ASYMMETRIC MINIMA
OF
INDEFINITE TERNARY QUADRATIC FORMS

by

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A Thesis submitted for the Degree of Doctor of Philosophy

in the University of Adelaide,

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May, 1966.

CONTENTS

SUMMARY

Let $f = f(\underline{x}) = f(x_1, x_2, \ldots, x_n)$ be an indefinite n-ary quadratic form of signature s and determinant ± 1; that is, $f(\underline{x}) = \underline{x}'A\underline{x}$ where A is a real symmetric matrix with determinant ± 1. Then when we say that f takes the value v we mean that there exists integral $\underline{x} \neq \underline{0}$ with $f(\underline{x}) = v$.

The problem of asymmetric minima is to find for each $t \geq 0$ the value $\phi_n^s(t)$ defined to be the infimum of the set of all positive α such that every form f takes a value in the closed interval $[-\alpha, t\alpha]$. The value $\phi_n^s(t)$ is thus a measure of the least closed interval $I = [-a, b]$ containing the origin and with asymmetry $b/a = t$ such that every form f takes a value in any open interval containing I.

For $n = 2$ Segre has given an upper bound on $\phi_2^0(t)$ which is best possible if and only if either t or $1/t$ is integral. However Tornheim has shown how to calculate $\phi_2^0(t)$ for any given $t > 0$ in terms of infinite chains $[g_i]$, $-\infty < i < \infty$, of positive integers and simple continued fractions associated with these chains, and it appears that $\phi_2^0(t)$ is an extremely complicated function.

In this thesis the function $\phi_3^1(t)$ is evaluated for all $t \geq 0$ and it is shown that $t\phi_3^1(t)$ is a continuous piecewise linear function of t. In fact constants α_i and β_i, $0 \leq i \leq 9$, are found such that

$$\phi_3^1(t) = \min_{0 \leq i \leq 9} \{ \max(\alpha_i, \beta_i/t) \} \qquad : t > 0.$$

This result is proved by showing that every indefinite ternary quadratic form of determinant -1 takes a value in each of the closed intervals $[-\alpha_i, \beta_i]$, and that there exist nine special forms F_i, $1 \leq i \leq 9$, with the property that F_i takes no value in the open interval $(-\alpha_i, \beta_{i-1})$, where the β_i are in descending order.

A further asymmetry problem concerning indefinite quadratic forms is the following. Let $m_+(f)$ and $m_-(f)$ denote the infimum of the non-negative values taken by the forms f and $-f$ respectively. Furthermore let $A(f)$ denote the ratio $m_-(f)/m_+(f)$ where this is defined. Restricting f to a given number of variables (n) and a given signature (s), the problem is for each integer $k \geq 1$ to determine the least value that the absolute value of the determinant of f may take if f satisfies $A(f) \geq k$.

This problem is dealt with in chapter 2 for two special cases, and the results so obtained are used later in the thesis.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief the thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

(R. T. Worley.)

ACKNOWLEDGEMENTS

It is a pleasure to thank my supervisor, Professor E. S. Barnes, for suggesting this asymmetric minimum problem for research. I should also like to thank him for the encouragement, helpful advice and criticism he has given throughout the duration of this work. Without his guidance and interest this thesis would not have been written.

I gratefully acknowledge the support the C.S.I.R.O. have given by awarding me a Senior Postgraduate Studentship.

INTRODUCTION

Part 1

In this section the terms and symbols to be used throughout this thesis will be introduced and some of the known results concerning quadratic forms will be given.

1 In this thesis we shall be concerned with real indefinite quadratic forms in n variables - that is, forms $f = f(\underline{x}) = \underline{x}'A\underline{x}$ where A is a real symmetric matrix - which have determinant $\det(f) = \det(A) \neq 0$. The signature of such forms is denoted by s.

As most results are concerned with $|\det(f)|$, we use $d(f)$, or d where it is not ambiguous, to denote $|\det(f)|$.

In the case of binary forms it is more usual to express results in terms of $\Delta = 2\sqrt{d}$, where $\Delta^2 = D$ is the discriminant of the form.

A form f will be called normalised if it has $d = 1$.

2 If there exists integral $\underline{x} \neq \underline{0}$ such that $f(\underline{x}) = v$ then v is called a value of the form f. If f does not take the value 0 it is called non-zero.

The quantities $m_+ = m_+(f)$ and $m_- = m_-(f)$ are defined by

$$m_+(f) = \inf\{v;\ v \geqslant 0 \text{ is a value of } f\},$$
$$m_-(f) = m_+(-f).$$

The problem of asymmetric minima is to find for each $t \geqslant 0$ the value $\phi_n^s(t)$ defined to be the infimum of the set of all positive α such that every normalised form f takes a value in the closed interval $[-\alpha, t\alpha]$. The value $\phi_n^s(t)$ is thus a measure of the least closed interval $I = [-a,b]$ containing the origin and with asymmetry $b/a = t$ such that every normalised form f takes a value in any open interval containing I.

3 In the theory of quadratic forms it is often convenient to pass from one form $f = \underline{x}'A\underline{x}$ to an equivalent form $g = \underline{x}'B\underline{x}$ where B is related to A in that there exists an integral unimodular matrix T such that $B = T'AT$. We use $f \sim g$ to denote that f is equivalent to g.

In passing to an equivalent form, d, n and s remain unchanged, and as equivalent forms take precisely the same values, m_+ and m_- are also unchanged.

If $v \neq 0$ is a value of f taken at a point $\underline{x} = (x_1, x_2, \ldots, x_n)$ where $\gcd(x_1, x_2, \ldots, x_n) = 1$ there exists a form g equivalent to f such that

$$g(1,0,..,0) = v.$$

4 The simple continued fraction $\alpha = (a_1, a_2, \ldots, a_n, \ldots)$ where all the a_i are positive integers is defined to have the value $\lim_{n\to\infty} p_n/q_n$ where

$$p_n/q_n = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_n}}}$$

$$= (a_1, a_2, \ldots, a_n).$$

The notation $(a_1, a_2, \ldots, a_r, \overline{a_{r+1}, \ldots, a_s})$ is used to denote the simple continued fraction

$$(a_1, \ldots, a_r, a_{r+1}, \ldots, a_s, a_{r+1}, \ldots, a_s, a_{r+1}, \ldots)$$

where the block $a_{r+1}, \ldots, a_s$ is repeated indefinitely.

If $\alpha = (a_1, a_2, \ldots, a_n, \ldots)$ and $\beta = (b_1, \ldots, b_n, \ldots)$ are two simple continued fractions then $\alpha > \beta$ if and only if the first non-zero signed difference $(-1)^{i-1}(a_i - b_i)$ is positive. Furthermore if $\alpha_j = (a_1, a_2, \ldots, a_j)$ and $\beta_j = (a_1, a_2, \ldots, a_j + 1)$, then $\alpha_j > \alpha > \beta_j$ if j is even, and $\alpha_j < \alpha < \beta_j$ if j is odd.

5 A non-zero indefinite binary quadratic form $f = ax^2 + bxy + cy^2$ is called reduced if

$$0 < \Delta - b < 2|a| < \Delta + b.$$

6 Lagrange's results: The following properties of non-zero indefinite binary quadratic forms are due to

Lagrange and are proved in Dickson [6].

(i) Every form is equivalent to at least one reduced form.

(ii) To every reduced form f there exists an infinite chain (f_i), $-\infty < i < \infty$, of reduced forms equivalent to f and such that

$$f_i = (-1)^i a_i x^2 + b_i xy + (-1)^{i+1} a_{i+1} y^2,$$

where the f_i are related by the following property: There exists a chain $[g_i]$ of positive integers g_i, $-\infty < i < \infty$, such that

(a) $(g_i, g_{i+1}, g_{i+2}, \ldots) = F_i$
$= (b_i + \Delta)/2a_{i+1}$,

(b) $(0, g_{i-1}, g_{i-2}, \ldots) = S_i$
$= (-b_i + \Delta)/2a_{i+1}$,

(c) $K_i = F_i + S_i = \Delta/a_{i+1}$, and so

(d) $f_i = (-1)^{i+1} a_{i+1} [y^2 + (-1)^{i+1}(F_i - S_i)xy - F_i S_i x^2]$
$= (-1)^{i+1} a_{i+1} (y \pm F_i x)(y \mp S_i x)$.

(iii) Every reduced form equivalent to f lies in the chain (f_i).

(iv) Every value v taken by f such that $|v| < \Delta/2$ occurs as one of the coefficients $(-1)^i a_i$ in the chain (f_i).

7 Segre's result: The following result, due basically to Segre [19] is proved in Cassels [3].

If $f = ax^2 + bxy + cy^2$ takes no values in the open interval $(-p,q)$ where $p > 0$ and $q > 0$, then

$$d \geq pq + \tfrac{1}{4}\max(p^2,q^2).$$

Furthermore equality is required if and only if either p/q or q/p is integral and f is equivalent to the form $-px^2 - \max(p,q)xy + qy^2$.

8 Tornheim's results: In his paper Tornheim [20] has used the continued fraction approach of Lagrange to extend Segre's result above in the case where either p/q or q/p is integral. Although the main result is not of use in this thesis a number of the minor results will be used. However before stating these results the necessary notation has to be introduced.

Let q be an indefinite binary quadratic form, and let Q denote the form $q/2\sqrt{d}$ so that Q has $\Delta = 1$. Let $P = m_+(Q)$, $N = m_-(Q)$ and for a given integer $k \geq 2$ let $A = \max(1/P, k/N)$ where this is defined (we shall not be interested in the cases where either $P = 0$ or $N = 0$). Let $[g_i]$ be the chain of integers associated with Q as in 6 above. Then the following are Tornheim's results.

(i) If any $g_{2i+1} \geq 2$ then $A \geq 2k$.

(ii) If any $g_{2i} > k$ then $A \geq k + \sqrt{5}$.

(iii) $A \geq \sqrt{k^2 + 4k}$ with equality if and only if Q is equivalent to a multiple of the form $x^2 - kxy - ky^2$.

(iv) If all $g_{2i+1} = 1$ and all $g_{2i} \leq k$ then $k/N \geq \sqrt{k^2 + 4k}$.

(v) If k is odd, all $g_{2i+1} = 1$, and $g_{2j} \geq k + 1$ for some j then $A \geq \sqrt{k^2 + 6k + 1}$.

(vi) If k is even and $A < \sqrt{k^2 + 6k + 1}$ then $g_{2i+1} = 1$ and $k/2 \leq g_{2i} \leq k$ for all i.

(vii) Either $A = A_1 = \sqrt{k^2 + 4k}$ with equality as in (iii) above or $A \geq A_2 = (k^2 + k + (3k - 1)A_1)/(4k - 2)$. Furthermore while $A = A_2$ only for one form Q (and its equivalent forms) there exist forms with A arbitrarily close, but not equal, to A_2.

This last result may be interpreted to give the following. If q is a binary form which takes no values in the open interval $(-l, kl)$ then either
(a) $d = l^2 A_1^2/4$ and $q \sim l(kx^2 - kxy - y^2)$, or
(b) $d \geq l^2 A_2^2/4$.

It should be noticed that the relations

$$1/P = \sup_{-\infty<i<\infty} K_{2i}, \qquad 1/N = \sup_{-\infty<i<\infty} K_{2i+1}$$

do not, as Tornheim appears to have assumed, follow directly from Lagrange's results quoted in 6 above in the cases where the suprema are less than 2. However the relations can still be proved in these cases.

Part 2

In this section some of the known results connected with the asymmetric minimum problem will be given.

1 The problem of the symmetric minimum, as it is sometimes called, is that of finding sup M(f) over all normalised n-ary quadratic forms f with signature s, where

$$M(f) = \min \{m_+(f), m_-(f)\}$$
$$= \inf \{|v| ; v \text{ is a value of } f\}.$$

Hence

$$\sup M(f) = \phi_n^s(1).$$

For convenience we shall use ϕ_n^s to denote $\phi_n^s(1)$.

In 1879 Markoff [8] showed not only that $\phi_2^0 = \sqrt{4/5} = m_1$, but that there exists an infinite sequence $m_1, m_2, m_3, \ldots$ of successive minima m_i with limit 2/3, and a sequence of forms $f_1, f_2, f_3, \ldots$ such that $M(f_i) = m_i$, with the property that $M(f) \leqslant 2/3$ for all normalised forms not equivalent to one of the forms f_i.

It is suspected that a similar sequence of successive minima m_i occurs for $n = 3$ and $n = 4$, that $\lim_{i\to\infty} m_i = 0$ for $n = 3$ and $n = 4$, and that $\phi_n^s = 0$ for $n \geq 5$. However there is no conclusive evidence that this is so.

For $n = 3$ Markoff [9] has shown that $\phi_3^1 = \sqrt[3]{2/3}$, and Venkov [21] has shown the existence of at least eleven successive minima.

For $n = 4$, $s = 0$, Oppenheim [13] has shown that $\phi_4^0 = \sqrt[4]{4/9}$ and that there exists a sequence of at least eight successive minima.

For $n = 4$, $s = \pm 2$, Oppenheim [14] has shown that $\phi_4^2 = \phi_4^{-2} = \sqrt[4]{4/7}$ and that there exists a sequence of at least three successive minima. In this particular case there are two non-equivalent forms $f_2^{(1)}$ and $f_2^{(2)}$ with $M(f) = m_2$.

2 A problem that is sometimes known as the asymmetric minimum problem to distinguish it from the above problem is that of finding $\sup M_+(f)$ over all normalised n-ary forms f with signature s, where

$$M_+(f) = \inf \{v;\ v > 0 \text{ is a value of } f\}.$$

For $n = 2$ the solution of this problem, due to Mahler and proved in Cassels [3], is that $M_+(f) \leq 2$ for all f, that equality is necessary only when f is equivalent to $x_1 x_2$, and that for any $\epsilon > 0$ there exist infinitely many non-equivalent forms with $M_+(f) > 2 - \epsilon$. Hence in contrast to the symmetric minimum problem there exists no sequence of successive minima.

For $n = 3$ Davenport [5] has shown that $\sup M_+(f) = \sqrt[3]{4}$ for forms of signature 1 and that $\sup M_+(f) = \sqrt[3]{27/4}$ for forms of signature -1. Oppenheim [15] has extended these results by showing the existence of several successive minima.

For $n = 4$ Oppenheim [16] has shown that $\sup M_+(f) = 2, \sqrt[4]{16/3}, \sqrt[4]{256/27}$ for forms of signature 0, 2, and -2 respectively. He has also shown the existence of several successive minima.

For $n \geq 5$ it is suspected that $M_+(f) = 0$ for all forms f.

3 Another one-sided problem concerning indefinite quadratic forms, similar to the above, is that of finding $\sup m_+(f)$ over all normalised forms f. This problem differs from the above problem only in that zero forms are virtually excluded from consideration. It is easily seen that $\sup m_+(f) = \phi_n^{-s}(0)$.

For $n = 2$ it is easily seen that $\overset{o}{\phi}_2(0) = 2$, for $\overset{o}{\phi}_2(0) \leq \sup M_+(f) = 2$, while taking the limit as $p/q \to \infty$ in Segre's work shows that $\overset{o}{\phi}_2(0) \geq 2$.

For $n = 3, 4$ Barnes [1] and Barnes and Oppenheim [2] have shown that $\phi_3^{1}(0) = \sqrt[3]{16/5}$, $\phi_3^{-1}(0) = \sqrt[3]{4/3}$, $\phi_4^{0}(0) = \sqrt[4]{64/81}$, $\phi_4^{-2}(0) \leq \sqrt[4]{32/27}$ and $\phi_4^{2}(0) \leq \sqrt[4]{64/27}$.

For $n \geq 5$ it is suspected that $\phi_n^{s}(0) \in 0$.

4 There are a number of results on the asymmetric minimum of binary forms.

For $t \geq 1$ Segre [19] and others [7],[10],[11],[12] and [17], have shown that $\phi_2^{0}(t) \leq 2(t^2 + 4t)^{-\frac{1}{2}}$, with equality only if t is integral, in which case the form

$$f^{(t)} = 2(tx^2 - txy - y^2)/\sqrt{t^2 + 4t}$$

takes no values in the open interval $(-\phi_2^{0}(t), t\phi_2^{0}(t))$. By using the relation

$$\phi_n^{s}(t) = \frac{1}{t}\phi_n^{-s}(1/t) \quad ; \qquad t > 0$$

a corresponding result for $t \leq 1$ may be deduced.

Sawyer [18] has proved that for integral $k \geq 1$ every normalised form f, not equivalent to $f^{(k)}$, takes a value in the closed interval $I = [-\psi(k), k\psi(k)]$ where $\psi(k) = 2(k^2 + 4k + 2 + k^{-2})^{-\frac{1}{2}}$, and that furthermore the interval I may be opened for $k \geq 2$.

Tornheim [20] has shown that for integral $k \geq 2$ every normalised form f not equivalent to $f^{(k)}$ or another form $f_1^{(k)}$ takes a value in the open interval

$J = (-\chi(k), k\chi(k))$ where

$$\chi(k) = [(k^2 + k + (3k - 1)\sqrt{k^2 + 4k})/(8k - 4)]^{-\frac{1}{2}}.$$

Furthermore he has shown that $f_1^{(k)}$ takes the value $k\chi(k)$ and that for arbitrarily small $\epsilon > 0$ there exist infinitely many non-equivalent normalised forms taking no values in the interval $(-\chi(k) + \epsilon, k\chi(k) - k\epsilon)$.

For non-integral $t > 0$ it follows from the work of Tornheim that $\overset{\circ}{\phi}_2(t)$ can be found as follows. Let $[g_i]$, $-\infty < i < \infty$, be an arbitrary chain of positive integers and let K_i be defined as in section 6 of part 1 of this introduction. Let

$$p = \sup_i K_{2i}, \qquad n = \sup_i K_{2i+1},$$

and let $A([g_i]) = \max(p, tn)$. Then

$$\overset{\circ}{\phi}_2(t) = \inf A([g_i])$$

where the infimum is taken over all possible chains $[g_i]$.

CHAPTER 1

Results on the asymmetric minima of indefinite ternary quadratic forms.

The complete answer to the problem of the asymmetric minima of indefinite ternary quadratic forms, that is, the evaluation of $\phi_3^{1}(t)$ and $\phi_3^{-1}(t)$ for all $t \geqslant 0$, follows from the following theorem.

Theorem A

Every normalised indefinite ternary quadratic form of signature 1 takes a value in each of the following closed intervals:

I_0 : $[0, \sqrt[3]{4/3}]$

I_1 : $[-\sqrt[3]{1/48}, \sqrt[3]{54/49}]$

I_2 : $[-\sqrt[3]{2/49}, \sqrt[3]{8/9}]$

I_3 : $[-\sqrt[3]{1/9}, \sqrt[3]{125/144}]$

I_4 : $[-\sqrt[3]{3/16}, \sqrt[3]{2/3}]$

I_5 : $[-\sqrt[3]{2/3}, \sqrt[3]{27/112}]$

I_6 : $[-\sqrt[3]{125/112}, \sqrt[3]{2/9}]$

I_7 : $[-\sqrt[3]{16/9}, \sqrt[3]{1/24}]$

I_8 : $[-\sqrt[3]{8/3}, \sqrt[3]{2/135}]$

I_9 : $[-\sqrt[3]{16/5}, 0]$.

Furthermore if we define:

$$
\begin{aligned}
f_1 &= (x + \tfrac{1}{2}z)^2 && - \tfrac{1}{2}(z^2 - 2yz - 2y^2) \\
f_2 &= (x + \tfrac{1}{6}y + \tfrac{1}{2}z)^2 && - \tfrac{7}{12}(z^2 - 2yz - \tfrac{5}{3}y^2) \\
f_3 &= (x + \tfrac{1}{2}y + \tfrac{1}{2}z)^2 && - \tfrac{3}{4}(z^2 - 2yz - y^2) \\
f_4 &= (x + \tfrac{4}{5}y + \tfrac{2}{5}z)^2 && - \tfrac{24}{25}(z^2 - yz - y^2) \\
f_5 &= (x + \tfrac{1}{2}y + \tfrac{1}{2}z)^2 && - \tfrac{5}{4}(z^2 - \tfrac{6}{5}yz - \tfrac{3}{5}y^2) \\
f_6 &= (x + \tfrac{1}{3}y)^2 && - \tfrac{8}{3}(z^2 - yz - \tfrac{1}{3}y^2) \\
f_7 &= (x + \tfrac{1}{2}y)^2 && - 3(z^2 - yz - \tfrac{1}{4}y^2) \\
f_8 &= x^2 && - 8(z^2 - yz - \tfrac{1}{8}y^2) \\
f_9 &= (x + \tfrac{1}{2}y)^2 && - 15(z^2 - yz - \tfrac{1}{20}y^2),
\end{aligned}
$$

and let F_i, $1 \leqslant i \leqslant 9$, denote that multiple of f_i which has determinant -1, then for $0 \leqslant i \leqslant 8$ closure is required on the left of interval I_{i+1} and on the right of interval I_i only for forms equivalent to F_{i+1}.

Clearly the closure conditions of this theorem imply that if I is any interval about the origin in which every normalised indefinite ternary quadratic form of signature 1 takes a value then I must contain an interval I_i for some i with $0 \leqslant i \leqslant 9$. Thus in particular for every $t \geqslant 0$ the interval $[-\phi_3^1(t), t\phi_3^1(t)]$ must have an end-point in common with an interval I_i. From this it follows that as t increases from zero, $\phi_3^1(t)$ and $t\phi_3^1(t)$ remain fixed alternately, so that the graph of $t\phi_3^1(t)$ is piecewise linear and continuous. Thus if we let $I_i = [-\alpha_i, \beta_i]$, we have that

$$\phi_3^{1}(t) = \begin{cases} \min\limits_{0 \leq i \leq 9} \{ \max(\alpha_i, \beta_i/t) \} & : \quad t > 0 \\ \alpha_9 & : \quad t = 0 , \end{cases}$$

with a similar expression for $\phi_3^{-1}(t)$.

It is of interest to note that the forms f_i have rational coefficients. The following table gives $m_-(f_i)$ and $d(f_i)$, while $m_+(f_i) = 1$ for all i.

Table 1.1

i	1	2	3	4	5	6	7	8	9
$m_-(f_i)$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{5}$	1	$\frac{5}{3}$	2	4	6
$d(f_i)$	$\frac{3}{4}$	$\frac{42}{54}$	$\frac{9}{8}$	$\frac{144}{125}$	$\frac{3}{2}$	$\frac{112}{27}$	$\frac{9}{2}$	24	$\frac{135}{2}$

The proof of theorem A occupies most of this thesis. After some preliminary results in Chapter 2. the forms F_i are considered in detail in Chapter 3. In Chapter 4 theorem A is broken down into ten separate sub-theorems and in the following chapters these sub-theorems are proven.

CHAPTER 2

2.1 In this chapter we prove results concerning the determinant of indefinite binary and ternary quadratic forms f which have asymmetry

$$A(f) = \frac{m_-(f)}{m_+(f)} \geq k$$

for integral $k \geq 2$. The following are the theorems proved.

Theorem 2.1

For integral $k \geq 2$ there exists a positive constant $c(k)$ such that whenever an indefinite binary quadratic form $q = q(x,y)$ satisfies

$$0 < (1 - c)m_+(q) \leq m_-(q)/k$$

for some c with $0 \leq c < c(k)$ it may be concluded that either

(i) $q \sim m_+(q)\,(x^2 - kxy - ky^2)$ and $A(q) \doteq k$, or

(ii) $d(q) \geq [m_+(q)]^2(1 - c)^2(k^2 + 6k + 1)/4$.

Theorem 2.2

Let $k \geq 2$ be integral and define

$$K = k^2 + 6k + 1,$$

$$t(S) = K^2(1 + 4/S)/64,$$

$$d_1 = (K^2 + 12K)/64,$$

$$d_2 = \max\ (\ \min\{t(S), 9(S + \sqrt{5})^2/64\}),$$

where the maximum is taken over all positive integers S, and let S^* denote the S at which the maximum is attained. For positive integers r and s let $q(r,s;y,z)$ denote the indefinite binary quadratic form

$$y^2 - \frac{s(r+2)}{rs+r+s}\,yz - \frac{r+2}{rs+r+s}\,z^2,$$

and for integral l, $0 \leqslant l < s$, let $f(r,s,l;x,y,z)$ denote the indefinite ternary quadratic form

$$(x + \tfrac{k}{2}y + \tfrac{l}{s}z)^2 - \tfrac{1}{4}(k^2 + 4k)q(r,s;y,z).$$

Let $f = f(x,y,z)$ be an indefinite ternary quadratic form of signature 1 with $d(f) = d$ such that (i) $m_+(f) = 1$ and this value is attained by f, and (ii) $m_-(f) \geqslant k$. Then either

(a) $d \geqslant \min(d_1, d_2)$, or

(b) $m_-(f) = k$ and $f \sim f(r,s,l;x,y,z)$ for some r and s such that $r \leqslant s \leqslant S^*$.

Theorem 2.1 may be used to obtain information about indefinite binary quadratic forms q that have asymmetry $A(q)$ slightly below an integer k. It is clear from the statement of the theorem that if q is an indefinite binary quadratic form with

$$k(1 - c(k)) < A(q) < k$$

then setting $c = 1 - A(q)/k$ yields that

$$d(q) \geq [m_-(q)]^2(k^2 + 6k + 1)/4k^2.$$

In addition, the following corollary to theorem 2.1 should be noted.

Corollary to Theorem 2.1

If $k \geq 2$ is integral and if $q = q(x,y)$ is an indefinite binary quadratic form with $m_+(q) = 1$ and $m_-(q) \geq k + l$ where $l > 0$, then

$$d(q) \geq \tfrac{1}{4}(k^2 + 6k + 1) + l.$$

It should be noted that the condition that f should attain the value $m_+(f) = 1$ can be removed to make theorem 2.2 apply to all forms f with $m_+(f) = 1$ and $m_-(f) \geq k$. This is easily done with the help of theorem 4.1 in exactly the same way as it is shown that theorem C_i implies theorem B_i (see chapter 4).

It should also be noticed that not all forms $f(r,s,l;x,y,z)$ have $m_+(f) = 1$ and $m_-(f) = k$. In fact it appears to be the exception rather than the rule that a form shall satisfy this condition. Calculations performed on the CSIRO's C.D.C. "3200" computer in Adelaide have shown that for $k = 7,10,11$ and 12 not one of the forms has $m_+(f) = 1$, $m_-(f) = k$ and $d(f) < \min(d_1, d_2)$, while for $k = 2,3,4,5,6,8$ and 9

the forms listed in table 2.1 were found to be the only ones satisfying these constraints (note: for simplicity in the table the transformation $x \to x - [\frac{k}{2}]y$ has been performed, where $[\frac{k}{2}]$ denotes the integer part of $k/2$).

For comparison with the determinants of the forms listed in table 2.1, $\min(d_1, d_2)$ is listed in table 2.2.

Table 2.1

k	form	r,s,l,d(f)
2	$(x + \frac{1}{2}z)^2 - 3(y^2 - yz - \frac{1}{4}z^2)$	$4,4,2,4\frac{1}{2}$
2	$(x + \frac{1}{2}z)^2 - 3(y^2 - yz - \frac{1}{2}z^2)$	$2,2,1,6\frac{3}{4}$
2	$x^2 - 3(y^2 - \frac{4}{3}yz - \frac{1}{3}z^2)$	1,4,0,7
3	$(x + \frac{1}{2}y)^2 - \frac{21}{4}(y^2 - \frac{8}{7}yz - \frac{2}{7}z^2)$	$2,4,0,16\frac{7}{8}$
4	$x^2 - 8(y^2 - yz - \frac{1}{8}z^2)$	8,8,0,24
5	$(x + \frac{1}{2}y + \frac{1}{2}z)^2 - \frac{45}{4}(y^2 - \frac{6}{5}yz - \frac{1}{15}z^2)$	3,18,9,54
6	$(x + \frac{1}{2}z)^2 - 15(y^2 - yz - \frac{1}{20}z^2)$	$20,20,10,67\frac{1}{2}$
6	$(x + \frac{1}{2}z)^2 - 15(y^2 - yz - \frac{1}{6}z^2)$	$6,6,3,93\frac{3}{8}$
6	$x^2 - 15(y^2 - \frac{6}{5}yz - \frac{1}{15}z^2)$	3,18,0,96
8	$x^2 - 24(y^2 - yz - \frac{1}{9}z^2)$	9,9,0,208
9	$(x + \frac{1}{2}y + \frac{1}{2}z)^2 - \frac{117}{4}(y^2 - \frac{14}{13}yz - \frac{1}{39}z^2)$	9,42,21,270

Table 2.2

k	2	3	4	5	6	7
$\min(d_1,d_2)$	7.5..	17.5	33.7..	59.3..	96.7..	149.3..

k	8	9
$\min(d_1,d_2)$	220.5..	314.1..

It will be noticed that the forms in table 2.1 which have least determinant for $k = 2,4$ and 6 are equivalent to the multiples of the forms F_7, F_8 and F_9 which have $m_+ = 1$. This may be related to the fact that these forms have determinant in absolute value much less than the corresponding value of $\min(d_1, d_2)$.

We shall now prove theorems 2.1 and 2.2 and the corollary to theorem 2.1

Proof of the Corollary to Theorem 2.1

Taking $c = 0$ in theorem 2.1 we find that

$$d = d(q) \geqslant (k^2 + 6k + 1)/4.$$

Suppose that $d < \frac{1}{4}(k^2 + 6k + 1) + 1$. Then as $m_+(q) = 1$ we may write for arbitrarily small $\delta \geqslant 0$

$$q \sim \frac{1}{1 - \delta}(x + \lambda y)^2 - d(1 - \delta)y^2,$$

and so by choosing x such that

$$(k^2 + 2k + 1)/4 \leqslant (x + \lambda)^2 \leqslant (k^2 + 4k + 4)/4$$

we obtain a value of q which for sufficiently small δ lies in the open interval $(-k - 1, 1)$. This contradicts either $m_+ = 1$ or $m_- \geqslant k + 1$.

Proof of Theorem 2.1

The proof of this theorem depends upon the work of Tornheim [20]. We let

$$Q(x,y) = q(x,y)/2\sqrt{d(q)},$$

so that Q is an indefinite binary quadratic form with discriminant $\Delta^2 = 1$. We define

$$M = m_+(q),$$

$$N = m_-(q),$$

$$A = \max(1/M, k/N),$$

$$A_1 = \sqrt{k^2 + 4k},$$

$$A_2 = [k^2 + k + (3k - 1)A_1]/(4k - 2),$$

$$c^*(k) = 1 - A_1/A_2 > 0.$$

Then Tornheim has shown that either

(a) $A = A_1$ and $N = kM$, in which case

$$Q \sim M(x^2 - kxy - ky^2), \quad \text{or}$$

(b) $A \geqslant \sqrt{k^2 + 6k + 1}$, or

(c) $M \geqslant 1/A_1$ and $N \leqslant k/A_2$.

Consider firstly the third alternative. This implies that

$$N/kM \leqslant A_1/A_2 = 1 - c^*(k),$$

and so

$$m_-(q)/k \leqslant (1 - c^*(k))m_+(q).$$

Hence if we set $c(k) = c^*(k)$ we have, for $0 \leqslant c < c(k)$, that

$$m_-(q)/k < (1 - c)m_+(q),$$

which contradicts the given. It remains to show that, with $c(k) = c^*(k)$, the conclusions (i) and (ii) of the theorem follow from the alternatives (a) and (b) above. Since (a) clearly implies (i) we need only show that (b) implies (ii).

From (b) we have that

$$\max(1/M, k/N) \geqslant \sqrt{k^2 + 6k + 1},$$

and so

$$2\sqrt{d(q)} \geqslant \sqrt{k^2 + 6k + 1}\,\min(m_+(q), m_-(q)/k).$$

Using the given it follows that

$$d(q) \geqslant \tfrac{1}{4}[m_+(q)]^2(1 - c)^2(k^2 + 6k + 1)$$

as required.

In order to prove theorem 2.2 we need the following lemma on indefinite binary quadratic forms.

Lemma 2.1

Let $q(x,y)$ be an indefinite binary quadratic form with $\Delta = 1$ and let $[g_i]$ be the chain of positive integers associated with the chain of reduced forms equivalent to q. Suppose that the elements g_{2i} of the chain are bounded above by the integer S, and let

$$C(S) = 3\sqrt{5}/40(S + 1)^2.$$

Let M, N denote $m_+(q), m_-(q)$ respectively. Let $k \geqslant 2$ be integral and let c_1 and c_2 be small positive numbers with $c_1 < C(S)$ such that for each negative value $-n$ taken by q either

$$n/N \leqslant 1 + c_1$$

or

$$n/N \geqslant (k^2 + 6k + 1)/(k^2 + 4k) - c_2.$$

Then either

(i) $1/N \geqslant 2$, or

(ii) $1/N \geqslant \sqrt{1 + 4/S}[(k^2 + 6k + 1)/(k^2 + 4k) - c_2]$, or

(iii) There exist integers r and s, both at most S, such that for all integers i,

$$g_{2i+1} = 1, \quad g_{4i} = r, \quad g_{4i+2} = s.$$

Proof

If $g_{2i+1} \geqslant 2$ for any i, then as indicated in the introduction (part 1 section 6) q takes the value $-n$ where

$$1/n = (g_{2i+1}, g_{2i+2}, g_{2i+3}, \ldots) + (0, g_{2i}, g_{2i-1}, \ldots).$$

Hence $1/n \geqslant 2$, and as $n \geqslant N$ it follows that $1/N \geqslant 2$.

We now suppose that $g_{2i+1} = 1$ for all i, and in addition that the chain is not of the form given in the third alternative. Clearly the proof of the lemma will be complete when we show that alternative (ii) must hold.

As the chain is not of the form in alternative (iii) there must exist an i for which $g_{2i} \neq g_{2i+4}$. Let

$$s = \max (g_{2i}, g_{2i+4}),$$

$$t = \min(g_{2i}, g_{2i+4}), \quad \text{and}$$

$$r = g_{2i+2}.$$

Let

$$\begin{aligned} 1/n &= (1,r,1,t,1,\ldots) + (0,s,1,\ldots) \\ &= (1,r,1,\lambda) + (0,\mu) \end{aligned}$$

and

$$\begin{aligned} 1/n_1 &= (1,r,1,s,1,\ldots) + (0,t,1,\ldots) \\ &= (1,r,1,\mu) + (0,\lambda), \end{aligned}$$

where the ... indicates the continuation of the chain in the expected manner, so that $-n$ and $-n_1$ are values taken by q. Consider the function

$$\begin{aligned} f(x) &= (0,r,x) - x \\ &= x/(1 + rx) - x. \end{aligned}$$

Then the derivative $f'(x)$ of $f(x)$ is given by

$$f'(x) = 1/(1 + rx)^2 - 1,$$

and so $f'(x) < -\frac{3}{4}$ for $r \geqslant 1$ and $x > 1$. Now by the mean value theorem of calculus, as $f(x)$ is continuous and differentiable for $r \geqslant 1$ and $x > 1$, we have for $r \geqslant 1$ and $1 < x_2 < x_1$ that

$$f(x_1) - f(x_2) = (x_1 - x_2)f'(\alpha)$$

for some α with $x_2 < \alpha < x_1$. Substituting $x_1 = (1,\lambda)$ and $x_2 = (1,\mu)$ and simplifying, noting that $f'(\alpha) < -\frac{3}{4}$, gives that

$$1/n - 1/n_1 < -\tfrac{3}{4}(1/\lambda - 1/\mu). \qquad (2.1$$

Now

$$\mu - \lambda = (s,1,\ldots) - (t,1,\ldots)$$
$$\geqslant (s - t) + (0,\overline{1}) - (0,\overline{1,S})$$
$$> 1 + \tfrac{1}{2} - 1$$
$$= \tfrac{1}{2}$$

and $\lambda\mu < (S + 1)^2$ as λ and μ are each at most $S + 1$. Hence

$$1/\lambda - 1/\mu > 1/2(S + 1)^2.$$

Using this in (2.1) yields that

$$1/n - 1/n_1 < -3/8(S + 1)^2.$$

Now as $1/N \geqslant 1/n_1$ it follows that

$$n/N - 1 > 3n/8(S + 1)^2,$$

from which, as $1/n \leqslant (1,\overline{1}) + (0,\overline{1}) = \sqrt{5}$, we can deduce that

$$n/N > 1 + 3\sqrt{5}/40(S + 1)^2 = 1 + C(S) > 1 + c_1.$$

Hence, using the given conditions, we must have

$$n/N \geqslant (k^2 + 6k + 1)/(k^2 + 4k) - c_2.$$

Now

$$1/n \geqslant (\overline{1,S}) + (0,\overline{S,1}) = \sqrt{1 + 4/S},$$

and so we can conclude that

$$1/N \geqslant \sqrt{1 + 4/S}[(k^2 + 6k + 1)/(k^2 + 4k) - c_2],$$

which is alternative (ii) as required.

The Proof of Theorem 2.2

Let f be an indefinite ternary quadratic form of signature 1 such that $m_+(f) = 1$, $m_-(f) \geqslant k$, and let f attain the value 1. By passing to a suitable equivalent form we may assume f to be given in the form

$$f = (x + \lambda y + \mu z)^2 + q(y,z), \qquad (2.2$$

where q is an indefinite binary quadratic form. Let e denote $m_-(q)$, so that for arbitrarily small $\rho \geqslant 0$ we may write

$$q(y,z) \sim q_\rho(y,z) = \frac{-e}{1-\rho}(y + \delta_\rho z)^2 + \frac{d(1-\rho)}{e}z^2,$$

where δ_ρ depends on ρ and satisfies $|\delta_\rho| \leqslant \frac{1}{2}$. Then for arbitrarily small $\rho \geqslant 0$ there exists a form f_ρ such that

$$f \sim f_\rho = (x + \lambda_\rho y + \mu_\rho z)^2 + q_\rho(y,z),$$

where λ_ρ and μ_ρ depend on ρ.

Consider the section

$$t(x,y) = (x + \lambda_\rho y)^2 - ey^2/(1-\rho)$$

of f_ρ. Clearly

$$m_+(t) = 1, \quad m_-(t) \geqslant m_-(f) \geqslant k.$$

Hence we may apply theorem 2.1, with $\delta = 0$, to t to conclude that either

(i) $t \sim x^2 - kxy - ky^2$, or

(ii) $d(t) \geqslant (k^2 + 6k + 1)/4$.

Now one of these possibilities must be true for arbitrarily small ρ. If the second possibility holds for arbitrarily small ρ, we have that

$$e/(1 - \rho) \geqslant (k^2 + 6k + 1)/4 = K/4$$

for arb. small ρ and so $e \geqslant K/4$. Now q cannot take any value in the open interval $(0,3/4)$, else by choosing x suitably we could obtain a value of f contradicting $m_+(f) = 1$. Hence as $m_-(q) = e$, q can take no values in the open interval $(-e,3/4)$. Then by a result of Segre mentioned in the introduction

$$d(q) \geqslant 3e/4 + \tfrac{1}{4}\max(9/16, e^2),$$

i.e. $$d \geqslant 3K/16 + K^2/64 = d_1.$$

We now consider the case that the first possibility above, namely $t \sim x^2 - kxy - ky^2$, occurs for arbitrarily small ρ. This implies that

$$d(t) = e/(1 - \rho) = (k^2 + 4k)/4$$

for arb. small ρ. Hence our "arb. small ρ" must be $\rho = 0$, and so

$$t = (x + \lambda_0 y)^2 - \tfrac{1}{4}(k^2 + 4k)y^2.$$

As this is equivalent to $x^2 - kxy - ky^2$, a form with integral coefficients, we must have $\lambda_0 \equiv k/2 \pmod 1$.

Suppose that q_0 takes a value in the open interval

$$I = (-(k^2 + 6k + 1)/4, -(k^2 + 4k)/4),$$

say at the point $(y,z) = (Y,Z)$. Then choosing x such that $(x + \lambda_0 Y + \mu_0 Z)^2$ lies in the closed interval

$$[(k^2 + 2k + 1)/4, (k^2 + 4k + 4)/4]$$

would give a value of f_0 lying in the open interval $(-k,1)$, which, as $f \sim f_0$, contradicts either $m_+(f) = 1$ or $m_-(f) \geq k$. Hence q_0 can take no values in the interval I.

Suppose for the moment that the integers g_{2i} of the chain $[g_i]$ associated with q_0 (as in lemma 2.1) are bounded above by S^*. Then by applying lemma 2.1 to the form

$$Q_0(x,y) = q_0(x,y)/2\sqrt{d(q_0)},$$

taking $c_1 = \tfrac{1}{2}C(S^*)$ and c_2 arbitrarily small, we may conclude that one of the following holds:

(a) $2\sqrt{d(q_0)}/m_-(q_0) = 2\sqrt{d}/e \geq 2$,

(b) $2\sqrt{d(q_0)}/m_-(q_0) = 2\sqrt{d}/e \geq \sqrt{1 + 4/S^*}\left(\dfrac{k^2 + 6k + 1}{k^2 + 4k} - c_2\right)$,

(c) There exist integers r and s, both at most S^*, such that for all i,

$$g_{2i+1} = 1, \qquad g_{4i} = r, \qquad g_{4i+2} = s.$$

If however, $g_{2i} > S^*$ for at least one i, then either (a) above holds if $g_j \geq 2$ for at least one odd j, or $g_j = 1$ for all odd j and Q_0 takes a value m_1 where

$$1/m_1 \geqslant (S^* + 1,1,\ldots) + (0,1,1,\ldots).$$

This latter implies that

$$2\sqrt{d}/m_+(q_0) \geqslant S^* + \sqrt{5},$$

and so

$$2\sqrt{d} \geqslant m_+(q_0)(S^* + \sqrt{5}).$$

Thus if the possibility (c) above does not hold, either

(i) $2\sqrt{d} \geqslant 2e$, or

(ii) $2\sqrt{d} \geqslant e\sqrt{1 + 4/S^*}((k^2 + 6k + 1)/(k^2 + 4k) - c_2)$ for arbitrarily small c_2, or

(iii) $2\sqrt{d} \geqslant m_+(q_0)(S^* + \sqrt{5})$.

From these we conclude that either

(i') $d \geqslant e^2$, or

(ii') $d \geqslant K^2(1 + 4/S^*)/64$, or

(iii') $d \geqslant 9(S^* + \sqrt{5})^2/64$.

We shall now show that in each of these cases $d \geqslant d_2$. Clearly it is only necessary to show that $e^2 \geqslant d_2$. For $k = 2$, numerical evaluation shows that $S^* = 6$ and that

$$e^2 = 9 > \tfrac{5}{3}\cdot\tfrac{289}{64} = d_2.$$

As $t(S)$ (and hence S^*) is increasing with k it follows that $S^* \geqslant 6$ for $k \geqslant 2$, and so

$$d_2 \leqslant \tfrac{5}{3}\cdot\tfrac{1}{64}(k^2 + 6k + 1)^2.$$

Now for $k \geqslant 3$ it is a simple matter to verify that

$$\tfrac{5}{3}\cdot\tfrac{1}{64}(k^2 + 6k + 1)^2 < ((k^2 + 4k)/4)^2,$$

and hence $e^2 > d_2$ as required.

Thus, summarising, we have proved so far that if f satisfies the conditions of theorem 2.2 then either $d \geqslant \min(d_1, d_2)$ or f is equivalent to the form

$$f_0 = (x + \lambda_0 y + \mu_0 z)^2 + q_0(y,z),$$

where

$$\lambda_0 \equiv \tfrac{1}{2}k \pmod 1,$$

$$m\_(q_0) = e = (k^2 + 4k)/4,$$

$$q_0 = -e(y + \delta_0 z)^2 + dz^2/e,$$

and the chain of integers $[g_i]$ associated with q_0 has the property that there exist integers r and s, both at most S^*, such that $g_{2i+1} = 1$, $g_{4i} = r$, and $g_{4i+2} = s$ for all integers i. Clearly, to complete the proof of theorem 2.2, we need only show that f_0, with the above properties, must be equivalent to $f(r,s,l;x,y,z)$ for some $l < s$, and that $m\_(f) = k$.

If q_0 has the above properties then

$$q_0(y,z) \sim -e\{y^2 - \frac{s(r+2)}{rs+r+s}yz - \frac{r+2}{rs+r+s}z^2\}$$

$$\sim -e\{y^2 - \frac{r(s+2)}{rs+r+s}yz - \frac{s+2}{rs+r+s}z^2\}.$$

Hence by passing to an equivalent form if necessary we may take f_0 to be of the form

$$(x + \lambda y + \mu z)^2 - eq(r,s;y,z)$$

where we may assume without loss of generality that $r \leqslant s \leqslant S^*$. The congruence

$$\lambda \equiv \tfrac{1}{2}k \pmod 1$$

may be deduced in the same way that $\lambda_0 \equiv \frac{1}{2}k$ was deduced. Hence f is equivalent to the form

$$f^* = (x + \tfrac{1}{2}ky + \mu z)^2 - \tfrac{1}{4}(k^2 + 4k)q(r,s;y,z)$$

which takes the value $-k$ at $(x,y,z) = (0,1,0)$. Thus as $m_-(f) \geqslant k$ is given we must have $m_-(f) = k$. It now remains, to complete the proof of the theorem, to show that $\mu \equiv l/s \pmod 1$.

We have

$$f^*(x,1,-s) = (x + \tfrac{1}{2}k - \mu s)^2 - (k^2 + 4k)/4,$$

and so by choosing x such that

$$\tfrac{1}{2}(k + 1) \leqslant |x + \tfrac{1}{2}k - \mu s| \leqslant \tfrac{1}{2}(k + 2)$$

we obtain a value of f^* contradicting either $m_+(f) = 1$ or $m_-(f) \geqslant k$ unless

$$\tfrac{1}{2}k - \mu s \equiv \tfrac{1}{2}k \pmod 1.$$

That is,

$$\mu \equiv l/s$$

for some l with $0 \leqslant l < s$. Hence

$$f \sim f^* \sim (x + \tfrac{1}{2}ky + lz/s)^2 - \tfrac{1}{4}(k^2 + 4k)q(r,s;y,z)$$

as required. This completes the proof of theorem 2.2.

2.2 Further information about the relationship between r, s and l for those forms $f(r,s,l;x,y,z)$ which do in fact have $m_-(f) = k$ and $m_+(f) = 1$ may be obtained by applying various automorphs of $q(r,s;y,z)$ and by applying various x-y transformations. The following theorem gives some of these relationships.

Theorem 2.3

Let $k \geq 2$ be integral and let d_1, d_2, S^* and $f = f(r,s,l;x,y,z)$ be defined as in theorem 2.2. Let

$$B = s(r + 2)/(rs + r + s),$$

$$e = (k^2 + 4k)/4,$$

$$E = B(1 + \tfrac{1}{2}k) + 2l/s, \quad \text{and}$$

$$F = (l/s)^2 - B(1 - l(1 + \tfrac{1}{2}k))/s + eB^2/4.$$

Then if $d(f) < \min(d_1, d_2)$ and if $m_+(f) = 1$ and $m_-(f) = k$ the following conditions must be satisfied:

(i) $r(k/2 + l/s) \equiv 0 \pmod 1$,

(ii) The fraction seB, when reduced to its lowest form, has denominator at most S^*,

(iii) There exist positive integers r' and s', both at most S^*, and an integer b such that

$$E = 2b \pm B'$$

and

$$-F = -b^2 \pm bB' + B'/s',$$

where we have used B' to denote the fraction

$$s'(r' + 2)/(r's' + r' + s'), \quad \text{and}$$

(iv) For this r' and s',

$$B^2/4 + B/s = (B')^2/4 + B'/s'.$$

Proof

(i) Considering the section

$$f(x, r+1, r) = (x + \tfrac{1}{2}k(r+1) + rl/s)^2 - (k^2 + 4k)/4$$

in the same way that the section $f^*(x, 1, -s)$ was considered in the proof of theorem 2.2 yields that

$$\tfrac{1}{2}k(r+1) + rl/s \equiv \tfrac{1}{2}k \pmod 1.$$

This clearly implies that

$$r(k/2 + l/s) \equiv 0 \pmod 1.$$

(ii) Applying the transformation

$$(x, y, z) \to (X, X - Y, Z)$$

to f yields the equivalent form

$$(X + \tfrac{1}{2}kY + DZ)^2 - e(Y^2 + EYZ + FZ^2), \quad (2.3$$

where

$$2D = l(k+2)/s + eB. \quad (2.4$$

Repeating the argument of theorem 2.2 we find that

$$Y^2 + EYZ + FZ^2 \sim q(r', s'; \bar{y}, \bar{z})$$

for some r' and s' satisfying $r' \leq s' \leq S^*$. We now proceed to find out further information about the transformations yielding this equivalence.

Let h be an integer such that

$$0 \leq |E + 2h| \leq 1,$$

and consider the transformation

$$\left.\begin{aligned} Y &\to \bar{y} + h\bar{z} \\ Z &\to \bar{z}. \end{aligned}\right\} \quad (2.5$$

This sends the form $Y^2 + EYZ + FZ^2$ into the form

$$\begin{aligned} q_1(\bar{y},\bar{z}) &= \bar{y}^2 + (E + 2h)\overline{yz} + (h^2 + Eh + F)\bar{z}^2 \\ &= \bar{y}^2 - E_1\overline{yz} - F_1\bar{z}^2. \end{aligned}$$

By changing the sign of $\bar{y}$ if necessary we may assume E_1 to be non-negative. Let d denote $d(q) = d(q_1)$. Then as $s \geqslant r$ we have $B \geqslant 1$ and so

$$E_1{}^2 + 4F_1 = 4d = B^2 + 4B/s > 1.$$

Hence as $E_1{}^2 \leqslant 1$ we find that $F_1 > 0$. We shall now show that $q_1(\bar{y},\bar{z})$ is either $q(r',s';\bar{y},\bar{z})$ or $q(s',r';\bar{y},\bar{z})$.

Suppose that

$$F_1 \geqslant \sqrt{d}. \quad (2.6$$

Then $d = E_1{}^2/4 + F_1 \geqslant \sqrt{d}$, and so $d \geqslant 1$. However this leads to a contradiction as follows:

(a) If $s \geqslant 2$, then

$$\begin{aligned} d &= 1 - \frac{3r^2s^2 - 4rs + 4rs^2 - 8s + 4sr^2 - 8r}{4(rs + r + s)^2} \\ &< 1 \end{aligned}$$

which contradicts $d \geqslant 1$.

(b) If $r = s = 1$, the only other possibility, $d = 5/4$ and E,F and thus F_1 are integral.

However

$$5/4 = d \geqslant F_1 \geqslant \sqrt{5}/2,$$

and this is clearly insoluble in integers F_1.

From the above considerations it follows that $F_1 < \sqrt{d}$, and so, from a theorem of Lagrange mentioned in the introduction, $-F_1$ occurs as a coefficient in one of the reduced forms equivalent to q_1 (and hence $q(r',s';\overline{y},\overline{z})$). From the nature of the chain of integers $[g_i]$ associated with q_1 it follows that either

$$F_1 = (r' + 2)/(r's' + r' + s')$$

or

$$F_1 = (s' + 2)/(r's' + r' + s').$$

Upon calculating E_1 from d in terms of r' and s', it immediately becomes clear that q_1 is either $q(r',s';\overline{y},\overline{z})$ or $q(s',r';\overline{y},\overline{z})$. By dropping the assumption that $r' \leqslant s'$ we may assume that

$$q_1 = q(r',s';\overline{y},\overline{z}).$$

Applying the transformation (2.5) to the form (2.3) yields the equivalent form

$$(X + \tfrac{1}{2}k\overline{y} + (D + \tfrac{1}{2}kh)\overline{z})^2 - eq(r',s';\overline{y},\overline{z}).$$

Considering this form as in theorem 2.2 yields that

$$D + \tfrac{1}{2}kh \equiv l'/s' \pmod 1.$$

Hence

$$2D = l(k + 2)/s + eB \equiv 2l'/s' \pmod 1$$

and so

$$2sD \equiv eBs \equiv 2sl'/s' \pmod 1.$$

Thus the denominator of the reduced form of the fraction seB divides s' and hence is at most S^*.

(iii) As $q_1 = q(r', s'; \bar{y}, \bar{z})$ it follows upon sorting out the relations between h, E, F, E_1 and F_1 that

$$B' = E_1 = \pm(E + 2h)$$

and

$$B'/s' = F_1 = -(h^2 + Eh + F).$$

The required integer b is then given by $b = -h$.

(iv) Equating the determinants of q and q_1 yields that

$$B^2/4 + B/s = (B')^2/4 + B'/s'.$$

It should be noticed that condition (iv) of theorem 2.3 is highly restrictive. Calculations performed on the CSIRO's C.D.C. "3200" computer in Adelaide have shown that for $S^* \leq 200$, the couple (r', s') must be either (r, s) or (s, r).

2.3 As the result of theorem 2.2 for $k = 2$ will be used later in chapter 11, we will now show that the forms $f(r, s, l; x, y, z)$ given in table 2.1 for $k = 2$

are the only forms with $m_+ = 1$, $m_- = 2$ and determinant at most 7.5 in absolute value.

The numerical calculations involved in showing that $S^* = 6$ and that $\min(d_1, d_2) > 7.5$ (for $k = 2$) are straightforward, and hence will be omitted. In table 2.3 below the values of seB for $r \leqslant s \leqslant 6$ have been listed.

Table 2.3

s	6						5				
r	6	5	4	3	2	1	5	4	3	2	1
B	1	$\frac{42}{41}$	$\frac{18}{17}$	$\frac{10}{9}$	$\frac{6}{5}$	$\frac{18}{13}$	1	$\frac{30}{29}$	$\frac{25}{23}$	$\frac{20}{17}$	$\frac{15}{11}$
seB	18	$\frac{756}{41}$	$\frac{324}{17}$	20	$\frac{108}{5}$	$\frac{324}{13}$	15	$\frac{450}{29}$	$\frac{375}{23}$	$\frac{300}{17}$	$\frac{225}{11}$

s	4				3			2		1
r	4	3	2	1	3	2	1	2	1	1
B	1	$\frac{20}{19}$	$\frac{8}{7}$	$\frac{4}{3}$	1	$\frac{12}{11}$	$\frac{9}{8}$	1	$\frac{6}{5}$	1
seB	12	$\frac{240}{19}$	$\frac{96}{7}$	16	9	$\frac{108}{11}$	$\frac{81}{8}$	6	$\frac{36}{5}$	3

Using condition (ii) of theorem 2.3 we need only consider those r and s where the denominator of seB is at most 6. These are $(s,r) = (6,6), (6,3), (6,2), (5,5), (4,4), (4,1), (3,3), (2,2), (2,1), (1,1)$.

We may exclude $(s,r) = (2,1)$ or $(1,1)$ as in these cases $f(r,s,l;x,y,z)$ has $d(f) > 7.5$. In table 2.4 the remaining (s,r) possibilities are listed together with the corresponding l which are not excluded by condition (i) of theorem 2.3.

Table 2.4

(s,r)	(6,6)	(6,3)	(6,2)	(5,5)	(4,4)
allowable l	0,1,2,3,4,5	0,2,4	0,3	0,1,2,3,4	0,1,2,3

(s,r)	(4,1)	(3,3)	(2,2)
allowable l	0	0,1,2	0,1

As $f(r,s,l;x,y,z) \sim f(r,s,s-l;x,y,z)$ we only need to consider those allowable l with $0 \leqslant l \leqslant s/2$. In table 2.6 the forms $q_1(\bar{y},\bar{z})$, as defined in theorem 2.3, are listed. These must be one of the forms $q(r',s';y,z)$ or $q(s',r';y,z)$, for allowable r' and s', which are listed in table 2.5.

Table 2.5

(s',r')	$q(r',s';y,z)$	$q(s',r';y,z)$
(6,6)	$y^2 - yz - \frac{1}{6}z^2$	$y^2 - yz - \frac{1}{6}z^2$
(6,3)	$y^2 - \frac{10}{9}yz - \frac{5}{27}z^2$	$y^2 - \frac{8}{9}yz - \frac{8}{27}z^2$
(6,2)	$y^2 - \frac{6}{5}yz - \frac{1}{5}z^2$	$y^2 - \frac{4}{5}yz - \frac{2}{5}z^2$
(5,5)	$y^2 - yz - \frac{1}{5}z^2$	$y^2 - yz - \frac{1}{5}z^2$
(4,4)	$y^2 - yz - \frac{1}{4}z^2$	$y^2 - yz - \frac{1}{4}z^2$
(4,1)	$y^2 - \frac{4}{3}yz - \frac{1}{3}z^2$	$y^2 - \frac{2}{3}yz - \frac{2}{3}z^2$
(3,3)	$y^2 - yz - \frac{1}{3}z^2$	$y^2 - yz - \frac{1}{3}z^2$
(2,2)	$y^2 - yz - \frac{1}{2}z^2$	$y^2 - yz - \frac{1}{2}z^2$

Table 2.6

(s,r)	l	E	F	$q_1(\overline{y},\overline{z})$
(6,6)	0	2	$\frac{7}{12}$	$\overline{y}^2 - \frac{5}{12}\overline{z}^2$
	1	$2\frac{1}{3}$	$\frac{17}{18}$	$\overline{y}^2 - \frac{1}{3}\overline{yz} - \frac{7}{18}\overline{z}^2$
	2	$2\frac{2}{3}$	$\frac{49}{36}$	$\overline{y}^2 - \frac{2}{3}\overline{yz} - \frac{11}{36}\overline{z}^2$
	3	3	$\frac{11}{6}$	$\overline{y}^2 - \overline{yz} - \frac{1}{6}\overline{z}^2$
(6,3)	0	$\frac{20}{9}$	$\frac{20}{27}$	$\overline{y}^2 - \frac{2}{9}\overline{yz} - \frac{13}{27}\overline{z}^2$
	2	$\frac{26}{9}$	$\frac{43}{27}$	$\overline{y}^2 - \frac{8}{9}\overline{yz} - \frac{8}{27}\overline{z}^2$
(6,2)	0	$\frac{12}{5}$	$\frac{22}{25}$	$\overline{y}^2 - \frac{2}{5}\overline{yz} - \frac{13}{25}\overline{z}^2$
	3	$\frac{17}{5}$	$\frac{233}{100}$	$\overline{y}^2 - \frac{3}{5}\overline{yz} - \frac{47}{100}\overline{z}^2$
(5,5)	0	2	$\frac{11}{20}$	$\overline{y}^2 - \frac{9}{20}\overline{z}^2$
	1	$2\frac{2}{5}$	$\frac{99}{100}$	$\overline{y}^2 - \frac{2}{5}\overline{yz} - \frac{41}{100}\overline{z}^2$
	2	$2\frac{4}{5}$	$\frac{151}{100}$	$\overline{y}^2 - \frac{4}{5}\overline{yz} - \frac{29}{100}\overline{z}^2$
(4,4)	0	2	$\frac{1}{2}$	$\overline{y}^2 - \frac{1}{2}\overline{z}^2$
	1	$2\frac{1}{2}$	$\frac{17}{16}$	$\overline{y}^2 - \frac{1}{2}\overline{yz} - \frac{7}{16}\overline{z}^2$
	2	3	$\frac{7}{4}$	$\overline{y}^2 - \overline{yz} - \frac{1}{4}\overline{z}^2$
(4,1)	0	$\frac{8}{3}$	1	$\overline{y}^2 - \frac{2}{3}\overline{yz} - \frac{2}{3}\overline{z}^2$
(3,3)	0	2	$\frac{5}{12}$	$\overline{y}^2 - \frac{7}{12}\overline{z}^2$
	1	$2\frac{2}{3}$	$\frac{43}{36}$	$\overline{y}^2 - \frac{2}{3}\overline{yz} - \frac{17}{36}\overline{z}^2$
(2,2)	0	2	$\frac{1}{4}$	$\overline{y}^2 - \frac{3}{4}\overline{z}^2$
	1	3	$\frac{3}{2}$	$\overline{y}^2 - \overline{yz} - \frac{1}{2}\overline{z}^2$

It is easily seen that the only r,s and l for which $q_1(\overline{y},\overline{z})$ is one of the forms in table 2.5 are

$$(r,s,l) = (4,4,2),\ (1,4,0),\ (2,2,1).$$

Hence the only possible forms $f(r,s,1;x,y,z)$ which have $m_+(f) = 1$, $m_-(f) = 2$ and $d(f) < 7.5$ are the following:

$$f_1 = f(4,4,2;x,y,z) \sim (x + \tfrac{1}{2}z)^2 - 3(y^2 - yz - \tfrac{1}{4}z^2),$$
$$f_2 = f(1,4,0;x,y,z) \sim x^2 - 3(y^2 - \tfrac{4}{3}yz - \tfrac{1}{3}z^2),$$
$$f_3 = f(2,2,1;x,y,z) \sim (x + \tfrac{1}{2}z)^2 - 3(y^2 - yz - \tfrac{1}{2}z^2).$$

It is now a simple exercise in congruences to verify that these forms do in fact have $m_+(f) = 1$ and $m_-(f) = 2$. The following facts are sufficient to show this.

(i) The coefficients of f_1, f_2 and $4f_3$ are integers.

(ii) Taking congruences mod 3 it can be seen that f_1 cannot take the value -1, while taking congruences mod 9 shows that it cannot take the value 0 for relatively prime x, y and z, and hence that it cannot take the value 0 at all.

(iii) Taking congruences mod 8, as

$$f_2 \equiv x^2 + y^2 + (z + 2y)^2 \pmod 8,$$

it can be seen that f_2 can take neither the value -1 nor the value 0 for relatively prime x, y and z.

(iv) Taking congruences mod 8, as

$$4f_3 \equiv (2x + z)^2 + (2y + z)^2 + 5z^2 \pmod 8,$$

it can be seen that $4f_3$ cannot take the values $1, \pm 2, \pm 3, -5, -6$ and -7. In addition, taking congruences mod 3 shows that $4f_3$ cannot take the values -4 or -1. Furthermore, taking congruences mod 3,9 and 27 in turn shows that $4f_3$ cannot take the value 0 for relatively prime x,y and z.

CHAPTER 3

In this chapter we consider the special forms F_i and show that the closure conditions of the intervals I_i are necessary.

The forms F_i are considered in separate lemmas, each giving $m_+(F_i)$ and $m_-(F_i)$ for some i.

Lemma 3.1

$$m_+(F_1) = \sqrt[3]{4/3}, \quad m_-(F_1) = \sqrt[3]{1/48}.$$

Proof (Due to Barnes [1])

$$F_1 = \sqrt[3]{4/3}\{(x + \tfrac{1}{2}z)^2 - \tfrac{1}{2}(z^2 - 2yz - 2y^2)\}.$$

For the proof we consider the integral form

$$G_1(x,y,z) = 4\sqrt[3]{3/4}\, F_1(x,y,z)$$

i.e. $$G_1(x,y,z) = 4x^2 + 4xz - z^2 + 4yz + 4y^2.$$

Then we must prove that $m_+(G_1) = 4$ and $m_-(G_1) = 1$. Since G_1 clearly takes the values 4 and -1, we only need to show that G_1 cannot take the values $3,2,1$ or 0. Taking congruences mod 8, as

$$G_1 = (2x + z)^2 + (2y + z)^2 - 3z^2,$$

it is clear that G_1 cannot take the values $3,2$ or 1.

To eliminate the value 0, we suppose to the contrary that $G_1(x,y,z) = 0$ has a non-trivial solution. Then there exist relatively prime X,Y,Z with

$$4X^2 + 4XZ - Z^2 + 4YZ + 4Y^2 = 0.$$

Clearly congruences mod 4 give $Z = 2t$ for some integer t. Then we must have

$$(X + t)^2 + (Y + t)^2 + t^2 \equiv 0 \pmod 4,$$

which can only be satisfied if

$$X + t \equiv Y + t \equiv t \equiv 0 \pmod 2.$$

This gives 2 as a common divisor of X, Y, Z, contrary to the assumption that X, Y, Z were relatively prime. This shows that G_1 cannot take the value 0 and completes the proof of the lemma.

Lemma 3.2

$$m_+(F_2) = \sqrt[3]{54/49}, \quad m_-(F_2) = \sqrt[3]{2/49}.$$

Proof

$$F_2 = \sqrt[3]{54/49}\{(x + \tfrac{1}{6}y + \tfrac{1}{2}z)^2 - \tfrac{7}{12}(z^2 - 2yz - \tfrac{5}{3}y^2)\}.$$

For the proof we consider the integral form

$$G_2(x,y,z) = 3\sqrt[3]{49/54}\, F_2(x - 4z, y, z)$$
$$= 3(x - y)^2 - 21xz + 35z^2 + 7xy.$$

Then we must prove that $m_+(G_2) = 3$ and $m_-(G_2) = 1$. Since G_2 takes the values 3 and -1 at $(1,0,0)$ and $(4,0,1)$ respectively, and as taking congruences mod 7 shows that G_2 cannot take the values 1 or 2, we only need to show that G_2 cannot take the value 0.

Suppose to the contrary that $G_2(x,y,z) = 0$ has a non-trivial solution. Then there exist relatively prime X, Y, Z with

$$3(X - Y)^2 - 21XZ + 35Z^2 + 7XY = 0.$$

This implies that $X \equiv Y \pmod 7$. Setting $X = Y + 7t$ and taking congruences mod 49 yields that

$$(Y + 2Z)^2 + Z^2 \equiv 0 \pmod 7.$$

This can have only the solution $Y + 2Z \equiv Z \equiv 0 \pmod 7$, which implies that 7 is a common divisor of X, Y, Z, contrary to the assumption that X, Y, Z were relatively prime. This contradiction shows that G_2 cannot take the value 0 and completes the proof of the lemma.

Lemma 3.3

$$m_+(F_3) = \sqrt[3]{8/9}, \qquad m_-(F_3) = \sqrt[3]{1/9}.$$

Proof

$$F_3 = \sqrt[3]{8/9}\{(x + \tfrac{1}{2}y + \tfrac{1}{2}z)^2 - \tfrac{3}{4}(z^2 - 2yz - y^2)\}.$$

For the proof we consider the integral form

$$G_3(x,y,z) = 2\sqrt[3]{9/8}\cdot F_3(x - y, y, x + y - z)$$
$$= 3x^2 + 3y^2 - z^2.$$

Then we have to show that $m_+(G_3) = 2$ and $m_-(G_3) = 1$. Since G_3 clearly takes the values 2 and -1, and as taking congruences mod 3 eliminates the value 1, we only need to show that G_3 does not take the value 0.

As usual we assume to the contrary that there exist relatively prime X, Y, Z with

$$3X^2 + 3Y^2 - Z^2 = 0. \qquad (3.1$$

However taking congruences mod 9 shows that any solution of this equation satisfies

$$X \equiv Y \equiv Z \equiv 0 \pmod 3,$$

and so (3.1) has no relatively prime solution. This shows that G_3 cannot take the value 0 and completes the proof of the lemma.

Lemma 3.4

$$m_+(F_4) = \sqrt[3]{125/144}, \quad m_-(F_4) = \sqrt[3]{3/16}.$$

Proof

$$F_4 = \sqrt[3]{125/144}\{(x + \tfrac{4}{5}y + \tfrac{2}{5}z)^2 - \tfrac{24}{25}(z^2 - yz - y^2)\}.$$

For the proof we consider the integral form

$$G_4(x,y,z) = 5\sqrt[3]{144/125}\, F_4(x,y,z)$$
$$= 5x^2 + 8xy + 4xz + 8yz + 8y^2 - 4z^2.$$

Then we must show that $m_+(G_4) = 5$ and $m_-(G_4) = 3$. As G_4 clearly takes the values 5 and -3, and as

$$G_4 \equiv 5(x + 2z)^2 \equiv 0,5 \text{ or } 4 \pmod 8)$$

and

$$G_4 \equiv 2(x + 2y - 2z)^2 \equiv 0 \text{ or } 2 \pmod 3)$$

it is clear that we only have to prove that $G_4(x,y,z) = 0$ has no non-trivial solution.

As usual we assume to the contrary that there exist relatively prime X,Y,Z with

$$5X^2 + 8XY + 4XZ + 8YZ + 8Y^2 - 4Z^2 = 0.$$

Taking congruences mod 4 yields that $X = 2t$ for some integer t. Then as X,Y,Z are relatively prime either Z is odd or Z is even and Y is odd. We consider these two cases separately.

(a) Z odd: We have

$$0 = 20t^2 + 16tY + 8tZ + 8Y(Y + Z) - 4Z^2,$$

which implies that

$$4t^2 + 8t - 4 \equiv 0 \pmod{16}$$

which is impossible.

(b) Y odd, Z even: Putting $Z = 2s$ we have

$$0 = 20t^2 + 16tY + 16ts + 8Y^2 + 16Ys - 16s^2,$$

which implies that

$$4t^2 + 8 \equiv 0 \pmod{16}$$

which is also impossible.

Thus G_4 cannot take the value 0. This completes the proof of the lemma.

<u>Lemma 3.5</u> (**Due to Markoff [9]**)

$$m_+(F_5) = m_-(F_5) = \sqrt[3]{2/3}.$$

<u>Proof</u>

$$F_5 = \sqrt[3]{2/3}\{(x + \tfrac{1}{2}y + \tfrac{1}{2}z)^2 - \tfrac{5}{4}(z^2 - \tfrac{6}{5}yz - \tfrac{3}{5}y^2)\}.$$

For the proof we consider the integral form

$$G_5(x,y,z) = \sqrt[3]{3/2}\ F_5(x,y,z)$$

$$= x^2 + xz + xy + 2yz + y^2 - z^2.$$

Then we must show that $m_+(G_5) = m_-(G_5) = 1$. This clearly follows once we show that G_5 cannot take the value 0.

Suppose to the contrary that there exist relatively prime X,Y,Z with

$$X^2 + X(Y + Z) + (Y + Z)^2 - 2Z^2 = 0.$$

Then taking congruences mod 2 gives that

$X \equiv Y + Z \equiv 0 \pmod 2$ and taking congruences mod 4 gives that Z is even. Hence X, Y, Z cannot be relatively prime. This shows that G_5 cannot take the value 0 and completes the proof of the lemma.

Lemma 3.6

$$m_+(F_6) = \sqrt[3]{27/112}, \qquad m_-(F_6) = \sqrt[3]{125/112}.$$

Proof

$$F_6 = \sqrt[3]{27/112}\{(x + \tfrac{1}{3}y)^2 - \tfrac{8}{3}(z^2 - yz - \tfrac{1}{3}y^2)\}.$$

For the proof we consider the integral form

$$G_6(x,y,z) = 3\sqrt[3]{112/27}\, F_6(x + y, y, z)$$
$$= 3x^2 + 8xy + 8y^2 - 8z^2 + 8yz.$$

Then we must show that $m_+(G_6) = 3$ and $m_-(G_6) = 5$. As G_6 takes the values 3 and -5, and as taking congruences mod 8 shows that G_6 cannot take the values $2, 1, -1, -2$ and -3, we only need to show that G_6 cannot take the values -4 and 0.

If $G_6(X,Y,Z) = -4$, then taking congruences mod 8 shows that $X \equiv 2 \pmod 4$. Setting $X = 2t$ and taking congruences mod 16 gives

$$12t^2 + 8Y^2 - 8Z^2 + 8YZ \equiv 12 \pmod{16}.$$

Now as t is odd this yields that

$$Y^2 + YZ - Z^2 \equiv 0 \pmod 2,$$

which only has the solution $Y \equiv Z \equiv 0 \pmod 2$.

Thus X,Y,Z are all even. However this implies that G_6 takes the value -1 at the point $(X/2,Y/2,Z/2)$, which we know is impossible.

If $G_6(X,Y,Z) = 0$ where X,Y,Z are relatively prime then taking congruences mod 8 shows that $X \equiv 0 \pmod 4$. Setting $X = 4t$ and taking congruences mod 16 gives that

$$Y^2 + YZ - Z^2 \equiv 0 \pmod 2.$$

Hence, as above, we are led to the contradiction that X,Y,Z are all even. This shows that G_6 cannot take the values 0 and -4, and completes the proof of the lemma.

Lemma 3.7

$$m_+(F_7) = \sqrt[3]{2/9}, \qquad m_-(F_7) = \sqrt[3]{16/9}.$$

Proof

$$F_7 = \sqrt[3]{2/9}\{(x + \tfrac{1}{2}y)^2 - 3(z^2 - yz - \tfrac{1}{4}y^2)\}.$$

For the proof we consider the integral form

$$\begin{aligned} G_7(x,y,z) &= \sqrt[3]{9/2}\, F_7(x + y,y,z) \\ &= x^2 + 3xy + 3y^2 - 3z^2 + 3yz. \end{aligned}$$

Then we must show that $m_+(G_7) = 1$ and $m_-(G_7) = 2$. As G_7 takes the values 1 and -2, and as taking congruences mod 3 shows that G_7 cannot take the value -1, we only need to show that G_7 cannot take the value 0.

As usual we suppose to the contrary that there exist relatively prime X,Y,Z with

$$X^2 + 3XY + 3Y^2 - 3Z^2 + 3YZ = 0.$$

Clearly taking congruences mod 3 shows that $X = 3t$ for some integer t. Then taking congruences mod 9 gives that

$$Y^2 + YZ - Z^2 \equiv 0 \pmod 3,$$

$$\text{i.e.} \quad (Y - Z)^2 + Z^2 \equiv 0 \pmod 3.$$

Hence $Z \equiv Y - Z \equiv 0 \pmod 3$. Then 3 is a common divisor of X,Y,Z, contrary to our assumption that X,Y,Z were relatively prime. This contradiction shows that G_7 cannot take the value 0 and completes the proof of the lemma.

Lemma 3.8

$$m_+(F_8) = \sqrt[3]{1/24}, \qquad m_-(F_8) = \sqrt[3]{8/3}.$$

Proof

$$F_8 = \sqrt[3]{1/24}\{x^2 - 8(z^2 - yz - \tfrac{1}{8}y^2)\}.$$

For the proof we consider the integral form

$$G_8(x,y,z) = \sqrt[3]{24}\, F_8(x,y,z).$$

Then we must show that $m_+(G_8) = 1$ and $m_-(G_8) = 4$. As G_8 takes the values 1 and -4, and as taking congruences mod 8 shows that G_8 cannot take the values -2 or -1, we only need to show that G_8 cannot take the values -3 or 0.

Suppose to the contrary that there exist integers X, Y, Z with

$$X^2 + Y^2 + 8YZ - 8Z^2 = 0 \quad \text{or} \quad -3,$$

i.e. $$X^2 + (Y + 4Z)^2 \equiv 0 \pmod 3.$$

This implies that $X = 3t$ and $Y + 4Z = 3s$ for some integers t and s. Then

$$9t^2 + 9s^2 - 24Z^2 = 0 \text{ or } -3,$$

i.e. $$8Z^2 \equiv 0 \text{ or } 1 \pmod 3.$$

This implies that $Z \equiv 0 \pmod 3$, and so 3 must divide each of X, Y, Z. Hence G_8 cannot take the values 0 or -3 for relatively prime X, Y, Z. This is sufficient to show that G_8 cannot take the values 0 or -3, and completes the proof of the lemma.

Lemma 3.9

$$m_+(F_9) = \sqrt[3]{2/135}, \qquad m_-(F_9) = \sqrt[3]{16/5}.$$

Proof (Due to Barnes and Oppenheim [2])

$$F_9 = \sqrt[3]{2/135}\{(x + \tfrac{1}{2}y)^2 - 15(z^2 - yz - \tfrac{1}{20}y^2)\}.$$

For the proof we consider the integral form

$$G_9(x,y,z) = \sqrt[3]{135/2}\, F_9(x + 5z, y - 10z, y - 11z)$$
$$= x^2 + xy + y^2 - 90z^2.$$

Then we must show that $m_+(G_9) = 1$ and $m_-(G_9) = 6$. As G_9 takes the values 1 and -6 at $(1,0,0)$ and $(8,2,1)$ respectively we only need to show that G_9 cannot take the values $0, -1, -2, -3, -4$ or -5.

Now

$$4G_9 = (2x + y)^2 + 3y^2 - 360z^2 \qquad (3.2$$

and so taking congruences mod 3 shows that G_9 cannot take the values -1 or -4. Furthermore if $G_9(X,Y,Z)$ were -3 we would have to have $2X + Y \equiv 0 \pmod 3$. Setting $2X + Y = 3t$ we have

$$3t^2 + Y^2 - 120Z^2 = -4,$$

which is impossible modulo 3.

If $G_9(X,Y,Z) = -2$ then taking congruences mod 2 gives

$$X^2 + XY + Y^2 \equiv 0 \pmod 2)$$

which implies that $X \equiv Y \equiv 0 \pmod 2$. Setting $X = 2t$ and $Y = 2s$ we have that

$$2t^2 + 2s^2 + 2ts - 45Z^2 = -1.$$

Thus Z must be odd and

$$2t^2 + 2ts + 2s^2 \equiv 44 \pmod 8,$$

which is impossible.

If $G_9(X,Y,Z) = -5$ then taking congruences mod 5 in (3.2) yields that

$$(2X + Y)^2 + 3Y^2 \equiv 0 \pmod 5.$$

This has only the solution $2X + Y \equiv Y \equiv 0 \pmod 5$. Setting $2X + Y = 5t$ and $Y = 5s$ we have that

$$5t^2 + 5s^2 + 3Z^2 \equiv -4 \pmod 5,$$

which is impossible.

This leaves only the value 0 to eliminate.

Suppose to the contrary that there exist relatively prime X, Y, Z with

$$(2X + Y)^2 + 3Y^2 - 360Z^2 = 0.$$

Then taking congruences mod 5 yields that $2X + Y \equiv Y \equiv 0 \pmod 5$, and taking congruences mod 25 yields that $Z \equiv 0 \pmod 5$. This gives 5 as a common divisor of X, Y, Z, contrary to the assumption that X, Y, Z were relatively prime. This shows that G_9 cannot take the value 0 and completes the proof of the lemma.

CHAPTER 4

In this chapter we establish the general method of proof of theorem A.

We first break down the theorem into ten sub-theorems which when combined together are equivalent to theorem A. Each of these sub-theorems takes the following form for some i, $0 \leq i \leq 9$, where a_i, b_i, I_i and F_i are as in theorem A.

Theorem A_i

Every normalised indefinite ternary quadratic form of signature 1 takes a value in the closed interval

$$I_i = [-\sqrt[3]{a_i}, \sqrt[3]{b_i}].$$

Furthermore (for $0 \leq i \leq 8$) closure is required on the right only for forms equivalent to F_{i+1}, and (for $1 \leq i \leq 9$) closure is required on the left only for forms equivalent to F_i.

We now take the theorems A_i and try to reduce them to a form in which they are more easily proven. Consider, for $1 \leq i \leq 8$, in place of theorem A_i the theorm B_i as follows.

Theorem B_i

If g is any indefinite ternary quadratic form of signature 1 and with $d(g) = d$ where

$$0 < d \leq 1/b_i ,$$

and if $m_+(g) = 1$, then either

$$m_-(g) < \sqrt[3]{a_i d}$$

or g is equivalent to a multiple of either F_i or F_{i+1}.

It is easily seen that theorem A_i follows from theorem B_i, for if f is any normalised form with $m_+(f) = m$ then

(a) If $0 \leq m < \sqrt[3]{b_i}$, f clearly takes a value in the interior $I_i^{\,o}$ of I_i.

(b) If $m \geq \sqrt[3]{b_i}$, consider the form

$$g(x,y,z) = f(x,y,z)/m.$$

This has

$$d = d(g) = 1/m^3 \leq 1/b_i ,$$

and applying theorem B_i gives that either

(i) $m_-(g) < \sqrt[3]{a_i d}$, from which it follows that $m_-(f) < \sqrt[3]{a_i}$, and so f takes a value in $I_i^{\,o}$, or

(ii) g is equivalent to a multiple of either F_i or F_{i+1}, from which it follows, on comparing determinants, that f is equivalent to either F_i or F_{i+1}.

(c) The closure conditions follow automatically from the results of Chapter 3.

Thus if we can establish theorems A_0 and A_9 and prove theorems $B_1, B_2, \ldots, B_8$ we will have proved theorem A.

Theorems B_i for $1 \leqslant i \leqslant 8$, or more specifically theorems C_i, stated below, from which theorems B_i follow, will be considered in later chapters. For the present we will consider theorems A_0 and A_9.

Proof of Theorem A_0

Barnes [1] has proved the following.

"Every indefinite ternary quadratic form of signature 1 with $d(f) \neq 0$ takes a value v satisfying

$$0 \leqslant v \leqslant \sqrt[3]{4d(f)/3}.$$

Furthermore equality on the right is necessary if and only if the form is equivalent to a multiple of

$$h_1 = -x^2 + 8(y^2 + yz + z^2)."$$

Theorem A_0 follows immediately on setting $d(f) = 1$ and observing that

$$F_1(x,y,z) = \tfrac{1}{4}h_1(z - 2x - 2y, x, y)\sqrt[3]{4/3}.$$

Proof of Theorem A_9

Barnes and Oppenheim [2] have proved the following.

"Every indefinite ternary quadratic form of signature -1 with $d(f) \neq 0$ takes a value v satisfying

$$0 \leqslant v \leqslant \sqrt[3]{16d(f)/5}.$$

Furthermore equality on the right is necessary if and only if the form is equivalent to a multiple of

$$h_2 = -x^2 - xy - y^2 + 90z^2."$$

Theorem A_9 follows immediately on multiplying the forms by -1, setting $d(f) = 1$, and observing that

$$F_9(x,y,z) = -h_2(x - 5z, y + 10z, -z)\sqrt[3]{2/135}.$$

In order to simplify the theorems B_i we need the following theorem.

Theorem 4.1

Let f be an indefinite ternary quadratic form of signature 1 and such that both $m_+(f)$ and $m_-(f)$ are non-zero. Then if f does not attain the value $m_+(f)$ we can associate with f another indefinite ternary quadratic form f' with the following properties.

(i) $\det(f') = \det(f)$.

(ii) $m_+(f') = m_+(f)$; $m_-(f') \geqslant m_-(f)$.

(iii) f' attains the value $m_+(f)$.

(iv) f' is not a multiple of a form with integral coefficients.

Proof

As $m_+(f)$ is not attained by f we can find, for each integer $n \geqslant 2$, relatively prime x_n, y_n, z_n such that

$$m_+(f) < f(x_n, y_n, z_n) \leqslant (1 + 1/n)m_+(f).$$

Let

$$f(x_n, y_n, z_n) = (1 + \delta_n) m_+(f)$$

where $0 < \delta_n \leqslant 1/n$. Then we can find a form g_n equivalent to f such that

$$g_n = m_+(f)(1 + \delta_n)[(x + \lambda_n y + \mu_n z)^2 + q_n(y,z)].$$

Now q_n is an indefinite binary quadratic form and it cannot take any value in the open interval

$$(-m_-(f)/2m_+(f), 1/4) \qquad (4.1$$

as otherwise by choosing x such that $(x + \lambda_n y + \mu_n z)^2 \leqslant 1/4$ we would obtain a value v of f satisfying

$$-(1 + \delta_n) m_-(f)/2 < v < (1 + \delta_n) m_+(f)/2,$$

which, as $\delta_n \leqslant 1/2$, contradicts the definition of either $m_+(f)$ or $m_-(f)$. Hence there exists a chain of reduced forms, as described in the introduction, all equivalent to q_n. We take one of these reduced forms and denote it by

$$c_n y^2 + d_n yz + e_n z^2.$$

Then by passing to an equivalent form we have

$$g_n \sim h_n = m_+(f)(1 + \delta_n)[(x + \alpha_n y + \beta_n z)^2 + c_n y^2 + d_n yz + e_n z^2].$$

We may assume without loss of generality that

$$|\alpha_n| \leqslant 1/2, \qquad |\beta_n| \leqslant 1/2,$$

as if this were not so, by using a suitable parallel transformation on x we could pass to a further equivalent form where this condition would be satisfied.

Clearly as q_n cannot take any values in the open interval (4.1) both $|c_n|$ and $|e_n|$ must be bounded away from zero by $\min\{1/4, m_-(f)/2m_+(f)\}$. Then as

$$4d(f) = (1 + \delta_n)^3(m_+(f))^3(d_n^2 + 4|c_n e_n|) \qquad (4.2$$

it is clear that the sequences $\{c_n\}$ $\{d_n\}$ and $\{e_n\}$ are bounded sequences. As $\{\alpha_n\}$ and $\{\beta_n\}$ are also bounded sequences we can choose a sub-sequence $\{\gamma_n\}$ of $\{1/n\}$ such that the corresponding subsequences of $\{c_n\}, \{d_n\}, \{e_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ converge to limits c, d, e, α and β respectively. We shall show that

$$f' = m_+(f)[(x + \alpha y + \beta z)^2 + cy^2 + dyz + ez^2]$$

has the desired properties.

By taking limits of the subsequences corresponding to $\{\gamma_n\}$ in (4.2) we have

$$4|\det(f)| = (m_+(f))^3(d^2 + 4|ce|).$$

Then property (i) follows as the right hand side of this equation is $-4\det(f')$ and as f' must clearly have signature 1.

Property (iii) is trivial.

Property (ii) clearly follows on showing that f takes values arbitrarily close to any value taken by f'. If f' takes the value v at X, Y, Z, let

$B = \max(|X|, |Y|, |Z|)$. From the definitions of c, d, e, α and β it is clear that for any $\sigma > 0$ we can choose N such that the coefficients of x^2, y^2, z^2, xy, xz and yz in h_N differ from the corresponding coefficients in f' by at most σ.
[For example, if $K \geq 1$ denotes a common upper bound of $m_+(f)$ and the elements of the sequences $\{|c_n|\}$, $\{|d_n|\}, \{|e_n|\}$, choose N such that $1/N$ is in $\{\gamma_n\}$, $3K^3/N < \sigma/4$, and each of $|c - c_N|, |d - d_N|, |e - e_N|$, $|\alpha - \alpha_N|$ and $|\beta - \beta_N|$ is less than $\sigma/8K^2$.]

Then

$$|h_N(X,Y,Z) - f'(X,Y,Z)| \leq 6\sigma B^2,$$

i.e.

$$|h_N(X,Y,Z) - v| \leq 6\sigma B^2.$$

As $h_N \sim f$, and as $\sigma > 0$ is arbitrary, it is clear that f takes values arbitrarily close to any value taken by f'.

Using the notation that f is in the ϵ-neighbourhood (abbreviated nhd) $N_g(\epsilon)$ of g if the coefficients of x^2, y^2 etc. in f differ by at most ϵ from the corresponding coefficients in g, then we have seen above that for any $\epsilon > 0$ we can choose n such that h_n is in $N_{f'}(\epsilon)$.

In order to show that f' cannot be a multiple of a form with integral coefficients we refer to the result of Cassels and Swinnerton-Dyer [4] concerning the isolation of indefinite ternary quadratic forms with integral coefficients. This result is that if g is such a form and (μ,η) is any open interval there exists a nhd $N_g(\epsilon)$ such that any form lying in $N_g(\epsilon)$, not a multiple of g, takes a value in (μ,η). If we assume kf' to be integral for some number k, and take $(\mu,\eta) = (0,\frac{1}{2}km_+(f))$, then the above isolation theorem shows that there exists $N_{f'}(\epsilon)$ such that every form g in $N_{f'}(\epsilon)$ with $m_+(g) \geqslant \frac{1}{2}m_+(f)$ is a multiple of f'. As there exists n such that h_n is in $N_{f'}(\epsilon)$, h_n, and thus f, must be equivalent to a multiple of f'. However this implies, using properties (ii) and (iii), that f takes the value $m_+(f)$, in contradiction to the given. This shows property (iv) and completes the proof of the theorem.

We may now simplify the theorems B_i as follows. Suppose that theorem B_i is false. Then there exists a form g of signature 1, with $d(g) = d$ where $0 < d \leqslant 1/b_i$, with $m_+(g) = 1$, such that g is not equivalent to a multiple of F_i or F_{i+1} and such that

$$m_-(g) \geqslant \sqrt[3]{a_i d}.$$

If $m_+(g)$ is not attained by g, then by the above theorem there exists g', not a multiple of an integral form (and hence not equivalent to a multiple of F_i or F_{i+1}) with $d(g') = d$, $m_+(g') = 1$ attained by g', and such that $m_-(g') \geqslant m_-(g) \geqslant \sqrt[3]{a_i d}$.

Hence if theorem B_i is false it still remains false if we insert the extra condition that $m_+(g)$ is attained by g.

Let theorem C_i denote theorem B_i with this extra assumption. Then clearly theorem B_i will follow once we have established theorem C_i.

For the proofs of theorems C_i we use a chain of forms (g_i), $-\infty < i < \infty$, equivalent to and associated with a given ternary form f.

Let f be an indefinite ternary quadratic form of signature 1 taking the value $m_+(f) = 1$. Then we can find an equivalent form

$$g = (x + \lambda y + \mu z)^2 + q(y,z).$$

Now q is an indefinite binary quadratic form with $d(q) = d(f) \neq 0$, and it cannot take a value in the open interval

$$(-m_-(g) - 1/4,\ 3/4) \tag{4.3}$$

as otherwise we could choose x suitably (i.e. such that $(x + \lambda y + \mu z)^2 \leqslant 1/4$ if the value of q were non-negative, otherwise such that $1/4 \leqslant (x + \lambda y + \mu z)^2 \leqslant 1$) to obtain a value of g

that contradicts the definition of either $m_-(f)$ or $m_+(f) = 1$. Hence, as in the introduction, there exists a chain of reduced forms

$$q_i = (-1)^i a_i y^2 + b_i yz + (-1)^{i+1} a_{i+1} z^2, \quad -\infty < i < \infty$$

each equivalent to q. By applying a suitable y-z transformation we may replace $q(y,z)$ in g by any one of the $q_i(y,z)$ giving

$$g_i' = (x + \alpha_i y + \beta_i z)^2 + q_i(y,z)$$

equivalent to f. Then by changing the sign of y if necessary and by applying a suitable parallel transformation to x we obtain, using the relations of the introduction, a chain of forms

$$g_i = (x + \lambda_i y + \mu_i z)^2 + (-1)^{i+1}(z - F_i y)(z + S_i y)a_{i+1}$$

with $|\lambda_i| \leqslant \frac{1}{2}$ and $|\mu_i| \leqslant \frac{1}{2}$ such that each form g_i of the chain is equivalent to f. We shall call such a chain an "equivalence chain" for f. It should be noted that there may be a number of distinct equivalence chains for a given f, depending on the initial choice of g.

CHAPTER 5

In this chapter we prove theorem C_1. The proof makes use of the following results.

Lemma 5.1

Let $k \geqslant 2$ be integral and let q be an indefinite binary quadratic form. Define

$$A = [k^2 + k + (3k - 1)\sqrt{k^2 + 4k}]/(4k - 2),$$

$$B = \min (4k^2, k^2 + 6k + 1),$$

$$d = \min \{A^2 m_-^2/4k^2, Bm_+^2/4, Bm_-^2/4k^2\}$$

where $m_+ = m_+(q)$ and $m_- = m_-(q)$. Then either q is equivalent to a multiple of $x^2 - kxy - ky^2$ or $d(q) \geqslant d$.

Proof

The proof of this result depends on the work of Tornheim [20]. Put

$$Q(x,y) = q(x,y)/2\sqrt{d(q)}$$

so that Q has discriminant $\Delta^2 = 1$ and let

$$\left.\begin{aligned} M &= m_+(Q) = m_+(q)/2\sqrt{d(q)} \\ N &= m_-(Q) = m_-(q)/2\sqrt{d(q)} \\ P &= \max (1/M, k/N). \end{aligned}\right\} \quad (5.1$$

Then Tornheim has shown that either

(a) $P \geqslant 2k$, or

(b) $P = \sqrt{k^2 + 4k}$ and Q is equivalent to

$$M(x^2 - kxy - ky^2) = N(x^2 - kxy - ky^2)/k, \quad \text{or}$$

(c) From the proof of lemma 7 of his paper,

$$N \leqslant k/A, \qquad \text{or}$$

(d) From his lemmas 8 and 10 the chain of g_i for Q contains at least one $(k + 1)$ and

$$P \geqslant \sqrt{k^2 + 6k + 1}.$$

Now (a) and (d) give

$$1/M \text{ or } k/N \geqslant \min\,(2k, \sqrt{k^2 + 6k + 1}) = \sqrt{B}$$

from which, using (5.1), we have that either

$$d(q) \geqslant m_+^{\,2} B/4 \geqslant d, \quad \text{or}$$

$$d(q) \geqslant m_-^{\,2} B/4k^2 \geqslant d.$$

Similarly (c) gives that

$$d(q) \geqslant m_-^{\,2} A^2/4k^2 \geqslant d.$$

The lemma now follows on observing that the alternative (b) implies that q is equivalent to a multiple of $x^2 - kxy - ky^2$.

Lemma 5.2

Both $h_1(x) = x^3 - \frac{1}{16}(x + \frac{1}{4})^2$ and $h_2(x) = x^3 - \frac{1}{18}(x + \frac{1}{4})^2$ have only one real root.

Proof

Evaluation of the roots of the derivatives of h_1 and h_2 shows that these roots are at most $1/8$ in absolute value (in fact they are $(1/8 \pm \sqrt{1/64 + 3/8})/6$ and $(1/9 \pm \sqrt{1/81 + 1/3})/6$ and the maximum of these is $1/8$). Then h_1 and h_2 are negative at these points, and so their graphs have both turning points below the x-axis. This implies that $h_1(x)$ and $h_2(x)$ have only one real root.

The Proof of Theorem C_1

We are now in a position to prove theorem C_1 which for reference is re-stated.

"If g is any indefinite ternary quadratic form of signature 1, with $d(g) = d$ where

$$0 < d \leqslant 49/54,$$

and if $m_+(g) = m_+ = 1$ is attained by g then either

(a) $m_-(g) < \sqrt[3]{d/48}$, or

(b) g is equivalent to a multiple of either F_1 or F_2."

As indicated at the end of Chapter 4 we consider in place of g an equivalence chain (g_i) of forms equivalent to g. We have

$$g_i = (x + \lambda_i y + \mu_i z)^2 + (-1)^{i+1} a_{i+1}(z - F_i y)(z + S_i y)$$

where as indicated in the introduction

$$a_i = a_{i+1} F_i S_i$$

$$F_i = (p_i, p_{i+1}, p_{i+2}, \ldots)$$

$$S_i = (0, p_{i-1}, p_{i-2}, \ldots)$$

$$K_i = F_i + S_i$$

$$a_{i+1} K_i = \Delta; \quad \Delta^2 = 4d. \tag{5.2}$$

Since $(-1)^{i+1} a_{i+1}(z - F_i y)(z + S_i y)$ cannot take any values in the interval (4.3) we have, assuming that

$$m_-(g) = m_- \geqslant \sqrt[3]{d/48}, \qquad (5.3$$

the following:

$$a_i \geqslant 3/4 \quad (i \text{ even}), \qquad (5.4$$

$$a_i \geqslant m_- + 1/4 \geqslant \sqrt[3]{d/48} + 1/4 \quad (i \text{ odd}). \qquad (5.5$$

Using (5.2) and setting

$$d = 49\beta/54, \quad 0 < \beta \leqslant 1 \qquad (5.6$$

we obtain

$$K_i = 7\sqrt{6\beta}/9a_{i+1}.$$

Then using the bounds (5.4) and (5.5) we find that

$$K_i \leqslant 28\sqrt{6\beta}/27 < 2.5403\sqrt{\beta} \quad (i \text{ odd}), \qquad (5.7$$

$$K_i \leqslant 7\sqrt{6\beta}[9\sqrt[3]{\beta}(\tfrac{1}{4} + \sqrt[3]{\tfrac{49}{2592}})]^{-1} < 3.6893\sqrt[6]{\beta} \ (i \text{ even}). \ (5.8$$

As $p_i < F_i < K_i$ we conclude that

$$p_i \leqslant 2 \quad (i \text{ odd}); \quad p_i \leqslant 3 \quad (i \text{ even}).$$

The proof is now presented as a series of lemmas, each eliminating various possibilities for combinations of p_i occurring in the chain $[p_i]$. In these lemmas the following property will be used.

"If the sequence $(r,s,\ldots,t) = (p_i, p_{i+1}, \ldots, p_{i+j})$ cannot occur in the chain $[p_i]$ then neither can the sequence $(t,\ldots,s,r) = (p_{k-j},\ldots,p_{k-1},p_k)$ where $k \equiv i \pmod 2$."

This follows from the fact that replacing y by $-y$ reverses the order of the chain $[p_i]$ without affecting the values taken by the form.

[In fact, if $q(y,z) = (-1)^{i+1} a_{i+1}(z - F_i y)(z + S_i y)$ then the transformation $\bar{z} = z + p_i y$, $\bar{y} = -y$ gives

$$q(y,z) \sim \bar{q}(\bar{y},\bar{z}) = (-1)^{i+1} a_{i+1}(\bar{z} - \psi_i \bar{y})(\bar{z} + \phi_i \bar{y})$$

where $\phi_i = (F_i - p_i) = (0, p_{i+1}, p_{i+2}, \ldots)$ and $\psi_i = (S_i + p_i) = (p_i, p_{i-1}, p_{i-2}, \ldots)$. Clearly this reverses the order of the chain.]

For simplicity, λ and μ will replace λ_i and μ_i in the local considerations of the chain $[p_i]$ in the following work.

Lemma 5.3

The chain cannot contain either $p_i = 3$ with i even or $p_i = 2$ with i odd.

Proof

Let $p_i = 2$ with i odd and suppose that one of p_{i-1}, p_{i+1} is not 3. Then

$$K_i > 2 + (0,2,1) + (0,3,1) = 2\tfrac{7}{12}$$

which contradicts (5.7). Thus if $p_i = 2$ with i odd then $p_{i-1} = p_{i+1} = 3$.

Let $p_i = 3$ with i even and suppose that one of p_{i+1}, p_{i-1} is not 2. Then

$$K_i > 3 + (0,1,1) + (0,2,1) = 3\tfrac{5}{6}$$

which contradicts (5.8). Thus we must have $p_{i-1} = p_{i+1} = 2$, and so $p_{i-2} = p_{i+2} = 3$. Then

$$K_i > 3 + 2(0,2,3,3) = 89/23$$

which again contradicts (5.8). Since $p_i = 3$ (i even) leads to a contradiction and $p_i = 2$ (i odd) implies $p_{i+1} = 3$ the lemma follows.

From this lemma we can conclude that

$$p_i = 1 \quad (i \text{ odd}); \qquad p_i \leqslant 2 \quad (i \text{ even}).$$

Lemma 5.4

The chain cannot have $p_{i-1} = p_{i+1} = 1$ where i is odd.

Proof

Suppose that $p_{i-1} = p_{i+1} = 1$ with i odd. Then

$$F_i \geqslant (1,1,\overline{1,2}) = 1 + 1/\sqrt{3} > 1.57735. \qquad (5.9$$

Similarly $S_i > .57735$, and so $K_i > 2.1547$. Using (5.7) we can obtain that $\beta > .71944$ and combining this with (5.3) and (5.6) we find that

$$m_- > .2386. \qquad (5.10$$

Now

$$\left.\begin{aligned} F_i &\leqslant (1,1,\overline{1}) = (\sqrt{5} + 1)/2 \\ \text{and} \quad S_i &\leqslant (0,1,\overline{1}) = (\sqrt{5} - 1)/2. \end{aligned}\right\} \qquad (5.11$$

Using these bounds together with the lower bounds (5.9) we obtain that

$$\left.\begin{aligned} .91068 &< F_iS_i \leqslant 1 \\ .91068 &< (F_i - 1)(S_i + 1) \leqslant 1. \end{aligned}\right\} \qquad (5.12$$

In addition we have, with regard to (5.4),(5.2) and

(5.6), that

$$.75 \leqslant a_{i+1} = 7\sqrt{6\beta}/9K_i < .8844.$$

Suppose, contrary to what we wish to prove, that $a_{i+1} \leqslant .81$. Then as $m_+ = 1$, choosing x so that $(x + \mu)^2 \leqslant \frac{1}{4}$, it is clear that we must have the value $(x + \mu)^2 + a_{i+1} \geqslant 1$. Therefore

$$(x + \mu)^2 \geqslant 1 - a_{i+1} \geqslant .19.$$

This implies that

$$\|\mu - \tfrac{1}{2}\| < .0642, \qquad (5.13$$

where $\|t\|$ denotes the distance from t to the nearest integer. Choosing x so that $1/4 \leqslant (x + \lambda)^2 \leqslant 1$ gives g_i the value $(x + \lambda)^2 - a_{i+1}F_iS_i$ which is less than 1. Then

$$(x + \lambda)^2 \leqslant a_{i+1}F_iS_i - m_-.$$

Using (5.10) and (5.12) gives that

$$(x + \lambda)^2 < .5714$$

and so $\|\lambda - \frac{1}{2}\| < .256$. Combining this with (5.13) yields that $\|\lambda - \mu\| < .3202$, so we can choose x such that $(x + \lambda - \mu)^2 < .103$. However using the bounds (5.9) and (5.11) we find that

$$a_{i+1}(1 + F_i)(1 - S_i) < .8963,$$

giving

$$(x + \lambda - \mu)^2 + a_{i+1}(1 + F_i)(1 - S_i) < .9993.$$

This is a value of g_i contradicting $m_+ = 1$, and shows that we cannot have $a_{i+1} \leqslant .81$. Thus we have

$$.81 < a_{i+1} < .8844. \qquad (5.14$$

In the following values of g_i we choose x such that the square lies between 1 and 2.25 inclusive:

$$(x + \lambda)^2 - a_{i+1}F_iS_i,$$

$$(x + \lambda + \mu)^2 - a_{i+1}(F_i - 1)(S_i + 1).$$

Equations (5.12) show that these values are non-negative, so they must be at least 1 $(=m_+)$. Thus

$$(x + \lambda)^2 \geqslant 1 + a_{i+1}F_iS_i. \qquad (5.15$$

Then using (5.12 and (5.14) we have $(x + \lambda)^2 > 1.73856$, which yields that $\|\lambda - \frac{1}{2}\| < .182$. Similarly $\|\lambda + \mu - \frac{1}{2}\| < .182$. Thus $\|\mu\| < .364$, so we can choose x such that

$$(x + \mu)^2 < (.364)^2 < .1325.$$

In order that the value $(x + \mu)^2 + a_{i+1}$ shall not contradict $m_+ = 1$, we must have $a_{i+1} > .8675$. Using this instead of (5.14) in (5.15) and repeating the argument gives that $\|\mu\| < .326$, so we can choose x such that

$$(x + \mu)^2 + a_{i+1} < (.326)^2 + .8844 < 1.$$

This contradicts $m_+ = 1$ and completes the proof of the lemma.

Lemma 5.5

The chain cannot have $p_{i-3} = p_{i-1} = 2$, $p_{i+1} = 1$ where i is odd.

<u>Proof</u>

Suppose to the contrary that such an i was in the chain. Then the previous lemma implies that $p_{i+3} = 2$, and so

$$F_{i-1} = (2,1,1,1,2,1,\ldots) \geqslant (\overline{2,1,1,1}) > 2.6329,$$

$$S_{i-1} = (0,1,2,1,\ldots) \geqslant (0,\overline{1,2,1,1}) > .7247.$$

Thus $K_i > 3.3576$. Using (5.2),(5.3) and (5.5) we find that

$$m_- \geqslant \sqrt[3]{K_i^2(m_- + 1/4)^2/3}\,/4,$$

and inserting the above bound for K_i gives that

$$m_- > \sqrt[3]{3.7578(m_- + 1/4)^2}\,/4.$$

By iterating on this, commencing with $m_- \geqslant 0$, we eventually obtain that $m_- > .242$.

The following bounds on F_i and S_i may be easily obtained.

$$1.57735 < (1,1,\overline{1,2}) \leqslant F_i \leqslant (\overline{1,1,1,2}) < 1.580,$$

$$.366 < (0,2,\overline{1,2}) \leqslant S_i \leqslant (0,2,\overline{1,2,1,1}) < .36702.$$

Then $K_i > 1.9433$, and using (5.2) and (5.6) we can deduce that $a_{i+1} < .9804$. Combining this with the bounds for F_i and S_i yields that

$$\left.\begin{aligned} a_{i+1}F_iS_i &< .5686, \\ a_{i+1}(1 + 3F_i)(3S_i - 1) &< .5686. \end{aligned}\right\} \quad (5.16$$

Choosing x with $1/4 \leqslant (x + \lambda)^2 \leqslant 1$ gives, by the same method as in the previous lemma, that

$$(x + \lambda)^2 \leq a_{i+1}F_iS_i - m_- .$$

Using the above bounds for m_- and $a_{i+1}F_iS_i$ gives

$$(x + \lambda)^2 < .3266 < (.5716)^2,$$

and so $\|\lambda - \frac{1}{2}\| < .0716$. Similarly we can prove that $\|3\lambda - \mu - \frac{1}{2}\| < .0716$, and so $\|4\lambda - \mu\| < .1432$. Now

$$a_{i+1}(7.309)(.464) < a_{i+1}(1 + 4F_i)(4S_i - 1) < 3.36,$$

and we can choose x such that $3.4 < (x + 4\lambda - \mu)^2 \leq 4$. This gives a positive value

$$(x + 4\lambda - \mu)^2 - a_{i+1}(1 + 4F_i)(4S_i - 1)$$

of g_i, so in order not to contradict $m_+ = 1$ we must have

$$a_{i+1}(7.309)(.464) \leq 3.$$

Thus $a_{i+1} < .8847$. This enables us to revise the bounds in (5.16), and repeating the analysis yields that $\|\lambda - \frac{1}{2}\| < .021$ and that $\|3\lambda - \mu - \frac{1}{2}\| < .021$. Then $\|\mu\| < .084$, so we can choose x such that

$$0 < (x + \mu)^2 + a_{i+1} < (.084)^2 + .8847 < 1.$$

This contradiction to $m_+ = 1$ completes the proof of the lemma.

It follows from the above lemmas that the chain $[p_i]$ must be one of the following:

(a) $\infty(1,2)\infty$, i.e. for all j, $p_{2j} = 2$, $p_{2j+1} = 1$.

(b) $\infty(1,1,1,2)\infty$, i.e. for all j,

$$p_{4j-1} = p_{4j} = p_{4j+1} = 1, \quad p_{4j+2} = 2.$$

We now consider these special cases in turn.

Lemma 5.6

If the chain $[p_i]$ is $\infty(1,2)\infty$, then $g \sim F_1\sqrt[3]{3/4}$.

Proof

If the chain is $\infty(1,2)\infty$, we have for i even

$$g_i = (x + \lambda_i y + \mu_i z)^2 - a_{i+1}(z^2 - 2yz - 2y^2).$$

Since $g_i \sim g$ there is no loss of generality in dropping the suffixes and taking g_i to be g. Then

$$d = d(g) = 3a^2 \leqslant 49/54,$$

and so $$a \leqslant 7\sqrt{2}/18 < .55.$$

In addition, $d/48 = a^2/16$, and so (5.3) and (5.5) yield that

$$m_-^3 \geqslant (m_- + 1/4)^2/16,$$

i.e. $$h_1(m_-) \geqslant 0.$$

By using lemma 5.2, noting that $h_1(1/4) = 0$, we have

$$m_- \geqslant 1/4; \quad a \geqslant 1/2.$$

Consider the binary quadratic form

$$t(x,z) = az^2 - (x + \mu z)^2,$$

the negative of a section of g. This must have

$$m_+(t) \geqslant 1/4, \quad m_-(t) = 1.$$

Then taking $k = 4$ in lemma 5.1 we have that either

(a) $t \sim (z^2 - 4xz - 4x^2)/4$ and $a = d(t) = 1/2$, or

(b) $a = d(t) > .5389$.

For the moment let us consider the second possibility. This gives

$$m_- \geq \sqrt[3]{a^2/16} > .26.$$

Choosing, without loss of generality, $0 \leq \mu \leq \frac{1}{2}$, we have in the section $-t(x,z)$ with $x = -z = 1$ that

$$(1 - \mu)^2 - a < .5.$$

Then this value must be at most $-m_-$, and so

$$(1 - \mu)^2 \leq a - m_- < .29 < (.5386)^2,$$

from which we can deduce that $.4614 < \mu \leq .5$. Then in the value $-t(1,3)$ we have that

$$5.66 < (1 + 3\mu)^2 \leq 6.25,$$

$$4.85 < 9a < 4.95.$$

In order not to contradict $m_+ = 1$ we must have $(1 + 3\mu)^2 > 5.85$, giving $.4728 < \mu \leq .5$. In the value $-t(5,-4)$ we have that

$$9 \leq (5 - 4\mu)^2 < 9.67,$$

$$8.622 < 16a < 8.8.$$

Then as $m_+ = 1$ we must have $16a < 8.67$. In the value $-t(1,4)$ we have that

$$8.35 < (1 + 4\mu)^2 \leq 9,$$

$$8.622 < 16a < 8.67.$$

Then as $m_+ = 1$, $m_- > .26$ we have that

$$8.35 < (1 + 4\mu)^2 < 8.67 - .26 = 8.41 = (2.9)^2.$$

Hence we must have that

$$.4728 < \mu < .475. \tag{5.17}$$

By an identical treatment applied to the sections

$$-t_1 = (x + (\lambda - \mu)z_1)^2 - az_1^2 : \quad y = -z = z_1$$

$$-t_2 = (x + (\lambda + 3\mu)z_2)^2 - az_2^2 : \quad z = 3z_2 = 3y$$

we can derive that

$$.4728 < \lambda - \mu < .475 \text{ or } .525 < \lambda - \mu < .5272 \quad (5.18$$

$$.4728 < \lambda + 3\mu < .475 \text{ or } .525 < \lambda + 3\mu < .5272 \quad (5.19$$

(modulo 1). These inequalities (5.17),(5.18) and (5.19) can be shown to be inconsistent by adding 4 times (5.17) to (5.18).

This eliminates the possibility that $a > .5389$ and leaves $a = \frac{1}{2}$. In this case

$$t \sim \tfrac{1}{2}z^2 - (x + \tfrac{1}{2}z)^2,$$

$$t_1 \sim \tfrac{1}{2}z_1^2 - (x + \tfrac{1}{2}z_1)^2.$$

This yields on considering the types of forms equivalent to $\frac{1}{2}z^2 - (x + \frac{1}{2}z)^2$ that $\mu \equiv \lambda - \mu \equiv \frac{1}{2}$ (mod 1), from which it follows that g is equivalent to

$$(x + \tfrac{1}{2}z)^2 - \tfrac{1}{2}(z^2 - 2yz - 2y^2) = F_1\sqrt[3]{3/4}.$$

Lemma 5.7

If the chain $[p_i]$ is $\infty(1,1,1,2)\infty$ then $g \sim F_2\sqrt[3]{49/54}$.

Proof

If the chain is $\infty(1,1,1,2)\infty$ we have for i even and $p_i = 2$ that

$$g_i = (x + \lambda_i y + \mu_i z)^2 - a_{i+1}(z^2 - 2yz - \tfrac{5}{3}y^2).$$

Since $g_i \sim g$ there is no loss of generality in dropping the suffixes and taking g_i to be g. Then

$$d = d(g) = 8a^2/3 \leqslant 49/54,$$

and so $$a \leqslant 7/12.$$

In addition $d/48 = a^2/18$ and so as in the previous lemma we obtain that $h_2(m_-) \geqslant 0$. Since $h_2(.23) < 0$ we must have $m_- > .23$. By the same method as in the previous lemma it can be shown that either

(a) $az^2 - (x + \mu z)^2 \sim \frac{1}{2}z^2 - (x + \frac{1}{2}z)^2$, or

(b) $a > .5389$.

For the moment let us consider the first possibility. In this case $a = \frac{1}{2}$. If we set $y = 3z_3$, $z = -2z_3$ then we must have

$$az_3{}^2 - (x + (3\lambda - 2\mu)z_3)^2 \sim \tfrac{1}{2}z_3{}^2 - (x + \tfrac{1}{2}z_3)^2,$$

which yields, taking (a) into consideration as well, that $\mu \equiv 3\lambda - 2\mu \equiv \frac{1}{2} \pmod 1$. From this we can deduce that $\lambda \equiv \frac{1}{2}$ or $\pm\frac{1}{6} \pmod 1$. However $\lambda \equiv \pm\frac{1}{6}$ gives the section $(x + \lambda)^2 + \frac{5}{6}$ the value $\frac{31}{36}$, and $\lambda \equiv \frac{1}{2}$ gives the section $(x + \lambda - \mu)^2 - \frac{1}{2}(1 + 2 - \frac{5}{3})$ the value $\frac{1}{3}$, in each case contradicting $m_+ = 1$.

This eliminates the possibility that $a = \frac{1}{2}$, leaving $a > .5389$, from which we obtain that

$$m_- \geqslant \sqrt[3]{a^2/18} > .252.$$

Choosing x with $1/4 \leqslant (x + \mu)^2 \leqslant 1$ in the section

$(x+\mu)^2 - a$ gives a value less than 1, so this value is at most $-m_-$. Therefore

$$(x+\mu)^2 < 7/12 - .252 < .3314 < (.5757)^2.$$

This yields that $\|\mu - \tfrac{1}{2}\| < .0757$. Thus we can choose x such that

$$5.0 < (x+3\mu)^2 \leqslant 6.25.$$

Then as $4.85 < 9a \leqslant 5.25$ this gives $(x+3\mu)^2 - 9a$ a value greater than $-m_-$, so this value is at least 1. This implies that

$$(x+3\mu)^2 > 5.8501 > (2.4178)^2,$$

from which it follows that

$$\|\mu - \tfrac{1}{2}\| < .0274. \qquad (5.20$$

The value $(x+\lambda-\mu)^2 - 4a/3$ with x chosen such that $1 \leqslant (x+\lambda-\mu)^2 \leqslant 9/4$ yields, as $.718 < 4a/3 \leqslant 7/9$, a positive value of g. This value must be at least 1, so

$$(x+\lambda-\mu)^2 > 1.718 > (1.31)^2,$$

which yields that

$$\|\lambda - \mu - \tfrac{1}{2}\| < .19. \qquad (5.21$$

Since $5a/3 \leqslant 35/36$ it is clear from the sections

$$(x+\lambda)^2 + 5a/3; \qquad (x+2\lambda-\mu)^2 + 5a/3;$$
$$(x+\lambda+2\mu)^2 + 5a/3; \qquad (x+2\lambda+5\mu)^2 + 5a/3;$$

that we must have

$\lambda, \lambda+2\mu, 2\lambda-\mu, 2\lambda+5\mu$ each at least $\tfrac{1}{6}$ from 0 (5.22

(modulo 1).

It is easily verified that the only solutions to the congruence inequalities (5.20),(5.21) and (5.22) are $\lambda \equiv \pm\frac{1}{6}$, $\mu \equiv \frac{1}{2}$ (mod 1). Then in order that the section $(x + \lambda)^2 + 5a/3$ shall not take a value contradicting $m_+ = 1$ we must have $a = 7/12$. Thus we must have

$$g \sim (x + \tfrac{1}{6}y + \tfrac{1}{2}z)^2 - \tfrac{7}{12}(z^2 - 2yz - \tfrac{5}{3}y^2) = F_2\sqrt[3]{49/54}$$

as required.

Combining the lemmas proven we have shown that if $m_-(g) \geqslant \sqrt[3]{d/48}$ then g is equivalent to a multiple of either F_1 or F_2. This is clearly equivalent to proving theorem C_1.

CHAPTER 6

In this chapter we prove theorem C_2 which for reference is re-stated.

"If g is any indefinite ternary quadratic form of signature 1, with $d(g) = d$ where

$$0 < d \leqslant 9/8,$$

and if $m_+(g) = m_+ = 1$ is attained by g then either

(a) $m_-(g) < \sqrt[3]{2d/49}$, or

(b) g is equivalent to a multiple of either F_2 or F_3."

The Proof of Theorem C_2

As in the previous chapter we consider, in place of g, an equivalence chain (g_i) of forms equivalent to g. For simplicity we use the same notation as in the previous chapter, renaming (5.2) as (6.1), i.e.

$$a_{i+1}K_i = \Delta; \qquad \Delta^2 = 4d, \tag{6.1}$$

and replacing (5.3) by the assumption that

$$m_-(g) = m_- \geqslant \sqrt[3]{2d/49}. \tag{6.2}$$

Similarly (5.4) and (5.5) become

$$a_i \geqslant 3/4 \quad (i \text{ even}) \tag{6.3}$$

$$a_i \geqslant m_- + 1/4 \geqslant \sqrt[3]{2d/49} + 1/4 \quad (i \text{ odd}), \tag{6.4}$$

from which, using (6.1) and setting

$$d = 9\beta/8, \quad 0 < \beta \leqslant 1, \tag{6.5}$$

we obtain that

$$K_i = 3\sqrt{2\beta}/2a_{i+1}. \tag{6.6}$$

Then using the bounds (6.3) and (6.4) we find that

$$K_i \leqslant 2\sqrt{2\beta} < 2.82843\sqrt{\beta} \quad \text{(i odd)} \qquad (6.7$$

$$K_i \leqslant 3\sqrt{2\beta}\left[2\sqrt[3]{\beta}\left(\tfrac{1}{4} + \sqrt[3]{\tfrac{15}{392}}\right)\right]^{-1} < 3.48859\sqrt[6]{\beta} \quad \text{(i even).} \qquad (6.8$$

As $p_i < F_i < K_i$ we can conclude that

$$p_i \leqslant 2 \quad \text{(i odd)}; \qquad p_i \leqslant 3 \quad \text{(i even)}.$$

The proof is now presented as a series of lemmas, with the use of λ, μ for λ_i, μ_i respectively for simplicity.

<u>Lemma 6.1</u>

The chain $[p_i]$ cannot contain $p_i = 3$ for i even.

<u>Proof</u>

If $p_i = 3$ with i even then

$$F_i > (3,2,1) = 10/3; \quad S_i > (0,2,1) = 1/3,$$

and so $K_i > 11/3$ which contradicts (6.8).

<u>Lemma 6.2</u>

The chain $[p_i]$ cannot contain $p_i = 2$ with i odd unless $p_{i-1} = p_{i+1} = 2$.

<u>Proof</u>

Suppose that $p_i = 2$ with i odd and with one of p_{i-1}, p_{i+1} not 2. Then

$$K_i > 2 + (0,2,1) + (0,1,1) = 17/6$$

which contradicts (6.7).

Lemma 6.3

If in the chain $[p_i]$ there is an odd j with $p_j = 1$ then either (a) $p_{j-2} = p_{j+2} = 1$ or (b) $p_{j-1} = p_{j+1} = 2$ and one of p_{j-2}, p_{j+2} is 1.

Proof

Suppose that $p_j = 1$ with j odd and that one of p_{j-2}, p_{j+2} is not 1. Then lemma 6.2 shows that there are in effect two possible cases where (b) does not hold, viz

(i) $p_{j-2} = p_{j+2} = 2$; the chain is $..,2,2,2,\underset{j}{1},2,2,2,..$

(ii) $p_{j-2} = 1$, $p_{j+2} = 2$; the chain is $..,1,1,\underset{j}{1},2,2,2,..$

It should be noticed that the reverse situation to (ii), i.e. $p_{j-2} = 2$, $p_{j+2} = 1$, is equivalent to (ii) - this was observed in chapter 5.

We now take $i = j + 1$ and consider these two cases together, for the actual method of proof is the same although the bounds may differ. Where the bounds do differ, those given will be those for case (ii) with those for case (i) following in square brackets.

We have that

$$F_i \geqslant (2,\overline{2}) = 1 + \sqrt{2} > 2.41421$$

and that

$$S_i \geqslant (0,\overline{1}) > .618 \quad [(0,1,\overline{2}) > .7071].$$

Hence $K_i > 3.0322$ [3.1213], and using (6.6) yields that

$$a_{i+1} < 3/(3.0322\sqrt{2}) < .69961 \quad [.68].$$

Now we also have that

$$F_i \leqslant (2,2,\overline{2,1}) < 2.42265$$

and that

$$S_i \leqslant (0,1,1,\overline{1,2}) < .634 \; [(0,1,2,2,\overline{2,1}) < .7079].$$

Consideration of the section $y = 1, z = 2$ of g_i, with $(x + \lambda + 2\mu)^2 \leqslant 1/4$, yields that $a_{i+1} > .67369$ [.6553] and that

$$\|\lambda + 2\mu - \tfrac{1}{2}\| < .03. \tag{6.9}$$

In addition, as $m_-^3 \geqslant a_{i+1}^2 K_i^2/98$, we obtain in each case that

$$m_- > .349.$$

Considering the section $(x + \mu)^2 - a_{i+1}$ with $1/4 \leqslant (x + \mu)^2 \leqslant 1$ yields that

$$\|\mu - \tfrac{1}{2}\| < .0925 \quad [.0754].$$

Then using (6.9) we obtain that

$$\|\lambda - \mu\| < .31 \quad [.257].$$

By choosing x such that $1 \leqslant (x + \lambda - \mu)^2 < 1.72$ [$.552 < (x + \lambda - \mu)^2 \leqslant 1$] in the section

$$(x + \lambda - \mu)^2 - a_{i+1}(1 + F_i)(1 - S_i)$$

we obtain a value of g_i in the open interval (.085,.879) [(−.138,.35)] which contradicts either $m_+ = 1$ or $m_- > .349$.

Lemma 6.4

If in the chain $[p_i]$ there is an odd j with $p_j = 1$ then we cannot have $p_{j-2} = 1$, $p_{j-1} = p_{j+1} = 2$.

Proof

Suppose that such a situation occurred. Then setting $i = j - 1$ we have that

$$F_i = (2,1,2,..) \geqslant (2,1,\overline{2}) > 2.7071,$$

$$S_i = (0,1,..) \geqslant (0,\overline{1}) > .618.$$

Hence $K_i > 3.3251$, and so $a_{i+1} < .638$ follows from (6.6). Combining (6.1),(6.2) and (6.4) yields that

$$m_-^3 \geqslant a_{i+1}^2 K_i^2/98 \geqslant K_i^2(m_- + 1/4)^2/98,$$

and inserting the bound for K_i gives that

$$98m_-^3 > 11.05629(m_- + 1/4)^2.$$

Iterating on this, commencing from $m_- \geqslant 0$, eventually gives that $m_- > .339$, $a_{i+1} > .589$.

Consideration of $(x + \mu)^2 - a_{i+1}$ with $1/4 \leqslant (x + \mu)^2 \leqslant 1$ gives that $\|\mu - \frac{1}{2}\| < .04681$. Then in the value $(x + 3\mu)^2 - 9a_{i+1}$ we have that $5.301 < 9a_{i+1} < 5.742$ and we can choose x such that $5.567 < (x + 3\mu)^2 \leqslant 6.25$. This gives a value of g_i that contradicts either $m_+ = 1$ or $m_- > .339$.

Lemma 6.5

If in the chain $[p_i]$ there is an odd j with $p_j = 1$ then $p_i = 1$ for all odd i.

Proof

We only need to show that $p_{j-2} = p_{j+2} = 1$ if $p_j = 1$ with j odd. This follows from lemma 6.3, using lemma 6.4 to eliminate the possibility (b).

Lemma 6.6

If in the chain $[p_i]$ there is an odd j with $p_j = 1$, $p_{j+1} = 2$ then $p_{j-1} = p_{j+3} = 1$.

Proof

This is a direct consequence of lemmas 6.4 and 6.5.

Lemma 6.7

If in the chain $[p_i]$ there is an odd j with $p_j = 1$, $p_{j+1} = 2$ then $p_{j-3} \neq 1$.

Proof

Suppose to the contrary that there was an odd j with $p_{j-3} = p_j = 1$, $p_{j+1} = 2$. Then setting $i = j + 1$ we have that

$$2.618 < (2,\overline{1}) \leqslant F_i < (2,1,1,1,2,1,2) = 79/30,$$

$$.618 < (0,\overline{1}) \leqslant S_i \leqslant (0,1,1,\overline{1,1,1,2}) < .62021,$$

and that $K_i > 3.236$. By using a similar method to that used in lemma 6.4 we obtain that $a_{i+1} < .6556$ and that

$$98m_-^3 > 10.47169(m_- + 1/4)^2,$$

from which $m_- > .33$ follows by iteration.

Consider the form

$$t = a_{i+1}z^2 - (x + \mu z)^2,$$

the negative of a section of g_i. This has

$$m_-(t) = m_+(g_i) = 1,$$

$$m_+(t) \geqslant m_-(g_i) > .33.$$

Now using lemma 5.1 with $k = 3$ we find that either

(i) $d(t) > .6577$, which contradicts the previous bound $a_{i+1} < .6556$, or

(ii) $d(t) = a_{i+1} = 7/12$.

Then as $K_i < 3.2536$ we have that

$$d = a_{i+1}{}^2K_i{}^2/4 < 49/54$$

and the result follows from theorem C_1.

<u>Lemma 6.8</u>

$p_i = 2$ for at least one i.

<u>Proof</u>

If $p_i = 1$ for all i then for i even

$$g_i = (x + \lambda_i y + \mu_i z)^2 - a_{i+1}(z^2 - yz - y^2).$$

Since $g_i \sim g$ there is no loss of generality in dropping the suffixes. Then

$$d = d(g) = 5a^2/4 \leqslant 9/8,$$

and so $\qquad a < .9487. \qquad$ (6.10

If $a \leqslant .852$ then $d < 49/54$ and the result follows from theorem C_1. Hence it is sufficient to assume that $a > .852$.

Considering the sections

$$(x + \lambda - \mu)^2 - a; \quad (x + \mu)^2 - a$$

with $1 \leqslant (x + \mu)^2 \leqslant 9/4$, $1 \leqslant (x + \lambda - \mu)^2 \leqslant 9/4$ we find that each square must be greater that 1.852, from which it follows that $\|\mu - \frac{1}{2}\| < .14$ and that $\|\lambda - \mu - \frac{1}{2}\| < .14$. Hence $\|\lambda\| < .28$, and so we can find x such that

$$0 < (x + \lambda)^2 + a < .0784 + a.$$

Thus as $m_+ = 1$ we must have $a > .921$. Repeating this argument using this new bound gives that $a > .947$, and a further iteration gives that $a > .95$, contradicting (6.10). This completes the proof of the lemma.

Suppose that $p_j = 1$ for all odd j. Then from lemma 6.8 we must have $p_{i+1} = 2$ for some odd i, and lemmas 6.6 and 6.7 applied to $[p_i]$ and the reverse chain show that

$$p_{i+5} = p_{i-3} = 2, \quad p_{i-1} = p_{i+3} = 1.$$

Repetition of the argument yields that

$$p_j = 2 \quad (j \equiv i + 1 \pmod 4),$$
$$p_j = 1 \quad (\text{otherwise}),$$

and hence the chain $[p_i]$ is $\infty(1,1,1,2)\infty$.

Suppose that $p_j = 2$ for some odd j. Then lemmas 6.2 and 6.5 show that $p_j = 2$ for all j, and so the chain is $\infty(2)\infty$.

We shall now consider these remaining two possibilities.

<u>Lemma 6.9</u>

If the chain $[p_i]$ is $\infty(1,1,1,2)\infty$ then $g \sim F_2\sqrt[3]{49/54}$.

<u>Proof</u>

If the chain is $\infty(1,1,1,2)\infty$ we have for i even and $p_i = 2$ that

$$g_i = (x + \lambda_i y + \mu_i z)^2 - a_{i+1}(z^2 - 2yz - \tfrac{5}{3}y^2).$$

Since $g_i \sim g$ there is no loss of generality in dropping the suffixes and taking g_i to be g. Then

$$d = d(g) = 8a^2/3 \leqslant 9/8,$$

and so $$a < .65.$$

By the usual method we obtain that

$$m_-^3 \geqslant 16(m_- + 1/4)^2/147.$$

Hence

$$147m_-^3 - 16m_-^2 - 8m_- - 1 \geqslant 0,$$

and so

$$(3m_- - 1)(49m_-^2 + 11m_- + 1) \geqslant 0.$$

Then as $m_- \geqslant 0$ we must have $m_- \geqslant 1/3$, $a \geqslant 7/12$. Now considering the section $(x + \mu)^2 - a$ with $1/4 \leqslant (x + \mu)^2 \leqslant 1$ as in lemma 6.7 we find that

$$\|\mu - \tfrac{1}{2}\| < .041.$$

Then considering the section $(x + 3\mu)^2 - 9a$ with $5.65 < (x + 3\mu)^2 \leqslant 6.25$ we find, if $a > 7/12$, that g takes a value in the open interval $(-.2,1)$, contradicting either $m_+ = 1$ or $m_- \geqslant 1/3$. Hence $a = 7/12$. Then $d = 49/54$ and applying theorem C_1 shows that $g \sim F_2\sqrt[3]{49/54}$.

Lemma 6.10

If the chain $[p_i]$ is $\infty(2)\infty$ then $g \sim F_3\sqrt[3]{9/8}$.

Proof

If the chain is $\infty(2)\infty$ we have for i even that

$$g_i = (x + \lambda_i y + \mu_i z)^2 - a_{i+1}(z^2 - 2yz - y^2).$$

As $g_i \sim g$ there is no loss of generality in dropping the suffixes and taking g_i to be g. Then

$$d = d(g) = 2a^2 \leqslant 9/8,$$

and so $$a \leqslant 3/4.$$

As $m_+ = 1$, considering the sections

$$(x + \lambda)^2 + a, \quad (x + \lambda + 2\mu)^2 + a$$

we find that $a = 1/4$ and $\lambda \equiv \lambda + 2\mu \equiv \frac{1}{2} \pmod 1$. Then as $\mu \equiv 0$ implies that g takes the value $1 - 3/4 = 1/4$ when $y = 0$, $z = 1$, contradicting $m_+ = 1$, we must have $\mu \equiv \frac{1}{2}$. Hence

$$g \sim (x + \tfrac{1}{2}y + \tfrac{1}{2}z)^2 - \tfrac{3}{4}(z^2 - 2yz - y^2) = F_3\sqrt[3]{9/8}.$$

CHAPTER 7

This chapter is devoted to proving the following result:

Theorem D

If g is any indefinite ternary quadratic form of signature 1, with $d(g) = d$ where

$$0 < d \leqslant 3/2,$$

and if $m_+(g) = m_+ = 1$ is attained by g, then either

(a) $m_-(g) < \sqrt[3]{d/9}$, or

(b) g is equivalent to a multiple of either F_3, F_4 or F_5.

This theorem is stronger than either theorem C_3, which makes the stronger assumption that $d \leqslant 144/125$, or theorem C_4, which has the weaker conclusion that $m_-(g) < \sqrt[3]{3d/16}$. Thus theorems C_3 and C_4 will follow when we prove theorem D.

Applying theorem D to normalised forms, in the way that theorems A_i are deduced from theorems B_i, it can be seen that every normalised indefinite ternary quadratic form of signature 1, not equivalent to F_4, takes a value in the closed interval $[-\sqrt[3]{1/9}, \sqrt[3]{2/3}]$, the intersection of intervals I_3 and I_4.

Proof of Theorem D

As usual we consider, in place of g, an equivalence chain (g_i) of forms equivalent to g. Assuming that $m_-(g) \geqslant \sqrt[3]{d/9}$ and using the same notation as in the previous chapters we have that

$$a_{i+1}K_i = \Delta; \qquad \Delta^2 = 4d, \tag{7.1}$$

$$m_-(g) = m_- \geqslant \sqrt[3]{d/9}, \tag{7.2}$$

$$a_i \geqslant 3/4 \quad (i \text{ even}), \tag{7.3}$$

$$a_i \geqslant m_- + 1/4 \geqslant \sqrt[3]{d/9} + 1/4 \quad (i \text{ odd}), \tag{7.4}$$

$$d = 3\beta/2, \quad 0 < \beta \leqslant 1, \tag{7.5}$$

and

$$K_i = \sqrt{6\beta}/a_{i+1}. \tag{7.6}$$

Using the bounds (7.3) and (7.4) in (7.6) we obtain that

$$K_i \leqslant 4\sqrt{6\beta}/3 < 3.266\sqrt{\beta} \quad (i \text{ odd}), \tag{7.7}$$

$$K_i \leqslant \sqrt{6\beta}[\sqrt[3]{\beta}(\sqrt[3]{\tfrac{1}{6}} + \tfrac{1}{4})]^{-1} < 3.062\sqrt[6]{\beta} \quad (i \text{ even}). \tag{7.8}$$

Hence we must have $p_i \leqslant 3$ for all i. If however $p_i = 3$ for some i we would have

$$K_i > (3,4,1) + (0,4,1) = 3.4$$

which contradicts the relevant one of (7.7) and (7.8). Thus we must have $p_i \leqslant 2$ for all i.

We now present the proof as a series of lemmas.

Lemma 7.1

If $p_i = 2$ with i even then $p_{i-1} = p_{i+1} = 2$.

Proof

Let $p_i = 2$ where i is even and suppose that one of p_{i-1}, p_{i+1} is 1. Then we have that

$$K_i \geqslant 2 + (0,1,\overline{1,2}) + (0,2,\overline{1,2}) > 2.943,$$

and comparing this with (7.8) yields that $\sqrt[6]{\beta} > .96113$, i.e. $\sqrt[3]{\beta} > .92377$. Hence as $m_- \geqslant \sqrt[3]{\beta/6}$ we have that $m_- > .508$, and so $a_{i+1} > .758$. In addition, using the above bound for K_i in (7.6), we find that $a_{i+1} < .8324$. However applying lemma 5.1 with $k = 2$ to the form

$$t = a_{i+1}z^2 - (x + \mu z)^2,$$

the negative of a section of g_i (and thus $m_-(t) = 1$, $m_+(t) > .508$), yields that either

(a) $t \sim \frac{1}{2}(x^2 - 2xz - 2z^2)$, with $d(t) = a_{i+1} = 3/4$, or

(b) $a_{i+1} = d(t) > .944$,

in either case contradicting $.758 < a_{i+1} < .8324$.

Lemma 7.2

If $p_i = 2$ with i even then $p_i = 2$ for all i and $g \sim F_3\sqrt[3]{9/8}$.

Proof

Let $p_i = 2$ where i is even. Then we have that

$$K_i \geqslant 2 + (0,\overline{2,1}) + (0,\overline{2,1}) = 1 + \sqrt{3}.$$

Now combining (7.1),(7.2) and (7.4) we have that

$$m_-^3 \geq K_i^2(m_- + 1/4)^2/36,$$

and inserting the above bound for K_i yields that

$$m_-^3 \geq 7.4641(m_- + 1/4)^2/36.$$

From this, by iteration, we obtain that $m_- > .478$, and hence the form

$$t = a_{i+1}z^2 - (x + \mu z)^2$$

has $m_-(t) = 1$, $m_+(t) > .478$. Applying lemma 5.1 with $k = 2$ yields that either

(a) $a_{i+1} = 3/4$, or

(b) $a_{i+1} > .913$.

However as (b) implies that $d = a_{i+1}^2 K_i^2/4 > 1.55$ which contradicts the given we must have $a_{i+1} = 3/4$.

Now we have, using the previous lemma, that

$$F_i \leq (\overline{2}) \; ; \quad S_i \leq (0,\overline{2}) = 1/(\overline{2}),$$

and so $F_i S_i \leq 1$ with equality if and only if $p_i = 2$ for all i. However as $F_i S_i < 1$ implies that $a_{i+1} F_i S_i < 3/4$ which contradicts (7.3) we must have $p_i = 2$ for all i. Thus

$$g_i = (x + \lambda_i y + \mu_i z)^2 - \tfrac{3}{4}(z^2 - 2yz - y^2),$$

and so $d = d(g) = 9/8$ and the lemma follows from theorem C_2.

For the remainder of the proof of theorem D we may assume that $p_i = 1$ for all even i.

Lemma 7.3

If $p_i = 2$ with i odd then $p_{i-2} = p_{i+2} = 1$.

Proof

Let $p_i = 2$ with i odd and suppose that one of p_{i-2}, p_{i+2} is 2. Then

$$K_i \geqslant 2 + (0,1,2,\overline{1}) + (0,\overline{1}) > 3.31$$

which contradicts (7.7).

Lemma 7.4

If $p_i = 2$ with i odd then $p_{i-4} = p_{i+4} = 2$.

Proof

Let $p_i = 2$ with i odd and suppose that one of p_{i-4}, p_{i+4} is 1. Then by considering the reverse chain if necessary we may assume that $p_{i-4} = 1$. This gives the following bounds:

$$2.618 < (2,\overline{1}) \leqslant F_i \leqslant (\overline{2,1,1,1}) < 2.633,$$

$$.618 < (0,\overline{1}) \leqslant S_i \leqslant (0,1,1,\overline{1,1,1,2}) < .62021.$$

Hence $K_i \geqslant 1 + \sqrt{5} > 3.236$, and using this in (7.6) gives that $a_{i+1} < .757$. In addition, $a_{i+1} \geqslant .75$ follows from (7.3), so combining (7.1) and (7.2) with the above bound for K_i yields that

$$m\_^3 \geqslant (3 + \sqrt{5})/32 > (.545)^3,$$

i.e. $$m\_ > .545.$$

Considering the value $(x + \mu)^2 + a_{i+1}$ of g_i yields that

$$\|\mu - \tfrac{1}{2}\| < .0071. \qquad (7.9$$

Furthermore choosing x such that $1 \leqslant (x+\lambda)^2 \leqslant 9/4$ in the section $(x+\lambda)^2 - a_{i+1}F_iS_i$ of g_i yields, as $1.212 < a_{i+1}F_iS_i < 1.237$, that $\|\lambda - \tfrac{1}{2}\| < .013$. Combining this with (7.9) shows that $\|5\lambda - 3\mu\| < .09$, and hence we can choose x such that $1 \leqslant (x + 5\lambda - 3\mu)^2 < 1.19$. However

$$1.08 < a_{i+1}(3 + 5F_i)(5S_i - 3) < 1.24,$$

and so g_i takes a value in the open interval $(-.24, .11)$, contradicting either $m_+ = 1$ or $m_- > .545$. This contradiction completes the proof of the lemma.

It follows from the above lemmas that if $g \not\sim F_3\sqrt[3]{9/8}$ and if $p_i = 2$ for some i then the chain must have

$$p_j = 2 \quad (j \equiv i \pmod 4),$$
$$p_j = 1 \quad (\text{otherwise}),$$

and so the chain is $\infty(1,1,2,1)\infty$.

<u>Lemma 7.5</u>

If the chain $[p_i]$ is $\infty(1,1,2,1)\infty$ then $g \sim F_5\sqrt[3]{3/2}$.

<u>Proof</u>

If the chain is $\infty(1,1,2,1)\infty$ then for odd i with $p_i = 2$ we have that

$$g_i = (x + \lambda_i y + \mu_i z)^2 + a_{i+1}(z^2 - 2yz - \tfrac{5}{3}y^2).$$

Since $g_i \sim g$ there is no loss of generality in dropping the subscripts and taking g_i to be g. Then

$$d = d(g) = 8a^2/3 \leqslant 3/2,$$

and so $$a \leqslant 3/4.$$

However as $a \geqslant 3/4$ by (7.3) we must have $a = 3/4$, and so considering the sections $(x + \mu)^2 + 3/4$ and $(x + \lambda)^2 - 5/4$ we find that $\mu \equiv \lambda \equiv \frac{1}{2}$ (mod 1). Hence

$$g \sim (x + \tfrac{1}{2}y + \tfrac{1}{2}z)^2 + \tfrac{3}{4}(z^2 - 2yz - \tfrac{5}{3}y^2)$$
$$\sim (x + \tfrac{1}{2}y + \tfrac{1}{2}z)^2 - \tfrac{5}{4}(z^2 - \tfrac{6}{5}yz - \tfrac{3}{5}y^2),$$

i.e. $g \sim F_5\sqrt[3]{3/2}$.

There is only one further possibility left for the chain $[p_i]$, namely $p_i = 1$ for all i. We now consider this case.

Lemma 7.6

If the chain $[p_i]$ is $\infty(1)\infty$ then $g \sim F_4\sqrt[3]{144/125}$.

Proof

If the chain is $\infty(1)\infty$ then for i even we have

$$g_i = (x + \lambda_i y + \mu_i z)^2 - a_{i+1}(z^2 - yz - y^2).$$

Since $g_i \sim g$ there is no loss of generality in dropping the suffixes and taking g_i to be g. Then

$$d = d(g) = 5a^2/4 \leqslant 3/2,$$

and so $$a < 1.096.$$

In addition $a \geqslant 3/4$ follows from considering the section $(x + \lambda)^2 + a$. In fact we must have

$$a \geqslant 1 - \|\lambda\|^2,$$

and as

$$\|\lambda\| \leqslant \|\mu - \tfrac{1}{2}\| + \|\lambda - \mu - \tfrac{1}{2}\|,$$

it follows that

$$a \geqslant 1 - (\|\mu - \tfrac{1}{2}\| + \|\lambda - \mu - \tfrac{1}{2}\|)^2. \qquad (7.10$$

Now the section $(x + \mu)^2 - a$ with x chosen such that $1 \leqslant (x + \mu)^2 \leqslant 9/4$ gives a value of g lying in the half-open interval $(-.096, 1.5]$. Then as

$$m_- \geqslant \sqrt[3]{d/9} \geqslant \sqrt[3]{5/4} > .427$$

this value of g is at least 1, and so $(x + \mu)^2 \geqslant 1+a$. Hence

$$\|\mu - \tfrac{1}{2}\| \leqslant 3/2 - \sqrt{1 + a}.$$

Similarly it can be shown that

$$\|\lambda - \mu - \tfrac{1}{2}\| \leqslant 3/2 - \sqrt{1 + a}.$$

Inserting these in (7.10) yields that

$$a \geqslant 1 - (3 - 2\sqrt{1 + a})^2,$$

which on simplification implies that $(25a - 24)a \geqslant 0$. Then as $a \geqslant 3/4$ we must have $a \geqslant 24/25$.

Considering the sections $(x + \mu)^2 - a$, $(x + \lambda - \mu)^2 - a$ and $(x + \lambda + 2\mu)^2 - a$ with the squares chosen the closed interval $[1, 9/4]$ we find that

$$\|\mu - \tfrac{1}{2}\| \leqslant .1, \qquad (7.11$$

$$\|\lambda - \mu - \tfrac{1}{2}\| \leqslant .1 \qquad (7.12$$

and $$\|\lambda + 2\mu - \tfrac{1}{2}\| \leqslant .1. \qquad (7.13$$

Subtracting (7.12) from (7.13) gives $\|3\mu\| \leqslant .2$, while multiplying (7.11) by 3 gives $\|3\mu - \tfrac{1}{2}\| \leqslant .3$, i.e. $\|3\mu\| \geqslant .2$. Hence $\|3\mu\| = .2$, and combining this with (7.11) we find that $\mu \equiv .4$ or $.6 \pmod 1$. Without loss of generality we may take $\mu = .4$. Then (7.12) and (7.13) imply that $\lambda \equiv .8 \pmod 1$.

As $3.84 \leqslant 4a < 4.4$ the value

$$(3 - 2\mu)^2 - 4a = 4.84 - 4a$$

will contradict $m_+ = 1$ unless $a = 24/25$. Hence

$$g \sim (x + \tfrac{4}{5}y + \tfrac{2}{5}z)^2 - \tfrac{24}{25}(z^2 - yz - y^2)$$

$$= F_4\sqrt[3]{144/125}.$$

This completes the proof of theorem D.

CHAPTER 8

The proof of Theorem C_5.

For reference the theorem is re-stated.

Theorem C_5

If g is any indefinite ternary quadratic form of signature 1, with $d(g) = d$ where

$$0 < d \leqslant 112/27,$$

and if $m_+(g) = m_+ = 1$ is attained by g then either

(a) $m_-(g) < \sqrt[3]{2d/3}$, or

(b) g is equivalent to a multiple of either F_5 or F_6.

Both this theorem and theorem C_6 may be deduced from the work of Venkov [21]. This will be shown in chapter 10. For the sake of completeness, however, theorems C_5 and C_6 will be proved first by methods similar to those of the previous chapters.

Proof of Theorem C_5

As usual we consider in place of g an equivalence chain (g_i) of forms equivalent to g. Assuming that $m_-(g) \geqslant \sqrt[3]{2d/3}$ and using the same notation as in the previous chapters we have that

$$a_{i+1}K_i = \Delta; \qquad \Delta^2 = 4d, \tag{8.1}$$

$$m_-(g) = m_- \geqslant \sqrt[3]{2d/3}, \tag{8.2}$$

$$a_i \geq 3/4 \quad (i \text{ even}), \tag{8.3}$$

$$a_i \geq m_- + 1/4 \geq \sqrt[3]{2d/3} + 1/4 \quad (i \text{ odd}), \tag{8.4}$$

$$d = 112\beta/27, \quad 0 < \beta \leq 1, \tag{8.5}$$

and

$$K_i = 8\sqrt{7\beta}/3\sqrt{3}a_{i+1}. \tag{8.6}$$

Then using the bounds (8.3) and (8.4) in (8.6) we obtain that

$$K_i \leq 8\sqrt{7\beta}[3\sqrt{3}\ \sqrt[3]{\beta}(\sqrt[3]{224/81} + 1/4)]^{-1} < 2.4643\sqrt[6]{\beta} \tag{8.8}$$

for i even and that

$$K_i \leq 32\sqrt{7\beta}/9\sqrt{3} < 5.43121\sqrt{\beta} \quad (i \text{ odd}). \tag{8.7}$$

Hence we must have

$$p_i \leq 5 \quad (i \text{ odd}); \quad p_i \leq 2 \quad (i \text{ even}).$$

Now suppose that $p_i = 5$ for some odd i. Then

$$K_i > 5 + 2(0,2,1) = 17/3$$

which contradicts (8.7). Hence $p_i \leq 4$ for all odd i.

We now present the proof as a series of lemmas.

Lemma 8.1

If $p_i = 2$ with i even then $p_{i-1} = p_{i+1} = 4$.

Proof

Let $p_i = 2$ with i even and suppose that one of p_{i-1}, p_{i+1} is not 4. Then

$$K_i \geq 2 + (0,\overline{4,1}) + (0,3,\overline{1,4})$$
$$> 2.468$$

which contradicts (8.8).

Lemma 8.2

$p_i = 1$ for all even i.

Proof

Let $p_i = 2$ with i even. Then $p_{i-1} = p_{i+1} = 4$ and so

$$K_i \geqslant 2 + 2(0,\overline{4,1}) > 2.414.$$

Hence using (8.6) we obtain that $a_{i+1} < 1.688$.

Now combining (8.1),(8.2) and (8.4) gives that

$$m_-^3 \geqslant K_i^2(m_- + 1/4)^2/6, \qquad (8.9$$

and inserting the bound on K_i yields that

$$m_- > \sqrt[3]{.97123(m_- + 1/4)^2}.$$

By iteration, commencing with $m_- \geqslant 0$, we eventually obtain that $m_- > 1.35$, and so (8.4) yields that $a_{i+1} > 1.60$. However we can choose x such that $1 \leqslant (x + \mu)^2 \leqslant 9/4$, and so g_i takes the value $(x + \mu)^2 - a_{i+1}$ lying in the open interval $(-.69,.65)$. This contradicts either $m_+ = 1$ or $m_- > 1.35$.

Lemma 8.3

If $p_i = 4$ with i odd then $p_{i+2} \geqslant 2$ and $p_{i-2} \geqslant 2$.

Proof

Let $p_i = 4$ with i odd and suppose that $p_{i-2} = 1$. Then

$$K_i \geqslant 4 + 2(0,\overline{1}) > 5.236.$$

Using (8.1),(8.2) and (8.3) we obtain that $m_{\_} > 1.369$. Now it follows on consideration of $(x + \mu_{i-1})^2 - a_i$, choosing the square suitably in the closed intervals [1/4,1] and [1,9/4], that if $m_{\_} > 1$ then a_i must be at least $1 + m_{\_}$ for odd i, i.e.

$$a_i \geqslant 1 + m_{\_} \quad \text{if} \quad m_{\_} > 1 \quad (i \text{ odd}). \tag{8.10}$$

Hence as $m_{\_} > 1.369$ we must have $a_i > 2.369$.

Now

$$K_{i-1} \geqslant (\overline{1,4}) + (0,1,\overline{1,4}) > 1.751,$$

and so

$$d = a_i^2 K_{i-1}^2/4 > 4.3$$

which contradicts (8.5). Hence $p_{i-2} \geqslant 2$.

Similarly (by considering the reverse chain) we must have $p_{i+2} \geqslant 2$.

Lemma 8.4

$p_i \leqslant 3$ for all odd i.

Proof

Let $p_i = 4$ with i odd. Then $p_{i-2} \geqslant 2$ and $p_{i+2} \geqslant 2$ from the previous lemma, and so

$$K_i \geqslant 4 + 2(0,1,2,\overline{1}) > 5.447,$$

which contradicts (8.7).

Lemma 8.5

If $p_i = 3$ with i odd then $p_{i-2} \geqslant 2$ and $p_{i+2} \geqslant 2$.

Proof

Let $p_i = 3$ with i odd and suppose that one of p_{i-2}, p_{i+2} is 1. Then by taking the reverse chain if necessary we may assume that $p_{i-2} = 1$. By a similar method to that used in lemma 8.3 we obtain that $K_i > 4.236$, $m_- > 1.189$ and $a_i > 2.189$. In addition,

$$K_{i-1} \geqslant (\overline{1,3}) + (0,1,\overline{1,3}) > 1.822,$$

and using this in (8.6) yields that $a_i < 2.24$. Then considering the value $(x + \mu_{i-1})^2 - a_i$ with $1 \leqslant (x + \mu_{i-1})^2 \leqslant 9/4$ we find that $\|\mu_{i-1}\| < .0255$. Hence $\|2\mu_{i-1}\| < .051$ and so we can choose x such that $8.6 < (x + 2\mu_{i-1})^2 \leqslant 9$. However this leads to a value $(x + 2\mu_{i-1})^2 - 4a_i$ which contradicts either $m_+ = 1$ or $m_- > 1.189$.

Lemma 8.6

If $p_i = 3$ for some odd i then $p_i = 3$ for all odd i and $g \sim F_6\sqrt[3]{112/27}$.

Proof

Let $p_i = 3$ with i odd and suppose that $p_{i-2} = 2$. Then

$$K_i \geqslant (3,1,2,\overline{1}) + (0,1,2,\overline{1}) > 4.447,$$

and so using (8.1),(8.2) and (8.3) we obtain that $m_- > 1.228$. Hence $a_i > 2.228$ follows from (8.10).

In addition,

$$K_{i-1} \geqslant (\overline{1,3}) + (0,2,\overline{1,3}) > 1.622,$$

and so $a_i < 2.513$ follows from (8.6). Now combining (8.10) with (8.1) and (8.2) gives that

$$a_i = a_{i+1}F_iS_i \geqslant 1 + \sqrt[3]{2(a_{i+1}^2K_i^2/4)/3},$$

and so

$$a_{i+1}K_i \geqslant [1 + \sqrt[3]{a_{i+1}^2K_i^2/6}](1/F_i + 1/S_i). \qquad (8.11$$

As

$$3.7236 < (3,1,2,\overline{1}) \leqslant F_i \leqslant (\overline{3,1}) < 3.7913$$

and

$$.7236 < (0,1,2,\overline{1}) \leqslant S_i \leqslant (0,1,2,\overline{1,3}) < .73624,$$

it follows on consideration of (8.11) that

$$a_{i+1}K_i > [1 + \sqrt[3]{a_{i+1}^2K_i^2}/1.8172](1.6220).$$

By iterating on this, starting with $a_{i+1}K_i \geqslant 0$, we eventually obtain that $a_{i+1}K_i > 3.79$, which implies that $m_- > 1.337$ and $a_i > 2.337$.

Considering the value $(x + \mu_{i-1})^2 - a_i$ where $1 \leqslant (x + \mu_{i-1})^2 \leqslant 9/4$ we find that $\|\mu_{i-1}\| < .086$. Thus we can choose x such that $9 \leqslant (x + 2\mu_{i-1})^2 < 10.1$. However this gives the section $(x + 2\mu_{i-1})^2 - 4a_i$ a value in the open interval $(-1.1, .8)$, contradicting either $m_+ = 1$ or $m_- > 1.337$.

This contradiction shows that $p_i = 3$ with i odd implies that $p_{i-2} = 3$, and by consideration of the reverse chain, that $p_{i+2} = 3$. Hence, clearly, $p_i = 3$ for all odd i, and so, for odd i, we have

$$g_i = (x + \lambda_i y + \mu_i z)^2 + a_{i+1}(z^2 - 3yz - 3y^2).$$

As $g_i \sim g$ there is no loss of generality in dropping the suffixes and taking g_i to be g. Then

$$d = d(g) = 21a^2/4 \leqslant 112/27,$$

and so $$a \leqslant 8/9.$$

Now

$$m_- \geqslant \sqrt[3]{7a^2/2} \geqslant \sqrt[3]{63/32} > 1.253,$$

and so, considering the section $(x + \lambda)^2 - 3a$, we find that

$$3a \geqslant 1 + m_- > 2.253$$

and $$1 \leqslant (x + \lambda)^2 \leqslant 3a - m_-. \tag{8.12}$$

If $3a < 2.419$ then (8.12) implies that $\|\lambda\| < .08$, and so we can choose x such that $9 \leqslant (x + 2\lambda)^2 < 10$. However this implies that g takes the value $(x + 2\lambda)^2 - 12a$ lying in the open interval $(-.7,1)$, contradicting either $m_+ = 1$ or $m_- > 1.253$. Hence we must have $3a > 2.419$, and so $m_- > 1.314$.

Consider the value $(x + 2\lambda)^2 - 12a$ of g, where $9 \leqslant (x + 2\lambda)^2 \leqslant 12.25$. As $9.67 < 12a \leqslant 32/3$ we have two possibilities: either

(a) $(x + 2\lambda)^2 - 12a \geqslant 1$, or

(b) $(x + 2\lambda)^2 - 12a \leqslant -m_-$.

Consider the first of these two. In this case $(x + 2\lambda)^2 > 10.67$, from which it follows that

$\|\lambda\| > .133$, and so we can choose x such that $1.283 < (x+\lambda)^2 \leqslant 9/4$. Then considering the value $(x+\lambda)^2 - 3a$ we find that $3a > 2.597$, and hence that $12a > 10.388$. Repeating the analysis shows that we can choose x such that $1.39 < (x+\lambda)^2 \leqslant 9/4$. As $2.597 < 3a \leqslant 8/3$ this value of x leads to a value of g contradicting either $m_+ = 1$ or $m_- > 1.314$.

Thus we have eliminated the first possibility, leaving the possibility (b). This implies that $12a > 10.314$, and so $m_- > 1.37$. As $(x+2\lambda)^2 \leqslant 12a - m_-$ and $12a \leqslant 32/3$ we can deduce that $\|2\lambda\| < .05$. Considering the section $(x+\lambda)^2 - 3a$ we find that $\|\lambda - \frac{1}{2}\| > .025$, so we must have $\|\lambda\| < .025$.

Similarly it can be shown that $\|\lambda + 3\mu\| < .025$ and so, by subtraction, it follows that

$$\|3\mu\| < .05. \qquad (8.13$$

We may assume, without loss of generality, that $0 \leqslant \mu \leqslant \frac{1}{2}$. Then as consideration of the section $(x+\mu)^2 + a$ shows that $\|\mu - \frac{1}{2}\| \leqslant \frac{1}{6}$ it follows from (8.13) that

$$\mu = r + 1/3$$

where $0 \leqslant r < .02$. As $\mu^2 + a \geqslant 1$ we must have

$$a \geqslant 8/9 - 2r/3 - r^2 \geqslant 8/9 - r.$$

Hence

$$23.46 < 24 - 27r \leqslant 27a \leqslant 24.$$

Now as $\|\lambda\| < .025$ we can choose x such that

$$24.25 < (x + 3\lambda)^2 \leqslant 25.$$

Hence g takes a value $(x + 3\lambda)^2 - 27a$ which is at least .25. Thus this value is at least 1, and so

$$(x + 3\lambda)^2 \geqslant 1 + 27a \geqslant 25 - 27r \geqslant (5 - 3r)^2.$$

Hence $\|3\lambda\| \leqslant 3r$, and as $\|\lambda\| < .025$ we must have $\|\lambda\| \leqslant r$. Similarly it may be shown that $\|\lambda + 3\mu\| \leqslant r$, and so, by subtraction, $\|3\mu\| \leqslant 2r$. However as $\|3\mu\| = 3r$ this implies that $r = 0$. Thus we must have $\mu = 1/3$, $a = 8/9$ and $\lambda \equiv 0 \pmod 1$, and so

$$\begin{aligned} g &\sim (x + \tfrac{1}{3}z)^2 + \tfrac{8}{9}(z^2 - 3yz - 3y^2) \\ &\sim (x + \tfrac{1}{3}y)^2 - \tfrac{8}{3}(z^2 - yz - \tfrac{1}{3}y^2) \\ &= F_6\sqrt[3]{112/27}. \end{aligned}$$

This completes the consideration of chains containing $p_i = 3$ for some odd i. For the rest of the proof we may assume that $p_i \leqslant 2$ for all odd i.

<u>Lemma 8.7</u>

If $p_{i-1} = 2$ with i even then $p_{i+1} = p_{i-3} = 1$.

<u>Proof</u>

Suppose that $p_{i-1} = p_{i+1} = 2$ where i is even.

Then

$$K_i \geq 1 + 2(0,\overline{2,1}) = \sqrt{3}$$

and

$$K_{i+1} \geq 2 + (0,1,2,\overline{1}) + (0,\overline{1}) > 3.3416.$$

Now $a_{i+2} \geq 3/4$, and so

$$d = a_{i+2}{}^2 K_{i+1}{}^2/4 > 9(3.3416)^2/64.$$

Hence $m_- > 1.015$ and $a_{i+1} \geq 1 + m_- > 2.015$. This yields that

$$m_- \geq \sqrt[3]{a_{i+1}{}^2 K_i{}^2/6} > 1.266,$$

and so $a_{i+1} > 2.266$. Repeating the procedure gives that $a_{i+1} > 2.368$, and so

$$d = a_{i+1}{}^2 K_i{}^2/4 > 4.205,$$

which contradicts the given condition that $d \leq 112/27$. This proves that, given $p_{i-1} = 2$, we must have $p_{i+1} = 1$. Upon replacing i by $i - 2$ in the above proof it can be seen that we must also have $p_{i-3} = 1$.

<u>Lemma 8.8</u>

If $p_{i-1} = 1$ with i even then $p_{i+1} = p_{i-3} = 2$.

<u>Proof</u>

Suppose that $p_{i-1} = p_{i+1} = 1$ where i is even. Then

$$2.1547 < 1 + 2(0,1,\overline{1,2}) \leq K_i \leq 1 + 2(0,\overline{1}) < 2.2361$$

and

$$2.236 < 1 + 2(0,\overline{1}) \leq K_{i+1} \leq 1 + (0,\overline{1,2}) + (0,1,1,\overline{1,2})$$
$$< 2.366.$$

Hence as $a_{i+2} \geqslant 3/4$ we must have $m_-^3 > .4687$, i.e. $m_- > .776$. Thus $a_{i+1} \geqslant m_- + 1/4 > 1.026$, and so $m_- \geqslant \sqrt[3]{a_{i+1}^2 K_i^2/6} > .93$. Repeating the argument a number of times gives that $m_- > 1.02$. This implies that $a_{i+1} > 2.02$ and so

$$d = a_{i+1}^2 K_i^2/4 > 4.7,$$

which contradicts the given bound on d.

From the above lemmas it is easily seen that the only possibility left for the chain $[p_i]$ is the chain $\infty(1,1,2,1)\infty$.

<u>Lemma 8.10</u>

If the chain $[p_i]$ is $\infty(1,1,2,1)\infty$ then $g \sim F_5\sqrt[3]{3/2}$.

<u>Proof</u>

If the chain $[p_i]$ is $\infty(1,1,2,1)\infty$ then for i odd and $p_i = 2$ we have

$$g_i = (x + \lambda_i y + \mu_i z)^2 + a_{i+1}(z^2 - 2yz - \tfrac{5}{3}y^2).$$

As $g_i \sim g$ there is no loss of generality in dropping the suffixes and taking g_i to be g. Then

$$d = d(g) = 8a^2/3 \leqslant 112/27,$$

and so $$a \leqslant \sqrt{14/9} < 1.2473. \qquad (8.14$$

Now $a \geqslant 3/4$. Suppose, contrary to what we wish to prove, that $a > 3/4$. Then $m_-^3 > \frac{2}{3}\cdot\frac{9}{16}\cdot\frac{8}{3} = 1$, and so $5a/3 \geqslant 1 + m_- > 2$. Hence $a > 6/5$ and so $m_- > 1.36$. However this implies that $5a/3 > 2.36$, i.e. $a > 1.416$ which contradicts (8.14). Hence $a = 3/4$ and $m_- \geqslant 1$. Then considering the sections $(x + \mu)^2 + a$ and $(x + \lambda)^2 - 5a/3$ it is clear that $\lambda \equiv \mu \equiv \frac{1}{2} \pmod 1$. Hence

$$g \sim (x + \tfrac{1}{2}y + \tfrac{1}{2}z)^2 + \tfrac{3}{4}(z^2 - 2yz - \tfrac{5}{3}y^2)$$
$$\sim (x + \tfrac{1}{2}y + \tfrac{1}{2}z)^2 - \tfrac{5}{4}(z^2 - \tfrac{6}{5}yz - \tfrac{3}{5}y^2)$$
$$= F_5\sqrt[3]{3/2}.$$

This completes the proof of theorem C_5.

CHAPTER 9

The proof of Theorem C_6.

For reference the theorem is re-stated.

Theorem C_6

If g is any indefinite ternary quadratic form of signature 1, with $d(g) = d$ where

$$0 < d \leqslant 9/2,$$

and if $m_+(g) = m_+ = 1$ is attained by g then either

(a) $m_-(g) < \sqrt[3]{125d/112}$, or

(b) g is equivalent to a multiple of either F_6 or F_7.

Proof

As usual we consider in place of g an equivalence chain (g_i) of forms equivalent to g. Assuming that $m_-(g) \geqslant \sqrt[3]{125d/112}$ and using the same notation as was used in the previous chapters we have that

$$a_{i+1}K_i = \Delta; \quad \Delta^2 = 4d, \tag{9.1}$$

$$m_-(g) = m_- \geqslant \sqrt[3]{125d/112}, \tag{9.2}$$

$$a_i \geqslant 3/4 \quad (i \text{ even}), \tag{9.3}$$

$$a_i \geqslant m_- + 1/4 \geqslant \sqrt[3]{125d/112} + 1/4 \ (i \text{ odd}), \tag{9.4}$$

$$d = 9\beta/2, \quad 0 < \beta \leqslant 1, \tag{9.5}$$

and

$$K_i = 3\sqrt{2\beta}/a_{i+1}. \tag{9.6}$$

Now if $d \leqslant 112/27$ it follows from chapter 3 and theorem C_5 that $g \sim F_6\sqrt[3]{112/27}$. Hence for the remainder of the proof we may assume that

$$112/27 < d \leqslant 9/2. \qquad (9.7$$

From (9.2) it follows that $m_- > 5/3$, and using (8.10) we may replace (9.4) by

$$a_i \geqslant m_- + 1 \geqslant \sqrt[3]{125d/112} + 1 \quad (i \text{ odd}). \qquad (9.8$$

Then using the bounds (9.3) and (9.8) in (9.6) we obtain that

$$K_i \leqslant 4\sqrt{2\beta} < 5.657\sqrt{\beta} \quad (i \text{ odd}) \qquad (9.9$$

and

$$K_i \leqslant 3\sqrt{2\beta}[\sqrt[3]{\beta}(1 + \sqrt[3]{\tfrac{1125}{224}})]^{-1} < 1.565\sqrt[6]{\beta} \ (i \text{ even}).(9.10$$

From (9.9) and (9.10) it follows that

$$p_i \leqslant 5 \quad (i \text{ odd}); \quad p_i = 1 \quad (i \text{ even}).$$

Now if $p_i = 5$ for some odd i then

$$K_i \geqslant 5 + (0,2) + (0,2) = 6,$$

which contradicts (9.9). Hence $p_i \leqslant 4$ (i odd). In addition $p_i \geqslant 2$ for i odd for if $p_i = 1$ we would have

$$K_{i-1} \geqslant 1 + (0,2) + (0,5) = 1.7$$

which contradicts (9.9).

The proof is now continued as a series of lemmas.

Lemma 9.1

$p_i \geqslant 3$ for all odd i.

Proof

Suppose that $p_i = 2$ for some odd i. Then

$$F_{i-1} \geqslant (1,2,\overline{1,4}) = 1 + \sqrt{2}/4,$$
$$S_{i-1} \geqslant (0,\overline{4,1}) = \tfrac{1}{2}(\sqrt{2} - 1),$$

and so

$$K_{i-1} \geqslant (2 + 3\sqrt{2})/4 > 1.56.$$

Using this in (9.6) gives that $a_i < 2.72$. In addition, as $K_{i-1} < 1.565\sqrt[6]{\beta}$, we must have $\sqrt[6]{\beta} > .9968$, and so

$$m_- \geqslant \sqrt[3]{1125\beta/224} > 1.7.$$

Thus using (8.10) we have that $a_i > 2.7$. Hence the value $(x + \mu_{i-1})^2 - a_i$ of g_{i-1}, where $1 \leqslant (x + \mu_{i-1})^2 \leqslant 9/4$, lies in the open interval $(-m_-,-.35)$ unless

$$(x + \mu_{i-1})^2 \leqslant a_i - m_- < 1.02.$$

This implies that $\|\mu_{i-1}\| < .01$ and so we can choose x such that $24.7 < (x + 3\mu_{i-1})^2 \leqslant 25$. However $24.3 < 9a_i < 24.48$. Hence g_{i-1} takes a value contradicting $m_+ = 1$.

Lemma 9.2

If $p_i = 3$ with i odd then $p_i = 3$ for all odd i and the chain $[p_i]$ is $\infty(1,3)\infty$.

Proof

Let $p_j = 3$ with j odd and suppose that one of p_{j-2}, p_{j+2} is 4. By taking the reverse chain if necessary we may assume that $p_{j+2} = 4$. Then setting

$i = j + 1$ we have that

$$1.207 < (\overline{1,4}) \leqslant F_i \leqslant (1,4,\overline{1,3}) < 1.209$$

and

$$.261 < (0,3,\overline{1,4}) \leqslant S_i \leqslant (0,\overline{3,1}) < .264.$$

Hence $K_i > 1.468$, and using this in (9.6) yields that $a_{i+1} < 2.891$.

Consider the section

$$(x + \lambda + \mu)^2 + a_{i+1}(F_i - 1)(S_i + 1).$$

We have

$$0 < a_{i+1}(F_i - 1)(S_i + 1) < .764,$$

and so in order not to contradict $m_+ = 1$ we must have

$$a_{i+1}(F_i - 1)(S_i + 1) \geqslant .75.$$

This implies that $a_{i+1} > 2.839$. Hence we must have

$$m_- \geqslant \sqrt[3]{125a_{i+1}{}^2K_i{}^2/448} > 1.69.$$

By choosing x such that $9/4 \leqslant (x + \mu)^2 \leqslant 4$ in the section $(x + \mu)^2 - a_{i+1}$ we obtain a value of g_i that is greater than $-.7$. Hence in order not to contradict either $m_+ = 1$ or $m_- > 1.69$ we must have

$$(x + \mu)^2 > 1 + a_{i+1} > 3.839,$$

and so $\|\mu\| < .041$. Hence we may choose x such that

$$25 \leqslant (x + 3\mu)^2 < (5.123)^2 < 26.25.$$

However

$$25.5 < 9a_{i+1} < 26.1,$$

and so g_i takes a value in the open interval

$(-1.1, .75)$, contradicting either $m_+ = 1$ or $m_- > 1.69$.

It follows from the above that $p_j = 3$ with j odd implies that $p_{j-2} = p_{j+2} = 3$ and so $p_j = 3$ for all odd j.

Lemma 9.3

The chain $[p_i]$ cannot be $\infty(1,3)\infty$.

Proof

If the chain $[p_i]$ is $\infty(1,3)\infty$ we have for i even that

$$g_i = (x + \lambda_i y + \mu_i z)^2 - a_{i+1}(z^2 - yz - \tfrac{1}{3}y^2).$$

Now as $g_i \sim g$ there is no loss of generality in dropping the suffixes and taking g_i to be g. Then

$$d = d(g) = 7a^2/12.$$

Using this in (9.7) we find that

$$8/3 < a \leqslant \sqrt{54/7} < 2.78.$$

By choosing x such that $1 \leqslant (x + \mu)^2 \leqslant 9/4$ in the section $(x + \mu)^2 - a$ we can contradict the bound $m_- > 5/3$ found earlier unless

$$(x + \mu)^2 < 2.78 - 1.66 = 1.12.$$

Hence $\|\mu\| < .06$. It can be shown similarly that $\|3\lambda - \mu\| < .06$ and $\|3\lambda + 4\mu\| < .06$. By subtracting these we obtain $\|5\mu\| < .12$ which, as $\|\mu\| < .06$, implies that $\|\mu\| < .024$. Hence we can choose x such that

$$24.28 < (4.928)^2 < (x + 3\mu)^2 \leqslant 25.$$

However as $24 < 9a < 25.02$ the value $(x + 3\mu)^2 - 9a$ of g contradicts either $m_+ = 1$ or $m_- > 5/3$. This completes the proof of the lemma.

The only remaining possibility for $[p_i]$ is the chain $\infty(1,4)\infty$ where $p_i = 1$ (i even) and $p_i = 4$ (i odd). We now consider this case.

<u>Lemma 9.4</u>

If the chain $[p_i]$ is $\infty(1,4)\infty$ then $g \sim F_7\sqrt[3]{9/2}$.

<u>Proof</u>

Let the chain $[p_i]$ be $\infty(1,4)\infty$. Then for i even we have that

$$g_i = (x + \lambda_i y + \mu_i z)^2 - a_{i+1}(z^2 - yz - \tfrac{1}{4}y^2).$$

As $g_i \sim g$ there is no loss of generality in dropping the suffixes and taking g_i to be g. Then

$$d = d(g) = a^2/2 \leqslant 9/2,$$

and so

$$a \leqslant 3.$$

As $m_+ = 1$ and g takes the values $(x + \lambda)^2 + a/4$ and $(x + \lambda + \mu)^2 + a/4$ it follows that $a = 3$ and $\|\lambda\| = \|\lambda + \mu\| = \frac{1}{2}$. Hence

$$g \sim (x + \tfrac{1}{2}y)^2 - 3(z^2 - yz - \tfrac{1}{4}y^2)$$
$$= F_7\sqrt[3]{9/2}.$$

This completes the proof of theorem C_6.

CHAPTER 10

Deduction of Theorems C_5 and C_6 from the work of Venkov.

The result of Venkov that we refer to is the following.

"Let f be an indefinite ternary quadratic form with determinant $d > 0$ (and hence signature -1) and define

$$M(f) = \min\{m_+(f),\ m_-(f)\}.$$

Then either

$$M(f) < \sqrt[3]{2d/9}$$

or f is equivalent to a multiple of one of the following forms:

$$l_1 = -x^2 - xy - y^2 + 2z^2,$$

$$l_2 = x^2 + xy - y^2 - 2z^2,$$

$$l_3 = -x^2 - y^2 + 3z^2,$$

$$l_4 = -x^2 - xy + y^2 - \tfrac{5}{2}z^2,$$

$$l_5 = -x^2 - \tfrac{6}{7}xy - y^2 + \tfrac{2}{7}xz + \tfrac{18}{7}yz + \tfrac{17}{7}z^2,$$

$$l_6 = -x^2 - y^2 - xz - yz + 3z^2,$$

$$l_7 = -x^2 - xy - y^2 + 5z^2,$$

$$l_8 = -\tfrac{7}{5}x^2 + 2xy - \tfrac{11}{5}y^2 + \tfrac{9}{5}xz + \tfrac{1}{5}yz + z^2,$$

$$l_9 = -x^2 + \tfrac{2}{3}xy - y^2 + \tfrac{8}{3}yz + \tfrac{8}{3}z^2,$$

$$l_{10} = -x^2 + xy - y^2 + 3yz + \tfrac{21}{8}z^2,$$

$$l_{11} = -x^2 + xy - y^2 + 2xz + 2yz + 2z^2.$$

Furthermore $M(l_i) = 1$ for $1 \leq i \leq 11$."

We can use this to prove theorems C_5 and C_6 as follows.

Proof of Theorem C_5

Let g be an indefinite ternary quadratic form of signature 1, with $d(g) = d$ where $0 < d \leq 112/27$, and let $m_+(g) = 1$ be attained by g. Furthermore let $m_-(g) \geq \sqrt[3]{2d/3}$. Then

$$M(-g) \geq \min\{\sqrt[3]{2d/3}, 1\}$$
$$\geq \min\{\sqrt[3]{2d/3}, \sqrt[3]{27d/112}\}$$
$$> \sqrt[3]{2d/9}.$$

Hence $-g$ is equivalent to a multiple of l_i for some i, $1 \leq i \leq 11$. Now this is a positive multiple as l_i and g have opposite signatures and so we can let

$$g = -kl_i, \quad k > 0.$$

Thus

$$1 = m_+(g) = m_-(kl_i) = km_-(l_i).$$

Now as $M(l_i) = 1$ and each l_i clearly takes the value -1 it follows that $g = -l_i$ for some i. Hence

$$m_+(l_i) = m_-(g) \geq \sqrt[3]{2d/3} \tag{10.1}$$

and

$$d(l_i) = d \leq 112/27. \tag{10.2}$$

As l_i takes the value 1 for $i = 2,3,4,6,7$ and 8, and l_5 takes the value $8/7$ at $(-1,0,1)$, it can be shown that

$$m_+(l_i) < \sqrt[3]{d/2}$$

for $2 \leqslant i \leqslant 8$. In addition $d(l_{11}) > d(l_{10}) > 112/27$, so the only values of i for which l_i satisfies (10.1) and (10.2) are 1 and 9.

As

$$-l_1(x, y + z, z) = \sqrt[3]{3/2}\; F_5(x,y,z)$$

and

$$-l_9(x,-y,z) = \sqrt[3]{112/27}\; F_6(x,y,z) \qquad (10.3$$

theorem C_5 may now be deduced from the results of lemmas 3.5 and 3.6.

<u>Proof of Theorem C_6</u>

Let g be an indefinite ternary quadratic form of signature 1, with $d(g) = d$ where $0 < d \leqslant 9/2$, and let $m_+(g) = 1$ be attained by g. Furthermore let $m_-(g) \geqslant \sqrt[3]{125d/112}$. Then

$$\begin{aligned} M(-g) &\geqslant \min\{\sqrt[3]{125d/112}, 1\} \\ &\geqslant \min\{\sqrt[3]{125d/112}, \sqrt[3]{2d/9}\} \\ &= \sqrt[3]{2d/9}. \end{aligned}$$

Hence, as in the above proof of theorem C_5, $g = -l_i$ for some i.

As l_i takes the value 1 for $i = 1,2,3,4,6,7$ and 8, and as l_5 takes the value 8/7 at $(-1,0,1)$ and l_{10} takes the value 3/2 at $(3,0,2)$, it can be shown that $m_+(l_i) \leqslant \sqrt[3]{4d/5}$ unless $i = 9$ or 11.

As (10.3) shows that $-l_9$ is equivalent to a multiple of F_6, and as

$$-l_{11}(x + z, -y, z) = \sqrt[3]{9/2}\, F_7(x, y, z),$$

theorem C_6 may now be deduced from the results of lemmas 3.6 and 3.7.

It should be noticed that it can be deduced, in a similar way, that theorems C_5 and C_6 may be replaced by the following:

"Let g be an indefinite ternary quadratic form of signature 1, with $d(g) = d$ where $0 < d \leqslant 9/2$, and let $m_+(g) = 1$ be attained by g. Then either

$$m_-(g) < \sqrt[3]{2d/3}$$

or g is equivalent to one of $-l_1, -l_9, -l_{10}$ or $-l_{11}$."

CHAPTER 11

The Proof of Theorem C_7

For reference theorem C_7 is re-stated.

Theorem C_7

If g is any indefinite ternary quadratic form of signature 1, with $d(g) = d$ where

$$0 < d \leqslant 24,$$

and if $m_+(g) = m_+ = 1$ is attained by g then either

(a) $m_-(g) < \sqrt[3]{16d/9}$, or

(b) g is equivalent to a multiple of either F_7 or F_8.

Proof

As usual we consider in place of g an equivalence chain (g_i) of forms equivalent to g. Assuming that $m_-(g) \geqslant \sqrt[3]{16d/9}$ and using the same notation as in the previous chapters we have that

$$a_{i+1}K_i = \Delta; \qquad \Delta^2 = 4d, \tag{11.1}$$

$$m_-(g) = m_- \geqslant \sqrt[3]{16d/9}, \tag{11.2}$$

$$a_i \geqslant 3/4 \quad (i \text{ even}), \tag{11.3}$$

$$a_i \geqslant m_- + 1/4 \geqslant \sqrt[3]{16d/9} + 1/4 \ (i \text{ odd}), \tag{11.4}$$

$$d = 24\beta, \quad 0 < \beta \leqslant 1, \tag{11.5}$$

and

$$K_i = 4\sqrt{6\beta}/a_{i+1}. \tag{11.6}$$

Now if $\beta \leqslant 3/16$ we have $d \leqslant 9/2$ and using the results of theorem C_6 and lemma 3.6 it can be seen that g must be equivalent to a multiple of F_7.

Hence we may assume from now on that $\beta > 3/16$.

Then

$$m_- > \sqrt[3]{\tfrac{16}{9}\times 24\times\tfrac{3}{16}} = 2,$$

and so using the results of chapter 2 we find that

either (i) $g \sim (x + \frac{1}{2}z)^2 - 3(y^2 - yz - \frac{1}{4}z^2)$, or

(ii) $g \sim (x + \frac{1}{2}z)^2 - 3(y^2 - yz - \frac{1}{2}z^2)$, or

(iii) $g \sim x^2 - 3(y^2 - \frac{4}{3}yz - \frac{1}{3}z^2)$, or

(iv) $d \geqslant 7.5$.

Now possibility (i) may be eliminated as $\beta \not> 3/16$ for this form, and possibilities (ii) and (iii) may be eliminated as these have $m_- = 2 < \sqrt[3]{16d/9}$, contradicting equation (11.2). Hence $d \geqslant 7.5$, and so $\beta \geqslant 5/16$. Using this in equation (11.2) we find that

$$m_- \geqslant \sqrt[3]{40/3} > 2.37.$$

Applying the corollary to theorem 2.1 to the sections

$$(x + \mu_i z)^2 - a_{i+1}z^2$$

of g_i (where i is even) we find that $a_{i+1} \geqslant 4.62$ for all even i, and hence that $q_i(y,z)$ can take no values in the open interval $(-4.62, .75)$. By a result of Segre [19] it follows that

$$d = d(q_i) \geqslant \{(4.62)^2 + 3(4.62)\}/4 > 8.8.$$

We may now use this in (11.2) to obtain that $m_- > 2.5$. Repeating the above process yields that

$$m_- > 2.53; \quad a_{i+1} > 4.78 \quad (i \text{ even}). \tag{11.7}$$

For the present we shall assume that

$$d \leqslant 243/16. \tag{11.8}$$

Then $\beta \leqslant 81/128$, and using this in (11.6) we obtain that $K_i < 1.631$ for i even. Thus $p_i = 1$ for even i.

Combining (11.3),(11.6) and (11.8) we obtain that $K_i < 10.393$ for odd i, and so $p_i \leqslant 10$ for i odd.

For i even we have that

$$F_i > (1,10,1,11), \qquad S_i > (0,10,1,11),$$

and so $K_i > 155/131$. Using this in (11.6) we obtain that $a_{i+1} < 6.6$ for even i. Now suppose that, for some even i,

$$5.25 < a_{i+1} < 6.6.$$

Then in the section $(x + \mu)^2 - a_{i+1}$ of g_i, choosing x such that $4 \leqslant (x + \mu)^2 \leqslant 6.25$, we obtain a value of g_i contradicting either $m_+ = 1$ or $m_- > 2.53$ unless $a_{i+1} > 6.53$. However in this case, as $K_i > 155/131$, combining (11.1) and (11.2) yields that $m_- > 2.8$, while the value $(x + \mu)^2 - a_{i+1}$ lies between -2.6 and $-.28$. This contradiction shows that we must have

$$4.78 < a_{i+1} \leqslant 5.25 \quad (i \text{ even}) \tag{11.9}$$

We may now improve our bound on K_i, for even i, as follows. From the corollary to theorem 2.1, as $m_- > 2$, we have that

$$a_{i+1} \geqslant 4.25 + (m_- - 2) \quad (i \text{ even}), \quad (11.10)$$

and combining this with (11.1) and (11.2) yields that

$$\phi_i(m_-) = m_-^3 - \tfrac{4}{9}K_i^2(2.25 + m_-)^2 \geqslant 0 \quad (i \text{ even}). \quad (11.11)$$

Now from (11.9) and (11.10) it is clear that $m_- \leqslant 3$, hence the inequality (11.11) must be satisfied for some $m_- \leqslant 3$. However using the known bounds on m_- and K_i it can easily be seen that the derivative

$$\phi_i'(m_-) = 3m_-^2 - \tfrac{8}{9}K_i^2(2.25 + m_-) > 0,$$

and so (11.11) must be satisfied with $m_- = 3$. Hence

$$K_i \leqslant 6\sqrt{3}/7 < 1.485 \quad (i \text{ even}). \quad (11.12)$$

The proof is now continued as a series of lemmas eliminating all possibilities for the chain $[p_i]$.

<u>Lemma 11.1</u>

$p_j \leqslant 8$ for all odd j.

<u>Proof</u>

Let $p_i \geqslant 9$ for some odd i. Then p_i is either 9 or 10, and so

$$12/131 = (0,10,1,11) < S_i < (0,9,2) = 2/19.$$

Now using (11.3) we have that

$$a_{i+2}F_{i+1}S_{i+1} = a_{i+1} \geqslant 3/4,$$

and so, using (11.9), it can be seen that

$$F_{i+1} \geqslant \tfrac{3}{4}\times\tfrac{4}{21}\times\tfrac{19}{2} > 1.357. \quad (11.13$$

Considering (11.12) and noting that

$$1 + 1/(1 + p_{i+2}) < F_{i+1} < 1 + 1/p_{i+2}$$

it follows that $p_{i+2} = 2$. Thus

$$S_{i+3} > (0,2,1,11) = 12/35,$$

and so in order that (11.12) may be satisfied we must have $F_{i+3} < 1.1422$. However as

$$F_{i+3} > 1 + 1/(1 + p_{i+4})$$

it follows that $p_{i+4} \geqslant 7$, and so

$$F_{i+1} < (1,2,1,7) = 31/23 < 1.35.$$

This contradiction to (11.13) completes the proof of the lemma.

Lemma 11.2

$p_j \leqslant 7$ for all odd j.

Proof

Let $p_{i-1} = 8$ with i even. Then

$$10/89 = (0,8,1,9) < S_i < (0,8,2) = 2/17,$$

and so using (11.12) we find that $F_i < 1.373$. In addition, considering the relation $a_{i+1}F_iS_i = a_i \geqslant 3/4$, we obtain the bound $F_i > 1.214$.

Now $a_{i+1}F_iS_i < .85$, so by choosing x such that $(x + \lambda)^2 \leqslant 1/4$ we obtain a value of g_i contradicting $m_+ = 1$ unless

$$\|\lambda - \tfrac{1}{2}\| < .113. \quad (11.14$$

In addition, by choosing x such that $4 \leq (x+\mu)^2 \leq 6.25$, we obtain a value $(x+\mu)^2 - a_{i+1}$ of g_i which contradicts either $m_+ = 1$ or $m_- > 2.53$ unless $\|\mu - \frac{1}{2}\| < .096$. Combining this with (11.14) we find that $\|\lambda - \mu\| < .209$, so we can choose x such that $9 \leq (x + \lambda - \mu)^2 < 10.3$. However using the known bounds on a_{i+1}, F_i and S_i we can show that

$$9.3 < a_{i+1}(1+F_i)(1-S_i) < 11.1,$$

and so g_i takes a value in the open interval $(-2.1, 1)$. This contradicts either $m_+ = 1$ or $m_- > 2.53$.

<u>Lemma 11.3</u>

$p_j \leq 6$ for all odd j.

<u>Proof</u>

Let $p_{i-1} = 7$ with i even. Then

$$9/71 = (0,7,1,8) < S_i < (0,7,2) = 2/15,$$

and so using (11.12) we find that $F_i < 1.359$. Now $F_i > (1,7,1,8) = 80/71$, hence

$$8.81 < a_{i+1}(1+F_i)(1-S_i) < 10.82,$$

and so we can obtain a bound on $\|\lambda - \mu - \frac{1}{2}\|$ as follows.

(a) If $8.81 < a_{i+1}(1+F_i)(1-S_i) \leq 9.89$ then by choosing x such that $6.25 \leq (x+\lambda-\mu)^2 \leq 9$ we obtain a value of g_i which contradicts either $m_+ = 1$ or $m_- > 2.53$ unless

$$\|\lambda - \mu - \tfrac{1}{2}\| < .213. \qquad (11.15$$

(b) If $9.89 < a_{i+1}(1 + F_i)(1 - S_i) < 10.82$ then by choosing x such that $9 \leqslant (x + \lambda - \mu)^2 \leqslant 12.25$ we obtain a value of g_i which contradicts either $m_+ = 1$ or $m_- > 2.53$ unless $\|\lambda - \mu - \frac{1}{2}\| < .2$. Thus in each of cases (a) and (b) we have (11.15) holding.

In addition we can show, as in the proof of the previous lemma, that

$$\|\mu - \tfrac{1}{2}\| < .096. \tag{11.16}$$

Hence combining this with (11.15) we find that $\|\lambda\| < .309$, so we can choose x such that $(x + \lambda)^2 < .096$. This implies that

$$a_{i+1}F_iS_i > .904$$

in order that the value $(x + \lambda)^2 + a_{i+1}F_iS_i$ shall not contradict $m_+ = 1$. This yields, using the known bounds on a_{i+1} and S_i, that

$$F_i > 1.291, \tag{11.17}$$

and so $K_i > 1.417$. Now as $\phi_i^I(m_-) > 0$ the inequality (11.11) yields, if $m_- \leqslant 2.85$, that $K_i < 1.416$, which is impossible. Hence

$$m_- > 2.85; \quad a_{i+1} > 5.10. \tag{11.18}$$

Using the bounds (11.17) and (11.18) it can be shown that $a_{i+1}(1 + F_i)(1 - S_i) > 10.12$, and so by a method similar to that used in (b) above it follows that

$$\|\lambda - \mu - \tfrac{1}{2}\| < .166. \tag{11.19}$$

Now by using (11.18) we can refine (11.16) to

$$\|\mu - \tfrac{1}{2}\| < .031,$$

and combining this with (11.19) gives that $\|\lambda\| < .197$. Hence, as $a_{i+1}F_iS_i < .96$, g_i takes a value

$$(x + \lambda)^2 + a_{i+1}F_iS_i < .96 + .04 = 1.$$

This contradiction to $m_+ = 1$ completes the proof of the lemma.

<u>Lemma 11.4</u>

$p_j \leqslant 5$ for all odd j.

<u>Proof</u>

Let $p_{i-1} = 6$ with i even. Then

$$8/55 = (0,6,1,7) < S_i < (0,6,2) = 2/13,$$

and so using (11.12) we find that $F_i < 1.34$. Hence we must have $p_{i+1} \geqslant 3$, as otherwise

$$F_i > (1,2,1,7) = 31/23 > 1.34.$$

Similarly, by considering the reverse chain, we have $p_{i-1} \geqslant 3$ and so

$$S_i < (0,6,1,3) = 4/27.$$

Now $F_i > 63/55$; hence

$$8.73 < a_{i+1}(1 + F_i)(1 - S_i) < 10.5,$$

and so as in the proof of the previous lemma we have that

$$\|\lambda - \mu - \tfrac{1}{2}\| < .213. \qquad (11.20$$

If $a_{i+1} \geqslant 4.96$, then by choosing x such that $4 \leqslant (x + \mu)^2 \leqslant 6.25$ we obtain a value $(x + \mu)^2 - a_{i+1}$ of g_i that contradicts either $m_+ = 1$ or $m_- > 2.53$ unless $\|\mu - \frac{1}{2}\| < .06$. In addition, if $a_{i+1} < 4.96$, then by choosing x such that $2.25 \leqslant (x + \mu)^2 \leqslant 4$ we obtain a value of g_i contradicting $m_- > 2.53$ unless $\|\mu - \frac{1}{2}\| < .06$. Hence as one of these two alternatives must hold we must have

$$\|\mu - \tfrac{1}{2}\| < .06. \qquad (11.21$$

From the relations between a_i, a_{i+1}, F_i and S_i it may be shown that $a_{i-1} = a_{i+1}(1 + 6F_i)(1 - 6S_i)$ and so by a similar analysis to that used above it can be shown that $\|6\lambda - \mu - \frac{1}{2}\| < .06$. Combining this with (11.21) we find that $\|6\lambda\| < .12$ and so $\|\lambda - 1/6\| < .02$ for some integer 1 with $0 \leqslant 1 \leqslant 5$. As (11.20) and (11.21) together imply that $\|\lambda\| < .273$ it follows that $1 = 0, 1$ or 5 and so $\|\lambda\| < .187$.

As the value $\|\lambda\|^2 + a_{i+1}F_iS_i$ of g_i must be at least 1 we must have $a_{i+1}F_iS_i > .965$ and so, using the known bounds on a_{i+1} and S_i we find that $F_i > 1.24$. Now if $p_{i+1} \geqslant 4$ then

$$F_i < (1,4,2) = 11/9$$

which contradicts the above bound. Hence as $p_{i+1} \geqslant 3$ we must have $p_{i+1} = 3$. Thus

$$F_i > (1,3,1,7) = 39/31 > 1.258$$

and so $K_i > 1.403$.

Now as $\phi_i(m_-) > 0$ the inequality (11.11) yields, if $m_- \leqslant 2.8$, that $K_i < 1.4$ which is impossible. Hence $m_- > 2.8$ and $a_{k+1} > 5.05$ for all even k. By choosing x such that $4 \leqslant (x + \mu)^2 \leqslant 6.25$ we obtain a value $(x + \mu)^2 - a_{i+1}$ of g_i which contradicts either $m_+ = 1$ or $m_- > 2.8$ unless $\|\mu - \frac{1}{2}\| < .041$. Similarly, as

$$a_{i+3} = a_{i+1}(4 - 3F_i)(4 + 3S_i),$$

we find that $\|3\lambda + 4\mu - \frac{1}{2}\| < .041$ and so

$$\|3\lambda + 6\mu - \frac{1}{2}\| < .123. \qquad (11.22$$

However as $F_i < (1,3,2) = 9/7$ we have that

$$7.7 < a_{i+1}(2 - F_i)(2 + S_i) < 8.4,$$

so by choosing x such that $6.25 \leqslant (x + \lambda + 2\mu)^2 \leqslant 9$ we obtain a value of g_i which contradicts either $m_+ = 1$ or $m_- > 2.8$ unless $\|\lambda + 2\mu\| < .051$. Clearly this is incompatible with (11.22).

<u>Lemma 11.5</u>

$p_j \leqslant 4$ for all odd j.

<u>Proof</u>

Let $p_{i-1} = 5$ with i even. Then

$$7/41 = (0,5,1,6) < S_i < (0,5,2) = 2/11,$$

and so using (11.12) we find that $F_l < 1.315$. Hence, as in the proof of the previous lemma, $p_{l+1} \geqslant 3$ and so

$$48/41 = (1,5,1,6) < F_l < (1,3,2) = 9/7.$$

By considering the sections $(x + \mu)^2 - a_{l+1}$ and $(x + 5\lambda - \mu)^2 - a_{l-1}$ in the same way that the sections $(x + \mu)^2 - a_{l+1}$ and $(x + 6\lambda - \mu)^2 - a_{l-1}$ were considered in the proof of the previous lemma we may deduce that $\|\mu - \frac{1}{2}\| < .06$, $\|5\lambda - \mu - \frac{1}{2}\| < .06$ and $\|5\lambda\| < .12$. Hence

$$\|\lambda - 1/5\| < .024 \qquad (11.23$$

for some integer 1 with $0 \leqslant 1 \leqslant 4$.

Now

$$8.48 < a_{l+1}(1 + F_l)(1 - S_l) < 9.96,$$

and so by the same method as was used in the proof of lemma 11.3 we can show that $\|\lambda - \mu - \frac{1}{2}\| < .213$. As $\|\mu - \frac{1}{2}\| < .06$ this implies that $\|\lambda\| < .28$ and so we have from (11.23) that $\|\lambda\| < .224$. Thus, again using the inequality $\|\mu - \frac{1}{2}\| < .06$, we have that

$$\|\lambda + 2\mu\| < .344, \qquad (11.24$$

and so we can choose x such that $7.05 < (x + \lambda + 2\mu)^2 \leqslant 9$. However as

$$7.41 < a_{l+1}(2 - F_l)(2 + S_l) < 9.5$$

this yields a value of g_l contradicting either $m_+ = 1$ or $m_- > 2.53$ unless both

$$a_{i+1}(2 - F_i)(2 + S_i) \leq 8 \qquad (11.25$$

and

$$\|\lambda + 2\mu\| \leq .1. \qquad (11.26$$

Using the known bounds on a_{i+1} and S_i, (11.25) yields that $F_i > 1.228$. Hence as $F_i < (1,4,2) = 11/9$ if $p_{i+1} \geq 4$ it follows that $p_{i+1} = 3$ and so $F_i > (1,3,1,6) = 34/27$. Hence $K_i > 1.403$ and so by the same method as was used in the proof of the previous lemma we have that $\|3\lambda + 6\mu - \frac{1}{2}\| < .123$. Clearly this is incompatible with (11.26).

Lemma 11.6

$p_j \leq 3$ for all odd j.

Proof

Let $p_{i-1} = 4$ with i even. Then

$$6/29 = (0,4,1,5) < S_i < (0,4,2) = 2/9,$$

and so using (11.12) we find that $F_i < 1.279$. Hence as in the proof of lemma 11.4 we find that $p_{i+1} \geq 3$, $p_{i-3} \geq 3$ and so $S_i < (0,4,1,3) = 4/19$.

As $p_{i+1} \leq 4$ we have $F_i > 35/29$ and so $K_i > 1.412$. Hence by the same method as was used in the proof of lemma 11.4 we have that $a_{i+1} > 5.05$, $m_- > 2.8$ and

$$\|\mu - \tfrac{1}{2}\| < .041. \qquad (11.27$$

Using the known bounds on F_i and S_i we find that

$$8.03 < a_{i+1}(2 - F_i)(2 + S_i) < 9.215$$

and

$$8.79 < a_{i+1}(1 + F_i)(1 - S_i) < 9.49.$$

However we may choose x_1 and x_2 such that $6.25 \leqslant (x_1 + \lambda + 2\mu)^2 \leqslant 9$ and $6.25 \leqslant (x_2 + \lambda - \mu)^2 \leqslant 9$. Hence g_i takes a value lying in the open interval $(-3.24, .97)$ in contradiction to either $m_+ = 1$ or $m_- > 2.8$ unless both $\|\lambda + 2\mu - \frac{1}{2}\| < .1$ and $\|\lambda - \mu - \frac{1}{2}\| < .1$. Subtracting these yields that $\|3\mu\| < .2$ which is incompatible with (11.27).

As a consequence of the preceding lemmas p_{i-1} can only be 1,2 or 3 for i even. Hence for i even we have that

$$K_i > (1,3,1,4) + (0,3,1,4) = 29/19 > 1.485$$

which contradicts (11.12). From this contradiction we can deduce that the assumption (11.8) was false. Thus from now on we may assume that

$$d > 243/16. \qquad (11.28$$

Inserting this into (11.2) we find that $m_- > 3$.

By an obvious modification of the corollary to lemma 2.1 applied to the sections $(x + \mu_i z)^2 - a_{i+1}z^2$ of g_i it follows that, for all even i,

$$a_{i+1} \geqslant 7 + (m_- - 3) > 7. \qquad (11.29$$

Hence the binary form $q_i(y,z)$ can take no values in the open interval $(-7,3/4)$, so by the result of Segre it follows that

$$d = d(q_i) \geqslant (49 + 21)/4 = 35/2.$$

We now use this in place of (11.28) and repeat the above analysis to obtain the bounds $m_- > 3.145$, $a_{i+1} > 7.145$ and $d > 18.12$. Repeating the iteration a number of times yields that $m_- > 3.19$ and that $a_{i+1} > 7.19$ for all even i.

Using (11.2) and (11.29) in (11.6) we find that, for i even,

$$K_i \leqslant 4\sqrt{6\beta}[4 + 4\sqrt[3]{2\beta/3}]^{-1} < 1.308\sqrt[6]{\beta}, \qquad (11.30$$

while using $a_{i+1} \geqslant 3/4$ for i odd in (11.6) yields that

$$K_i \leqslant 16\sqrt{6\beta}/3 < 13.07\sqrt{\beta} \quad (i \text{ odd}). \qquad (11.31$$

Hence as $\beta \leqslant 1$ we can conclude that

$$p_i = 1 \ (i \text{ even}); \quad p_i \leqslant 13 \ (i \text{ odd}).$$

Now if $p_i \leqslant 3$ with i odd we would have

$$K_{i-1} > (1,4) + (0,14) > 1.32$$

which contradicts (11.30). Hence $p_i \geqslant 4$ for odd i. In addition, if $p_i \geqslant 12$ with i odd we would have

$$K_i > (12,1,4) + (0,1,4) = 13.6$$

which contradicts (11.31). Hence we must have

$$4 \leqslant p_i \leqslant 11 \quad (i \text{ odd}).$$

For i even we have

$$K_i > (1,12) + (0,12) = 7/6,$$

and so using (11.6) we find that $a_{i+1} < 8.4$ for all even i. If however $a_{i+1} > 8$ for some even i then by choosing x such that $6.25 \leqslant (x + \mu)^2 \leqslant 9$ we obtain a value $(x + \mu)^2 - a_{i+1}$ of g_i that contradicts either $m_- > 3.19$ or $m_+ = 1$. Hence

$$7.1 < a_{i+1} \leqslant 8 \quad (i \text{ even}).$$

We now obtain an improved lower bound on a_{i+1} for i even and a bound on $\|\mu_i\|$. For a given even j we may assume without loss of generality that $0 \leqslant \mu_j \leqslant \frac{1}{2}$. Then in order not to contradict either $m_+ = 1$ or the definition of m_- it is clear that we must have

$$(2 + \mu_j)^2 - a_{j+1} \leqslant -m_- \qquad (11.32$$

and

$$(3 - \mu_j)^2 - a_{j+1} \geqslant 1. \qquad (11.33$$

Subtracting (11.33) from (11.32) yields that $10\mu_j \leqslant 4 - m_-$ and hence that $\mu_j < .081$. Thus

$$25 \leqslant (5 + 2\mu_j)^2 < 26.7,$$

and as

$$28.72 < 4a_{j+1} \leqslant 32,$$

we must have, in order not to contradict the definition of m_-, that

$$(5 + 2\mu_j)^2 - 4a_{j+1} \leqslant -m_-.$$

By multiplying this last inequality by $9/4$ and rearranging we obtain that

$$(8 + 3\mu_j)^2 - 9a_{j+1} \leqslant 7.75 + 3\mu_j - 9m_-/4$$
$$< .83.$$

Hence in order not to contradict either $m_+ = 1$ or the definition of m_- we must have

$$(8 + 3\mu_j)^2 - 9a_{j+1} \leqslant -m_-. \qquad (11.34$$

By subtracting 9 times (11.33) from this we find that $102\mu_j < 8 - m_-$, and so

$$\mu_j < .048. \qquad (11.35$$

In addition rearranging (11.34) yields that

$$9a_{j+1} \geqslant (8 + 3\mu_j)^2 + m_- > 67.19$$

and so

$$a_{j+1} > 7.46. \qquad (11.36$$

As j was chosen arbitrarily we may deduce from (11.35) and (11.36) that for all even i both

$$\|\mu_i\| < .048 \qquad (11.37$$

and $$a_{i+1} > 7.46. \qquad (11.38$$

Using this new bound on a_{i+1} we find, by repeating the argument immediately following (11.29), that $m_- > 3.26$. This enables (11.38) to be refined to

$$a_{i+1} > 7.47 \quad \text{(all even } i\text{)}. \qquad (11.39$$

We now proceed to eliminate all possible chains

$[p_i]$ except the one required to give g as equivalent to a multiple of F_8.

Lemma 11.7

$p_i < 11$ for some odd i.

Proof

Let $p_i = 11$ for all odd i. Then for even j we have that

$$q_j = (x + \lambda_j y + \mu_j z)^2 - a_{j+1}(z^2 - yz - \tfrac{1}{11}y^2),$$

and so q_j takes the value

$$\|\lambda_j\|^2 + a_{j+1}/11 \leqslant 1/4 + 8/11 < 1,$$

contradicting $m_+ = 1$.

Lemma 11.8

$p_j \leqslant 10$ for all odd j.

Proof

Let $p_{i+1} = 11$ with i even and suppose that $4 \leqslant p_{i-1} \leqslant 10$. Then

$$168/155 = (1,11,1,12) < F_i < (1,11,1,4) = 64/59,$$

$$13/142 = (0,10,1,12) < S_i < (0,4,1,4) = 5/24,$$

and so

$$a_{i+1}(F_i - 1)(S_i + 1) < .8193.$$

Hence in order that the value $(x + \lambda + \mu)^2 + a_{i+1}(F_i - 1)(S_i + 1)$ shall not contradict $m_+ = 1$ for any x we must have $\|\lambda + \mu - \tfrac{1}{2}\| < .075$.

Combining this with (11.37) we find that

$\|\lambda - \mu - \frac{1}{2}\| < .171$ and so we can choose x_1 and x_2 such that

$$11.07 < (x_1 + \lambda - \mu)^2 \leqslant 12.25$$

and

$$12.25 \leqslant (x_2 + \lambda - \mu)^2 < 13.48.$$

However

$$12.3 < a_{i+1}(1 + F_i)(1 - S_i) < 15.2,$$

and so g_i takes at least one value contradicting either $m_+ = 1$ or $m_- > 3.26$ (The value involving x_1 if $a_{i+1}(1 + F_i)(1 - S_i) < 14.33$, otherwise the value involving x_2).

Hence $p_{i+1} = 11$ implies that $p_{i-1} = 11$. Repeating this argument indefinitely to both the original and the reverse chains shows that $p_j = 11$ for all odd j, in contradiction to the result of lemma 11.7.

<u>Lemma 11.9</u>

If $p_{i-1} = 10$ with i even then $p_{i+1} \leqslant 6$.

<u>Proof</u>

Let $p_{i-1} = 10$ with i even and suppose that $p_{i+1} \geqslant 7$. Then

$$143/131 = (1,10,1,11) < F_i < (1,7,1,4) = 44/39,$$

$$12/131 = (0,10,1,11) < S_i < (0,10,1,4) = 5/54,$$

and so

$$14.15 < a_{i+1}(1 + F_i)(1 - S_i) < 15.47.$$

However we can choose x such that $12.25 \leqslant (x + \lambda - \mu)^2 \leqslant 16$ and so g_i takes a value contradicting either $m_+ = 1$ or $m_- > 3.26$ unless $\|\lambda - \mu\| < .108$. Combining this with (11.37) we find that $\|\lambda\| < .156$ and so g_i takes the value

$$\|\lambda\|^2 + a_{i+1}F_iS_i < .025 + .836 < 1$$

contradicting $m_+ = 1$.

<u>Lemma 11.10</u>

$p_j \leqslant 9$ for all odd j.

<u>Proof</u>

Let $p_{i-1} = 10$ with i even. Then using the previous lemmas we have that

$$95/83 = (1,6,1,11) < F_i < (1,4,1,4) = 29/24,$$

$$12/131 = (0,10,1,11) < S_i < (0,10,1,4) = 5/54,$$

and so $K_i > 1.235$. Using (11.37), application of the steps

$$\left.\begin{aligned} d &= (a_{i+1}K_i)^2/4, \\ m_- &\geqslant \sqrt[3]{16d/9}, \end{aligned}\right\} \quad (11.40$$

yields that $m_- > 3.35$.

Now $a_{i+1}F_iS_i < .896$, so in order that the value $\|\lambda\|^2 + a_{i+1}F_iS_i$ shall not contradict $m_+ = 1$ we must have $\|\lambda - \frac{1}{2}\| < .178$. Combining this with (11.37) we find that $\|\lambda + 2\mu - \frac{1}{2}\| < .274$ and so we can choose x_1 and x_2 such that

$$10.4 < (x_1 + \lambda + 2\mu)^2 \leq 12.25$$

and

$$12.25 \leq (x_2 + \lambda + 2\mu)^2 < 14.3.$$

However

$$12.3 < a_{i+1}(2 - F_i)(2 + S_i) < 14.4,$$

and so g_i takes at least one value contradicting either $m_+ = 1$ or $m_- > 3.35$ (the value involving x_1 if $a_{i+1}(2 - F_i)(2 + S_i) < 13.75$, otherwise the value involving x_2).

<u>Lemma 11.11</u>

If $p_{i-1} = 9$ with i even then $p_{i+1} \leq 5$.

<u>Proof</u>

Let $p_{i-1} = 9$ with i even and suppose that $p_{i+1} \geq 6$. Then

$$120/109 = (1,9,1,10) < F_i < (1,6,1,4) = 39/34,$$

$$11/109 = (0,9,1,10) < S_i < (0,9,1,4) = 5/49,$$

and so $a_{i+1}F_iS_i < .94$. Hence in order that the value $\|\lambda\|^2 + a_{i+1}F_iS_i$ shall not contradict $m_+ = 1$ we must have $\|\lambda - \frac{1}{2}\| < .26$. Combining this with (11.37) we find that $\|\lambda - \mu - \frac{1}{2}\| < .308$, so we can choose x such that

$$12.25 \leq (x + \lambda - \mu)^2 < 14.51.$$

However

$$14.0 < a_{i+1}(1 + F_i)(1 - S_i) < 15.45,$$

and so g_i takes a value contradicting either $m_+ = 1$ or $m_- > 3.26$.

<u>Lemma 11.12</u>

If $p_{i-1} = 9$ with i even then $p_{i+1} = 4$.

<u>Proof</u>

Let $p_{i-1} = 9$ with i even. Then we only have to show that $p_{i+1} = 5$ is impossible. If $p_{i+1} = 5$ then we have

$$76/65 = (1,5,1,10) < F_i < (1,5,1,4) = 34/29,$$

$$11/109 = (0,9\ 1,10) < S_i < (0,9,1,4) = 5/49,$$

and so $K_i > 1.27$. Using this in steps (11.40) we find that $m_- > 3.419$. Now

$$14.55 < a_{i+1}(1 + F_i)(1 - S_i) < 15.63,$$

and so in order that the value $(x + \lambda - \mu)^2 - a_{i+1}(1 + F_i)(1 - S_i)$ shall not contradict either $m_+ = 1$ or $m_- > 3.419$ for any x we must have $\|\lambda - \mu\| < .057$ and $a_{i+1}(1 + F_i)(1 - S_i) \leqslant 15$. Using (11.37) and the known bounds on F_i and S_i these inequalities imply that $\|\lambda\| < .105$ and $a_{i+1} < 7.71$. Hence g_i takes the value

$$\|\lambda\|^2 + a_{i+1}F_iS_i < .02 + .93 < 1,$$

in contradiction to $m_+ = 1$.

<u>Lemma 11.13</u>

$p_j \leqslant 8$ for all odd j.

<u>Proof</u>

Let $p_{i-1} = 9$ with i even. Then from the above lemmas it follows that $p_{i+1} = 4$. Hence

$$65/54 = (1,4,1,10) < F_i < (1,4,1,4) = 29/24,$$

$$11/109 = (0,9,1,10) < S_i < (0,9,1,4) = 5/49,$$

and so $K_i > 1.304$. Using this in steps (11.40) in conjunction with the bounds $a_{i+1} > 7.47$, $a_{i+1} \geqslant 7.922$ and $a_{i+1} \geqslant 7.992$ we obtain that $m_- > 3.48$, $m_- > 3.619$ and $m_- > 3.641$ respectively. Now

$$14.78 < a_{i+1}(1 + F_i)(1 - S_i) < 15.884,$$

where the upper bound may be reduced to 15.869 or 15.73 according as the upper bound on a_{i+1} is reduced to 7.992 or 7.922 respectively. Hence in order that the value $(x + \lambda - \mu)^2 - a_{i+1}(1 + F_i)(1 - S_i)$ shall not contradict either $m_+ = 1$ or the bound on m_- for any x we must have $\|\lambda - \mu\| < .03$ and $a_{i+1}(1 + F_i)(1 - S_i) \leqslant 15$. Thus $\|\lambda\| < .078$ and $a_{i+1} < 7.59$. Hence g_i takes the value

$$\|\lambda\|^2 + a_{i+1}F_iS_i < .01 + .94 < 1,$$

in contradiction to $m_+ = 1$.

<u>Lemma 11.14</u>

$p_j \geqslant 5$ for all odd j.

Proof

Let $p_j = 4$ with j odd. Then

$$K_{j+1} > (0,5) + (1,9) > 1.31,$$

which contradicts (11.30).

Lemma 11.15

If $p_j = 8$ for all odd j then $g \sim F_8\sqrt[3]{24}$.

Proof

Let $p_j = 8$ for all odd j. Then for i even we have that

$$g_i = (x + \lambda_i y + \mu_i z)^2 - a_{i+1}(z^2 - yz - \tfrac{1}{8}y^2).$$

As $g_i \sim g$ we may drop the subscripts without loss of generality. Then from earlier work we know that

$$\|\mu\| < .048 \qquad (11.41$$

and that $7.47 < a \leqslant 8$. In addition,

$$m_- \geqslant \sqrt[3]{16d/9} = \sqrt[3]{2a^2/3} > 3.33.$$

Now $14.0 < 15a/8 \leqslant 15$ and so in order that the value $(x + \lambda - \mu)^2 - 15a/8$ shall not contradict either $m_+ = 1$ or $m_- > 3.33$ for any x we must have

$$\|\lambda - \mu\| < .13. \qquad (11.42$$

Combining this with (11.41) yields that $\|2\lambda - \mu\| < .31$, and so we can choose x such that

$$16 \leqslant (x + 2\lambda - \mu)^2 < 18.6.$$

However as $18.6 < 5a/2 \leqslant 20$, g takes a value contradicting $m_- > 3.33$ unless $\|2\lambda - \mu\| < .083$.

Hence $\|2\lambda\| < .131$ and so either $\|\lambda\| < .066$ or $\|\lambda - \frac{1}{2}\| < .066$. Clearly the second of these is incompatible with (11.41) and (11.42), hence

$$\|\lambda\| < .066. \qquad (11.43$$

Now as the transformations $(y,z) \to (y + 8z, -z)$ and $(y,z) \to (y + 8z, y + 9z)$ send g into

$$(x + \lambda y + (8\lambda - \mu)z)^2 - a(z^2 - yz - \tfrac{1}{8}y^2)$$

and

$$(x + (\lambda + \mu)y + (8\lambda + 9\mu)z)^2 - a(z^2 - yz - \tfrac{1}{8}y^2)$$

respectively it is clear that any bound for $\|\mu\|$ must hold for $\|8\lambda - \mu\|$ and $\|8\lambda + 9\mu\|$. Hence taking the bound $\|\mu\| < r$ we have $\|8\lambda - \mu\| < r$ and $\|8\lambda + 9\mu\| < r$ which yield that

$$\|8\lambda\| < 2r \qquad (11.44$$

and $\|10\mu\| < 2r$. From the second of these we have that $\|\mu - l/10\| < r/5$ for some integer l with $0 \leqslant l \leqslant 9$. Using (11.37) it is clear, if $r \leqslant .048$, that l must be 0 and that $\|\mu\| < r/5$. Repeating this argument indefinitely yields, if $r \leqslant .048$, that $\|\mu\| < r/25, r/125$, etc., and so, for $r \leqslant .048$, the only possibility is that $\|\mu\| = 0$. Similarly, for $r \leqslant .048$, (11.44) may be replaced by $\|8\lambda\| < 2r/5$, $2r/25$, etc., and so $\|8\lambda\| = 0$.

Now considering (11.41) we may take $r = .048$

and so we must have $\|\mu\| = 0$ and $\|8\lambda\| = 0$. The second of these yields, taking (11.43) into consideration, that $\|\lambda\| = 0$, and so

$$g \sim x^2 - a(z^2 - yz - \tfrac{1}{8}y^2).$$

Thus as the value $a/8$ contradicts $m_+ = 1$ unless $a = 8$ we must have

$$g \sim x^2 - 8(z^2 - yz - \tfrac{1}{8}y^2) = F_8\sqrt[3]{24}.$$

Lemma 11.16

If $p_{i-1} = 8$ with i even then $p_{i+1} \neq 7$.

Proof

Let $p_{i-1} = 8$ with i even and suppose that $p_{i+1} = 7$. Then

$$89/79 = (1,7,1,9) < F_i < (1,7,1,5) = 53/47,$$

$$10/89 = (0,8,1,9) < S_i < (0,8,1,5) = 6/53,$$

and so $K_i > 1.2389$. Using this in (11.1) and (11.5) it follows that $a_{i+1} < 7.91$. Hence

$$14.08 < a_{i+1}(1 + F_i)(1 - S_i) < 14.94,$$

and so in order that the value $(x + \lambda - \mu)^2 - a_{i+1}(1 + F_i)(1 - S_i)$ shall not contradict either $m_+ = 1$ or $m_- > 3.26$ we must have $\|\lambda - \mu\| < .117$. Thus $\|\lambda\| < .165$ and so in order that the value $\|\lambda\|^2 + a_{i+1}F_iS_i$ shall not contradict $m_+ = 1$ we must have $a_{i+1} > 7.61$. Using this new bound in the above analysis yields that

$a_{i+1}(1 + F_i)(1 - S_i) > 14.35$, $\|\lambda - \mu\| < .083$, $\|\lambda\| < .131$

and $a_{i+1} > 7.69$. Hence

$$16 \leqslant (x + 2\lambda + 3\mu)^2 < (4 + .262 + .144)^2$$
$$< 19.42$$

for suitable x and

$$18.46 < a_{i+1}(3 - 2F_i)(3 + 2S_i) < 19.06.$$

This implies that g_i takes a value contradicting either $m_+ = 1$ or $m_- > 3.26$.

<u>Lemma 11.17</u>

If $p_{i-1} = 8$ with i even then $p_{i+1} \neq 6$.

<u>Proof</u>

Let $p_{i-1} = 8$ with i even and suppose that $p_{i+1} = 6$. Then

$$79/69 = (1,6,1,9) < F_i < (1,6,1,5) = 47/41,$$
$$10/89 = (0,8,1,9) < S_i < (0,8,1,5) = 6/53,$$

and so $K_i > 1.257$. Using this in (11.1) and (11.5) it follows that $a_{i+1} < 7.795$, while steps (11.40) yield that $m_- > 3.39$. Following the method of proof of the previous lemma we have

$14.2 < a_{i+1}(1 + F_i)(1 - S_i) < 14.84$, $\|\lambda - \mu\| < .102$,

and so $\|2\lambda - \mu\| < .254$. Hence we can choose x such that $16 \leqslant (x + 2\lambda - \mu)^2 < 18.1$. However as

$$19.0 < a_{i+1}(1 + 2F_i)(1 - 2S_i) < 19.9$$

it can be seen that g_i takes a value contradicting $m_- > 3.39$ unless $\|2\lambda - \mu\| < .064$ and $a_{i+1} > 7.59$. These two inequalities imply that

$$16 \leqslant (x + 2\lambda + 3\mu)^2 < 18.2$$

for suitable x and that

$$17.3 < a_{i+1}(3 - 2F_i)(3 + 2S_i) < 17.9.$$

Hence g_i takes a value in the open interval $(-1.9, .9)$, contradicting either $m_+ = 1$ or $m_- > 3.39$.

Lemma 11.18

If $p_{i-1} = 8$ with i even then $p_{i+1} \neq 5$.

Proof

Let $p_{i-1} = 8$ with i even and suppose that $p_{i+1} = 5$. Then

$$69/59 = (1,5,1,9) < F_i < (1,5,1,5) = 41/35,$$

$$10/89 = (0,8,1,9) < S_i < (0,8,1,5) = 6/53,$$

and so $K_i > 1.2818$. Using this in (11.1) and (11.5) it follows that $a_{i+1} < 7.644$, while steps (11.40) yield that $m_- > 3.44$. Using the same method as in the proof of lemma 11.16 we have

$14.37 < a_{i+1}(1 + F_i)(1 - S_i) < 14.74$, $\|\lambda - \mu\| < .08$, and so $\|2\lambda + 3\mu\| < .4$. Hence there exists x such that $12.96 < (x + 2\lambda + 3\mu)^2 \leqslant 16$. However

$$15.82 < a_{i+1}(3 - 2F_i)(3 + 2S_i) < 16.33$$

and so g_i takes a value contradicting either $m_+ = 1$

or $m_- > 3.44$.

From the above work it is clear that if $p_i = 8$ for any odd i then $p_j = 8$ for all odd j and $g \sim F_8\sqrt[3]{24}$. Hence we may assume, for the rest of the proof, that $p_j \leqslant 7$ for all odd j.

Lemma 11.19

If $p_{i-1} = 7$ with i even then $p_{i+1} = 7$.

Proof

Let $p_{i-1} = 7$ with i even and suppose that $p_{i+1} \neq 7$. Then p_{i+1} is either 5 or 6. Hence

$$71/62 = (1,6,1,8) < F_i < (1,5,1,5) = 41/35,$$

$$9/71 = (0,7,1,8) < S_i < (0,7,1,5) = 6/47,$$

and so $K_i > 1.2719$. Using this in (11.1) and (11.5) it follows that $a_{i+1} < 7.704$, while steps (11.40) yield that $m_- > 3.42$. Using the same method as in the proof of lemma 11.17 we have that

$$13.97 < a_{i+1}(1 + F_i)(1 - S_i) < 14.61,$$

$$\|\lambda - \mu\| < .131,$$

$$\|2\lambda - \mu\| < .310,$$

$$16 \leqslant (x + 2\lambda - \mu)^2 < 18.6 \quad \text{for suitable } x,$$

$$18.3 < a_{i+1}(1 + 2F_i)(1 - 2S_i) < 19.25,$$

and so g_i takes a value contradicting either $m_+ = 1$ or $m_- > 3.42$.

Lemma 11.20

$p_j = 6$ for all odd j.

Proof

Let $p_{i-1} = 7$ with i even. Then by applying the above lemma indefinitely to both the original and the reverse chain we find that $p_j = 7$ for all odd j, and so

$$g_i = (x + \lambda_i y + \mu_i z)^2 - a_{i+1}(z^2 - yz - \tfrac{1}{7}y^2).$$

As $g_i \sim g$ we may drop the suffixes without any loss of generality. As $d = 11a^2/28$ and $a > 7.47$, equations (11.2) and (11.5) imply that $a < 7.82$ and $m_- > 3.39$. Following the method of proof of the above lemma we have that

$$13.87 < 13a/7 < 14.53,$$
$$\|\lambda - \mu\| < .144,$$
$$\|2\lambda - \mu\| < .336,$$
$$16 \leqslant (x + 2\lambda - \mu)^2 < 18.81 \quad \text{for suitable } x,$$
$$18.1 < 17a/7 < 19.0,$$

and so g takes a value contradicting either $m_+ = 1$ or $m_- > 3.39$. Hence $p_j \leqslant 6$ for all odd j. However if $p_i \leqslant 5$ for some odd i then

$$K_{i-1} > (1,5,1,7) + (0,6,1,7)$$
$$> 1.308$$

which contradicts (11.30). Hence $p_j = 6$ for all odd j.

Lemma 11.21

$p_i \neq 6$ for some odd i.

Proof

Let $p_j = 6$ for all odd j. Then for i even we have that

$$g_i = (x + \lambda_i y + \mu_i z)^2 - a_{i+1}(z^2 - yz - \tfrac{1}{6}y^2).$$

As $g_i \sim g$ we may drop the suffixes without any loss of generality. As $d = 5a^2/12$ and $a > 7.47$, equations (11.2) and (11.5) imply that $a < 7.59$ and $m_- > 3.45$. Following the method of proof of lemma 11.19 we have that

$$13.69 < 11a/6 < 13.92,$$

$$\|\lambda - \mu\| < .168,$$

$$\|2\lambda - \mu\| < .384,$$

$$16 \leqslant (x + 2\lambda - \mu)^2 < 19.22 \quad \text{for suitable } x,$$

$$17.43 < 7a/3 < 17.71,$$

and so g takes a value contradicting either $m_+ = 1$ or $m_- > 3.45$ unless $\|2\lambda - \mu\| > .293$. Combining this with (11.37) we find that $.245 < \|2\lambda - 2\mu\| < .336$ and so there exists x such that

$$52.49 < (x + 2\lambda - 2\mu)^2 < 53.82.$$

However as

$$54.76 < 22a/3 < 55.68$$

this implies that g takes a value contradicting $m_- > 3.45$.

The result of theorem C_7 now follows as we have shown that if $0 < d \leqslant 24$ and if $m_-(g) \geqslant \sqrt[3]{16d/9}$ then g is equivalent to a multiple of either F_7 or F_8.

CHAPTER 12

The Proof of Theorem C_8.

For reference theorem C_8 is restated.

Theorem C_8

If g is any indefinite ternary quadratic form of signature 1, with $d(g) = d$ where

$$0 < d \leqslant 67.5,$$

and if $m_+(g) = m_+ = 1$ is attained by g then either

(a) $m_-(g) < \sqrt[3]{8d/3}$, or

(b) g is equivalent to a multiple of either F_8 or F_9.

Proof

As usual we consider in place of g an equivalence chain (g_i) of forms equivalent to g. Assuming that $m_-(g) \geqslant \sqrt[3]{8d/3}$ and using the same notation as in the previous chapters we have that

$$a_{i+1}K_i = \Delta; \qquad \Delta^2 = 4d, \tag{12.1}$$

$$m_-(g) = m_- \geqslant \sqrt[3]{8d/3}, \tag{12.2}$$

$$a_i \geqslant m_- + 1/4 \geqslant \sqrt[3]{8d/3} + 1/4 \quad (i \text{ odd}), \tag{12.3}$$

$$a_i \geqslant 3/4 \quad (i \text{ even}), \tag{12.4}$$

$$d = 135\beta/2, \quad 0 < \beta \leqslant 1, \tag{12.5}$$

and

$$K_i = 3\sqrt{30\beta}/a_{i+1}. \tag{12.6}$$

Now if $d \leqslant 24$ then using theorem C_7 we have that $m_- \leqslant \sqrt[3]{16d/9}$ unless g is equivalent to a multiple of F_8.

Thus we may assume from now on that $d > 24$ and try to show that g is equivalent to a multiple of F_9.
Under this assumption we have that

$$m_- \geqslant \sqrt[3]{8d/3} > 4.$$

Applying theorem 2.1 to the sections

$$(x + \mu_i z)^2 - a_{i+1}z^2 \qquad (12.7$$

of g_i (where i is even) we find that $a_{i+1} \geqslant 10.25$ for all even i, and so $q_i(y,z)$ can take no values in the open interval $(-10.25,.75)$. By the result of Segre it follows that

$$d = d(q_i) \geqslant \{(10.25)^2 + 3(10.25)\}/4,$$

and using this in (12.2) we find that $m_- > 4.49$.
We now use this and apply the corollary to theorem 2.1 to the sections (12.7) of g_i to show that $a_{i+1} > 10.74$ for all even i. Repetition of the above process yields, after a few iterations, that $d > 37.87$, $m_- > 4.65$ and

$$a_{i+1} > 10.9 \quad (i \text{ even}). \qquad (12.8$$

For the present we shall assume that

$$8d/3 \leqslant 125. \qquad (12.9$$

Using this in (12.5) we obtain a bound on β which in conjunction with (12.4),(12.6) and (12.8) yields that

$$K_i < 18.267 \quad (i \text{ odd}) \qquad (12.10$$

and

$$K_i < 1.257 \quad (i \text{ even}). \qquad (12.11$$

From these we have, as $K_i > p_i$, that $p_i = 1$ for even i and $p_i \leqslant 18$ for i odd.

Now if $p_k \leqslant 3$ for some odd k then

$$K_{k-1} > (1,4) + (0,19) > 1.3$$

which contradicts (12.11). Hence $p_i \geqslant 4$ for all odd i. Using this we can improve our upper bound on p_i to $p_i \leqslant 16$ for all odd i, for if $p_k \geqslant 17$ for some odd k then

$$K_k > (17,1,4) + (0,1,4) = 18.6$$

which contradicts (12.10). This enables us to show that $p_i \geqslant 5$ for all odd i, for if $p_k = 4$ for some odd k then

$$K_{k-1} > (1,5) + (0,17) > 1.258$$

which contradicts (12.11). Thus we have shown that

$$p_i = 1 \quad (i \text{ even}); \qquad 5 \leqslant p_i \leqslant 16 \quad (i \text{ odd}). \tag{12.12}$$

Before commencing to eliminate various $[p_i]$ chains we shall first obtain upper bounds on a_{i+1} and $\|\mu_i - \frac{1}{2}\|$ for even i. From the bounds (12.12) we have that

$$K_i > (1,16,1,17) + (0,16,1,17) = 341/305$$

for i even, and so using (12.1) and (12.9) we find that $a_{i+1} < 12.26$ for i even. For a fixed even i consider the values $(3 + \|\mu\|)^2 - a_{i+1}$ and $(3 - \|\mu\|)^2 - a_{i+1}$ of g_i. In the first of these values we have that $10.9 < a_{i+1} < 12.26$ and $9 \leq (3 + \|\mu\|)^2 \leq 12.25$. Hence in order not to contradict either $m_+ = 1$ or $m_- > 4.65$ we must have

$$(3 + \|\mu\|)^2 - a_{i+1} \geq 1, \tag{12.13}$$

and so $a_{i+1} \leq 11.25$. Considering the second of the two values, as $(3 - \|\mu\|)^2 \leq 9$, it is clear that

$$(3 - \|\mu\|)^2 - a_{i+1} \leq -m_-.$$

Subtracting this from (12.13) we find that $12\|\mu\| > 1 + m_-$, and so

$$\|\mu\| > (1 + m_-)/12. \tag{12.14}$$

As $m_- > 4.65$ this implies that $\|\mu\| > .47$, i.e.

$$\|\mu - \tfrac{1}{2}\| < .03. \tag{12.15}$$

From the corollary to theorem 2.1, as $m_- > 4$, we have that

$$a_{i+1} \geq 10.25 + (m_- - 4) \quad (i \text{ even}). \tag{12.16}$$

Combining this with (12.1) and (12.2) yields that

$$\psi_i(m_-) = m_-^3 - 2K_i^2(6.25 + m_-)^2/3 \geq 0 \quad (i \text{ even}). \tag{12.17}$$

Now from (12.16), as $a_{i+1} \leq 11.25$, it is clear that $m_- \leq 5$ and so the inequality (12.17) must be satisfied for some $m_- \leq 5$. However using the known bounds on m_- and K_i it can easily be shown that the derivative

$$\psi_i'(m_-) = 3m_-^2 - 4K_i^2(6.25 + m_-)/3 > 0,$$

and so (12.17) must be satisfied with $m_- = 5$. Hence

$$K_i \leq \sqrt{40/27} < 1.2172 \quad (i \text{ even}). \tag{12.18}$$

Furthermore as $\psi_i'(m_-) > 0$ it is clear that if

$\psi_i(x) < 0$ for all allowable values of K_i then $m_- > x$.

The bound (12.18) allows us to improve the bounds (12.12) on p_i for i odd to $6 \leqslant p_i \leqslant 16$, for if $p_k = 5$ for some odd k then

$$K_{k-1} > (1,6) + (0,17) > 1.22$$

which contradicts (12.18).

The proof is now continued as a series of lemmas eliminating all possibilities for the chain $[p_i]$.

Lemma 12.1

$p_j \leqslant 14$ for all odd j.

Proof

Let $15 \leqslant p_{i-1} \leqslant 16$ for some even i. Then

$$18/17 = (1,17) < F_i < (1,6,1,6) = 55/48,$$

$$1/17 = (0,17) < S_i < (0,15,1,6) = 7/111,$$

and so $a_{i+1}F_iS_i < .813$. Hence in order that the value $\|\lambda\|^2 + a_{i+1}F_iS_i$ shall not contradict $m_+ = 1$ we must have $\|\lambda - \frac{1}{2}\| < .07$. Combining this with (12.15) we find that $\|2\lambda - \mu - \frac{1}{2}\| < .17$, and so there exists x such that

$$28.4 < (x + 2\lambda - \mu)^2 \leqslant 30.25.$$

However

$$29.6 < a_{i+1}(1 + 2F_i)(1 - 2S_i) < 32.68,$$

and so g_i takes a value contradicting either $m_+ = 1$ or $m_- > 4.65$.

Lemma 12.2

$p_j \leqslant 9$ for all odd j.

Proof

Let $10 \leqslant p_{i-1} \leqslant 14$ for some even i. Then

$$255/239 = (1,14,1,15) < F_i < (1,6,1,6,1,6) = 433/378,$$

$$16/239 = (0,14,1,15) < S_i < (0,10,1,6) = 7/76,$$

and so

$$20.4 < a_{i+1}(1 + F_i)(1 - S_i) < 22.53$$

and

$$19.25 < a_{i+1}(2 - F_i)(2 + S_i) < 21.97.$$

Hence in order that the values

$$(4 + \|\lambda - \mu\|)^2 - a_{i+1}(1 + F_i)(1 - S_i)$$

and

$$(4 + \|\lambda + 2\mu\|)^2 - a_{i+1}(2 - F_i)(2 + S_i)$$

shall not contradict either $m_+ = 1$ or $m_- > 4.65$ we must have $\|\lambda - \mu\| < .23$ and $\|\lambda + 2\mu\| < .17$. However by subtraction these yield that $\|3\mu\| < .4$ which is in contradiction with thrice (12.15).

Lemma 12.3

$p_j > 9$ for at least one odd j.

Proof

Let $p_j \leqslant 9$ for all odd j. Then for any even i we have that

$$120/109 = (1,9,1,10) < F_i < (1,6,1,6) = 55/48,$$

$$11/109 = (0,9,1,10) < S_i < (0,6,1,6) = 7/48,$$

and so $K_i > 1.2018$. Hence as $\psi_i(4.93) < 0$ for

$K_i > 1.2018$ we must have $m_- > 4.93$ and $a_{i+1} > 11.18$.

Consider the values

$$(4 + \|\lambda - \mu\|)^2 - a_{i+1}(1 + F_i)(1 - S_i)$$

and

$$(4 + \|\lambda + 2\mu\|)^2 - a_{i+1}(2 - F_i)(2 + S_i).$$

As

$$20.0 < a_{i+1}(1 + F_i)(1 - S_i) < 21.8$$

and

$$20.0 < a_{i+1}(2 - F_i)(2 + S_i) < 21.8$$

it follows that g_i takes a value contradicting either $m_+ = 1$ or $m_- > 4.93$ unless $\|\lambda - \mu\| < .12$ and $\|\lambda + 2\mu\| < .12$. However subtracting these yields that $\|3\mu\| < .24$ which is in contradiction with thrice (12.15).

From the contradiction of lemmas 12.2 and 12.3 it is clear that the assumption (12.9) is false. Hence

$$8d/3 > 125 \qquad (12.19$$

and so, using (12.2), we have that $m_- > 5$.

By an obvious modification of the corollary to lemma 2.1 applied to the sections $(x + \mu_i z)^2 - a_{i+1}z^2$ of g_i it follows that

$$a_{i+1} \geqslant 14 + (m_- - 5) > 14 \qquad (12.20$$

for all even i. Hence the binary form $q_i(y,z)$ can take no values in the open interval $(-14, .75)$, and so by the result of Segre it follows that

$$d = d(q_i) \geqslant (196 + 42)/4 = 59.5.$$

We now use this in place of (12.19) to obtain new bounds on m_- and a_{i+1}. Repeating this iterative process a number of times yields that $m_- > 5.538$ and that $a_{i+1} > 14.538$ for all even i. Combining this with (12.4) and (12.6) we find that

$$K_i < 21.911 \quad (i \text{ odd}); \quad K_i < 1.1304 \quad (i \text{ even}). \quad (12.21$$

We may obtain a tighter bound on K_i for i even as follows. As

$$a_{i+1} \geqslant 9 + m_- \geqslant 9 + \sqrt[3]{2\Delta^2/3}$$

we have that

$$K_i \leqslant \Delta/[9 + \sqrt[3]{2\Delta^2/3}].$$

Now the RHS of this inequality has positive derivative with respect to Δ (over the allowable range) and so as $\Delta \leqslant \sqrt{270}$ we have that

$$K_i \leqslant \sqrt{270}/(9 + \sqrt[3]{180}) < 1.1222 \quad (i \text{ even}). \quad (12.22$$

From (12.21) and (12.22) we have immediately that $p_i = 1$ for all even i and that $p_i \leqslant 21$ for all odd i. Now suppose that $p_k = 21$ for some odd k. Then

$$K_k > (21,2) + (0,2) = 22$$

which contradicts (12.21). Hence $p_i \leqslant 20$ for all odd i. Suppose that $p_k \leqslant 12$ for some odd k. Then

$$K_{k-1} > (1,13) + (0,21) > 1.124$$

which contradicts (12.22). Hence $p_i \geqslant 13$ for all odd i.

We now find upper bounds on a_{i+1} and $\|\mu_i\|$ for even i. As

$$K_i > (1,20,1,21) + (0,20,1,21) = 505/461,$$

using (12.6) yields that $a_{i+1} < 15.01$. Now consider the values

$$(4 - \|\mu\|)^2 - a_{i+1}$$

and

$$(3 + \|\mu\|)^2 - a_{i+1}$$

of g_i. Clearly we must have

$$(4 - \|\mu\|)^2 - a_{i+1} \geqslant 1$$

and

$$(3 + \|\mu\|)^2 - a_{i+1} < -5.538$$

in order not to contradict either $m_+ = 1$ or $m_- > 5.538$. From the first of these it follows that $a_{i+1} \leqslant 15$, while subtracting the first from the second yields that $\|\mu\| < .033$. Hence

$$\|\mu_i\| < .033 \quad (\text{all even } i). \qquad (12.23$$

We are now in a position to work on the $[p_i]$ chain, eliminating all possibilities except that which gives g as equivalent to a multiple of F_9.

<u>Lemma 12.4</u>

If $p_j = 20$ for all odd j then $g \sim F_9\sqrt[3]{135/2}$.

<u>Proof</u>

Let $p_j = 20$ for all odd j. Then for any even

i we have that

$$g_i = (x + \lambda y + \mu z)^2 - a_{i+1}(z^2 - yz - \tfrac{1}{20}y^2).$$

As $\|\lambda\|^2 + a_{i+1}/20 \geq 1$ in order not to contradict $m_+ = 1$ it is clear that we must have $a_{i+1} = 15$ and $\|\lambda\| = \frac{1}{2}$. Similar treatment of the value $\|\lambda + \mu\|^2 + a_{i+1}/20$ yields that $\|\lambda + \mu\| = \frac{1}{2}$ and so $\|\mu\| = 0$. Hence

$$g \sim g_i \sim (x + \tfrac{1}{2}y)^2 - 15(z^2 - yz - \tfrac{1}{20}y^2)$$

$$= F_9\sqrt[3]{135/2}$$

as required.

In order to eliminate the other possibilities for the chain $[p_i]$ we shall suppose from now on that $p_i < 20$ for at least one odd i.

<u>Lemma 12.5</u>

If $p_{i-1} = 20$ with i even then $p_{i+1} \neq 19$.

<u>Proof</u>

Let $p_{i-1} = 20$ and $p_{i+1} = 19$ where i is even. Then

$$461/439 = (1,19,1,21) < F_i < (1,19,1,13) = 293/279,$$

$$22/461 = (0,20,1,21) < S_i < (0,20,1,13) = 14/293,$$

and so $a_{i+1}F_iS_i < .757$. Hence in order that the value $\|\lambda\|^2 + a_{i+1}F_iS_i$ shall not contradict $m_+ = 1$ we must have $\|\lambda - \frac{1}{2}\| < .003$ and $a_{i+1} > 14.94$. This implies that $\|\mu\| < .008$ as otherwise the value $(4 - \|\mu\|)^2 - a_{i+1}$ will contradict either $m_+ = 1$ or $m_- > 5.538$.

Using the above bounds on $\|\lambda - \frac{1}{2}\|$ and $\|\mu\|$ we find that $\|8\lambda + 9\mu\| < .096$, and so there exists x such that $81 \leqslant (x + 8\lambda + 9\mu)^2 < 82.8$. However

$$83.3 < a_{i+1}(9 - 8F_i)(9 + 8S_i) < 84.4,$$

and so g_i takes a value contradicting $m_- > 5.538$.

Lemma 12.6

$p_j \leqslant 19$ for all odd j.

Proof

Let $p_k = 20$ for some odd k. Then by the above lemma there must occur, either in the original or the reverse chain, an even i such that $p_{i-1} = 20$ and $13 \leqslant p_{i+1} \leqslant 18$. Then

$$441/419 = (1,18,1,21) < F_i < (1,13,1,13) = 209/195,$$

$$22/461 = (0,20,1,21) < S_i < (0,20,1,13) = 14/293,$$

and so $a_{i+1}F_iS_i < .7685$. Hence in order that the value $\|\lambda\|^2 + a_{i+1}F_iS_i$ shall not contradict $m_+ = 1$ we must have $\|\lambda - \frac{1}{2}\| < .019$ and $a_{i+1} > 14.638$.

Now combining the above bound on $\|\lambda - \frac{1}{2}\|$ with (12.23) we find that $\|2\lambda + 3\mu\| < .137$, and so there exists x such that $36 \leqslant (x + 2\lambda + 3\mu)^2 < 38$.

However

$$38 < a_{i+1}(3 - 2F_i)(3 + 2S_i) < 41.534$$

and so g_i takes a value contradicting $m_- > 5.538$.

Lemma 12.7

$p_j \leqslant 18$ for all odd j.

Proof

Let $p_{i-1} = 19$ with i even. Then

$$440/419 = (1,19,1,20) < F_i < (1,13,1,13) = 209/195,$$

$$21/419 = (0,19,1,20) < S_i < (0,19,1,13) = 14/279,$$

and so $K_i > 1.1002$. Using this in (12.6) we find that $a_{i+1} < 14.936$. Hence $a_{i+1}F_iS_i < .8033$, and so in order that the value $\|\lambda\|^2 + a_{i+1}F_iS_i$ shall not contradict $m_+ = 1$ we must have $\|\lambda - \frac{1}{2}\| < .06$. Combining this with (12.23) we find that $\|2\lambda + 3\mu\| < .22$, and so there exists x such that

$$36 \leqslant (x + 2\lambda + 3\mu)^2 < 38.7.$$

However

$$38.5 < a_{i+1}(3 - 2F_i)(3 + 2S_i) < 41.666,$$

and so g_i takes a value contradicting either $m_+ = 1$ or $m_- > 5.538$ unless $a_{i+1} > 14.8$, $3 - 2F_i > .897$ and $\|2\lambda + 3\mu\| < .011$. Combining this with (12.23) we find that $\|8\lambda + 9\mu\| < .143$, and so there exist x_1 and x_2 such that $81 \leqslant (x_1 + 8\lambda + 9\mu)^2 < 83.6$ and $78.4 < (x_2 + 8\lambda + 9\mu)^2 \leqslant 81$. However

$$81 < a_{i+1}(9 - 8F_i)(9 + 8S_i) < 84.2$$

and so g_i takes a value contradicting either $m_+ = 1$ or $m_- > 5.538$.

Lemma 12.8

$p_j \geqslant 14$ for all odd j.

Proof

Let $p_{i-1} = 13$ with i even. Then

$$K_i > (1,18,1,19) + (0,13,1,19) > 1.1222,$$

which contradicts (12.22).

Lemma 12.9

$p_j \leqslant 17$ for all odd j.

Proof

Let $p_{i-1} = 18$ with i even. Then

$$399/379 = (1,18,1,19) < F_i < (1,14,1,14) = 239/224,$$

$$20/379 = (0,18,1,19) < S_i < (0,18,1,14) = 15/284,$$

and so $K_i > 1.1055$. Using this in (12.6) we find that $a_{i+1} < 14.862$. Hence $a_{i+1}F_iS_i < .838$, and so in order that the value $\|\lambda\|^2 + a_{i+1}F_iS_i$ shall not contradict $m_+ = 1$ we must have $\|\lambda - \frac{1}{2}\| < .1$. Combining this with (12.23) we find that $\|2\lambda + 3\mu\| < .3$, and so there exists x such that

$$36 \leqslant (x + 2\lambda + 3\mu)^2 < 39.7.$$

However

$$39 < a_{i+1}(3 - 2F_i)(3 + 2S_i) < 41.3$$

and so g_i takes a value contradicting either $m_+ = 1$ or $m_- > 5.538$.

Lemma 12.10

$p_j \geq 15$ for all odd j.

Proof

Let $p_{i-1} = 14$ with i even. Then

$$K_i > (1,17,1,18) + (0,14,1,18) > 1.1222$$

which contradicts (12.22).

Lemma 12.11

$p_j \leq 16$ for all odd j.

Proof

Let $p_{i-1} = 17$ with i even. Then

$$360/341 = (1,17,1,18) < F_i < (1,15,1,15) = 271/255,$$

$$19/341 = (0,17,1,18) < S_i < (0,17,1,15) = 16/287,$$

and so $K_i > 1.1114$. Using this in (12.6) we find that $a_{i+1} < 14.785$. Hence $a_{i+1}F_iS_i < .876$, and so in order that the value $\|\lambda\|^2 + a_{i+1}F_iS_i$ shall not contradict $m_+ = 1$ we must have $\|\lambda - \frac{1}{2}\| < .148$. Combining this with (12.23) we find that $\|2\lambda - \mu\| < .33$ and so there exists x such that $36 \leq (x + 2\lambda - \mu)^2 < 40.1$. However $40.1 < a_{i+1}(1 + 2F_i)(1 - 2S_i) < 41.1$, and so g_i takes a value contradicting $m_- > 5.538$.

Lemma 12.12

$p_j \leqslant 15$ for all odd j.

Proof

Let $p_{i-1} = 16$ with i even. Then

$$323/305 = (1,16,1,17) < F_i < (1,15,1,15) = 271/255,$$

$$18/305 = (0,16,1,17) < S_i < (0,16,1,15) = 16/271,$$

and so $K_i > 1.118$. Using this in (12.6) we find that $a_{i+1} < 14.7$. Hence

$$39.97 < a_{i+1}(1 + 2F_i)(1 - 2S_i) < 40.6.$$

Now by choosing x_1 such that $36 \leqslant (x_1 + 2\lambda - \mu)^2 \leqslant 42.25$ we obtain a value of g_i contradicting either $m_+ = 1$ or $m_- > 5.538$ unless $\|2\lambda - \mu - \frac{1}{2}\| < .1$. Hence $\|2\lambda - \mu\| > .4$, and combining this with (12.23) we find that $\|2\lambda\| > .367$. Thus $\|\lambda - \frac{1}{2}\| > .183$, and so $\|\lambda - \mu - \frac{1}{2}\| > .15$. Hence there exists x_2 such that $25 \leqslant (x_2 + \lambda - \mu)^2 < 29$. However

$$28.1 < a_{i+1}(1 + F_i)(1 - S_i) < 28.6,$$

and so g_i takes a value contradicting either $m_+ = 1$ or $m_- > 5.538$.

Lemma 12.13

$p_j > 15$ for at least one odd j.

Proof

Let $p_j \leqslant 15$ for all odd j. Then for all even i we have that

$$K_t > (1,15,1,16) + (0,15,1,16) > 1.1222$$

which contradicts (12.22).

From the contradiction inherent in lemmas 12.12 and 12.13 it is clear that we have eliminated all possible $[p_t]$ chains which have $p_j \leqslant 19$ for at least one odd j. Thus the only possible $[p_t]$ chain is that giving g as equivalent to a multiple of F_9. This completes the proof of Theorem C_8.

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