1, the difference between the above integral over B and that over the whole space is

$$\ll \Lambda^{(\frac{s}{h}-1)(R-1)} P^{n-sR+(s-k)R-\tau(\frac{s}{h}-1)}$$

(3.5.3) $$\ll \Lambda^{(\frac{s}{h}-1)(R-1)} P^{n-Rk-\tau(\frac{s}{h}-1)} .$$

Hence we can now replace our integral by

$$\lim_{\phi\to\infty} \int_{-\phi}^{\phi} \cdots \int_{-\phi}^{\phi} \prod_{j=1}^{n} I(\gamma_j)\, d\beta_1 \cdots d\beta_R ,$$

where γ_j $(1 \leqslant j \leqslant n)$ are linear forms in $\beta_1,\ldots,\beta_R$ as in (3.4.6) and $I(\gamma_j)$ as in (2.2.2). In $I(\gamma_j)$ $(1 \leqslant j \leqslant n)$ we put $\xi = y^{1/k}$ and then by the standard arguments (as in Lemma 30 of D-L), the above limit can be written as

(3.5.4) $$\frac{1}{k^n} \lim_{\phi\to\infty} \int_{\mathcal{B}(P^k)} (y_1 \cdots y_n)^{-1+1/k} \prod_{i=1}^{R} \frac{\sin 2\pi\phi z_i}{\pi z_i}\, d\underline{y} ,$$

where $\underline{y} = (y_1,\ldots,y_n)$, $z_i(\underline{y})$ are linear forms

(3.5.5) $$z_i(\underline{y}) = \sum_{j=1}^{n} \lambda_{ij} y_j , \quad (1 \leqslant i \leqslant R)$$

and $\mathcal{B}(P^k)$ is as in (3.1.6) with P replaced by P^k .

Now we make a change of variables $f : \underline{y} \to \underline{z}$ defined by

$$z_i = z_i(\underline{y}), \quad (1 \leqslant i \leqslant R)$$
$$z_i = y_i \quad , \quad (R+1 \leqslant i \leqslant n).$$

The Jacobian J of this transformation satisfies

$$|J| = \left|\frac{\partial \underline{y}}{\partial \underline{z}}\right| = \frac{1}{\left|\frac{\partial \underline{z}}{\partial \underline{y}}\right|} \gg \Lambda^{-R} ,$$

since we always take the submatrix with the first R columns of (λ_{ij}) to be non-singular. (The implied constant depends only on $\mathcal{R}$.) By the change of variable theorem, the expression in (3.5.4) is

(3.5.6)
$$\gg k^{-n}\Lambda^{-R} \lim_{\phi\to\infty} \int_{f(\mathcal{B}(P^k))} (y_1 \cdots y_R)^{-1+1/k}(z_{R+1} \cdots z_n)^{-1+1/k} \prod_{i=1}^{R} \frac{\sin 2\pi\phi z_i}{\pi z_i}\, d\underline{z}$$

where $y_1,\ldots,y_R$ are uniquely determined by (3.5.5) with $y_i = z_i$, $(R+1 \leqslant i \leqslant n)$.

In Lemma 2.2.10, if we take M to be our coefficient matrix (λ_{ij}), then $m = \Lambda$ and all the hypotheses of Lemma 2.2.10 are satisfied with $Q = P^k$. So by this lemma, it follows that for all $(z_1,\ldots,z_R)$ such that $|z_i| \leqslant \dfrac{1}{(R-1)!\Lambda^{nR}}$, $(1 \leqslant i \leqslant R)$, there exist $(z_{R+1},\ldots,z_n)$, $(= (y_{R+1},\ldots,y_n))$ such that $(z_1,\ldots,z_n)$ is in $f(\mathcal{B}(P^k))$. Therefore by Fourier's integral theorem applied R times, the value of the expression in (3.5.6) is

(3.5.7)
$$\gg k^{-n}\Lambda^{-R} \int_{\mathcal{L}} (y_1\cdots y_R z_{R+1}\cdots z_n)^{-1+1/k}\, d(z_{R+1},\ldots,z_n),$$

where $\mathcal{L}$ is the region in $(n-R)$-dimensional space given by

$$(0,\ldots,0,z_{R+1},\ldots,z_n) = f(\underline{y}), \quad \text{for some } \underline{y} \text{ in } \mathcal{B}(P^k)$$
$$(= f(y_1,\ldots,y_R,z_{R+1},\ldots z_n)).$$

And for this $z_{R+1},\ldots,z_n$, we have

$$1 \leqslant |z_i| \leqslant P^k, \quad (R+1 \leqslant i \leqslant n)$$

and there exist unique $(y_1,\ldots,y_R)$ such that

$$\sum_{j=1}^{n} \lambda_{ij}\, y_j = 0, \quad (1 \leqslant i \leqslant R)$$

and

$$1 \leqslant |y_i| \leqslant P^k, \quad (1 \leqslant i \leqslant R);$$

also, since k is odd,

$$(y_1\cdots y_R z_{R+1}\cdots z_n)^{-1+1/k} = |y_1\cdots y_R z_{R+1}\cdots z_n|^{-1+1/k} > 0.$$

Now taking the particular case $z_i = 0$, $(1 \leqslant i \leqslant R)$ in (2.2.22) of Lemma 2.2.10, we notice that the set $S(P^k)$ obtained there, is contained in the above set $\mathcal{L}$ and the integrand in (3.5.7) is non-negative and so (3.5.7) is

$$\gg \Lambda^{-R} \int_{S(P^k)} P^{kn(-1+1/k)} \, d(z_{R+1}, \ldots, z_n)$$

$$\gg \Lambda^{-R} P^{-kn+n} \mu(S(P^k)) ,$$

where μ denotes the measure. By part (ii) of Lemma 2.2.10, with $Q = P^k$ and $m = \Lambda$, this is

$$\gg \Lambda^{-R} P^{-kn+n} \cdot \frac{P^{k(n-R)}}{\Lambda^{(3 \cdot n) R^2}} ,$$

$$\gg \Lambda^{-(3nR+1)R} P^{n-Rk} .$$

From this and (3.5.3) the result (3.5.2) follows.

6. SINGULAR SERIES

In this section, we shall obtain a lower bound for the singular series which is a summation extended from $\mathfrak{S}(P^\omega)$ as in (3.4.18). For convenience, I shall restate and relabel it here.

$$(3.6.1) \qquad \mathfrak{S}(P^\omega) = \sum_{Q \leqslant P^\omega} \sum_{\underline{A}} \prod_{j=1}^{n} q_j^{-1} S(a_j, q_j),$$

where $\underline{A} = (A_1, \ldots, A_R)$ is over $0 \leqslant A_1, \ldots, A_R \leqslant Q$ such that $(Q, A_1', \ldots, A_R) = 1$. We shall make use of the results of §4 of Chapter 2 .

Lemma 3.6.1. Let $\mathfrak{S}(P^\omega)$ be as in (3.6.1). Then we have ,

(i)

(3.6.2) $$\mathfrak{S}(P^{\omega}) = \mathfrak{S} + O(\Lambda^{R(s/k-1)}\ P^{-\omega(s/k-2-\epsilon_0)})$$

where

(3.6.3) $$\mathfrak{S} = \sum_{Q=1}^{\infty} \sum_{\underline{A}} Q^{-n}\ S_0(\underline{A},Q) = \sum_{Q=1}^{\infty} \sum_{\underline{A}} \prod_{j=1}^{n} q_j^{-1} S(a_j, q_j),$$

(3.6.4) $$S_0(\underline{A},Q) = \sum_{x_1=1}^{Q} \cdots \sum_{x_n=1}^{Q} e\left(\frac{A_1F_1+\ldots+A_RF_R}{Q}\right),$$

where F_i's are as in (3.1.2); and

(ii) $\mathfrak{S}$ is a positive number and is

(3.6.5) $$\gg \Lambda^{-\epsilon_2}, \quad \epsilon_2 = MR(\tfrac{n}{h} - R)$$

where Λ is as in (3.1.4) and $\epsilon_0 = (R-1)\epsilon$, $\epsilon > 0$,

Proof. The result is a standard one and it follows as in Lemma 29 of D-L, with the error term

$$= \sum_{Q>P^{\omega}} \sum_{\underline{A}} \prod_{j=1}^{n} q_j^{-1}\ S(a_j, q_j),$$

$$\ll \sum_{Q>P^{\omega}} \sum_{\underline{A}} \prod_{j=1}^{n} q_j^{-1/k},$$

by Lemma 2.2.4. As in the proof of Lemma 3.4.1, we apply the result of Lemma 2.3.2 and Hölder's inequality to the inner sum and thus the above sum is

$$\ll \Lambda^{R(s/k-1)}\ P^{-(s/k-2-\epsilon_0)\omega},$$

where $s = [9k^2 R \log 3Rk]$.

To prove the result (ii), we shall first recollect the notations of §4 of Chapter 2. The $\mathfrak{S}$ is in fact (by (2.4.3)),

(3.6.6) $$\mathfrak{S} = \prod_{p} \chi(p)$$

where p runs through all the primes and $\chi(p)$ is as in (2.4.4) which can also be written as

$$\chi(p) = 1 + \sum_{\nu=1}^{\infty} \sum_{\underline{A}} (p^{\nu})^{-n} S_0(\underline{A}, p^{\nu})$$

where S_0 is as in (3.6.4). We also have by expanding $S_0(\underline{A}, p^{\nu})$ and applying Lemma 2.2.4, and using our Hypothesis H, that if $\varpi \nmid \Theta$, then

$$|(p^{\nu})^{-n} S_0(\underline{A}, p^{\nu})| \ll p^{-\nu\delta/k} .$$

[For these ϖ, if $Q = \varpi^{\nu}$, at least δ of the $\sum \lambda_{ij} A_i$ in (3.4.5) are relatively prime to ϖ.]

And so the inner sum in the right hand side of the above expression is

$$\ll p^{-\nu\delta/k + R\nu} \ll p^{-\nu(\delta/k - R)} .$$

Hence

$$|\chi(p) - 1| \ll \sum_{\nu=1}^{\infty} p^{-\nu(\delta/k - R)}$$

$$\ll p^{-(\delta/k - R)} .$$

We note that the exponent $-(\frac{\delta}{k} - R) < -1$.

Hence it follows that there exists a p_0 such that for $p > p_0$,

(3.6.7) $$\prod_{\substack{p > p_0 \\ \varpi \nmid \Theta}} \chi(p) > \tfrac{1}{2} .$$

Since our $F_1, \ldots, F_R$ are normalised linearly independent forms, by Lemma 3.2.8 for each prime p, we have, after re-ordering the variables if necessary,

$$F_i(\underline{x}) = F_{i0}(x_1, \ldots, x_m) + p\, F_{i1}(x_{m+1}, \ldots, x_n), \quad (1 \leqslant i \leqslant R)$$

where the coefficients of each variable $x_1, \ldots, x_m$ in F_{i0} are not divisible by p, $m \geqslant \frac{n}{k}$ and the number of variables q in any prescribed linear combination (as in Lemma 3.2.8) satisfies

$$q \geqslant \frac{n}{Rk} .$$

Thus the m, q in Lemma 2.4.3 also satisfy these inequalities. Now since (3.1.3) holds for n, (2.4.12) is satisfied. And so we can apply the

Lemmas 2.4.3 to 2.4.5 and by these lemmas, we have that for $p > k$, $\chi(p) \gg p^{-\varepsilon_0}$. Hence we have

$$\prod_{\substack{p>k \\ p \mid \Theta}} \chi(p) \gg |\Theta|^{-\varepsilon_0'} \gg \Lambda^{-\varepsilon_1}, \tag{3.6.8}$$

where ε_0' is the largest value of $\varepsilon_0 = \varepsilon_0(h)$ for $h \mid \Theta$, as $|\Theta| < \Lambda^{MR}$, where M is the number of products in Θ and Λ, Θ are as in (3.1.4), (3.2.3) respectively and $0 < \varepsilon_1 = MR\varepsilon_0 \leqslant MR(\frac{n}{2k} - R)$.

Also by Lemma 3.2.9, since our forms $F_1, \ldots, F_R$ are normalised, the congruences (2.4.5) have non-singular solutions (mod p^s), for all p and $s \geqslant \gamma$. And so in particular, for $p \leqslant \max(k, p_0)$. Thus $N(p^{\gamma}) \geqslant 1$, according to the notation in Lemma 2.4.2. And so going back to §4 of Chapter 2, by (2.4.6) and (2.4.7), we have for $p \leqslant \max(k, p_0)$,

$$\chi(p) \geqslant p^{-\gamma(n-R)} \gg 1. \tag{3.6.9}$$

Hence combining the results (3.6.7) to (3.6.9), we have

$$\prod_{p} \chi(p) \gg \Lambda^{-\varepsilon_1},$$

and from this, by (3.6.6), the result (3.6.5) follows. Thus the proof of the lemma is completed.

7. CONCLUSION

In this section, we shall complete the proof of the Theorem 2, which is stated in §2 and then Theorem 1 will follow. We do this by choosing P suitably such that

$$\Lambda^{A} \ll P \ll \Lambda^{A}, \tag{3.7.1}$$

for some (suitable) A such that the main term $C_0\, \mathfrak{S}\, \Lambda^{-(3nR+1)R} P^{n-Rk}$, dominates all the error terms and all the assumptions used while determining the estimates are satisfied. Then there will be a solution inside the box $P\mathcal{B}$ and thus this $\underline{x}$ will satisfy

$$(3.7.2) \qquad 0 < |\underline{x}| \ll P \ll \Lambda^{A} ,$$

the required result.

To choose P, we need to determine all the small constants used, namely τ, ω, δ . From (3.3.5), (3.4.9), (3.4.13), (3.4.21), (3.5.2), (3.6.2), the total error term is

$$(3.7.3) \qquad \ll P^{n-Rk-\delta_0} \Lambda^{s(R-1)+1} ,$$

where

$$(3.7.4) \qquad \delta_0 = \min\left\{\sigma,\ \omega\left(\frac{s}{k} - 2 - \varepsilon_0\right),\ \tau(s-1),\ 1 - 2\delta - (R+1)\omega - \tau R\right\} > 0,$$

$\varepsilon_0 = (R-1)\varepsilon$, $\varepsilon > 0$.

And the main term is (as given above)

$$C_0 \mathfrak{S} \Lambda^{-(3nR+1)R} P^{n-Rk} ,$$

where C_0 is a constant independent of Λ (depending only on n) and

$$\mathfrak{S} \gg \Lambda^{-\varepsilon_2}, \qquad \varepsilon_2 = MR\left(\frac{n}{2} - R\right),$$

by (3.6.5). Also the conditions (3.3.4), (3.4.2), (3.4.8), (3.4.16) and (3.5.1) must be satisfied by P and we need to choose ω, τ, δ so small that they satisfy (3.4.2), (3.4.7) and (3.4.15), namely

$$\omega + \tau < k\sigma = \frac{1}{8k\log k} ,$$

$$k\sigma + \delta < 1 , \qquad \sigma < \delta < \tfrac{1}{3},$$

and

$$\omega + \tau < \delta .$$

And also

$$\tau R + (R+1)\omega + 2\delta < 1 ;$$

so that (3.7.4) will be satisfied, since $s = [9k^2 R\log 3Rk]$.

It is possible to choose such ω, τ, δ . For example

$$\delta < \min\left\{\frac{1}{8k\log k}, \frac{2}{2R+5}\right\}, \quad \tau = \omega < \frac{\delta}{2}$$

is such a choice. Now we choose P satisfying (3.7.1) with

$$A = \max\left\{\frac{R-1}{h_m - 2h_0\alpha}, \frac{R}{1-\delta-k\sigma}, \frac{1}{\delta-\omega-\tau}, \frac{2R^2}{k},\right.$$

$$\left.\frac{(3nR+1)R+s(R-1)+1+MR(\frac{n}{h_m}-R)}{\delta_0}\right\},$$

for δ_0 as in (3.7.4). (The first 4 values arose from (3.3.4), (3.4.8), (3.4.16), (3.5.1).) Then all the conditions needed will be satisfied and the main term also will dominate all the error terms. And so (3.7.2) will hold. Thus Theorem 2 holds and so this completes the proof of Theorem 2. Then Theorem 1 follows by Lemmas 3.2.1 and 3.2.5.

We notice that the value of s in the hypothesis H and the condition (3.2.15) on the number of variables for p-adic solvability suggest the condition (3.1.3) on n, for our theorem.

CHAPTER 4

A SYSTEM OF EQUATIONS AND INEQUALITIES

1. INTRODUCTION

In this chapter, we shall work towards our aim on R-diagonal inequalities, by first considering, for $R \geqslant 1$ and $0 \leqslant r \leqslant R$, a system of r diagonal equations and $R - r$ inequalities

$$(4.1.1) \qquad \begin{cases} F_i(\underline{x}) = 0 & , \quad (i \in S_r), \\ |F_i(\underline{x})| < 1 & , \quad (i \in T_{R-r}), \end{cases}$$

where S_r and T_{R-r} are disjoint subsets of integers in $[1,R]$ having r and $R - r$ elements respectively such that $S_r \cup T_{R-r} = [1,R]$, and all the $F_i(\underline{x})$'s are of the type (1.1.2), namely

$$(4.1.2) \qquad F_i(\underline{x}) = \sum_{j=1}^{n} \lambda_{ij} x_j^k , \quad (1 \leqslant i \leqslant R),$$

whose coefficients λ_{ij}'s are all integers for $i \in S_r$ and reals for $i \in T_{R-r}$ and $k \geqslant 3$ is a fixed odd integer.

For these F_i's, we write

(4.1.3)

$$\Delta_F(j_1,\ldots,j_R) = \Delta(j_1,\ldots,j_R) = \det(\lambda_{ij}), \quad 1 \leqslant i \leqslant R, \quad j = j_1,\ldots,j_R,$$

$$(4.1.4) \qquad \Lambda = \max_{1 \leqslant i \leqslant R} |F_i| ,$$

and

$$(4.1.5) \qquad \Delta = \min|\Delta(j_1,\ldots,j_R)| ,$$

where the minimum is taken for all $j_1, \ldots, j_R$ for which $\Delta(j_1, \ldots, j_R) \neq 0$. We also define the following hypothesis, which conditions the coefficients λ_{ij} of the system:

The forms $f_1, \ldots, f_R$ are linearly independent

<u>HYPOTHESIS H.</u> Let $s = [9k^2 R \log 3Rk]$. If any non-singular $R \times R$ submatrix of the coefficient matrix (λ_{ij}) is specified, then there exist $s - 1$ other non-singular $R \times R$ submatrices which together with the given one are all disjoint.

The main part of this chapter is devoted to proving the following theorem:

<u>THEOREM 3.</u> Let $F_i(\underline{x})$, $(1 \leqslant i \leqslant R)$ be forms as in (4.1.2) of odd degree $k \geqslant 3$, in

(4.1.6) $$n \geqslant [9R^3 k^2 \log 3Rk]$$

variables and suppose for all i, j, $|\lambda_{ij}| \geqslant 1$ whenever $\lambda_{ij} \neq 0$, $\Delta \geqslant 1$ and the hypothesis H holds, where Δ is as in (4.1.5). Then there exist constants $B_1(R,k)$, $B_2(n,R,k)$ such that the system (4.1.1) for $0 \leqslant r \leqslant R$ has an integral solution $\underline{x}$ satisfying

(4.1.7) $$0 < |\underline{x}| \leqslant B_2 \Lambda^{B_1} ,$$

where Λ is as in (4.1.4).

As a consequence, we shall obtain the solvability of R diagonal inequalities $|F_i(\underline{x})| < \varepsilon$, $(1 \leqslant i \leqslant R)$ of odd degree provided n satisfies (4.1.6).

For preliminary materials of this chapter please see Chapters 1 and 2.

OUTLINE OF THE METHOD

The approach is an extension of the modification of Hardy-Littlewood method, in R-dimensions (as used by Pitman [33] for the particular case $R = 2$) as mentioned in Chapter 1. The notations are the same as in Chapter 3, in particular the box $\mathcal{B}(P)$, the trigonometric sum $T(\Lambda)$ are as in (3.1.6), (3.1.7) respectively. Then the number $N_1(P)$, say, of integral solutions of (4.1.1) in $\mathcal{B}(P)$ satisfies

$$N(P) \geqslant \int_V \prod_{j=1}^{n} T(\Lambda_j) \, . \prod_{j \in \mathcal{T}_{R-r}} K(\alpha_j) \, d\underline{\alpha} \, , \tag{4.1.8}$$

where the Λ_j 's are as in (3.1.10), that is

$$\Lambda_j = \sum_{i=1}^{R} \lambda_{ij} \alpha_i \quad , \quad (1 \leqslant j \leqslant n) \; ,$$

K is a kernel satisfying (1.3.4) and V is the R-dimensional region

$$V = \{\underline{\alpha} = (\alpha_1, \ldots, \alpha_R); \; 0 < \alpha_i < 1 \; , \; \text{for} \; i \in S_r \quad \text{and} \; -\infty < \alpha_i < \infty \; , \; \text{for} \; i \in \mathcal{T}_{R-r} \}. \tag{4.1.9}$$

So to prove our theorem, it is sufficient to prove the existence of P bounded in terms of Λ (as in(4.1.4))such that the integral in the right hand side of (4.1.8) is positive. We follow the usual procedure of dissecting V suitably, into subregions and estimating their contributions to the above integral, using the tools available, as far as this is possible. When this is not possible, we shall use some special methods and apply the result of Theorem 1 on R equations and an inductive argument.

In order to deal with the "singular series" (which arises due to the r equations of the system), we shall consider a "normalised system" which we shall describe in §2. In §3, we shall describe the dissection of the region of integration V and estimate some error terms. In §4, we shall deal with the "basic set" (a subset of V), the contribution of which to the integral in (4.1.8) forms the main term. Then in §5, we shall consider the case for which our basic approach fails and in §6, the proof of Theorem 3 is completed. Finally in §7, we deduce a theorem on R inequalities using the result of Theorem 3.

2. REDUCTION

In this section, we shall introduce a normalisation technique and show that it is sufficient to consider a "normalised system" of forms $F_1, \ldots, F_R$, to prove Theorem 3. Then we work towards proving Theorem 3 for a normalised system.

NORMALISATION

The p-operations are defined as in the same way of §2, Chapter 3, with $R = r$ and we perform these to the forms F_i, for $i \in S_r$. But whenever we do a p-operation to F_i, $i \in S_r$, we multiply the coefficients of one of the forms F_i, $i \in T_{R-r}$, by the same p, so that the property $\Delta \geqslant 1$ is conserved, where Δ is as in (4.1.5).

A system $F_1, \ldots, F_R$ is said to be *normalised* if no p-operation is possible on these forms. Thus if a system $F_1, \ldots, F_R$ is normalised, then the system F_i, $i \in S_r$ is normalised in the sense of Chapter 3 and $\Delta \geqslant 1$ holds for $F_1, \ldots, F_R$.

Lemma 4.2.1. Let $F_1(\underline{x}), \ldots, F_R(\underline{x})$ be forms as in (4.1.2) and let $F_1'(\underline{x}), \ldots, F_R'(\underline{x})$ be a normalised system of the system $F_1(\underline{x}), \ldots, F_R(\underline{x})$. Suppose Theorem 3 holds for the $F_1'(\underline{x}), \ldots, F_R'(\underline{x})$. Then it holds for $F_1(\underline{x}), \ldots, F_R(\underline{x})$.

<u>Proof</u>. Similar to Lemma 3.2.5, since $F_i'(\underline{x})$, $i \in S_r$ are normalised in the sense of Chapter 3, and since for any i such that $i \in T_{R-r}$, $|F_i'(\underline{x})| < 1$ implies $|F_i(\underline{x})| < 1$ as $F_i'(\underline{x}) = p^{t_i}F_i(\underline{x})$, $t_i \geqslant 0$; and the coefficients of F_i's, $(i \in T_{R-r})$, λ_{ij}' say, satisfy $|\lambda_{ij}'| \geqslant |\lambda_{ij}| \geqslant 1$.

Thus, from Lemma 4.2.1 we see that to prove Theorem 3, it is enough to prove the same for a normalised system of forms. So from now on we shall assume that the system $F_1(\underline{x}),\ldots,F_R(\underline{x})$ is normalised. We may also assume, by reordering if necessary, that $\Delta = |\Delta(1,\ldots,R)|$.

3. SOME ERROR TERMS

In order to achieve our goal of estimating the integral in (4.1.8), as in the last chapter, we shall first dissect the region of integration V (given by (4.1.9)) suitably, into disjoint subregions. In this section, we shall describe this dissection and estimate some error terms which are contributions of some of these subregions to the above integral.

From now on, we shall take, for convenience, without loss of generality, $S_r = \{1,\ldots,r\}$ and so $T_{R-r} = \{r+1,\ldots,R\}$, whenever S_r is non-empty, (we note that when S_r is empty, $r = 0$ and so T_{R-r} $(= T_R) = \{1,\ldots,R\})$.

<u>Definition 4.3.1</u>. Let σ be as in §2 of Chapter 2 (see Lemma 2.2.8) and ω, η, δ be given small (positive) constants and P is a positive integer such that

$$(4.3.1) \qquad 0 < \eta < \tfrac{1}{4}, \quad 0 < \delta < 1, \quad 0 < \omega \leqslant \eta, \quad P^2 \geqslant 2\Delta^{2R-1}(R+1)!,$$

and whose explicit values will be decided at the end of this chapter. Let $Q, A_{i_1},\ldots,A_{i_r}$ be integers (where $S_r = \{i_1,\ldots,i_r\}$) such that

$$(4.3.2) \qquad 0 \leqslant A_{i_1},\ldots,A_{i_r} \leqslant Q, \quad (A_{i_1},\ldots,A_{i_r},Q) = 1$$

and let $M(\underline{A},Q)$, called a *major arc*, be the set of $\underline{\alpha}$ in V (see (4.1.9)) such that

$$(4.3.3) \qquad \begin{cases} \alpha_i = \dfrac{A_i}{Q} + \beta_i \ , & \text{for } i \in S_r \ , \\ 1 \leqslant Q < P^{\omega} \ , & |\beta_i| < P^{-k+\eta} \Lambda^{2R-1}(R+1)! \end{cases}$$

and

$$(4.3.4) \qquad |\alpha_i| < \Lambda^{2R-1} P^{-k+\eta}(R+1)! \ , \ i \in T_{R-r} \ .$$

Then the union of these major arcs $M(\underline{A},Q)$ for all the integers Q, A_i $(i \in S_r)$ satisfying (4.3.2) is defined as the *basic set* and denoted by M. The complement of M in V is dissected into the *Supplementary set* S, the *Tail* T and the *Residual set* $\mathcal{R}$ which are defined as follows:

$$(4.3.5) \qquad S = \{\underline{\alpha} \, ; \underline{\alpha} \text{ in } V - M, \ |\alpha_i| < P^{\delta}, \ (i \in T_{R-r}) \text{ and for some } j \text{ in } [1,n], \ (4.3.6) \text{ holds}\},$$

where (4.3.6) is the condition (2.2.5), that is

$$(4.3.6) \qquad |T(\Lambda_j)| << P^{1-\rho+\epsilon} \ ,$$

with ρ, ε are as in Lemma 2.2.1.

$$(4.3.7) \qquad T = \{\underline{\alpha} \, ; \underline{\alpha} \text{ in } V - (M \cup S) \text{ and } |\alpha_i| \geqslant P^{\delta} \text{ for some } i \in T_{R-r} \} \ ,$$

and

$$(4.3.8) \qquad \mathcal{R} = V - (M \cup S \cup T) \ .$$

For $\underline{\alpha}$ in $\mathcal{R}$, by its definition, $|\alpha_i| < P^{\delta}$, $(i \in T_{R-r})$ and since (4.3.6) does not hold for any j in $[1,n]$, the alternative (i) of Lemma 2.2.1 holds for all j. That is, for $\underline{\alpha}$ in $\mathcal{R}$, for each j, $(1 \leqslant j \leqslant n)$ there exist integers a_j, q_j such that

$$\Lambda_j = \frac{a_j}{q_j} + \gamma_j ,$$

with

$$(a_j , q_j) = 1, \quad 1 \leqslant q_j \leqslant P^{\eta}, \quad |\gamma_j| < q_j^{-1} \, P^{-k+\eta} .$$

~~Also as in Lemma 17 of D-L, we see that when (4.3.6) holds, we have~~

~~(4.3.9)~~ $$\cancel{|T(\Lambda_j)| \ll P^{1-\sigma},}$$

~~where~~ $\cancel{\sigma = \frac{1}{8k^2 \log k}}$, ~~(since (4.3.6) is (2.2.5)).~~

Since (4.3.1) holds, it is easily checked that the major arcs are disjoint. We also note that when the set S_r is empty, M is the interval given by (4.3.4) where $T_{R-r} = \{1,\ldots,R\}$ ($= T_R$) .

<u>Lemma 4.3.1.</u> Let S be as in (4.3.5) and suppose the hypothesis H holds. Then the contribution of S to the integral in (4.1.8) satisfies

(4.3.10) $$\int_S \prod_{j=1}^{n} |T(\Lambda_j)| \, . \prod_{i \in T_{R-r}} |K(\alpha_i)| \, d\underline{\alpha} \ll \Lambda^R \, P^{n-Rk-\rho+R\delta+\varepsilon(R+1)},$$

where the implied constant depends only on n, k, R and δ .

<u>Proof</u>. For $\underline{\alpha}$ in S , by definition, (4.3.6) holds for some j such that $1 \leqslant j \leqslant n$. Let S_ℓ be the subset of S for which (4.3.6) holds for $j = \ell$. Then S is the union of the S_ℓ's for $\ell = 1,\ldots,n$. So to prove (4.3.10), we shall first estimate the left hand side of (4.3.10) integrating over S_ℓ and then sum it over ℓ from 1 to n .

Since we are dealing with independent forms $F_1,\ldots,F_R$, the rank of the coefficient matrix (λ_{ij}) is R and so there is an $R \times R$ non-singular submatrix of (λ_{ij}). We can always choose one such submatrix having the above ℓ-th column as one of its column. Then by the hypothesis H , there are $s - 1$ more disjoint non-singular submatrices, where s is as given in the hypothesis.

Proceeding as in the proof of Lemma 3.3.1, by using the trivial estimate P to the $T(\Lambda_j)$'s, for the j's not corresponding to the above $s - 1$ sets and those corresponding to the first set with $j \neq \ell$, and using the estimate (4.3.6) for $T(\Lambda_\ell)$, it follows that

$$\int_{S_\ell} \prod_{j=1}^{n} |T(\Lambda_j)| \cdot \prod_{i \in T_{R-r}} |K(\alpha_i)| \, d\underline{\alpha} \ll P^{n-(s-1)R-1} \cdot P^{1-\rho+\varepsilon} \tag{4.3.11}$$

$$\cdot \int_{S_\ell} \prod_{(j_1,\ldots,j_R)} |T(\Lambda_{j_1}) \ldots T(\Lambda_{j_R})| \cdot 1 \cdot d\underline{\alpha} \; ,$$

since $|K(\alpha)| \ll 1$, by (1.3.4), and where the product is over the $s - 1$ sets of R, j's .

Now, we make change of variables from $\underline{\alpha}$ to $\Lambda_{j_1}, \ldots, \Lambda_{j_R}$, where

$$\Lambda_j = \sum_{i=1}^{R} \lambda_{ij} \alpha_i \; , \quad (\det|\lambda_{ij}| \neq 0) \; ,$$

for $j = j_1, \ldots, j_R$, in the integral

$$\int_{S_\ell} |T(\Lambda_{j_1}) \ldots T(\Lambda_{j_R})|^{s-1} d\underline{\alpha} \; . \tag{4.3.12}$$

Then the Jacobian J of this transformation satisfies

$$|J| = \left| \frac{1}{\Delta(j_1, \ldots, j_R)} \right| \leqslant 1 \; .$$

For $\underline{\alpha}$ in S_ℓ, the $\Lambda_{j_1}, \ldots, \Lambda_{j_R}$ are in S'_ℓ, say, where

$$S'_\ell = \{(\Lambda_{j_1}, \ldots, \Lambda_{j_R}) ; \; |\Lambda_j| \leqslant R\Lambda P^{\delta} , \; j = j_1, \ldots, j_R\} \; .$$

Then the integral (4.3.12) is

$$\leq \int_{S'_\ell} \prod_{j=j_1,\ldots,j_R} |T(\Lambda_j)|^{s-1} \, d\underline{\Lambda}, \quad \underline{\Lambda} = (\Lambda_{j_1}, \ldots, \Lambda_{j_R})$$

$$= \prod_{j=j_1,\ldots,j_R} \int_{S'_\ell} |T(\Lambda_j)|^{s-1} \, d\Lambda_j,$$

$$\ll \prod_{j=j_1,\ldots,j_R} R\Lambda P^{\delta} \cdot P^{s-1-k+\varepsilon},$$

by Lemma (2.2.7), since $T(\Lambda)$ is periodic of period 1 in Λ. And this is

$$< (R\Lambda P^{\delta})^R \, P^{R(s-1-k)+R\varepsilon}.$$

Using this and Hölder's inequality, the integral in the right hand side of (4.3.11) is

$$\ll \Lambda^R \, P^{(s-1-k+\delta)R+R\varepsilon}.$$

From this and (4.3.11) and then summing over ℓ, the result (4.3.10) follows.

Lemma 4.3.2. Let T be as in (4.3.7) and t be a given positive integer such that (1.3.4) holds with $r = t$. Suppose the hypothesis H holds. Then we have

$$(4.3.13) \qquad \int_T \prod_{j=1}^{n} |T(\Lambda_j)| \cdot \prod_{i \in T_{R-r}} |K(\alpha_i)| \, d\underline{\alpha} \ll P^{n-t\delta},$$

where the constant implied by $\ll$ depends on t, n, R, r and δ. (The value of the integer t will be determined later in the chapter.)

Proof. We take the trivial estimate P, for all $|T(\Lambda_j)|$'s. Then the left hand side of (4.3.13) is

$$\ll \int_T P^n \prod_{i \in T_{R-r}} |K(\alpha_i)| \, d\underline{\alpha},$$

$$\ll P^n \int_T \prod_{i \in T_{R-r}} \min\{|\alpha_i|^{-1}, 1\} \, d\underline{\alpha},$$

by (1.3.4) with r replaced by t . This integral factorises and it is

$$\ll P^n \left(\prod_{i=1}^{r} \int_0^1 d\alpha_i \right) . \left(\int_{P^\delta}^{\infty} |\alpha|^{-t-1} \, d\alpha \right) \left(\int_{-\infty}^{\infty} \min(1, |\alpha|^{-t-1}) \, d\alpha \right)^{R-r}$$

$$\ll P^n \,.\, 1 \,.\, P^{-t\delta} \,.\, 1 \,,$$

(since $|\alpha_i| \geqslant P^\delta$, for at least one $i \in T_{R-r}$, for $\underline{\alpha}$ in T) and thus (4.3.13) follows.

4. BASIC SET

We now consider the region M , the basic set, the main contribution of which to the integral in (4.1.8) consists of a product of two terms, one is a sum called the *singular series* and the other is an integral called the *singular integral* . In this section, we shall estimate these with some error terms (due to M). We now recall that the set M is the union of all the major arcs $M(\underline{A},Q)$ given by (4.3.2) to (4.3.4).

For the $\underline{\alpha}$'s in $M(\underline{A},Q)$, (4.3.3) and (4.3.4) hold and so we have

$$\Lambda_j = \sum_{i=1}^{R} \lambda_{ij} \, \alpha_i \,, \quad (1 \leqslant j \leqslant n)$$

(4.4.1) $$= \sum_{i \in S_r} \lambda_{ij} \frac{A_i}{Q} + \sum_{i=1}^{R} \lambda_{ij} \, \beta_i' = \frac{a_i}{q_j} + \gamma_j \,, \text{ (say)},$$

where

(4.4.2) $$\beta_i' = \begin{cases} \beta_i \,, & i \in S_r \,, \\ \alpha_i \,, & i \in T_{R-r} \,, \end{cases}$$

(4.4.3) $$\frac{a_j}{q_j} = \frac{1}{Q} \,.\, \sum_{i \in S_r} \lambda_{ij} \, A_i = \frac{C_j}{Q} \,, \quad (1 \leqslant j \leqslant n), \text{ say},$$

and

(4.4.4) $$\gamma_j = \sum_{i=1}^{R} \lambda_{ij} \beta_i', \quad (1 \leq j \leq n).$$

Then

(4.4.5) $$q_j \leq Q < P^{\omega}$$

and

(4.4.6) $$|\gamma_j| \leq R\Lambda P^{-2R-k+\eta}(R+1)!,$$

using (4.3.3) and (4.3.4), where ω, η are as in (4.3.1).

(We note that when the set S_r is empty, $a_j = 0$, $q_j = 1$, $(1 \leq j \leq n)$; and $r = 0$.)

<u>Lemma 4.4.1</u>. When S_r is non-empty (that is when $S_r = \{1,\ldots,r\}$), let $A_1,\ldots,A_r,Q$ be integers such that

$$(A_1,\ldots,A_r,Q) = 1, \quad 0 \leq A_1,\ldots,A_r < Q,$$

(that is (4.3.2) holds) and $\omega,\eta,M(\underline{A},Q)$ and M be as in Definition 4.3.1. Let a_j, q_j, γ_j $(1 \leq j \leq n)$ be as in (4.4.1) to (4.4.6), and $S(a_j,q_j)$, $I(\gamma_j)$ be as in (2.2.2), (2.2.3) respectively, for the above a_j, q_j, γ_j's. Suppose

(4.4.7) $$P > 4R^2k^2\Lambda^{4R}(R!)^2.$$

Then we have

(4.4.8) $$\int_{M} \prod_{j=1}^{n} T(\Lambda_j). \prod_{i=r+1}^{R} K(\alpha_i)d\underline{\alpha} = \mathfrak{S}(P^{\omega})J(P^{-k+\eta}) + E_1,$$

where

(4.4.9) $$\mathfrak{S}(P^{\omega}) = \sum_{Q<P^{\omega}} \sum_{\underline{A}} \prod_{j=1}^{n} q_j^{-1} S(a_j,q_j),$$

(4.4.10) $$J(P^{-k+\eta}) = \int_{M(\underline{A},Q)} \prod_{j=1}^{n} I(\gamma_j) \cdot \prod_{i=r+1}^{R} K(\alpha_i)\, d\underline{\beta}' ,$$

for $\underline{\beta}' = (\beta_1', \ldots, \beta_R')$ (with β_1' as in (4.4.2)) and

(4.4.11) $$|E_1| \ll \Lambda^{-R+2/R^2} P^{n-Rk-1+R\eta+(r+2)\omega} ,$$

$\underline{A}$ in the summation is over all the integers $A_1, \ldots, A_r$ satisfying (4.3.2).

Proof. For $\underline{\alpha}$ in $M(\underline{A},Q)$, (4.3.1), (4.4.1), (4.4.5) and (4.4.6) hold, and since (4.4.7) also holds, all the hypotheses of Lemma 2.2.2 are satisfied. And so by the same lemma, we have

(4.4.12) $$T(\Lambda_j) = q_j^{-1} S(a_j, q_j) I(\gamma_j) + O(q_j), \quad (1 \leqslant j \leqslant n) .$$

As in the proof of the Lemma 3.4.3, the main term of (4.4.8) is obtained by multiplying all the main terms of (4.4.12) and integrating and summing over $\underline{A}$ and Q as then it factorises into (4.4.9) and (4.4.10).

Proceeding as in the same lemma (that is Lemma 3.4.3), our error term becomes

$$\ll \sum_{Q<P^{\omega}} \sum_{\underline{A}} \int_{M(\underline{A},Q)} P^{n-1} Q \prod_{i=r+1}^{R} |K(\alpha_i)|\, d\underline{\beta}' ,$$

where $\underline{\beta}'$ as in (4.4.2). The integral in the above expression factorises and since $|K(\alpha_i)| \ll 1$, β_i $(1 \leqslant i \leqslant r)$ satisfy (4.3.3) and α_i $(r+1 \leqslant i \leqslant R)$ satisfy (4.3.4), it is

$$\ll P^{n-1} Q (\Lambda^{2R-1} P^{-k+\eta})^R .$$

Therefore the required error term is

$$\ll \Lambda^{-R+2/R^2} P^{n-Rk-1+R\eta} \sum_{Q<P^{\omega}} \sum_{\underline{A}} Q$$

$$\ll \Lambda^{-R+2/R^2} P^{n-Rk-1+R\eta+(r+2)\omega} ,$$

which is (4.4.11). Thus the proof of the lemma is completed.

(We note that when S_r is empty, $(r = 0)$, the series in (4.4.9) is just 1.)

Lemma 4.4.2. Suppose the hypothesis H holds and let A_i, $(i \in S_r)$, Q be integers satisfying (4.3.2). Suppose (4.3.3), (4.3.4) and (4.4.1) to (4.4.6) hold. Then

(4.4.13) $$\sum_{Q \geq P^{\omega}} \sum_{\underline{A}} \prod_{j=1}^{n} q_j^{-1/k} \ll \Lambda^r P^{-\omega((n-1)/k - r - \epsilon)},$$

where $\varepsilon > 0$.

Proof. Suppose S_r is empty (that is $r = 0$). Then $q_j = 1$, $(1 \leqslant j \leqslant n)$ and $Q = 1$; and so (4.4.13) clearly holds. Therefore it is sufficient to consider S_r non-empty and so we may take $S_r = \{1,\ldots,r\}$ and $T_{R-r} = \{r+1,\ldots,R\}$. We have from (4.4.3) that for $1 \leqslant j \leqslant n$, $\frac{a_j}{q_j} = \frac{C_j}{Q}$, where

(4.4.14) $$C_j = \sum_{i=1}^{r} \lambda_{ij} A_i ,$$

and so $q_j | Q$ and let

$$u_j = (Q, C_j), \quad d = (u_1,\ldots,u_n).$$

Then $d | C_j$, $(1 \leqslant j \leqslant n)$ and $q_j = \frac{Q}{u_j}$. Therefore

(4.4.15) $$\sum_{\underline{A}} \prod_{j=1}^{n} q^{-1/k} = \sum_{\underline{A}} Q^{-n/k} \prod_{j=1}^{n} u_j^{1/k} .$$

Since the forms $F_1,\ldots,F_R$ and so $F_1,\ldots,F_r$ are independent forms, by rearranging the variables if necessary, we may take rank (λ_{ij}), $1 \leqslant i, j \leqslant r$, to be r. We follow a process similar to the proof of Lemma 2.3.2, with minor alteration, because here the summation is over $A_1,\ldots,A_r$. From (4.3.2) and (4.4.14), $|C_j| \leqslant r\Lambda Q$ and then as in Lemma 2.3.2, for a given $u_1,\ldots,u_n$ (all the divisors of Q), the number of possibilities for $C_1,\ldots,C_r$ is $\ll \frac{\Lambda^r Q^r}{u_1 \ldots u_r}$.

Since the rank (λ_{ij}), $1 \leqslant i, j \leqslant r$, is r, for a given set $C_1, \ldots, C_r$ there exist a unique set $A_1, \ldots, A_r$, obtained by solving (4.4.14) for $j = 1, \ldots, r$. For these values of $C_1, \ldots, C_r$ the values of $C_{r+1}, \ldots, C_n$ are fixed as they are obtained from (4.4.14) for $j = r+1, \ldots, n$. Thus for a fixed $C_1, \ldots, C_r$, the $C_{r+1}, \ldots, C_n$ are linear combinations of $C_1, \ldots, C_r$. That is, if we write

(4.4.16) $$C_j = u_j \; x_j \;, \quad (1 \leqslant j \leqslant n),$$

there exist integers t_{ij} $(r+1 \leqslant i \leqslant n, \; 1 \leqslant j \leqslant r)$ such that

(4.4.17) $$\sum_{j=1}^{r} t_{ij} \; u_j \; x_j \equiv 0 \pmod{u_i}, \quad (r+1 \leqslant i \leqslant n) \quad .$$

Thus for given $x_1, \ldots, x_{r-1}$, the value of $x_r \bmod (u_{r+1} \ldots u_n)$ is determined with only a bounded number of possibilities and the number of values for x_r (for given $x_1, \ldots, x_{r-1}$) is $\ll \frac{\Lambda Q}{u_r} \cdot \frac{1}{u_{r+1} \ldots u_n}$. The number of possibilities for $x_1, \ldots, x_{r-1}$ is $\ll \frac{\Lambda^{r-1} \; Q^{r-1}}{u_1 \ldots u_{r-1}}$. Thus the number of possibilities for $x_1, \ldots, x_r$ is $\ll \frac{\Lambda^r \; Q^r}{u_1 \ldots u_n}$.

From (4.4.16) and (4.4.17) both for $1 \leqslant j \leqslant r$, it follows that the number of possibilities for $x_1, \ldots, x_r$ is the same as that for $A_1, \ldots, A_r$, and so the number of possibilities for $A_1, \ldots, A_r$ is $\ll \frac{\Lambda^r \; Q^r}{u_1 \ldots u_n}$. Hence the right hand side of (4.4.15), as in the proof of Lemma 2.3.2, is

$$\ll \Lambda^r \; Q^{r-n/k} \sum_{u_1, \ldots, u_n | Q} (u_1 \; \ldots \; u_n)^{-1+1/k}$$

$$\ll \Lambda^r \; Q^{r-n/k} \; Q^{-1+1/k+\epsilon}$$

(4.4.18) $$\ll \Lambda^r \; Q^{r-1-(n-1)/k+\epsilon} \quad .$$

Thus the result (4.4.13) follows, by summing this over $Q \geqslant P^{\omega}$.

<u>Lemma 4.4.3</u>. Let $\mathfrak{S}(P^{\omega})$, $J(P^{-k+\eta})$ be as in (4.4.9), (4.4.10) respectively. Then

$$\mathfrak{S}(P^{\omega})\ J(P^{-k+\eta}) = \mathfrak{S}J + E_2 + E_3 , \tag{4.4.19}$$

where

$$\mathfrak{S} = \sum_{Q=1}^{\infty} \sum_{\underline{A}} \prod_{j=1}^{n} q_j^{-1}\ S(a_j , q_j), \tag{4.4.20}$$

$$J = \int_W \prod_{j=1}^{n} I(\gamma_j) . \prod_{i \in T_{R-r}} K(\alpha_i)\ d\underline{\beta}' , \tag{4.4.21}$$

$$|E_2| \ll \Lambda^{Rs/k+2R^2-R} P^{n-Rk-\eta(s/k-R)} , \tag{4.4.22}$$

$$|E_3| \ll \Lambda^{R(s/k+2R-2)} P^{n-Rk-\omega(s/k-2-(R-1)\varepsilon)+R\eta} , \tag{4.4.23}$$

where $s = [9k^2 R \log 3Rk]$ and the constants implied by $\ll$ depend on k, R and n, and $\varepsilon > 0$.

<u>Proof</u>. The main term $\mathfrak{S}J$ is obtained by extending the range of integration $\int J(P^{-k+\eta})$ to the whole of R-dimensional space W, and the range of Q to all the integers in $[1,\infty]$. We shall estimate the error in doing this.

We easily see that the left hand side of (4.4.19) is

$$\mathfrak{S}J - \mathfrak{S}J_0 - \mathfrak{S}_0 J(P^{-k+\eta}) + \mathfrak{S}_0 J_0 , \tag{4.4.24}$$

where J_0 is the integral J with W replaced by $W - M(\underline{A},Q) = W_0$, say, and $\mathfrak{S}_0$ is the $\mathfrak{S}$ with the range of Q replaced by Q in $[P^{\omega},\infty)$. Then for $\underline{\beta}'$ in W_0, $\max|\beta_i'| > \Lambda^{2R-1} P^{-k+\eta} (R+1)!$, The errors denoted by E_2, E_3 in (4.4.19) are $|\mathfrak{S}_0 J_0| + |\mathfrak{S}J_0|$, $|\mathfrak{S}_0 J(P^{-k+\eta})|$ respectively.

We shall now find an upper bound for J_0. We have

$$|J_0| \leqslant \int_{W_0} \prod_{j=1}^{n} |I(\gamma_j)| . \prod_{i \in T_{R-r}} |K(\alpha_i)| \, d\underline{\beta}' ,$$

$$\ll \int_{W_0} \prod_{j=1}^{n} |I(\gamma_j)| \; . \; 1 \; . \; d\underline{\beta}' , \qquad (4.4.25)$$

since $|K(\alpha_i)| \ll 1$ by (1.3.4), (where the implied constant depends only on n). Since the hypothesis H holds and the forms $F_1,\dots,F_R$ are independent, as in Lemma 4.3.1, we can find s disjoint sets of R subscripts in $[1,n]$ such that the $R \times R$ matrices of (λ_{ij}) with columns corresponding to each of these s sets are all non-singular. Let $j_1,\dots,j_R$ be one of these s sets. Because of the non-singularity, it follows from (4.4.4) for $j = j_1,\dots,j_R$, that

$$\max_{j=j_1,\dots,j_R} |\gamma_j| \geqslant \frac{1}{(R+1)!} \Lambda^{-2R+1} |\Delta(j_1,\dots,j_R)| . \max_{1 \leqslant i \leqslant R} |\beta_i'| .$$

Thus, by Lemma 2.2.5,

$$\Big| \prod_{j=j_1,\dots,j_R} I(\gamma_j) \Big| \ll P^{R-1} \Lambda^{(2R-1)/k} |\Delta(j_1,\dots,j_R)|)^{-1/k} (\max_{1 \leqslant i \leqslant R} |\beta_i'|)^{-1/k}$$

$$\ll P^{R-1} \Lambda^{(2R-1)/k} (\max|\beta_i'|)^{-1/k} ,$$

since $|\Delta| \geqslant 1$, where implied constant depends on k and R. Using this for each of the above s sets and taking the trivial estimate P for the rest of the $I(\gamma_j)$'s in (4.4.25), we have

$$|J_0| \ll P^{n-sR+s(R-1)} \Lambda^{s(2R-1)/k} \int_{W_0} (\max|\beta_i'|)^{-s/k} \, d\underline{\beta}'$$

$$\ll P^{n-s} \Lambda^{s(2R-1)/k} \int_{\Lambda^{2R-1} P^{-k+\eta}} x^{-s/k+R-1} \, dx$$

$$\ll \Lambda^{R(2R-1)} P^{n-Rk-\eta(s/k-R)} . \qquad (4.4.26)$$

Also by Lemma 2.2.4, for any $1 \leqslant j \leqslant n$,

$$|q_j^{-1} S(a_j, q_j)| << q_j^{-1/k} ,$$

(the implied constant depends only on k). Thus

$$|\mathfrak{S}| << \sum_{Q=1}^{\infty} \sum_{\underline{A}} \prod_{j=1}^{n} q_j^{-1/k}$$

$$<< \sum_{Q=1}^{\infty} \Lambda^{Rs/k} Q^{-(s/k-1-(R-1)\varepsilon)} \ll \Lambda^{Rs/k},$$

using Lemma 2.3.2 and applying Hölder's inequality, since $s/k - 1 - (R-1)\varepsilon > 1$. Also the same upper bound certainly holds for $\mathfrak{S}_0$. Thus combining this and (4.4.26), the error (4.4.22) follows.

The error E_3 is $|\mathfrak{S}_0 J(P^{-k+\eta})|$ and we shall now find an upper bound for this. We recall that in $M(\underline{A},Q)$,

$$|\beta_i| \ll \Lambda^{2R-1} P^{-k+\eta}, \ (i \in S_r); \quad |\alpha_i| \ll \Lambda^{2R-1} P^{-k+\eta}, \ (i \in T_{R-r}).$$

Therefore we have ,

$$|J(P^{-k+\eta})| << P^n . \qquad .(\Lambda^{2R-1} P^{-k+\eta})^F$$

$$<< \Lambda^{R(2R-1)} P^{n-Rk+R\eta} ,$$

by taking the trivial estimate P for all $|I(\gamma_j)|$, $(1 \leqslant j \leqslant n)$ and since $|K(\alpha)| << 1$. As in Lemma 3.6.1 ,

$$|\mathfrak{S}_0| \ll \Lambda^{R(\frac{s}{k}-1)} P^{-\omega((\frac{s}{k}-2)-(R-1)\varepsilon)},$$

Therefore, multiplying these two estimates, the error $|E_3|$, that is (4.4.23), follows. Thus the result (4.4.19) follows from (4.4.24). This completes the proof of the lemma.

Lemma 4.4.4. Let J be as in (4.4.21). Suppose

(4.4.27) $$P^k > 18(3nR!)^{2R^2}\Lambda^{2R^2} .$$

Then we have

(4.4.28) $$J \gg \Lambda^{-(3nR \ \ +1)R} P^{n-Rk} ,$$

where the implied constant depends on n and k.

Proof. As before we take $S_r = \{1,\ldots,r\}$, whenever S_r is non-empty (that is $r \neq 0$); and rewrite J

$$J = \int_W \prod_{j=1}^{n} |I(\gamma_j)| \prod_{i \in T_{R-r}} K(\alpha_i)\, d\underline{\beta}' ,$$

where $\underline{\beta}' = (\beta_1', \ldots, \beta_R')$, β_i' as in (4.4.2), and γ_j, $(1 \leqslant j \leqslant n)$ are linear forms as in (4.4.4) and W is the whole of R-dimensional space. We proceed as in the singular integral for equations, in Lemma 3.5.1. Then we get, in this case, that

$$J = \frac{1}{k^n} \lim_{\phi \to \infty} \int_{\mathcal{B}(P^k)} (y_1 \ldots y_n)^{-1+1/k} \prod_{i=1}^{r} \frac{\sin 2\pi\phi z_i}{\pi z_i}$$

$$\cdot \int_{W_0} \prod_{i=r+1}^{R} e(\alpha_i z_i) K(\alpha_i)\, d\alpha_{r+1} \ldots d\alpha_R\, d\underline{y} ,$$

where W_0 is whole of $(R - r)$-dimensional space and

(4.4.29) $$z_i(\underline{y}) = \sum_{j=1}^{n} \lambda_{ij} y_j , \quad (1 \leqslant i \leqslant R) ,$$

and $\mathcal{B}(P^k)$ as in (3.1.6) with P replaced by P^k.

Now we make the same change of variable, as in the proof of Lemma 3.5.1, that $f : \underline{y} \to \underline{z}$, where

$$z_i = z_i(\underline{y}), \quad (1 \leqslant i \leqslant R),$$
$$z_i = y_i \quad , \quad (R+1 \leqslant i \leqslant n).$$

For the same reason as there, we have by the change of variable theorem,

$$J \gg k^{-n} \Lambda^{-R} \lim_{\phi\to\infty} \int_{f\mathcal{B}(P^k)} (y_1 \ldots y_R)^{-1+1/k} (z_{R+1} \ldots z_n)^{-1+1/k}$$

$$\cdot \prod_{i=1}^{r} \frac{\sin 2\pi\phi z_i}{\pi z_i} \left(\prod_{i=r+1}^{R} \int_{-\infty}^{\infty} e(\alpha_i z_i) K(\alpha_i) d\alpha_i \right) d\underline{z} \ ;$$

since the second integral factorises and where the factor Λ^{-R} is due to the change of variables as the $|\text{Jacobian}|$ is $\gg \Lambda^{-R}$ and the $y_1, \ldots, y_R$ are determined (uniquely) by (4.4.29) and $y_j = z_j$, $(R+1 \leqslant j \leqslant n)$, in terms of z_j's .

By the hypothesis of Theorem 3 and since (4.4.27) holds,we can apply the Lemma 2.2.10 with $Q = P^k$. And so by reasoning as in the proof of Lemma 3.5.1, and by Fourier's integral theorem applied r times, to the above expression, we get

$$J \gg k^{-n} \Lambda^{-R} \int_{\mathcal{L}} |y_1 \ldots y_R \, z_{R+1} \ldots z_n|^{-1+1/k}$$

$$\cdot \left(\prod_{i=r+1}^{R} \int_{-\infty}^{\infty} e(\alpha_i z_i) K(\alpha_i) d\alpha_i \right) d(z_{r+1}, \ldots, dz_n) \quad ,$$

where $\mathcal{L}$ is the region in (n-r)-dimensional space given by

$$(0, \ldots, 0, z_{r+1}, \ldots, z_n) = f(\underline{y}), \quad \text{for some } \underline{y} \text{ in } \mathcal{B}(P^k) \ ,$$
and introduction of the modulus sign is justified as on page 64.

The set $S(P^k)$ obtained by Lemma 2.2.10, for the particular case when $z_i = 0$, $(1 \leqslant i \leqslant r)$ and $|z_i| < \dfrac{1}{(R-1)!\Lambda^{nR}}$, $(r+1 \leqslant i \leqslant R)$, is contained in the above region $\mathcal{L}$. For these z_i , $(r+1 \leqslant i \leqslant R)$, since $|z_i| < \frac{1}{3}$ hold, by Lemma 1.3.1, each

$$\int_{-\infty}^{\infty} e(\alpha_i z_i) K(\alpha_i) d\alpha_i = 1 , \quad (r+1 \leqslant i \leqslant R),$$

and $|y_1 \ \dots \ y_R \ z_{R+1} \ \dots \ z_n| \leqslant P^{nk}$. *Also* the integrand is > 0 in $\mathcal{L}$ + so on $S(P^k)$. Also by the same lemma (Lemma 2.2.10), the measure of $S(P^k) \geqslant \dfrac{P^{k(n-R)}}{\Lambda^{(3n)R^2}}$. Thus

$$J >> \Lambda^{-R} \int_{S(P^k)} |y_1 \ \dots \ y_R \ z_{R+1} \ \dots z_n|^{-1+1/k} \ . \ 1 \ . \ d(z_{r+1}, \dots, dz_n)$$

$$>> \Lambda^{-R} \ P^{nk(-1+1/k)} \ . \int_{S(P^k)} d(z_{r+1}, \dots, dz_n) .$$

$$>> \Lambda^{-R} \ P^{nk(-1+1/k)} \ . \ \Lambda^{-(3n)R^2} \ P^{k(n-R)}$$

$$>> \Lambda^{-(3nR+1)R} \ P^{n-Rk} .$$

Thus the proof of the lemma is completed.

Lemma 4.4.5. Suppose $\mathfrak{S}$ is as in (4.4.20). Then it is positive and

(4.4.30) $$\mathfrak{S} \gg \Lambda^{-\varepsilon_2} ,$$

where $\varepsilon_2 = Mr(\frac{n}{h} - r)$.

Proof. We easily see that (4.4.30) holds when S_r is empty. So we consider $S_r = \{1, \dots, r\}$. The sum $\mathfrak{S}$ given by (4.4.20) is clearly the singular series corresponding to the system of r equations

$$F_i(\underline{x}) = 0, \quad (1 \leqslant i \leqslant r);$$

(in the sense of Chapter 3). Therefore it is enough to show that all the hypotheses of Theorem 2 are satisfied with R replaced by r, for the above r equations.

From our definition of normalisation (see §2), since the forms $F_1, \ldots, F_R$ are normalised, it is clear that the forms $F_1, \ldots, F_r$ are normalised, in the sense of Chapter 3. Now we shall show that the block condition is satisfied for $F_1, \ldots, F_r$. For each $R \times R$ non-singular submatrix of (λ_{ij}), it is possible to pick up r columns in it such that the corresponding $r \times r$ submatrix with the first r rows of these r columns is non-singular; and this $r \times r$ matrix is a submatrix of the coefficient matrix of $F_1, \ldots, F_r$. Therefore, the hypothesis H holds for $F_1, \ldots, F_r$, as it holds for $F_1, \ldots, F_R$ and since $s = [9k^2 \log 3Rk] \geqslant [9k^2 \log 3rk]$ is true. Thus all the conditions of Theorem 2 are satisfied with $R = r$, as required. Now since

$$\max_{1 \leqslant i \leqslant r} |F_i| \leqslant \max_{1 \leqslant i \leqslant R} |F_i| = \Lambda ,$$

our result (4.4.30) follows, by §6 of Chapter 3.

5. RESIDUAL SET

In this section, we shall discuss the residual set $\mathcal{R}$ as defined by (4.3.8), and prove the theorem (by induction) when this set $\mathcal{R}$ is non-empty. So from now on, we shall assume that $\mathcal{R}$ is non-empty and shall first study some properties of $\underline{\alpha}$ in $\mathcal{R}$.

We noticed in §3 that for every $\underline{\alpha}$ in $\mathcal{R}$,

$$|\underline{\alpha}| \leqslant P^{\delta} , \tag{4.5.1}$$

and there exist integers a_j, q_j, $(1 \leqslant j \leqslant n)$, such that

$$\Lambda_j = \frac{a_j}{q_j} + \gamma_j , \quad (1 \leqslant j \leqslant n), \quad (a_j, q_j) = 1 , \tag{4.5.2}$$

$$1 \leqslant q_j \leqslant P^{\eta}, \quad |\gamma_j| < q_j^{-1} P^{-k+\eta} , \tag{4.5.3}$$

where η is as in (4.3.1), and not all the a_j's are zero (as otherwise $\underline{\alpha} \in M$ with $\Lambda_i = 0$, $i \leqslant i \leqslant r$, $Q = 1$).

Now for $\underline{\alpha}$ in $\mathcal{R}$, we have

$$\sum_{i=1}^{R} \alpha_i \ F_i(\underline{x}) = \sum_{j=1}^{n} \left(\sum_{i=1}^{R} \alpha_i \ \lambda_{ij} \right) x_j^k \ ,$$

by (4.1.2) and then changing the order of summation. And this, by the definition of Λ_j (as in (3.1.10)), is

$$= \sum_{j=1}^{n} \Lambda_j \ x_j^k$$

$$= \sum_{j=1}^{n} \frac{a_j}{q_j} x_j^k + \sum_{j=1}^{n} \gamma_j \ x_j^k \ ,$$

by (4.5.2). Let $Q = \prod_{j=1}^{n} q_j$. Then we have

(4.5.4) $$(R+1)!\,4^R \Lambda_0^R Q \sum_{i=1}^{R} \alpha_i \ F_i(\underline{x}) = G_0(\underline{x}) + H(\underline{x}) \ ,$$

where $\Lambda_o = [\Lambda]$ (the integral part of Λ),

(4.5.5) $$G_0(\underline{x}) = \sum_{j=1}^{n} (R+1)!\,4^R \frac{Q}{q_j} \Lambda_o^R a_j \ x_j^k \ ,$$

and

(4.5.6) $$H(\underline{x}) = \sum_{j=1}^{n} (R+1)!\,(4\Lambda_o)^R Q \ \gamma_j \ x_j^k \ .$$

And so $G_o(\underline{x})$ is an integral form and $H(\underline{x})$ is a form with small real coefficients; and by (4.5.3) and since

$$\left|\frac{a_j}{q_j}\right| = |\Lambda_j - \gamma_j| \leqslant R \ \Lambda \ P^{\delta} + q_j^{-1} \ P^{-k+\eta} < (R+1) \ \Lambda \ P^{\delta} \ ,$$

we have

(4.5.7) $$|G_0| \leqslant 4^R (R+1)! \ \Lambda_o^R \ P^{n\eta+\delta} (R+1)\Lambda \ ,$$

(For the meaning of $|G_0|$, see the table of notation, page 3.)

We now recollect that $S_r = \{1,\ldots,r\}$, whenever S_r is non-empty. We also note that Theorem 3 for $r = R$ (that is when T_{R-r} is empty), follows from Theorem 2, which has been already proved in Chapter 3.

And so it is enough to consider $\underline{0 \leqslant r \leqslant R-1}$. Without loss of generality, we may take

$$|\alpha_{r+1}| = \max_{r+1 \leqslant i \leqslant R} |\alpha_i| \ .$$

Now let

$$(4.5.8) \qquad \begin{cases} G_i(\underline{x}) = 4\, F_i(\underline{x}), & 1 \leqslant i \leqslant R,\ i \neq r+1 \ , \\ G_{r+1}(\underline{x}) = G_0(\underline{x}) \ , & \end{cases}$$

where the $F_i(\underline{x})$'s are as in (4.1.2) and $G_0(\underline{x})$ is as in (4.5.5). Thus $G_i(\underline{x})$'s are all integral for $1 \leqslant i \leqslant r+1$, and real for $r+2 \leqslant i \leqslant R$.

In this section, we shall consider the new system

$$(4.5.9) \qquad \begin{cases} G_i(\underline{x}) = 0 \ , & 1 \leqslant i \leqslant r+1 \ , \\ |G_i(\underline{x})| < \frac{1}{2R}, & r+2 \leqslant i \leqslant R \ . \end{cases}$$

~~If all the $G_i(\underline{x})$'s are not independent, then $G_{r+1}(\underline{x})$ is dependent on the others as the latter are in fact $F_i(\underline{x})$, $i \neq r+1$ and are already known to be independent. Therefore any solution of our original system will be a solution of the present system also. Thus it is enough to consider the $G_i(\underline{x})$'s to be independent.~~

We can prove that the Gi's are linearly independent.

Lemma 4.5.1. Let $G_i(\underline{x})$, $(1 \leqslant i \leqslant R)$ be as in (4.5.9). Suppose η, P are such that

$$(4.5.10) \qquad 0 < \eta < \frac{k}{n}, \quad \omega > n\eta$$

and

$$(4.5.11) \qquad P >> \Lambda^{(2R-1)/(k-n\eta)} \ .$$

Let $\Delta_F(i_1, \ldots, i_R)$, $\Delta_G(i_1, \ldots, i_R)$ be as in (4.1.3). Then the Gi's are linearly independent whenever

$$\Delta_F(i_1, \ldots, i_R) \neq 0, \quad |\Delta_G(i_1, \ldots, i_R)| \geqslant 1 \ ;$$

and the non-zero coefficients of $G_i(\underline{x})$, $(1 \leq i \leq R)$ have absolute value at least one.

<u>Proof</u>. Let $i_1,\ldots,i_R$ be any R-tuple such that $\Delta_F(i_1,\ldots,i_R) \neq 0$. Let $M(i_1,\ldots,i_R)$ be the submatrix of (λ_{ij}) corresponding to the columns $i_1,\ldots,i_R$. Then we can write

$$\frac{1}{(R+1)!}\Delta_G(i_1,\ldots,i_R) = \sum_i 4^R\Lambda_o^R \; Q \frac{a_i}{q_i} D_{r+1,i} \quad ,$$

where the summation is over $i = i_1,\ldots,i_R$, and $D_{r+1,i}$ is the cofactor of $\lambda_{r+1,i}$ in $M(i_1,\ldots,i_R)$. By (4.5.2), this is

$$= \sum_{i=1}^{R} 4^R\Lambda_o^R \; Q\,(A_i - \gamma_i)D_{r+1,i}$$

$$= 4^R\Lambda_o^R \; Q \sum_{i=1}^{R}\sum_{j=1}^{R} \lambda_{ji}\,\alpha_j\, D_{r+1,i} - 4^R\Lambda_o^R \; Q \sum_i \gamma_i\, D_{r+1,i}$$

$$= 4^R\Lambda_o^R \; Q \sum_{i=1}^{R} \alpha_{r+1}\lambda_{r+1,i}\,D_{r+1,i} - 4^R\Lambda_o^R \; Q \sum_i \gamma_i\, D_{r+1,i}$$

(4.5.12) $$= 4^R\Lambda_o^R \; Q\,\alpha_{r+1}\,\Delta_F(1\ldots,R) - 4^R\Lambda_o^R \; Q \sum_i \gamma_i\, D_{r+1,i} \quad .$$

Now,

$$\left|4^R\Lambda_o^R \; Q \sum_i \gamma_i\, D_{r+1,i}\right| < 4^R\Lambda_o^R \; Q\,(R-1)!\,\Lambda^{R-1}\,P^{-k+\eta} \sum_{i=1}^{R} q_i^{-1}$$

(4.5.13) $$< 4^R R!\,\Lambda_o^R\,\Lambda^{R-1} Q\,P^{-k+\eta} \quad ,$$

by (4.5.3); and this is < 1, since (4.5.10) and (4.5.11) hold.

Since the F_i's are linearly independent, we may assume that $|\Delta_F(i_1,\ldots,i_R)| \geq 1$. Now if $\Delta_G(i_1,\ldots,i_R) = 0$, then by (4.5.12) and (4.5.13), it follows that

(4.5.14) $$|\alpha_{r+1}| < R!\,\Lambda^{R-1}\,P^{-k+\eta} \quad .$$

This holds for all α_i with $r+1 \leq i \leq R$ too. Also solving the r equations of the R equations

$$\Lambda_j = \frac{a_j}{q_j} + \gamma_j = \sum_i^R \lambda_{ij}\alpha_i \quad , \quad 1 \le j \le r,$$

for $\alpha_1,\ldots,\alpha_r$ and using (4.5.14), we have that each α_i is within $(R+1)!\,\Lambda^{R+r-2}\,P^{-k+\eta}$ of a rational number with denominator atmost $\Lambda^r Q$ which atmost $\Lambda^r P^{r\eta}$. Thus $\underline{\alpha}$ is in the major arc, by (4.5.10), which contradicts the fact that $\underline{\alpha} \in \mathbb{R}$. Hence $\Delta_G(i_1,\ldots,i_R) \neq 0$ and all the G_i's are linearly independent.

Now since $G_1,\ldots,G_{r+1}$ are linearly independent, we can choose $r+1$, j's such that j th column of the coefficient matrix of $G_1,\ldots,G_r$, G'_{r+1} where $G'_{r+1} = G_{r+1}/(4^R\Lambda_0^R(R+1)!\,Q) = \sum_j \frac{a_j}{q_j} x_j^k$, are linearly independent and so the determinant of the matrix of these columns is

$$\geqslant (q_1\cdots q_{r+1})^{-1} \geqslant Q^{-1}, \tag{4.5.14'}$$

since the λ_{ij}'s, $i=1,\ldots,r$ for the above j's are integral.

Consider the r+1 equations of the equations of (4.5.2), for the above chosen j's, as equations in $\alpha_1,\ldots,\alpha_r,1$:

$$\sum_{i=1}^r \lambda_{ij} - \frac{a_j}{q_j} = b_j$$

Where $|b_j| < q_j^{-1} P^{-k+\eta} + (R-r)\Lambda|\alpha_{r+1}|$. On solving these equations for the variable 1, we get

$$1 < Q(r+1)!\,\Lambda^R(P^{-k+\eta} + (R-r)\Lambda|\alpha_{r+1}|),$$

since the determinant of the above r+1 equations, by (4.5.14'), is $\geqslant Q^{-1}$. And so, by (4.5.11)

$$|\alpha_{r+1}| > (2(R+1)!\,Q\Lambda^R)^{-1}.$$

Also since $\Delta_F(i_1,\ldots,i_R) \neq 0$, it's absolute value is $\geqslant 1$. therefore

$$|4^R\Lambda_0^R(R+1)!\,Q\,\alpha_{r+1}\Delta_F(i_1,\ldots,i_R)| \geqslant 2^{R-1} \geqslant 2. \tag{4.5.15}$$

Thus using (4.5.15) and (4.5.13) in (4.5.12), the second part of our result follows. The last part is clear from (4.5.8), since $|G_0| \geqslant 4\Lambda \geqslant 1$.

<u>Lemma 4.5.2.</u> Suppose the residual set $\mathcal{R}$ is non-empty and the hypothesis H holds for the forms $F_1, \ldots, F_R$. Also suppose Theorem 3 holds for all systems of $r + 1$ equations and $R - r - 1$ inequalities of the type (4.1.2). Then it is possible to choose the constants η, P so that (4.5.10) and (4.5.11) hold and the system (4.1.1) has a solution $\underline{x}$ satisfying (4.1.7).

<u>Proof.</u> For $\underline{\alpha}$ in $\mathcal{R}$, we have proved in (4.5.4) that

$$(R+1)!\, 4^R \Lambda_0^R Q \sum \alpha_i \; F_i(\underline{x}) = G_0(\underline{x}) + H(\underline{x}) , \tag{4.5.4}$$

where $Q = \prod_{j=1}^{n} q_j$ and $G_0(\underline{x})$, $H(\underline{x})$ are forms as in (4.5.5), (4.5.6) respectively, the coefficients of $G_0(\underline{x})$ satify (4.5.7). We shall consider the system

$$\begin{cases} G_i(\underline{x}) = 0 \quad , & (1 \leqslant i \leqslant r+1), \\ |G_i(\underline{x})| < \dfrac{1}{2R} \quad , & (r+2 \leqslant i \leqslant R), \end{cases} \tag{4.5.9}$$

where $G_i(\underline{x})$'s are independent forms which are integral for $1 \leqslant i \leqslant r+1$ and real for $r+2 \leqslant i \leqslant R$ and are as in (4.5.8).

Suppose (4.5.10) and (4.5.11) hold. Then we can apply Lemma 4.5.1 amd so by the same lemma, we have that $|\Delta_G(i_1, \ldots, i_R)| \geqslant 1$, whenever $\Delta_F(i_1, \ldots, i_R) \neq 0$, where $F_1, \ldots, F_R$ are as in (4.1.1). Therefore any $R \times R$ submatrix of the coefficient matrix of the system (4.5.9) corresponding to a non-singular $R \times R$ submatrix of (λ_{ij}) is non-singular having absolute value at least 1 . And so $\Delta_G \geqslant 1$, (Δ_G is as in (4.1.5) for the system (4.5.9)). Thus, the hypothesis H holds for the system (4.5.9) as it holds for $F_1, \ldots, F_R$. Also from (4.5.5), we see that the absolute values of the non-zero coefficients of $G_0(\underline{x})$ is at least $(4\Lambda_0^R)^{(R+1)!} \geqslant 4$ and so from (4.5.8), we have $|\lambda'_{ij}| \geqslant 4 > 1$, whenever $\lambda'_{ij} \neq 0$, where (λ'_{ij}) is the coefficient matrix of $G_1, \ldots, G_R$. Therefore all the hypotheses of Theorem 3 hold for the system (4.5.9) which consists of $r + 1$ equations and $R - r - 1$ inequalities. And so by the hypothesis of our lemma, there exist constants $B_3(R,k)$, $B_4(n,R,k)$ such that the system (4.5.9) has a solution $\underline{X}$ satisfying

$$(4.5.16) \qquad 0 < |\underline{X}| \leq B_4 (4^R (R+1)! (R+1) \Lambda^{R+1} P^{n\eta+\delta})^{B_3} \quad ,$$

by (4.5.7), (4.5.8) and (4.1.4).

Let

$$(4.5.17) \qquad \eta_0 = k - k B_3 \delta - (B_3 k+1) n \eta .$$

We choose η, δ such that $\eta_0 > 0$ and (4.5.10) holds and P such that

$$(4.5.18) \qquad P^{\eta_0} > 2n(4R+4)^{B_3 k} B_4^k \Lambda^{2R+(R+1)B_3 k} (R+1)!^{B_3 k} .$$

Then for the above $\underline{X}$, by (4.5.8), $G_0(\underline{X}) = G_{r+1}(\underline{X}) = 0$ and $G_i(\underline{X}) = F_i(\underline{X}) = 0$, $(1 \leq i \leq r)$. And so using these and (4.5.8) in (4.5.4), we have

$$|F_{r+1}(\underline{X})| \leq \frac{1}{|\alpha_{r+1}|} \left(\sum_{i=r+2}^{R} |\alpha_i| . |G_i(\underline{X})| + \frac{(4\Lambda_0)^{-R}}{(R+1)!} Q^{-1} |H(\underline{X})| \right)$$

$$\leq \tfrac{1}{2} + n(4R+4)^{B_3 k} B_4^k \Lambda^{2R+(R+1)B_3 k} P^{-\eta_0} (R+1)!^{B_3 k}$$

$$< 1 ,$$

since $|\alpha_{r+1}| = \max|\alpha_i|$ and (4.5.18) holds where $\eta_0 > 0$ is given by (4.5.17). Also $|F_i(\underline{X})| = |\tfrac{1}{4} G_i(\underline{X})| < 1$, $(r+2 \leq i \leq R)$. Thus $\underline{X}$ is a solution of (4.1.1).

Suppose we restrict P such that it satisfies

$$\Lambda^{2R+(R+1)B_3 k} << P^{\eta_0} << \Lambda^{2R+(R+1)B_3 k} .$$

This restriction is possible and for such P, the above solution $\underline{X}$ satisfies

$$0 < |\underline{X}| < C_2 \Lambda^{C_1} ,$$

where $C_2 = C_2(B_4, R, n, k)$ and $C_1 = C_1(B_3, R, k)$ are constants. Thus the conclusion of our lemma is proved.

6. COMPLETION OF THEOREM 3

In this section, we shall complete the proof of Theorem 3, using the results of §3 to §6 by choosing the constants δ, ε, η, t, P, which appeared in the previous sections, appropriately and by applying induction on $R - r$.

Suppose Theorem 3 holds for all systems of $r + 1$ equations and $R - (r+1)$ inequalities, (a system (4.1.1) with r replaced by $r + 1$) of the type (4.1.2), in n variables, for n satisfying (4.1.6). Consider now a system as in (4.1.1), of r equations and $R - r$ inequalities (of the type (4.1.2)), satisfying the hypotheses of Theorem 3. We deal separately with the situation when the residual set is empty and when it is non-empty.

Suppose the residual set $\mathcal{R}$, as in (4.3.8) is empty. We shall prove the possibility of choosing the above constants so that the main term of the integral in (4.1.8) dominates all its error terms and to satisfy all the conditions of §3 to §5 . The main term (see (4.4.19), (4.4.28), (4.4.30)) is

$$\gg \Lambda^{-(3nR+1)R-\epsilon_2} P^{n-Rk}, \quad \varepsilon_2 = Mr\left(\frac{n}{k}-r\right)$$

and the error is, by (4.3.10), (4.3.13), (4.4.11), (4.4.22) and (4.4.23),

$$\ll \Lambda^{R\frac{s}{k}+2R^2 s-R} P^{n-Rk-\delta_0},$$

where

$$(4.6.1) \qquad \delta_0 = \min\{2^{-k+1}\eta - R\delta - (R+1)\varepsilon, \omega - Rk, 1 - R\eta - (r+2)\omega, \eta\left(\frac{s}{k} - R\right),$$

$$\omega\left(\frac{s}{k} - 2^{-(R-1)}\varepsilon\right) - R\eta\},$$

$$s = [9k^2 R \log 3Rk] .$$

The relations (4.4.27), (4.5.18) need to be satisfied by P and η, δ must satisfy

$$\eta_0 = k - kB_3\delta - (B_3k + 1)n\eta > 0,$$

(see (4.5.17)). Then clearly (4.4.7), (4.5.10) and (4.5.11) hold. If $\delta_0 > 0$ (as in (4.6.1)), and

$$\Lambda^K \ll P \ll \Lambda^K ,$$

where

$$K = \max\left\{\frac{(3n+2)R^2 + 9kR^2\log 3Rk + Mr\left(\frac{n}{k} - R\right)}{\delta_0}, \frac{2R+(R+1)B_3k}{\eta_0}\right\}$$

(B_3 is the constant appearing in (4.5.16)), then the main term is dominated by all the error terms and so there is a solution $\underline{x}$ to (4.1.1) inside the box $\mathcal{B}P$ and so it satisfies

$$0 < |\underline{x}| \leqslant P \ll \Lambda^K ,$$

which is (4.1.7) with $B_1 = K$.

Suppose the set $\mathcal{R}$ is non-empty. For the above choice (of the constants), all the conditions of §5 are satisfied (including (4.5.10), (4.5.11)). Therefore by Lemma 4.5.2 and our induction hypothesis, there exist constants $C_1 = (R,n,k)$, $C_2 = (R,k)$ such that (4.1.1) has a solution $\underline{x}$ satisfying

$$0 < |\underline{x}| < C_2\,\Lambda^{C_1} .$$

Thus, in either case, a solution to (4.1.1) exists satisfying (4.1.7), for some constants B_1, B_2 (as in the theorem). That is Theorem 3 holds under our inductive hypothesis. When $R - r = 0$ (that is, when $r = R$), the required system is a system of R equations satisfying all the conditions of Theorem 2 (Chapter 3). And so by Theorem 2, Theorem 3 holds for $R - r = 0$. Hence the truth of Theorem 3 follows, by induction on $R - r$, provided the above choice is possible.

Now we shall show that a choice for the constants as described above is possible. We take, in particular, $\omega = \eta$. Then since $n \geqslant [9R^3k^2 \log 3Rk]$ and $s = [9k^2R \log 3Rk]$, the last two terms in the right hand side of (4.6.1) are clearly positive. Let (η_1, δ_1) be the coordinates of some point in the region obtained by the inequalities

$$0 < \eta < \frac{1}{R+(r+2)},$$

$$0 < \delta < \frac{\eta}{R\,3^{k-1}}$$

and

$$kB_3\delta + (kB_3+1)n\eta < k .$$

(This region is non-empty, since $\frac{1}{R+(r+2)}$, $\frac{1}{R}$, kB_3 are all positive.) Then the values $\eta = \omega = \eta_1$, $\delta = \delta_1$ and $t = t_0$, where t_0 is some integer greater than $\frac{Rk}{\delta}$ is a possible choice (though not the best possible). This completes the proof of Theorem 3.

7. MAIN THEOREM

This section is concerned with a system of R diagonal Diophantine inequalities; and we shall deduce a result on this (stated below), which is the main result of this part of the thesis, from Theorem 3.

THEOREM 4. Let $F_1(\underline{x}), \ldots, F_R(\underline{x})$ be R independent diagonal forms of odd degree $k \geqslant 3$, in n variables with real coefficients, as in (4.1.2). Then for any arbitrarily small $\varepsilon > 0$, the system of inequalities

$$|F_i(\underline{x})| < \varepsilon, \quad (1 \leqslant i \leqslant R) \tag{4.7.1}$$

is solvable non-trivially in integers, for

$$(4.1.6) \qquad n \geqslant [9R^3k^2 \log 3Rk] \quad .$$

We prove this by induction on R. When $R = 1$, if the block condition (hypothesis H) is not satisfied, then there exist zero coefficients and so we get a non-trivial solution $\underline{x}$ by giving the value $x_j = 1$ for j's corresponding to the zero coefficients and the value $x_j = 0$ for the remaining x_j's. If the block condition is satisfied, the result follows from Theorem 3.

Suppose Theorem 4 holds for $R - 1$ forms, and consider R independent forms $F_1, \ldots, F_R$, as in Theorem 4, with n satisfying (4.1.6). If every linear combination of these R forms has more than $(s-1)R$ variables, where $s = [9k^2R \log 3Rk]$, then as $F_1, \ldots, F_R$ are independent, by Lemma 2.2.9, hypothesis H holds for these forms. Also it is easily seen that there exists a positive integer a_0, say, such that for any integer $a \geqslant a_0$, the forms $aF_1, \ldots, aF_R$ satisfying $\Delta' \geqslant 1$ and $|\lambda'_{ij}| \geqslant 1$, whenever $\lambda'_{ij} \neq 0$, where Δ' is as in (4.1.5) for these forms, and λ'_{ij} are the coefficients of aF_i's. Since a is a positive integer, $a \geqslant 1$ and the hypothesis H holds for $F_1, \ldots, F_R$, it also holds for $aF_1, \ldots, aF_R$. Thus all the hypotheses of Theorem 3 are satisfied for forms $aF_1, \ldots, aF_R$ (since n satisfies (4.1.6)), and so by Theorem 3, the system

$$|aF_i| < 1, \quad (1 \leqslant i \leqslant R)$$

is solvable non-trivially in integers. Hence by taking a sufficiently large, we obtain a solution for $|F_i(\underline{x})| < \varepsilon$, $(1 \leqslant i \leqslant R)$.

Suppose there exists a linear combination $a_1F_1 + \ldots a_RF_R$, not all $a_i = 0$ ($a_t \neq 0$, say), with at most $(s-1)R$ variables and we give the value 0 to all the variables appearing in this particular linear combination. Then

$$(4.7.2) \qquad a_1F_1 + \ldots + a_RF_R = 0, \quad a_t \neq 0.$$

For $A = \max\limits_{1 \leq i \leq R} |a_i|$, consider the system

$$(4.7.3) \qquad |A.R.F_i(\underline{x})| < \varepsilon, \quad 1 \leq i \leq R, \quad i \neq t .$$

This is a system of $R - 1$ inequalities in

$$\geq n - (s - 1)R$$
$$> [9(R-1)^3k^2 \log 3(R-1)k]$$

variables, since n satisfies (4.1.6). Therefore by our inductive supposition, this system has a non-trivial solution. From (4.7.2) and (4.7.3), we see that this solution is also a solution of $|F_t| < \varepsilon$. Thus we have obtained a solution for (4.7.1), and so in either case, the system (4.1.1) has a solution under the inductive hypothesis. Now since Theorem 4 holds for $R = 1$, the truth of our main theorem follows by induction on R.

It could be possible to improve the value of n in the theorems of this part of the thesis, by using diminishing ranges for the variables.

PART II

QUADRATIC AND CUBIC INEQUALITIES

CHAPTER 1

INTRODUCTION TO PART II

1. INTRODUCTION

In this part of the thesis, we consider homogeneous quadratic and cubic inequalities of the type

$$(1.1) \qquad |F(\underline{x})| < 1 ,$$

where

$$(1.2) \qquad F(\underline{x}) = \lambda_1 F_1(\underline{x}) + \ldots + \lambda_R F_R(\underline{x}),$$

$\lambda_1,\ldots,\lambda_R$ are real numbers which are linearly independent over the rationals and $F_1(\underline{x}),\ldots,F_R(\underline{x})$ are homogeneous forms in n variables, either all quadratic or all cubic with integer coefficients. The main aim is to find conditions on n which, together with indefiniteness of $F(\underline{x})$ in the quadratic case, ensure that (1.1) always has a non-trivial integral solution, and hence that $|F(\underline{x})| < \varepsilon$ is solvable for any $\varepsilon > 0$.

In this chapter, I shall begin, in §2 and §3, by giving the relevant available results on quadratic and cubic equations and inequalities, concentrating on the general (non-diagonal) case, and relying on Part I for the diagonal case. I shall also discuss, in §4, the system of Diophantine equations

$$F_i(\underline{x}) = 0, \quad (1 \leqslant i \leqslant R) \quad ,$$

where the F_i's are integral, since the tools needed for this sytem are interconnected with our problems (1.1). In the last two sections, I shall return to these problems and discuss various possible approaches.

In the case of an integral quadratic form, the extension of a result from the diagonal case to the general case is immediate by "completing the square". However, results about the diagonal case for real quadratic forms or for forms of higher degree (whether integral or not) cannot be extended in a similar way because there is no corresponding diagonalisation process.

2. QUADRATIC EQUATIONS AND INEQUALITIES

QUADRATIC EQUATIONS

The historical starting point of the subject of Diophantine equations in several variables is the Meyer's Theorem [26], on the solvability of quadratic equations in five variables:

RESULT 1. If $F(\underline{x})$ is an indefinite quadratic form in 5 or more variables, then

$$F(\underline{x}) = 0$$

is always solvable.

(Result 1 of Chapter 1, Part I, is when $F(\underline{x})$ is a diagonal quadratic.)

In 1955, Cassels [5] proved the following bounded result using the methods of Geometry of Numbers:

RESULT 2. Consider the equation

$$(2.1) \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} \, x_i \, x_j = 0,$$

where the f_{ij}'s are integers, $n \geqslant 2$ and let $F = \max\limits_{1 \leqslant i,j \leqslant n} |f_{ij}|$. If

(2.1) is solvable, then it has a solution $\underline{x}$ satisfying

$$0 < |\underline{x}| < C\, F^{n-1/2}, \quad C = C(n) .$$

He also had shown in [5] that the above exponent $\frac{n-1}{2}$ is the best possible.

QUADRATIC INEQUALITIES

In 1929, Oppenheim conjectured [28] that if $F(\underline{x})$ is any real indefinite quadratic form in at least five variables, then for any $\varepsilon > 0$, the inequality

$$(2.2) \qquad |F(\underline{x})| < \varepsilon ,$$

is solvable non-trivially. The result of Davenport and Heilbronn [18] is a particular case of this conjecture, when $F(\underline{x})$ is diagonal.

By a series of papers culminating in [22] , as a result of the joint efforts of Birch, Davenport and Ridout, we have the following result:

RESULT 3. Let $F(\underline{x})$ be an indefinite quadratic form in n variables, having real coefficients. For $n \geqslant 21$ and arbitrary $\varepsilon > 0$, the inequality (2.2) is solvable non-trivially. Further, if the coefficients are not all in rational ratios, the values assumed by $F(\underline{x})$, for integral $\underline{x}$ are everywhere dense.

The last part of this result was proved using a result of Oppenheim [29]. A result similar to the result 3 was proved independently for an indefinite non-singular form with the same number of variables, by Watson [41].

The basic method used by Birch, Davenport and Ridout to obtain result 3 was the modified form of the Hardy-Littlewood method as described in §3 of Chapter 1, Part I. However, here too, there were difficulties with the possible failure of the analytic method, and these were dealt with by two special methods, the choice depending on the rank and signature of the form. Both special methods involved a change of variables yielding a form which was approximately equal to a form of a special type, and then application of a suitable bound result to this second form. For one method (see, for example, Davenport [13]), the second form was integral and the required bound result was Cassels's Theorem (result 2 above); for the other (see Birch and Davenport [4]), the second form was a real diagonal form and the required bound was provided by the case $k = 2$ of result 6 of Chapter 1, Part I (proved in Birch and Davenport [3]).

The problems on Diophantine inequalities are interconnected with problems on fractional parts. For results on the latter, for quadratics, see [12], [35].

3. CUBIC EQUATIONS AND INEQUALITIES

CUBIC EQUATIONS

It can be shown (see [27]) that $n \geqslant 10$ is a necessary condition for the solvability of a homogeneous integral cubic equation

$$C(\underline{x}) = 0 \tag{3.1}$$

in n variables. This problem involves difficulties rather similar to those arising in connection with quadratic inequalities. The best result to date was proved by Davenport [16] and is as follows:

RESULT 4. Let $C(\underline{x})$ be a homogeneous cubic form in n variables, where the coefficients are integers. If $n \geqslant 16$, then (3.1) is always solvable.

This result is an improvement of his earlier results [14], [15] proved for $n \geqslant 32$, $n \geqslant 29$. The improvement was due to a more effective treatment of the bilinear equations associated with $C(\underline{x})$. Using the method of proof used by Davenport in [17] to prove the result for $n \geqslant 17$, with some refinements and a non-singularity condition, Lloyd [25] proved the following bound result.

RESULT 5. If $C(\underline{x})$ is a non-singular integral cubic form in $n = 17$ variables then the equation (3.1) has an integer solution $\underline{x}$ satisfying

$$0 < |\underline{x}| \ll M^{8\times 10^8} ,$$

where $M = |C|$ (see table of notation in page 3 for the meaning of $|C|$).

Similar results, but with weaker bounds, could be obtained by his methods for $n > 17$. The non-singularity condition, which was used to obtain a lower bound for the singular series appears to be difficult to eliminate.* Even so, one might reasonably hope to apply the result in the above form to the solvability of at least some cubic inequalities by using ideas similar to those discussed at the end of §2 above.

CUBIC INEQUALITIES

The idea of reduction to almost diagonal form which was applied to the general quadratic inequality by Birch and Davenport (see §2) can be modified so as to apply to the general cubic inequality. Recently Schlickewei [37] used this approach together with the result 4 [38] of Chapter 1, Part I, on small zeros of additive forms applied to $k = 3$, to obtain the following result on general cubics.

RESULT 6. Let $C(\underline{x})$ be a cubic form in n variables, with real coefficients. Then for any $\varepsilon > 0$, there are constants $c(\varepsilon)$, $c_0(\varepsilon)$ such that for $n > c(\varepsilon)$, the inequality

$$|C(\underline{x})| < 1$$

has an integral solution $\underline{x}$ satisfying

$$0 < |\underline{x}| \leqslant c_0(\varepsilon)\, M^{1/3+\varepsilon}$$

where $M = |C|$.

* I have subsequently been informed that a bound result of the required type was obtained by P.D.T. Elliott in his unpublished Cambridge Ph.D. thesis.

This result is an improvement of that of Pitman [30] who obtained $\frac{25}{6}$ instead of $\frac{1}{3}$ for the exponent of M, by the same method but by using [34] Pitman and Ridout, but giving an explicit (extremely large) value for n.

For the results on fractional parts of cubics, see [10].

4. A SYSTEM OF GENERAL FORMS

In 1962, Birch [1] gave a far-reaching extension of the method developed by Davenport for handling the cubic equation and obtained the following general result on a system of equations

$$F_i(\underline{x}) = 0, \quad (1 \leqslant i \leqslant R),$$

of arbitrary degree.

RESULT 7. Let $F_1(\underline{x}),\ldots,F_R(\underline{x})$ be rational forms of degree d in n variables where $n \geqslant R \geqslant 1$. Let $V(\mu)$ be the variety

$$V(\mu) : F_1(\underline{x}) = \mu_1,\ldots,F_R(\underline{x}) = \mu_R,$$

and let V^* be the union of the loci of singularities of the $V(\mu)$. Then if

$$\dim V(0) = n - R,$$

and if $V(0)$ has non-singular points in $\mathbb{R}$ and in every p-adic completion of the rationals and

$$n > R(R+1)(d-1)2^{d-1} + \dim V^*, \tag{4.1}$$

then $V(0)$ is solvable non-trivially in integers.

The method used was again the modified Hardy-Littlewood method; Birch found that the case when this method does not apply (which was the main difficulty in the work on the cubic) was connected with the dimension of the associated singular locus V^* , and so was able to eliminate this case by including the term $\dim V^*$ in (4.1).

In 1975, Dr. Pitman observed that the estimates obtained by Birch in this paper for the relevant exponential sum

$$\sum_{\underline{x}\in\mathcal{B}} e\left(\sum_{i=1}^{R} \alpha_i \; F_i(\underline{x})\right)$$

were immediately applicable to the exponential sum

$$\sum_{\underline{x}\in\mathcal{B}} e\left(\alpha \sum_{i=1}^{R} \lambda_i \; F_i(\underline{x})\right) \quad ,$$

which is the one needed for applying the modified Hardy-Littlewood method to the inequality (1.1) for $F(\underline{x})$ as in (1.2). She did some preliminary work on this problem with $F_1(\underline{x}),\ldots,F_R(\underline{x})$ of odd degree d , and found that this problem is solvable if

$$n > R(d-1)2^d + \dim V^*,$$

for V^* as in result 7. Also she obtained better results if the forms are disjoint and non-singular.

The analytic work in this part of the thesis depends on the ideas in this preliminary work of Dr. Pitman (which therefore in turn depends on Birch [1]) and on improvements which were developed jointly with her. On the basis of this work, it should be possible to replace $R(d-1)2^d$ in the above condition by $(R+1)(d-1)2^{d-1}$. Since the hypothesis involving the singular locus was not very satisfactory for this problem, Dr. Pitman suggested that I seek to eliminate this kind of condition in the cubic case, and this led to the investigation of the corresponding problem in the quadratic case, which is in some respects easier. In the light of Part I, it also seemed desirable to aim at bound results, if possible.

5. PROBLEMS OF PART II

The results finally obtained in this part are as follows:

<u>THEOREM ON QUADRATIC INEQUALITY</u>. Let $F(\underline{x})$ be an indefinite quadratic form as in (1.2), where $F_1(\underline{x}),\ldots,F_R(\underline{x})$ are integral quadratic forms in n variables, with symmetric coefficients. Suppose that $n \geqslant 2R+6$, $|F| \geqslant 1$, $\max\limits_{1\leqslant t\leqslant R} |\lambda_t| \leqslant 1$. Then there exists a computable constant $K = K(n,R)$ such that the inequality (1.1) has an integer solution $\underline{x}$ satisfying

$$0 < |\underline{x}| \ll M^K ,$$

where $M = \max\limits_{1\leqslant t\leqslant R} |F_t|$.

<u>THEOREM ON CUBIC INEQUALITY</u>. Let $F(\underline{x})$ be as in (1.2), where $F_1(\underline{x}),\ldots,F_R(\underline{x})$ are integral cubic forms in n variables, having symmetric coefficients. The inequality (1.1) is solvable non-trivially, if $n \geqslant 4R(R+3)$.

The theorem on quadratic is proved in Chapter 2 and that on cubic, in Chapter 3.

6. VARIOUS APPROACHES

The main approach for these kinds of problems is to apply the modified Hardy-Littlewood method (as mentioned in §4) as far as the method is effective. The difficulty arises thereafter. Mostly this difficult case is concerned with a condition on the nature of the forms. We need *either* to impose additional conditions on the forms (as in result 7) to dispose of the possibility of this difficulty *or* to find some other methods which may prove the result in this case. This is the cause of the involvement of dim V^* in the expression (4.1) of result 7 (as seen in §4).

I shall now briefly outline the various approaches that I have tried in dealing with these alternate (difficult) cases.

APPROACH 1. Approximating a certain (suitable) multiple of $F(\underline{x})$ to an almost integral form and applying the available results on "bounded solutions" of the corresponding equations, for the new (integral) form.

APPROACH 2. We shall call this a rank approach as it involves the rank of some matrix. The difficult case arises when the rank of a matrix associated with a certain multilinear form is $\leqslant R - 1$. This multilinear form is bilinear for cubics and linear for quadratics. We may consider each value $(\leqslant R - 1)$ of the rank separately and for each case the form can be approximated to one which is a linear combination of r forms, where r $(0 \leqslant r \leqslant R-1)$ is the rank mentioned above. Then we may be able to prove the solvability by induction on r. The rank zero case will turn out to be the "difficult case" associated with the corresponding system of R equations.

APPROACH 3. This is by induction on R. If the difficult case occurs for a given value of R, the problem is equivalent to one for which one of the forms vanishes on a large subspace of n-dimensional rational space, and so is equivalent to a problem involving $R - 1$ forms in fewer variables.

We may use any one or a combination of more than one of these approaches. We may need additional hypotheses on the forms depending on the problem and the availability of the preliminary results. The approach 3 will increase the number of variables needed, considerably.

We use approach 1 in Chapter 2 to prove our theorem on quadratics; and approach 2 also applies to this case. Unfortunately, for various reasons (see §7 of Chapter 3), we were unable to apply the approaches 1 and 2 for our cubic problem. Approach 3 is used in this case and this causes the larger value of n (larger than one would expect), and also gives rise to difficulties in obtaining a bound result, as explained in §7, Chapter 3. Although a bound result has not achieved in the cubic case, I give most results in Chapter 3, in a form showing dependence on coefficients (though this detail was not needed for the present result), enabling one to apply them later, in problems on "bounded solution" of cubic inequalities.

The notations given in page 3 are valid for this part also and throughout this part, the constants implied by the notations $\ll$, $\gg$, O are all independent of the coefficients of the forms but may (ultimately) depend on n, R and k (and independent of the constants appearing in Part I).

CHAPTER 2

A QUADRATIC INEQUALITY

1. INTRODUCTION

In this chapter, I consider the inequality

(1.1) $$|Q(\underline{x})| < 1 \,,$$

for an indefinite real quadratic form of the type

(1.2) $$Q(\underline{x}) = \lambda_1 Q_1(\underline{x}) + \lambda_2 Q_2(\underline{x}) + \ldots + \lambda_R Q_R(\underline{x}) \,,$$

where

(1.3) $$Q_t(\underline{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{(t)} x_i x_j = \underline{x}' A^{(t)} \underline{x}, \quad (1 \leqslant t \leqslant R),$$

$A^{(1)}(\underline{x}), \ldots, A^{(R)}(\underline{x})$ are symmetric $n \times n$ matrices with integer components and $\lambda_1, \ldots, \lambda_R$ are real numbers which are linearly independent over the rationals.

We also write

(1.4) $$Q(\underline{x}) = \underline{x}' A \underline{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \,,$$

where A is a symmetric matrix with real components. I shall prove that under suitable conditions, the inequality (1.1) has a non-trivial integral solution bounded by an explicit function of the $|Q_t|$'s (for the meaning of $|Q_t|$, see the table of notations in the general introduction).

In this section, I shall state the theorem proved in this Chapter and give an outline of the method of its proof.

For $Q(\underline{x})$, $Q_1(\underline{x}),\ldots,Q_R(\underline{x})$ as in (1.2) to (1.4), we write

(1.5) $$M = \max_{1\leqslant t\leqslant R} |Q_t| ,$$

(1.6) $$m = |Q| ,$$

and

(1.7) $$\Lambda = \max_{1\leqslant t\leqslant R} |\lambda_t| .$$

<u>THEOREM</u>. For an indefinite real quadratic form $Q(\underline{x})$ as in (1.2) to (1.4) and M, m, Λ as in (1.5) to (1.7), suppose that $n \geqslant 2R + 6$, $m \geqslant 1$, $\Lambda \leqslant 1$, $\lambda_1 = 1$ and $a_{11}a_{22} < 0$. Then there exists a computable constant $K = K(n,R)$ such that the inequality (1.1) has an integer solution $\underline{x}$ satisfying

(1.8) $$0 < |\underline{x}| << M^K ,$$

where the constant implied by $<<$ depends on n and R but not on the coefficients of $Q(\underline{x})$ and so not on M and Λ.

The value of K is very large and is determined explicitly in §5.

We note that any indefinite $Q(\underline{x})$ as in (1.1) to (1.4) can be normalised to satisfy the hypotheses of the theorem (by using indefiniteness and a suitable change of variable to obtain $a_{11}a_{22} < 0$, dividing by the largest $|\lambda_i|$, and multiplying by a suitable constant to satisfy the other conditions). By applying the theorem to a multiple of the normalised form, we see that if $Q(\underline{x})$ is any indefinite form as in (1.1) to (1.4), $\lambda_1,\ldots,\lambda_R$ are linearly independent over the rationals and $n \geqslant 2R + 6$, then the inequality $|Q(\underline{x})| < \varepsilon$ is non-trivially solvable in integers, for arbitrarily small $\varepsilon > 0$. For $R \leqslant 7$, fewer than 21 variables are required, so this gives an improvement, for quadratic inequalities of this particular type, of the theorem of Davenport, Birch and Ridout on quadratic inequalities which is given as Result 3 of Chapter 1.

We also note that $\Delta M \geqslant m \geqslant 1$ and so if either $|a_{11}| < 1$ or $|a_{22}| < 1$, then (1.1) is clearly solvable with $0 < |\underline{x}| \leqslant 1 \ll M$ (by taking x_1 or x_2 as 1 and the other x_i's as 0). Hence by relabelling, if necessary, we can assume from now on that

$$(1.9) \qquad a_{11} \geqslant 1, \quad a_{22} \leqslant -1 .$$

Outline of the method.

The approach is basically the adaptation by Davenport, of the Hardy-Littlewood method that was used in Part I of this thesis. To give a brief outline of the method, I need the following definition:

Definition 1.1. Let $\mathcal{B}$ be the box in n dimensional space (a cartesian product of n intervals) with side length $\frac{1}{2}$ and centre at the origin $\underline{0} = (0,\dots,0)$. Let P be a large positive integer, α a real number and let

$$(1.10) \qquad S(\alpha) = \sum_{\underline{x}} e(\alpha Q(\underline{x})),$$

where the summation is over all the integer points $\underline{x}$ inside the box $P\mathcal{B}$. Also let $N(P)$ be defined as the number of integer solutions inside the box $P\mathcal{B}$, of (1.1).

Let r be a *fixed* integer whose value will be decided in §5. Let $K(\alpha)$ be the real valued function, corresponding to r, given by the Lemma 1.3.1 of Chapter 1, Part I, so that, in particular

$$(1.11) \qquad |K(\alpha)| \ll \min(1, |\alpha|^{-r-1}),$$

where the constant implied by $\ll$ depends only on r.

For $S(\alpha)$ defined by (1.10), let

$$(1.12) \qquad J(P) = \int_{-\infty}^{\infty} S(\alpha)K(\alpha)d\alpha .$$

Then by the Lemma 1.3.1 of Chapter 1, Part I, we have

(1.13) $$N(P) + 1 \geqslant J(P) .$$

It is therefore sufficient to prove that $J(P) > 1$, for some P bounded by an explicit power of M. In order to do this, we dissect the interval $(-\infty,\infty)$ of integration of $J(P)$ into smaller sets, namely, the basic interval, the tail, the supplementary set and the residual set. The Hardy-Littlewood approach fails if the residual set is non-empty and so, in this case, we use a different approach.

Throughout this chapter, δ, θ, ε are small positive constants and the constants implied by the symbols 0, $<<$, $>>$, K_i $(i = 1,2, \text{etc.})$ are independent of P and of the coefficients of $Q(\underline{x})$, but may depend on δ, θ, ε, n and R.

In §2, I shall give some preliminary lemmas, some of which are needed for the estimate of $J(P)$ (whenever it is estimable). In §3, the dissection of $(-\infty,\infty)$ and the estimate of the contribution to $J(P)$ of all the sets but the residual set are given. In §4, the residual set is discussed and the existence of a solution of (1.1), when this set is non-empty, is proved. And finally in §5, we find values of P for which the main term, (which is the contribution from the basic interval), dominates the error terms (which include the contribution from the supplementary set and the tail). Also the values of δ, θ, ε, r are determined explicitly in terms of n and R and the proof of the theorem is completed by combining the results of §3 and §4.

We shall also see that $n \geqslant 2R + 3$ is sufficient for the theorem to hold for forms $Q(\underline{x})$, for which the residual set is always empty.

2. PRELIMINARY RESULTS

In this section, I shall give a result for a quadratic of a special type and state some known results which are needed in the later sections of this chapter.

Lemma 2.1. Suppose

$$Q(\underline{x}) = \sum_{i=1}^{5} \sum_{j=1}^{5} b_{ij} x_i x_j$$

is an indefinite quadratic form in five variables such that all the components b_{ij}, except possibly b_{55}, are integers. Then the inequality

$$(2.1) \qquad |Q(\underline{x})| < 1$$

has a non-trivial integral solution $\underline{x}$ such that

$$0 < |\underline{x}| \leqslant K_1 |Q|^{63},$$

where K_1 is an absolute constant (and so independent of the coefficients of $Q(\underline{x})$).

Proof. We note that

$$|Q| \geqslant \max_{(i,j)\neq(5,5)} |b_{ij}| \geqslant 1,$$

since otherwise $Q(\underline{x})$ would not be indefinite.

If $Q(x_1,x_2,x_3,x_4,0) = 0$ for some integers x_1,x_2,x_3,x_4 not all zero, then the result follows from Meyer's Theorem and Cassel's Theorem (Results 1 and 2 of Chapter 1). Hence from now on we can assume that

$$(2.2) \qquad Q(x_1,x_2,x_3,x_4,0) \neq 0, \text{ for integral } x_1,x_2,x_3,x_4, \text{ not all zero.}$$

In particular, $b_{11} \neq 0$, and it is easily checked that completing the square yields

$$Q(\underline{x}) = \frac{1}{b_{11}}(b_{11}x_1+b_{12}x_2+b_{13}x_3+b_{14}x_4 + b_{15}x_5)^2 + \frac{1}{b_{11}} Q_1(x_2,x_3,x_4,x_5),$$

where

$$Q_1(x_2,x_3,x_4,x_5) = \sum_{i=2}^{5} \sum_{j=2}^{5} b'_{ij} x_i x_j$$

has all the components b'_{ij} ,except possibly b'_{55} , integral and

$$|Q_1| \ll |Q|^2 ,$$

(the constant implied by $\ll$ is an absolute constant). Moreover, $b'_{22} \neq 0$, since otherwise we easily obtain a contradiction to (2.2).

Repeating this process, we obtain

$$Q(\underline{x}) = \frac{1}{B_1} L_1^2 + \frac{1}{B_1B_2} L_2^2 + \frac{1}{B_1B_2B_3} L_3^2 + \frac{1}{B_1B_2B_3B_4} L_4^2 + \frac{\mu}{B_1B_2B_3B_4} x_5^2 , \tag{2.3}$$

where B_1,B_2,B_3,B_4 are non-zero integers, μ is real, the L_i's are integral linear forms of the shape

$$L_i = L_i(x_i,\ldots,x_5) = B_i x_i + \sum_{j=i+1}^{5} \beta_{ij} x_j ,$$

with determinant

$$\Delta = B_1B_2B_3B_4 ,$$

and

$$|B_i| \ll |Q|^{2^{i-1}}, \quad |L_i| \ll |Q|^{2^{i-1}}, \quad |\mu| \ll |Q|^{2^4} .$$

We now consider the inequality

$$\left|\frac{\Delta^2}{B_1} y_1^2 + \frac{\Delta^2}{B_1B_2} y_2^2 + \frac{\Delta^2}{B_1B_2B_3} y_3^2 + \frac{\Delta^2}{B_1B_2B_3B_4} y_4^2 + \frac{\mu\Delta^2}{B_1B_2B_3B_4} y_5^2\right| < 1 . \tag{2.4}$$

All the coefficients, except possibly the last one, of (2.4) have absolute value at least one, and they cannot all have the same sign as $Q(\underline{x})$ is indefinite. If $|\mu\Delta^2/B_1B_2B_3B_4| < 1$, then $(0,0,0,0,1)$ is a non-trivial solution of (2.4). Otherwise, the last coefficient also has absolute value at least one, and so we can apply the bound result of Birch and Davenport (Result 6 of Chapter 1, part I) with $\theta = 2/125$, say. In either case, we obtain an integral solution $\underline{y} = (y_1,\ldots,y_5)$ of (2.4) such that

$$(2.5) \qquad 0 < |y_i| \ll |Q|^{48+2^{i-1}}, \ (1 \leqslant i \leqslant 4); \quad 0 < |y_5| \ll |Q|^{48},$$

where the constant implied by $\ll$ is an absolute constant (as it depends only on the value $2/125$).

It is then easily checked that the unique $\underline{x}$ such that

$$L_i(\underline{x}) = \Delta y_i, \quad (1 \leqslant i \leqslant 4), \quad x_5 = \Delta y_5$$

is an integral solution of (2.1) such that

$$0 < |\underline{x}| \ll |Q|^{63} \ .$$

Hence the lemma is proved.

<u>Definition 2.1.</u> Let $Q(\underline{x}), Q_1(\underline{x}),\ldots,Q_R(\underline{x})$ be quadratic forms as in (1.4) and (1.3). For j such that $1 \leqslant j \leqslant n$, we define linear forms $L_j(\underline{x}), L_j^{(1)}(\underline{x}),\ldots,L_j^{(R)}(\underline{x})$ corresponding to the forms $Q(\underline{x}), Q_1(\underline{x}),\ldots,Q_R(\underline{x})$ as follows:

$$L_j(\underline{y}) = \sum_{k=1}^{n} a_{jk}y_k = A_j\underline{y}$$

and

$$L_j^{(t)}(\underline{y}) = \sum_{k=1}^{n} a_{jk}^{(t)}y_k = A_j^{(t)}\underline{y}, \quad (1 \leqslant t \leqslant R),$$

where A_j, $A_j^{(t)}$ denote the j-th rows of the matrices A, $A^{(t)}$ $(1 \leqslant t \leqslant R)$ respectively.

<u>Lemma 2.2</u>. Let θ be a real number independent of P such that $0 < \theta < 1$. Suppose

(2.6) $$|S(\alpha)| > P^{n-(R+5/4)\theta+\epsilon} .$$

Then the number of integer points $\underline{y}$ such that

(2.7) $$|\underline{y}| < P^{\theta},\ \|2\alpha L_j(\underline{y})\| < P^{-2+\theta},\quad (1 \leqslant j \leqslant n)$$

is

(2.8) $$\gg P^{(n-2R-5/2)\theta+\epsilon} .$$

<u>Proof</u>. This is Lemma 2.4 of Birch [1] with $d = 2$, $k = (R + \frac{5}{4})\theta - \varepsilon$. The proof uses a result on geometry of numbers, see Lemma 28 of Davenport [17], where the implied constant is independent of the lattices and so our lemma follows.

<u>Definition 2.2</u>. For given θ such that $0 < \theta < 1$, let $E(\theta)$ be the set of α satisfying

(2.9) $$|\alpha| < P^{\delta}$$

and

(2.10) $$|S(\alpha)| > P^{n-(R+5/4)\theta+\epsilon} .$$

<u>Lemma 2.3</u>. For $1 \leqslant j \leqslant n$, let $L_j(\underline{x})$ be the linear forms as in Definition 2.1, corresponding to $Q(\underline{x})$. Suppose that for given $P_0 > 0$ and θ such that $0 < \theta < 1$, we have

that there is a non-zero integral vector $\underline{x}$ such that

(2.12) $$0 < |\underline{x}| < P_o^{\theta}, \quad |L_j(\underline{x})| \leq \frac{1}{n} P_o^{-\theta}, \quad (1 \leq j \leq n).$$

Then this $\underline{x}$ is a solution of (1.1).

Proof. The proof is trivial, since

$$Q(\underline{x}) = \sum_{j=1}^{n} L_j(\underline{x})x_j, \quad \underline{x} = (x_1, \ldots, x_n).$$

We notice that (2.8) is of no value if $n < 2(R + \frac{5}{4})$. Hence from now on we take

(2.13) $$n \geq 2R + 3 .$$

Lemma 2.4. For given θ such that $0 < \theta < 1$, let $E(\theta)$ be as given by Definition 2.2. Suppose there is no non-zero integral $\underline{x}$ satisfying (2.12) with $P_o = P$. Then,

(i) for all α in $E(\theta)$, there exist integers $b, q_1, \ldots, q_R$ such that

(2.14) $$0 \leq q_1, \ldots, q_R < n\, MP^{\theta} ,$$

(2.15) $$|q_1 + \lambda_2 q_2 + \ldots + \lambda_R q_R| > \frac{1}{n} P^{-\theta} ,$$

and

(2.16) $$|\alpha(q_1 + \lambda_2 q_2 + \ldots + \lambda_R q_R) - b| < P^{-2+\theta} ;$$

(ii) we have for $P \geq 2$,

(2.17) $$\mu E(\theta) \ll M^R P^{-2+(R+1)\theta+\delta} .$$

Proof. (i) Since α is in $E(\theta)$, (2.10) holds. Therefore (2.14) and (2.16) will follow immediately from Lemma 2.2. For example, we may take $q_t = 2L_j^{(t)}(\underline{y})$, $(1 \leq t \leq R)$ for some j, $(1 \leq j \leq n)$, where $\underline{y}$ is any solution of (2.7), b will then be the integer closest to $2\alpha L_j(\underline{y})$ and (2.15) is the immediate consequence of the hypothesis of our lemma.

(ii) When $b \neq 0$, by (2.16), since $0 < \theta < 1$, we have, for $P \geqslant 2$

$$|\alpha(q_1 + \lambda_2 q_2 + \ldots + \lambda_R q_R) - b| < \tfrac{1}{2} \leqslant \tfrac{1}{2}|b| .$$

Therefore

$$|\alpha(q_1 + \lambda_2 q_2 + \ldots + \lambda_R q_R)| \geqslant \tfrac{1}{2}|b| .$$

Thus by (2.9),

$$1 \leqslant |b| \leqslant 2P^{\delta}|q_1 + \lambda_2 q_2 + \ldots + \lambda_R q_R| . \tag{2.18}$$

And so the contribution to $\mu E(\theta)$ from the α's for which $b \neq 0$ is

$$<< \sum_{q_1} \ldots \sum_{q_R} \sum_{b} |q_1 + \lambda_2 q_2 + \ldots + \lambda_R q_R|^{-1} P^{-2+\theta},$$

summed over $q_1, \ldots, q_R$ such that (2.14) holds. Thus the above contribution is

$$<< M^R P^{-2+(R+1)\theta+\delta} .$$

When $b = 0$,

$$|\alpha| << P^{-2+2\theta} < M^R P^{-2+(R+1)\theta+\delta} ,$$

by (2.16) and (2.15). This proves (2.17).

3. ESTIMATION OF J(P)

In this, I shall state the dissection of the range of α and estimate the main term and the error terms of $J(P)$ defined by (1.12). This involves simultaneous rational approximations of $\alpha, \lambda_2\alpha, \ldots, \lambda_R\alpha$.

We recall the conventions about the parameters δ and ε.

Let $\theta_0, \theta_1, \ldots, \theta_h$ be real numbers such that

$$0 < \theta_0 < \theta_1 < \ldots < \theta_h < 1 \tag{3.1}$$

and

$$\theta_g - \theta_{g-1} = \frac{\delta}{R+1}, \quad (1 \leqslant g \leqslant h). \tag{3.2}$$

<u>The dissection of the interval $(-\infty,\infty)$.</u>

Let $I = (-\infty,\infty)$ and θ_0 be as in (3.1); its precise value will be decided only in §5. We dissect I into disjoint subsets as follows:

<u>Definition 3.1</u>. Let $E(\theta_0)$ be the set given by Definition 2.2 with $\theta = \theta_0$. We define

$$B = \{\alpha : |\alpha| < M^{R-1} P^{-2+R\theta_0}\}, \tag{3.3}$$

$$T = \{\alpha : |\alpha| \geqslant P^{\delta}\}, \tag{3.4}$$

$$S = I - (B \cup T \cup E(\theta_0)), \tag{3.5}$$

$$\mathcal{R} = I - (B \cup T \cup S) = E(\theta_0) - (B \cup T). \tag{3.6}$$

Then,

$$I = B \cup T \cup S \cup \mathcal{R}.$$

We call B, the basic interval; T, the tail; S, the supplementary set and $\mathcal{R}$, the residual set.

I shall estimate the contribution to $J(P)$ from the sets B, T and S, in this section.

Lemma 3.1. The contribution from T to $J(P)$ satisfies

$$(3.7) \qquad \int_T |S(\alpha)K(\alpha)|\,d\alpha \ll P^{n-r\delta} ,$$

where r is the fixed integer as given in (1.11) and the constant implied by $\ll$ depends only on r.

Proof. We use the trivial estimate of $S(\alpha)$, that for any α

$$|S(\alpha)| < P^n .$$

Then by (1.11), (where the implied constant depends only on r),

$$\int_T |S(\alpha)K(\alpha)|\,d\alpha \ll \int_{|\alpha| \geqslant P^\delta} P^n |\alpha|^{-r-1}\,d\alpha$$

$$\ll P^{n-r\delta} ,$$

thus the lemma is proved.

Lemma 3.2. Suppose that for all θ such that $\theta_0 \leqslant \theta \leqslant \theta_h$, there does not exist any integral $\underline{x}$ satisfying (2.12) with $P_0^0 = P$. Then,

$$(3.8) \qquad \int_S |S(\alpha)K(\alpha)|\,d\alpha \ll M^R P^{n-2-\delta_0} ,$$

where

$$(3.9) \qquad \delta_0 = \min\{-2 + (R + \tfrac{5}{4})\theta_h - \delta - \varepsilon,\ \tfrac{1}{2}\theta_0 - \varepsilon - 2\delta\}.$$

Proof. From the definition of S (see (3.5)), we see that it is contained in the set of all the α for which $|\alpha| < P^\delta$ and α is *either* not in $E(\theta_h)$ *or* in $E(\theta_h) - E(\theta_0)$. For those α in $E(\theta_h)$, since (2.9), (2.10) hold with $\theta = \theta_h$, it follows, from Definition 2.2, that (2.10) cannot hold with $\theta = \theta_h$ for the α's for which $|\alpha| < P^\delta$ and α not in $E(\theta_h)$. Therefore the contribution to the integral in (3.8) from these α's is

$$< \int P^{n-(R+5/4)\theta_h+\epsilon} |K(\alpha)|\,d\alpha$$

$$< \int P^{n-(R+5/4)\theta_h+\epsilon} .1.d\alpha ,$$

by (1.11); and this is

$$< P^{n-(R+5/4)\theta_h+\epsilon+\delta} . \tag{3.10}$$

The set $E(\theta_h) - E(\theta_0)$ is the union of the sets $E(\theta_g) - E(\theta_{g-1})$ for $g = 1,\ldots,h$. By Definition 2.2, for α in $E(\theta_g) - E(\theta_{g-1})$, (2.10) does not hold for $\theta = \theta_{g-1}$. Hence from this, (1.11) and Lemma 2.4 (ii) with $\theta = \theta_g$ for $g = 1,\ldots,h$, the contribution to the integral in (3.8) from $E(\theta_h) - E(\theta_0)$ is, as in Lemma 37 of Davenport [17],

$$<< \sum_{g=1}^{h} M^R P^{n-2+(R+1)\theta_g-(R+5/4)\theta_{g-1}+\epsilon+\delta}$$

$$<< \sum_{g=1}^{h} M^R P^{n-2-\frac{1}{4}\theta_{g-1}+\epsilon+2\delta} ,$$

by (3.2); and this is

$$<< M^R P^{n-2-\frac{1}{4}\theta_0+\epsilon+2\delta} ,$$

by (3.1). Hence from this and (3.10), the conclusion of the lemma follows.

<u>Lemma 3.3</u>. For α in B, given by (3.3), we have

$$S(\alpha) = I(\alpha) + O(M^R P^{n-1+R\theta_0}), \tag{3.11}$$

where

$$I(\alpha) = \int_{PB} e(\alpha Q(\underline{\xi}) d\underline{\xi} . \tag{3.12}$$

(The proof follows similar steps to that of Lemma 4, Davenport [17] with $a = 0$, $q = 1$.)

<u>Proof</u>. The definition of $S(\alpha)$ is given by (1.10). The main term of the lemma is obtained by replacing $\underline{x}$ by a continuous variable $\underline{\eta}$

and so the sum in $S(\alpha)$ is replaced by an integral. Then the error is

$$\left| \sum_{\underline{x}} e(\alpha Q(\underline{x})) - \int_{\underline{\eta}} e(\alpha Q(\underline{\eta}))d\underline{\eta} \right| ,$$

where the range of each of $\underline{x}$, $\underline{\eta}$ is such that they are in $P\mathcal{B}$. Thus the discrepancy between the number of integer points in the box for $\underline{x}$ and the volume of the box for $\underline{\eta}$ is $<< P^{n-1}$. Hence the above error is

$$<< P^{n-1} + P^n \max \left| \alpha \frac{\partial Q(\underline{\eta})}{\partial \eta_i} \right|$$

$$<< P^{n-1} + P^n M^{R-1} P^{-2+R\theta_0} .MP$$

$$<< M^R \quad P^{n-1+R\theta_0} .$$

Thus the lemma is proved.

<u>Lemma 3.4</u>. Suppose $n \geq 2R + 3$, $r \geq 2$, $\varepsilon < \frac{5}{4R}$, $(R+\frac{13}{4})\theta_0 < \frac{1}{2}$, $P^{\frac{1}{2}} > 2M^{2R-1}$, and that for $\theta_0 \leq \theta \leq \frac{1}{2}$ and $P^{\theta_0} \leq P \leq M^R P^{2R+5}$ there is no integral solution of (2.12). Then

(3.13) $$\int_B S(\alpha)K(\alpha)d\alpha = \int_{-\infty}^{\infty} I(\alpha)K(\alpha)d\alpha + E_1 + E_2 ,$$

where $I(\alpha)$ is as in (3.12) and

(3.14) $$E_1 = O(M^{2R-1} P^{n-3+2R\theta_0}), \quad E_2 = O(P^{n-2-(5/4-R\epsilon)\theta_0}).$$

<u>Proof</u>. We apply Lemma 3.3, and we get

$$\int_B S(\alpha)K(\alpha)d\alpha = \int_B (I(\alpha) + O(M^R \quad P^{n-1+R\theta_0}))K(\alpha)d\alpha$$

$$= \int_{\mathcal{B}} I(\alpha)K(\alpha)d\alpha + E_1 \quad ,$$

where

Note We use the following

Lemma; Suppose that for any θ, P such that $\theta_0 \le \theta \le \frac{1}{2}$ and $P^{\frac{1}{2}} > 2n^2 M$ there does not exist any non-zero integral $\underset{\sim}{x}$ satisfying (2.12) with $P = P_0$. Then for $nP^{R\theta_0} < |\gamma| < nP$

$$\int_{\mathcal{B}} e(\gamma Q(\underset{\sim}{\xi}))\, d\underset{\sim}{\xi} \ll \gamma^{-\frac{(R+5/4)}{R}+\varepsilon}$$

Proof. Suppose $nP^{-2+R\theta_0} < |\alpha| < nP^{-1}$.

By writing $|\alpha P^2| = nP^{R\theta}$, where $\theta_0 \le \theta \le \frac{1}{2}$; and using $|\alpha| < P^{\delta}$ (since $P > n$), we can easily prove that $\alpha \notin E(\theta)$ and so

$$|S(\alpha)| \le P^{n-(R+\frac{5}{4})\theta+\varepsilon} \ll P^{n+\varepsilon} |\alpha P^2|^{-\frac{(R+\frac{5}{4})}{R}}.$$

As in Lemma 3.3 we can also prove that

$$\int_{\mathcal{B}} e(\alpha P^2 Q(\underset{\sim}{\xi}))\, d\underset{\sim}{\xi} = P^{-n} S(\alpha) + O(M^R |\alpha P^2| P^{-1}).$$

Now using the above estimate for $S(\alpha)$ in this and writing $\gamma = \alpha P^2$ and $P = \gamma^{1+\frac{R+5/4}{R}} M^R$, our result follows.

$$|E_1| = O\Big(\int M^R \quad P^{n-1+R\theta_0}\, |K(\alpha)|\, d\alpha\Big)$$

$$= O(M^{2R-1} P^{n-3+2R\theta_0}).$$

The main term of (3.13) is obtained by extending the interval of integration of the above integral to the whole of the real line. The error in doing so is E_2, say. We write $\underline{\xi} = P\underline{\eta}$ in $I(\alpha)$, then

$$I(\alpha) = \int_{\mathcal{B}} e(\alpha P^2 \sum_1^R \lambda_i Q_i(\underline{\eta}))\, P^n d\underline{\eta} \quad .$$

By an argument similar to Lemma 5.2 of Birch [1], the main steps of which are outlined on the opposite page, it can be shown that if $M^{R-1}P^{R\theta_0} \le |\alpha P^2| \le P^3$, then

$$I(\alpha) \ll P^n(\alpha P^2)^{-(R+\frac{5}{4})/R \;+\varepsilon}$$
$$\ll P^{n-2(R+\frac{5}{4})/R\;+2\varepsilon}|\alpha|^{-(R+\frac{5}{4})/R\;+\varepsilon}$$

Then E_2 is given by

$$|E_2| \leqslant \int_H |I(\alpha)K(\alpha)|\, d\alpha \ ,$$

where the range H of α is $|\alpha| \geqslant M^{R-1}P^{-2+R\theta_0}$. And so, using Lemma 1.3.1 of Part I for $|\alpha| \geqslant P$,

$$|E_2| \ll \int_H P^{n-2-5/2R+2\epsilon}\, |\alpha|^{-1-5/4R+\epsilon} .1.d\alpha \;+\; \int_P^\infty P^n \alpha^{-\nu-1} d\alpha$$

$$\ll P^{n-2-5/2R+2\epsilon}(M^{R-1}P^{-2+R\theta_0})^{-5/4R+\epsilon} \;+\; P^{n-\nu}$$

$$\ll M^{(-5/4R+\epsilon)(R-1)}P^{n-2-(5/4-R\epsilon)\theta_0} + P^{n-\nu}$$

$$\ll P^{n-2-(5/4-R\epsilon)\theta_0}\ ,$$

since $\varepsilon < \frac{5}{4R}$, and using our hypotheses on θ_0 and ν. This completes the proof of the lemma.

Lemma 3.5. Suppose $P \geqslant 11$. Then we have

(3.15) $$\int_{-\infty}^{\infty} I(\alpha)K(\alpha)d\alpha >> M^{-n}P^{n-2} ,$$

where the constant implied by $>>$ depends only on n.

Proof. Let

$$J = \int_{-\infty}^{\infty} I(\alpha)K(\alpha)d\alpha .$$

Then we have,

$$J = \int_{-\infty}^{\infty}\int_{P\mathcal{B}} e(\alpha Q(\underline{\xi}))K(\alpha)d\underline{\xi}d\alpha .$$

By the definition of $e(\alpha)$ and by (1.11), the absolute value of the integrand of J is bounded by $\min\{1,|\alpha|^{-r-1}\}$ which is integrable on $(-\infty,\infty) \times P\mathcal{B}$. Hence we may use the substitution $\underline{\xi} = P\underline{\eta}$ (and so $d\underline{\xi} = P^n d\underline{\eta}$), and interchange the order of integration. We then obtain that

(3.16) $$J = P^n \int_{\mathcal{B}}\int_{-\infty}^{\infty} e(\alpha P^2 Q(\underline{\eta}))K(\alpha)d\alpha d\underline{\eta} .$$

We shall apply the change of variable $f : \underline{\eta} \to \underline{y}$ defined by

$$y_1 = P^2 Q(\underline{\eta}),$$

$$y_i = \eta_i , \quad (2 \leqslant i \leqslant n)$$

to a suitable subset of $\mathcal{B}$. Let $\mathcal{B}_1$ be the box defined by

(3.17) $$0 < y_1 < \frac{1}{3}, \quad \frac{1}{16m} < |y_2| < \frac{1}{8m} , \quad |y_i| < \frac{1}{32mn} , \quad (3 \leqslant i \leqslant n).$$

We investigate the $\underline{\eta}$ such that $\underline{y} = f(\underline{\eta})$ and $\underline{y}$ is in the box $\mathcal{B}_1$.

Let $\underline{y}$ be in $\mathcal{B}_1$ and $\eta_i = y_i$, $(2 \leqslant i \leqslant n)$. Consider

(3.18) $$g(\eta) = Q(\eta,\eta_2,\ldots,\eta_n) - \frac{y_1}{P^2},$$

$$= a_{11}\eta^2 + F\eta + G - \frac{y_1}{P^2},$$

where

$$F = \sum_{i=2}^{n} a_{1i}\eta_i, \quad G = \sum_{i=2}^{n}\sum_{j=2}^{n} a_{ij}\eta_i\eta_j .$$

Then since $a_{22} \leqslant -1$ (by (1.9)), we have

(3.19) $$|F| \leqslant \frac{5}{2^5}, \quad |G| < \frac{21}{2^{10}m}, \quad G < -\frac{11}{2^{10}m}$$

and since $P \geqslant 11$ and $y_1 < 3$, we easily see that

(3.20) $$\frac{y_1}{P^2} < \frac{3}{2^{10}} .$$

Hence for $\underline{y}$ in $\mathcal{B}_1$, it follows from (3.18), (3.19) and (3.20), since $a_{11} \geqslant 1$ (by (1.9)), that $g(-\frac{1}{4}) > 0$, $g(\frac{1}{4}) > 0$ and $g(0) < 0$. Thus $g(\eta)$ has exactly one zero between 0 and $\frac{1}{4}$, and in fact this zero lies between η_0 and $\frac{1}{4}$, where $\eta_0 = \max(0, -F/(2a_{11}))$, and $g'(\eta) > 0$ for $\eta_0 < \eta < \frac{1}{4}$.

It follows that there is an open set $\mathcal{B}_2 \subseteq \mathcal{B}$ such that f is one-to-one on $\mathcal{B}_2$, $f(\mathcal{B}_2) = \mathcal{B}_1$ and the Jacobian $|\partial\underline{y}/\partial\underline{\eta}|$ of f on $\mathcal{B}_2$ satisfies

(3.21) $$0 < \left|\frac{\partial\underline{y}}{\partial\underline{\eta}}\right| = P^2\left|\frac{\partial Q}{\partial\eta_1}\right| = P^2|g'(\eta_1)| \ll P^2.m ,$$

where the implied constant depends only on n. And so the change of variable theorem is applicable to f on $\mathcal{B}_2$. By (3.17) and the Lemma 1.3.1 of Chapter 1, Part I, the inner integral of J in (3.16) is non-negative. Therefore we have

$$J \geqslant P^n \int_{\mathcal{B}_2}\int_{-\infty}^{\infty} e(\alpha P^2 Q(\underline{\eta}))K(\alpha)d\alpha d\underline{\eta} ;$$

substituting $\underline{y} = f(\underline{\eta})$ and applying the change of variable theorem to the above integral, we get

$$J \geqslant P^n \int_{\mathcal{B}_1} \int_{-\infty}^{\infty} e(\alpha y_1) K(\alpha) d\alpha \left|\frac{\partial \underline{\eta}}{\partial \underline{y}}\right| d\underline{y}$$

$$\gg P^n \int_{\mathcal{B}_1} \int_{-\infty}^{\infty} e(\alpha y_1) K(\alpha) d\alpha . m^{-1} P^{-2} d\underline{y} ,$$

by (3.21); and this is

$$\gg P^n \int_{\mathcal{B}_1} 1.m^{-1} P^{-2} d\underline{y} ,$$

since $0 < y_1 < \frac{1}{3}$ for all $\underline{y}$ in $\mathcal{B}_1$, by Lemma 1.3.1 of Chapter 1, Part I; and this is

$$\gg m^{-1} P^{n-2} . \frac{1}{m^{n-1}} ,$$

by the definition of $\mathcal{B}_1$ (see (3.17)). From this, the result (3.15) follows, since $m \ll M$.

4. RESIDUAL SET

The set $\mathcal{R}$ is as defined in Definition 3.1. In this section, I shall prove that when this set $\mathcal{R}$ is non-empty, there exists a solution of (1.1) satisfying a bound in terms of M and P. This is done by using approximation to a form to which Lemma 2.1 can be applied.

For α in $\mathcal{R}$, we have

$$M^{R-1} P^{-2+R\theta_0} < |\alpha| < P^{\delta} \tag{4.1}$$

and there are N_{θ_0} integer points $\underline{x}$ satisfying (2.7) with $\theta = \theta_0$, where N_{θ_0} satisfies

(4.2) $$N_{\theta_0} \gg P^{(n-2R-5/2)\theta_0 - \epsilon} .$$

In this section, by points I shall mean integer points.

Lemma 4.1. Suppose α belongs to the residual set $\mathcal{R}$ and let V be the corresponding set of N_{θ_0} points mentioned in (4.2),

(4.3) $$n > 2R + \frac{11}{2} + \frac{\varepsilon}{\theta_0} + \varepsilon,$$

and

(4.4) $$P > K_2(n) ,$$

for a constant K_2 depending on n. Then V contains at least four linearly independent points.

Proof. Let v be the dimension of the subspace spanned by V. Then the number of points $\underline{x}$ in V is $\ll P^{v\theta_0}$, since $|\underline{x}| < P^{\theta_0}$ for $\underline{x}$ in V. But by the definition of V, the number of points in V satisfies (4.2). Therefore, we have

$$P^{v\theta_0} \gg P^{(n-2R-5/2)\theta_0 - \epsilon} .$$

And since (4.4) holds, we have

$$v > n - 2R - \frac{5}{2} - \frac{\varepsilon}{\theta_0} .$$

Thus

$$v > 3 + \varepsilon ,$$

by (4.3). Since v, being the dimension, is an integer and $\varepsilon > 0$, we have

$$v \geqslant 4.$$

<u>Lemma 4.2</u>. Suppose that the residual set $\mathcal{R}$ is non-empty, that there does not exist an integral $\underline{x}$ satisfying (2.12) with $\theta = \theta_0$ and that (4.3) holds. Also suppose that θ_0, δ, P satisfy

(4.5) $$3(127\theta_0 + 42\delta) < 2$$

and

(4.6) $$P \geqslant K_3 M^{K_4},$$

where

$$K_4 = 253/(2-3(127\theta_0+42\delta))$$

and K_3 is a constant depending only on n. Then (1.1) has an integral solution $\underline{x}$ such that

(4.7) $$0 < |\underline{x}| < K_5 P^2 ,$$

where K_5 is a constant depending only on n.

<u>Proof</u>. Since $\mathcal{R}$ is non-empty, it contains an element, α, say. It follows from our hypothesis regarding (2.12), together with Lemma 2.4 and (4.1), that there exist integers $b, q_1, \ldots, q_R$ satisfying (2.14), (2.15) and (2.16) with $\theta = \theta_0$ and $b \neq 0$. Taking

$$p = [|q_1 + \lambda_2 q_2 + \ldots + \lambda_R q_R|] + 1 ,$$

we see that p is integral,

(4.8) $$|\alpha p| > \tfrac{1}{2}$$

and

(4.9) $$|p| \ll MP^{\theta_0} .$$

By choosing K_3 appropriately, we can ensure that the hypothesis of Lemma 4.1 are satisfied and hence the set V defined there contains four linearly independent points $\underline{x}^{(1)}$, $\underline{x}^{(2)}$, $\underline{x}^{(3)}$, $\underline{x}^{(4)}$, say. We shall apply the linear transformation

(4.10) $$\underline{x} = \sum_{i=1}^{5} \ell_i \, \underline{x}^{(i)} ,$$

to the form $\alpha p Q(\underline{x})$, where ℓ_i are integers and $\underline{x}^{(5)}$ is a fifth integral vector, which we choose as follows:

If $Q\left(\sum_{i=1}^{4} \ell_i \underline{x}^{(i)}\right)$ is positive definite, then, by (1.9), the vector $\underline{X}$, say, with $X_2 = 1$, $X_i = 0$ for $i \neq 2$ cannot be a linear combintation of $\underline{x}^{(1)}$, $\underline{x}^{(2)}$, $\underline{x}^{(3)}$, $\underline{x}^{(4)}$ since $Q(\underline{X}) < 0$, so we take $\underline{x}^{(5)} = \underline{X}$. Similarly if $Q\left(\sum_{i=1}^{4} \ell_i \underline{x}^{(i)}\right)$ is negative definite, we take the vector $\underline{Y}$, say, with $Y_1 = 1$, $Y_i = 0$ for $2 \leqslant i \leqslant n$ as $\underline{x}^{(5)}$. Otherwise we take as $\underline{x}^{(5)}$ one of the unit vectors (with one $x_i = 1$ and the others all zero) which is linearly independent of $\underline{x}^{(1)}$, $\underline{x}^{(2)}$, $\underline{x}^{(3)}$, $\underline{x}^{(4)}$; such a vector exists as $n > 4$.

With this choice of $\underline{x}^{(5)}$, the form

(4.11) $$\alpha p Q_0(\underline{\ell}) = \alpha p Q\left(\sum_{i=1}^{5} \ell_i \underline{x}^{(i)}\right), \quad \underline{\ell} = (\ell_1, \ldots, \ell_5)$$

is indefinite and the vectors $\underline{x}^{(1)}$ to $\underline{x}^{(5)}$ are linearly independent integral vectors with

(4.12) $$|\underline{x}^{(i)}| < p^{\theta_0}, \ (1 \leqslant i \leqslant 4); \quad |\underline{x}^{(5)}| \leqslant 1 .$$

Now,

$$Q_0(\underline{\ell}) = \sum_{\sigma=1}^{5} \sum_{\rho=1}^{5} \phi_{\sigma\rho} \ell_\sigma \ell_\rho ,$$

where

$$\phi_{\sigma\rho} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i^{(\sigma)} x_j^{(\rho)} .$$

From (2.7) on Lemma 2.2, (for $L_j(\underline{x})$, $1 \leqslant j \leqslant n$, as in Definition 2.1), we see that for every vector $\underline{x}$ in V (V as in Lemma 4.1), there exist integers b_j and real numbers δ_j, $(1 \leqslant j \leqslant n)$ such that

$$2\alpha L_j(\underline{x}) = b_j + \delta_j, \quad (1 \leqslant j \leqslant n),$$

(4.13) $$|b_j| < nMP^{\delta+\theta_0}, \quad |\delta_j| < P^{-2+\theta_0}.$$

And so by our choice of $\underline{x}^{(1)}$, $\underline{x}^{(2)}$, $\underline{x}^{(3)}$, $\underline{x}^{(4)}$ (in V) and $\underline{x}^{(5)}$, we easily see that

$$2\alpha p\phi_{\sigma\rho} = b_{\sigma\rho} + \delta_{\sigma\rho}, \quad (1 \leqslant \sigma, \rho \leqslant 5, (\sigma,\rho)\neq(5,5))$$

and

$$\alpha p\phi_{55} = \alpha p\phi(\underline{x}^{(5)}),$$

where $b_{\sigma\rho}$ are integers and $\delta_{\sigma\rho}$ are real numbers satisfying

(4.14) $$|b_{\sigma\rho}| << M^2 P^{3\theta_0+\delta}, \quad |\alpha p\phi_{55}| << M^2 P^{\theta_0+\delta}$$

and

(4.15) $$|\phi_{\sigma\rho}| << M P^{-2+3\theta_0},$$

by (4.9), (4.12) and (4.13). The constants implied by $<<$ depends only on n.

Hence, we have

(4.16) $$2\alpha pQ_0(\underline{\ell}) = Q_1(\underline{\ell}) + Q_2(\underline{\ell}) + 2\alpha pQ(\underline{x}^{(5)})\ell_5^2,$$

where $Q_1(\underline{\ell})$ is integral with no terms in ℓ_5^2 and $Q_2(\underline{\ell})$ has small real coefficients. Thus the hypothesis of the Lemma 2.1 holds for $Q_1(\underline{\ell}) + \alpha pQ(\underline{x}^{(5)})\ell_5^2$ and so by Lemma 2.1, there is an integral $\underline{\ell}$ such that

(4.17) $$|Q_1(\underline{\ell}) + \alpha pQ(\underline{x}^{(5)})\ell_5^2| < \tfrac{1}{4}$$

and

(4.18) $$0 < |\underline{\ell}| \leqslant K_6 (M^2 \ P^{3\theta_0 + \delta})^{63} ,$$

using (4.14). For this $\underline{\ell}$, we have

(4.19) $$|Q_2(\underline{\ell})| < K_6^2 \ M \ P^{-2+3\theta_0} (M^2 P^{3\theta_0 + \delta})^{2.63} < \tfrac{1}{4},$$

using (4.15), (4.18) and then by (4.6) for suitable K_3 . And hence the $\underline{x}$ corresponding to this $\underline{\ell}$ satisfies (1.1), by (4.11), (4.16), (4.17), (4.19) and (4.8); and is such that

$$0 < |\underline{x}| \ll P^{\theta_0} . (M^2 \ P^{3\theta_0 + \delta})^{63} ,$$

$$\ll M^{126} \ P^{190\theta_0 + 63\delta} ,$$

by (4.10), (4.12) and (4.18). This is

$$\ll P^2 ,$$

since (4.5) and (4.6) hold, where the constant implied by $\ll$ depends only on K_3, K_4, n and so only on n . Thus the lemma is proved.

5. PROOF OF THE THEOREM

In this section, I shall give a choice of the small constants δ, ε, θ_0 and of P , which appeared in the previous sections, so that the theorem holds. The existence of a solution with a bound in terms of P has been proved in Lemma 4.2, under certain conditions, when the set $\mathcal{R}$ given by (3.6) is non-empty. Now we prove the theorem, by considering all the various possibilities.

We recall that $\underline{n \geqslant 2R + 6}$ and so (4.3) holds if

(5.1) $$\frac{\varepsilon}{\theta_0} + \varepsilon < \tfrac{1}{2} .$$

Now we consider the integral $J(P)$ given by (1.12). The Lemma 3.5 gives the main term of $J(P)$ and the error terms are given by (3.7), (3.8) with (3.9), together with E_1, E_2 given by (3.14). We shall choose P, δ, ε, θ_0 to make the main term dominate the above mentioned error terms and to satisfy (5.1) and all the conditions used in §4. We shall do this now.

The main term is

$$\gg M^{-n} P^{n-2}$$

and the errors are (from Lemmas 3.1, 3.2 and 3.4)

$$\ll M^{K_7} P^{n-2-\delta_1} ,$$

where

$$K_7 = \max\{R, 2R - 1\} ,$$

$$\delta_1 = \min\{r\delta - 2, (R + \tfrac{5}{4})\theta_h - 2 - \varepsilon - \delta, \tfrac{1}{4}\theta_0 - \varepsilon - 2\delta,$$

$$1 - 2R\theta_0 , (\tfrac{5}{4} - R\varepsilon)\theta_0 \}.$$

Let Δ be a small positive constant to be determined and θ_0 satisfy $(R+\frac{13}{4})\theta_0 < \frac{1}{2}$ and let

$$\delta = \varepsilon = \frac{\Delta}{12} .$$

Then the main term dominates all the error terms, if

$$\delta_1 > \Delta .$$

It is possible to find values of Δ (and so δ, ε), θ_0, θ_h, r to satisfy the above and the conditions given by (5.1), (4.5). For example,

$$\Delta = \frac{1}{1000}, \quad \theta_0 = \frac{2}{395} , \quad \text{for} \quad R \leqslant 95,$$

or

$$\Delta = \frac{1}{10(R+4)}, \quad \theta_0 = \frac{1}{2R+7}, \quad \text{for} \quad R \geqslant 96,$$

and $\theta_h = \frac{2}{3}$, for $R \geqslant 2$ *or* $\theta_h = \frac{9}{10}$, for $R = 1$ and r is any integer such that

$$r > 12\left(1 + \frac{2}{\Delta}\right),$$

depending on the value of R; is a particular choice. (Of course, this is not the best possible.) For this choice of Δ, θ_0, θ_h and r, let P be an integer such that

$$K_0 M^{K^*} < P < 2K_0 M^{K^*}, \tag{5.3}$$

where

$$K^* > \max\{(K_7 + n)/\Delta, \; K_4\} \tag{5.4}$$

and

$$K_0 > \max\{K_2(n), K_3\}, \tag{5.5}$$

(K_2, K_3, K_4 are as in Lemmas 4.1 and 4.2, and K_7 as in (5.2)). Then all the conditions including (4.4), (4.6) are satisfied.

If for some θ such that $\theta_0 \leqslant \theta \leqslant \theta_h$, there exists an $\underline{x}$ such that (2.12) holds for some P_0 with $P^{\theta_0} \leq P_0 \leq M^R P^{2R+2}$, then by Lemma 2.3, there exists a solution $\underline{x}$ of (1.1) such that

$$0 < |\underline{x}| < P_0^{\theta} \leqslant P_0^{\theta_h} \leqslant M^{2R/3}(2K_0 M^{K^*})^{(2R+2)2/3} < M^{2(R+(2R+2)K^*)/3}(2K_0)^{4(R+1)/3}, \tag{5.6}$$

using (5.3).

Hence from now on, we assume that for θ, P_0 such that $\theta_0 \leqslant \theta \leqslant \theta_h$, and $P^{\theta_0} \leq P_0 \leq M^R P^{2R+2}$, there is no $\underline{x}$ such that (2.12) holds; in particular there is no $\underline{x}$ such that (2.12) holds with $\theta = \theta_0$ and so since $n \geqslant 2R + 6$, it follows from Lemma 4.2 that if the residual set is non-empty, there exists a solution $\underline{x}$ of (1.1) such that

$$(5.7) \qquad 0 < |\underline{x}| \ll P^2 \ll M^{2K^*},$$

using (5.4).

Hence we may now assume that the residual set is empty. Then the analytic argument shows that the main term dominates the error terms and so $N(P) > 1$ and there exists a solution $\underline{x}$ in the box $\mathcal{B}P$ such that (1.1) holds and so

$$(5.8) \qquad 0 < |\underline{x}| < \tfrac{1}{2}P \leqslant K_0 M^{K^*} .$$

Thus we always have a solution $\underline{x}$ of (1.1) satisfying one of (5.6), (5.7) or (5.8) and so the result of the theorem follows.

CHAPTER 3

A CUBIC INEQUALITY

1. INTRODUCTION

In this chapter, I consider the inequality

$$|C(\underline{x})| < 1 \tag{1.1}$$

for a cubic form $C(\underline{x})$ of the type

$$C(\underline{x}) = \lambda_1 C_1(\underline{x}) + \lambda_2 C_2(\underline{x}) + \ldots + \lambda_R C_R(\underline{x}) , \tag{1.2}$$

where

$$C_t(\underline{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} c_{ijk}^{(t)} x_i x_j x_k , \quad (1 \leqslant t \leqslant R) \tag{1.3}$$

for each integer t, $(1 \leqslant t \leqslant R)$, $c_{ijk}^{(t)}$ are integers and symmetric functions in i, j, k and $\lambda_1,\ldots,\lambda_R$ are real numbers which are linearly independent over the rationals. We also write

$$C(\underline{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} c_{ijk} x_i x_j x_k , \tag{1.4}$$

where c_{ijk} are real numbers which are symmetric functions in i, j, k.

The aim will be to prove the solvability of (1.1) without making any assumptions of non-singularity, and the result finally obtained will be the following theorem:

<u>THEOREM</u>. Let $C(\underline{x})$ be defined as in (1.2) to (1.4), and suppose that the number of variables n is such that $n \geqslant 4R(R+3)$. Then (1.1) has a non-trivial solution in integers.

As in the case of quadratics, it follows from this theorem that $|C(\underline{x})| < \varepsilon$ is solvable for arbitrarily small $\varepsilon > 0$, if $n \geqslant 4R(R+3)$.

From now on, $C(\underline{x})$ is a form as in (1.2) to (1.4). We note that if $|c_{111}| < 1$, then (1.1) is certainly solvable (with $x_1 = 1$, $x_i = 0$, $i \neq 1$). Hence we assume from now on that $|c_{111}| \geqslant 1$. Moreover, replacing $C(\underline{x})$ by $-C(\underline{x})$ if necessary, we can assume that

(1.5) $$c_{111} \geqslant 1.$$

We also introduce the following notations:

(1.6) $$M = \max_{1 \leqslant t \leqslant R} |C_t|, \quad m = |C|$$

and

(1.7) $$\Lambda = \max_{1 \leqslant t \leqslant R} |\lambda_t| .$$

(For the meanings of $|C_t|$, $|C|$, see the table of notations, in the general introduction - page 3.)

We now notice that

(1.8) $$1 \leqslant m \ll \Lambda M ,$$

where the constant implied by the symbol $\ll$ depends only on n.

Outline of the method

For this problem too, the basic idea of the approach is the same as that used in Part I and in our quadratic problem. But the alternate method used whenever the basic approach fails, is different.

In this chapter, δ and ε are small positive constants, r a fixed positive integer and P a large positive integer, all of which will be fixed throughout the argument. In addition, θ is a parameter such that $0 < \theta < 1$, and the constants θ_i $(0 \leqslant i \leqslant h)$ are particular values of θ. The values of all the above constants will be determined explicitly, later in the chapter. Unless otherwise mentioned, the constants implied by the symbols O, $\ll$, $\gg$, K_1, K_2 etc. are independent of P and of the coefficients of $C(\underline{x})$ and so on M, Λ; but may depend on n, R, δ, θ and ε.

(We note that these constants are independent of the ones that appeared in the other chapters.)

Definition 1.1. Let $\mathcal{B}$ be the fixed box in n-dimensional space with side length $\frac{1}{2}$ and having its centre at the origin $\underline{O}$. Let α be a real number and

$$(1.9) \qquad S(\alpha) = \sum_{\underline{x}} e(\alpha C(\underline{x})),$$

summed over all the integer points $\underline{x}$ inside the box $P\mathcal{B}$. Also, let $N(P)$ be the number of non-zero integer solutions of (1.1) inside the box $P\mathcal{B}$.

Let $K(\alpha)$ be the real valued function corresponding to the integer r, given by the lemma 1.3.1 of Chapter 1, Part I and so satisfying the inequality (1.11) of Chapter 2, Part II.

As in the other cases, we define

$$(1.10) \qquad J(P) = \int_{-\infty}^{\infty} S(\alpha)\, K(\alpha)\, d\alpha,$$

where $S(\alpha)$ is defined by (1.9). Then,

$$N(P) + 1 > J(P),$$

where the 1 on the lefthand side is for the trivial solution $(0,\ldots,0)$. We follow the usual method, the process of estimating $J(P)$ whenever it is estimable and showing that there exist values of P for which $J(P)$ is greater than one. In the alternate case, we prove the theorem under a certain hypothesis on linear combinations of $R - 1$ cubics. Then we prove the theorem by induction on R, and the above hypothesis turns out to be the inductive hypothesis for this proof.

In §2, I give some preliminary lemmas concerning $S(\alpha)$ and bilinear forms (Definition 2.1) and in §3, two lemmas on bilinear equations are given. In §4, I give the estimations of some error terms and in §5, that of the main term is given, the values of the constants θ_0, θ_h, δ, ε, r are determined and the theorem is proved when the alternate case does not occur. Then in §6, the alternate case is discussed and finally in §7, the proof of the theorem is completed. Also in §7, some other methods that one would think of to deal with the alternate case (namely the residue set with alternative (ii) - see §6 for details) are discussed in brief, and the reasons which led to the choice of the present method are explained.

2. BILINEAR FORMS

In this section, I shall give some known results concerning the estimate of $S(\alpha)$ and some bilinear forms which are needed later on.

Definition 2.1. Let $\underline{x}$, $\underline{y}$ be any two points. Then for each of the cubic forms $C(\underline{x})$, $C_1(\underline{x}),\ldots,C_R(\underline{x})$ given as in (1.2) to (1.4), we define a set of n bilinear forms:

$$B_j(\underline{x}|\underline{y}) = \sum_{i=1}^{n}\sum_{k=1}^{n} c_{ijk}\, x_i\, y_k \;, \quad (1 \leqslant j \leqslant n), \tag{2.1}$$

$$B_j^{(t)}(\underline{x}|\underline{y}) = \sum_{i=1}^{n}\sum_{k=1}^{n} c_{ijk}^{(t)}\, x_i\, y_k \;, \quad (1 \leqslant j \leqslant n), \tag{2.2}$$

for $1 \leqslant t \leqslant R$.

We notice that by (1.2),

$$B_j(\underline{x}|\underline{y}) = \sum_{t=1}^{R} \lambda_t\, B_j^{(t)}(\underline{x}|\underline{y}), \quad (1 \leqslant j \leqslant n). \tag{2.3}$$

Lemma 2.1. Suppose that θ is a real number such that $0 < \theta < 1$ and is independent of P. Then *either*

(i) $|S(\alpha)| < P^{n-K\theta+\epsilon}$

or

(ii) there are $\gg P^{(2n-4K)\theta+\epsilon}$ pairs $(\underline{x},\underline{y})$ of integer points such that

(2.4) $$|\underline{x}| < P^{\theta}, \quad |\underline{y}| < P^{\theta}$$

and

(2.5) $$\| 6\alpha B_j(\underline{x}|\underline{y})\| < P^{-3+2\theta}, \quad (1 \leqslant j \leqslant n).$$

Proof. This result follows from Lemma 31, of Davenport [17] with K replaced by $K\theta - \varepsilon$, where the constant implied by $\gg$ is independent of $C(\underline{x})$ (and hence of M, Λ) as it uses a result (Lemma 28, Davenport [17]) on geometry of numbers where the implied constant is independent of the lattices.

Definition 2.2. For given θ such that $0 < \theta < 1$, let $E(\theta)$ be the set of α satisfying

(2.6) $$|\alpha| < P^{\delta}$$

and

(2.7) $$|S(\alpha)| > P^{n-K\theta+\epsilon}.$$

Lemma 2.2. For all α in $E(\theta)$, there exists integers $q_1,\ldots,q_R$, b such that

(2.8) $$0 \leqslant |q_1|,\ldots,|q_R| \ll MP^{2\theta},$$

and

(2.9) $$|\alpha(\lambda_1 q_1+\ldots+\lambda_R q_R) - b| < P^{-3+2\theta},$$

where the constant implied by $\ll$ in (2.8) depends only on n.

Proof. For α in $E(\theta)$, by definition, (2.7) holds and so alternative (ii) of Lemma 2.1 holds. Thus (2.8) and (2.9) follow from (2.4) and (2.5), by taking $q_t = 6B_j^{(t)}(\underline{x}|\underline{y})$, $(1 \leqslant t \leqslant R)$, for some solution pair $(\underline{x},\underline{y})$ and for some j in $1 \leqslant j \leqslant n$.

Lemma 2.3. For given θ such that $0 < \theta < 1$, at least one of the following two alternatives holds:

Alternative (i). For each α in $E(\theta)$, $\lambda_1\alpha,\ldots,\lambda_R\alpha$ have simultaneous rational approximations $\frac{a_1}{q},\ldots,\frac{a_R}{q}$, say, such that for $1 \leqslant t \leqslant R$,

$$\lambda_t\alpha = \frac{a_t}{q} + \beta_t, \tag{2.10}$$

where $(a_1,\ldots,a_R,q) = 1$,

$$|\beta_t| \leqslant q^{-1}P^{-3+2R\theta} M^{R-1} \tag{2.11}$$

and

$$1 \leqslant q \ll M^R P^{2R\theta}. \tag{2.12}$$

Alternative (ii). For some number N_θ such that

$$N_\theta \gg P^{(2n-4K)\theta+\epsilon}, \tag{2.13}$$

there exist N_θ integer pairs $(\underline{x},\underline{y})$ satisfying (2.4) such that if A is the $nN_\theta \times R$ matrix whose i^{th} column consists of the components $B_j^{(i)}(\underline{x}|\underline{y})$ for $1 \leqslant j \leqslant n$ and for all the N_θ pairs $(\underline{x},\underline{y})$, then

$$\operatorname{rank} A \leqslant R - 1. \tag{2.14}$$

Proof. This is a slight variation of Lemma 2.5 of Birch [1] with $d = 3$ and $k = K\theta - \varepsilon$.

Suppose α is in $E(\theta)$. Then alternative (ii) of Lemma 2.1 holds. Let N_θ be the number of points $(\underline{x},\underline{y})$ therein, and let A be the matrix of the corresponding $B_j^{(t)}(\underline{x}|\underline{y})$'s. Our conclusion corresponds to his alternatives (ii) and (iii) but is obtained by considering the whole matrix A instead of the submatrix corresponding to a particular pair $(\underline{x},\underline{y})$. If A has rank R, then arguing in the same way as Birch, we obtain approximations $\frac{a_1}{q},\ldots,\frac{a_R}{q}$, where q is the absolute value of a non-zero $R \times R$ submatrix of A. Thus

$$q = \left|\left(B_j^{(t)}(\underline{x}|\underline{y})\right)_{R\times R}\right| \ll M^R P^{2R\theta},$$

since for each j and t,

$$|B_j^{(t)}(\underline{x}|\underline{y})| \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n} |c_{ijk}^{(t)}|\,|x_i|\,|y_k| \ll MP^{2\theta},$$

by (2.4) and (1.6). The implied constant here depends only on n and so is the case for the constant in (2.12).

If for any α in $E(\theta)$, A does not have rank R, then we obtain our alternative (ii), and hence the lemma is proved.

3. BILINEAR EQUATIONS

In this section, I shall give results concerning algebraic varieties with many integer points. The main result (Lemma 3.2) generalises the result of Lemma 3 of Davenport [16] and provides the tool needed to deal with the problems arising from alternative (ii) of Lemma 2.3.

By "points", I shall always mean integer points.

Lemma 3.1. Let n, Q, N be given positive integers and v a non-integral positive real number, for $\underline{x} = (x_1,\ldots,x_n)$, let $f_1(\underline{x}),\ldots,f_N(\underline{x})$ be homogeneous polynomials with integer coefficients and suppose that N and the degrees of $f_1(\underline{x}),\ldots,f_N(\underline{x})$ are bounded in terms of n. Suppose there are $\gg Q^{n-v}$ integer points $\underline{x}$ such that

$$|\underline{x}| < Q, \quad f_i(\underline{x}) = 0, \quad (1 \leqslant i \leqslant N),$$

where Q satisfies

(3.1) $$Q > K_0,$$

for some constant K_0 depending on n, the bound on the degrees of $f_i(\underline{x})$, $(1 \leqslant i \leqslant N)$ and the constant implied by the notation $\gg$. Then there exists one of these points for which the rank of the matrix

$$\frac{\partial f_i}{\partial x_\nu} \; ; \quad 1 \leqslant i \leqslant N \; ; \quad 1 \leqslant \nu \leqslant n$$

is at most $[v]$.

Proof. This is Lemma 34 of Davenport [17], with R, $r-\varepsilon$ replaced by Q, v respectively and so $r-1$ is then $[v]$.

Lemma 3.2. Suppose $B_j^{(t)}(\underline{x}|\underline{y})$ for $1 \leqslant j \leqslant n$ are bilinear forms corresponding to the cubic forms $C_t(\underline{x})$ as in (1.3), for $1 \leqslant t \leqslant R$. For given non-integral, real $v > 0$ and an integer s such that $1 \leqslant s \leqslant n$, and Q satisfying (3.1) for the appropriate $K_0 = K_0(n,R,s)$, suppose there are $\gg Q^{n+v-s}$ points $\underline{x}$ with

(3.2) $$|\underline{x}| < Q,$$

for which

(3.3) $$B_j^{(t)}(\underline{x}|\underline{y}) = 0, \quad (1 \leqslant j \leqslant n, \quad 1 \leqslant t \leqslant \ell),$$

where $\ell \leqslant R$, have exactly s linearly independent solutions in $\underline{y}$. Then there exists a sublattice of $\mathbb{Z}^n$ of dimension $[v] + 1$ such that all the integer vectors in this sublattice are solutions of

(3.4) $$C_t(\underline{x}) = 0, \quad (1 \leqslant t \leqslant \ell).$$

Proof. The proof follows similar steps to those for Lemma 3 of Davenport [16].

For each $\underline{x}$, the equations (3.3) are linear equations in $\underline{y}$ with determinant $D(\underline{x})$, say, given by

$$D(\underline{x}) = \det\Big(\sum_i c_{ijk}^{(t)} x_i\Big), \quad (1 \leqslant j,\ k \leqslant n; \quad 1 \leqslant t \leqslant \ell).$$

Let X_s be the set of points $\underline{x}$ for which the equations (3.3) have exactly s linearly independent solutions in $\underline{y}$. Then for each $\underline{x}$ in X_s, the rank of $D(\underline{x})$ is $n - s$, and so X_s consists of all the points for which all the subdeterminants of $D(\underline{x})$ of order $n - s + 1$ are zero and at least one subdeterminant of order $n - s$ is non-zero.

For any $\underline{x}$ in X_s, we can construct a set $\underline{y}^{(1)},\dots,\underline{y}^{(s)}$, say, of linearly independent solutions of (3.3) by taking certain particular subdeterminants of $D(\underline{x})$ of order $n - s$ as their coordinates. Each such determinant is a homogeneous polynomial of degree $n - s$ in $\underline{x}$ with integer coefficients. Then for any point $\underline{x}$ in $\mathbb{R}^n$,

$$B_j^{(t)}(\underline{x}|\underline{y}^{(p)}) = \sum_i \sum_k c_{ijk}^{(t)} x_i y_k^{(p)} = \Delta_{j,p}^{(t)}(\underline{x}), \tag{3.5}$$

for $1 \leqslant j \leqslant n;\ 1 \leqslant p \leqslant s;\ 1 \leqslant t \leqslant \ell$, where $\Delta_{j,p}^{(t)}(\underline{x})$ for each j, p and t are certain subdeterminants of $D(\underline{x})$ of order $n - s + 1$.

In (3.5), $\underline{y}^{(1)},\dots,\underline{y}^{(s)}$ are functions of $\underline{x}$, and $x_1,\dots,x_n$ are independent variables in (3.5) and so we can differentiate (3.5) partially with respect to x_ν, $(1 \leqslant \nu \leqslant n)$. Then we get,

$$\sum_k c_{\nu jk}^{(t)} y_k^{(p)} + \sum_i \sum_k c_{ijk}^{(t)} x_i \frac{\partial y_k^{(p)}}{\partial x_\nu} = \frac{\partial}{\partial x_\nu}(\Delta_{j,p}^{(t)}), \tag{3.6}$$

for $1 \leqslant \nu,\ j \leqslant n;\ 1 \leqslant p \leqslant s;\ 1 \leqslant t \leqslant \ell$. Let $P_1,\dots,P_s$ be constants, to be determined at the end of the lemma and let

$$\underline{Y} = P_1\underline{y}^{(1)} + \dots + P_s\underline{y}^{(s)}. \tag{3.7}$$

Then from (3.6), we obtain

$$(3.8) \qquad \sum_{k} c_{\nu j k}^{(t)} Y_k + \sum_{i} \sum_{k} c_{i j k}^{(t)} x_i \frac{\partial Y_k}{\partial x_\nu} = \sum_{p=1}^{s} P_p \frac{\partial}{\partial x_\nu} (\Delta_{j,p.}^{(t)}),$$

for all $i \leqslant j, \nu \leqslant n$ and $1 \leqslant t \leqslant \ell$. From (3.5) and (3.7), we also have

$$(3.9) \qquad \sum_{j} \sum_{i} c_{i j k}^{(t)} x_i Y_j = \sum_{p=1}^{s} P_p \Delta_{k,p.}^{(t)}, \quad (1 \leqslant k \leqslant n; \quad 1 \leqslant t \leqslant \ell).$$

Multiplying (3.8) by Y_j and summing over j and using (3.9), we get

$$(3.10) \qquad \sum_{j} \sum_{k} c_{\nu j k}^{(t)} Y_j Y_k + \sum_{k} \sum_{p=1}^{s} P_p \Delta_{k,p.}^{(t)} \frac{\partial Y_k}{\partial x_\nu} = \sum_{j} \sum_{p=1}^{s} Y_j P_p \frac{\partial}{\partial x_\nu} (\Delta_{j,p.}^{(t)}),$$

for $1 \leqslant \nu \leqslant n$ and $1 \leqslant t \leqslant \ell$.

Now, since X_s consists of all the points for which the subdeterminants of $D(\underline{x})$ of order $n - s + 1$ are zero and by the hypothesis of this lemma, the hypothesis of Lemma 3.1 are satisfied with v replaced by $s - v$, for $f_1, \ldots, f_N$ taking all the above subdeterminants. Therefore, by Lemma 3.1, where now K_0 depends only on n, R and s, there is a point $\underline{x} = \underline{X}$, say, in X_s for which

$$\Delta_{j,p.}^{(t)}(\underline{X}) = 0, \quad (1 \leqslant j \leqslant n; \quad 1 \leqslant p \leqslant s; \quad 1 \leqslant t \leqslant \ell)$$

and

$$\operatorname{rank}\left(\frac{\partial}{\partial x_\nu} \Delta_{j,p.}^{(t)}\right) \leqslant [s-v] = s - 1 - [v] \quad ,$$

since s is an integer and v is not an integer. Then, there exist some functions $T_{j,p,\rho.}^{(t)}(\underline{X})$ and $U_{\rho,\nu.}(\underline{X})$ for $1 \leqslant j, \nu \leqslant n$; $1 \leqslant p \leqslant s$; $1 \leqslant \rho \leqslant s-1-[v]$; $1 \leqslant t \leqslant \ell$ such that

$$\frac{\partial}{\partial X_\nu} (\Delta_{j,p.}^{(t)}(\underline{X})) = \sum_{\rho=1}^{s-1-[v]} T_{j,p,\rho.}(\underline{X}) \, U_{\rho,\nu.}(\underline{X}), \quad (1 \leqslant t \leqslant \ell) ,$$

where T, U are rational since $\frac{\partial}{\partial x_\nu}(\Delta_{j,p})$ are integers as $\underline{X}$ is integral.

For this particular $\underline{X}$, (3.10) becomes

$$\sum_j \sum_k c_{\nu j k}^{(t)} Y_j Y_k = \sum_j \sum_{p=1}^{s} Y_j P_p \sum_{\rho=1}^{s-1-[v]} T_{j,p,\rho}^{(t)} U_{\rho,\nu}$$

$$= \sum_{\rho=1}^{s-1-[v]} W_\rho^{(t)} U_{\rho,\nu} \text{ , say,}$$

for $1 \leqslant t \leqslant \ell$ and $1 \leqslant \nu \leqslant n$. Multiplying this by Y_ν and summing over ν and using (3.7), we get

(3.11) $$\sum_\nu \sum_j \sum_k c_{\nu j k}^{(t)} Y_\nu Y_j Y_k = \sum_\rho W_\rho \Big(\sum_\nu \sum_p P_p \, y_\nu^{(p)} \, U_{\rho,\nu}\Big)$$

summed over $1 \leqslant \rho \leqslant s - 1 - [v]$ and $1 \leqslant p \leqslant s$.

Now we choose $P_1, \ldots, P_s$ to satisfy

(3.12) $$\sum_{p=1}^{s} P_p \Big(\sum_{\nu=1}^{n} y_\nu^{(p)} \, U_{\rho,\nu}\Big) = 0, \quad (1 \leqslant \rho \leqslant s - 1 - [v]).$$

These are $s - 1 - [v]$ equations in s variables $P_1, \ldots, P_s$ with rational coefficients. We write this as

$$U\underline{P} = 0 \text{ ,}$$

where $\underline{P} = (P_1, \ldots, P_s)$ and U is the $s \times (s - 1 - [v])$ matrix whose coefficients are $\sum_{\nu=1}^{n} y_\nu^{(p)} U_{\rho,\nu}$, for $1 \leqslant \rho \leqslant s - 1 - [v], 1 \leqslant p \leqslant s$. Thus,

$$\text{rank } U \leqslant s - 1 - [v] \text{ .}$$

The dimension of the solution set of (3.12) is then

$$s - \text{rank } U \geqslant [v] + 1 \text{ .}$$

Since the coefficients of U are rational, the above solutions determine a sublattice L, say, of $\mathbb{Z}^n$, whose dimension is at least $[v] + 1$. Then by (3.12) and (3.11), any $\underline{Y}$ in L is a solution of (3.4). Hence the lemma.

We note that the result is a trivial one for $s = n$. Because if $s = n$ then any $\underline{y}$ in $\mathbb{Z}^n$ is a solution (3.3) with $\underline{x}$ taking any of the Q^{n+v-s} points mentioned in the hypothesis and so $(\underline{x},\underline{x})$ for such $\underline{x}$ is a solution of (3.3). Therefore

$$C_t(\underline{x}) = \sum_{j=1}^{n} x_j B_j^{(t)}(\underline{x}|\underline{x}) = 0, \quad (1 \leqslant t \leqslant \ell).$$

Thus $\underline{x}$ is a solution of (3.4). The dimension of the set of these $\underline{x}$ is $[n+v-s]+1 = [v]+1$, since $s = n$.

Lemma 3.3. For given non-integral, real $v > 0$, suppose there are $\gg Q^{n+v}$ integer solutions $(\underline{x},\underline{y})$ of (3.3) satisfying

$$(3.13) \qquad |\underline{x}| < Q, \quad |\underline{y}| < Q.$$

If

$$(3.14) \qquad Q > K^*,$$

where K^* depends on n, v, R and the constant implied by $\gg$, then there exists a sublattice L of $\mathbb{Z}^n$, of dimension $[v] + 1$ such that all the integral points of L are solutions of (3.4).

Proof. As in Lemma 3.2, let X_r be the set of points $\underline{x}$ with $|\underline{x}| < Q$ for which the equations (3.3) have exactly r linearly independent solutions in $\underline{y}$. Then it follows from our hypothesis that there is some s such that $1 \leqslant s \leqslant n$ for which there are at least $\frac{K_1}{n} Q^{n+v}$ pairs $(\underline{x},\underline{y})$ satisfying (3.3) and (3.13) with $\underline{x}$ in X_s, where K_1 is the constant implied by $\gg$. Write

$$v = [v] + h ,$$

where $h > 0$ since v is non-integral. For each $\underline{x}$ in X_s , since there are exactly s corresponding linearly independent $\underline{y}$'s , the total number of $\underline{y}$'s such that $|\underline{y}| < Q$ and (3.3) holds is at most $K_2 Q^s$, where K_2 depends only on n . It follows that if $Q > K_3$, for suitable K_3 depending on K_1, K_2 and h , then the number of $\underline{x}$ in X_s is

$$\gg \frac{K_1}{nK_2} Q^{n+v-s} > Q^{n+v-\frac{1}{2}h-s} .$$

Since $[v-\frac{1}{2}h] = [v]$, the result now follows from Lemma 3.2 with $v-\frac{1}{2}h$ in place of v .

4. DISSECTION AND ERROR TERMS

In this section, I shall give a dissection of $(-\infty,\infty)$ into four smaller subsets and find the estimate of $J(P)$ for α in two of these sets. We recall the conventions regarding the parameters δ and ε .

Let $\theta_0, \theta_1, \ldots, \theta_h$ be real numbers such that

$$0 < \theta_0 < \theta_1 < \ldots < \theta_h < 1 \tag{4.1}$$

and

$$\theta_g - \theta_{g-1} = \frac{\delta}{R+1} , \quad (1 \leqslant g \leqslant h). \tag{4.2}$$

Dissection of the interval $I = (-\infty,\infty)$.

Let θ_0 be as in (4.1) and *fixed* from now on. We dissect I into disjoint subsets as follows:

Definition 4.1. Let $E(\theta_0)$ be as in Definition 2.2. Then let

(4.3) $$B = \{\alpha : |\alpha| \ll P^{-3+2R\theta_0} M^{a-1}\Lambda^{-1}\} ,$$

the implied constant being the same as that in (2.11),

(4.4) $$T = \{\alpha : |\alpha| \geqslant P^{\delta}\} ,$$

(4.5) $$S = I - (B \cup T \cup E(\theta_0)) ,$$

and

(4.6) $$\mathcal{R} = I - (B \cup T \cup S) = E(\theta_0) - (B \cup T).$$

Then,

$$I = B \cup T \cup S \cup \mathcal{R} .$$

We call these sets B, T, S, $\mathcal{R}$, the basic interval, the tail, the supplementary set and the residual set respectively.

The basic interval contributes the main term of $J(P)$ and the contributions from the supplementary set and the tail form a part of the error term. The residual set is referred to as the alternate case which is dealt with by different methods. In this section, we estimate $J(P)$ for the sets T and S.

We notice that (2.13) is of no value if $n < 4K$, because there are $\gg P^{n\theta}$ trivial solutions with $\underline{x} = 0$ or $\underline{y} = 0$ of (2.5). Therefore we take $n \geqslant 4K$. Also from now on, we take

(4.7) $$K = 2(R+1) + \tfrac{1}{4} .$$

Lemma 4.1. For α in T, we have

(4.8) $$\int_T |S(\alpha)K(\alpha)|\,d\alpha \ll P^{n-r\delta} ,$$

where r is the fixed integer given in § 1.

<u>Proof</u>. For any α, the trivial estimate for $S(\alpha)$ is

$$|S(\alpha)| \leqslant P^n .$$

Therefore,

$$\int_T |S(\alpha)K(\alpha)|d\alpha \leqslant \int_{|\alpha|\geqslant P^\delta} P^n |K(\alpha)|d\alpha$$

$$\ll \int_{|\alpha|\geqslant P^\delta} P^n |\alpha|^{-r-1} d\alpha ,$$

by (1.11) of Chapter 2 of this part

$$\ll P^{n-r\delta} .$$

<u>Lemma 4.2</u>. Suppose $P \geqslant 2$, $0 < \theta < 1$ and the alternative (ii) of Lemma 2.3 does not hold for this value of θ. Then

$$\mu(E(\theta)) \ll M^R P^{-3+2(R+1)\theta+\delta} . \tag{4.9}$$

<u>Proof</u>. For α in $E(\theta)$, (2.9) holds with (2.8). The integer b in (2.9) is zero only when

$$|6\alpha B_j(\underline{x}|\underline{y})| < P^{-3+2\theta}, \quad (1 \leqslant j \leqslant n),$$

for all the pairs $(\underline{x},\underline{y})$ mentioned in alternative (ii) of Lemma 2.1. Then alternative (i) of Lemma 2.3 will hold with $a_1 = a_2 = \ldots = a_R = 0$ and $q = 1$. Hence by (2.10), (2.11) and using (1.8), we have

M^{R-1}

$$|\alpha| \leqslant \Lambda^{-1} P^{-3+2R\theta} \leqslant M^R P^{-3+2(R+1)\theta+\delta} . \tag{4.10}$$

Now suppose that <u>$b \neq 0$</u>. Then $|b| \geqslant 1$ and since $P \geqslant 2$, $0 < \theta < 1$, by (2.9) we have,

$$|\alpha(\lambda_1 q_1 + \ldots + \lambda_R q_R) - b| < P^{-3+2\theta} < P^{-1} \leqslant \tfrac{1}{2} \leqslant \tfrac{1}{2}|b| .$$

Thus,

$$\tfrac{1}{2}|b| \leqslant |\alpha(\lambda_1 q_1 + \ldots + \lambda_R q_R)| \leqslant \tfrac{3}{2}|b| .$$

And so

$$(4.11) \qquad |b| \leqslant 2|\alpha(\lambda_1 q_1 + \ldots + \lambda_R q_R)| << P^{\delta} |\lambda_1 q_1 + \ldots + \lambda_R q_R| .$$

Thus the contribution to $\mu(E(\theta))$ from these α is

$$<< \sum_{q_1} \ldots \sum_{q_R} \sum_{b} |\lambda_1 q_1 + \ldots + \lambda_R q_R|^{-1} P^{-3+2\theta} ,$$

summed over b satisfying (4.11) and over $q_1, \ldots, q_R$ satisfying (2.8). Therefore it is

$$<< M^R P^{-3+2(R+1)\theta+\delta} .$$

Hence this and (4.10) give (4.9). This completes the proof of the lemma.

Lemma 4.3. Suppose that the alternative (ii) of Lemma 2.3 does not hold for any θ such that $\theta_0 \leqslant \theta < 1$. Then the contribution to $J(P)$ from the supplementary set S defined by (4.5) satisfies

$$(4.12) \qquad \int_S |S(\alpha)K(\alpha)| d\alpha << M^R P^{n-3-\delta_0} ,$$

where

$$(4.13) \qquad \delta_0 = \min\left\{\tfrac{1}{4}\theta_0 - 3\delta - \varepsilon, (2R + \tfrac{9}{4})\theta_h - 3 - \delta - \varepsilon\right\} .$$

Proof. For α in S, α is *either* in $E(\theta_h) - E(\theta_0)$ and not in B *or* in $I - (T \cup E(\theta_h))$. We consider these two cases separately.

For α in $E(\theta_g) - E(\theta_{g-1})$, (2.7) will not hold for $\theta = \theta_{g-1}$ with (4.7), also $E(\theta_h) - E(\theta_0)$ is the union of $E(\theta_g) - E(\theta_{g-1})$ for $1 \leqslant g \leqslant h$. Therefore, the contribution to the integral in (4.12) from all α in $E(\theta_h) - E(\theta_0)$ but not in B is

$$\ll \sum_{g=1}^{h} \int P^{n-(2R+2+\frac{1}{4})\theta_{g-1}+\epsilon} \; . \; 1 \; . \; d\alpha$$

$$\ll \sum_{g} P^{n-(2R+2+\frac{1}{4})\theta_{g-1}+\epsilon} \; . \; \mu(E(\theta_g) - E(\theta_{g-1}) - B)$$

$$\ll \sum_{g} P^{n-(2R+2+\frac{1}{4})\theta_{g-1}+\epsilon} \; . \; \mu E(\theta_g)$$

$$\ll \sum_{g} M^R \; P^{n-3-(2R+2+\frac{1}{4})\theta_{g-1}+2(R+1)\theta_g+\epsilon+\delta} \quad ,$$

by (4.9).

$$\ll \sum_{g} M^R \; P^{n-3-\frac{1}{4}\theta_{g-1}+3\delta+\epsilon} \quad ,$$

by (4.2).

$$\ll M^R \; P^{n-3-\frac{1}{4}\theta_0+3\delta+\epsilon} \quad . \tag{4.14}$$

For α in $I - (T \cup E(\theta_h))$, (2.7) will not hold for $\theta = \theta_h$ but (2.6) holds. Therefore, the contribution from this α to the integral in (4.12) is

$$\ll \int_{|\alpha|<P^{\delta}} P^{n-(2R+2+\frac{1}{4})\theta_h+\epsilon} \; . \; 1 \; . \; d\alpha$$

$$\ll P^{n-(2R+2+\frac{1}{4})\theta_h+\delta+\epsilon} \quad .$$

Using this and (4.14), we get the result (4.12) with (4.13). Hence the lemma.

5. BASIC INTERVAL

In this section, I shall give the estimate of $J(P)$ for the basic interval.

Definition 5.1. For $\underline{a} = (a_1, \ldots, a_R)$, $\underline{\beta} = (\beta_1, \ldots, \beta_R)$, $q > 0$ where $a_1, \ldots, a_R$, q are integers, let

$$S(\underline{a}, q) = \sum_{\underline{x} (\bmod q)} \cdots \sum e\left(\sum_t \frac{a_t}{q} C_t(\underline{x})\right)$$

and let

$$I(\underline{\beta}) = \int_{P\mathcal{B}} e\left(\sum_t \beta_t C_t(\underline{\xi})\right) d\underline{\xi} ,$$

where in both the cases , t is summed from 1 to R .

Lemma 5.1. For α in $E(\theta_0)$ for which the alternative (i) of Lemma 2.3 holds with $\theta = \theta_0$, let $S(\underline{a},q)$, $I(\underline{\beta})$ be as in Definition 5.1 with $a_1, \ldots, a_R$, $\beta_1, \ldots, \beta_R$, q given by (2.10) satisfying (2.11) and (2.12) with $\theta = \theta_0$. Then,

$$S(\alpha) = q^{-n} S(\underline{a},q) I(\underline{\beta}) + O(M^{dR+1} P^{n-1+4R\theta_0}) . \tag{5.1}$$

Proof. By (1.2), (1.9) and (2.10), we have

$$\begin{aligned} S(\alpha) &= \sum_{\underline{x} \in P\mathcal{B}} e\left(\sum_t \lambda_t \alpha C_t(\underline{x})\right) \\ &= \sum_{\underline{z}} e\left(\sum_t \frac{a_t}{q} C_t(\underline{z})\right) \cdot \sum_{\underline{y}} e\left(\sum_t \beta_t C_t(q\underline{y}+\underline{z})\right) , \end{aligned} \tag{5.2}$$

where $\underline{x} = q\underline{y} + \underline{z}$, $0 \leqslant z_1, \ldots, z_n < q$. The main term is obtained as in Lemma 5.1 of Birch [1] by replacing the second sum in the right hand side of (5.2) by the corresponding integral. The error in doing so is (as there)

$$\ll \sum_{\underline{z}(q)} \left(\left(\frac{P}{q}\right)^{n-1} + \left(\frac{P}{q}\right)^{n} \sum_t |\beta_t| MqP^2 \right)$$

$$\ll \sum_{\underline{z}(q)} M^{R} P^{n-1+2R\theta_0} q^{1-n} ,$$

by (2.11).

$$\ll M^{3R} \quad P^{n-1+4R\theta_0} \quad ,$$

by (2.12). This proves (5.1).

Lemma 5.2. Suppose that $(2R+K)\theta_0 < 1$ and alternate (ii) of Lemma 2.3 does not hold for any P, θ such that $\theta_0 \le \theta < 1$ and $P^{\theta_0} > K^*$, where K^* is as in Lemma 3.3. Then for the box $\mathcal{B}$ defined as in definition 1.1, we have

(5.3) $$\int_{\mathcal{B}} e(\gamma C(\underline{\xi}))\,d\underline{\xi} \le K_1 \min(1, |\gamma|^{-(K/2R)+\varepsilon}),$$

where K, as before, $K = 2(R+1)+\frac{1}{4}$, and K_1 depends on M, Λ, θ_0 and R.

Proof. By writing $|\Lambda\alpha| = CP^{-3+2R\varphi}M^{R-1}$, where $\theta_0 \le \varphi$ and $P^{2R\varphi}\ M^{-(2R-1)/2}\ P^{3/2}$, using the fact that only alternative (i) of Lemma 2.3 can hold for $\theta = \varphi$, and arguing as in Birch's Corollary to Lemma 4.3 ([1]), one shows that for α as specified

$$|S(\alpha)| \ll P^{n+\varepsilon}(M^{-R+1}P^3\Lambda\alpha)^{-K/2R} .$$

By applying this with $\gamma = P^3\alpha$, $P = |\gamma|^{1+(K/2R)}$ and using exactly the same sort of argument as in Birch's Lemma 5.2, one obtains (5.3).

Lemma 5.3. For B as defined by (4.3), under the hypothesis of Lemma 5.2 we have

$$\int_B S(\alpha)K(\alpha)\,d\alpha = \int_{-\infty}^{\infty} I(\underline{\alpha})K(\alpha)\,d\alpha + E_1 + E_2 ,$$

where $I(\underline{\alpha})$ is as in Definition 5.1, with $\underline{\alpha} = (\lambda_1\alpha, \ldots, \lambda_R\alpha)$,

(5.4) $$|E_1| \ll K_1 \quad P^{n-3-(9/4-2R\epsilon)\theta_0}\, M^{-(R-1)(\frac{9}{8R}-\varepsilon)}\, \Lambda^{\frac{9}{8R}-\varepsilon},$$

and

(5.5) $$|E_2| \ll M^{3R} \quad P^{n-4+6R\theta_0} ,$$

where K_1 is as in Lemma 5.2.

Proof. For α in B, the alternative (i) of Lemma 2.3 holds with

$a_1 = a_2 = \ldots = a_R = 0$; $q = 1$; $\theta = \theta_0$ and so $\lambda_t\alpha = \beta_t$, $(1 \le t \le R)$.

Therefore by Lemma 5.1, since $S(0,1) = 1$, we have

$$\int_B S(\alpha)K(\alpha)d\alpha = \int_B I(\underline{\alpha})K(\alpha)d\alpha + O(M^{2R} \quad P^{n-1+4R\theta_0} \quad P^{-3+2R\theta_0} M^{R-1}\Lambda)$$

$$= \int_{-\infty}^{\infty} I(\underline{\alpha})K(\alpha)d\alpha - \int_G I(\underline{\alpha})K(\alpha)d\alpha + O(M^{3R} \quad P^{n-4+6R\theta_0}),$$

where the last term is E_2 and

$$G = \{\alpha \; ; \; |\alpha| \gg P^{-3+2R\theta_0} M^{R-1}\Lambda^{-1}\},$$

with implied constant as in (2.11).

Now, by putting $P\underline{\eta} = \underline{\xi}$ in $I(\underline{\alpha})$, we get

$$I(\underline{\alpha}) = \int_{P\mathcal{B}} e(\alpha C(\underline{\xi}))d\underline{\xi} \; ,$$

$$= \int_{\mathcal{B}} e(\alpha P^3 C(\underline{\eta}))P^n d\underline{\eta},$$

$$\ll P^n K_1 \min\{1, \; |P^3\alpha|^{-(R+1+1/8)/R+\epsilon}\},$$

by Lemma 5.2.

$$\ll K_1 P^{n-3(R+9/8)/R+3\epsilon} |\alpha|^{-(R+9/8)/R+\epsilon},$$

by (1.8). Hence for G given above, we have

$$\left| \int_G I(\underline{\alpha})K(\alpha)d\alpha \right| \ll \int_G |I(\underline{\alpha})K(\alpha)|d\alpha$$

$$\ll \int_G |I(\underline{\alpha})|d\alpha$$

$$\ll K_1 P^{n-3(R+9/8)/R+3\epsilon} \; P^{(-9/8R+\epsilon)(-3+2R\theta_0)} (\Lambda^{-1}M^{R-1})^{(\frac{-9}{8R}+\epsilon)}$$

$$\ll K_1 P^{n-3-(9/4-2R\epsilon)\theta_0} \; M^{-(R-1)(\frac{9}{8R}-\epsilon)} \Lambda^{\frac{9}{8R}-\epsilon}.$$

This gives E_1 . Thus the lemma is proved.

Lemma 5.4. We have,

$$\int_{-\infty}^{\infty} I(\underline{\alpha})K(\alpha)d\alpha \gg \Lambda^{-n} M^{-n} P^{n-3} , \tag{5.6}$$

where $I(\underline{\alpha})$ is as in Definition 5.1, with $\underline{\alpha} = (\lambda_1\alpha,\ldots,\lambda_n\alpha)$ and the constant implied by $\gg$ depends only on n.

Proof. Let

$$J = \int_{-\infty}^{\infty} I(\underline{\alpha})K(\alpha)d\alpha .$$

Then we have,

$$J = \int_{-\infty}^{\infty} \int_{P\mathcal{B}} e(\alpha C(\underline{\xi}))K(\alpha)d\underline{\xi}d\alpha ,$$

where by the definition of $e(\alpha)$ and by (1.11) of Chapter 2, Part II, the absolute value of the integrand is bounded by the $\min\{1,|\alpha|^{-\tau-1}\}$ which is integrable on $(-\infty,\infty) \times P\mathcal{B}$. Hence we may substitute $\underline{\xi} = P\underline{\eta}$ (and so $d\underline{\xi} = P^n d\underline{\eta}$) and interchange the order of integration, obtaining

$$J = P^n \int_{\mathcal{B}} \int_{-\infty}^{\infty} e(\alpha P^3 C(\underline{\eta}))K(\alpha)d\alpha d\underline{\eta} . \tag{5.7}$$

We shall apply the change of variable $f : \underline{\eta} \to \underline{y}$ defined by

$$y_1 = P^3 C(\underline{\eta})$$

$$y_i = \eta_i , \quad (2 \leqslant i \leqslant n)$$

to a suitable subset of $\mathcal{B}$. We first consider the box $\mathcal{B}_1$ defined by

$$|y_1| < \tfrac{1}{3}, \quad |y_i| < \frac{1}{10mn} , \quad (2 \leqslant i \leqslant n) \tag{5.8}$$

and investigate the $\underline{\eta}$ such that $\underline{y} = f(\underline{\eta})$.

Let $\underline{y}$ be in $\mathcal{B}_1$ and $\eta_i = y_i$, $(2 \leqslant i \leqslant n)$. Consider

$$g(\eta) = C(\eta, \eta_2, \ldots, \eta_n) - \frac{y_1}{P^3} \tag{5.9}$$

$$= c_{111}\eta^3 + F\eta^2 + G\eta + H - \frac{y_1}{P^3} ,$$

where

$$F = \sum_{i=2}^{n} c_{11i}\,\eta_i ,$$

$$G = \sum_{i=2}^{n} \sum_{j=2}^{n} c_{1ij}\,\eta_i\,\eta_j$$

and

$$H = \sum_{i=2}^{n} \sum_{j=2}^{n} \sum_{k=2}^{n} c_{ijk}\,\eta_i\,\eta_j\,\eta_k .$$

And so

$$|F| < \frac{1}{10}, \quad |G| < \frac{1}{10^2}, \quad |H| < \frac{1}{10^3} . \tag{5.10}$$

We also note that for $P \geqslant 5$,

$$\left|\frac{y_1}{P^3}\right| < \frac{1}{3.5^3} . \tag{5.11}$$

Since $c_{111} \geqslant 1$, it follows from (5.9), (5.10) and (5.11) that $g(-\frac{1}{4}) < 0$ and $g(\frac{1}{4}) > 0$ and hence $g(\eta)$ has at least one zero in the interval $(-\frac{1}{4}, \frac{1}{4})$. Hence there is at least one $\underline{\eta}$ in $\mathcal{B}$ such that $\underline{y} = f(\underline{\eta})$, for each element $\underline{y}$ in $\mathcal{B}_1$. Therefore we have

$$\mathcal{B}_1 \subseteq f(\mathcal{B}) .$$

It can be checked that the extreme points, if any, of g lie in $(-\frac{1}{10}, \frac{1}{10})$ and so in $(-\frac{1}{4}, \frac{1}{4})$. And using this fact, one can determine an open set $\mathcal{B}_2$ such that

$$\mathcal{B}_1 = f(\mathcal{B}_2) \cup \mathcal{N}, \quad \mathcal{B}_2 \subseteq \mathcal{B} ,$$

where $\mathcal{N}$ is the set of measure zero consisting of the points $\underline{y}$ of $\mathcal{B}_1$ for which $g(\eta)$ has a triple zero.

and f is one-to-one on $\mathcal{B}_2$ and the Jacobian of f on $\mathcal{B}_2$ satisfies

$$(5.12) \qquad 0 < \left|\frac{\partial \underline{y}}{\partial \underline{\eta}}\right| = P^3\left|\frac{\partial C}{\partial \eta_1}\right| = P^3|g'(\eta_1)| << P^3.m,$$

where the implied constant depends only on n . Thus, the change of variable theorem is applicable to f on $\mathcal{B}_2$.

Since

$$|P^3C(\underline{\eta})| = |y_1| < \frac{1}{3},$$

by Lemma 1.3.1 of Chapter 1, Part I, the inner integral of J in (5.7) is non-negative, we have

$$J \geqslant P^n \int_{\mathcal{B}_2}\int_{-\infty}^{\infty} e(\alpha P^3C(\underline{\eta}))K(\alpha)d\alpha d\underline{\eta} \ .$$

Substituting $\underline{y} = f(\underline{\eta})$ in this integral and applying the change of variable theorem, we obtain

$$J \geqslant P^n \int_{\mathcal{B}_1}\int_{\infty}^{\infty} e(\alpha y_1)K(\alpha)d\alpha\left|\frac{d\underline{\eta}}{d\underline{y}}\right|d\underline{y} \ ,$$

$$>> P^n \int_{\mathcal{B}_1}\int_{\infty}^{\infty} e(\alpha y_1)K(\alpha)d\alpha \ m^{-1}P^{-3}d\underline{y},$$

by (5.12).

$$>> P^n \int_{\mathcal{B}_1} 1.m^{-1}P^{-3}d\underline{y} \ ,$$

since $|y_1| < \frac{1}{3}$ for all $\underline{y}$ in $\mathcal{B}_1$, by Lemma 1.3.1 of Chapter 1, Part I

$$>> m^{-1} \ P^{n-3} . \frac{1}{m^{n-1}} \quad ,$$

using (5.8), (that is the definition of $\mathcal{B}_1$). From this,(5.6) follows by (1.8).

Lemma 5.5. Suppose $n \geqslant 8R + 9$, the residual set $\mathcal{R}$ given by (4.6) is empty and the alternative (ii) of Lemma 2.3 does not hold for any θ such that $\theta_0 \leqslant \theta < 1$. Then there exist positive constants $D_1 = D_1(n,R)$, $D_2 = D_2(n,R)$, $D_3 = D_3(n,R)$ such that if

(5.13) $$P \gg \max\{\Lambda^{D_1} M^{D_2} K_1^{D_3}, (K^*)^{1/\theta_0}\},$$

where K_1 is as in Lemma 5.2,

then

$$J(P) > 1 .$$

Proof. Since the alternative (ii) of Lemma 2.3 does not hold for any θ, $(\theta_0 \leqslant \theta < 1)$, the hypothesis of Lemma 4.3 is satisfied. Now as $\mathcal{R}$ is empty, using Lemmas 4.1, 4.3 and 5.3, we have

$$J(P) = D + E ,$$

where

$$D \gg \Lambda^{-n} M^{-n} P^{n-3} ,$$

by (5.6),

$$|E| \ll M^{3R} P^{n-3-\delta_1} \Lambda^{\frac{9}{8R}},$$

with

$$\delta_1 = \min\{r\delta-3,\ \tfrac{1}{4}\theta_0-3\delta-\varepsilon,\ (2R+\tfrac{9}{4})\theta_h-3-\delta-\varepsilon,\ (\tfrac{9}{4}-2R\varepsilon)\theta_0,\ 1-6R\theta_0\} ,$$

by (4.8), (4.12), (4.13), (5.4) and (5.5).

We choose

$$\theta_0 = \frac{1}{2(3R+1)},\quad \delta = \varepsilon = \frac{1}{96(R+1)},\quad \theta_h = \frac{3}{4} \quad \text{and} \quad r = 288(R+1) + 1 .$$

Then θ_0 satisfies the hypothesis of Lemma 5.2 and

(5.14) $$\delta_1 = \min\left\{\frac{1}{96(R+1)}, \frac{1}{12(R+1)(3R+1)}\right\} .$$

And so if P satisfies (5.13) with

(5.15) $$D_1 = \frac{w + \frac{9}{5R}}{\delta_1}\quad D_2 = \frac{w+3R}{\delta_1}, \quad D_3 = \frac{1}{\delta_1}, \text{ where}$$

δ_1 is given by (5.14), then

$$J(P) > 1 .$$

The constants implied by $\ll, \gg$ depend only on $n, R, \delta, \varepsilon$. Hence the lemma.

We see therefore that (1.1) has a solution under the hypotheses of this lemma.

6. RESIDUAL SET

The residual set $\mathcal{R}$ is given by (4.6). In this section, I shall show that under certain hypotheses, (1.1) is solvable when $\mathcal{R}$ is non-empty.

From the definition of $\mathcal{R}$ and by Lemma 2.3, we see that for α in $\mathcal{R}$,

either (i) the alternative (i) of Lemma 2.3 holds with $\theta = \theta_0$ and $(a_1,\ldots,a_R) \neq 0$,

or (ii) the alternative (ii) of Lemma 2.3 holds with $\theta = \theta_0$.

We prove first that the set $\mathcal{R}$ with alternative (i) is empty for suitable choice of P . Then in the second lemma we prove the solvability of (1.1) under certain hypotheses on $R - 1$ cubics, when the alternative (ii) of Lemma 2.3 holds for some θ such that $\theta_0 \leq \theta < 1$.

We do this by rearranging $C(\underline{x})$ and using the results on bilinear equations. It is this case which forces us to take our final value of n much larger than the value $n = 8R + 9$ required by Lemma 5.5.

Lemma 6.1. Suppose the alternative (ii) of Lemma 2.3 does not hold for any θ such that $\theta_0 \leq \theta < 1$. Suppose θ_0, δ satisfy

$$(6.1) \qquad 2R\theta_0 + \delta < \tfrac{1}{2}$$

and P is a denominator greater than $4\Lambda^2 M^{2R}$ of a convergent to the continued fraction of $\frac{\lambda_i}{\lambda_j}$ for some i, j, $(1 \leqslant i,j \leqslant R)$ such that $\frac{\lambda_i}{\lambda_j}$ is irrational. Then the residual set $\mathcal{R}$ is empty.

Proof. We prove the result by contradiction. Without loss of generality, we may take the irrational $\frac{\lambda_i}{\lambda_j}$ to be $\frac{\lambda_1}{\lambda_2}$. Suppose the *residual set* $\mathcal{R}$ *is non-empty.* Let α be in $\mathcal{R}$. Then since the alternative (ii) of Lemma 2.3 does not hold, there exist numbers $a_1, \ldots, a_R$, q, $\beta_1, \ldots, \beta_R$ satisfying (2.10), (2.11) and (2.12) for $\theta = \theta_0$. We then have $a_1 \neq 0$, $a_2 \neq 0$. Because if $a_1 = 0$, we can take $q = 1$ and so by (2.10), (2.11)

$$|\alpha| = |\lambda_1|^{-1} |\beta_1|$$
$$< M P^{-3+2R\theta_0} .$$

That is α is in B, a contradiction. Similarly $a_2 \neq 0$. Thus,

$$|a_1|, \ |a_2| \geqslant 1 .$$

Also by (2.10) and (2.12) since $|\alpha| \leqslant P^{\delta}$,

$$(6.2) \qquad |a_1|, \ |a_2| \ll \Lambda M^R P^{2R\theta_0 + \delta} .$$

Let $\frac{a_0}{q_0}$ be a convergent to the continued fraction of $\frac{\lambda_1}{\lambda_2}$ such that

(6.3) $$q_0 > 4\Lambda^2 M^{2R} .$$

Then

(6.4) $$\left|\frac{\lambda_1}{\lambda_2} - \frac{a_0}{q_0}\right| < \tfrac{1}{2} q_0^{-2} .$$

Let $P = q_0$. This limits P to an infinite sequence of values. By (2.10), (2.11) and (2.12) for $\theta = \theta_0$, we have

$$\frac{\lambda_1}{\lambda_2} = \frac{\lambda_1\alpha}{\lambda_2\alpha} = \frac{\frac{a_1}{q}(1+0(P^{-3+2R\theta_0}))}{\frac{a_2}{q}(1+0(P^{-3+2R\theta_0}))}$$

(6.5) $$= \frac{a_1}{a_2}(1+0(P^{-3+2R\theta_0})),$$

since $\left|\frac{a_1}{a_2}\right|$ is bounded above, by (6.2); where the constant implied by the symbol 0 is independent on the coefficients of $C(\underline{x})$. Since $P = q_0$ and that (6.1) holds, (6.4) and (6.5) give

(6.6) $$\left|\frac{a_0}{q_0} - \frac{a_1}{a_2}\right| < \tfrac{1}{2} q_0^{-2} + P^{-3+2R\theta_0}$$

$$< q_0^{-2} ,$$

since (6.1) holds and $q_0 \geqslant 4$, by (6.3) and (1.8).

Also we have that

$$\left|\frac{a_0}{q_0} - \frac{a_1}{a_2}\right| \geqslant |q_0|^{-1} |a_2|^{-1}$$

$$\geqslant 2q_0^{-2} ,$$

by (6.2) and (6.3). This contradicts (6.6). Hence $\mathcal{R}$ is empty. Thus the proof of the lemma is completed.

Lemma 6.2. Suppose that all cubic inequalities of the type,

$$(6.7) \qquad |\lambda_1 C_1(\underline{x}) + \dots + \lambda_{R-1} C_{R-1}(\underline{x})| < 1 ,$$

where $C_1(\underline{x}),\dots,C_{R-1}(\underline{x})$ are cubic forms of the type (1.3) and $\lambda_1,\dots,\lambda_{R-1}$ are real numbers, linearly independent over the rationals, are solvable if $n \geqslant N(R-1)$. Suppose $\rho > (K^*)^{1/\theta_0}$ and there exists a θ such that $\theta_0 \leqslant \theta < 1$ and that the alternative (ii) of Lemma 2.3 holds for this value of θ. Then (1.1) is solvable provided

$$(6.8) \qquad n \geqslant N(R-1) + 8(R+1) .$$

Proof. By the alternative (ii) of Lemma 2.3, there are

$$(2.13) \qquad \gg (P^{\theta})^{(2n-4K)+\epsilon/\theta}$$

integral solutions $(\underline{x},\underline{y})$ of (2.5), satisfying (2.4) such that (2.14) holds, where K is given by (4.7). Without loss of generality, we may take that for these $(\underline{x},\underline{y})$, $B_j^{(R)}(\underline{x}|\underline{y})$ (see Definition 2.1) depends on $B_j^{(t)}(\underline{x}|\underline{y})$, $(1 \leqslant t \leqslant R-1)$ for all j, $(1 \leqslant j \leqslant n)$. Then for these $(\underline{x},\underline{y})$, we have

$$(6.9) \qquad B_j^{(R)}(\underline{x}|\underline{y}) = \sum_{t=1}^{R-1} r_t \, B_j^{(t)}(\underline{x}|\underline{y}), \quad (1 \leqslant j \leqslant n),$$

where r_t, $(1 \leqslant t \leqslant R-1)$ are rational numbers.

We write,

$$(6.10) \qquad C(\underline{x}) = \sum_t (\lambda_t + r_t \lambda_R) C_t(\underline{x}) + \lambda_R C^*(\underline{x}) ,$$

where

$$(6.11) \qquad C^*(\underline{x}) = C_R(\underline{x}) - \sum_t r_t C_t(\underline{x}),$$

in both the cases, t is summed over $1 \leqslant t \leqslant R-1$. Thus $C^*(\underline{x})$ is a rational cubic. Let $B_j^{(*)}(\underline{x}|\underline{y})$, $(1 \leqslant j \leqslant n)$ be the bilinear forms corresponding to $C^*(\underline{x})$. Then by (6.9) and (6.11), for all the points $(\underline{x},\underline{y})$, mentioned in (2.13), we have

$$B_j^{(*)}(\underline{x}|\underline{y}) = 0, \quad (1 \leqslant j \leqslant n).$$

Thus the hypothesis of Lemma 3.3 is satisfied for $B_j^{(*)}(\underline{x}|\underline{y})$ (that is for $\ell = 1$) with

$$Q = P^{\theta}, \quad v = n-4(2R+2+\tfrac{1}{4}) + \frac{\varepsilon}{\theta},$$

and so $[v]+1$ is at least $n-8(R+1)$. Hence by Lemma 3.3, there exists a sublattice L, say, of $\mathbb{Z}^n$ of dimension at least $n-8(R+1)$ such that all the integral points of this sublattice are solutions of

(6.12) $$C^*(\underline{x}) = 0.$$

Then for any $\underline{x}$ in L, by (6.12) and (6.10), we have

$$C(\underline{x}) = \sum_{t=1}^{R-1} (\lambda_t + r_t\lambda_R)C_t(\underline{x}).$$

Thus for these $\underline{x}$,

$$|C(\underline{x})| < 1$$

is of the type (6.7), and so by the hypothesis of the lemma, this is solvable if

$$\dim L \geqslant N(R-1).$$

That is if

$$n-8(R+1) \geqslant N(R-1).$$

Hence (1.1) is solvable if

$$n \geqslant N(R-1) + 8(R+1).$$

This is (6.8).

7. CONCLUSION

In this section, first I shall complete the proof of the theorem and then discuss the alternate approaches.

Completion of the proof of the theorem.

We now prove the theorem by induction on R. By the result 4 of Chapter 1, Part II (Davenport [16]), a single cubic equation is solvable if $n \geqslant 16$. For $R = 1$, (1.1) becomes a single cubic equation and so by the above result of Davenport [16], (1.1) is solvable for $n \geqslant 4.1.(1+3)$. That is, the theorem holds for $R = 1$.

Suppose that the theorem holds for $R - 1$, that is

$$|\lambda_1 C_1(\underline{x}) + \ldots + \lambda_{R-1} C_{R-1}(\underline{x})| < 1$$

is solvable if $n \geqslant 4(R-1)(R-1+3)$. We now consider

$$C(\underline{x}) = \lambda_1 C_1(\underline{x}) + \ldots + \lambda_R C_R(\underline{x}),$$

as in (1.2) for $n \geqslant 4R(R+3)$. Suppose P is a denominator of a convergent to the continued fraction of an irrational λ_i/λ_j such that this denominator $> 4n^2M^{2R}$ and satisfies (5.13), where D_1, D_2, D_3 are as in (5.15) with (5.14) and K^* is as in (3.14). We also choose θ_0, θ_h, δ, ε as in Lemma 5.5.

If for any θ such that $0 < \theta < 1$, the alternative (ii) of Lemma 2.3 holds, then by our inductive hypothesis, the hypotheses of Lemma 6.2 are satisfied with $N(R-1) = 4(R-1)(R+2)$. Therefore by Lemma 6.2, (1.1) is solvable since

$$n \geqslant 4(R-1)(R+2) + 8(R+1) = 4R(R+3)$$

that is (6.8) holds. Hence we may now suppose that the alternative (ii) of Lemma 2.3 does not hold for any θ such that $\theta_0 \leqslant \theta < 1$. By our choice of P, θ_0, δ the hypotheses of the Lemma 6.1 are satisfied and so by Lemma 6.1, the residue set $\mathcal{R}$ is empty. Then the hypotheses of Lemma 5.5 are satisfied as

$$n \geqslant 4R(R+3) > 8R+9, \quad \text{for} \quad R \geqslant 2 .$$

And so by Lemma 5.5, $J(P) > 1$. Hence (1.1) is solvable.

Thus the truth of the theorem for linear combinations of $R-1$ cubics implies its truth for linear combinations of R cubics. And since the theorem is true for a single cubic $(R = 1)$, the truth of the theorem for all R follows by induction.

This completes the proof of the theorem.

Alternate methods

I describe here briefly why a bounded solution could not be obtained by the methods used for the quadratic problem of Chapter 2, Part II and how the approach 2 (mentioned in the introduction of Part II) was applied to the alternate case of the problem of this chapter.

As we can see from the definition of $\mathcal{R}$, the alternative (ii) of the alternate case does not concern α and it gives a condition on the forms. This happens when the rank of the matrix in (2.14) is $\leqslant R-1$. We need to consider the rank r, $(0 \leqslant r \leqslant R-1)$. The possibility $r = 0$ can be disposed of by assuming $C_1(\underline{x}) = \ldots = C_R(\underline{x}) = 0$ is not solvable non-trivially.

Now consider rank r, $(1 \leqslant r \leqslant R-1)$. Then by rearrangement of $C(\underline{x})$, we can write

$$C(\underline{x}) = \mu_1 C_1'(\underline{x}) + \ldots + \mu_R C_R'(\underline{x}),$$

where $\mu_1, \ldots, \mu_R$ are real numbers (having at least one irrational ratio) and $C_1'(\underline{x}), \ldots, C_R'(\underline{x})$ are cubics of the type (1.3) such that at all the points that define the matrix A in (2.14), the bilinear forms corresponding to $R - r$ of the $C_t'(\underline{x})$'s, say, $C_{r+1}'(\underline{x}),\ldots,C_R'(\underline{x})$ are zero and those of $C_1'(\underline{x}),\ldots,C_r'(\underline{x})$ have rank r. Then by Lemma 3.3, we can find a subspace of $\mathbb{Z}^n$ in which $C_{r+1}'(\underline{x}) = \ldots = C_R'(\underline{x}) = 0$. So the problem reduces to solving

$$|\mu_1 C_1'(\underline{x}) + \ldots + \mu_r C_r'(\underline{x})| < 1 ,$$

in this subspace.

Let $N(R)$ be the number of variables needed for our theorem. Since r in the above inequality will go up to $R - 1$, for the solvability of this, the number of variables need to go up to $N(R-1)$. That is, the dimension of that subspace mentioned, need to be at least $N(R-1)$.

We may proceed by induction on r , with the result 5 of Chapter 1, Part II (Paul Lloyd [25]), (for $r = 1$), and obtain a "bounded solution" provided the new form is non-singular on the above subspace. Unfortunately, the new form need not necessarily be non-singular. So this idea will not, at present, yield a "bounded solution". It would work if the non-singular restriction in the result 5 of Chapter 1, part II, could be removed, but this seems to be difficult.*

For the alternative (i) of the alternate case, by (2.10) and (1.2),

$$\alpha q C(\underline{x}) = a_1 C_1(\underline{x}) + \ldots + a_R C_R(\underline{x}) + C'(\underline{x}),$$

where $C'(\underline{x})$ is a cubic with small real coefficients. When $C(\underline{x})$ is non-singular, we can prove that the integral cubic $a_1 C_1(\underline{x}) + \ldots + a_R C_R(\underline{x})$ is non-singular. And so we can apply result 5 of Chapter 1, Part II, on bounded solution to

$$a_1 C_1(\underline{x}) + \ldots + a_R C_R(\underline{x}) = 0 .$$

* See note on page 108.

The existence of a bounded solution of (1.1) for this case then follows as the coefficients of $C'(\underline{x})$ are very small.

Hence the essential difficulty is the alternative (ii) of the alternate case. A possible approach is to use approximation to a linear combination of $R - r + 1$ integral cubics, but there are difficulties if the alternative (ii) of the alternate case occurs for this linear combination, as may well happen.

A comment.

We note that if n is very much larger than R, we could try to solve (1.1) by solving the system of equations

$$C_t(\underline{x}) = 0, \quad (1 \leqslant t \leqslant R). \tag{7.1}$$

However, the condition of solvability of this as given by Birch (case $d = 3$ of [1] on general forms) is

$$n \geqslant 8R(R+1) + \dim V^*$$

and that the system has a real non-singular solution, where V^* is the algebraic variety: $\operatorname{rank}\left(\frac{\partial C_t}{\partial x_j}\right) < R$. Thus, although the number of variables $n = 4R(R+3)$ is larger than one would like, our result is not covered by Birch's results.

We also note that the results of §3 should yield further information about (7.1) in the singular case.

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