OFF - SHELL EFFECTS IN PION - NUCLEUS INTERACTIONS

By

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ABSTRACT

One of the fundamental questions in few and many-body physics is to what extent are differences in the off-shell form of two particle amplitudes observable in the properties of composite systems. This question is particularly relevant in the field of pion-nucleus interactions where short range properties of the pion-nucleon interaction are poorly determined.

In this thesis we have studied two different problems to examine the dependence of physically observable quantities of the pion-nucleus many-body system on the off-shell behaviour of the pion-nucleon amplitudes. In the first part of the thesis we investigate the possible sources of off-shell dependence in the energy shift of the pion-deuteron system. In the second part of the thesis we calculate the differential scattering cross-section of the pion-carbon system using the second-order optical potential.

A consistent off-shell behaviour of the pion-nucleon scattering amplitude has been obtained from the Yamaguchi type separable potential for the pion-nucleon interaction. This particular type of two-body interaction also facilitates numerical calculations.

In Chapter II we provide different separable models for the pion-nucleon interaction. The parameters of the interaction have been determined from the experimental data. For the resonance (P33) and absorption (P11) channels a simple Yamaguchi type interaction with energy dependent strength has been used to fit appropriate experimental phase shifts (up to $\sim$ 300 MeV), scattering lengths, the position of the resonance and the position of the pole at the nucleon mass. The models for the

other pion-nucleon channels also fit the experimental phase shifts and scattering lengths correctly.

By treating the strong interaction contribution as a perturbation to the Coulomb interaction, the energy shift of the 1S level of the pion-deuteron system has been calculated with the aid of the Faddeev theory from the knowledge of the two-body pion-nucleon scattering amplitudes. Having derived our exact expression for the energy shift in Chapter III, we have investigated critically the approximate nature of Deser *et al's* formula, relating the energy shift to the pion-nucleus scattering length. Our calculation of the energy shift of the pion-deuteron system demonstrates the importance of off-shell contributions due to

1. the momentum distribution of the pionic wave function

and

2. the momentum variation of the pion-nucleon amplitude at negative energies.

Deser *et al* assumed that these off-shell contributions were negligible in their original calculation. Our results suggest that if the experimental value of the energy shift is refined further, as seems likely in the near future, extraction of the scattering length from the energy shift will require a more careful treatment, taking account of the corrections indicated by the present calculation.

In Chapter IV the second-order optical potential for the pion-carbon system has been constructed in terms of the two-body pion-nucleon scattering amplitude and the nucleon-nucleon correlation function in order to investigate the effect of the nucleon-nucleon correlation in pion-carbon scattering at intermediate energies. The main part of the

nucleon-nucleon correlation is generated by the strong and repulsive nature of the short-range part of the nucleon-nucleon forces. The dependence of the differential scattering cross-section on the range parameters of the pion-nucleon interaction and on the nucleon-nucleon correlation length has been examined by using different sets of the pion-nucleon interaction potentials and varying the correlation length.

We find that different correlation lengths alter the numerical results only for the large angle scattering. The best fit is obtained for unrealistic values of the correlation length. The implication of this result is discussed in Section 4.8.

The differential cross-sections have been calculated from the second-order optical potential using two sets of pion-nucleon interaction potentials to check the off-shell dependence. We find that the scattering cross-section is sensitive (~ 10 - 15%) to the off-shell dependence of the pion-nucleon scattering amplitudes and to the range parameters of the pion-nucleon interaction for the different channels. However, it is difficult to state to what extent the scattering cross-section depends on the individual range parameters of the model.

Although the second-order optical potential gives a more complete description of the microscopic processes, in our calculations the addition of the second-order optical potential does not improve the agreement of the theoretical results obtained from the first-order optical potential with the experimental data. The basic assumptions of the model and possible results for this discrepancy are discussed in the final section.

STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University, and, to the best of the candidate's knowledge and belief, the thesis contains no material previously published or written by any other person, except where due reference is made in the test of the thesis.

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C O N T E N T S

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CHAPTER 1

INTRODUCTION

The basic ingredient for the microscopic theories dealing with pion-nucleus scattering is the two-body pion-nucleon scattering amplitude or t-matrix. This amplitude is a function of the three independent variables, the initial and final momenta of the initial and final scattering states and the total energy (E) of the system. We say that the scattering amplitude is on-shell if both the magnitudes of the incident (k_i) and final (k_f) momenta are equal to the momentum (k_E) corresponding to the total energy (E) of the system

$$k_i = k_f = k_E \ (= \sqrt{2\mu E}), \qquad (1.1)$$

where μ is the reduced mass of the system. If one of the two momenta is independent of the momentum k_E, i.e.

$$k_i = k_E \neq k_f \quad \text{or} \quad k_i \neq k_E = k_f$$

then the amplitude is called half off-shell, and when both are independent of the momentum k_E, i.e.

$$k_i \neq k_E \neq k_f$$

the amplitude is said to be fully off-shell. For microscopic theory we need both on-shell and off-shell amplitudes of the two-body scattering. The physical relevance of the off-shell amplitude may be seen from the following.

If nuclear scattering involving more than two particles is viewed as a succession of two-body scatterings, then energy and momentum are not

necessarily conserved in any two-particle collision because momentum can be transferred to the other particles. The virtual processes, where energy and momentum are not conserved quantities, can only be described by means of the off-shell scattering amplitudes or off-shell t-matrix elements. These virtual processes have a very significant influence on the physically observable quantities, such as the scattering length, cross-section and energy-shift. Consequently, the knowledge of the off-shell behaviour of the t-matrix is basic to the understanding of nuclear scattering involving a many-particle system. The physically observable quantities of the nuclear scattering depend not only on the on-shell t-matrix elements, but also on the off-shell t-matrix elements. The total contributions of these off-diagonal* elements are known as off-shell contributions. Now, the effects of the off-shell two-body t-matrix are more easily observable in an N-body system, so the three-body system, being the simplest one with $N > 2$, offers an unique opportunity to test the importance of the off-shell contribution.

The two-body pion-nucleon scattering data are fairly well-known over a wide energy range from zero to 700 MeV, and as we shall discuss in the next chapter, can be reproduced, more or less accurately, by a variety of potential models. An advantage of the potential model approach is that for a hermitian potential the corresponding scattering amplitude satisfies the requirements of time reversal invariance and off-shell unitarity.

It is well known that the Lippmann-Schwinger equation for the two-body t-matrix embodies all the dynamics of the two-particle scattering problem. At least formally this equation is adequate for the

* The transition operator t is a matrix in the momentum space. The on-shell elements of t are the diagonal elements ~~while the off-shell ones are the off-diagonal elements in this representation.~~ satisfying (1.1).

treatment of two-particle scattering. The scattering cross-section can be determined easily from the solution of this equation.

Unfortunately, this simple prescription ceases to apply when three or more particles are present because of the possibility of formation of bound states. In two-particle scattering this causes no essential difficulty because conservation of energy forbids the formation of bound states. However, if there are three or more particles participating in a process, then two or more may form a bound state while the rest of the particles may go off separately. Because of this, the corresponding Lippmann-Schwinger equation for the three-particle problem is not a well behaved integral equation. More fundamentally, the difficulties in the three-particle problem may be seen as arising from a situation in which two of the particles interact while the third propagates through unmodified. Mathematically, the unmodified propagation is given in terms of Dirac delta functions which occur in the kernel of the three-body Lippmann-Schwinger integral equation. The presence of the delta functions makes the kernel neither square integrable nor compact. However, Faddeev (1) showed, in principle, how to write the three-body amplitude in terms of the off-shell two-body t-matrix. Faddeev's equations provide a good insight into the physics of the problem, but the equations still do not admit an easy numerical solution. Subsequently, the work of Mitra (2), Amado (3) and Lovelace (4) demonstrated that the solution of the Faddeev equations could be greatly simplified by the assumption of separability for the two-particle potential, but only at the expense of the physical interpretation, because such potentials are non-local. By introducing the two-body separable interaction we simplify the two-body interaction so that the three-body problem can be solved exactly. From the point of view of the three-body problem this approximation introduces two major simplifications:

(i) the two-body t-matrix for any partial wave can be obtained and therefore one can construct the three-body kernel easily;

(ii) the three-body equations with the separable interaction can be written in terms of the one-dimensional integral equations for each three-body partial wave and these can be easily solved numerically.

The recent studies (5) on the pion-deuteron problem in terms of the Faddeev equations including pion-absorption and the pion-nucleon resonance have shown quite good agreement with the experimental pion-deuteron differential cross-section up to energy 240 MeV. Such calculations are necessarily limited by the separable structure of the input potential or the two-body t-matrix. The three-body pion-deuteron model calculation includes the contributions from the nuclear-structure of the deuteron and the effects of absorption and resonance on the multiple scattering series of the pion-deuteron scattering. Afnan and Thomas (6) have observed that the multiple scattering series diverges when the pion-absorption channel is taken into account.

On the other hand, the results of pion-nucleus scattering calculated in terms of the optical potential (7) contain information about the many-body contributions such as the size of the proton and neutron distributions, the nucleon-nucleon correlations and the fermi motion of the nucleons. The existence of the pion-nucleon resonance makes the pion-nucleus scattering amplitude sensitive to the energy variation of the pion-nucleus scattering. Without having a complete many-body theory it is difficult to estimate the many-body contributions arising from the nuclear matter to the pion-nucleon interaction inside the nucleus. Therefore the agreement with the experimental data may not necessarily yield any

fundamental information about the basic pion-nucleon interaction.

Before constructing a model for the two-body t-matrix or potential, we cannot avoid asking the questions: what guides us in our search for the two-body interaction, and how can we incorporate as much of the physical properties of the interaction as possible in a model suitable for describing complex nuclear reactions ? For example, the pion-nucleon and nucleon-nucleon interactions which are the building blocks for all microscopic theories have some well known features, such as the energy variation of the scattering amplitude, charge symmetry, saturation character, the existence of resonances, nucleon-nucleon correlations, and absorption processes. Any theoretical model we may use to describe a particular process should ideally exhibit the corresponding features of the basic interaction. In practice however, it may not be possible to include all of them in any single model because the interaction is necessarily quite complicated. It is also necessary to understand clearly the relationship between the mathematical structure of the model in question and the physical features of the two-body interaction at different energies, which it seeks to describe.

The problem of constructing a realistic two-particle t-matrix for the microscopic theories from the knowledge of the two-particle data does not have a simple unique solution. Actually, several models may reproduce two-body scattering data with varying precision. Each interaction depends not only on fitting phase-shifts but also on assumptions about the two-body off-shell behaviour of the model, so under these circumstances it is not easy to decide how far the predictions of a given model may be attributed to the nature of the interaction, as embodied in the potential, or how far they depend on the approximations of the calculational technique.

Part of this ambiguity may be eliminated by considering the off-shell behaviour of the models having similar on-shell properties (known as phase equivalent models).

Theoretically the circumstances in which the off-shell effects should be important can be appreciated by considering Beg's theorem (8). The theorem states that : for fixed scatterers, if the projectile-nucleon interaction range is finite and if the nucleons are separated by at least twice the range of the projectile-nucleon range, then the nuclear scattering amplitude will depend only on the on-shell projectile-nucleon scattering amplitude.

To understand the implication of this theorem, let us assume that :

(i) the nucleus consists of fixed scattering centres and the respective co-ordinate of each scatterer is x_i. This is the frozen nucleus approximation;

(ii) the range $(r_{\pi n})$ of the pion-nucleon interaction is finite $(0.3 \leqslant r_{\pi n} \leqslant 0.7$ fm) and the interaction regions never overlap, so that

$$V_{\pi n}(r) = 0 \quad \text{for} \quad r > r_{\pi n} ,$$

and $$|x_i - x_j| > 2\, r_{\pi n}.$$

In this picture it is clear that between two successive scattering the pion passes through a force free region where the momentum of the pion is on-shell. This picture fits very well in the limit of low nuclear density (9). However, in an actual nucleus the situation is more complex and

simple arguments above may be modified. We observe that :

(i) the energy binding the scattering nucleon to the rest of the nucleus is one of the sources of the off-shell dependence in the low-energy pion-nucleon scattering amplitude for the frozen nucleus approximation;

(ii) inside the nucleus there exists a strong repulsive nucleon-nucleon correlation when two nucleons come within the correlation length r_c ($\sim$ 0.5 fm). If the range $r_{\pi n}$ is such that $r_c > 2\, r_{\pi n}$, the pion-nucleon interaction regions will never overlap. Thus the repulsive nucleon-nucleon correlations will reduce, if not annul completely, the off-shell effects. Since a pion must interact with one nucleon before it undergoes off-shell scattering with a second nucleon, it seems that there is an intimate relation between the off-shell dependence of the pion-nucleon t-matrix, the short range nucleon-nucleon correlations, and the pion-nucleon interaction range;

(iii) for every local interaction there should be a well defined range for the corresponding potential. For any practical calculation one assumes that the potential goes to zero beyond the interaction range. The typical value of the nucleon-nucleon correlation length is about 0.4 to 0.5 fm and from the theoretical studies (10), the corresponding ranges of the pion-nucleon interactions are about 0.3 to 0.7 fm. But there are some ambiguities about the values of the pion-nucleon range (9). From these standard values of the nucleon-nucleon and pion-nucleon ranges it seems that Beg's theorem is not strictly applicable in the pion-nucleus scattering. Nevertheless, it helps us to understand how the off-shell contributions enter into any calculation and when the off-shell contributions become unimportant.

The scattering of a pion by a free nucleon is completely described by the on-shell values of the pion-nucleon t-matrix, $t_{\pi n}(\underline{k}', \underline{k}, E)$, but the pion-nucleus scattering also depends on the off-shell values of $t_{\pi n}(\underline{k}', \underline{k}\, E)$. These are not measurable in a two-body experiment, so the off-shell dependence of $t_{\pi n}(\underline{k}', \underline{k}, E)$ will always introduce some uncertainty into any calculation. But this knowledge about the off-shell two-body t-matrix is equivalent to knowing the potential or the wave function inside the scattering region. Therefore we may get in principle more information about the two-body forces from the many-body experiment than from the two-body experiment.

The relative merits of two phase equivalent models, having the same on-shell properties but different off-shell behaviour , can be assessed from the off-shell properties of their two-body t-matrices. The exact off-shell behaviour of the pion-nucleon t-matrix is unknown since a complete covariant theory of pion-nucleon scattering is yet to be formulated. There are different ways to generate model two-body pion-nucleon t-matrices. Several attempts (11) have also been made to develop a covariant theory for the pion-nucleon scattering on the basis of the Bethe-Salpeter's equation. In that approach the interaction potential becomes non-local and energy dependent. It is very difficult to obtain the numerical solution for the two-body pion-nucleon t-matrix (12). For non-relativistic scattering we generally construct the two-body t-matrix from the solution of the ordinary Lippmann-Schwinger equation with an energy independent real interaction. Because of this simple choice of interaction the corresponding scattering amplitude satisfies the unitarity condition.

For an accurate estimation of the off-shell contribution in the pion-nucleus scattering, the precise information about the pion-nucleon range

is needed. It is possible to define an overall range of the pion-nucleon interaction from the scattering data but to give an exact estimate of the range is theoretically difficult, since we cannot construct a local interaction potential for every possible interaction associated with the pion-nucleon scattering at the various energies. Theoretically the range of the interaction depends on the position of the nearest left hand cut of the t-matrix. The location of the left hand cut is determined by the mass of the exchanged particle. Therefore, when we fit the parameters of the potential model to reproduce the experimental data we basically approximate this left hand structure (dynamical information) of the t-matrix by a finite number of poles or a finite cut and as a consequence of this approximation we lose exact crossing symmetry.

It is not clear whether the potential model which approximates the left hand structure of the scattering amplitude, is the best way to construct the pion-nucleon potential. In recent years several attempts have been made to derive the correct analytic structure of the P11 and P33 channel interactions. The P11 channel is responsible for the absorption and the pion-nucleon resonance is in the P33 channel. A critical analysis of the analytic structure of the pion-nucleon t-matrix has been considered by Hamilton (10).

The Chew-Low (13) field theoretical model provides a fundamental description of the P-wave pion-nucleon interaction. This model can generate successfully the P33 resonance with the right energy and width. But it is not very successful in describing the other P-wave interactions. This may be due to the neglect of

(i) the recoil of the nucleon (which is the consequence of the static limit approximation);

(ii) the possible direct meson-meson interaction. The neglect of the nucleon recoil makes the Chew-Low scattering amplitude factorable which permits one to go from the off-shell scattering to the on-shell scattering situation in a simple manner (14, 15).

The scattering amplitude in the lowest order, consists of two distinct types of diagrams (or processes), (i) the diagram in which the incident and final meson lines do not cross (direct scattering) with each other; (ii) the diagram in which the initial and final lines cross with each other (exchange scattering). To determine the appropriate off-shell behaviour of the scattering amplitude we need to know the actual analytic structure of the pion-nucleon t-matrix in the complex energy-plane. The success of the Chew-Low model suggests that we consider its analytic structure for the pion-nucleon scattering amplitude as correct and appropriate.

According to Chew-Low theory, the pion nucleon transition matrix has the following analytic structure:

(i) the scattering amplitude has a simple pole at $E = 0$ (corresponds to the crossed Born graph in the Chew-Low model), and it goes to zero like $1/E$ for large E;

(ii) it has branch points and cuts along the real axis for $E > m_\pi$ and $E < -m_\pi$, where m_π is the mass of the pion;

(iii) the amplitude satisfies:

(a) crossing symmetry which provides the connection between the process being studied and the potential which determines it;

(b) reality (due to time reversality);

(c) unitarity ($t^{+}_{ij}(E)\, t_{ji}(E) = 1$) which provides a basic constraint on the amplitude, i.e. conservation of number of particles;

where i and j are the appropriate quantum numbers required to specify the scattering process. These analytic properties provide the practical basis for solving the pion-nucleon problem in the Chew-Low theory. We also assume that the inelastic scattering cross-sections are small compared to that of the elastic one (one meson-state approximation).

Therefore, when we construct a model to represent the pion-nucleon scattering, we shall expect its analytic properties to be consistent with that of the scattering amplitude obtained from the Chew-Low model.

Now we shall discuss briefly the analytic behaviour of the two-body pion-nucleon models available in the literature to describe the pion-nucleus scattering and corresponding ranges obtained from those models. The Chew-Low model predicts that the pion-nucleon range should be finite (~ 0.3 fm) for the P33 interaction.

Kisslinger (16) first generalised the on-shell low energy pion-nucleon t-matrix to approximate its off-shell behaviour and used the form

$$t_{\pi N}(\underline{k}', \underline{k}, E) = \langle \underline{k}' | t_{\pi N}(E) | \underline{k} \rangle = a(E) + b(E)\, \underline{k}' . \underline{k},$$

where E is the scattering energy, a (E) represents S-wave scattering and the second term is related to the P-wave scattering. Using this model different authors (17) have calculated the pion-nucleus optical potential which is non-local in co-ordinate space and from which they have determined the energy-shift and width of the pion-nucleus bound state problem. But Kisslinger's model does not reproduce the resonant behaviour of the pion-nucleon scattering amplitude and it is an essentially a zero range inter-

action model which is unrealistic. The reason for these shortcomings may be related to the inappropriate analytic properties of the model, which is divergent linearly at the far off-shell region and has no resonance structure.

One may again assume that the pion-nucleon t-matrix depends on the momentum transfer $\underline{q} = (\underline{k}' - \underline{k})$ and has the form

$$t_{\pi N}(\underline{k}', \underline{k}, E) = \langle \underline{k}' | t_{\pi N}(E) | \underline{k} \rangle = a(E) + b(E)\, \underline{q}^2 .$$

The corresponding potential in the co-ordinate space is generally known as the Laplacian potential. This model also possesses the unwanted analytic behaviour of the Kisslinger potential and it is divergent quadratically for large values of $\underline{k}$ and $\underline{k}'$ and gives unphysical zero-range pion-nucleon interaction. At low energy both these models are good, but at higher energy they are inferior to the separable models which we shall use to describe the pion-nucleus scattering. For these singular, zero range interactions, the pion-nucleon interaction potential overlaps completely with the nucleon-nucleon potential, thereby inducing high unphysical off-shell sensitivity for these two models. The corresponding optical potentials also exhibit the unphysical behaviour.

Separable models are generally constructed either from the inverse scattering theory (18) or from fitting the parameters of a supplied form of the interaction (19). The separable interaction for each reaction channel is determined from the experimental data corresponding to that channel. Therefore, this model should reproduce the actual properties of each channel for all values of the momentum, such as the resonance structure in the P33 channel, the change of the sign of the phase shift in the P11 channel at $E_\pi \sim 150$ MeV. It also goes to zero smoothly for large values of the momentum and predicts a finite range (0.3 to 0.7 fm) for the pion-nucleon interaction. The analytic properties of the separable t-matrix are

also consistent with that obtained from the Chew-Low model. However, a one-term separable interaction (19) fails to reproduce the crossed Born graph which has a pole at zero energy in the Chew-Low model for P-wave interaction.

This consistency of the analytic properties of the separable t-matrix means that the potential model provides a reasonable off-shell extrapolation of the pion-nucleon t-matrix. The separable interaction replaces the left hand cut of the pion-nucleon t-matrix by the double pole. But the structure of the model is non-unique and therefore its off-shell behaviour is also non-unique. The off-shell behaviour of the $t_{\pi N}(\underline{k}', \underline{k}, E)$ computed from the Yamaguchi (20) type form factor is reasonable for the proper off-shell behaviour. Therefore Yamaguchi type form factors have become our natural choice to represent the various pion-nucleon interactions.

In this thesis we shall present the results of two different calculations regarding the pion-nucleus scattering. The first problem is to estimate the off-shell effects in the energy shifts of the pion-deuteron system. The second problem is to calculate the differential cross-section of the pion-carbon system at intermediate energies using the optical potential model. In the optical potential calculation we have included the second-order term of the optical potential. To provide the basis for our work we shall discuss in the next chapter some of the essential features of the pion-nucleon interaction and present different separable models to determine the various pion-nucleon reaction channels.

In Chapter III we will consider the shift in the energy levels of the pion-deuteron system (mesic atom) induced by the strong interaction. A mesic atom is formed when a negative pion is captured in an atomic orbit. The typical value of the Bohr radius (a_0) of the mesic atom is

$$a_0 = \frac{197.732}{Z} \text{ fm} \quad ,$$

where Z is the atomic number of the atom. Therefore the pion is well inside the atomic electrons and has little overlap with the nucleus. The mesic atom can be well represented by a hydrogenic type model. The pion in the atomic orbit interacts with the nucleus via strong as well as electromagnetic interactions, shifting and broadening the atomic levels of the unperturbed atom. Different groups (17) have calculated level shifts and widths of the various mesic atoms to determine the pion-nucleus scattering length and absorption rate of the pion. The standard procedure in this scheme is to prepare an optical potential to describe the interaction of the atomic pion with the nucleus. The eigenvalue of the Klein-Gordon equation with this potential gives the shift and width. An extensive theoretical study on the mesic atom was made on the basis of the optical potential by Ericson-Ericson (17). The pionic atoms have also been studied experimentally over the whole periodic table by Backenstoss (21). It has been observed that the resulting shifts and widths are small compared to the atomic level spacing due to the Coulomb interaction. Their magnitudes increase with increasing Z-value. For a given Z , lower levels show greater shifts and widths.

However, the important point is that the effect of the strong interaction relative to the level spacing is small and thus one may treat the strong interaction contribution as a perturbation to the Coulomb interaction. The approximate Bohr energy of the mesic atom is

$$\varepsilon_{\pi} = (- 3.7 \text{ KeV}) \, Z^2 .$$

Information about the pion-nucleus strong interaction is contained in the shifts and widths of the energy levels of the mesic atom. Therefore it is necessary to develop a rigorous microscopic theory for the

energy shift calculation on the basis of the two-body pion-nucleon t-matrix with suitable off-shell behaviour. The first attempt was made by Deser, Goldberger, Baumann and Thirring* (22) to relate the pion-nucleus scattering length (on-shell quantities) to its energy shift. They calculated the energy shift of the pion-hydrogen system from the pion-nucleon scattering length. A question which naturally arises is then how the energy shift depends on the off-shell part of the pion-nucleon t-matrix ? To investigate this problem we have calculated the energy shift with the aid of the Faddeev theory from the knowledge of the two-body pion-nucleon t-matrix. It has been shown that one can re-derive the formula of DGBT, relating the energy shift to the scattering length, starting from our exact expression for the energy shift. Their formula is widely used to calculate the scattering length from the experimentally observed energy shift. However, we note that they neglected the following important factors in their calculation:

(i) the momentum distribution of the pionic wave function;

(ii) the momentum variation of the pion-nucleon amplitude at negative energies;

(iii) the overlap of the pion-nucleon and nucleon-nucleon interaction potentials.

We have assessed separately the relative importance of these contributions to the energy shift.

In Chapter IV we shall consider the pion-nucleus scattering in terms of the optical potential. The exact optical potential contains all possible contributions of the many-body scattering, namely, the fermi motion of the nucleons, nucleon-nucleon correlations etc.. It also involves the structure both of the nuclear target, through the nuclear form factor, and of the elementary pion-nucleon interaction through its off-shell

* Future references to these authors are indicated by the abbreviation DGBT.

dependence. Neither of these quantities can be determined in a model independent way from the experimental data. No one has been able, to date, to evaluate the complete form of the optical potential. In our model calculation we have considered the second-order term of the optical potential to study the carbon-pion scattering at the intermediate energies. Once the optical potential is computed, we can calculate the elastic pion-nucleus scattering cross-section by solving Lippmann-Schwinger equation. The main purpose of studying pion-nucleus scattering in terms of the optical potential is to understand the importance of the pion-nucleon resonance, absorption and fermi motion of the nucleons and nucleon-nucleon correlations.

Until recently only the first-order optical potential has been calculated for pion-nucleus scattering at intermediate energies. Landau *et al* (7) calculated the first-order optical potential in momentum space for pion-carbon scattering including fermi averaging and the angle transformation (importance of the angle transformation was pointed out by Mach (23)). Their work was the first improvement over the Kisslinger type potential which is singular for large momentum. Their results are quite good for the intermediate energies but not impressive for very low energies. The elastic scattering cross-section has been over estimated to some extent in the region below the resonance. Recently, Landau and Thomas (24) have suggested that the apparent disagreement with the low energy data may be reduced by a new choice of the effective pion-nucleon collision energy, i.e. three-body energy choice.

In our calculation we have evaluated the contribution of the second-order multiple scattering correction to the optical potential which is generally called the second-order potential. We have followed the

conventional multiple scattering theory of Kerman, McManus and Thalar (25) and of Watson (26) in the pion-carbon nucleus to compute the optical potential. Recently, Lee and Chakravorty (27) and Wakamatsu (28) have also evaluated the second-order optical potential for the pion-Helium scattering. Lee and Chakravorty (27) have calculated the nucleon-nucleon correlation contribution in the second-order optical potential elaborately, but Wakamatsu has also incorporated along with the nucleon-nucleon correlation , the spin and iso-spin degrees of freedom of the pion-nucleon amplitude explicitly. He has taken the pion-bound-nucleon t-matrix in place of the standard free pion-nucleon t-matrix. He has concluded that the contribution of the pion-bound-nucleon t-matrix is important particularly when the incident pion is less energetic. This contribution is known as the binding correction.

However, in our model calculation we have discarded the binding correction, since within the impulse approximation, the difference between a nucleon in a nucleus and a free nucleon, is negligible at the high energy. Moreover the second-order optical potential not only includes the nucleon-nucleon correlations, but it also accounts for the departure from the coherent scattering. In coherent scattering the nucleus remains in its ground state during the scattering period. The departure from it means that the nucleus can go to a higher excited state during the collision and finally returns back to the ground state. In calculating the first-and second-order optical potential we have used:

(i) a spin and iso-spin averaged pion-nucleon t-matrix for the S - and P - wave interactions, and

(ii) relativistic kinematics for the pion and nucleon. We have derived two sets of separable form factors for the S- and P -wave interactions. We have also checked the

importance of fermi motion of the nucleons. The final results of this calculation are given in Chapter IV.

Before the end of this Chapter, we believe that a few words are in order to state briefly the current experimental information about the pion-nucleus scattering at different energies. In early experimental work the use of the pion as a probing projectile to study the nuclear structure was not very successful. This was largely caused by the low intensity of the pion beams, the poor energy resolution in comparison with the conventional particles and the very small lifetime of the pion. Therefore the results and the conclusions of the early pion-nucleus experiments were not conclusive. Consequently, the microscopic theory of the pion-nucleus scattering was only tested qualitatively. But in recent years the situation has changed greatly, mainly due to the availability of the new experimental data. Very high intensity accelerators are now being installed which have better particle counting systems with highly sensitive detection facilities. Therefore the precise measurements of the energy shift (30), angular distribution of the cross-sections, polarisation, differential cross-section (31) at below and above the resonance energy are possible. Hence it would be interesting to estimate theoretically the interplay of the various aspects of the two-body pion-nucleon interaction, as mentioned earlier, and their effects on the experimentally measured quantities (32).

CHAPTER II

SEPARABLE MODELS FOR PION-NUCLEON INTERACTIONS

2.1 Introduction

In this chapter we will give a brief account of the nature of the pion-nucleon interaction. Using the experimental pion-nucleon phase shifts and scattering lengths, we construct separable models for the two-body pion-nucleon t-matrices in the various reaction channels. These t-matrices are required in the later chapters of the thesis to calculate the energy shift in a pionic atom and to derive the optical potential for pion-nucleus scattering at intermediate energies.

The phenomenological properties of the pion-nucleon interaction are generally obtained from the study of the two-body pion-nucleon scattering process over various energy ranges. The pion-nucleon cross-section shows considerable energy variation. The most interesting phenomenon is the P33 resonance at the pion lab. kinetic energy $E_\pi \sim 180.0$ MeV, with orbital angular momentum $l = 1$, total iso-spin $I = 3/2$, and total angular momentum of the pion-nucleon system $J = 3/2$. This resonant channel dominates the pion-nucleus scattering in the intermediate energies ($E_\pi \sim 100 - 325$ MeV). The energy dependence of the pion-nucleon scattering is quite unlike the energy dependence of nucleon-nucleon scattering, where the scattering cross-section decreases monotonically with energy without resonances at intermediate energies. Furthermore, at low energies ($E_\pi < 50$ MeV) the magnitudes of the differential cross-section for nucleon-nucleon scattering is in barns, while that for the pion-nucleon scattering is in millibarns, but at high energies both the scattering cross-sections are in barns (32). Because of the existence of the P33 resonance, the pion-nucleon scattering amplitude exhibits both forward and backward peaking of the elementary amplitude nicely.

It is known (7) that the microscopic theory provides a better description of pion-nucleus scattering than nucleon-nucleus scattering. This can be attributed to the first of the following properties of the pion and the pion-nucleon interaction:

(i) the pion is the lightest of all hadrons, so the pion controls the long range part of the nuclear force (range ~ m_π^{-1}) and thus influence strongly low energy nuclear scattering. The smallness of the pion mass may be explained by chiral invariance (33), which also implies that the low energy pion-nucleon interaction is very weak. This probably explains the rapid convergence of the pion-nucleon multiple scattering series. On the other hand the pion-~~nucleon interaction~~ absorption is very strong in the intermediate energy region again leading to rapid convergence, so the pion is a good probe for studying the distribution of the nuclear matter (in contrast to the electron, which probes the distribution of the protons only);

(ii) the pion is a spinless, pseudo scalar particle ($J^\pi = 0^-$); where J is the spin of the pion and π is its parity . The simpler pion-nucleon spin structure $0x\frac{1}{2}$ restricts the number of initial and final states in any scattering process and simplifies the phase-shift analysis considerably compared with nucleon-nucleon scattering. Moreover, the spinflip part of the interaction may provide some information about the nuclear spin densities;

(iii) since the pion is an iso-vector, it participates in single and double charge exchange reactions, which can be used to investigate the isobaric analog states of the nucleus;

(iv) the existence of the resonance at E_π ~ 180.0 MeV. The interaction of the P33 resonance with other nucleons in a pion-nucleus

scattering can give insight into the nature of the strong interaction as well as the contribution of the nuclear many-body scatterings;

(v) the absorption of a pion by two nucleons provides information about the nucleon-nucleon correlations;

(vi) for the pion kinetic energy $\leqslant$ 300 MeV the contributions of D- and F-waves are negligible and may be discarded safely;

(vii) the pion-nucleon scattering lengths ($a_{\pi N}$) are very small compared to the average separation between the nucleons ($a_{\pi N} < r^{-1} > \sim 0.05$). Therefore the multiple scattering series will converge rapidly. Hence it is possible to write the pion-nucleus scattering in terms of the multiple scattering series at low energy (9).

In formulating a microscopic description of the interactions of a pion with a nucleus the two-body pion-nucleon t-matrices for the different reaction channels are the essential ingredients. The strength and the range of the individual interaction potential for each channel are determined from the experimental data at the various energies. We shall now discuss several models of two-body pion-nucleon t-matrices, which include different aspects of the pion-nucleon interaction mentioned earlier. In order to simplify the comparison with experimental results we follow the notation of Koltun (32) to represent the pion-nucleon elastic scattering amplitude $f_{\pi N}(\underline{k}', \underline{k}, E)$ in terms of the rotationally invariant amplitudes $f_{2I,2\ell\pm1=2J}(k', k, E)$ with iso-spin (I) and total angular momentum (J) by

$$f_{\pi N}(\underline{k}', \underline{k}, E) = \sum_I Q_I \sum_\ell \{[\ell f_{2I,2\ell-1}(k',k,E) + (\ell+1) f_{2I,2\ell+1}(k',k,E)] P_\ell(\hat{k}'\cdot\hat{k}) - i\,\underline{\sigma}\cdot\underline{n}\,[f_{2I,2\ell-1}(k',k,E) - f_{2I,2\ell+1}(k',k,E)] P_\ell'(\hat{k}'\cdot\hat{k})\} \quad (2.1.1)$$

where $\underline{n} = \hat{k}' \times \hat{k}$ and $\underline{\sigma}$ is the spin of the nucleon. Q_I is the projection operator on to a state with total iso-spin I. We will label the channels of the pion-nucleon system by

$$\alpha \equiv {}^{\ell}_{2I,\ 2\ell \pm 1} \qquad (2.1.2)$$

so, for example, the S31 channel has $\ell = 0$, $I = +3/2$ and $J = 1/2$. For the α-channel the partial waves amplitude $f_\alpha (k', k, E)$ on the energy-shell limit ($k = k'$, $E_k = E_{k'}$) reads

$$f_\alpha (k, k, E_k) = (\eta_\alpha \exp(2i\,\delta_\alpha) - 1)/2ik, \qquad (2.1.3)$$

where δ_α is the real phase-shifts for the channel α and $\eta_\alpha \leq 1$ is the corresponding inelasticity parameter. The value of

$$f_\alpha (k, k, E_k) \Big|_{k \to o} = a_\alpha$$

is the scattering length, from which it is possible to infer the attractive or the repulsive nature of the basic two-body interaction.† The effective range theory predicts that the phase shifts at the low energy limit should be related to the scattering length by

$$\lim_{k \to 0} k^{2\ell + 1} \cot \delta_\alpha = \frac{1}{a_\alpha} , \qquad (2.1.4)$$

where $a_\alpha \equiv a^{\ell}_{2I,\ 2J}$. This low energy behaviour of the scattering amplitude (value of the scattering amplitude at the zero energy is the scattering length) is one of the criteria that the model two-body amplitude should satisfy. For the elastic scattering ($k' = k$, $E_{k'} = E_k$), the scattering amplitude and the t-matrix are

$$f_{\pi N} (k, k, E) = - (2\pi)^2 \left[E_\pi^{-1} + E_N^{-1} \right] t_{\pi N} (k', k, E) ,$$

† This is only true provided that it is known that the interaction does not support a bound state

where $E_\pi = \sqrt{k^2 + m_\pi^2}$ and $E_N = \sqrt{k^2 + m_N^2}$ are the pion and nucleon relativistic energies. In the non-relativistic limit the energy factor becomes simply the pion-nucleon reduced mass; m_π and m_N are the masses of pion and nucleon respectively.

The two-body pion-nucleon t-matrix is generally computed from the Lippmann-Schwinger equation as a function of the two-body interaction potential. The actual form of the pion-nucleon interaction is not completely known, but both energy dependent and energy independent (18, 19) Yamaguchi type separable interactions approximate reasonably the correct low energy pion-nucleon interaction properties over a wide energy range for the different partial waves. As mentioned earlier, the analytic structure of $t_{\pi N}(k', k, E)$ predicted from the one term separable potential is very similar to that obtained from the field theoretical model except that this interaction cannot reproduce the contribution due to the crossed Born term, which has a pole at $E = 0$ in the Chew-Low model. The separable interaction preserves the unitarity but it violates crossing symmetry. For the pion-nucleon and pion-nucleus scattering at intermediate energies one might expect unitarity to be a more important restriction on the scattering amplitude. The relatively large mass of the nucleon ensures that crossing singularities are "far" away and hence (hopefully) not very important. Moreover the ability to reproduce the on-shell pion-nucleon data more or less correctly is a measure of the success of the separable interaction. This interaction also preserves the off-shell unitarity and generates a reasonable off-shell behaviour for the pion-nucleon t-matrix. In fact, the separable approximation for the two-body interaction is exact in the neighbourhood of a sharp resonance. An excellent example is the

resonance in the P33 channel.

Before going further into the details of the calculations for the separable models, we would like to discuss the range of the pion-nucleon interactions for the S- and P-waves scattering. Knowledge about these ranges is very important for the correct estimation of the off-shell contribution in the pion-nucleus scattering. Theoretically, the range of any interaction consists of the contributions coming from the different processes involved in that interaction.

The S-wave part of the pion-nucleon scattering consists of the three-types of interactions :

(i) the exchange of two-pion in the iso-spin zero state (which is known as σ-exchange, with range $1/2m_\pi \sim 0.7$ fm);

(ii) the exchange of two-pion in the iso-spin 1 state (which is known as ρ-exchange with a range $1/5.4m_\pi \sim 0.3$ fm);

(iii) the hard core repulsive interaction.

At very low energy scattering, Hamilton (10) has shown theoretically that the contributions of the σ-exchange and the hard core repulsion cancel each other. Therefore the S-wave interaction is mainly due to the exchange of the ρ-meson and has an approximate range ≤ 0.5 fm. The recent models given by Thomas (19) for the S-wave interaction do not predict unambiguously the exact range of the interaction because of the nonlocal structure of the form factor. But his models can reproduce nicley the experimental data. Similarly the models we have used in our calculations are not free from this ambiguity (see Table 1.), and they can also reproduce the phase shifts and scattering lengths correctly. There is no centrifugal barrier to keep the S-wave out from the central region of the interaction. Therefore the S-wave scattering amplitude is sensitive to the short range forces in the

pion-nucleon scattering. Because of the large cancellation of the iso-scalar part of the S-wave scattering†, an accurate determination of the pion-nucleon scattering lengths is very important and the corresponding off-shell correction should be significant for the S-wave scattering.

The situation for the P-wave is more complex since it is a strong interaction channel compared to $\ell = 0$ channel. In addition to the types of interactions necessary for the S-wave scattering, one should also consider:

(i) the nucleon exchange contribution between pion and nucleon interaction;

(ii) the nucleon pole contribution;

(iii) the pion re-scattering contribution.

The importance of the re-scattering is to generate a resonance.

An approximate value of the range of any quasi-particle pion-nucleon interaction with an angular momentum at the momentum p should be

$$r_{\pi N} \times p \geq \ell ;$$

for the P33 resonance, we have $\ell = 1$, p = 300 MeV/c therefore $r_{\pi N} \geq 0.7$ fm. The models of Landau *et al* (18) and Londergan *et al* (18) predict the value of the range for the P33 interaction is about 0.8 fm. The corresponding value obtained from Hamilton's (10) and Hüfner *et al* (34) calculations is ~ 0.7 fm and from Chew-Low model we get about ~ 0.35 fm.

The present uncertainties regarding the range of the interaction and its off-shell behaviour encourages one to look for different models. These models should give correct pion-nucleon scattering lengths and phase shifts for the appropriate channel and a realistic pion-nucleon interaction

† This is a result which follows from chiral models. See also remark on page 41.

range. Otherwise they may induce spurious off-shell contributions in the calculation of the pion-nucleus energy shift, scattering lengths and differential cross-section. It may be important to note that use of a more realistic and complicated two-body potential may not significantly improve the results which have been obtained from much simpler and cruder models. For example, the simple Amado model (35) for the neutron-deuteron scattering can reproduce qualitatively almost all the interesting properties of the scattering. Using a complicated realistic interaction (36) the results have improved slightly but the amount of complication is enormously high.

But at this stage of the pion-nucleus scattering one may try to understand qualitatively the physics of the problem including different phase equivalent models, containing the essential features of the pion-nucleon scattering. However, to keep the calculation simple and transparent we have chosen Yamaguchi type interaction with non-relativistic and relativistic kinematics to generate various off-shell behaviour . The estimation of possible off-shell effects in the energy shift of the pion-deuteron system and the influence of resonance, absorption and fermi motion in the pion-carbon scattering are our main points of interest.

2.2 Determination of the pion-nucleon form factor from the experimental data

The Lippmann-Schwinger equation for the two-body pion-nucleon system with the interaction potential $v_\alpha\ (\underline{p},\underline{p})$ takes the form

$$t_\alpha\ (\underline{p},\underline{p},\ E) = v_\alpha\ (\underline{p},\underline{p}) + \int \frac{d\underline{p}''\ v_\alpha\ (\underline{p}',\ \underline{p}'')\ t_\alpha\ (\underline{p}'',\ \underline{p},\ E)}{E^+ - E_N(p'') - E_\pi(p'')}\ , \qquad (2.2.1)$$

where E_N and E_π are the corresponding nucleon and pion intermediate

kinetic energies and $E^+ = E + i\varepsilon$ is the scattering energy. The scattering phase shift is related to the on-shell solution of the Lippmann-Schwinger equation by

$$t_\alpha^{on}(p, p, E = E_\pi(p) + E_N(p)) = \frac{-1}{\pi \mu_{\pi N}(p) p} \exp(i\delta_\alpha(p)) \sin\delta_\alpha(p), \quad (2.2.2)$$

where $$\mu_{\pi N}(p) = \frac{E_\pi(p)\, E_N(p)}{E_\pi(p) + E_N(p)} . \quad (2.2.3)$$

The simplest non-local approximation to the two-body pion-nucleon interaction is the separable interaction

$$v_\alpha = |\, g_\alpha > \lambda_\alpha < g_\alpha \,| \quad , \quad (2.2.4)$$

where

$$V = \sum_{m_J,\, m_I} |m_J, m_I, \alpha > v_\alpha < m_J, m_I, \alpha| \quad , \quad (2.2.5)$$

where λ_α is +1 or -1 according to the interaction being repulsive and $\alpha = \{ \underline{\ell}, \underline{S}, \underline{J}, \underline{I}, \pi \}$ labels the quantum numbers, such as angular momentum, spin, total angular momentum, total iso-spin of the pair and parity, necessary to denote the channel; m_J and m_I are the corresponding z-components of $\underline{J}$ and $\underline{I}$ respectively. For a model dependent calculation one generally supplies some analytic form for the form factor $|\, g_\alpha >$. For the separable type interaction given in Equation (2.2.4), the solution of the Lippmann-Schwinger equation may be written in a closed form

$$t_\alpha(E) = |\, g_\alpha > \zeta_\alpha^{-1}(E) < g_\alpha \,| \quad (2.2.6)$$

for which the two-body propagator is

$$\zeta_\alpha(E) = \lambda_\alpha^{-1} - \int_0^\infty \frac{p^2\, dp\, g_\alpha^{\,2}(p_\alpha)}{E^+ - E_p} . \quad (2.2.7)$$

Here we have adopted the momentum normalization for the plane wave states, so that

$$< \underline{p}' | \underline{p} > = \delta^3 (\underline{p}' - \underline{p}) \tag{2.2.8}$$

and $$E_p = \frac{p^2 \hbar^2}{2\mu_{\pi N}} \tag{2.2.9}$$

if the c.m. kinetic energy is non-relativistic, but

$$E_p = \sqrt{p^2 + m_\pi^2} + \sqrt{p^2 + m_N^2} \tag{2.2.10}$$

if it is relativistic, $\mu_{\pi N}$ is the reduced mass of the pion-nucleon system.

It is easier to work with R-matrix instead of t-matrix to determine the parameters of the interaction (3 7). Since $R_\alpha(E)$ is proportional to the inverse of $p\cot\delta_\alpha$, it has no cut along the real axis at least if the potential is suitably bounded and it is always real. For the separable interaction Equation (2.2.5), the R-matrix becomes

$$R_\alpha(E) = |g_\alpha > r_\alpha^{-1}(E) < g_\alpha| \ , \tag{2.2.11}$$

where $$r_\alpha(E) = \lambda_\alpha^{-1} - P \int_0^\infty \frac{p^2 \, dp \, g_\alpha^2(p)}{E - E_p} \tag{2.2.12}$$

and P denotes the Cauchy principal value integral. The phase shifts and $r_\alpha^{-1}(E)$ are related by

$$\tan\delta_\alpha(E) = - \frac{\pi \mu_{\pi N}(k) \ k \ g_\alpha^2(k)}{r_\alpha(E)} \ , \tag{2.2.13}$$

where k is the on-shell centre of mass momentum of the pion-nucleon system for the scattering energy

$$E = \sqrt{k^2 + m_\pi^2} + \sqrt{k^2 + m_N^2} \ . \tag{2.2.14}$$

With the help of the Equations (2.1.4) and (2.2.13) one may write

$$\lambda_\alpha^{-1} = \left[- \frac{\pi \mu_{\pi N}(k)\, k\, g_\alpha^{\,2}(k)}{a_\alpha} + P \int_0^\infty \frac{p^2\, dp\, g_\alpha^{\,2}(p)}{E - E_p} \right]\Big|_{k \to 0} \cdot \qquad (2.2.15)$$

Combining the Equations (2.2.13) and (2.2.15) together with experimental phase shifts, one can easily determine the parameters of the potential for a given channel. The pion-nucleon phase shifts over a wide range of energies (0 to 700 MeV) are now available in the recent literature. However, some of the phase shifts may be unreliable, in particular various CERN data, as reported by Herndon *et al* (38) as well as the data of Almehed-Lovelace (39). This has also been pointed out by Carter *et al* (40) in their recent analysis of the phase-shifts data.

For the S11 and S31 channels we have considered the data of Roper, Wright and Feld (R-W-F, 41) covering the range 0 to 700 MeV, as well as the data of Carter *et al* (40). It is very easy to fit R-W-F data to the different forms of the pion-nucleon interactions.

For the energy shift calculation, only S11 and S31 channels have been considered and we have used non-relativistic kinematics. The analytic forms for S11 and S31 channel interactions are

$$g_{\alpha = S_{11}, S_{31}}(p) = \frac{S_1}{p^2 + \alpha_1^{\,2}} + \frac{S_2}{p^2 + \alpha_2^{\,2}} \cdot \qquad (2.2.16)$$

Using these types of two-body interaction one may compute the phase shifts (model function $\underset{\sim}{\delta}(E, A)$) from the Equation (2.2.13) and then use the methods of least squares to fit the parameter $\underset{\sim}{A}$ [$\underset{\sim}{A}(s_1, s_2, \beta_1, \beta_2)$] appearing in the model function for the data $\delta_1, \delta_2 \ldots \delta_{10}$ observed at the energies $E_1, E_2 \ldots E_{10}$. We chose the best fit $\underset{\sim}{A}$, the minimiser of

$$\chi^2 = \sum_{i=1}^{10} \frac{1}{\sigma_i^{\,2}} \left[\underset{\sim}{\delta}_i - \underset{\sim}{\delta}(E_i, \underset{\sim}{A}) \right]^2, \qquad (2.2.17)$$

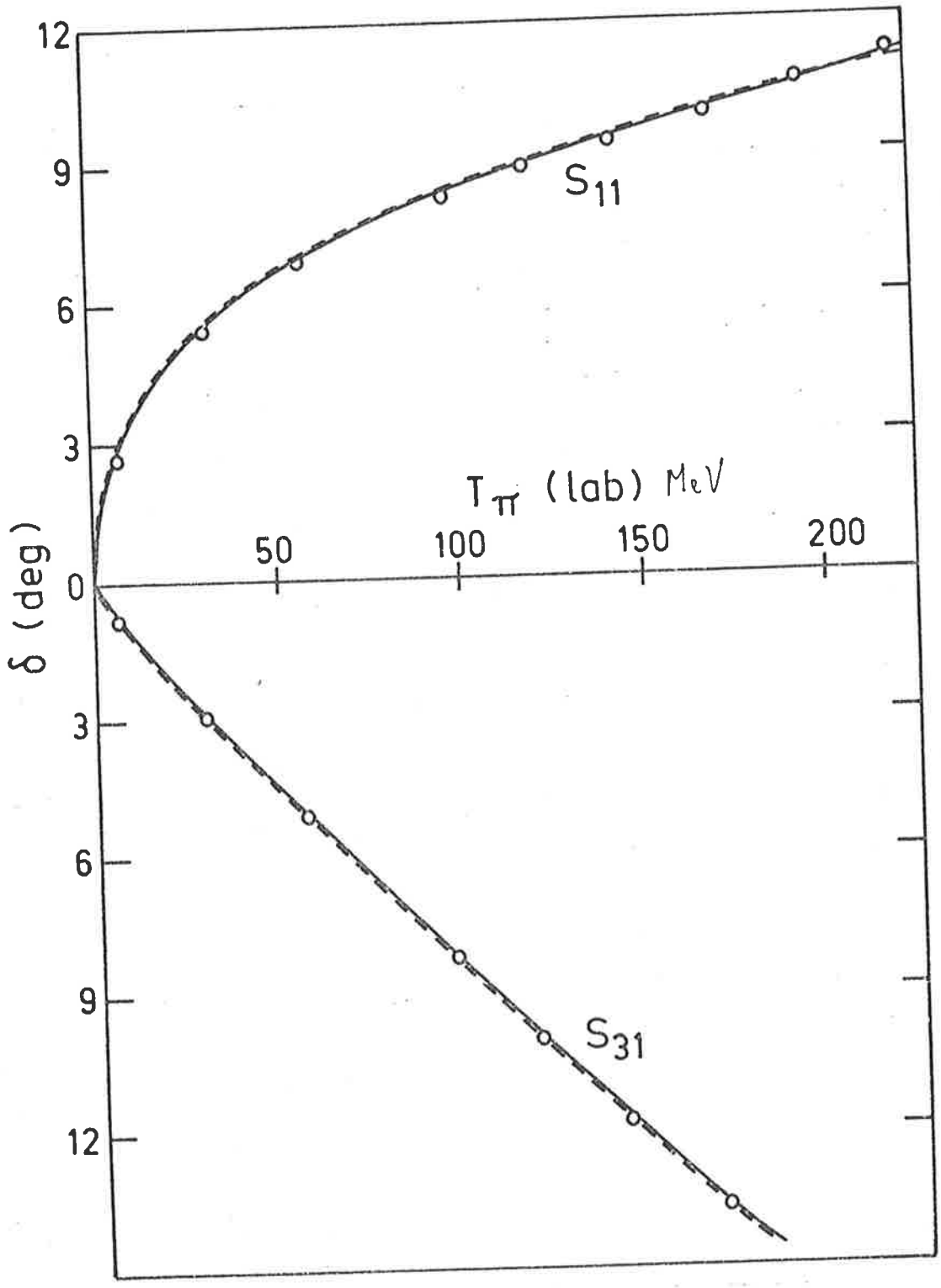


FIG 1

Theoretical phase shifts for the S11 and S31 pion-nucleon waves calculated from the parameters of Table 1a (solid curves) and Table 1b (dashed curves) respectively. Circles are the experimental points from Ref 41.

where σ_i^2 is the variance of the ith observation. An initial set of values of $\underset{\sim}{A}_0$ have been supplied and then minimised the Equation (2.2.17) with the algorithm of Marquardt (41a). The calculated values of the parameters are given in Table 1 and the fit is shown in Figure 1.

In the case of optical potential calculations we have considered S11, S31, P33, P31, P13 and P11 channels and used relativistic kinematics for the pion and nucleon to fix the parameters of these interactions, since we studied the pion-nucleus scattering at intermediate energies. The reason for re-calculating the potentials for the pion-nucleon interactions, although there are some models available in the literature, is that the kinematics of the other models are different from ours. It is also not clear how one can incorporate the peculiar behaviour of the P11 absorption channel for the separable Yamaguchi model type interaction. Assuming a two-potential model (41b) we have been able to reproduce the change of sign of the phase-shift at $E_\pi \sim 150$ MeV (lab) energy. The analytic forms for the S- and P- wave pion-nucleon interactions are :

(a) $$g_{\alpha=s_{11},s_{31}}(p) = \frac{S_1}{(p^2+\beta_1^2)^2} + \frac{S_2}{(p^2+\beta_2^2)^2} \qquad (2.2.18)$$

(b) $$g_{\alpha=s_{11},s_{31}}(p) = \frac{S_1}{(p^2+\beta_1^2)} + \frac{S_2}{(p^2+\beta_2^2)} \qquad (2.2.19)$$

(c) $$g_{\alpha=p_{33},p_{31},p_{13}}(p) = \frac{S_1 p}{(p^2+\beta_1^2)^2} + \frac{S^2\, p^2}{(p^2+\beta_2^2)^2} \qquad (2.2.20)$$

(d) $$g_{\alpha=p_{33},p_{31},p_{13}}(p) = \frac{S_1 p}{(p^2+\beta_1^2)^2} + \frac{S_2\, p^3}{(p^2+\beta_2^2)^2} \qquad (2.2.21)$$

All of them have the correct low energy behaviour consistent with the Equations(2.2.13) and (2.1.4) and we may calculate the appropriate model phase shifts function ($\underset{\sim}{\delta}$ (E, $\underset{\sim}{A}$)) for the respective channel and then

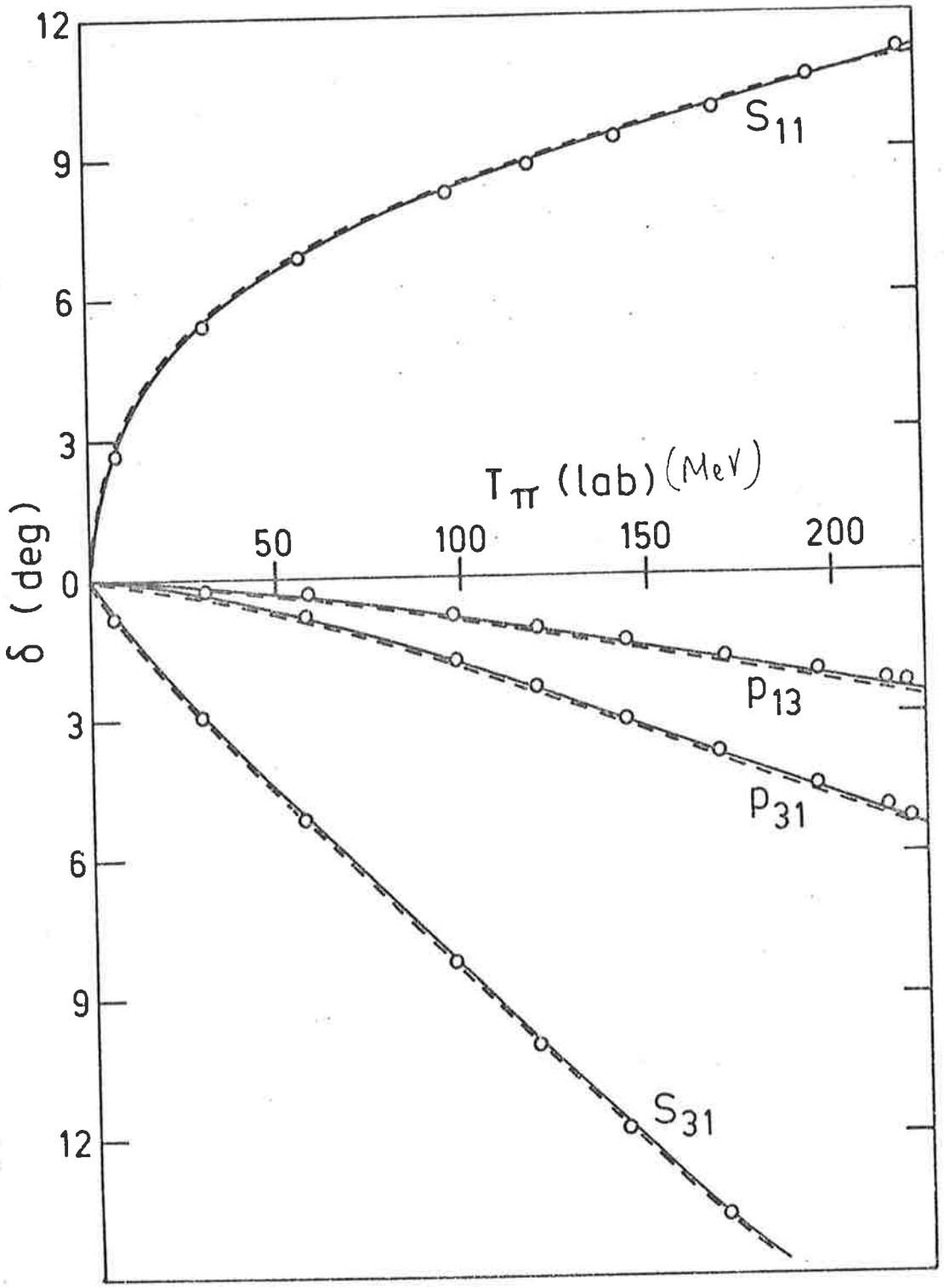


FIG. 2

Theoretical phase shifts for the S11, S31 P31 and P13 pion-nucleon waves calculated from the parametersof Table 2a (solid curves) and Table 2b (dashed curves) respectively. Circles are the experimental points from Refs. (40, 41).

follow the same technique as mentioned earlier to fix the strength, range and scattering length. The phase shifts of R-W-F and Carter's *et al* (40) scattering lengths have been used to adjust the parameters. Thomas (19) has also calculated the strengths and ranges of the pion-nucleon form factors for the S- and P-wave scattering using semi-relativistic kinematics. This is also one of the reasons why I have re-calculated again the Yamaguchi type pion-nucleon interaction models using relativistic kinematics for the pion and nucleon. The quality of the fit is given in Figure 2 and potential parameters are presented in Table 2.

There are large uncertainties in the P-wave scattering phase-shifts reported by the various groups (19). To illustrate these uncertainties we mention the P13 phase shifts at various energies.

TABLE 3 *

E^{lab} in MeV	R-W-F	CERN Theory	Salomon	Bugg	CERN (Kirsopp)
31	- .2				- .3
58	- .4	- .2 ± .2	- .3		- .2
98	- .9	- .6 ± .2	- .6		- .6
120	- 1.1	- .9 ± .13	- 1.0	-1.11 ± .02	- .6
144	- 1.5	- 1.3 ± .3	- 1.4	-1.7 ± .3	+ .9
195	- 2.1	- 2.4 ± .5	- 2.4	- 2.4 ± .6	- 2.6
310	- 3.7	- 4.6 ± .8	- 4.0	- 3.7 ± .4	- 4.8

P11 channel interaction for the pion-nucleon scattering

To describe the pion-nucleon interaction in the P11 channel the two term potential has been considered (41b)

$$v_\alpha = |g_\alpha^{(1)} > \lambda_1 < g_\alpha^{(1)}| + |g_\alpha^{(2)} > \lambda_2 < g_\alpha^{(2)}| , \qquad (2.2.22)$$

* This table has been taken from Ref. 19

where $\alpha = \{ \underline{\ell}, \underline{S}, \underline{J}, \underline{I} \}$ denotes the two-body P11 channel quantum numbers, $\lambda_{1,2} = +(-)$ according to the repulsive and attractive parts of the potential and ν_α is defined as before

$$V = \sum_{m_J m_I} |m_J, m_I, \alpha > \nu_\alpha < m_J, m_I, \alpha | \cdot \tag{2.2.23}$$

In terms of the matrix notation ν_α becomes

$$\nu_\alpha = |g_\alpha > \underset{\sim}{\lambda} < g_\alpha |, \tag{2.2.24}$$

where $$| g_\alpha > = [|\psi > |\phi >] , \tag{2.2.25}$$

and $$\underset{\sim}{\lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{2.2.26}$$

The corresponding two-body t-matrix may be written as

$$t_\alpha(E^+) = |g_\alpha > \tau_\alpha (E^+) < g_\alpha |, \tag{2.2.27}$$

for which the propagator is given by

$$\tau_\alpha(E^+) = [\underset{\sim}{\lambda}^{-1} - < g_\alpha | G_o(E^+) |g_\alpha >]^{-1} . \tag{2.2.28}$$

Here $G_o(E^+)$ is the two-body Green's function and E^+ is the scattering energy. The R-matrix on the energy-shell limit is related to the scattering phase shifts, Equation (2.2.13). The appropriate expression for the R-matrix in two-potential model is

$$R_\alpha(E) = | g_\alpha > \gamma_\alpha(E) < g_\alpha |, \tag{2.2.29}$$

where $$\gamma_\alpha(E) = [\underset{\sim}{\lambda}^{-1} - < g_\alpha | G_o^P(E) | g_\alpha >]^{-1} . \tag{2.2.30}$$

We may re-write Equation (2.2.29) explicitly as

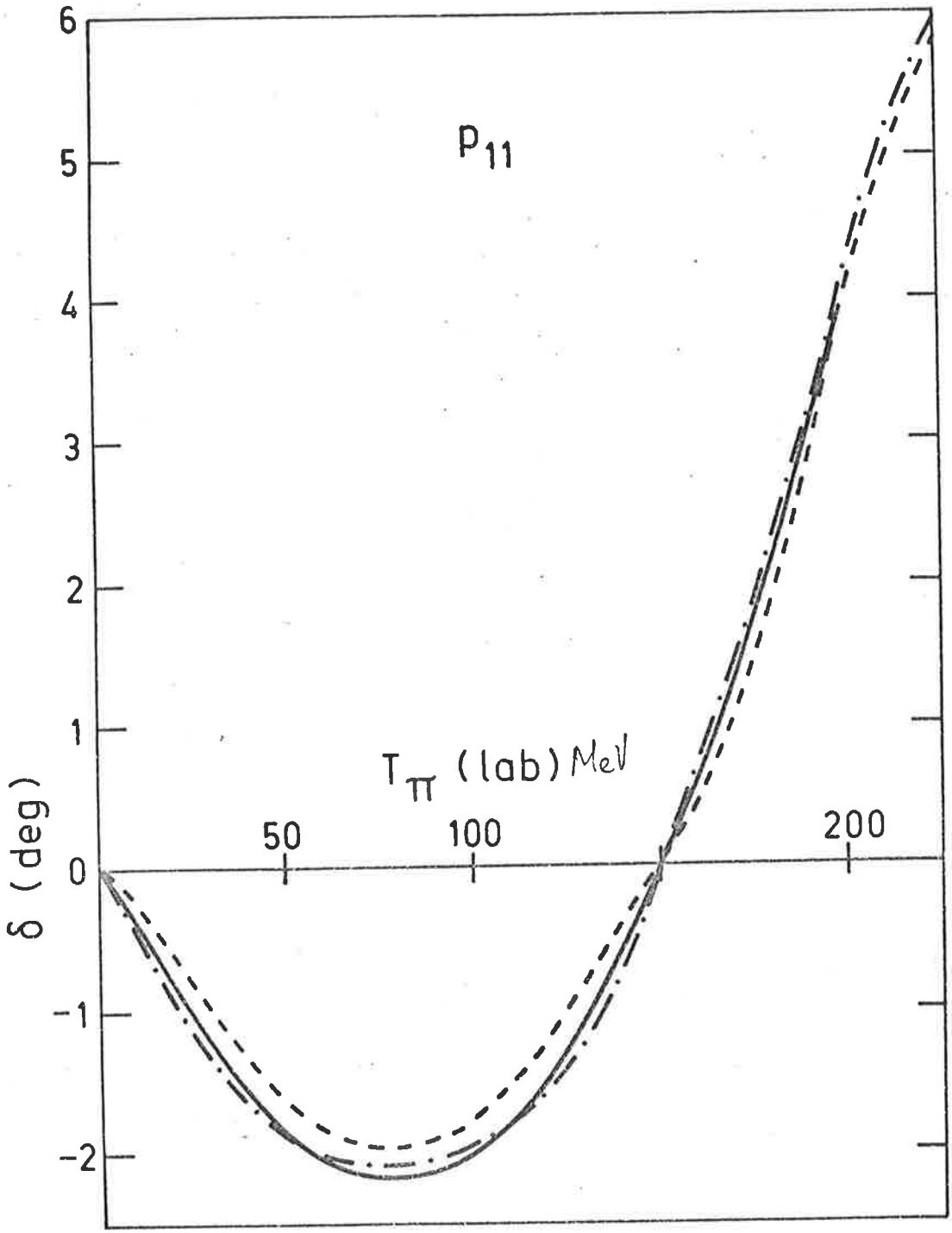


FIG. 3

Theoretical phase shifts for the P11 pion-nucleon wave calculated from the parameters of Table 2a (dashed-dotted curve) and Table 2b (dashed curve). The solid curve represents the experimental points (Refs 40, 41).

$$R_\alpha(k, k, E) = \frac{\psi^2(k)(\lambda_2^{-1} - I_{22}) + \psi(k)\,\phi(k)\,I_{12} + \phi^2(k)(\lambda_1^{-1} - I_{11})}{[(\lambda_1^{-1} - I_{11})(\lambda_2^{-1} - I_{22}) - (\frac{I_{12}}{2})^2]}$$

$$= \frac{-1}{\pi\, k\, \mu_{\pi N}(k)} \tan\delta_\alpha(k) \quad , \qquad (2.2.31)$$

where $I_{11} = \langle \psi \mid G_o^P \mid \psi \rangle$, $I_{22} = \langle \phi \mid G_o^P \mid \phi \rangle$, $I_{12} = 2\langle \psi \mid G_o^P \mid \phi \rangle$ (2.2.32)

and G_o^P is the principal value part of the Green's function. The experimental phase shift goes to zero at $E_\pi \sim 150$ MeV (lab). The corresponding momentum is donated by k_o (say). We may force this condition by setting

$$R_\alpha(k_o, k_o\, E) = 0$$

i.e. $$\psi^2(k_o)(\lambda_2^{-1} - I_{22}) + \psi(k_o)\,\phi(k_o)\,I_{12} + \phi^2(k_o)(\lambda_1^{-1} - I_{11}) = 0$$

or $$\lambda_2^{-1} = I_{22} - \frac{\phi(k_o)\,I_{12}}{\psi(k_o)} - \frac{\phi^2(k_o)(\lambda_1^{-1} - I_{11})}{\psi^2(k_o)} \qquad (2.2.33)$$

Combining Equations (2.2.31) and (2.2.33), it is now straightforward to determine the other parameters of the interaction by the method of least squares. The chosen analytic forms for $\mid \psi >$ and $\mid \phi >$ are

$$\mid \psi > = \frac{p}{(p^2 + \beta_1^2)^2} \quad , \quad \mid \phi > = \frac{p^3}{(p^2 + \beta_2^2)^3} \quad . \qquad (2.2.34)$$

The fit is presented in Figure 3, and the parameters are given in Table 2(b).

We now conclude this chapter by noting that one may also calculate the pion-nucleon vertex function for an individual channel by the direct solution of the inverse scattering problem (18). This is a very straightforward way to determine the potential for any channel where the phase shifts are known for all energies. But one encounters two serious problems

in this method for the case of pion-nucleon scattering, namely :

(i) the phase shifts are not known at all high energies;

(ii) at very high energies the inelastic pion production channel opens.

However, Landau and Tabakin (18) used this technique to determine the form factors appropriately, taking care of these two points. Afterwards, a multichannel separable model was introduced by Londergan *et al* (18) to get rid of these two unwarranted features of the inverse scattering technique. The two-body t-matrix derived by this method has the same sort of off-shell behaviour to that of the t-matrix calculated from the parameter fitting technique. In the parameter fitting method we get a mixture of the attractive and repulsive Yamaguchi type interactions with different strengths and ranges to describe the pion-nucleon interaction. The pion-nucleon vertex functions obtained from the inverse scattering technique are not convenient for Faddeev type calculations to study the pion-nucleon scattering. We prefer to use Yamaguchi type form directly for the pion-nucleon vertex, which makes the angular momentum reduction of Faddeev's equations easier and also facilitates their numerical solution greatly. The fall-off of the corresponding two-body t-matrix is smoother compared to that obtained from inverse scattering theory for large momentum.

TABLE 1 Results of the parameter search for the model pion-nucleon potentials. The inverse ranges α are in $f m^{-1}$, the strengths s in $(MeV/f m)^2$ and scattering lengths a are in units $f m$. The parameter set a(b) corresponds to the phase-shifts of R-W-F (Carter *et al* 40)

Set	Channel	Form factor	λ'	S_1	S_2	α_1	α_2	a
a	S_{11}	$\lambda'\left(\frac{S_1}{p^2+\alpha_1^2}+\frac{S_2}{p^2+\alpha_2^2}\right)$	9.5945	- 1.5099	5.9576	2.3688	2.9026	.2419
	S_{31}	$\lambda'\left(\frac{S_1}{p^2+\alpha_1^2}+\frac{S_2}{p^2+\alpha_2^2}\right)$	302.8913	1.0824	-.0815	3.6232	1.2594	-.1344
b	S_{11}	$\lambda'\left(\frac{S_1}{p^2+\alpha_1^2}+\frac{S_2}{p^2+\alpha_2^2}\right)$	9.218	-1.4012	6.4120	2.7508	3.1711	.2391
	S_{31}	$\lambda'\left(\frac{S_1}{p^2+\alpha_1^2}+\frac{S_2}{p^2+\alpha_2^2}\right)$	278.4231	1.0887	-.0811	3.5822	1.2382	-.1316

S_{11} (S_{31}) channel interaction is attractive (repulsive).

TABLE 2a

RESULTS OF THE PARAMETER SEARCH FOR THE MODEL PION-NUCLEON POTENTIALS

Channel	S_1	β_1^2	S_2	β_2^2	λ^{-1}	$a_{\pi N}$ (Theory)	$a_{\pi N}$ expt.	λ_0^{-1}
S_{11}	33.920	22.865	.0155	.599	- .0969	$.171 m_\pi^{-1}$	$.171 m_\pi^{-1}$	-
S_{31}	1.112	2.730	-12.330	7.1945	.1088	$-.092 m_\pi^{-1}$	$-.092 m_\pi^{-1}$	-
P_{31}	-.0832	.325	- 3.988	2.140	9.1822	$-.043 m_\pi^{-3}$	$-.043 m_\pi^{-3}$	-
P_{33}	4.078	3.332	-7.618	7.978	-	$.214 m_\pi^{-3}$	$.214 m_\pi^{-3}$	.241
P_{13}	-.0802	.3119	-3.443	2.197	15.479	$-.029 m_\pi^{-3}$	$-.029 m_\pi^{-3}$	-
P_{11}	2.463	3.650	-3.104	16.263	-	$-.056 m_\pi^{-3}$	$-.056 m_\pi^{-3}$	-.0267

For the S-waves the parameters refer to Equation (2.2.18) and for the P_{31}, P_{13}, P_{33} waves the parameters refer to Equation (2.2.20) and for P_{11} wave to Equation (4.5.11) The square of the inverse ranges (β^2) are in fm^{-2}. the S-waves strengths are in fm^{-3}. For the P_{31}, P_{13} and P_{33} waves the strength S_2 is in fm^{-1} and S_1 is in fm^{-2}. The P_{11} wave strengths are dimensionless. The parameter λ_0 for the P_{33} wave is in fm^{-2} and for the P_{11} wave it is dimensionless.

TABLE 2b

RESULTS OF THE PARAMETER SEARCH FOR THE MODEL PION-NUCLEON POTENTIALS

Channel	S_1	β_1^2	S_2	β_2^2	λ^{-1}	$a_{\pi N}$ theory	$a_{\pi N}$ expt.	λ_0^{-1}
S11	4.893	14.186	.097	.305	- 3.804	$.181 m_\pi^{-1}$	$.171 m_\pi^{-1}$	-
S31	5.491	21.282	- .023	.202	.0366	$-.092 m_\pi^{-1}$	$-.092 m_\pi^{-1}$	-
P31	- .258	.555	-1.574	.971	3.118	$-.043 m_\pi^{-3}$	$-.043 m_\pi^{-3}$	-
P33	- 2.511	1.333	-4.099	17.533		$.214 m_\pi^{-3}$	$.214 m_\pi^{-3}$	4.131
P13	-.226	.441	-1.657	.935	22.475	$-.029 m_\pi^{-3}$	$-.029 m_\pi^{-3}$	-
P11	-	3.992	-	6.776	$\lambda_1^{-1} = .40 \times 10^{-3}$ $\lambda_2^{-1} = - 2.956$	$-.032 m_\pi^{-3}$	$-.056 m_\pi^{-3}$	

For the S-waves the parameters refer to Equation (2.2.19) and for the P31, P13, P33 waves the parameters refer to Equation (2.2.21) and for P11 wave to Equation (2.2.34). The square of the inverse ranges (β^2) are in fm^{-2}, the S-waves strengths are in fm^{-1}. For P-waves the strengths S_2 dimensionless and S_1 is in fm^{-2}. The parameters λ_1^{-1} is in fm^4 and λ_2^{-1} is dimensionless for the P11 channel. The parameter λ_0 for the P33 channel is in fm^{-2}.

CHAPTER III

THREE-BODY PERTURBATIVE ESTIMATES OF THE ENERGY SHIFT IN THE PION-DEUTERON ATOM

3.1 Introduction

In this chapter we shall present our model based on three-body perturbation theory to calculate directly the energy shift for the K_α ($^2P - {}^1S$) transition in the pionic-deuteron system without evaluating the pion-deuteron scattering length by the usual multiple scattering formalism. Theoretically, the pionic deuterium system provides the simplest example of a pion in an atomic orbit interacting with a complex nucleus, exhibiting most of the phenomena relating to the pion-nucleus scattering process. Also the deuteron's loosely bound nature makes a non-relativistic three-body model appropriate for the problem and the well known structure of the deuteron allows one to appraise the nuclear structure effects on the pion-nucleon scattering series.

In a pimesic atom the bound pion interacts with the nucleus via strong as well as electromagnetic interactions. The experimental observations (21) show that the effects of the strong interaction on the Coulomb energy levels are small compared with the spacing between levels. This shift increases with the increase of the atomic number Z of the atom. The Bohr radius of the mesic atom is*

$$a_0 = (m_\pi Z \alpha)^{-1} \approx \frac{200}{Z} f m,$$

hence the pion is well inside the atomic electrons and interacts with the nucleus directly. The Bohr energy is

$$E_\pi^0 = -\tfrac{1}{2} m_\pi (Z\alpha)^2 = (-3.7 \text{ KeV}) Z^2 .$$

As the effect of the strong interaction is small one assumes that the

* m_π is the mass of the pion.

strong interaction acts as a perturbation on the Coulomb interaction. The ratio of the nuclear interaction distance to the mesic-atom Bohr radius is

$$(A^{1/3} m_\pi^{-1}) / (Z\alpha\, m_\pi)^{-1} \sim \frac{ZA^{1/3}}{137} ,$$

where α is the fine-structure constant, and A is the atomic weight. The smallness of this ratio justifies the above assumption. These ideas were first introduced by Deser, Goldberger, Baumann and Thiring (D G B T, 22) who showed that the strong interaction shift of the atomic level is related essentially to the pion-nucleus scattering length $(a_{\pi A})$ by the expression

$$\Delta E = \left(\frac{2\pi}{m_\pi}\right) \left| \psi_\pi (\underline{r} = 0) \right|^2 a_{\pi A} , \qquad (3.1.1)$$

where $\left| \psi_\pi(r = 0) \right|^2$ is the probability density for the atomic pion being at the nucleus. We would like to emphasize that DGBT made the following approximations in deriving this relation :

(i) the pionic density $\rho(\underline{r})$ is constant over the nuclear volume;

(ii) the atomic binding energy is small relative to the range of energy over which the pion-nucleus amplitude varies;

(iii) the off-shell dependence of the pion-nucleon amplitude is negligible.

We have investigated these approximations separately in Section 3.

Because of the weak nature of the pion-nucleon interaction, several attempts (5) have been made to calculate the pion-nucleus scattering length $(a_{\pi A})$ in terms of the multiple scattering series in the

low energy limit. The pion-nucleon scattering amplitude may be expressed as (32) :

$$f_{\pi N}(\underline{k}', \underline{k}\ E) = f_0(\underline{k}', \underline{k}, E) + f_1(\underline{k}', \underline{k}, E)\ \underset{\sim}{I} \cdot \underset{\sim}{i}\ ,$$

where f_0 and f_1 are the iso-scalar and iso-vector parts of the pion-nucleon scattering amplitude and depend on the spin, iso-spin, momentum of the pion before and after the collision; $\underset{\sim}{I}$ and $\underset{\sim}{i}$ are the iso-spin operators of the pion and nucleon respectively.

In terms of the single scattering approximation, the pion-nucleus transition operator (T-matrix) at low energy may be written as

$$T_{\pi A} = \sum_{j=1}^{A} t_{\pi N}(j), \tag{3.1.2}$$

where $$t_{\pi N}(j) = -(2\pi)^{-2}\left(\frac{E_\pi + E_N}{E_\pi \cdot E_N}\right) f_{\pi N}. \tag{3.1.3}$$

A small correction due to the P-wave scattering should be taken into account in this term. The pion-nucleus scattering length under this approximation is then given by (32)

$$a^0_{\pi A}(Z,N) = -(2\pi)^2\ m_\pi\ \left(1 + \frac{m_\pi}{A m_N}\right) \langle T_{\pi A} \rangle$$

$$= \frac{(1 + m_\pi/m_N)}{(1 + m_\pi/A m_N)}\ (A f_0 + (N - Z) f_1). \tag{3.1.4}$$

In particular, for the pion-deuteron system, we have

$$a^0_{\pi d} = \frac{2}{3}\left(\frac{m_\pi + m_N}{m_N + m_\pi/2}\right)(a_1 + 2a_3). \tag{3.1.5}$$

The mass factor arises from the transformation to the c. m. co-ordinates of pion-deuteron. a_1 and a_3 are the pion-nucleon scattering lengths corresponding to iso-spin 1/2 and 3/2 respectively. Soft pion theory (42) predicts that the iso-scalar combination $(a_1 + 2a_3)$ should be zero. Recent theoretical estimates (43) are in close agreement with

the prediction.

As a result of this small value of the iso-scalar part, the corrections due to off-shell contribution become important in pion-deuteron scattering. Hence one may question whether Equation (3.1.1), which is essentially an on-shell approximation, is sufficiently accurate to evaluate $a_{\pi d}$ from the experimentally observed energy shift. The aim of this chapter is to appraise the sensitivity of the energy shift to the off-shell dependence of the pion-nucleon amplitude and the momentum distribution of the atomic wave function, using a form of three-body perturbation theory which avoids the intermediate step of calculating the scattering length. Furthermore, proper treatment of the unitarity correction (44, 45), generated due to the three-body kinematics and the associated off-energy shell variation of the pion-nucleon amplitude in the negative energies appearing in the individual terms of the multiple scattering series, is essential if an accurate evaluation of the pion-deuteron energy shift is desired. These considerations should also be taken into account for the correct evaluation of the pion-deuteron scattering length. Fäldt (46) pointed out the importance of treating corrections to the simple impulse approximation consistently, taking all terms to a given order in the pion-nucleon scattering length into consideration. He has shown that for a loosely bound system the impulse approximation is quite good because of the mutual cancellation of the binding correction in the first three terms of the multiple scattering series. An excellent review of this problem has recently been given by Thomas and Landau (47). One of the ways to study the sensitivity of the experimentally observed quantities associated with pion-deuteron system to the off-shell variations of the pion-nucleon amplitudes is to solve the Faddeev equations which sum the

multiple scattering series to all orders. Although off-shell contributions are successfully incorporated in the Faddeev calculations, it is difficult to discuss their magnitudes in comparison with the estimates obtained from the multiple scattering series, since individual multiple scattering terms are not usually singled out explicitly. There is also the practical problem of repeating lengthy numerical calculations to test the sensitivity of the computed results to input amplitudes with different off-shell properties.

Our method of calculating the energy shift is based on the formal reduction of the three-body problem to an equivalent two-body problem with an effective potential acting between the pion and deuteron c. m. In this respect our method is similar to the calculation of energy shift in pionic atoms with heavier nuclei in terms of the optical potential (17) prepared from an assumed phenomenological form of the pion-nucleon interaction and nuclear form factor. Since the pion-nucleon range is relatively small in comparison with the nuclear size, the nuclear form factor is a sharply peaked function in momentum space, relative to the two-body pion-nucleon amplitude, and this suppresses partially the off-shell differences of the phase equivalent potentials. Moreover, the optical potential may show unphysical off-shell contributions to the energy shift if the pion-nucleon interaction has an unrealistic range.

But in our method the effective interaction is not phenomenological in the sense of the potential used by Ericson and Ericson (17), but it has been determined from the pion-nucleon amplitude. The essential feature of our derivation of the energy shift is the comparison of the actual three-body problem with a solvable model three-body problem with a displaced Coulomb interaction. The difference between

the true effective potential which includes the strong interaction between pion and nucleons and the effective potential of the comparison problem which gives unperturbed Coulomb energy levels may then be treated by simple perturbation theory. When the resulting first-and higher-order expressions for the energy shift are expanded in terms of the pion-nucleon amplitude, terms similar to those occurring in the multiple scattering expansion of the pion-deuteron scattering length are obtained. However, the perturbation expansion is expected to converge more rapidly due to the presence of different energy denominators in the intermediate states.

In the light of the calculations of $a_{\pi d}$ mentioned above, it is important to ask whether higher order terms in the perturbative approach can be neglected and whether in fact they might affect our observation about the role of the off-shell effects in the single scattering contribution to the energy shift. In this chapter we give estimates of higher order terms and confirm the relative importance of the off-shell correction to the DGBT's formula. In Section 2, our expression for the energy shift is derived and the various approximations needed to re-derive DGBT's formula have been considered. The first-order energy shift is evaluated in Section 3, using specific separable parametrizations of the pion-nucleon amplitudes. Only the S11 and S31 channels have been considered. We have also calculated the typical values of the weight function $F_{n_\alpha,d}(\underline{p}_\alpha)$ corresponding to Equations (3.3.13) and (3.3.16) for different values of $\underline{p}_\alpha$ to examine the importance of the momentum spread of the atomic wave function and off-shell dependence of the pion-nucleon t-matrix in these two channels. In Section 4, we derive the rotationally invariant deriving terms which are required to solve Faddeev-type equations to estimate the contributions of the multiple scattering part to the first-

order energy shift and the results have been described in Section 5. The values obtained for the total first-order shift are compared with those obtained from neglecting the off-shell dependence of the pion-nucleon amplitude and the momentum distribution of the bound pion. The numerical results are summarised in Table 7. Unlike the first-order shift, the second-order shift allows nucleon-nucleon re-scattering to occur in the intermediate states. In the calculations of the pion-deuteron scattering length, such process play an important role. Therefore in Section 6 we have determined an upper bound for the magnitude of the second-order shift and show it is reasonably small compared with the values calculated for the first order shift. Finally, the implications of the various approximations for the calculation of the energy shift are discussed and conclusions offered.

3.2 Theory

The three-body hamiltonian, describing the interaction of a negative pion bound in an atomic orbit with the deuteron nucleus, is given by

$$H = H_0 + V_\pi + v_{np} \quad . \tag{3.2.1}$$

Here H_0 is the total kinetic energy of the system, v_{np} is the potential binding the proton and neutron in the deuteron ϕ_d and V_π is the interaction between the pion and the constituents of the deuteron. The non-relativistic elastic scattering amplitude for the negative pion from the bound state of the deuteron is given by

$$T(E) = V_\pi + V_\pi G(E)\, V_\pi \, , \tag{3.2.2}$$

where $G(E) = (E-H)^{-1}$ is the full Green's function. To obtain a well-behaved integral equation satisfied by $T(E)$, we write

$$G(E) = G_1 [1 + \nu_{np} G(E)] \qquad (3.2.3a)$$

and $$G(E) = G_2 [1 + V_\pi \; G(E)] , \qquad (3.2.3b)$$

where $G_1(E) = (E - H_0 - V_\pi)^{-1}$ and $G_2(E) = (E - H_0 - \nu_{np})^{-1}$ are the operators acting in the three-particle space. Combining Equations (3.2.2) and (3.2.3b) we get

$$T(E) = V_\pi + V_\pi \; G_2(E) \; T(E). \qquad (3.2.4)$$

This integral equation has a singular kernel (48). Introducing Equation (3.2.3a) into Equation (3.2.2) we have

$$T(E) = V_\pi + V_\pi \; G_1(E) \; V_\pi + V_\pi \; G_1(E) \; \nu_{np} \; G(E) \; V_\pi. \qquad (3.2.5)$$

Replacing $G(E)$ from Equation (3.2.5) with the aid of Equation (3.2.3b) yields the appropriate integral equation for $T(E)$, i.e.

$$T(E) = T_\pi(E) + T_\pi \; G_0(E) \; t_{np}(E) \; G_0(E) \; T(E), \qquad (3.2.6)$$

where t_{np} and T_π are the standard transition operators for the interactions ν_{np} and V_π and G_0 is the three-body free-Green's operator $G_0 = (E - H_0)^{-1}$.

Assuming a separable type interaction between neutron and proton system

$$\nu_{np} = - \lambda \; | f_d > < f_d > , \qquad (3.2.7)$$

we may reduce Equation (3.2.6) to an equivalent two-body problem. Following Dodd and Strobel (48), the three-body transition operator corresponding to Equation (3.2.7) may be written in a form which separates out the dependence of the kinetic energy of the pion

$$t_{np}(E) = | f_d > \hat{h}^{\frac{1}{2}}(E) \; \hat{G}_0(E) \; \hat{h}^{\frac{1}{2}}(E) \; < f_d | . \qquad (3.2.8)$$

Here the caret indicates that the operator acts only in the subspace spanned by the kets $|\underline{p}>$, $\underline{p}$ being the momentum of the pion relative to the c. m. of the proton and neutron.

Explicitly, $\hat{h}^{\frac{1}{2}}$ and $\hat{G}_0$ are defined by

$$<\underline{p}\,|\,\hat{h}^{\frac{1}{2}}(E)\,|\,\underline{p}'> = <\underline{p}\,|\,\underline{p}'> \left[\frac{<\underline{p}\, f_d\,|G_0(-\varepsilon_d + \frac{p^2}{2M})\, G_0(E)\,|\, f_d \underline{p}'>}{<\underline{p}\,|\,\underline{p}'>}\right]^{\frac{1}{2}},$$

and

$$<\underline{p}\,|\,\hat{G}_0(E)\,|\,\underline{p}'> = <\underline{p}\,|\,\underline{p}'>\,(E + \varepsilon_d - \frac{p^2}{2M})^{-1}, \qquad (3.2.9)$$

where ε_d is the binding energy of the deuteron and

$$M = \frac{m_\pi\,(m_n + m_p)}{(m_\pi + m_n + m_p)}.$$

Substitution of (3.2.8) into (3.2.6), yields the Lippmann-Schwinger equation

$$\hat{T} = \hat{V} + \hat{V}\,\hat{G}_0\,\hat{T}, \qquad (3.2.10)$$

for an equivalent two-body operator

$$\hat{T} = \hat{h}^{\frac{1}{2}} <f_d\,|\,G_0\,T\,G_0\,|\,f_d>\,\hat{h}^{\frac{1}{2}}, \qquad (3.2.11)$$

for pion-deuteron scattering with an effective potential

$$\hat{V} = \hat{h}^{\frac{1}{2}} <f_d\,|\,G_0\,T_\pi\,G_0\,|\,f_d>\,\hat{h}^{\frac{1}{2}}. \qquad (3.2.12)$$

On the energy shell limit, $p^2 = 2M(E + \varepsilon_d) = p'^2$ the scattering amplitude is related to the effective two-body operator

$$<\underline{p}'\,|\,\hat{T}(E)\,|\underline{p}> = <\underline{p}'\,\phi_d\,|T(E)|\,\underline{p}\,\phi_d>,$$

so that the evaluation of $\hat{T}(E)$ on-shell yields the physical amplitude.

Let us now consider a similar situation where the pion interacts only with the Coulomb field of the deuteron with its charge concentrated at its centre of mass. Following the above procedure with V_π replaced by

the Coulomb potential v_c^0, we find an equivalent two-body amplitude

$$\hat{T}^0 = \hat{V}^0 + \hat{V}^0 \hat{G}_0 \hat{T}^0, \tag{3.2.13}$$

generated by the effective potential

$$\hat{V}^0 = \hat{h}^{\frac{1}{2}} \langle f_d \mid G_0 t_c^0 G_0 \mid f_d \rangle \hat{h}^{\frac{1}{2}}, \tag{3.2.14}$$

where t_c^0 is the two-body Coulomb t-matrix corresponding to v_c^0.

The model problem described by Equations (3.2.13) and (3.2.14) is exactly solvable, since the internal motion of the deuteron and the motion of the pion are independent. The ground state of the model problem is well represented by hydrogenic type wave functions. The corresponding ground state energy is $E_\pi^0 = -\varepsilon_d - \varepsilon_\pi$, where ε_π is the Coulomb binding energy of the pion with the deuteron considered as a point charge.

The ground state wave function $|\hat{\psi}_\pi\rangle$ of the actual three-body problem with ground state energy E_π and the model three-body ground state $|\hat{\psi}_\pi^0\rangle$ with the ground state energy E_π^0 satisfy the homogeneous equations

$$(1 - \hat{G}_0 (E_\pi) \hat{V}) |\hat{\psi}\rangle = 0, \tag{3.2.15}$$

and

$$(1 - \hat{G}_0 (E_\pi^0) \hat{V}^0) |\hat{\psi}_\pi^0\rangle = 0. \tag{3.2.16}$$

The model ground state wave function $|\hat{\psi}_\pi^0\rangle$ is constructed from the usual Coulomb wave function $|\psi_\pi^0\rangle$ (48)

$$|\hat{\psi}_\pi^0\rangle = \hat{h}^{-\frac{1}{2}} |\psi_\pi^0\rangle. \tag{3.2.17}$$

We consider the discrete part of the spectrum of the exact three-body hamiltonian (i.e. perturbed problem) corresponding to the scattering Equation (3.2.10) as non-degenerate. In fact, we have only one discrete state with energy eigen value E_π. In general there is also a continuum

part of the spectrum, but we are only concerned with computing the perturbation of the simple discrete state in this calculation. The exact three-body ground state wave function

$$\lim_{\Delta \hat{V} \to o} | \hat{\psi}_\pi > \to |\hat{\psi}_\pi^o > ,$$

where $\Delta \hat{V}$ is the perturbing potential and $|\hat{\psi}_\pi^o >$ is the unperturbed wave function (i.e. model wave function) .

The energy shift is defined as

$$\Delta E = E_\pi - E_\pi^o = \frac{< \hat{\psi}_\pi^o \mid \Delta \hat{V} \mid \hat{\psi}_\pi >}{< \hat{\psi}_\pi^o \mid \hat{\psi}_\pi >} . \qquad (3.2.18)$$

The next step is to obtain the solution of the perturbed three-body ground state wave function $| \hat{\psi}_\pi >$ in terms of the model wave function $| \hat{\psi}_\pi^o >$. We may re-write the Equation (3.2.15) as

$$(H_0 + \hat{V} - E_\pi) \mid \hat{\psi}_\pi > = 0$$

or
$$(H_0 + \hat{V}^o - E_\pi^o) \mid \hat{\psi}_\pi > = (\hat{V}^o - \hat{V} + E_\pi - E_\pi^o) \mid \hat{\psi}_\pi >$$
$$= (\Delta E - \Delta \hat{V}) \mid \hat{\psi}_\pi > . \qquad (3.2.19)$$

The homogeneous part of Equation (3.2.19) gives the solution of the unperturbed problem which is known to be the ordinary hydrogenic type wave function. The required solution of the perturbed problem is

$$| \hat{\psi}_\pi > = |\hat{\psi}_\pi^o > + \frac{Q}{H_0 - E_\pi^o + \hat{V}^o} (\Delta E - \Delta \hat{V}) \mid \hat{\psi}_\pi > , \qquad (3.2.20)$$

and
$$(H_0 + \hat{V}^o) \mid \hat{\psi}_\pi^o > = E_\pi^o \mid \hat{\psi}_\pi^o > , \qquad (3.2.21)$$

where Q is the projection operator $1 - | \hat{\psi}_\pi^o > < \hat{\psi}_\pi^o |$. The structure of the ground state energy shift is

$$\Delta E = < \hat{\psi}_\pi^o \mid \Delta \hat{V} \mid \hat{\psi}_\pi^o > + < \hat{\psi}^o \mid \Delta \hat{V} \frac{Q}{H_0 - E_\pi^o + \hat{V}} (\Delta E - \Delta \hat{V}) \mid \hat{\psi}_\pi > \qquad (3.2.22)$$

As the perturbative potential $\Delta\hat{V} = \hat{V} - \hat{V}^0$ is very weak, the series will converge rapidly and can be solved in principle by iteration. The successive terms of the iteration are labelled by the number of powers of $\Delta\hat{V}$ that are retained.

The first-order energy shift is

$$\Delta E^{(1)} = < \hat{\psi}_\pi^0 \mid \Delta\hat{V} \mid \hat{\psi}_\pi^0 > . \tag{3.2.23}$$

Similarly the first-order correction to $\mid \hat{\psi}_\pi >$ reads

$$\mid \hat{\psi}_\pi^{(1)} > = \mid \hat{\psi}_\pi^0 > + \frac{Q}{H_0 - E_\pi^0 + \hat{V}^0} \Delta\hat{V} \mid \hat{\psi}_\pi^0 > . \tag{3.2.24}$$

The second-order energy shift is given by

$$\begin{aligned} \Delta E^{(2)} &= < \hat{\psi}_\pi^0 \mid \Delta\hat{V} \mid \hat{\psi}_\pi^{(1)} > \\ &= < \hat{\psi}_\pi^0 \mid \Delta\hat{V} \left(1 + \frac{Q}{H_0 - E_\pi^0 + \hat{V}^0} \Delta\hat{V} \right) \mid \hat{\psi}_\pi^0 > . \\ &= < \hat{\psi}_\pi^0 \mid \Delta\hat{V} \mid \hat{\psi}_\pi^0 > + < \hat{\psi}_\pi^0 \mid \Delta\hat{V} \frac{Q}{H_0 - E_\pi^0 + \hat{V}^0} \Delta\hat{V} \mid \hat{\psi}_\pi^0 > . \end{aligned} \tag{3.2.25}$$

Combining Equations (3.2.12) and (3.2.14) together with the Equation (3.2.17) the first-order energy shift is given by

$$\Delta E^{(1)} = < \psi_\pi^0 f_d \mid G_0 (T_\pi - t_c^0) G_0 \mid f_d \psi_\pi^0 >. \tag{3.2.26}$$

The interaction of the pion and deuteron may be expressed as the sum of the separate pion-proton and pion-neutron interactions

$$V_\pi = v_{\pi n} + v_{\pi p}.$$

Therefore the pion-deuteron elastic transition operator T_π satisfies the coupled equations

$$T_\pi = T_{\pi n} + T_{\pi p} , \tag{3.2.27}$$

where

$$T_{\pi p} = t_{\pi p} + t_{\pi p} G_0 T_{\pi n}, \tag{3.2.28a}$$

and

$$T_{\pi n} = t_{\pi n} + t_{\pi n} G_0 T_{\pi p} \quad (3.2.28b)$$

The multiple scattering series for T_π is

$$T_\pi = t_{\pi n} + t_{\pi p} + t_{\pi n} G_0 t_{\pi p} + t_{\pi p} G_0 t_{\pi n} + \ldots \quad (3.2.29)$$

The first-order energy shift may be split into the single scattering contribution and the higher order re-scattering contribution of the pion from the nucleons without break up of the deuteron in the intermediate state

$$\Delta E^{(1)} = \Delta E_s^{(1)} + \Delta E_h^{(1)} , \quad (3.2.30)$$

with $$\Delta E_s^{(1)} = \langle \psi_\pi^0 f_d \mid G_0 (t_{\pi p} + t_{\pi n} - t_c^0) G_0 \mid \psi_\pi^0 f_d \rangle . \quad (3.2.31)$$

The Coulomb scattering contribution between the pion and the proton may be eliminated partially by introducing the Coulomb t-matrix t_c in the pion-proton subsystem

$$\Delta E_s^{(1)} = \langle \psi_\pi^0 f_d \mid G_0 (t_{\pi p} + t_{\pi n} - t_c) G_0 \mid \psi_\pi^0 f_d \rangle + \langle \psi_\pi^0 f_d \mid G_0 (t_c - t_c^0) G_0 \mid \psi_\pi^0 f_d \rangle . \quad (3.2.32)$$

The first term of the r.h.s. of Equation (3.2.32) gives the energy shift due to the strong interaction and the second term is the electromagnetic correction due to the finite size and polarizability of the deuteron. The magnitude of this correction is small and has been ignored.

It should be noted that in the first-order energy shift calculation there is no contribution from intermediate N-N re-scattering, since break up and recombination of deuteron occur only in initial and final states. However, the second-and higher-order terms in the expansion of the energy shift in powers of the perturbing potential $\Delta\hat{V} = \hat{V} - \hat{V}^0$, obtained by iterating Equation (3.2.20) contain the effects of N-N re-scattering, and these will be considered in Section 6.

3.3 Evaluation of the first-order energy shift

The evaluation of the first-order energy shift of Equation (3.2.31) is simplified by neglecting the Coulomb interaction in the multiple scattering terms of Equation (3.2.29) and assuming iso-spin invariance. The operator T_π may then be split into two parts, $T_\pi = T_{\pi 1} + T_{\pi 2}$ according to which of the identical nucleons labelled by 1 or 2 interacts last with the pion. If P is the operator which exchanges the observables of the nucleons $PT_{\pi 2} = T_{\pi 1} P$, the first-order energy shift may be expressed as

$$\Delta E_s^{(1)} = 2 < \psi_\pi^0 f_d \mid G_0 T_{\pi 1} G_0 \mid f_d \psi_\pi^0 > , \tag{3.3.1}$$

where $T_{\pi 1}$ may be determined by

$$T_{\pi 1} = t_{\pi 1} + t_{\pi 1} G_0 P^{-1} T_{\pi 1} P. \tag{3.3.2}$$

To evaluate the energy shift from Equation (3.3.1) we introduce a simple separable potential, acting in the Hilbert space of the three particle, for the interacting pion-nucleon pair. The corresponding two-body t-matrix acting in three-particle space is

$$t_{\pi 1}(E) = \sum_{n_\alpha} \int d^3 p_1 \mid \lambda_{n_\alpha}, \underline{p}_1, n_\alpha > \tau_{n_\alpha} \left(E - \frac{p_1^2}{2M}\right) < \lambda_{n_\alpha}, \underline{p}_1, n_\alpha \mid ,$$

$$\text{where} \quad \tau_{n_\alpha}^{-1}(E) = \lambda_{n_\alpha}^{-1} - \int \frac{d^3 q_\alpha \mid g_{n_\alpha}(\underline{q}_\alpha) \mid^2}{E^+ - E(q_\alpha)} , \tag{3.3.3}$$

and

$$< \underline{p}_\alpha, \underline{q}_\alpha \mid \lambda_{n_\alpha}, \underline{p}_1, n_\alpha > = g_{n_\alpha}(\underline{q}_\alpha)\, \delta^3 (\underline{p}_\alpha - \underline{p}_1).$$

Here λ_{n_α} is the strength of the pion-nucleon form factor $g_{n_\alpha}(\underline{q}_\alpha)$, $\underline{p}_\alpha$

is the momentum of the spectator particle (α), $\tau_{n_\alpha}(E)$ is propagator of the quasi-particle and n_α represents all other quantum numbers required to specify the three-particle state . The explitic structure of the three-particle state is (49)

$$| \lambda_{n_\alpha}, \underline{p}_\alpha, n_\alpha > = | \lambda_{n_\alpha}, \underline{p}_\alpha, (\underline{L}_\alpha(\underline{\bar{j}}_\alpha \underline{j}_\alpha) \underline{S}_\alpha) \underline{J}M; (\underline{\bar{i}}_\alpha \underline{i}_\alpha)\underline{I} M_I >$$

We generally couple the pair total angular momentum $\underline{\bar{j}}_\alpha$ to spectator spin $\underline{j}_\alpha$ to get the channel spin $\underline{S}_\alpha$ which is then added to the orbital angular momentum $\underline{L}_\alpha$ of the spectator with respect to the pair c. m. to obtain the total angular momentum $\underline{J}$ and its z-component M. Similarly, the pair iso-spin $\underline{\bar{i}}_\alpha$ is coupled to the free particle iso-spin $\underline{i}_\alpha$ to obtain total iso-spin $\underline{I}$ and its z-component M_I of the three-particle state. This coupling scheme is called channel coupling. In this model calculation we have considered iso-spin 1/2 and 3/2 channels of the pion-nucleon system only. The specific forms of the pion-nucleon interaction potential have been given in table 1 in Chapter II.

Substituting Equations (3.3.2) and (3.3.3) into Equation (3.3.1) the first-order energy shift contribution is

$$\Delta E_s^{(1)} = 2 \sum_{n_\alpha} \int F_{n_\alpha,d}(\underline{p}_\alpha)\, \tau_{n_\alpha}(-E_\pi^0 - \frac{p_\alpha^2}{2M})\, F_{n_\alpha,d}(\underline{p}_\alpha)\, d^3 p_\alpha, \tag{3.3.4}$$

with $$F_{n_\alpha,d}(\underline{p}_\alpha) = < \lambda_{n_\alpha}, \underline{p}_\alpha, n_\alpha \mid G_0(-E_\pi^0) \mid \psi_\pi^0 f_d > \tag{3.3.5}$$

and the multiple scattering contribution is

$$\Delta E_h^{(1)} = - \sum_{n_\alpha, n_\beta} \int F_{d,n_\beta}(\underline{p}_\beta)\, \tau_{n_\beta}(-E_\pi^0 - \frac{p_\beta^2}{2M}) \cdot A_{n_\beta,n_\alpha}(\underline{p}_\beta, \underline{p}_\alpha, -E_\pi^0)$$
$$\tau_{n_\alpha}(-E_\pi^0 - \frac{p_\alpha^2}{2M})\, F_{n_\alpha,d}(\underline{p}_\alpha)\, d^3 p_\alpha\, d^3 p_\beta. \tag{3.3.6}$$

The amplitude $A_{n_\beta,n_\alpha}(\underline{p}_\beta, \underline{p}_\alpha, E)$ satisfies the standard form of the Faddeev equations with separable potentials,

$$A_{n_\beta,n_\alpha}(\underline{p}_\beta,\underline{p}_\alpha,E) = B_{n_\beta,n_\alpha}(\underline{p}_\beta, \underline{p}_\alpha, E)$$

$$+ \sum_{n_\gamma} \int B_{n_\beta,n_\gamma}(\underline{p}_\beta, \underline{p}_\gamma, E)\tau_{n_\gamma}(E - \frac{p_\gamma^2}{2M}) A_{n_\gamma,n_\alpha}(\underline{p}_\gamma, \underline{p}_\alpha, E)d^3p_\gamma, \tag{3.3.7}$$

where $$B_{n_\beta,n_\alpha}(\underline{p}_\beta, \underline{p}_\alpha, E) = \langle \lambda_{n_\beta}, \underline{p}_\beta, n_\beta \mid G_0(E)P \mid \lambda_{n_\alpha}, \underline{p}_\alpha, n_\alpha \rangle , \tag{3.3.8}$$

describes the exchange of pion between quasi-particles.

The sensitivity of the energy shift on the off-shell dependence of the pion-nucleon t-matrix and the momentum distribution of the atomic wave function can be studied from the momentum dependence of $F_{n_\alpha,d}(\underline{p}_\alpha)$.

The ground state wave function for the model pion-deuteron system in the momentum space reads

$$\psi_\pi^0(\underline{q}) = (2\pi)^{-3/2} \int e^{i\,\underline{q}\cdot\underline{r}}\, \psi_\pi^0(\underline{r})\, d^3\underline{r}, \tag{3.3.9}$$

where

$$\psi_\pi^0(r) = (\pi a_0^3)^{-1/2} e^{-r/a_0} , \tag{3.3.10}$$

and $a_0 = \frac{\hbar^2}{m_\pi e^2} = 193.73$ fm is the Bohr radius of the pion-deuteron atom. An elementary integration yields

$$\psi_\pi^0(q) = \frac{2\sqrt{2}}{\pi} \frac{a_0^{3/2}}{(1 + a_0^2 q^2)^2} , \tag{3.3.11}$$

and $$\psi_\pi^0(\underline{r} = 0) = (\pi a_0^3)^{-1/2}. \tag{3.3.12}$$

The explicit expression of the weight function $F_{n_\alpha,d}(\underline{p}_\alpha)$ is

$$F_{n_\alpha,d}(\underline{p}_\alpha) = -\int \frac{g_{n_\alpha}\,(\underline{k} + \mu\underline{p}_\alpha)\, f_d\,(\underline{p}_\alpha + \lambda\,\underline{k})\,\psi_\pi^0\,(\underline{k})}{E_\pi^0 + p_\alpha^2/2M + (\underline{k} + \mu\underline{p}_\alpha)^2/2m}\, d^3k\,, \tag{3.3.13}$$

where in terms of the nucleon mass m_N and the pion mass m_π the reduced masses M and m are

$$M = m_N(m_N + m_\pi)/(2m_N + m_\pi), \quad m = m_N\, m_\pi/(m_N + m_\pi), \tag{3.3.14}$$

and the parameters λ and μ are

$$\lambda = m_N/(m_N + m_N), \quad \mu = m_\pi/(m_N + m_\pi).$$

The value of the atomic wave function of the pion at the c. m. of the deuteron

$$\psi_\pi^0\,(\underline{r} = 0) = \frac{1}{(2\pi)^{3/2}} \int \psi_\pi^0\,(\underline{k})\, d^3k \tag{3.3.15}$$

can be extracted from $F_{n_\alpha,d}(\underline{p}_\alpha)$ since the atomic bound-state wave function of the pion falls off much more rapidly in momentum space than the deuteron on pion-nucleon vertex function, and under this approximation we get

$$F_{n_\alpha,d}(\underline{p}_\alpha) = \psi_\pi^0\,(\underline{r} = 0)\,(2\pi)^{3/2}\, g_{n_\alpha}(\mu\,\underline{p}_\alpha)\,\bar{f}_d\,(\underline{p}_\alpha) \tag{3.3.16}$$

with

$$\bar{f}_d\,(\underline{p}_\alpha) = \left(E_\pi^0 + \frac{p_\alpha^2}{2M} + \frac{\mu^2\, p_\alpha^2}{2m}\right)^{-1} f_d\,(\underline{p}_\alpha). \tag{3.3.17}$$

Since the argument of the pion-nucleon form factor contains the mass ratio $\mu \sim 1/7$, a more adhoc approximation is to neglect the momentum dependence of $F_{n_\alpha,d}$ arising from the pion-nucleon vertex compared with that of the deuteron vertex f_d. Also by dropping ε_π from E_π^0 ($\varepsilon_\pi \ll \varepsilon_d$) in the denominator of Equation (3.3.17), the object $\bar{f}_d$ becomes the exact deuteron bound state ϕ_d and we get the simplest expression for

$F_{n_\alpha,d}(\underline{p}_\alpha)$,

$$F_{n_\alpha,d}(\underline{p}_\alpha) = (2\pi)^{3/2}\ \psi_\pi\ (\underline{r} = 0)\ g_{n_\alpha}(0)\ \phi_d\ (\underline{p}_\alpha) \quad . \qquad (3.3.18)$$

We have presented in Tables 4-5 the typical values of the weight function $F_{n_\alpha,d}(\underline{p}_\alpha)$, corresponding to the channels S11 and S31, obtained from the Equations (3.3.13) and (3.3.16) for different values of p_α. The important conclusion from these tabulated values of $F_{n_\alpha,d}$ is that the momentum variation of $F_{n_\alpha,d}$ arising from that of the pion-nucleon form factor and the atomic wave function is quite significant and should be taken into account properly for the evaluation of the energy shift. We also observe from the Tables 4-5 and Figures 4-5, that the values of $F_{n_\alpha,d}$ for small values of p_α corresponding to the Equation (3.3.16) are always greater than those obtained from the Equations (3.3.13).

With the approximation (3.3.16) the first-order energy shift due to the single scattering becomes

$$\Delta E_s^{(1)'} = (2\pi)^3\ |\ \psi_\pi^0\ (\underline{r} = 0)\ |^2 \int d^3\ p_\alpha\ \bar{f}_d^{\,2}\ (\underline{p}_\alpha)$$
$$< \mu\ \underline{p}_\alpha\ |\ t_{\pi_1}\ (-E_\pi^0 - \frac{p_\alpha^2}{2M})\ |\ \mu\ \underline{p}_\alpha > \qquad (3.3.19)$$

The deuteron nuclear form factor $\phi_\alpha^2(p)$ is a sharply peaked function in the momentum space compared with the pion-nucleon t-matrix since the size of the deuteron nucleus is appreciably larger ($\sim$ 4.0 fm) than pion-nucleon interaction range ($\sim \leqslant$ .7 fm). Therefore we may assume that the pion-nucleon amplitude is a slowly varying function of the off-shell energy $-E_\pi^0 - p_\alpha^2/2M$ over the momentum variation of the $\phi_d^2(\underline{p}_\alpha)$ and may replace the pion-nucleon amplitude by its value at the zero energy,

i.e. scattering length approximation. Under this approximation the integral over $\underline{p}_\alpha$ in Equation (3.3.19) is just the normalization integral for the deuteron bound state and the energy shift is related to the iso-scalar combination of the pion-nucleon scattering lengths

$$\Delta E_s^{(1)''} = \frac{2\pi}{m_\pi} \mid \psi_\pi^0 \ (r = 0) \mid^2 \left(\frac{2}{3} a_1 + \frac{4}{3} a_3 \right). \qquad (3.3.20)$$

This is the DGBT's formula relating the energy shift to the scattering length. We may consider the Equation (3.3.4) as the off-shell generalization of the DGBT's formula. The uncertain energy dependence of the actual pion-nucleon amplitude for the negative energies makes the approximations leading to Equation (3.3.20) difficult to justify. We have calculated the pion-nucleon scattering amplitude $f(0, 0, -E)$ for the negative energies, to show its off-shell dependence in the S11 and S31 channels (see Table 6). The off-shell contributions to the energy shift due to the pion-nucleon interaction is generated mainly from the negative energy dependence of the two-body pion-nucleon t-matrix. In relating the energy shift in terms of the pion-nucleus scattering amplitude, DGBT regarded as negligible ~~completely ignored~~ this negative energy dependence of the two-body scattering amplitude. To test these approximations, calculated values of the energy shifts defined by the Equations (3.3.4), (3.3.19) and (3.3.20) have been compared. The calculated shifts are presented in Table 7.

3.4 The Angular Momentum Reduction of $A_{n_\beta, n_\alpha}(\underline{p}', \underline{p}, E)$

To solve Equations (3.3.7) it is necessary to write them in the rotationally invariant form in the angular momentum and iso-spin space. The rotationally invariant amplitude is written in terms of the channel coupling scheme as mentioned earlier. In this coupling scheme we couple the total angular momentum of the interacting pair to the spectator

particle's spin to get the channel spin. Then we couple the channel spin to the angular momentum of the free particle relative to the pair c. m. to obtain the total angular momentum of the system. The main advantage of using the channel coupling scheme is that we can determine the total parity of the reaction easily. Similarly, the iso-spin of the interacting pair is coupled to the free particle's iso-spin to get the total iso-spin of the system. The total angular momentum and iso-spin are the constants of motion. Let us define our notations:

(i) $\underline{j}_\alpha(m_\alpha)$, $\underline{j}_\beta(m_\beta)$ and $\underline{j}_\gamma(m_\gamma)$ are the individual spin (z-component) of the particles α, β and γ respectively;

(ii) $\underline{l}_\alpha(\lambda_\alpha)$ is the relative angular momentum (z-component) of the interacting pair (α);

(iii) $\underline{j}_\beta + \underline{j}_\gamma = \underline{s}_\alpha(\sigma_\alpha)$ is the total spin (z-component) of the interacting pair (α);

(iv) $\underline{\ell}_\alpha + s_\alpha = \underline{\bar{j}}_\alpha(\bar{m}_\alpha)$ is the total angular momentum (z-component) of the pair (α);

(v) $\underline{\bar{j}}_\alpha + \underline{j}_\alpha = \underline{S}_\alpha(\Sigma_\alpha)$ is the total spin (z-component) of the channel;

(vi) $\underline{L}_\alpha(M_\alpha)$ is the relative angular momentum (z-component) of the free-particle relative to c. m. co-ordinate of the pair (α);

(vii) $\underline{L}_\alpha + \underline{S}_\alpha = \underline{J}(M)$ is the total angular momentum (z-component) of the three-particle state.

Under this coupling scheme the scattering amplitude is

$$A_{n_\beta,n_\alpha}(\underline{p}', \underline{p}, E) \equiv (\bar{j}_\beta \ \bar{m}_\beta, j_\beta m_\beta, \underline{p}', n_\beta \,|A|\, \bar{j}_\alpha \bar{m}_\alpha, j_\alpha m_\alpha, \underline{p}, n_\alpha >$$

$$= \sum < \bar{i}_\beta \ \bar{m}_{i\beta}, \ i_\beta m_{i\beta} \mid IM_I > < \bar{i}_\alpha \bar{m}_{i_\alpha}, \ i_\alpha m_\alpha \mid IM_I >$$

$$Im_I, S_\beta \ \Sigma_\beta, L_\beta \ M_\beta$$

$$J \ M, \ S_\alpha \ \Sigma_\alpha, L_\alpha \ M_\alpha$$

$$< \bar{j}_\alpha \bar{m}_\alpha, j_\alpha m_\alpha \mid S_\alpha \Sigma_\alpha > < L_\alpha M_\alpha, S_\alpha \Sigma_\alpha \mid JM >$$

$$< \bar{j}_\beta \bar{m}_\beta, j_\beta m_\beta \mid S_\beta \Sigma_\beta > < L_\beta \ M_\beta, S_\beta \Sigma_\beta \mid JM >$$

$$Y^*_{L_\beta,M_\beta}(\hat{p}') \ A^{JI}_{n_\beta,n_\alpha}(p', p, E) \ Y_{L_\alpha M_\alpha}(\hat{p}), \qquad (3.4.1)$$

where $A^{JI}_{n_\beta,n_\alpha}(p', p, E) = < L_\beta, S_\beta, \bar{j}_\beta, \bar{i}_\beta, n_\beta, p' | \ A^{JI} | L_\alpha, S_\alpha, \bar{j}_\alpha, \bar{i}_\alpha, n_\alpha, p >$.

The Clebsch-Gordon co-efficients and spherical harmonics are defined according to reference (50).

Here n_α contains all other discrete quantum numbers, i.e. parity of the individual particle and the quasi-particles, required to specify completely the corresponding three-particle state. Similarly we can also write the corresponding rotationally invariant form of the driving term in Equation (3.3.8) in terms of $B^{JI}_{n_\beta,n_\alpha}(p', p, E)$. The invariant amplitude $A^{JI}_{n_\beta,n_\alpha}$ is independent of M because of the Wigner-Eckart theorem and also diagonal since the hamiltonian commutes with $(\underline{J}, J_z)$. Substituting Equation (3.4.1) and the corresponding expression for B_{n_β,n_α} into Equation (3.3.7) and using orthogonality relations for the vector coupling co-efficient and spherical harmonics, it is easy to prove that the amplitude $A^{JI}_{n_\beta,n_\alpha}(p', p, E)$ satisfies the one-dimensional integral equation

$$A^{JI}_{n_\beta,n_\alpha}(p', p, E) = B^{JI}_{n_\beta,n_\alpha}(p', p, E) + \sum_{n_\gamma} \int_0^\infty q''^2 dq'' \ B^{JI}_{n_\beta,n_\gamma}(p', q'', E)$$

$$\tau_{n_\gamma}(E - \tfrac{q''^2}{2M}) \ A^{JI}_{n_\gamma,n_\alpha}(q'', p, E). \qquad (3.4.2)$$

This is the appropriate form for the evaluation of $A^{J\,I}_{n_\beta, n_\alpha}$ numerically. For every value of the total angular momentum J there will be a set of coupled equations. To solve these equations we supply $B^{J\,I}_{n_\beta, n_\alpha}(p', p, E)$ and τ_{n_γ} which are determined from the knowledge of the separable two-body interaction. In our model calculation it is relatively simple to solve Equation (3.4.2) since the energy variable $E = -E_\pi^0$ is below the threshold. Therefore no logarithmic singularity will appear from the kernel and the equations are non-singular. We replace the integral by the discrete sum and the matrix inversion method is used to obtain the numerical solution.

Now, we shall determine the rationally invariant structure of the driving term. To achieve this end we shall follow closely the steps given in References (51) and (52). The rotationally invariant form of the driving term is

$$B^{J}_{n_\beta, n_\alpha}(p_\beta, p_\alpha, E) = \hat{J}^{-2} \sum \langle j_\gamma m_\gamma, j_\alpha m_\alpha \mid s_\beta \sigma_\beta \rangle \langle j_\beta m_\beta, j_\gamma m_\gamma \mid s_\alpha \sigma_\alpha \rangle$$

$$\langle \ell_\beta \lambda_\beta, s_\beta \sigma_\beta \mid \bar{j}_\beta \bar{m}_\beta \rangle \langle \ell_\alpha \lambda_\alpha, s_\alpha \sigma_\alpha \mid \bar{j}_\alpha \bar{m}_\alpha \rangle \langle \bar{j}_\beta \bar{m}_\beta, j_\beta m_\beta \mid S_\beta \Sigma_\beta \rangle$$

$$\langle \bar{j}_\alpha \bar{m}_\alpha, j_\alpha m_\alpha \mid S_\alpha \Sigma_\alpha \rangle \langle L_\beta M_\beta, S_\beta \Sigma_\beta \mid JM \rangle \langle L_\alpha M_\alpha, S_\alpha \Sigma_\alpha \mid JM \rangle$$

$$\langle \lambda_{n_\beta}, M_\beta, L_\beta, n_\beta, p_\beta \mid G_0(E)\, P \mid \lambda_{n_\alpha}, M_\alpha, L_\alpha, n_\alpha, p_\alpha \rangle \quad . \qquad (3.4.3)$$

The summation includes all angular momentum projections. The object $\langle \lambda_{n_\beta}, M_\beta, L_\beta, n_\beta, p_\beta \mid G_0(E)\, P \mid \lambda_{n_\alpha}, M_\alpha, L_\alpha, n_\alpha, p_\alpha \rangle$ is written as (51)

$$\langle \lambda_{n_\beta}, M_\beta, L_\beta, n_\beta, p_\beta \mid G_0(E)\, P \mid \lambda_{n_\alpha}, M_\alpha, L_\alpha, n_\alpha, p_\alpha \rangle$$

$$= \int d\hat{p}_\alpha \, d\hat{p}_\beta \, dp'_\alpha \, dp'_\beta \, Y^*_{L_\beta, M_\beta}(\hat{p}_\beta) < \lambda_{n_\beta}, \underline{p}_\beta \, n_\beta \mid \underline{p}'_\beta \, \underline{q}'_\beta >$$

$$(E - \frac{p'^2_\beta}{2\mu_\beta} - \frac{q'^2_\beta}{2\mu_{\gamma\alpha}})^{-1} < \underline{p}'_\alpha, \underline{q}_\alpha' \mid \lambda_{n_\alpha}, \underline{p}_\alpha, n_\alpha > Y_{L_\alpha, M_\alpha}(\hat{p}_\alpha) \, , \tag{3.4.4}$$

where μ_β is the reduced mass of the particle β and the pair $(\alpha + \gamma)$; $\mu_{\gamma\alpha}$ is the reduced mass of the pair $(\alpha + \gamma)$. The form factors are written as

$$< \underline{p}_\alpha, \underline{q}_\alpha \mid \lambda_{n_\alpha}, \underline{p}'_\alpha, n_\alpha > = \delta^3 (\underline{p}_\alpha - \underline{p}'_\alpha) \, g_{n_\alpha}(\underline{q}_\alpha) \, Y_{\ell_\alpha \lambda_\alpha}(\hat{q}_\alpha) \, . \tag{3.4.5}$$

The relative momentum $\underline{q}_\alpha$ can be expressed in terms of the free momentum $\underline{p}_\alpha$ and $\underline{p}_\beta$ of the interacting particles as

$$\underline{q}_\alpha = \underline{p}_\beta + \rho_\alpha \underline{p}_\alpha \; ; \quad \underline{q}_\beta = - \underline{p}_\alpha - \rho_\beta \underline{p}_\beta ,$$

where

$$\rho_\alpha = m_\beta / (m_\beta + m_\gamma), \quad \rho_\beta = m_\alpha / (m_\alpha + m_\gamma), \tag{3.4.6}$$

and m_α, m_β and m_γ are the masses of the particles α, β, and γ respectively. Using the identity given in Reference (53) one can write the spherical harmonics $Y_{\ell_\alpha, \lambda_\alpha}(\hat{q}_\alpha)$ in terms of the spherical harmonics $Y_{a,ma}(\hat{p}_\alpha)$ and $Y_{b,mb}(\hat{p}_\beta)$

$$\underline{q}_\alpha{}^{\ell_\alpha} Y_{\ell_\alpha, \lambda_\alpha}(\hat{q}_\alpha) = \sum_{a \, m_a, \, b \, m_b} \delta_{\ell_\alpha, \, a+b} \left\{ \frac{4\pi \, (2\ell_\alpha + 1)!}{(2a+1)!(2b+1)!} \right\}^{\frac{1}{2}}$$

$$(\rho_\alpha p_\alpha)^a \, p_\beta{}^b \, Y_{a,m_a}(\hat{p}_\alpha) \, Y_{b,m_b}(\hat{p}_\beta) < a m_a, \, b m_b \mid \ell_\alpha \, \lambda_\alpha > \tag{3.4.7}$$

The partial wave decomposition of the product of form factors and Green's function is

$$\underline{q}^{-\ell_\beta}\, g_{n_\beta}{}^{*}(\underline{q}_\beta)\, \frac{1}{E - \frac{p_\alpha^2}{2m_\alpha} - \frac{p_\beta^2}{2m_\beta} - \frac{(\underline{p}_\alpha + \underline{p}_\beta)^2}{2m_\gamma}}\, \underline{q}^{-\ell_\alpha}\, g_{n_\alpha}(\underline{q}_\alpha)$$

$$= 4\pi \sum_{\mathcal{L}M} F^{\mathcal{L}}_{n_\beta, n_\alpha}(p_\beta, p_\alpha, E)\, Y_{\mathcal{L}M}{}^{*}(\hat{p}_\beta)\, Y_{\mathcal{L}M}(\hat{p}_\alpha), \qquad (3.4.8)$$

where

$$F^{\mathcal{L}}_{n_\beta, n_\alpha}(p_\beta, p_\alpha, E) = \frac{1}{2} \int_{-1}^{1} dx\, \frac{\underline{q}_\beta^{-\ell_\beta}\, g_{n_\beta}{}^{*}(\underline{q}_\beta)\, g_{n_\alpha}(\underline{q}_\alpha)\, \underline{q}_\alpha^{-\ell_\alpha}\, P_{\mathcal{L}}(x)}{E - p_\alpha{}^2/2m_\alpha - p_\beta{}^2/2m_\beta - \frac{(\underline{p}_\alpha + \underline{p}_\beta)^2}{2m_\gamma}}$$

(3.4.9)

Here x is the cosine of the angle between the unit vectors $\hat{p}_\alpha$ and $\hat{p}_\beta$. $F^{\mathcal{L}}_{n_\beta, n_\alpha}(p_\beta, p_\alpha, E)$ contains all the dynamics of the interaction.

Now combining the Equations (3.4.3) to (3.4.8) and using the following relations

(i)

$$Y_{\ell_1, m_1}(\hat{p})\, Y_{\ell_2, m_2}(\hat{p}) = \sum_{LM} \left[\frac{(2\ell_1 + 1)(2\ell_2 + 1)(2L + 1)}{\sqrt{4\pi}} \right]^{\frac{1}{2}} \begin{pmatrix} \ell_1 & \ell_2 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & L \\ m_1 & m_2 & M \end{pmatrix} Y^{*}_{L,M}(\hat{p}) \qquad (3.4.10)$$

(ii) Equations (2.20) and (3.21) of the Reference (54) or Equation (25.3) of the Reference (55), we may re-derive the final expression for the invariant driving term as

$$B^{J}_{n_\beta, n_\alpha}(p_\beta, p_\alpha, E) = (1 - \delta_{n_\beta, n_\alpha})\, p_\alpha{}^{\ell_\beta}\, p_\beta{}^{\ell_\alpha} \sum_{\mathcal{L}} F^{\mathcal{L}}_{n_\beta, n_\alpha}(p_\beta, p_\alpha, E) \times \sum_{a=0}^{\ell_\alpha} \sum_{b=0}^{\ell_\beta} \tilde{A}^{\mathcal{L}, a, b}_{n_\beta, n_\alpha} (p_\alpha/p_\beta)^{a-b}, \qquad (3.4.11)$$

where

$$\tilde{A}^{\mathcal{L},\,a,\,b}_{n_\beta, n_\alpha} = (-)^{R}\,\hat{\ell}_\alpha\,\hat{\ell}_\beta\,\hat{L}_\beta\,\hat{S}_\alpha\,\hat{S}_\beta\,\hat{\bar{j}}_\alpha\,\hat{\bar{j}}_\beta\,\hat{s}_\alpha\,\hat{s}_\beta\,\hat{\mathcal{L}}^2\,\hat{L}_\alpha$$

$$\rho_\alpha^{\,a}\;\rho_\beta^{\,b}\;\{(2\ell_\alpha+1)!\,(2\ell_\beta+1)!/(2a)!\,(2b)!\,(2\ell_\alpha-2a)!\,(2\ell_\beta-2b)!\}^{\frac{1}{2}}$$

$$\times \sum_{f\,\Lambda\Lambda'} [\hat{f}\,\hat{\Lambda}\hat{\Lambda}']^2 \begin{Bmatrix} S_\alpha & S_\beta & f \\ L_\beta & L_\alpha & J \end{Bmatrix} \begin{Bmatrix} j_\alpha & S_\alpha & S_\beta & j_\beta \\ \bar{j}_\alpha & f & \bar{j}_\beta & j_\gamma \\ s_\alpha & \ell_\alpha & \ell_\beta & s_\beta \end{Bmatrix} \times$$

$$\begin{Bmatrix} L_\alpha & L_\beta & f \\ \Lambda' & \Lambda & \mathcal{L} \end{Bmatrix} \begin{Bmatrix} \ell_\alpha & \ell_\beta & f \\ a & \ell_\beta - b & \Lambda \\ \ell_\alpha - a & b & \Lambda' \end{Bmatrix} \begin{Bmatrix} \Lambda' & \mathcal{L} & L_\beta \\ o & o & o \end{Bmatrix} \begin{Bmatrix} \Lambda & \mathcal{L} & L_\alpha \\ o & o & o \end{Bmatrix} \times$$

$$\begin{Bmatrix} \ell_\alpha - a & b & \Lambda' \\ o & o & o \end{Bmatrix} \quad \begin{Bmatrix} a & \ell_\beta - b & \Lambda \\ o & o & o \end{Bmatrix}$$

$$R = -J + L_\alpha + L_\beta + S_\alpha + S_\beta + \bar{j}_\alpha + \bar{j}_\beta + s_\alpha + \ell_\beta - j_\alpha\,, \qquad (3.4.12)$$

where $\hat{\ell} = \sqrt{2\ell+1}$.

The 3j, 6j and 9j symbols are defined according to Edmonds (50). Following Ord-Smith we can define 12j symbol. This complicated structure of the driving term was first derived by Stingle and Rinat (52) for the neutron-deuteron scattering. In their reference more detailed steps are given. The iso-spin contribution can be determined separately. The corresponding contribution to the invariant driving term is

$$\langle [(i_\gamma, i_\alpha)\,\bar{i}_\beta\,i_\beta]\,I \,|\, [(i_\beta, i_\gamma)\,\bar{i}_\alpha, i_\alpha]\,I \rangle$$

$$= (-)^{i_\alpha + i_\gamma - \bar{i}_\beta + 2I}\,\hat{\bar{i}}_\beta\,\hat{\bar{i}}_\alpha \begin{Bmatrix} i_\beta & i_\gamma & \bar{i}_\alpha \\ i_\alpha & I & \bar{i}_\beta \end{Bmatrix}\,, \qquad (3.4.13)$$

where $\underline{i}_\alpha$ is the iso-spin for the particle α ; and $\bar{\underline{i}}_\alpha$ is the iso-spin for the pair $(\beta + \gamma)$. For our calculation we have considered only S11 and S31 channels. For the S-wave pion-deuteron scattering the total angular

momentum J is 1 and the total parity of the system is - 1 and the total iso-spin I = 1.

3.5 The Specific Form of the Driving Term for a Free Exchange of a Pion and the Multiple Scattering Contributions

Following Stingle and Rinat we have also coupled the spin of the exchanged particle last to determine the appropriate form of the driving term for our calculation. The exact form of the rotationally invariant Born term will be simplified greatly because the pion has spin zero. When one of the angular momenta is zero the 12j symbol reduces to the 9j symbols or two 6j symbols (56). For a free exchange of a pion the value of the 12j symbol becomes

$$\begin{Bmatrix} j_\alpha & S_\alpha & S_\beta & j_\beta \\ \bar{j}_\alpha & f & \bar{j}_\beta & o \\ s_\alpha & \ell_\alpha & \ell_\beta & s_\beta \end{Bmatrix} = \frac{\delta(j_\beta, s_\beta)\ \delta(j_\alpha, s_\alpha)}{\hat{s}_\alpha\ \hat{s}_\beta}$$

$$\times \begin{Bmatrix} \bar{j}_\alpha & S_\alpha & j_\alpha \\ \ell_\alpha & f & \ell_\beta \\ j_\beta & S_\beta & \bar{j}_\beta \end{Bmatrix} \qquad (3.5.1)$$

We now substitute this value into the general expression to obtain the total contribution from the angular momentum part for a free exchange of a pion between the quasi-particles

$$\tilde{A}^{\mathcal{L},a,b}_{n_\beta,n_\alpha} = (-)^R \hat{\ell}_\alpha \hat{\ell}_\beta \hat{S}_\alpha \hat{S}_\beta \hat{\bar{j}}_\alpha \hat{\bar{j}}_\beta \hat{s}_\beta \hat{s}_\alpha \hat{\mathcal{L}}^2 \hat{L}_\alpha \hat{L}_\beta$$

$$\times \rho_\alpha^{\,a}\ \rho_\beta^{\,b} \{ (2\ell_\alpha + 1)!\ (2\ell_\beta + 1)!/(2a)!\ (2b)!\ (2\ell_\alpha - 2a)!\ (2\ell_\beta - 2b)! \}^{\frac{1}{2}}$$

$$\times \sum_{f\Lambda\Lambda'} \frac{[\hat{f}\ \hat{\Lambda}\ \hat{\Lambda}']^2}{\hat{s}_\alpha\ \hat{s}_\beta} \begin{Bmatrix} S_\alpha & S_\beta & f \\ L_\beta & L_\alpha & J \end{Bmatrix} \begin{Bmatrix} \bar{j}_\alpha & S_\alpha & j_\alpha \\ \ell_\alpha & f & \ell_\beta \\ j_\beta & S_\beta & \bar{j}_\beta \end{Bmatrix} \begin{Bmatrix} L_\alpha & L_\beta & f \\ \Lambda' & \Lambda & \mathcal{L} \end{Bmatrix}$$

$$\begin{Bmatrix} \ell_\alpha & \ell_\beta & f \\ a & \ell_\beta - b & \Lambda \\ \ell_\alpha - a & b & \Lambda' \end{Bmatrix} \begin{Bmatrix} \Lambda & \mathcal{L} & L_\beta \\ o & o & o \end{Bmatrix} \begin{Bmatrix} \Lambda & \mathcal{L} & L_\alpha \\ o & o & o \end{Bmatrix}$$

$$\times \begin{Bmatrix} \ell_\alpha - a & b & \Lambda' \\ o & o & o \end{Bmatrix} \begin{Bmatrix} a & \ell_\beta - b & \Lambda \\ o & o & o \end{Bmatrix} \qquad (3.5.2)$$

Finally, the specific form of the driving term for a free exchange of a pion is

$$B^{J\,I}_{n_\beta n_\alpha}(p_\beta, p_\alpha, E) = \{ (1 - \delta_{n_\beta, n_\alpha})\, p_\alpha^{\ell_\beta}\, p_\beta^{\ell_\alpha} \sum_{\mathcal{L}} F^{\mathcal{L}}_{n_\beta, n_\alpha}(p_\beta, p_\alpha, E)$$

$$\times \sum_{a=0}^{\ell_\alpha} \sum_{b=0}^{\ell_\beta} A^{\sim \mathcal{L}}_{n_\beta, n_\alpha} (p_\alpha/p_\beta)^{a-b} \}\; C(I, \bar{i}_\beta, \bar{i}_\alpha) \qquad (3.5.3)$$

The Equation (3.5.3) reduces to a very simple form for our calculation since $l_\alpha = l_\beta = 0$; $L_\beta = L_\alpha = 0$. The values of the 3j, 6j and 9j's have been determined from Rotenberg *et al's* (54) manual. Combining the Equations (3.5.3) and (3.4.2) we can solve $A^{J\,I}_{n_\beta, n_\alpha}$ by standard matrix inversion method. Now, introducing the solutions of $A^{J\,I}_{n_\beta, n_\alpha}$ into Equations (3.4.1) and then using Equation (3.3.6) it is straightforward to evaluate the higher-order re-scattering contributions. The multiple scattering contributions to the first-order energy shift has been evaluated for the parameter sets a and b of Table 1 in Chapter II. For the parameter sets a and b the calculated values of $\Delta E_h^{(1)}$ are - 1.28 eV to - 1.24 eV respectively. The corresponding values of $\Delta E_h^{(1)}$, when we neglect the momentum spread of the pionic wave function from the weight function $F_{n_\alpha, d}$, Equation (3.3.16), are - 1.42 eV to - 1.37 eV. Finally the scattering length approximation is made by using Equation (3.3.18) for the factor $F_{n_\alpha, d}$ and replacing $A_{n_\beta n_\alpha}(p', p, -E_\pi^0)$ by the zero energy on-

shell value $A_{n_\beta, n_\alpha}(o, o, -E_\pi^o)$. The values of $\Delta E_h^{(1)}$ obtained under this approximation are - 1.12 eV to - 1.07 eV for the parameter sets a and b. We note that off-shell corrections are also significant in the multiple scattering contribution to the first-order energy shift. The calculated values of the total first-order shift ΔE and its components under the various approximations are summarized in Table 7.

3.6 Contribution from N-N Re-Scattering

Nucleon-nucleon scattering does not occur in the first-order energy shift of our theory but it appears only in the second and higher orders of the perturbation $\delta \hat{V} = \hat{V} - \hat{V}^o$. Terms involving nucleon-nucleon re-scattering are however not negligible in calculations of the pion-deuteron scattering length using the multiple scattering series. Therefore it seems desirable to estimate the second-order shift to confirm the validity of our approach.

From Equations (3.2.22) the second order shift is given by

$$\Delta E^{(2)} = \sum_{i \neq 0} \frac{< \hat{\psi}_\pi^o \mid \delta \hat{V} \mid \hat{\psi}_\pi^i > < \hat{\psi}_\pi^i \mid \delta \hat{V} \mid \hat{\psi}_\pi^o >}{E_\pi^o - E_\pi^i} , \quad (3.6.1)$$

where we have introduced a complete set of states $\mid \hat{\psi}_\pi^i >$ with energies E_π^i of the model problem. These states are simply related to the usual atomic states $\mid \psi_\pi^i >$ of the pion bound to the c. m. of the deuteron by $\mid \hat{\psi}_\pi^i > = \hat{h}^{-\frac{1}{2}} (E_\pi^i) \mid \psi_\pi^i >$ and have energies $E_\pi^i = - \varepsilon_d - \varepsilon_\pi^i$ where ε_π^i is the Coulomb binding energy of the pion. The second-order energy shift should be small since the matrix elements

$$< \hat{\psi}_\pi^o \mid \delta \hat{V} \mid \hat{\psi}_\pi^i > = < \psi_\pi^o f_d \mid G_0 (T_\pi - t_c^o) G_0 \mid \psi_\pi^i f_d > \quad (3.6.2)$$

are of the order of $\sim$ eV and the energy denominators, $E_\pi^o - E_\pi^i = \varepsilon_\pi^i - \varepsilon_\pi^o$, are of the order $\sim$ KeV.

Replacing $E_\pi^{\,i}$ by $E_\pi^{\,1}$, the energy of the first excited state of the model pion-deuteron system and using closure, we have the standard bound

$$| \Delta E^{(2)} | \leq \frac{1}{E_\pi^{\,1} - E_\pi^{\,0}} \{ < \hat{\psi}_\pi^{\,0} \mid (\delta \hat{V})^2 \mid \hat{\psi}_\pi^{\,0} > - < \hat{\psi}_\pi^{\,0} \mid \delta \hat{V} \mid \hat{\psi}_\pi^{\,0} >^2 \}. \tag{3.6.3}$$

To estimate the second-order shift we have evaluated the right hand side of Equation (3.6.3) in the single scattering approximation for $\delta \hat{V}$.

$$\begin{aligned}(\delta \hat{V})^2 &= \hat{h}^{1/2} < f_d \mid G_0 (t_{\pi_1} + t_{\pi_2}) G_0 \mid f_d > \hat{h} < f_d \mid G_0 (t_{\pi_1} + t_{\pi_2}) \\ &\qquad G_0 \mid f_d > \hat{h}^{1/2} \\ &= \hat{h}^{1/2} < f_d \mid G_0 (t_{\pi_1} + t_{\pi_2}) G_0 \left(- E_\pi^{\,0} - \varepsilon_d - \frac{p^2}{2M} \right) t_{np} \left(- E_\pi^{\,0} - \frac{p^2}{2M} \right) \\ &\qquad G_0 (t_{\pi_1} + t_{\pi_2}) G_0 \mid f_d > \hat{h}^{1/2}. \end{aligned} \tag{3.6.4}$$

With the parameter set a, we obtained the bound 0.17 eV and for set b, 0.15 eV. These estimates seem to confirm the rapid convergence of the perturbation series.

Moreover, the modification of the unperturbed atomic state by the strong interaction is given to first order by

$$\mid \hat{\psi}_\pi > \; = \; \mid \hat{\psi}_\pi^{\,0} > \; + \sum_{i \neq 0} \frac{1}{E_\pi^{\,i} - E_\pi^{\,0}} \mid \hat{\psi}_\pi^{\,i} > < \hat{\psi}_\pi^{\,i} \mid \delta \hat{V} \mid \hat{\psi}_\pi^{\,0} > .$$

The admixture of excited Coulomb states is small, since

$$< \hat{\psi}_\pi^{\,i} \mid \delta \hat{V} \mid \hat{\psi}_\pi^{\,0} > \; \sim \text{ eV} \quad \text{and} \quad E_\pi^{\,i} - E_\pi^{\,0} \; \sim \text{ KeV}.$$

3.7 Conclusions

From Table 7, the calculated first-order energy shift, which includes multiple scattering of the pion on the nucleus is - 3.68 eV to

- 3.55 eV depending on the parameterization of the pion-nucleon interaction. The agreement between the calculated and present experimental measured shift ($-4.8^{+1.6}_{-2.0}$ eV) is quite good. The calculated energy-shift would increase slightly if we include (i) charge exchange type reaction $\pi^- p \rightarrow \pi^0 n$; (ii) P-wave contributions; we shall give rough estimates of these contributions later on. It is however important to note that because of the large cancellation in the iso-scalar part of the pion-nucleon scattering length, the first-order energy shift obtained from the exact expression Equation (3.3.4) is - 1.12 eV greater than that obtained from the scattering length approximation of DGBT., Equation (3.1.1). We interpret this contribution as the off-shell correction to the energy shift. When we neglect the momentum spread of the pionic wave function from the energy shift calculation the absolute values of the shifts for S11 and S31 channels become larger compared to those obtained from the exact evaluation of the corresponding energy shifts. This fact can be understood from the tabulated values of the weight function $F_{n_\alpha,d}(p_\alpha)$ for the respective channels. Similarly, the energy shifts obtained from the scattering length approximation for the S11 (S31) channel is greater (smaller) than that obtained from the Equation (3.3.16). This point can be explained from Figure 6 of the S11 and S31 channel propagators. From these sets of results we have been able to demonstrate the sensitivity of the energy shift to the momentum spread of the pion wave function and off-shall dependence of the pion-nucleon t-matrix. These factors have been ignored in DGBT's calculation. The importance of the off-shell behaviour of the pion-nucleon amplitude in the single scattering approximation which was previously noted in our earlier work (57) persists when the total

first-order shift is evaluated. The results of the Table 7 are slightly different from those of our published paper. There was one programming error in our earlier calculation (57). Bugg et al (43) have re-calculated the pion-nucleon scattering lengths from the low energy pion-nucleon angular distribution data using dispersion theory. They have found that the magnitude of the iso-scalar part of the pion-nucleon scattering length is really very small ($a_1 + 2a_3 = -0.014\ m_\pi^{-1}$). To determine the pion-deuteron scattering length, Bailey et al (30) have measured experimentally the strong interaction shift (ΔE_{1s}) of the ^1S level of the pion-deuteron system. Their method provides a check on the theoretical results obtained for pion-deuteron scattering at low energy. They derived the pion-deuteron scattering length following DGBT's formula

$$a_{\pi d} = a_0\ \Delta E_{1_S} / 4E_{1_S}$$

where a_0 is the pionic Bohr radius. They obtained

$$\Delta E_{1_S} \sim -4.8\ ^{+1.6}_{-2.0}\ \text{eV}\ .$$

The pion-deuteron scattering length can be written as

$$a_{\pi d} = a_{\pi d}^{\,o} + a_{\pi d}^{\,h} = -0.073\ ^{+.031}_{-.024}\ fm \quad \text{(experimental)},$$

where $$a_{\pi d}^{\,o} = \frac{m_\pi + m_N}{m_N + m_\pi/2}\ \left(\frac{2}{3}\ a_1 + \frac{4}{3}\ a_3\right).$$

The contribution of the higher order re-scattering ($a_{\pi d}^{h}$) of the pion to the pion-deuteron scattering length was first evaluated by Kolybasov and Kudryavtsev (KK; 5) summing the multiple scatter scattering series. They obtained $a_{\pi d}^{h} \sim -.047$ to $-.037$ fm. Afnan and Thomas (6) estimated $a_{\pi d}^{h}$ with the aid of Faddeev theory and found $a_{\pi d}^{h} \sim -.036$ fm The calculation of Kwon (58) provides an estimate of $a_{\pi d}^{h}$ about

$\sim$ - 0.045 fm. Recently authors Galimzianov et al (59) have calculated the energy shift and pion-deuteron scattering length from a variational technique and obtained the scattering length - 0.083 fm, and the energy shift - 6.09 eV. Therefore, the experimental measurement of the energy shift is of urgent interest. Unfortunately, the inherent error in Bailey et al's measurement of energy shift is too high to make any conclusive prediction. The fundamental importance of an accurate experimental determination of the energy shift is two fold :

(i) it will give independently, a bound on the pion-deuteron scattering length not relying on the approximations of the dispersion theory;

(ii) it will also provide a check to the convergence of a graph summing method in strong interaction of KK (5) and it will also assess the reliability of precise Faddeev calculations.

Recently Beer et al (30) are trying to determine the energy shift of the pion-deuteron system to an accuracy $\pm$ 0.2 eV using better experimental arrangements. This accurate result will give $a_{\pi d}$ or the energy shift to an accuracy required to test the importance of the theoretical calculation. Nevertheless, it would not be possible to separate out the off-shell contributions from the pion-deuteron scattering length if we use DGBT's formula to determine it.

The estimates of charge exchange and P-wave scattering to the energy shift can be fixed up in the following way. The contribution of the charge exchange scattering to the pion-deuteron scattering length is approximately (60)

$$a_{\pi d} = a_{\pi^- p \to \pi^0 n} \quad (< r^{-1} >_d) a_{\pi^0 n \to \pi^- p} \sim - .009\ 6 m_\pi^{-1} .$$

The total P-wave contribution is $\sim 0.006 m_\pi^{-1}$ (60). The net contribution of these two processes in the energy shift calculation will be ~ -1.12 eV. If we now add these contributions to the calculated energy shift, the total energy shift becomes -4.80 eV to -4.67 eV. We believe this is the appropriate result for the energy shift (theoretical).

The width of the ^1S level transition is very small (9) which implies that the absorption contribution will be negligible. Although the P-wave contribution is very important in pion-nucleus scattering it has a minor effect on $\Delta E^{(1)}$, as the pion orbits are large compared with the nucleus radius and a pion in a relative S-state to the nucleus, the S-wave interaction dominates. We also confirm that the iterative solution of $A^{J\,I}_{n_\beta, n_\alpha}$ converges rapidly as might be expected from the weakness of the pion-nucleon interaction (the second iteration gives matrix elements to an accuracy of better than $\sim 8\%$).

It appears from our calculations that the correction to the first-order shift from N-N re-scattering in intermediate states is small and insensitive to the choice of the pion-nucleon form factor. On the other hand, in multiple scattering calculations of $a_{\pi d}$ nucleon-nucleon scattering makes an appreciable contribution. To understand this apparent discrepancy it is necessary to consider carefully the differences between the present perturbative approach and the usual treatments. If Equation (3.2.10) is iterated, the terms of the usual multiple scattering series are obtained. However, we have not attempted to solve Equation (3.2.10) by calculating terms in the series solution, which would be the analogous procedure to that of the usual scattering length calculations. Instead using perturbative theory we have compared the solution of this equation with the solution of the Equation (3.2.13) which describes the solvable three-

body problem where the pion interacts with the deuteron only through the Coulomb potential at its c. m. Thus, although the higher-order terms in the perturbation $\delta \hat{V} = \hat{V} - \hat{V}^0$ include re-scattering of two nucleons, the effect of these terms cannot be compared directly with the multiple scattering terms associated with the iterative solution of Equation (3.2.13) which may converge slowly. In particular, the energy denominators of the perturbative series are much larger than the intermediate propagators in the multiple scattering expansion.

A necessary restriction on the form of the interaction in the present theory is separability of the nucleon-nucleon amplitude, which was introduced in order to reduce the three-body problem to an equivalent two-body problem. The form of the pion-nucleon interaction is, however, quite general. We can also use different types of pion-nucleon interaction potentials in our calculation.

Finally, we observe that the theoretical estimates of the energy shift, based either on the multiple scattering expansion, the Faddeev equations or the present perturbative theory are in reasonable agreement with experimental results. Indeed the naive impulse approximation gives a quite good estimate of the shift, for the reasons discussed by Fäldt (46). Nevertheless, our work does show that if the experimental value is refined further, as seems likely in the near future, theoretical analyses will require a more careful treatment of the relationship between the energy shift and the scattering length. Both the momentum spread of the pionic wave function and the overall off-shell dependence of the pion-nucleon amplitude should be taken into account properly.

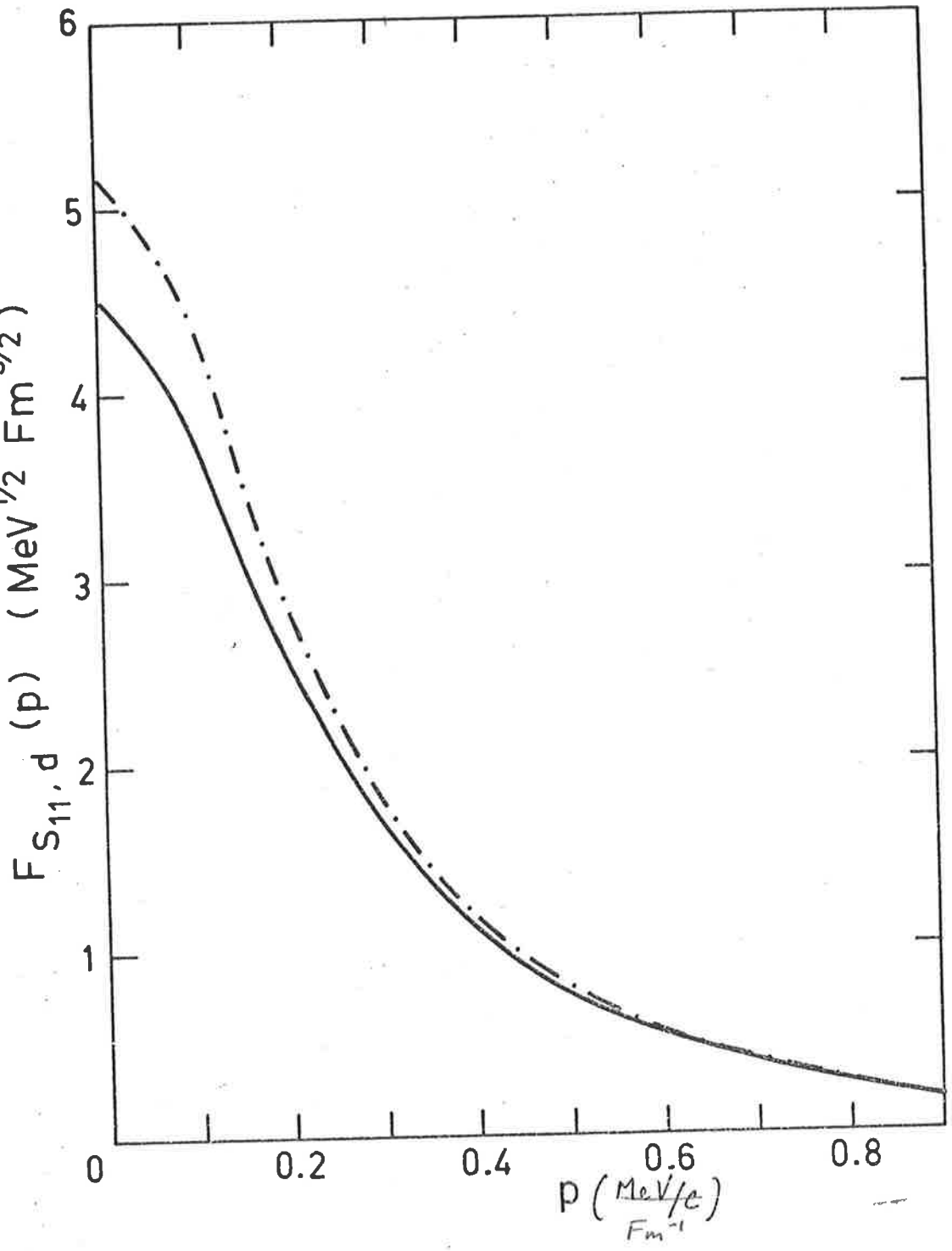


Fig. 4a Plots of the weight function $F_{S11,d}(p)$ with the parameter set a of Table 1.

p (MeV/c) Fm^{-1}	Equation (3.3.13) MeV$^{\frac{1}{2}}$ Fm$^{3/2}$	Equation (3.3.16) MeV$^{\frac{1}{2}}$ Fm$^{3/2}$
0.0	.4512 x 10^{-2}	.5143 x 10^{-2}
.1	.3816 x 10^{-2}	.4314 x 10^{-2}
.2	.2575 x 10^{-2}	.2890 x 10^{-2}
.3	.1675 x 10^{-2}	.1840 x 10^{-2}
.4	.1102 x 10^{-2}	.1198 x 10^{-2}
.5	.0751 x 10^{-2}	.0811 x 10^{-2}
.6	.0529 x 10^{-2}	.0568 x 10^{-2}
.7	.0384 x 10^{-2}	.0410 x 10^{-2}
.8	.0285 x 10^{-2}	.0304 x 10^{-2}
.9	.0216 x 10^{-2}	.0230 x .0^{-2}
1.0	.01667 x 10^{-2}	.0177 x 10^{-2}

TABLE 4a The values of the weight function $|F_{S11,d}(p)|$ for the parameter set a of Table 1.

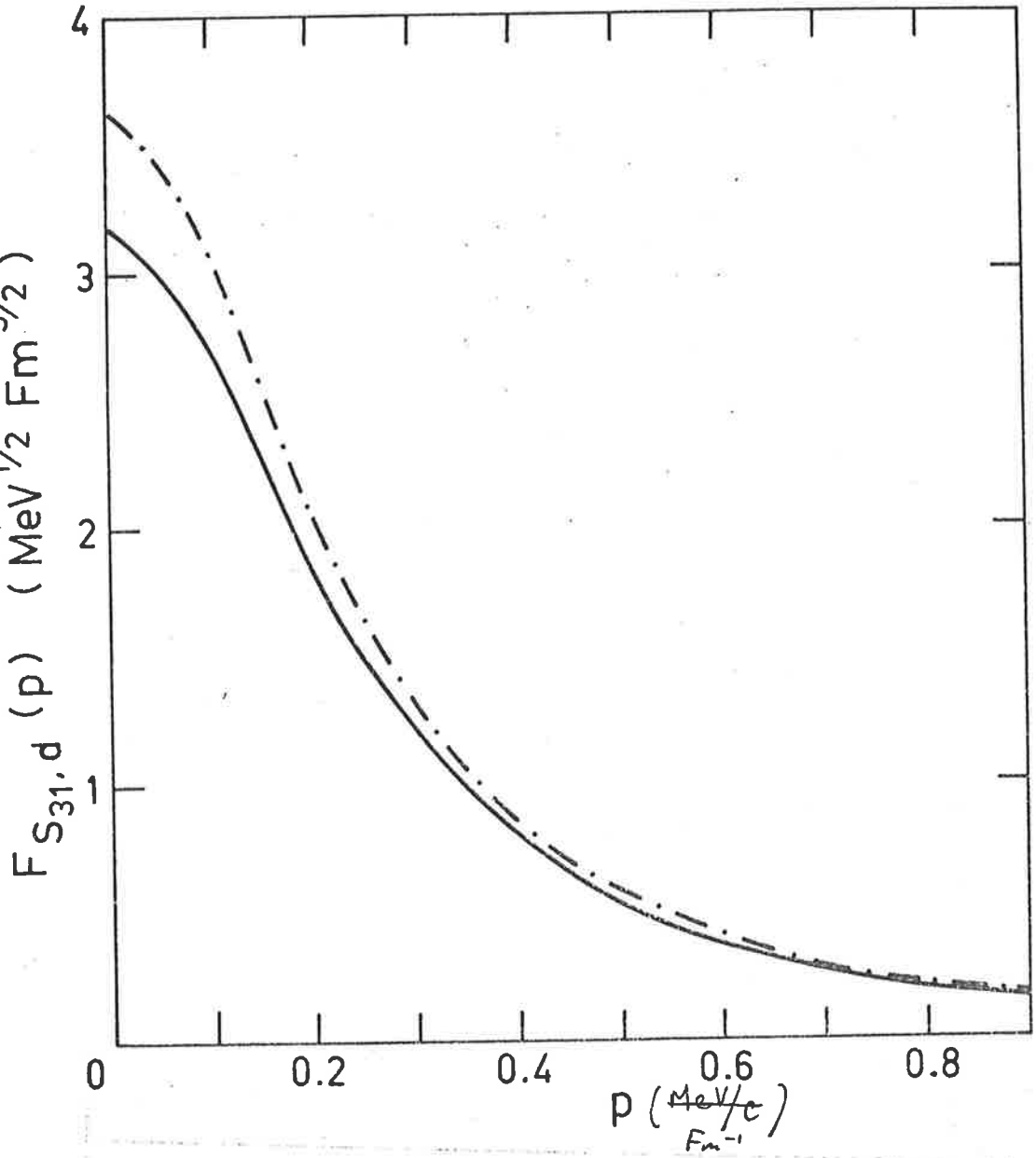


Fig. 4b Plots of the weight function $F_{S31,d}(p)$ with the parameter set a of Table 1.

p (MeV/c) Fm^{-1}	Equation (3.3.13) MeV$^{\frac{1}{2}}$ F$^{3/2}$	Equation (3.3.16) MeV$^{\frac{1}{2}}$ Fm$^{3/2}$
0.0	.3196 x 10^{-3}	.3644 x 10^{-3}
.1	.2708 x 10^{-3}	.3057 x 10^{-3}
.2	.1843 x 10^{-3}	.2049 x 10^{-3}
.3	.1191 x 10^{-3}	.1306 x 10^{-3}
.4	.0784 x 10^{-3}	.0849 x 10^{-3}
.5	.0535 x 10^{-3}	.0577 x 10^{-3}
.6	.0378 x 10^{-3}	.0405 x 10^{-3}
.7	.0275 x 10^{-3}	.0291 x 10^{-3}
.8	.0204 x 10^{-3}	.0217 x 10^{-3}
.9	.0155 x 10^{-3}	.0165 x 10^{-3}
1.0	.0120 x 10^{-3}	.0127 x 10^{-3}

TABLE 4b The values of the weight function $|F_{S31,d}(p)|$ for the parameter set a of Table 1.

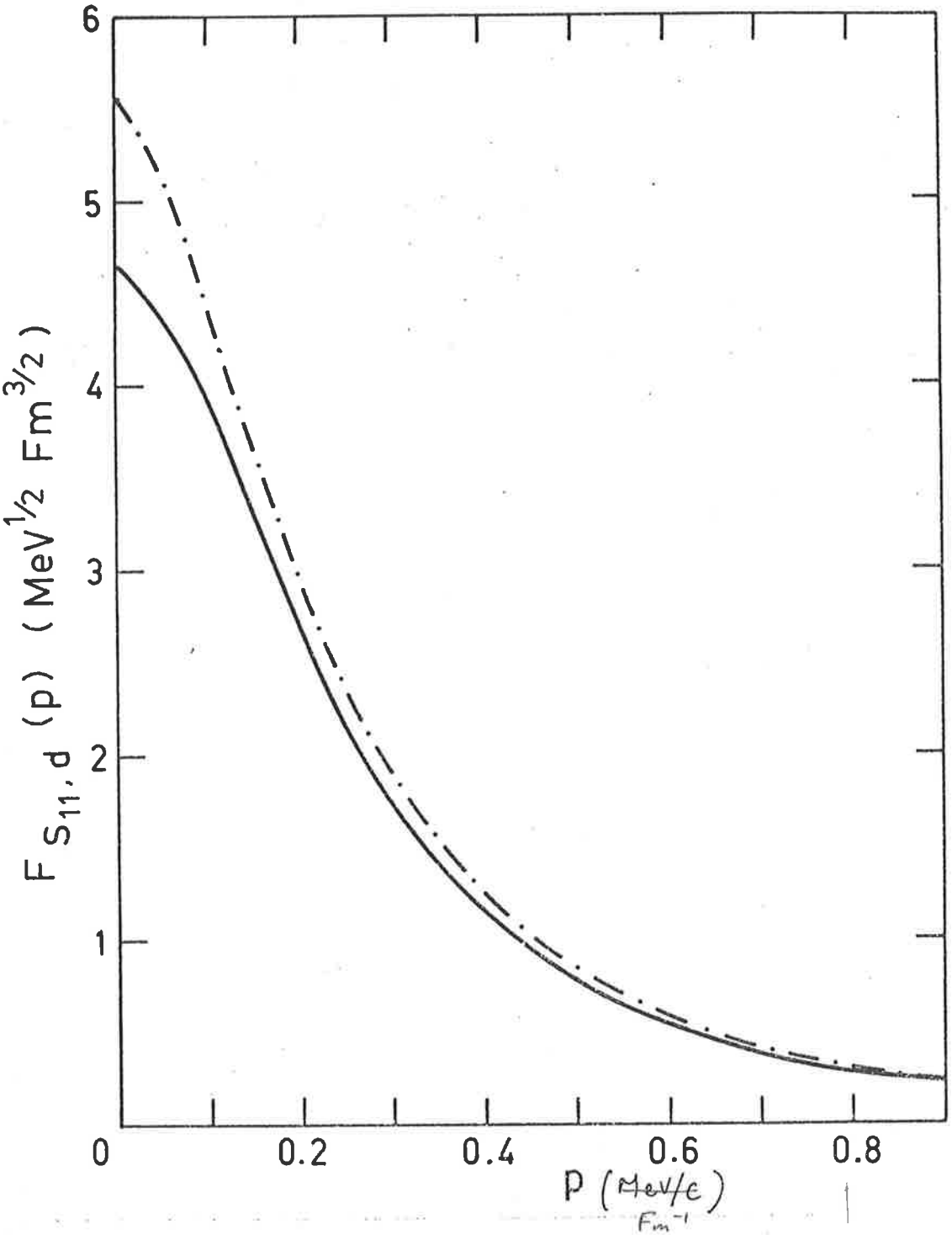


Fig. 5a Plots of the weight function $F_{S11,d}(p)$ with the parameter set b of Table 1.

p (MeV/c) Fm^{-1}	Equation (3.3.13) $MeV^{-\frac{1}{2}}$ $Fm^{3/2}$	Equation (3.3.16) $MeV^{-\frac{1}{2}}$ $Fm^{3/2}$
0	$.4644 \times 10^{-2}$	$.5313 \times 10^{-2}$
.1	$.3942 \times 10^{-2}$	$.4457 \times 10^{-2}$
.2	$.2681 \times 10^{-2}$	$.2985 \times 10^{-2}$
.3	$.1731 \times 10^{-2}$	$.1901 \times 10^{-2}$
.4	$.1138 \times 10^{-2}$	$.1238 \times 10^{-2}$
.5	$.0776 \times 10^{-2}$	$.08379 \times 10^{-2}$
.6	$.05469 \times 10^{-2}$	$.058751 \times 10^{-2}$
.7	$.0396 \times 10^{-2}$	$.0424 \times 10^{-2}$
.8	$.0294 \times 10^{-2}$	$.0314 \times 10^{-2}$
.9	$.0223 \times 10^{-2}$	$.02377 \times 10^{-2}$
1.0	$.0172 \times 10^{-2}$	$.0182 \times 10^{-2}$

TABLE 5a The values of the weight function $|F_{S11,d}(p)|$ for the parameter set b of Table 1.

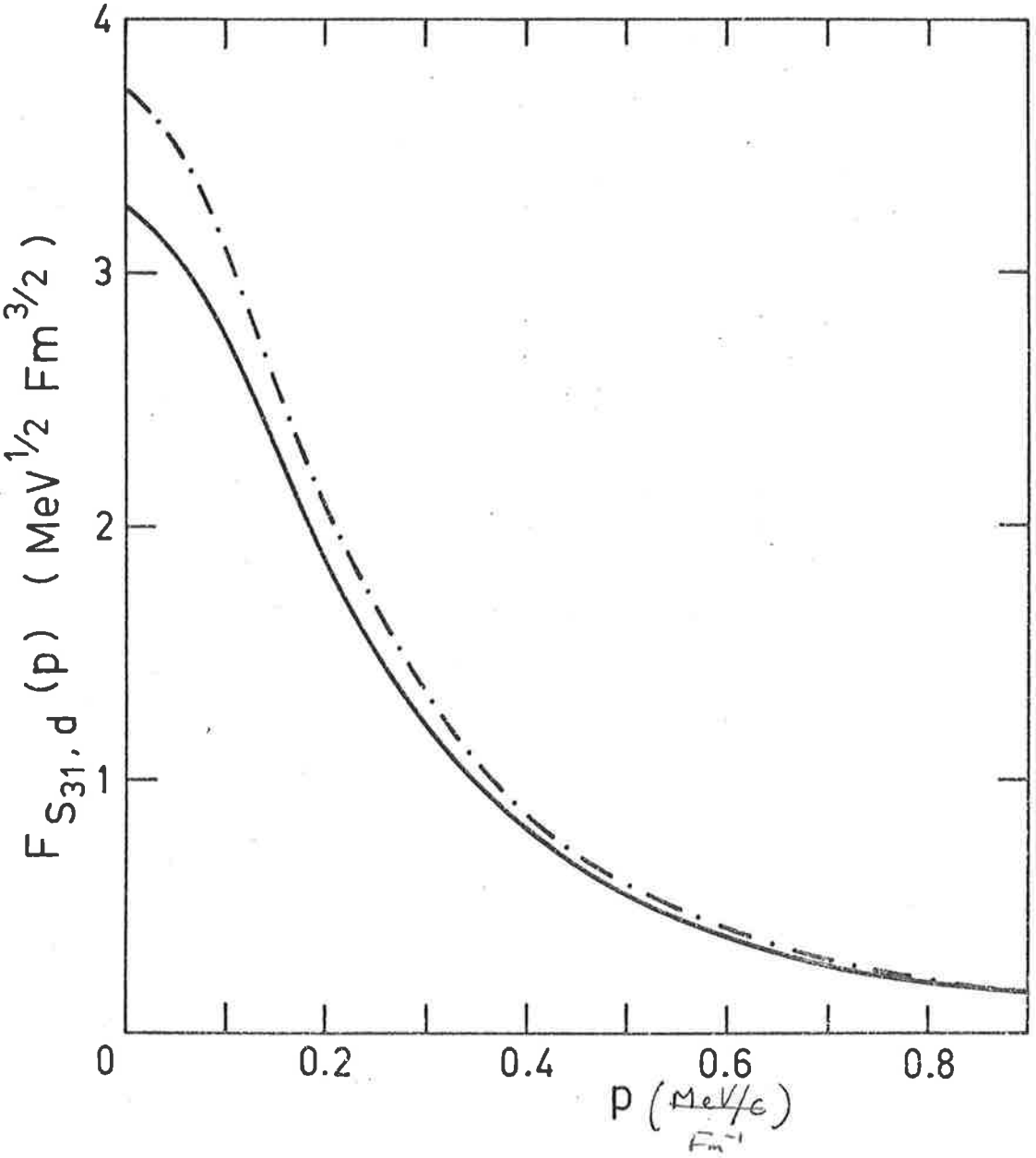


Fig. 5b Plots of the weight function $F_{S31,d}(p)$ with the parameter set b of Table 1.

p (~~MeV/c~~) Fm^{-1}	Equation (3.3.13) MeV$^{\frac{1}{2}}$ F$^{3/2}$	Equation (3.3.16) MeV$^{\frac{1}{2}}$ F$^{3/2}$
0	.3261 x 10^{-3}	.3727 x 10^{-3}
.1	.2769 x 10^{-3}	.3126 x 10^{-3}
.2	.1885 x 10^{-3}	.2095 x 10^{-3}
.3	.1218 x 10^{-3}	.1325 x 10^{-3}
.4	.08028 x 10^{-3}	.0871 x 10^{-3}
.5	.0548 x 10^{-3}	.0590 x 10^{-3}
.6	.0387 x 10^{-3}	.0414 x 10^{-3}
.7	.0281 x 10^{-3}	.0300 x 10^{-3}
.8	.02097 x 10^{-3}	.0222 x 10^{-3}
.9	.0159 x 10^{-3}	.0169 x 10^{-3}
1.0	.0123 x 10^{-3}	.01302 x 10^{-3}

TABLE 5b The values of the weight function $|F_{S31,d}(p)|$ for the parameter set b of Table 1.

E (MeV)	$f_1(0, 0, -E)$ (fm)	$f_3(o, 0, -E)$ (fm)
0	.24191	- .13409
3.0	.19275	- .20999
6.0	.186889	- .24887
9.0	.18390	- .27897
12.0	.18202	- .30410
15.0	.18069	- .32586
18.0	.17969	- .34513
21.0	.17891	- .36244
24.0	.17828	- .37818

TABLE 6 The off-shell dependence of the S11 and S31 channel pion-nucleon amplitudes calculated from the parameter set a, where f_1 and f_3 are the amplitudes for the channel S11 and S31 respectively.

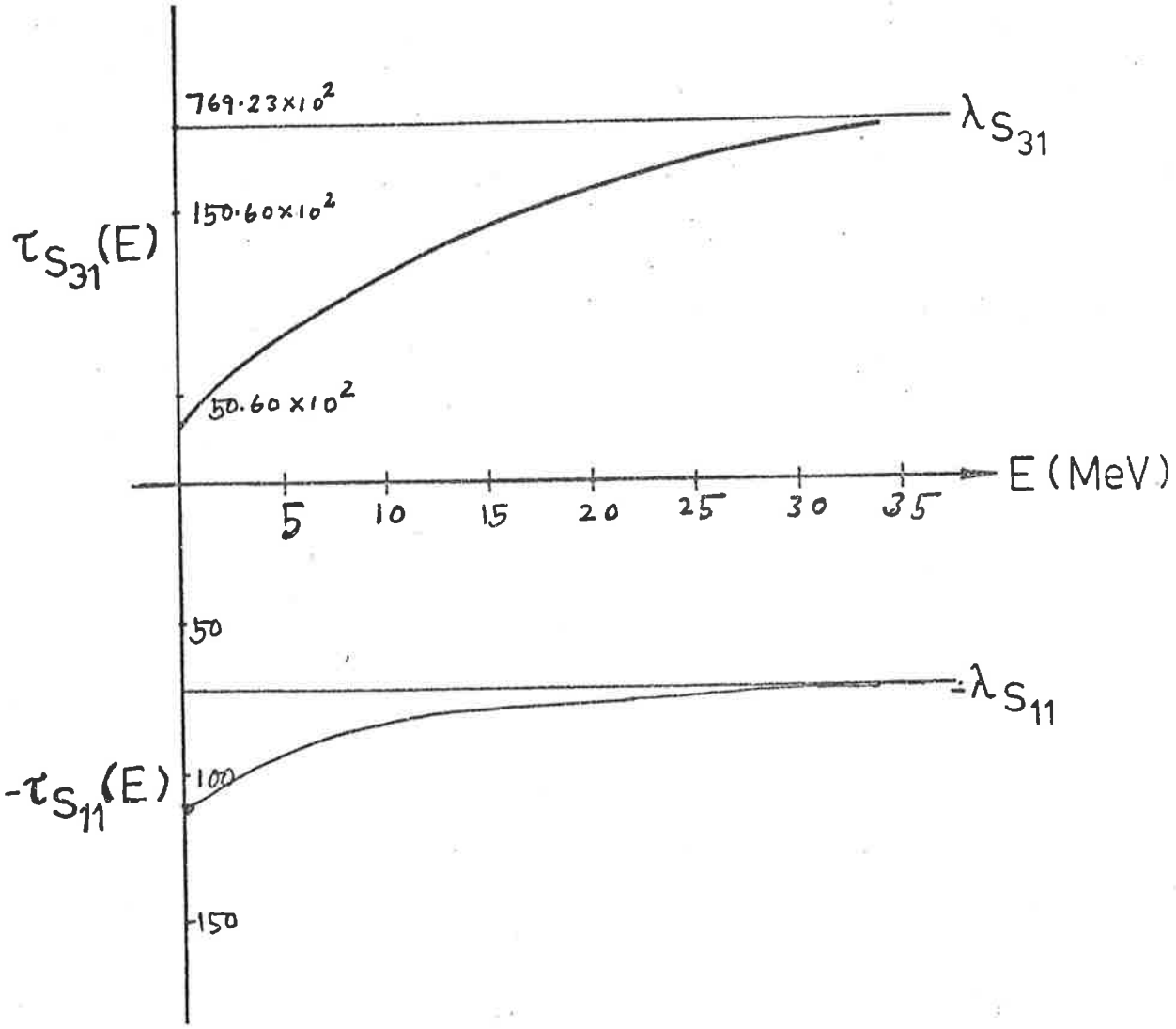


FIG. 6

The propagators of the S11 and S31 channel interaction for the different values of E.

	parameter set a			parameter set b		
	$\Delta E^{(1)''}$	$\Delta E^{(1)'}$	$\Delta E^{(1)}$	$\Delta E^{(1)''}$	$\Delta E^{(1)'}$	$\Delta E^{(1)}$
S11	+ 12.33	+ 11.69	+ 9.80	12.14	11.55	9.66
S31	- 13.66	- 14.49	- 12.22	- 13.38	- 14.19	- 11.97
SSC	- 1.33	- 2.80	- 2.40	- 1.24	- 2.64	- 2.31
MSC	- 1.12	- 1.42	- 1.28	- 1.07	- 1.37	- 1.24

TABLE 7 First-order energy shift in eV for the pion-nucleon parameter sets a and b given in Table 1, Chapter II. The total first-order shift $\Delta E^{(1)}$ is the sum of the single scattering contribution (SSC) of Equation 3.3.4, consisting of contributions from the S_{11} and S_{31} pion-nucleon channels, and the multiple scattering contribution (MSC) of Equation 3.3.6. The values $\Delta E^{(1)'}$ and $\Delta E^{(1)''}$ are approximate shifts obtained by neglecting the momentum spread of the pion bound state and additionally in the case of $\Delta E^{(1)''}$, the momentum dependence of the pion-nucleon amplitude.

CHAPTER IV

SECOND-ORDER OPTICAL POTENTIAL FOR THE PION-CARBON SYSTEM

4.1 Introduction

In this chapter we shall study the interaction of the pion with the carbon nucleus at intermediate energies by using an optical potential constructed from the two-body pion-nucleon interaction and the nuclear form factor. When this optical potential is introduced into the two-body Schrödinger equation, the solution of the equation correctly describes the multiple scattering of the pion from the various nucleons in the nucleus. We shall introduce some fundamental approximations to convert this many-body problem into an effective two-body pion-nucleus problem. In particular, the nucleus can be considered as "frozen" by means of the closure approximation on intermediate nuclear states. Then one calculates the pion scattering amplitude from a set of spatially fixed nucleons and averages these amplitudes over the nuclear form factor to determine the pion-nucleus scattering amplitude. Immediately one can make some corrections to this picture such as

(i) the two-body pion-nucleon amplitude used as input for the calculation may be fermi averaged;

(ii) the two-body pion-nucleon amplitude may be evaluated at a shifted energy to take partial account of the nucleon binding or the proper kinematics of the problem (61).

These two corrections are important since the pion-nucleon cross-section is very sensitive to the energy variation at intermediate energies. Again the existence of the pion-nucleon resonance has made the pion-nucleus scattering amplitude sensitive to the energy-variation of the pion-nucleon scattering amplitudes.

For the evaluation of the optical potential in our calculation, we shall use this "frozen" nucleus approximation. Otherwise the numerical calculations are too complicated to manage. But it is clear that this approximation neglects the nuclear dynamics during the intermediate stages of the scattering process and it is also difficult to give any precise estimate of the error incurred by this approximation.†

Unlike the first-order optical potential which is the product of the two-body pion-nucleon scattering amplitude and nuclear form factor, the second-order optical potential depends also on the pair correlation functions. In fact there are three types of pair correlations, viz.

(i) Centre-of-mass correlation due to the recoil of the nucleus during the scattering;

(ii) Pauli correlation because of the anti-symmetrization of nuclear wavefunctions;

(iii) The short-range dynamical correlation because of the nucleon-nucleon interaction.

The contributions of the first two-type correlations are small and have been neglected in our study. To include the short-range dynamical correlation we assume that the short-range part of the two-nucleon relative wave function of the independent particle model should be modified. This is achieved by suitably approximating the intrinsic two-body density function $\rho(\underline{r}_1, \underline{r}_2)$, which is the probability of observing a nucleon at point $\underline{r}_1$ if another nucleon is located at $\underline{r}_2$ (27).

The questions we would like to consider in our study of the pion-carbon system are the following:

† For this reason we cannot be sure that a significant part of the off-shell dependence is not discarded by making this approximation.

(i) What are the effects of including the second-order optical potential in the calculation of the differential cross-section ?

(ii) How does the differential cross-section change if we vary the nucleon-nucleon correlation length ?

(iii) Is the scattering cross-section sensitive to the off-shell behaviour of the two-body pion-nucleon t-matrix ?

Our calculation also confirms the observed (7, 27, 28) downward shift in the peak of the total cross-section of the pion-carbon scattering at intermediate energies. To answer these questions, we need information about the basic pion-nucleon and nucleon-nucleon interactions which are the building blocks for any pion-nucleus microscopic theory. As mentioned earlier, we have constructed separable models for the two-body interactions from the experimental data at different energies for all significant pion-nucleon reaction channels.

There are many models of pion-nucleus scattering, both above and below the resonance region. One of the more successful uses the multiple scattering theory of Watson (26) to write down the pion-nucleus optical potential in terms of the basic two-body interactions. Because it is better to use a theory valid over a wide energy region, rather than to simply extrapolate from either high or low energy models, we shall make physically reasonable approximations to consider the effects of the complicated nuclear structure of the target on our construction of the pion-nucleus optical potential.

In principle, we could try to solve the many-body Schrödinger equation for the pion-nucleus system, subject to the appropriate boundary

conditions, to determine the scattering amplitude. In practice this would be a very difficult procedure, since excited intermediate states are involved and the many-body effects caused by the target nucleus are poorly understood. However, we are primarily interested in the elastic scattering of pions by the nucleus, and this is basically a two-body reaction, even though the internal degrees of freedom of the target nucleus can be excited during the scattering process. Most of the nuclear structure effects can be included in the optical potential. We shall construct an approximate optical potential by using only part of the bound state information, such as nuclear density and nucleon-nucleon correlation functions. The aims in this chapter are

(i) to calculate the first-order optical potential $U^{(1)}$ and the second-order optical potential $U^{(2)}$ using the general Kerman-McManus-Thaler (KMT, 25) formulation;

(ii) to evaluate the differential cross-sections from $U^{(1)}$ and $U^{(1)} + U^{(2)}$ to estimate the importance of the nucleon-nucleon correlation;

(iii) to investigate systematically the off-shell dependence of the differential scattering cross-section using two sets of phase-equivalent potentials for the pion-nucleon interaction.

One of the important reasons to include the second-order optical potential in this calculation is the following. The contributions of the multiple scattering of pions from the carbon nucleus are significant at intermediate energies. The sum of all single pion-nucleon scatterings contributes $6\sigma(\pi^- n) + 6\sigma(\pi^- p)$ to the total cross-section, which is about twice the calculated cross-section at resonance (7).

Therefore, the double and higher order microscopic re-scattering processes

should be included and they actually cancel the large contribution obtained from the single scattering approximation. The first-order optical potential includes only the contributions of the single scattering processes. The solution of the Lippmann-Schwinger equation with $U^{(1)} + U^{(2)}$ includes a larger class of microscopic processes than those contained in the solution obtained from the first-order optical potential only. In principle the solution of the Lippmann-Schwinger equation with the second-order optical potential should give better results.

From the studies of Landau *et al* (7), Lee and Chakravorty (27) Wakamatsu (28) and Garcilazo (29) on pion-nucleus scattering at different energies, it is clear that the first-order optical potential is quite adequate for the qualitative description of the pion-nucleus elastic scattering at and above the resonance energy. The results are also consistent for the elastic differential-scattering cross-sections at small to moderate angles. It is also clear from their studies that at large angle scattering the differential cross-sections are very small and the momentum transfer is large, so the second-order optical potential should probably be included in the calculation. The success of the first-order optical potential model at intermediate energies is mainly due to the strong absorption in the resonance region, which actually makes the theoretical results to some extent model-independent (61). Below the resonance energy, where the effects of resonance are very weak, the predictions of these models are not as good, possibly because the models fail to take account of the inelastic contribution . When the kinetic energy of the pion is too low to excite the nucleus, the only source of inelasticity is associated with true absorption of pions, so the correct representation of the P11 channel, in which the absorption takes place, is very important. In practice the

optical potential model can only take absorption into account in a gross manner. For a better description of the real absorption one should look for a field theoretic approach. The optical potential calculated from the field theoretic approach shows some problems regarding the convergence of the higher-order terms of the optical potential series (62, 63). In our calculations we have considered two different models to represent the P11 interaction.

In Section 4.2, we introduce our work with a brief account of the formal theory of the optical potential, following the approach of Feshbach, Gal and Hüfner (FGH, 64) and Joachin (65). Next, in Section 4.3, we derive explicit formulae for the first-and second-order optical potentials. Following Landau *et al* (7), we shall express in Section 4.4, the pion-nucleon amplitudes in the reference frame of the centre of mass of the pion-nucleus system. These transformations are not unique. In Section 4.5 we shall present the separable model interactions for the P33 and P11 channels. The partial wave decompositions for the first-and second-order optical potentials are given in Sections 4.6 and 4.7 respectively. We solve the Lippmann-Schwinger equation for the reaction matrix R_ℓ (k', k, E) using the technique of Noyes and Kowalski(66) to determine the pion-nucleus differential scattering cross-sections.

Finally, in Section 4.8, we shall discuss the results of the differential scattering cross-sections evaluated from $U^{(1)}$ and $U^{(1)} + U^{(2)}$ to examine the importance of the nucleon-nucleon correlations, and the off-shell dependence of the cross-section will be appraised using two sets of pion-nucleon interaction potentials. Recently Lee and Chakravorty (27) and Wakamatsu (28) have calculated the second-order optical potential for the pion-Helium nucleus and subsequently have determined the pion-Helium differential cross-sections at different energies and scattering angles. Their models representing the pion-nucleon interactions are quite different

from those we have used in our calculations. They have also included the nucleon-nucleon correlation term and the contribution of the binding correction. Lee and Chakravorty have examined the influence of the variation of the P33 channel range parameter in pion-Helium scattering. Their calculational steps are also different from ours. Moreover our parameterizations for all pion-nucleon interaction channels are very consistent with the experimental results. We have investigated the pion-carbon scattering using two sets of phase equivalent potentials. The P11 channel has been included to compute the second-order optical potential. This channel contribution has not been considered in earlier calculations (27, 28).

4.2 Formal Theory for the Optical Potential

Watson's theory (26) of multiple scattering leads to a simple description of pion-nucleus scattering. The effects of all complicated interactions between the pion and target nucleus can be incorporated into an optical potential, through which the incident particle propagates. Once the optical potential has been determined, the scattering problem reduces to a one-body problem, namely, the scattering of a particle by a potential. However, the cost of this dramatic reduction from a many-body to a one-body problem is the introduction of a complex, non-local optical potential, whose determination is necessarily difficult. Therefore we must in practice work with an approximate model for the optical potential.

Following KMT (25 and 65), we may write the total hamiltonian of the system, made up of the incident pion and target nucleus, as

$$H = H_0 + V, \quad H_0 = H_N + K \quad , \tag{4.2.1}$$

where H_N and K stand for the hamiltonian of the target nucleus and kinetic energy of the incident pion, respectively. The potential V,

which may be considered as a perturbation, is the sum of all primary interactions between the pion and the nucleons of the nucleus.

$$V = \sum_{i}^{A} v_i . \tag{4.2.2}$$

The initial and final asymptotic states of the target nucleus plus the incident pion satisfy

$$\begin{aligned} H_0 \phi_a &= E_a \phi_a \\ H_0 \phi_b &= E_b \phi_b \ , \end{aligned} \tag{4.2.3}$$

where

$$\phi_a = \Phi_{a,\ \underline{k},\ \underline{\nu}_\pi = 0}\ e^{i\,\underline{K}.\underline{R}}\ \psi_{o,\ \underline{\nu}_T}\ (\underline{r}_1,\ \underline{s}_1,\ \underline{r}_2,\ \underline{s}_2; \ldots . \underline{r}_A,\ \underline{s}_A), \tag{4.2.4a}$$

and

$$\phi_b = \Phi_{b,\underline{k}',\ \underline{\nu}_\pi = 0}\ e^{i\,\underline{K}'.\,\underline{R}}\ \psi_{o,\ \underline{\nu}_T'}(\underline{r}_1,\ \underline{s}_1;\ \ \underline{r}_2,\ \underline{s}_2; \ldots\ \underline{r}_A,\ \underline{s}_A). \tag{4.2.4b}$$

We have assumed that there are A elementary scatterers inside the nucleus and $\underline{r}_1, \underline{r}_2, \ldots . \underline{r}_A$ are their spatial and $\underline{s}_1, \underline{s}_2, \ldots \underline{s}_A$ are their spin co-ordinates respectively. The target nucleus initially has momentum $\underline{K}$, spin $\underline{\nu}_T$, and bound state wave function $\psi_{o,\ \underline{\nu}_T}$. The corresponding quantities in the final state are $\underline{K}'$, $\underline{\nu}'_T$, and $\psi_{o,\ \underline{\nu}'_T}$. $\underline{R}$ is the centre-of-mass co-ordinate and $\Phi_{a,\underline{k},\underline{\nu}_\pi}$ and $\Phi_{b,\underline{k}',\underline{\nu}_\pi}$ are the wave functions for the incident and scattered pion with momenta $\underline{k}$ and $\underline{k}'$ respectively. The initial and final energies of the system are

$$\begin{aligned} E_a &= E_\pi(k) + \varepsilon_0 + \frac{K^2}{2M_T} \\ E_b &= E_\pi(k') + \varepsilon_0 + \frac{K'^2}{2M_T} \ , \end{aligned} \tag{4.2.5}$$

where ε_0 is the internal ground state energy, M_T is the mass of the target and E_π is the pion kinetic energy.

The pion-nucleus scattering state (ψ_a^+) is given by (65)

$$\psi_a^+ = \phi_a + \frac{1}{E_a^+ - H_o} V \psi_a^+ . \tag{4.2.6}$$

This many-body scattering state ψ_a^+ contains information about all processes between the pion and target nucleus. In fact the complete scattering solution ψ_a^+ embodies the description of elastic, inelastic absorption, and production reactions.

The elastic scattering component of the total many-body scattering state (ψ_a^+) is

$$\psi_{e,a}^+ = \Pi_0 \psi_a^+, \tag{4.2.7}$$

where we define Π_0 to be the projection operator on the state $\psi_{o,\underline{\nu}_T}$,

$$\Pi_0 = \sum_{\underline{\nu}_T'} | \psi_{o,\underline{\nu}_T'} > < \psi_{o,\underline{\nu}_T'} | . \tag{4.2.8}$$

The elastic scattering state $\psi_{e,a}^+$ satisfies the integral equation

$$\psi_{e,a}^+ = \phi_a + \frac{1}{E_a^+ - H_o} \tau_c \phi_a, \tag{4.2.9}$$

where τ_c is the corresponding elastic scattering component of the total scattering operator T.

We now define an operator F which reconstructs the total scattering state out of its elastic scattering component by

$$\psi_a^+ = F \psi_{e,a}^+ . \tag{4.2.10}$$

Multiplying by the operator Π_0 from the left of Equation (4.2.6) and using Equation (4.2.7), we have

$$\psi^+_{e,a} = \phi_a + \frac{1}{E_a^+ - H_o} \tilde{V}_{opt} \psi^+_{e,a} , \qquad (4.2.11)$$

where $$\tilde{V}_{opt} \psi^+_{e,a} = \Pi_0 V F \psi^+_{e,a}$$

or

$$\tilde{V}_{opt} = < 0 | V F | 0 > , \qquad (4.2.12)$$

with $$| 0 > \equiv \psi_{o,\underline{\nu}_T} .$$

For an explicit representation of the optical potential $\tilde{V}_{opt}$, we must know the exact structure of the operator F. Combining Equations (4.2.6), (4.2.10) and (4.2.11) we write

$$\begin{aligned} F \psi^+_{e,a} &= \psi^+_{e,a} - \frac{1}{E_a^+ - H_o} \tilde{V}_{opt} \psi^+_{e,a} \\ &\quad + \frac{1}{E_a^+ - H_o} V F \psi^+_{e,a} \\ &= \psi^+_{e,a} - \frac{1}{E_a^+ - H_o} \Pi_0 V F \psi^+_{e,a} + \frac{1}{E^+ - H_o} V F \psi^+_{e,a} \\ &= \{ 1 + \frac{1}{E_a^+ - H_o} (1 - \Pi_0) V F \} \psi^+_{e,a} . \end{aligned} \qquad (4.2.13)$$

Therefore

$$F = 1 + \frac{1}{E_a^+ - H_o} (1 - \Pi_0) V F. \qquad (4.2.14)$$

On combining Equations (4.2.12) and (4.2.14), we obtain the general form of the optical potential

$$\begin{aligned} \tilde{V}_{opt} &= < 0 | V | 0 > + < 0 | V \frac{1}{E_a^+ - H_o} (1 - \Pi_0) V | 0 > \\ &\quad + < 0 | V \frac{1}{E_a^+ - H_o} (1 - \Pi_0) V \frac{1}{E_a^+ - H_o} (1 - \Pi_0) V | 0 > + \ldots \end{aligned} \qquad (4.2.15)$$

The appearance of the pion-nucleus potential $V = \sum_i^A v_i$ in the optical potential is a great obstacle to its practical applications. Our intention is to re-write Equation (4.2.15) in terms of the two-body t-matrix. To achieve this aim, we introduce an auxiliary equation

$$\tau_j = v_j + v_j \frac{1}{E_a^+ - H_o} (1 - \Pi_0)\ \tau_j \ ; \tag{4.2.16}$$

and the operator F, in terms of Watson's equation, is

$$F = 1 + \sum_j \frac{1}{E_a - H_o} (1 - \Pi_0)\ \tau_j F_j \ , \tag{4.2.17}$$

with

$$F_j = 1 + \frac{1}{E_a^+ - H_o} \sum_{k(\neq j)=1}^{A} (1 - \Pi_0)\ \tau_k F_k \ . \tag{4.2.18}$$

Introducing Equation (4.2.17) into the definition of F in Equation (4.2.14) we write

$$F = 1 + \frac{1}{E_a^+ - H_o} (1 - \Pi_0)\ V + \frac{1}{E_a^+ - H_o} (1 - \Pi_0) V \frac{1}{E_a^+ - H_o} (1 - \Pi_0) \sum_{j=1}^{A} \tau_j F_j . \tag{4.2.19}$$

Now, it is easy to see that

$$\begin{aligned} \sum_j^A \tau_j F_j &= \sum_j^A v_j + \sum_j^A (\tau_j - v_j) F_j + \sum_j v_j (F_j - 1) \\ &= V + \sum_j^A v_j \frac{1}{E_a^+ - H_o} (1 - \Pi_0)\ \tau_j F_j \\ &\quad + \sum_j^A \sum_{k(\neq j)1}^{A} v_j \frac{1}{E_a^+ - H_o} (1 - \Pi_0)\ \tau_k F_k \\ &= V + \sum_j^A v_j \frac{1}{E_a^+ - H_o} (1 - \Pi_0) \sum_{k=1}^{A} \tau_k F_k , \\ &= VF \end{aligned} \tag{4.2.20}$$

where the Equations(4.2.16) and (4.2.18) have been used to derive Equation (4.2.20). It is straightforward to show from Equations (4.2.19) and (4.2.20) that

$$F = 1 + \frac{1}{E_a^+ - H_o}(1 - \Pi_0)V \{ 1 + \frac{1}{E^+ - H_o}(1 - \Pi_0)VF\}$$

$$= 1 + \frac{1}{E_a^+ - H_o}(1 - \Pi_0)VF \quad . \qquad (4.2.21)$$

Equation (4.2.21) explains thechoice of the auxiliary Equation (4.2.16). Substitution of Equation (4.2.20) into the definition of the optical potential, in Equation (4.2.12) yields

$$\tilde{V}_{opt} = < 0 \mid \sum_j^A \tau_j F_j \mid 0 > \quad . \qquad (4.2.22)$$

Now, Equation (4.2.22) is exact. But the coupled equations satisfied by F_j are very complicated, because the Green's function and the scattering operator τ_j are many-body operators and carry information about the structure of the target nucleus. We assume in our calculation that

(i) the pion-nucleon scattering matrix may be equated with that for scattering on the unbound nucleon t_j (i.e. impulse approximation) and

(ii) the energy transferred to the target at any stage during the multiple scattering process is negligible compared with the incident energy.

As mentioned earlier, the second approximation strongly influences the whole calculation.

On the basis of these assumptions we may set

$$\tau_j \simeq t_j \qquad (4.2.23)$$

and the equation for the optical potential becomes

$$\tilde{V}_{opt} = < 0 \mid \sum_j t_j F_j \mid 0 > , \qquad (4.2.24)$$

with

$$F_j = 1 + \frac{1}{E^+ - H_o} \sum_{k(\neq j)1}^{A} (1 - \Pi_0)\ t_k F_k . \qquad (4.2.25)$$

The validity of the first approximation is justifiable, if the energy of the incident particle is much greater than the average target particle binding energy. It is a quite good approximation for the pion-nucleus scattering at intermediate energies. We may imagine the pion to be scattered at a point with very high velocity, while the shock wave carrying the nuclear excitation will proceed from the same point at a much lower velocity. Before the meson reaches its next interaction point it will overrun the nuclear excitation and will find the medium again in its ground state. Therefore we may omit the effects of nuclear excitation from the argument of t-matrices (26). In this derivation, the incident particle is not identical with the bound target particles, but this condition may be relaxed with little modification to the structure of the optical potential equation (67).

4.3 Explicit Expressions for the First-and Second-Order Optical Potentials

Following the method of Feshbach - Gal and Hüfner (FGH, 64) or re-arranging the Equation (4.2.24) slightly the first-and second-order optical potentials may be written as

$$U(E) = U^{(1)}(E) + U^{(2)}(E) , \tag{4.3.1}$$

where

$$U^{(1)}(E) = (A - 1) \langle \psi_{o,\underline{\nu}_T} | \frac{1}{A} \sum_{i}^{A} t_i(\omega_0) | \psi_{o,\underline{\nu}_T} \rangle \tag{4.3.2}$$

and

$$U^{(2)}(E) = (A-1)^2 \left[\frac{1}{A(A-1)} \sum_{i \neq j} \langle \psi_{o,\underline{\nu}_T} | t_i(\omega_0) \frac{1}{E^+ - K_\pi - \bar{\varepsilon} - \bar{U}_0} t_j(\omega_0) | \psi_{o,\underline{\nu}_T} \rangle \right.$$

$$\left. - \langle \psi_{o,\underline{\nu}_T} | \frac{1}{A} \sum_{j}^{A} t_j(\omega_0) | \psi_{o,\underline{\nu}_T} \rangle \frac{1}{E^+ - K_\pi - \bar{\varepsilon} - \bar{U}_o} \langle \psi_{o,\underline{\nu}_T} | \frac{1}{A} \sum_{i}^{A} t_i(\omega_0) | \psi_{o,\underline{\nu}_T} \rangle \right] . \tag{4.3.3}$$

We use the closure approximation for the evaluation of the second-order optical potential. In fact we replace the original energy denominator $E^+ - K_\pi - \varepsilon_j - U_{j,j}$ by an average value $E^+ - K_\pi - \bar{\varepsilon} - \bar{U}_o$ which is independent of the intermediate nuclear states (27, 64). The term $\bar{\varepsilon}$ is the mean nuclear excitation energy and $\bar{U}_o$ is the average optical potential operator. K_π is the pion kinetic energy operator.

The free pion-nucleon operator $t_j(\omega_0) \equiv t_{\pi N}(\omega_0)$ is defined by

$$t_j(\omega_0) = v_j + v_j \frac{1}{\omega_0^+ - K_\pi - K_N} ,$$

where K_N is the nucleon kinetic energy operator and ω_0 is the pion-nucleon scattering energy. The choice of ω_0 is model dependent and arbitrary to some extent. Our particular choice is presented in Section 4.4.

Let us now focus on the first-order optical potential given in Equation (4.3.2). The spin-flip and spin-isospin flip parts of the two-

body pion-nucleon scattering amplitude do not contribute to the pion-nucleus optical potential for nuclei with zero total spin. The pion-nucleus scattering process is represented by the following diagram.

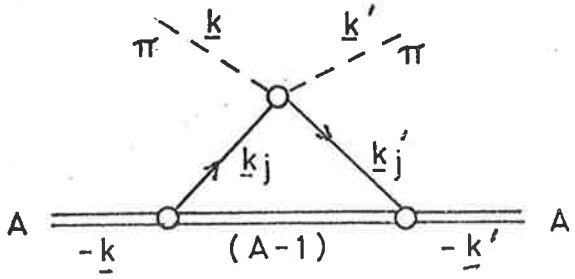


FIG. 7

For the elastic scattering of a pion from a nucleus, the standard form of the first-order optical potential is

$$\langle \underline{k}' \mid U^{(1)}(E) \mid \underline{k} \rangle$$

$$= \frac{(A-1)}{A} \sum_j \int d\underline{k}_j \, d\underline{k}'_j \, \delta(\underline{k} + \underline{k}_j - \underline{k}' - \underline{k}'_j)$$

$$\int d\underline{r}_j \, d\underline{r}'_j \, e^{-i\underline{k}_j \cdot \underline{r}_j} \, e^{i\underline{k}'_j \cdot \underline{r}_j'} \, \rho(\underline{r}_j', \underline{r}_j) \langle \underline{k}', \underline{k}'_j | t_j(\omega_0) | \underline{k}, \underline{k}_j \rangle \, , \tag{4.3.4}$$

where

$$\rho(\underline{r}_j', \underline{r}_j) = \int \psi^*_{o,\nu_T}(\underline{r}_j, \underline{r}_2 \ldots \underline{r}_A) \, \psi_{o,\nu_T}(\underline{r}_j, \underline{r}_2 \ldots \underline{r}_A) \, d\underline{r}_2 \ldots d\underline{r}_A . \tag{4.3.5}$$

$\underline{k}_j$ and $\underline{k}'_j$ are the momenta of the j th nucleon of the target nucleus before and after the collision. After integration with respect to $\underline{k}'_j$ Equation (4.3.4) becomes

$$= \frac{(A-1)}{(2\pi)^3} \int d\underline{k}_1 \, d\underline{r}_1 \, d\underline{r}'_1 \, e^{-i\underline{k}_1 \cdot (\underline{r}-\underline{r}'_1)} \, e^{i(\underline{k}-\underline{k}') \cdot \underline{r}'_1} \, \rho(\underline{r}_1, \underline{r}'_1)$$

$$\langle \underline{k}', \underline{k}_1 - \underline{q} \mid t(\omega_0) \mid \underline{k}, \underline{k}_1 \rangle \tag{4.3.6}$$

where $\underline{q} = \underline{k}' - \underline{k}$ is the momentum transferred by the pion in scattering from momentum $\underline{k}$ to momentum $\underline{k}'$. We now introduce a crucial assumption, namely that we ignore the dependence of the pion-nucleon scattering amplitude on $\underline{k}_1$, the momentum of the struck nucleon. This implies that the kinetic energy of the target nucleon is negligible compared with that of the incident pion so we may extract

$$< \underline{k}', \underline{k}_1 - \underline{q} \mid t\ (\omega_0) \mid \underline{k}, \underline{k}_1 >$$

from the integral sign and evaluate it at some average nucleon momentum, $\underline{k}_1 = \underline{k}_0\ (\approx \underline{k}/A)$. This procedure is known as the factorization approximation. The momentum $\underline{k}_1$ then appears only in the factor $\exp(- i\underline{k}_1.(\underline{r} - \underline{r}'))$ so the integration with respect to $\underline{k}_1$ yields

$$(2\pi)^3\ \delta\ (\underline{r}_1 - \underline{r}'),$$

and Equation (4.3.4) becomes

$$\langle \underline{k}' \mid U^{(1)}(E) \mid \underline{k} \rangle = (A-1)\ < \underline{k}', \underline{k}_0 - \underline{q} \mid t\ (\omega_0) \mid \underline{k}, \underline{k}_0 > \int d\underline{r} e^{i\,\underline{q}.\underline{r}} \rho(\underline{r})$$

$$= (A-1)\ < \underline{k}', \underline{k}_0 - \underline{q} \mid t\ (\omega_0) \mid \underline{k}, \underline{k}_0 > \rho(\underline{q}). \qquad (4.3.7)$$

Equation (4.3.7) may be generalised and for any nuclear state $\mid \psi_\beta >$

$$< \psi_\beta, \underline{k}' \mid \frac{1}{A} \sum_j t_j\ (\omega_0) \mid \psi_{o,\nu_T}, \underline{k} >$$

$$= F_{\beta,o}\ < \underline{k}', \underline{k}_0 - \underline{q} \mid t\ (\omega_0) \mid \underline{k}, \underline{k}_0 > , \qquad (4.3.8)$$

where $F_{\beta,0}$ is the appropriate nuclear form factor. For Gaussian type nuclear ground state wave functions, the appropriate value of $\underline{k}_0$ is

$$\underline{k}_0 = - \frac{\underline{k}}{A} + \frac{\underline{q}\ (A-1)}{A} \qquad (4.3.9)$$

The first-order optical potential in terms of the neutron and proton

t-matrices for a spherically symmetric nucleus becomes

$$< \underline{k}' \mid U^{(1)}(E) \mid \underline{k} >$$

$$= (A-1)\ \rho(\underline{q})\ < \underline{k}',\ \underline{k}_0 - \underline{q} \mid \frac{N}{A}\, t_{\pi N}(\omega_0) + \frac{Z}{A}\, t_{\pi p}(\omega_0) \mid \underline{k},\ \underline{k}_0 >$$

$$= (A-1)\ \rho(\underline{q})\ < \underline{k}'\ \underline{k}_0 - \underline{q} \mid (\frac{2Z}{3A})\ t_{1/2}(\omega_0) + (\frac{Z+3N}{3A})\ t_{3/2}(\omega_0) \mid \underline{k},\ \underline{k}_0 > ,$$

(4.3.10)

where we have written the pion-nucleon t-matrices in terms of the isospin 1/2 and 3/2 components.

4.4 Transformation of the Pion-Nucleon Scattering Amplitude from Pion-Nucleon Co-ordinates to Pion-Nucleus Co-ordinates

The frame of reference for the Equation (4.3.10) is the frame attached to the centre-of-mass of the pion-nucleus system, and so the two-body pion-nucleon amplitudes appearing in the equations must be expressed in these co-ordinates. The experimental data for the pion-nucleon scattering are usually given in the frame attached to the centre-of-mass (c.m.) of the pion-nucleon system, so we need the transformation relating the amplitudes in these two frames. There is a difficulty here. The on-shell amplitudes are translationally invariant, because energy and momentum are conserved in the reaction, and so the transformation is uniquely determined by Poincare' invariance. For the off-shell amplitudes, translation invariance is lost and it is not clear whether the transformation between the frames is unique or whether any non-uniqueness has observable effects. We will simply extend the transformation from on-shell to off-shell amplitudes.

If we assume Lorentz invariance of the transition probabilities, the required transformation is (37)

$$< \underline{k}'\ \underline{k}_0 - \underline{q} \mid t(\omega_0) \mid \underline{k},\ \underline{k}_0 > = \gamma\ < \underline{\kappa}' \mid \tilde{t}(\tilde{\omega}_0) \mid \underline{\kappa} > , \qquad (4.4.1)$$

where $\underline{\kappa}'$ and $\underline{\kappa}$ are the relative momenta in the pion-nucleon c.m. system and ω_0 is the on-shell pion-nucleon scattering energy in the pion-nucleus c.m. system. Thus, following Landau *et al* (7)

$$\omega_0 = E_\pi(k_0) + E_N(k_0/A) ,$$

where $$E_\pi = \sqrt{k_0^2 + m_\pi^2} \quad , \quad E_N(k_0) = \sqrt{k_0^2 + m_N^2} \quad . \tag{4.4.2}$$

Because the square of the total centre-of-mass energy is invariant, we may relate the corresponding on-shell energy $\tilde{\omega}_0$ and the on-shell momentum $\underline{\kappa}_0$ in the pion-nucleon c.m. system by

$$S \doteq \tilde{\omega}_0^{\,2} = (E_\pi(\kappa_0) + E_N(\kappa_0))^2$$

$$= (E_\pi(k_0) + E_N(k_0/A))^2 - k_0^2 (1 - \frac{1}{A})^2 . \tag{4.4.3}$$

For on-shell scattering γ is given by

$$\gamma = E_\pi(\kappa_0)\, E_N(\kappa_0)/E_\pi(k_0)\, E_N(k_0/A), \tag{4.4.4}$$

and the appropriate off-shell generalization of γ is

$$\gamma = \left(\frac{E_\pi(\kappa)\, E_\pi(\kappa')\, E_N(\kappa)\, E_N(\kappa')}{E_\pi(k)\, E_\pi(k')\, E_N(k/A)\, E_N(k'/A)}\right)^{\frac{1}{2}} . \tag{4.4.5}$$

We shall use Equation (4.4.3) to determine the pion-nucleon off-shell momenta, $\underline{\kappa}'$ and $\underline{\kappa}$ from the corresponding off-shell values $\underline{k}'$ and $\underline{k}$ in the pion-nucleus centre of mass system.

Simiarly the invariance of the square of the four momentum transfer $t = (P_\pi^{\text{initial}} - P_\pi^{\text{final}})^2$ suggests the appropriate transformation relating the scattering angles in the c.m. of the pion-nucleon system and the pion-nucleus system. For on-shell scattering, we have

$$\cos\theta_{\pi N} = \frac{\kappa_0^2 - k_0^2}{\kappa_0^2} + \frac{k_0^2}{\kappa_0^2} \cos\theta_{\pi\text{nucleus}} . \tag{4.4.6}$$

A corresponding off-shell generalization is

$$\cos\theta_{\pi N} = \frac{E_\pi(\kappa')\,E_N(\kappa) - E_\pi(k)\,E_\pi(k')}{\kappa\kappa'} + \frac{k\,k'}{\kappa'\kappa}\cos\theta_{\pi\text{nucleus}}$$

or $$P_\ell(\cos\theta_{\pi N}) = P_\ell(a + b\cos\theta_{\pi\text{nucleus}}) = \sum_{\ell' \leq \ell} d_{\ell\ell'} P_{\ell'}(\cos\theta_{\pi\text{nucleus}}). \qquad (4.4.7)$$

Therefore it is clear from Equation (4.4.7) that the P-wave scattering in the pion-nucleon c.m. influences $\ell' = 0, 1$ waves of the pion-nucleon scattering amplitude related to the pion-nucleus c.m. frame. The values of the co-efficient $d_{\ell\ell'}$ for different values of ℓ and ℓ' are given in Table 8 .

TABLE 8 (Ref 7)

The Angle transformation Coefficients $d_{\ell\ell'}$ for Legendre Polynomials

ℓ \ ℓ'	0	1	2
0	1	0	0
1	a	b	0
2	$\frac{(3a^2+b^2-1)}{2}$	3ab	b^2

4.5 P33 and P11 channel interactions

We present here two simple separable models to describe accurately the on-shell data, namely the phase shifts and the scattering lengths of the P33 and P11 channel interactions. First we shall consider the P33 channel.

It is very important to reproduce the resonance channel accurately for the calculation of pion-nucleus scattering. Assuming Breit-Wigner type form for resonance amplitude, one may well approximate the P33 channel in the neighbourhood of the resonance. But the Breit-Wigner

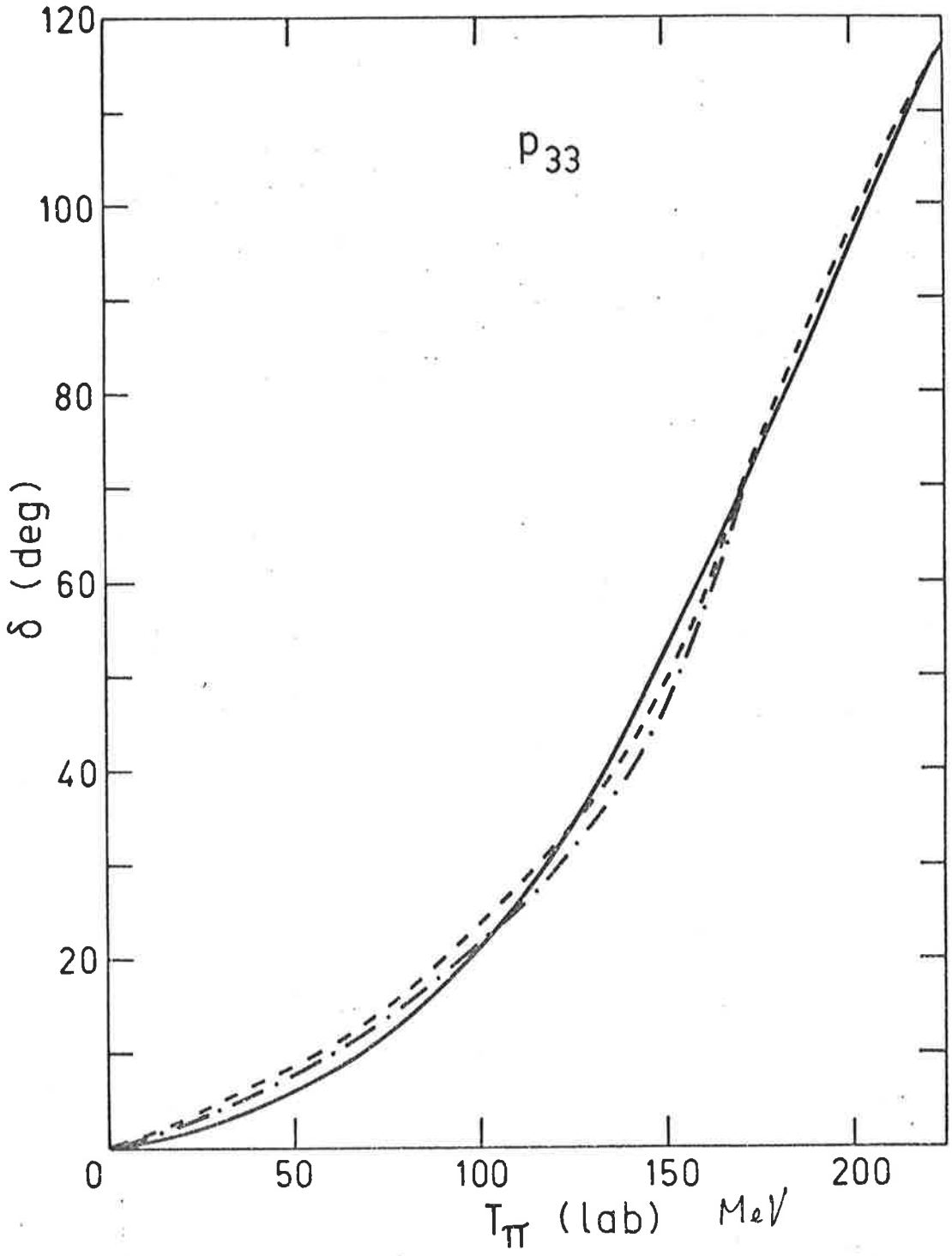


FIG. 9

Theoretical phase shifts for the P33 pion-nucleon wave calculated from the parameters of Table 2a (dashed curve) and Table 2b (dashed-dotted curve). The solid curve represents the experimental points (Refs 40, 41).

form does not give a good representation of the amplitude when the scattering energy is away from the resonance.

In this channel the pion-nucleon interaction proceeds by P-wave coupling to Δ isobar. Fig 8 represents the pion-nucleon amplitude in the P33 channel

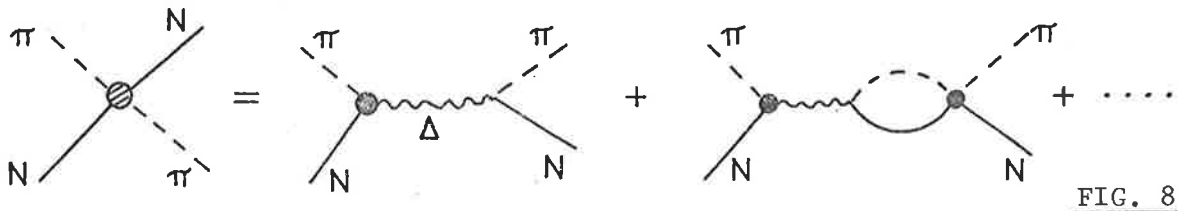


FIG. 8

The model interaction corresponding to the first diagram on right hand side in Fig. 8 , is

$$V'_\ell(p', p) = \lambda(E)\, V_\ell(p')\, V_\ell(p), \tag{4.5.1}$$

where $$\lambda(E) = \frac{\lambda_0}{E^2 - m_\Delta^2} . \tag{4.5.2}$$

According to Equation (2.2.11) the reaction matrix is

$$R_\ell(k', k, E) = \frac{V_\ell(k')\, V_\ell(k)}{\gamma_\ell(E)} \tag{4.5.3}$$

and

$$\gamma_\ell(E) = \frac{1}{\lambda} - P \int_0^\infty \frac{q^2\, V_\ell^2(q)}{E - E(q)}\, dq \tag{4.5.4}$$

Here m_Δ^2 may be interpreted as the square of the isobar mass. By making the strength of the model interaction energy-dependent, we have been able to include the basic form of the interaction obtained from the field theoretic approach (68). The unknown parameter m_Δ^2 of the model can be eliminated by the condition

$$\gamma_\ell(E \to E_r) = 0, \tag{4.5.5}$$

where E_r determines the position of the resonance (experimental data). On combining Equations (4.5.5) and (4.5.4) together with Equation (4.5.3) we

obtain

$$R_\ell (k', k, E) = \frac{V_\ell(k')\, V_\ell(k)}{\tilde{\gamma}_\ell(E)} ,$$

where

$$\tilde{\gamma}_\ell(E) = \frac{E^2 - E_r^2}{\lambda_0} + P\int_0^\infty \frac{q^2\, dq\, V_\ell^2(q)}{E_r - E(q)} - P\int_0^\infty \frac{q^2\, dq\, V_\ell^2(q)}{E - E(q)} ,$$

and $$E(q) = \sqrt{m_\pi^2 + q^2} + \sqrt{m_N^2 + q^2} . \qquad (4.5.6)$$

Now we shall follow our usual technique of Section 2.2, to fix up the parameters of the model interaction from the phase shifts and scattering length. The chosen forms of the potential

$$\text{(a)} \qquad V_\ell (p) = \frac{S_1\, p}{(p^2 + \beta_1^2)^2} + \frac{S_2\, p^2}{(p^2 + \beta_2^2)^2} \qquad (4.5.7a)$$

$$\text{(b)} \qquad V_\ell (p) = \frac{S_1\, p}{(p^2 + \beta_1^2)^2} + \frac{S_2\, p^3}{(p^2 + \beta_2^2)^2} \qquad (4.5.7b)$$

The calculated values of the parameters are given in Table 2 and the fit is displayed in Fig 9 .

For the P11 channel interaction, the scattering amplitude has a pole when the scattering energy becomes equal to the mass of the proton, to account for the emission and absorption of a pion. The phase shift of this channel also changes its sign at $E_0 \sim 150.0$ MeV (lab). One of the simple and economic ways to include these two features is to sec

$$\lambda = \frac{E_0 - E}{E - m_p^0}\, \lambda_0 . \qquad (4.5.8)$$

Here, the parameter m_p^0 can be eliminated by

$$\gamma_\ell (E \to m_p) = 0 , \qquad (4.5.8)$$

where m_p is the mass of the proton. The appropriate reaction matrix for this channel is

$$R_\ell (k', k, E) = \frac{V_\ell (k') V_\ell (k)}{\tilde{\gamma}_\ell} (E_0 - E) , \qquad (4.5.10)$$

where

$$\tilde{\gamma}_\ell (E) = \frac{E - m_p}{\lambda_0} + (E_0 - m_p) \int_0^\infty \frac{q^2 \, dq \, V_\ell^2 (q)}{m_p - E(q)}$$

$$- (E_0 - E) \, P \int_0^\infty \frac{q^2 \, V_\ell^2 (q) \, dq}{E - E(q)} .$$

The chosen form for $V_\ell (p)$ is

$$V_\ell (p) = \frac{S_1 \, p}{(p^2 + \beta_1^2)} + \frac{S_2 \, p}{(p^2 + \beta_2^2)} . \qquad (4.5.11)$$

The parameters of the interaction have been determined from the knowledge of the phase shifts and scattering length (41, 43) following the earlier technique of Section 2.2. The values of the parameters are presented in Table 2a and the corresponding fit is given in Fig 3 In our earlier model given in Section 2.2 for this channel, we have been able to reproduce only the phase shifts. A relativistic generalisation of the present model has been given by Schwarz *et al* (68).

To incorporate partially the contributions of the fermi motion of the nucleus we shall replace the propagators of the two-body t-matrix of every channel in this calculation by

$$\tau_{j\ell}^{I} = \int d^3 \, p \, | \, \phi \, (\underline{p}) \, |^2 \, \tau_{\ell j}^{I} \, (\tilde{\omega}_0), \qquad (4.5.12)$$

where $\phi(\underline{p})$ is the nuclear ground state wave function.

4.6 The Partial Wave Decomposition for the First-Order Optical Potential

The first-order optical potential, Equation (4.3.10) is the product of the two-body pion-nucleon scattering amplitude ($t_{\pi N} (\omega_0)$) and the nuclear form factor $\rho(\underline{q})$. The separable models for the two-body scattering amplitudes $\tilde{t}_{\pi N} (\tilde{\omega}_0)$ for all significant pion-nucleon interactions have been determined from the experimental pion-nucleon scattering

data. We determine $t_{\pi N}(\omega_0)$ (which is in the pion-nucleus centre-of-mass system) from the knowledge of $\tilde{t}_{\pi N}(\tilde{\omega}_0)$ by means of the transformation in Equation (4.4.1).

The standard form for the spin averaged two-body t-matrix, in the pion-nucleon centre-of-mass frame, for a definite iso-spin channel is given by

$$\langle \underline{\kappa}' | \tilde{t} \quad (\tilde{\omega}_0) | \underline{\kappa} \rangle \equiv \langle \underline{\kappa}' | \tilde{t}^{\,I}_{\pi N}(\tilde{\omega}_0) | \underline{\kappa} \rangle$$

$$= \sum_{\ell j} (j + \tfrac{1}{2})\, \tilde{t}^{\,I}_{\ell j}(\kappa', \kappa, \tilde{\omega}_0)\, P_\ell(\hat{\kappa}'.\, \hat{\kappa}) , \qquad (4.6.1)$$

where ℓ and j are the orbital angular momentum of that channel and its total angular momentum respectively. With the help of Equation (4.4.7) we may write

$$P_\ell(\hat{\kappa}'.\, \hat{\kappa}) = P_\ell(\cos\theta_{\pi N}) = \sum_{\ell' \leq \ell} d_{\ell\ell'}\, P_{\ell'}(\cos\theta_{\pi\text{Nucleus}}) . \qquad (4.6.2)$$

The nuclear form factor $\rho(\underline{q})$ in terms of partial waves reads

$$\rho(\underline{k}' - \underline{k}) = \rho(\underline{q}) = \sum_{\ell} (2\ell+1)\, \rho_\ell(k', k)\, P_\ell(\hat{k}'.\, \hat{k}) . \qquad (4.6.3)$$

The Fourier transform of the nuclear form factor is given by

$$\rho(\underline{q}) = \frac{1}{(2\pi)^3} \int e^{\,i\, \underline{q}.\, \underline{r}}\, \rho(\underline{r})\, d^3 r , \qquad (4.6.4)$$

where $$\rho(r) = \rho(o)\, [\, 1 + \alpha \left(\frac{r}{a}\right)^2] \exp(-r^2/a^2)\,] \qquad (4.6.5)$$

and $$\alpha = (A-4)/6.$$

The parameter a is determined from the electron scattering data.

The explicit expression for $\rho_\ell(k', k)$ is

$$\rho_\ell(k', k) = \frac{\exp(-x)}{2 + 3\alpha} [(2+3\alpha-2\alpha x)\, (2\ell+1) i_\ell(z) + 2\alpha z\, (\ell\, i_{\ell-1}(z) + (\ell+1) i_{\ell+1}(z))\,] , \qquad (4.6.6)$$

where $z = .5kk'a^2$; $x = .25(k^2 + k'^2)a^2$

and $i_\ell(z)$ is a modified spherical Bessel function of the first kind (69). If we now combine Equation (4.3.10) together with the Equations (4.6.1) - (4.6.6), we obtain the partial wave expansion for the first-order optical potential in the pion-nucleus centre of mass frame,

$$\langle \underline{k}' | U^{(1)}(E) | \underline{k} \rangle = \sum_L (2L + 1)\, U_L^{(1)}(k', k, E)\, P_L(\hat{k}'.\hat{k}) \quad , \quad (4.6.7)$$

where

$$U_L^{(1)}(k', k, E) = (A-1) \sum_{\ell,\ell',\ell'',j} (2\ell'' + 1)(j + \tfrac{1}{2}) \begin{pmatrix} L & \ell' & \ell'' \\ o & o & o \end{pmatrix}^2$$

$$\times \ddot{\gamma}\, d_{\ell\ell'}, \rho_{\ell''}(k', k) \left\{ \frac{2Z}{3A} \tilde{t}^{\,\frac{1}{2}}_{\ell j}(\kappa', \kappa, \tilde{\omega}_0) + \frac{3N+Z}{3A} \tilde{t}^{\,3/2}_{\ell j}(\kappa', \kappa, \tilde{\omega}_0) \right\} .$$

E is the scattering energy. This is the final expression for the first-order optical potential for L th pion-nucleus partial wave. The angle transformation factor $d_{\ell\ell'}$ mixes the pion-nucleon partial waves into each pion-nucleus partial wave (7). As a consequence the P33 resonance will contribute to every pion-nucleus scattering state. To keep the calculation transparent we have considered only S- and P-wave pion-nucleon scattering states. Now we shall concentrate on the second-order optical potential

4.7 The Partial wave Decomposition for the Second-Order Optical Potential

Introducing Equation (4.3.8) into Equation (4.3.3) we may re-write the second-order optical potential $U^{(2)}$ in the form

$$\langle \underline{k}' \mid U^{(2)}(E) \mid \underline{k} \rangle$$

$$= (A-1)^2 \int d\underline{k}''\, d\underline{k}'''\ \langle \underline{k}', \underline{p}_0' - \underline{q}' \mid t_{\pi N}(\omega_0) \mid \underline{k}'', \underline{p}_0' \rangle\, C(\underline{k}' - \underline{k}'', \underline{k}''' - \underline{k})$$

$$\langle \underline{k}'' \mid (E^+ - \bar{\epsilon} - K_\pi - \bar{U}_0)^{-1} \mid \underline{k}''' \rangle \langle \underline{k}''', \underline{p}_0 - \underline{q} \mid t_{\pi N}(\omega_0) \mid \underline{k}, \underline{p}_0 \rangle \ , \tag{4.7.1}$$

where $\underline{q} = \underline{k}''' - \underline{k}$, $\underline{p}_0 = -\frac{\underline{k}}{A} + \frac{\underline{q}(A-1)}{2A}$, $\underline{q}' = \underline{k}' - \underline{k}''$, (4.7.2)

$$\underline{p}_0' = -\frac{\underline{k}''}{A} + \frac{\underline{q}'(A-1)}{2A} \quad \text{and}$$

$$C(\underline{q}_1, \underline{q}_2) = \rho^{(2)}(\underline{q}_1, \underline{q}_2) - \rho(\underline{q}_1)\, \rho(\underline{q}_2) \tag{4.7.3}$$

is the two-particle correlation function in momentum space.

On the basis of earlier arguments we shall assume that the average nuclear fluctuation from the ground state is small during the propagation of the pion, and the average optical potential operator may be approximated by (27, 64).

$$\bar{U}_0 \simeq \langle \underline{k}_0 \mid U^{(1)}(E) \mid \underline{k}_0 \rangle \ , \tag{4.7.4}$$

where $\underline{k}_0$ is the incident momentum.

The mean nuclear excitation energy $(\bar{\epsilon})$ has been neglected in our calculation. Therefore the propagator in the Equation (4.7.1) becomes

$$\langle \underline{k}'' \mid (E^+ - \bar{U}_0 - K_\pi)^{-1} \mid \underline{k}''' \rangle$$

$$= \delta(\underline{k}'' - \underline{k}''') \left(E^+ - \bar{U}_0 - \sqrt{m_\pi^2 + \underline{k}''^2} - \sqrt{M_T^2 + \underline{k}''^2}\right)^{-1} . \tag{4.7.5}$$

Substitution of Equation (4.7.5) into Equation (4.7.1) yields

$$\langle \underline{k}' \mid U^{(2)}(E) \mid \underline{k} \rangle$$

$$= (A-1)^2 \int d\underline{k}''\ \langle \underline{k}', \underline{p}_0' - \underline{q}' \mid t_{\pi N}(\omega_0) \mid \underline{k}''\, \underline{p}_0' \rangle$$

$$C(\underline{k}' - \underline{k}'', \underline{k}'' - \underline{k})\ \langle \underline{k}'', \underline{p}_0 - \underline{q} \mid t_{\pi N}(\omega_0) \mid \underline{k}, \underline{p}_0 \rangle \times$$

M_T is the mass of the target nucleus.

$$(E^{+} - \bar{U}_0 \quad - \sqrt{m_\pi^2 + k''^2} - \sqrt{M_T^2 + k''^2}\ )^{-1} \qquad (4.7.6)$$

The term $\rho^{(2)}(\underline{q}_2, \underline{q}_1)$ is the momentum representation of the two-nucleon density function

$$\rho^{(2)}(\underline{r}_2, \underline{r}_1) = \int d\underline{r}_3 \ldots d\underline{r}_A \mid \psi_{o,\, \nu_T}(\underline{r}_1, \underline{r}_2 \ldots . r_A) \mid^2$$

and $$\rho^{(2)}(\underline{q}_2, \underline{q}_1) = \int \underline{dr}_1 \ \underline{dr}_2 \ e^{i\underline{q}_2 . \underline{r}_2} \rho^{(2)}(\underline{r}_2, \underline{r}_1) \ e^{i\underline{q}_1 . \underline{r}_1}.$$

If the nucleons inside the nucleus move independent of each other then the two-nucleon density function is simply the product of two single-particle densities. Consequently the second-order optical potential vanishes completely. Thus the second-order optical potential depends on the nucleon-nucleon correlations inside the nucleus. We have considered only the short-range dynamical correlation in our calculation, which occurs because of the comparatively strong and repulsive character of the short-range part of the nucleon-nucleon forces which are connected to the shell model wavefunction through the G-matrix (64, 70). To account for this correlation one should modify the intrinsic two-nucleon density function suitably. There is no direct experimental evidence about the shape of the two-nucleon correlation function. We assume that the two-nucleon density function becomes zero if the relative separation between the two nucleons is less than the hard core radius and the product of two single-particle densities for large relative separation. A particular model which fulfils these criteria is

$$\rho^{(2)}(\underline{r}_2, \underline{r}_1) = C_0 \rho(\underline{r}_1)\ \rho(\underline{r}_2)\ [1 + G(\mid \underline{r}_1 - \underline{r}_2 \mid)\ ]\ . \qquad (4.7.7)$$

The constant C_0 is determined by the normalization condition

$$\int d\underline{r}_1\ d\underline{r}_2\ \rho\ (\underline{r}_1, \underline{r}_2) = 1.$$

However, it can be shown that the constant C_0 is very close to one (64).

The function $G(|\underline{r}_1 - \underline{r}_2|)$ is only non-vanishing when the two nucleons come within a distance less than the correlation length. Therefore it may be reasonable to assume that the function $G(|\underline{r}_2 - \underline{r}_1|)$ is effective over a region much smaller than the dimension of the nucleus. Thus we take

$$\rho(\underline{r}_1)\ \rho(\underline{r}_2)\ G(|\underline{r}_2 - \underline{r}_1|) \approx \rho^2\ (\tfrac{1}{2}(\underline{r}_1 + \underline{r}_2))\ G(|\underline{r}_2 - \underline{r}_1|)\ . \tag{4.7.8}$$

With the help of Equations(4.7.7) and (4.7.8) the Fourier transform of the two-nucleon correlation function can be re-written

$$\begin{aligned} C(\underline{q}_1, \underline{q}_2) &\simeq \int d\underline{r}_1\ d\underline{r}_2\ e^{i\ (\underline{q}_1 .\ \underline{r}_1 + \underline{q}_2 .\ \underline{r}_2)}\ \rho(\underline{r}_1)\ \rho(\underline{r}_2) G(|\underline{r}_1 - \underline{r}_2|) \\ &\simeq \int d\underline{r}_1\ \ d\underline{r}_2\ e^{i(\underline{q}_1 .\underline{r}_1 + \underline{q}_2 .\ \underline{r}_2)}\ \rho^2\ (\tfrac{1}{2}(\underline{r}_1 + \underline{r}_2)) \\ &\quad \times\ G(|\underline{r}_2 - \underline{r}_1|) \end{aligned} \tag{4.7.9}$$

In terms of the relative and centre-of-mass co-ordinates, Equation (4.7.9) reads

$$\begin{aligned} C(\underline{k}' - \underline{k}'', \underline{k}'' - \underline{k}) &= \int d\underline{r}_1\ \ d\underline{r}_2\ e^{i(\underline{k}' - \underline{k}'').(\underline{R} + \frac{\underline{r}}{2})} \\ &\qquad \times\ e^{i(\underline{k}'' - \underline{k}).(\underline{R} - \frac{\underline{r}}{2})} \\ &\qquad \times \rho^2\ (\frac{\underline{r}_1 + \underline{r}_2}{2})\ G(|\underline{r}_1 - \underline{r}_2|) \\ &= \int d\underline{r}\ d\underline{R}\ e^{i\ (\underline{k}' - \underline{k}).\underline{R}}\ \rho^2\ (\underline{R})\ e^{-\ i(\underline{k}'' - \frac{\underline{k}+\underline{k}'}{2}).\underline{r}}\ G(|\underline{r}|) \\ &= \rho^{(2)}(\underline{k}' - \underline{k})\ G(\underline{k}'' - \frac{\underline{k}+\underline{k}'}{2}), \end{aligned} \tag{4.7.10}$$

where

$$\underline{r}_1 = \underline{R} + \frac{\underline{r}}{2} , \quad \underline{r}_2 = \underline{R} - \frac{\underline{r}}{2} ,$$

$$\rho^{(2)}(\underline{k}' - \underline{k}) = \int e^{i(\underline{k}'-\underline{k}).\underline{R}} \rho^2 (\underline{R}) \, d\underline{R} \qquad (4.7.11)$$

and

$$G(\underline{k}'' - \frac{\underline{k}+\underline{k}'}{2}) = \int e^{-i(\underline{k}'' - \frac{\underline{k}+\underline{k}'}{2}).\underline{r}} G(|\underline{r}|) \, d\underline{r} \qquad (4.7.12)$$

Substitution of Equation (4.7.10) in the second-order optical potential yields

$$< \underline{k}' \mid U^{(2)}(E) \mid \underline{k}, > = (A-1)^2 \rho^{(2)}(\underline{k}' - \underline{k})$$

$$\int d\underline{k}'' \, G(\underline{k}'' - \frac{\underline{k}+\underline{k}'}{2}) < \underline{k}', \underline{p}_0' - \underline{q}' \mid t_{\pi N}(\omega_0) \mid \underline{k}'', \underline{p}_0' >$$

$$\times (E^+ - \overline{U}_0 - \sqrt{m_\pi^2 + k''^2} - \sqrt{M_T^2 + k''^2})^{-1} < \underline{k}'', \underline{p}_0 - \underline{q} \mid t_{\pi N}(\omega_0) \mid \underline{k}, \underline{p}_0 > \qquad (4.7.13)$$

To proceed further we need a model for the correlation function. For this calculation, a gaussian type correlation function has been considered.

$$G(|\underline{r}|) = -e^{-r^2/\ell_c^2} \qquad (4.7.14)$$

The parameter ℓ_c is the correlation length.

The corresponding structure of $G(\underline{k}'' - \frac{\underline{k}'+\underline{k}}{2})$ in momentum representation is

$$G(\underline{k}'' - \frac{\underline{k}'+\underline{k}}{2}) = -(\frac{\pi}{\beta})^{3/2} \exp(-\alpha (k''^2 + \tfrac{1}{4}(k^2 + k'^2)))$$

$$\cdot \exp(-x \cos\theta) \exp(z \cos\theta_1) \exp(y \cos\theta_2),$$

where $x = \frac{\alpha kk'}{2}$, $y = \alpha kk''$, $z = \alpha k''k'$, $\alpha = \frac{1}{4\beta}$,

$$\theta_1 = \hat{k}''.\hat{k}, \quad \theta_2 = \hat{k}''.\hat{k}, \quad \theta = \hat{k}.\hat{k}', \quad \beta = \frac{1}{\ell_c^2} \qquad (4.7.15)$$

and so,

$$\langle \underline{k}' | U^{(2)}(E) \quad | \underline{k} \rangle = \rho^{(2)}(\underline{k}' - \underline{k})\ \tilde{A}\ (\underline{k}', \underline{k})\ (A-1)^2 \tag{4.7.16}$$

where

$$\tilde{A}\ (\underline{k}', \underline{k}) = \int d\underline{k}''\ G(\ \underline{k}'' - \frac{\underline{k}'+\underline{k}}{2}\ )\ \langle \underline{k}', \underline{p}_0' - \underline{q}' | t_{\pi N}\ (\omega_0) \ | \underline{k}'', \underline{p}_0' \rangle$$

$$\times (E^+ - \bar{U}_0 - \sqrt{m_\pi^{\,2} + k''^2} \ - \sqrt{M_T^{\,2} + k''^2}\ )^{-1} \ \langle \underline{k}'', \underline{p}_0 - \underline{q} | t_{\pi N}(\omega_0) | \underline{k}, \underline{p}_0 \rangle \ . \tag{4.7.17}$$

For the evaluation of the second-order optical potential we have used the closure approximation and the fixed scatterer approximation which make the pion-nucleon t-matrix dependent only on the pion momentum. To incorporate the fermi motion of the nucleons we have replaced the two-body propagator $\tau_{\ell j}^{\ I}\ (\tilde{\omega}_0)$ by the fermi averaged quantity, Equation (4.5.12). After some angular momentum algebra, the partial wave decomposition of the second-order optical potential can be written as

$$\langle \underline{k}' \ | U^{(2)}(E) \ | \underline{k} \rangle = \sum_L\ (2L+1)\ U_L^{\ (2)}(k', k, E)\ P_L(\hat{k}' \cdot \hat{k})$$

$$U_L^{\ (2)}(k', k, E) = (A-1)^2 \sum_{m,n} f_n\ (k', k)\ \tilde{A}_m(k', k)\ \begin{pmatrix} n & m & L \\ o & o & o \end{pmatrix}^2 (\hat{m})^1 (\hat{n}) \tag{4.7.18}$$

The term $f_n(k', k)$ is related to the square of the nuclear form factor and

$$\tilde{A}(\underline{k}', \underline{k}) = \sum_\delta\ (2\delta + 1)\ \tilde{A}_\delta(k', k)\ P_\delta(\hat{k}', \hat{k}), \tag{4.7.19}$$

where

$$\tilde{A}_\delta(k', k) = -\frac{1}{(2\pi)^3}\ (\frac{\pi}{\beta})^{3/2} \int_o^\infty dk''\ k''^2\ e^{-\alpha(k''^2 + \frac{1}{4}(k^2 + k'^2))}$$

$$\sum_{\substack{\ell, \ell_1, \\ \ell_3, \ell_4 \\ \Lambda, n}} \frac{\gamma_1\ (\kappa', \kappa'')\ \tilde{t}_\ell\ (\kappa', \kappa'', \tilde{\omega}_0)\ \gamma_2(\kappa'', \kappa)\ \tilde{t}_{\ell_1}\ (\kappa'', \kappa, \tilde{\omega}_0)}{E^+ - \bar{U}_0 \ - \sqrt{m_\pi^{\,2} + k''^2} \ - \sqrt{M_T^{\,2} + k''^2}}$$

$$(-)^n\ i_n(x)\ i_{\ell_3}\ (z)\ i_{\ell_4}\ (y)\ C_\delta\ (\ell, \ell_1, \ell_3, \ell_4, n, \Lambda) \tag{4.7.20}$$

and $$C_\delta(\ell, \ell_1, \ell_3, \ell_4, n, \Lambda) = (\hat{\ell}\ \hat{\ell}_1\ \hat{\ell}_3\ \hat{\ell}_4\ \hat{n}\ \hat{\Lambda})^2$$

$$\times \begin{pmatrix} \ell & \ell_3 & \Lambda \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{Bmatrix} \ell_1 & \ell_4 & \Lambda \\ 0 & 0 & 0 \end{Bmatrix}^2 \begin{Bmatrix} n & \Lambda & \delta \\ 0 & 0 & 0 \end{Bmatrix}^2 , \qquad (4.7.21)$$

where $\tilde{t}_\ell$ and $\tilde{t}_{\ell_1}$ are the appropriate spin and iso-spin averaged pion-nucleon t-matrices. The terms $\gamma_1(\kappa', \kappa'')$ and $\gamma_2(\kappa'', \kappa)$ will be determined from the transformation (4.4.1). There will be a great simplification in evaluation of Equation (4.7. 20) since we have considered only S- and P-wave pion-nucleon scattering. Because of the angle transformation factor $(d_{\ell\ell'})$ the possible values of ℓ and ℓ_1 are only o and 1. Here $\hat{\ell}$ is $\sqrt{2\ell + 1}$

We substituted the resulting matrix element of the optical potential

$$U_L(k', k) = U_L^{(1)}(k', k) + U_L^{(2)}(k', k) \quad \text{for} \qquad (4.7.22)$$

the integral equation satisfied by the reaction matrix $R_L(k', k, E)$

$$R_L(k', k, E) = U_L(k', k) + P \int_0^\infty \frac{U_L(k', k'')\, R_L(k'', k, E)\, k''^2}{E - \sqrt{m_\pi^2 + k''^2} - \sqrt{M_T^2 + k''^2}}\, dk'' \qquad (4.7.23)$$

and obtained

$$R_L(k, k, E) = \frac{U_L(k, k)}{1 + P \int_0^\infty \frac{U_L(k, k'')\, f_L(k'', E)\, k''^2\, dk''}{E - \sqrt{m_\pi^2 + k''^2} - \sqrt{M_T^2 + k''^2}}} \qquad (4.7.24)$$

This equation suggests how one might recover the on-shell reaction matrix R_L from the function f_L. The function f_L satisfies

$$f_L(k'', E) = \frac{U_L(k'', k)}{U_L(k, k)} + \int_0^\infty \Lambda_L(k'', q, E)\, f_L(q, E)\, dq \qquad (4.7.25)$$

whose kernel $\Lambda_L(k'', q, E)$ is

$$\Lambda_L(k'', q, E) = \frac{q^2}{q^2 - E}\left[\frac{U_L(k'', k)\, U_L(k, q)}{U_L(k, k)} - U_L(k'', q)\right]. \qquad (4.7.26)$$

Equation (4.7.19) is a well-behaved non-singular Fredholm integral equation when $E > 0$. For positive E, the denominator goes to zero when $q^2 \to E$, but at the same time the numerator in the square bracket also approaches zero. Consequently Λ_L is continuous in the neighbourhood of $q^2 \to E$. We converted Equation (4.7.25) to a system of linear equations and determined f_L numerically by matrix inversion technique.

Once we have obtained f_L, the phase shifts of the pion-nucleus scattering may be computed from

$$\tan \delta_L = \frac{-\pi k\, E_\pi(k)\, E_T(k)}{(E_\pi + E_T)}\, R_L(k, k, E), \qquad (4.7.27)$$

where

$$E_\pi(k) = \sqrt{m_\pi^2 + k^2} \quad \text{and} \quad E_T(k) = \sqrt{M_T^2 + k^2}\,.$$

The pion-nucleus centre of mass of scattering amplitude for a definite scattering angle θ

$$f(\theta) = \frac{A}{A-1} \sum_L (2L + 1)\, \frac{e^{i\delta_L(k)} \sin \delta_L(k)}{k}\, P_L(\cos \theta) \qquad (4.7.28)$$

The corresponding differential cross-section for the elastic pion-scattering is given by

$$\frac{d\sigma(\theta)}{d\Omega} = |f(\theta)|^2$$

In this model calculation we have ignored the effects of Coulomb interaction in the optical potential.

4.8 Results and Discussion

In this section we shall discuss our results on the pion-carbon differential cross-sections calculated from the first-and second-order optical potentials at intermediate energies, assessing the importance of the nucleon-nucleon correlation, P33 resonance, fermi motion and the off-shell dependence.

Two sets of phase-equivalent pion-nucleon interaction potentials for S11, S31, P11, P13, P31 and P33 channels have been considered to examine the off-shell dependence of the scattering cross-section. The parameters of these interactions are given in Table 2. The importance of the second-order optical potential, i.e. the contribution of the nucleon-nucleon correlation has been presented in Fig 10. The calculated values of the differential scattering cross-sectionsfor the parameter sets (2a) and (2b) are displayed in the Figs 12a - 12b respectively. The solid and dash and dash-dot curves in Fig 10 represent the corresponding values of the differential cross-sections obtained from experiment, $U^{(1)}$ and $U^{(1)} + U^{(2)}$ respectively. Our results are slightly larger in magnitude than the experimental results (10-20%). To incorporate partially the contribution of the fermi motion we have used the fermi averaged values of the appropriate propagators (of Equation (4.5.12)) for two-body t-matrices. From Fig 10 it is clear that the second-order optical potential induces a small additive contribution to the results of the first-order optical potential calculation at intermediate energies. The calculations of Lee and Chakravorty (27) and Wakamatsu (28) also suggest a similar conclusion. But they have noted in addition that at low energies $E_{\pi} < 75$ MeV , the second-order optical potential reduces slightly the forward scattering cross-sections. Therefore we may conclude that the inclusion of the second-order optical potential does not improve the numerical results. But in principle, it should have given a better fit to

the experimental data since the second-order optical potential contains a more complete description of the microscopic processes.

To examine the dependence of the cross-section on the nucleon-nucleon correlation length (ℓ_c), we have computed the differential cross-sections for $U^{(1)} + U^{(2)}$ using different values of ℓ_c in our model correlation function and the results are displayed in Fig 11. It has been found for a given set (say set a) of the pion-nucleon interactions and at a particular energy ($E_\pi \sim 180$ MeV) when ℓ_c is changed from 0.46 - 0.86 fm that the angular distribution for large angles becomes flatter. This conclusion remains unchanged when E_π takes different values. The different correlation lengths alter the numerical results only for large angle scattering. The best fit is obtained when the second-order optical potential is minimum. This is equivalent to assuming that the effect of correlation is vanishing, i.e. the correlation length should be minimum. In other words, the modification of the intrinsic two-nucleon density function is very small. This observation contradicts one of the important theoretical conclusions of nuclear matter calculations which suggests that the correlation length is non-vanishing (~ hard core radius).

Next, we have tried to investigate the following properties of the differential cross-section :

(i) It s sensitivity to the off-shell behaviour of the pion-nucleon t-matrices at different energies;

(ii) The change in the off-shell contribution induced by using different form factors and interaction ranges for the pion-nucleon interaction.

With regard to (i), the differential cross-sections have been calculated from $U^{(1)} + U^{(2)}$ using the parameter sets (a) and (b) for the pion-nucleon interactions respectively. The results are presented in Fig 12.

It seems that the results are not very sensitive to the detailed structure of the off-shell variations of the pion-nucleon t-matrices. The theoretical results do, however, depend on how the P33 channel interaction is approximated. The overall difference in the theoretical results calculated from these two sets of parameters is about ~ 15%. Without considering phase-equivalent interactions, Lee and Chakravorty (27) in their calculation on pion-Helium scattering have varied the range of the P33 channel interaction. Their observations are not dissimilar to our own. Actually one should use phase-equivalent potentials to examine this sensitivity. Otherwise, the variation of the range parameters only misrepresents the basic interaction, since the pion-nucleon interaction ranges are different for different channels. Another difficulty in discriminating between off-shell contributions of different models is that the nuclear form factor is a strongly peaked function in momentum space compared with the pion-nucleon t-matrices. Consequently, it suppresses a large portion of the off-shell contributions of any particular two-body interaction (7).

To understand the second point of the preceding page, we proceed as follows. From Fig 12 we see that the results obtained from two sets of potentials depend on the off-shell behaviour of the two-body interactions. In fact, the nature of the interactions is different and the range parameters are also quite different for every channel. Therefore, we conclude that the scattering cross-section is sensitive to the off-shell behaviour of the pion-nucleon t-matrices and the range parameters of the pion-nucleon interaction for the different channels. However, it is difficult to state to what extent the scattering cross-section depends on the individual range parameters in this model calculation. To obtain a better fit to the experimental data for individual channels we have increased the number of

parameters and this in turn forbids us to make unambiguous estimates of the off-shell contributions coming from the range parameters. Although model interactions with higher rank separable potentials may give better overall numerical results, it is not clear whether one learns anything more about the basic features of the two-particle interactions. The numerical results can be improved to some extent by varying the range parameters, nucleon-nucleon correlation function, correlation length and using different types of model potentials for the pion-nucleon interactions. However, very good agreement with the experimental results for both the forward and backward scattering does not seem possible. This is understandable, since we have made several simplifications which could lead to significant changes in the results, viz.

(i) fixed scatterer approximation. This appears intuitively to be a good approximation for intermediate pion-nucleus scattering (cf argument on page 90). Nevertheless, it suppresses any possible role for nuclear dynamics and minimizes off-shell contributions. The nucleons are static in a "frozen" nucleus. Therefore the velocity operator $\dot{x}_j$ for the j th nucleon in the Heisenberg picture is related to the intermediate nuclear hamiltonian (H_N) by

$$\dot{x}_j = i\,[H_N, x_j]$$

$$\text{and} \quad \langle \phi_0 \mid \dot{x}_j^2 \mid \phi_0 \rangle = \sum_n (E_n - E_o)^2 \left| \langle \phi_0 \mid x_j \mid \phi_n \rangle \right|^2 ,$$

where $(E_n - H_N) \mid \phi_n \rangle = 0, \quad (E_0 - H_0) \mid \phi_0 \rangle = 0.$

The frozen nucleus approximation demands $E_n = E_o$ for all intermediate states, i.e. all nuclear excitation energies are zero (9);

(ii) the neglect of the contribution of the medium on the two-body scattering amplitude. This is generally known as the local field correction and it is quite significant (62);

(iii) approximate description of the pion absorption. We have not considered the influence of delta (Δ(1236 MeV)) propagation;

(iv) difficulty in considering recoil effects;

(v) overestimates of multiple scattering contributions (47).

Further, we may improve the theoretical results slightly by making the following more straightforward corrections :

(i) coulomb correction;

(ii) a more realistic nuclear form factor and nucleon-nucleon correlation function.

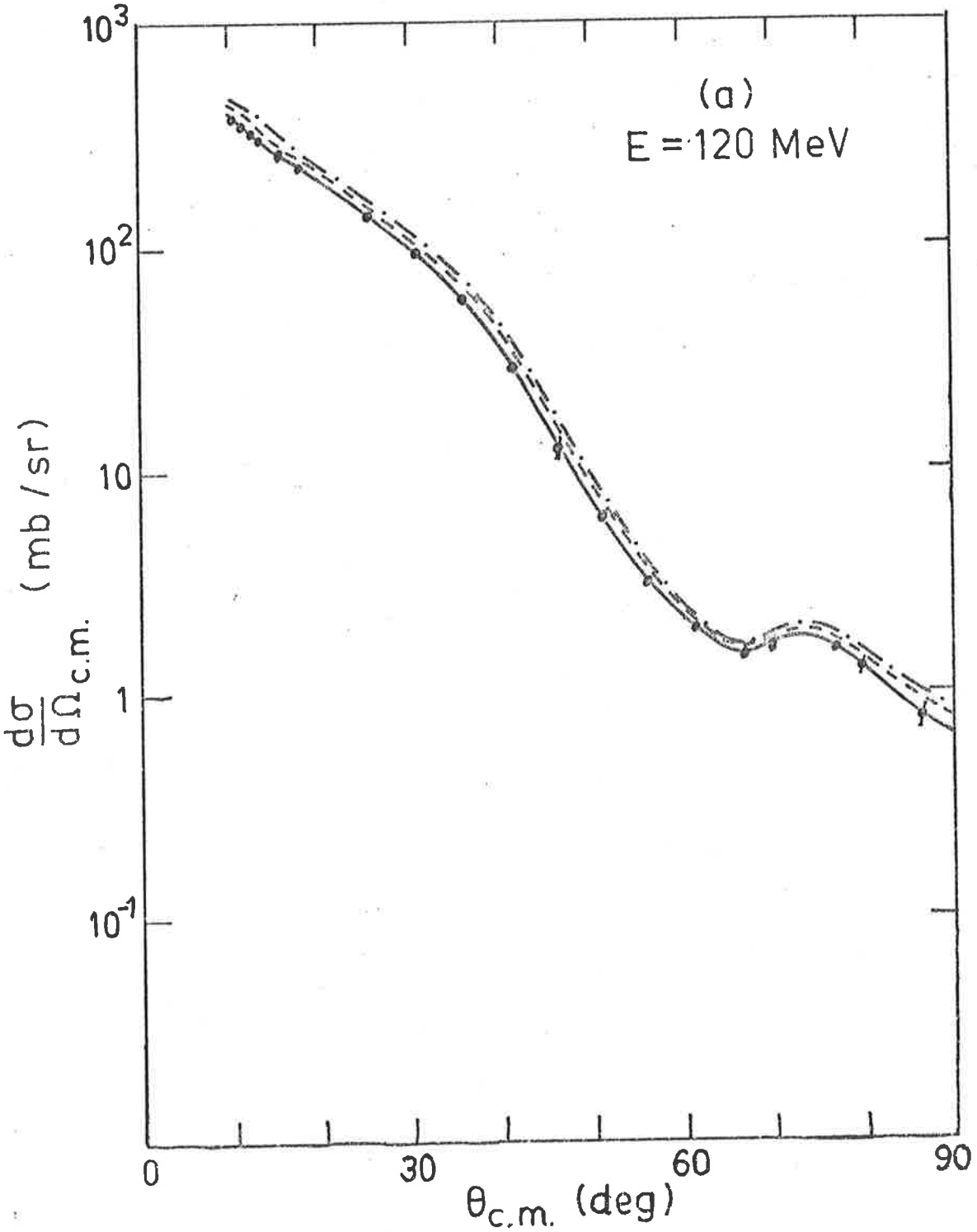
However, our observations agree well with those of Lee and Chakravorty (27) and Wakamatsu (28) obtained from their calculations on pion-Helium scattering. Wakamatsu has incorporated the contribution of binding correction. This model calculation shows that at intermediate energies the second-order term in fact overestimates the differential scattering cross-sections from the corresponding experimental data, and fails to give a fully convincing fit to the data. Therefore as a first step in testing the optical potential model approach to pion-nucleus scattering it is necessary to include the most accurate possible input information. The separable model interactions used in our calculation are quite good and they consistently reflect the dynamics of the different channels. The P11 channel interaction has not been considered previously in computing the second-order optical potential. Clearly one should take more rigorous

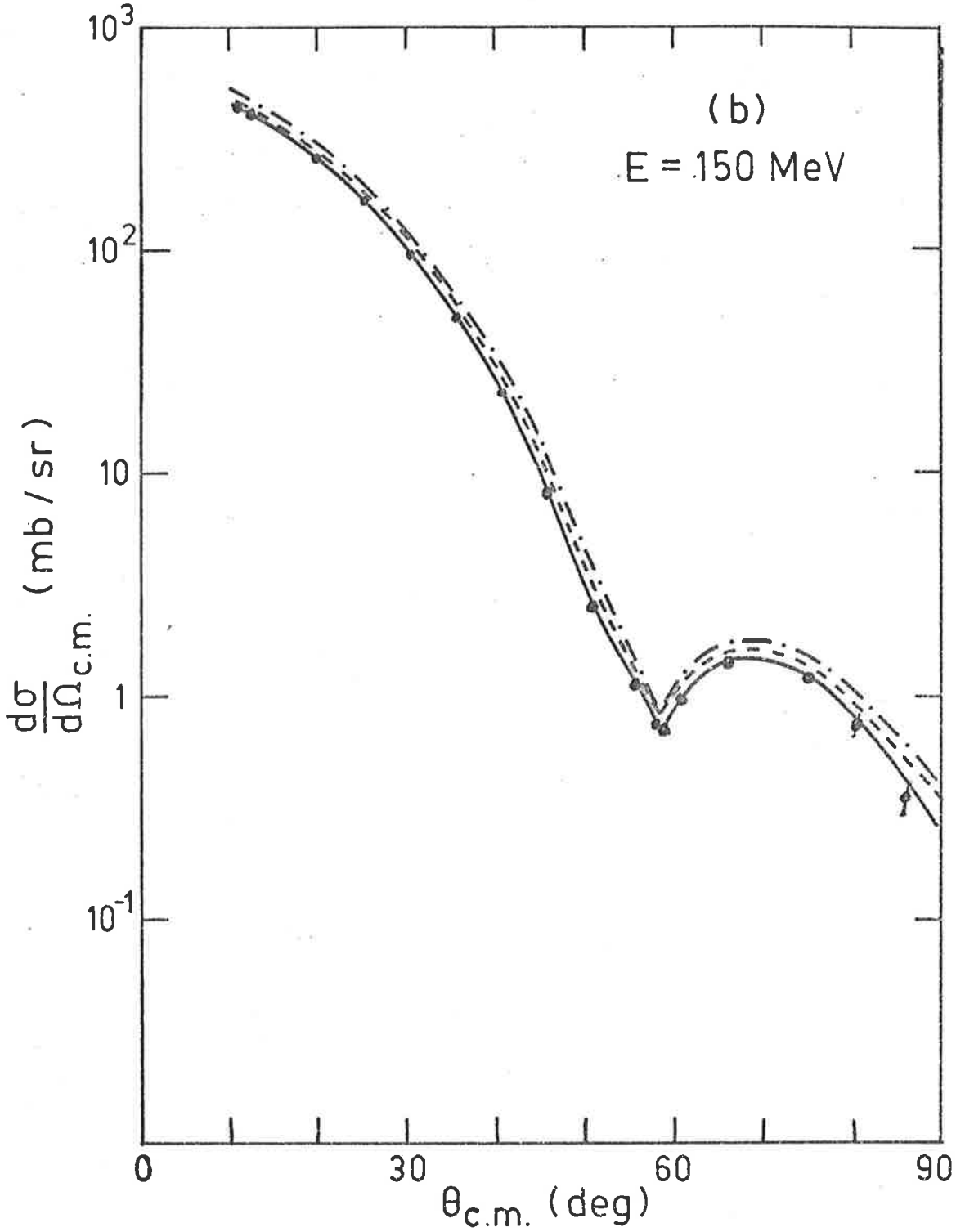
account of the contributions of the fermi motion and the Pauli blocking in terms of an effective pion-nucleon interaction. We also note that the differential scattering cross-sections at large angles ($\theta_{c.m.} > 90^{\circ}$), calculated from $U^{(1)} + U^{(2)}$, are roughly 2- 3 times greater than the data. We have not included them in any diagrams (71).

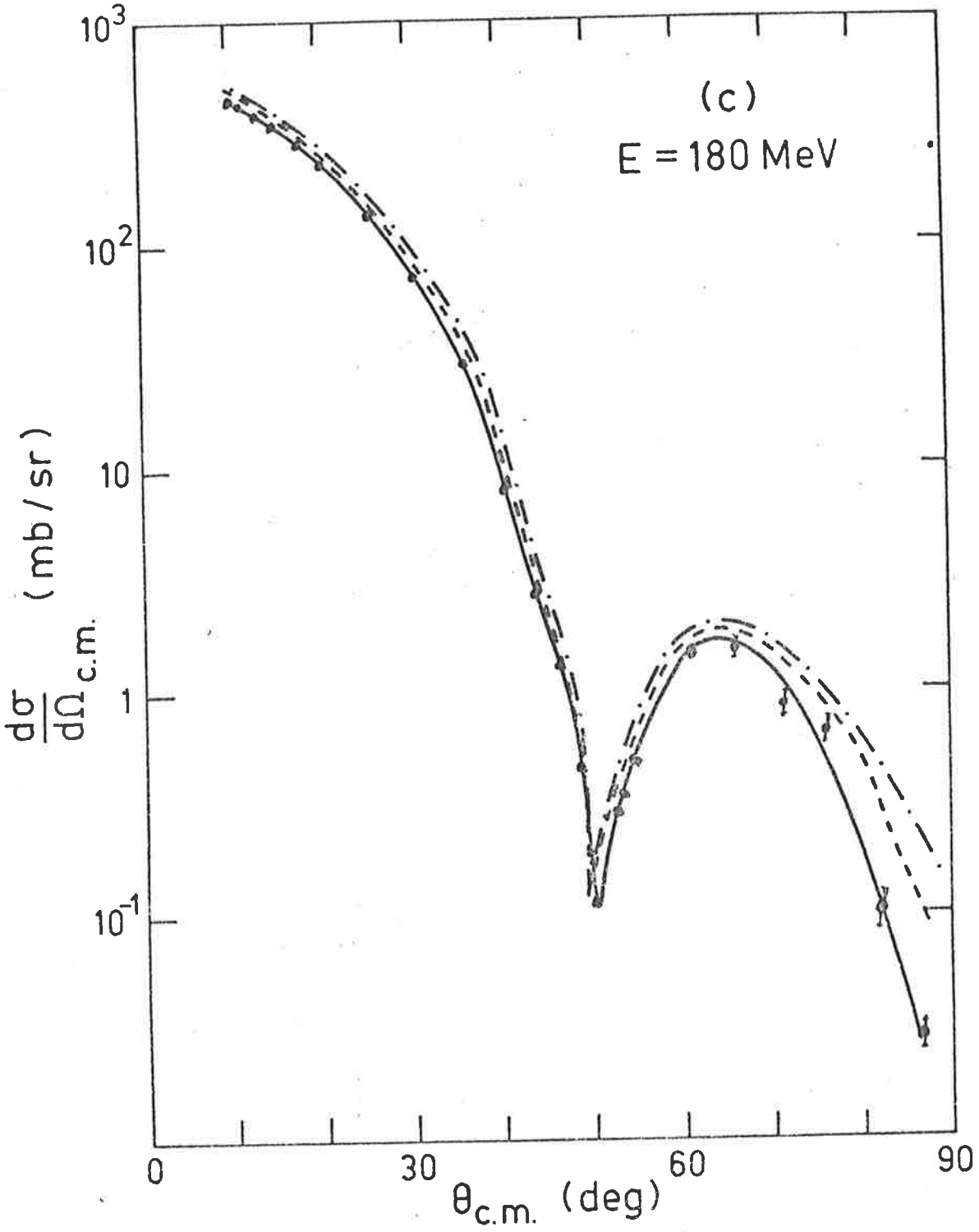
In general we believe that one can understand qualitatively some of the aspects of the pion-nucleus scattering in terms of the optical potential at intermediate energies. As suggested by some authors (62, 63) a more rigorous theoretical approach, which includes, appropriately the various many-body contributions, appears necessary for a more adequate description of pion-nucleus scattering.

FIG. 10

The effects of the nucleon-nucleon correlation on the pion-carbon elastic differential cross-sections. The differential cross-sections calculated from $U^{(1)}$ (dashed curves) and $U^{(1)} + U^{(2)}$ (dash-dotted curves) are compared. The experimental data (solid curves) are from Ref 31. The parameters of the pion-nucleon interaction potentials are given in Table 2a. The value of the correlation length (ℓ_c) is .46 fm.


(a)
E = 120 MeV
10^3
10^2
10
1
10^{-1}
$\frac{d\sigma}{d\Omega}_{c.m.}$ (mb/sr)
0
30
60
90
$\theta_{c.m.}$ (deg)


(b)
E = 150 MeV
10^3
10^2
10
1
10^{-1}
$\frac{d\sigma}{d\Omega}_{c.m.}$ (mb / sr)
0
30
60
90
$\theta_{c.m.}$ (deg)


(c)
E = 180 MeV
10^3
10^2
10
1
10^{-1}
$\frac{d\sigma}{d\Omega}_{c.m.}$ (mb/sr)
0
30
60
90
$\theta_{c.m.}$ (deg)

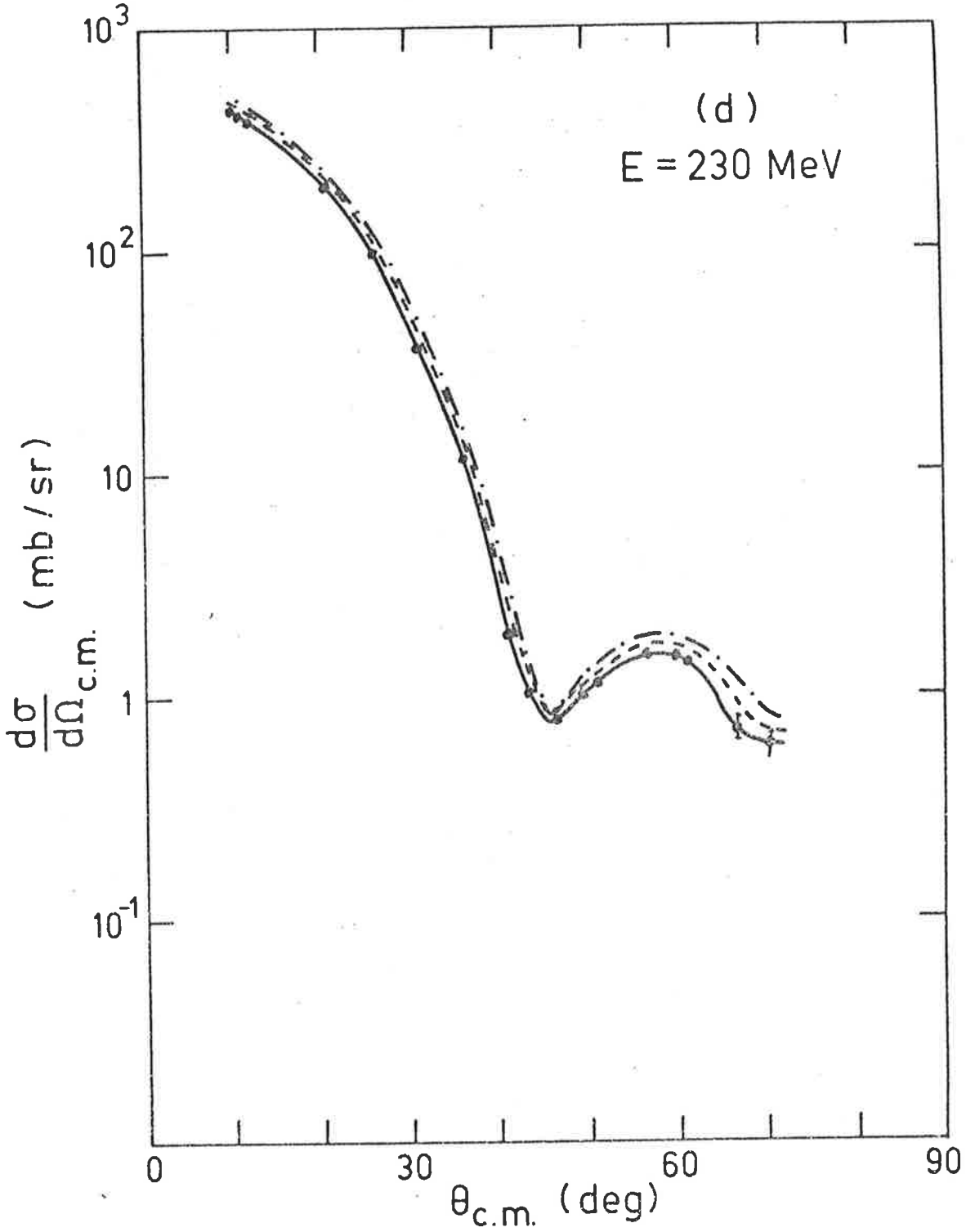
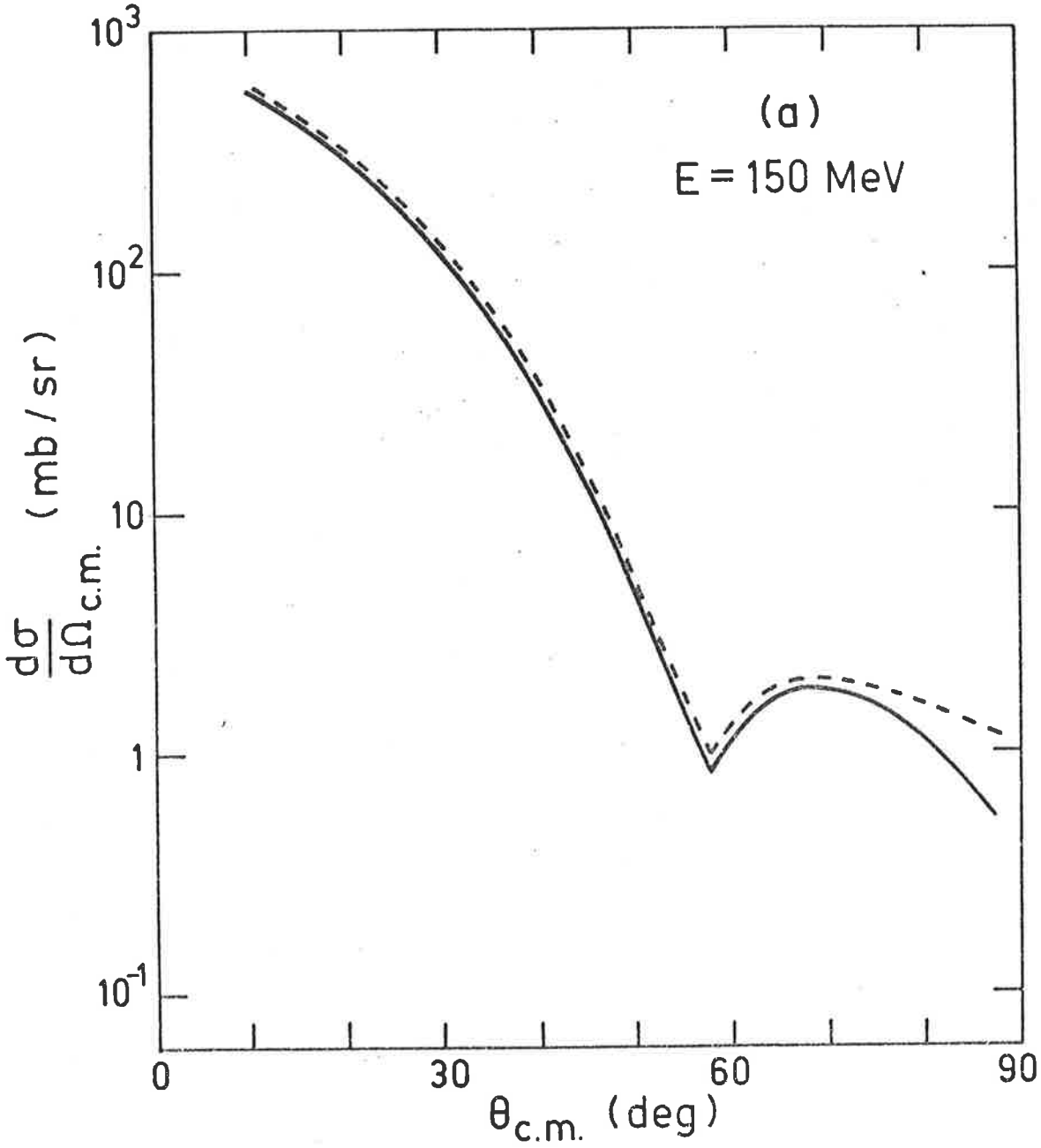
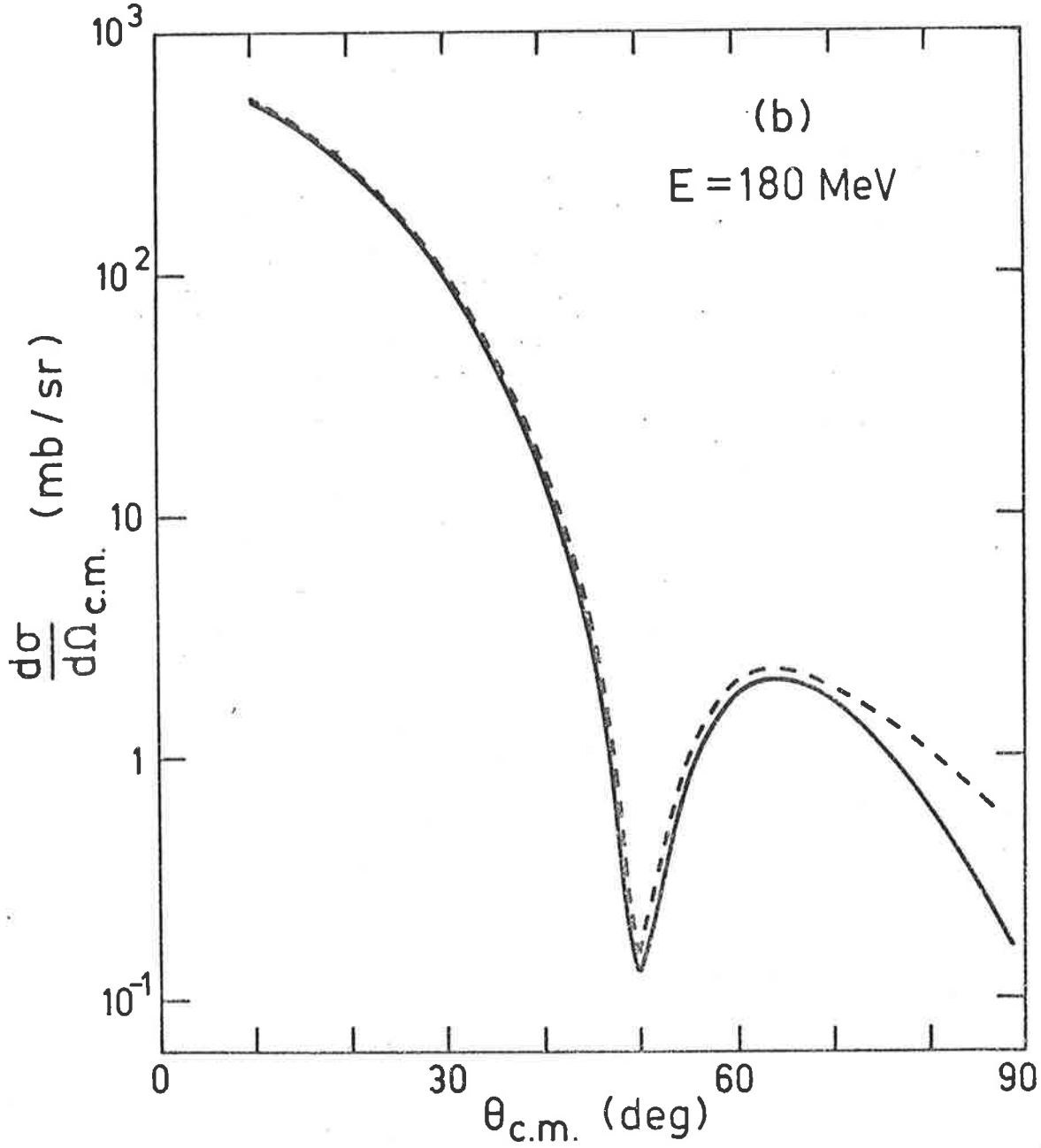

(d)
E = 230 MeV
10^3
10^2
10
1
10^{-1}
$\frac{d\sigma}{d\Omega}_{c.m.}$ (mb/sr)
0
30
60
90
$\theta_{c.m.}$ (deg)

FIG. 11

The elastic differential cross-sections for the pion-carbon scattering calculated from $U^{(1)} + U^{(2)}$ for $\ell_c = .46$ fm (solid curves) and $\ell_c = .86$ fm (dashed curves) are compared. The parameters of the pion-nucleon interactions are given in Table 2a.


(a)
E = 150 MeV
10^3
10^2
10
1
10^{-1}
$\frac{d\sigma}{d\Omega}_{c.m.}$ (mb / sr)
0
30
60
90
$\theta_{c.m.}$ (deg)


(b)
E = 180 MeV
10^3
10^2
10
1
10^{-1}
$\frac{d\sigma}{d\Omega}_{c.m.}$ (mb / sr)
0
30
60
90
$\theta_{c.m.}$ (deg)

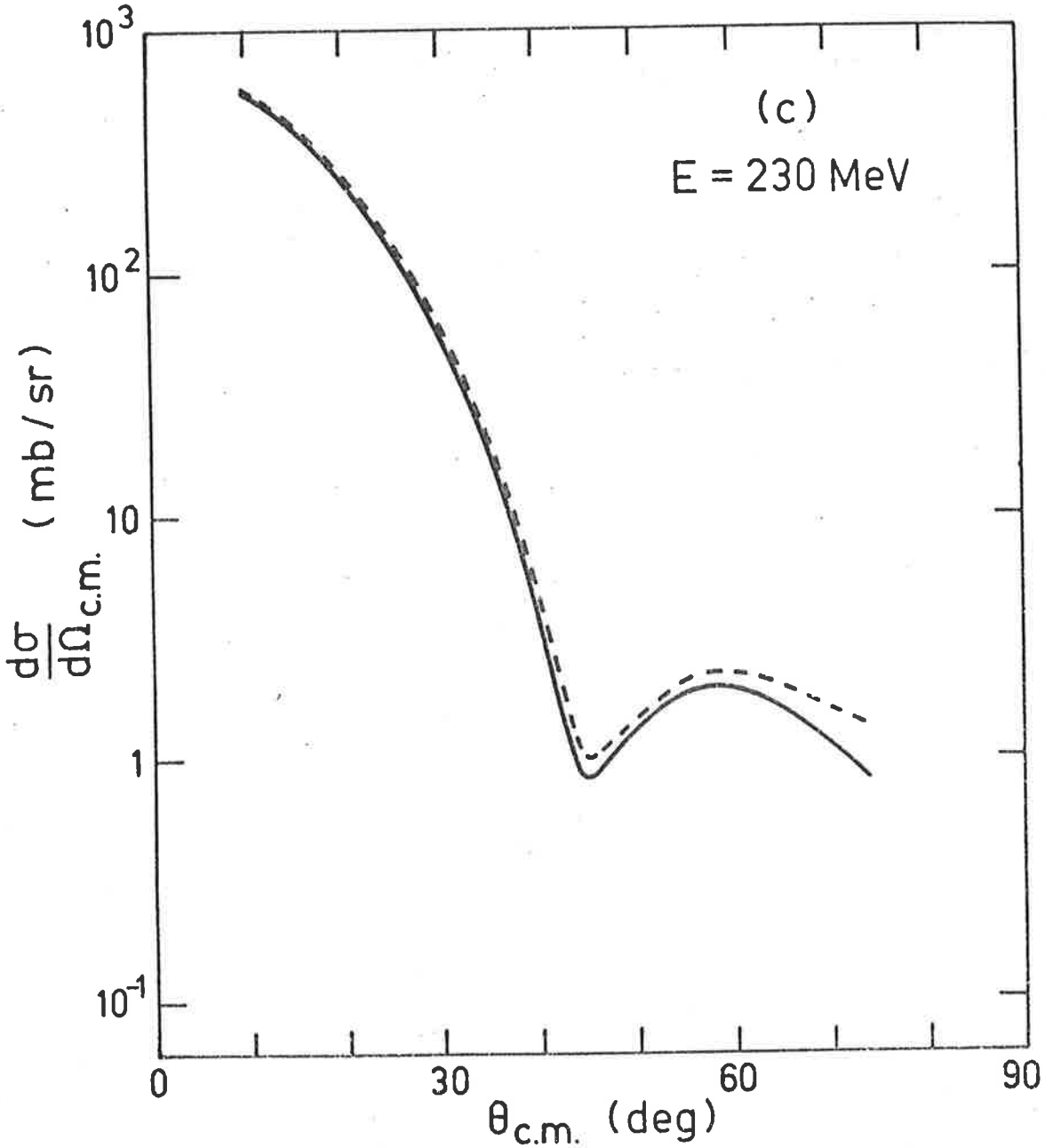
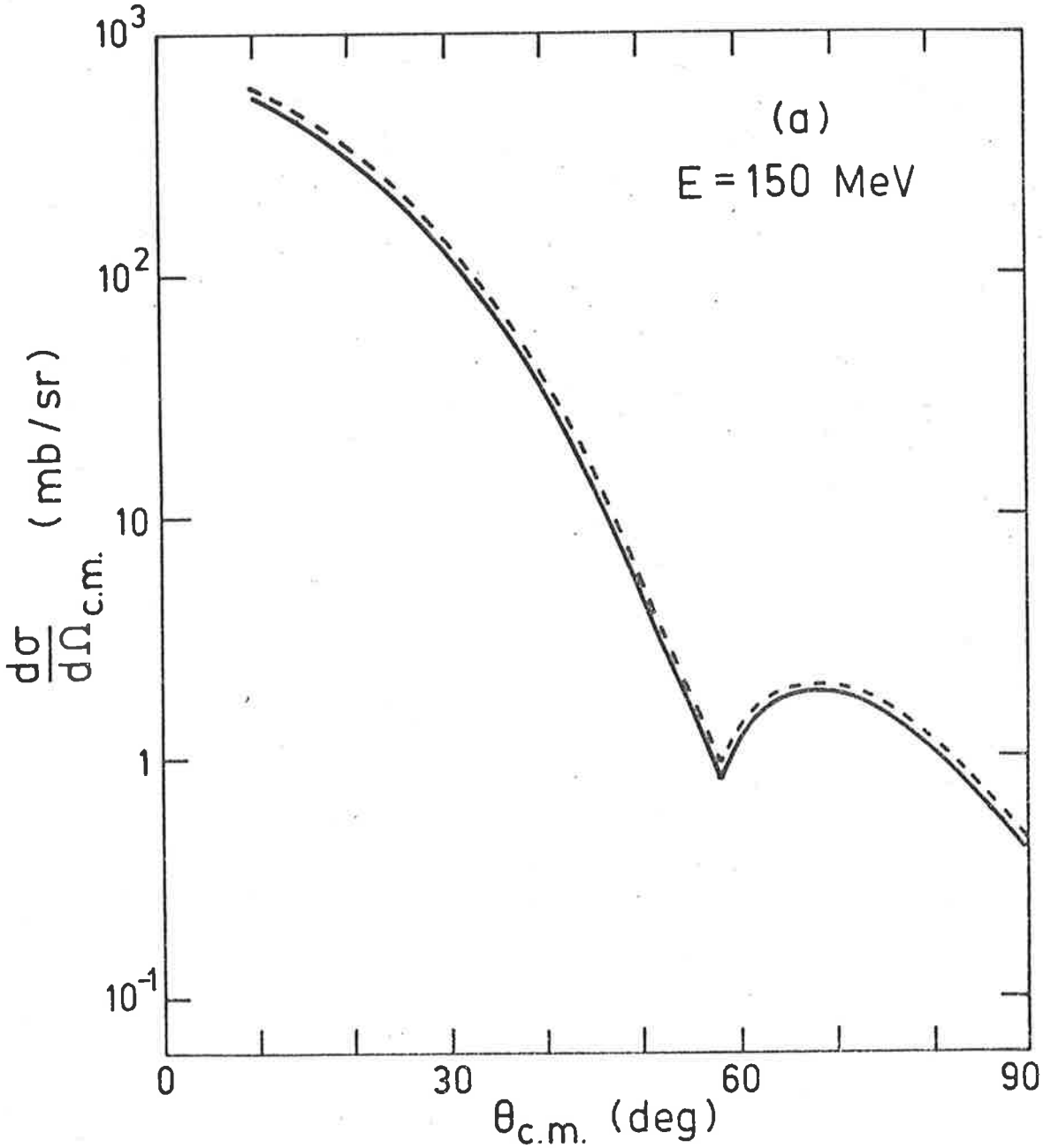
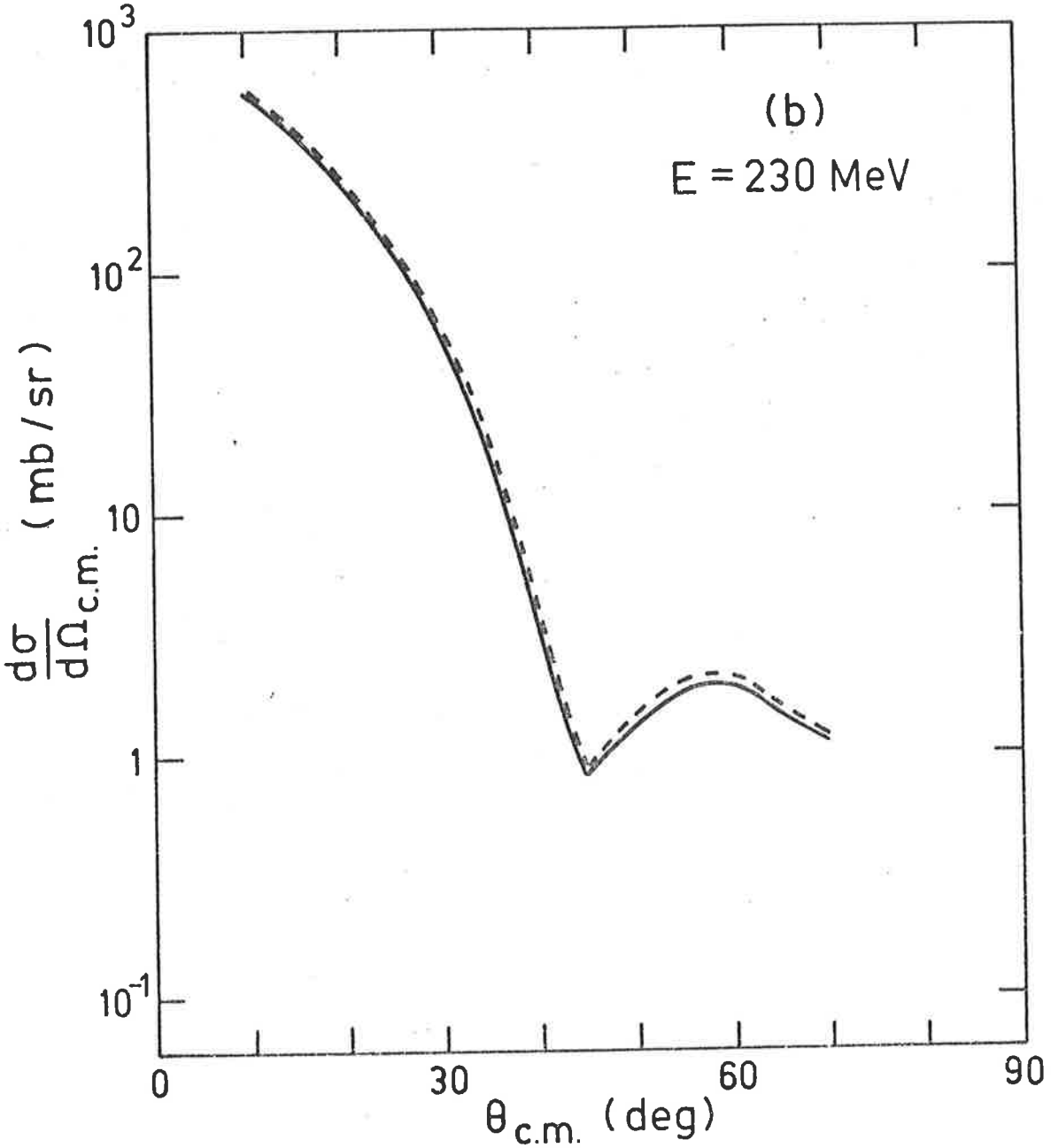

(c)
E = 230 MeV
10^3
10^2
10
1
10^{-1}
$\frac{d\sigma}{d\Omega}_{c.m.}$ (mb/sr)
0
30
60
90
$\theta_{c.m.}$ (deg)

FIG. 12

The elastic differential cross-sectionsfor pion-carbon scattering calculated from $U^{(1)} + U^{(2)}$ by using the parameters of pion-nucleon interactions from Table 2a (solid curves) and Table 2b (dashed curves) are compared. The chosen value of the correlation length (ℓ_c) is .46 fm.


(a)
E = 150 MeV
10^3
10^2
10
1
10^{-1}
$\frac{d\sigma}{d\Omega}_{c.m.}$ (mb/sr)
0
30
60
90
$\theta_{c.m.}$ (deg)


(b)
E = 230 MeV
10^3
10^2
10
1
10^{-1}
$\frac{d\sigma}{d\Omega}_{c.m.}$ (mb/sr)
0
30
60
90
$\theta_{c.m.}$ (deg)

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