DIFFERENTIAL GAMES WITH NO INFORMATION

by

David John Wilson
B.Sc.(Hons.), Adelaide University

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CONTENTS

SUMMARY

This thesis contains the account of an extensive examination of differential games in which the players are granted no information concerning the evolution of the game. Mostly, games considered herein are two person zero-sum deterministic differential games of fixed duration, but many of the methods developed for dealing with these games appear to be applicable to a much wider class of differential games.

Chapter 1 contains a general introduction, a brief subjective account of the historical development of the subject, and a list of notations and conventions used throughout the thesis.

The main theme is taken up in Chapter 2, with the definition of the games to be considered. The nature of, and the relation between, the pure strategies of differential games with perfect information, and those of imperfect information is then discussed, and the difficulties of defining strategies for games of perfect information is illustrated. The chapter concludes by showing that if a differential game with no information has a saddle point in pure strategies, then these strategies will constitute a saddle point of <u>any</u> differential game with the same dynamics, whether of perfect or imperfect information.

Games with open-loop saddle points are discussed in Chapters 3 and 4. In Chapter 3 is given a fairly general necessary condition for a pair of open-loop strategies to constitute a saddle point. Certain conditions, first stated (but not proved) by Fichefet, for the existence of these saddle points are proved in this chapter, and some other existence conditions of Fichefet are given much simpler proofs. The chapter concludes with the description of a procedure for deciding whether a differential game of a certain class has an open-loop saddle point.

By treating the payoffs as quadratic functions on a Hilbert space, Chapter 4 gives some necessary and sufficient conditions for an open loop strategy pair to be a saddle point of a linear-quadratic differential game. In the light of these results, and those of previous chapters, the traditional treatment of quadratic games with noise corrupted measurements is criticised.

The question of mixed strategies is taken up in Chapter 5. It is shown that any conceivable mixed strategy for a differential game with no information can be represented in an especially simple way. The existence of a value for a wide class of differential games, and of mixed strategy saddle points for a more restricted class is then established, and a general necessary condition for a mixed strategy pair to constitute a saddle point is given.

The thesis concludes with a discussion of the limitations and possible extensions of the results which it contains.

PREFACE

I hereby declare that this thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of my knowledge and belief, it contains no material previously published by another person, except where due reference is made in the text of the thesis.

D.J. Wilson

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CHAPTER 1

INTRODUCTION

1. General Introduction

One of the major limitations of the theory of differential games in its present state is its failure to deal adequately with situations of imperfect information. Until quite recently, almost no theoretical work had been carried out in this field, and even at present (July 1971) the few papers which have been published still leave the major portion of any comprehensive theory yet to be discovered.

Current activity seems to be concentrated on three classes of differential games in each of which the players are granted a different kind of information, which may consist of any one of the following:

i) Noise corrupted state measurements

ii) Delayed state measurements

iii) No information at all (except, of course, the 'rules' of the game).

Investigation of differential games with an information structure of the last type constitutes the major part of the research embodied in this thesis. The objectives of this reaearch were to describe the nature of strategies, both pure and mixed, which may be used in such games, to find conditions under which they have solutions, and to provide methods of determining these solutions when they do exist.

2. Historical Survey

In Chapter 12 of his book, 'Differential Games' [1], Rufus Isaacs outlines the few facts then known about differential games with imperfect information. Most of the games of a dynamic character which had been analysed at that time were discrete, multistage games with an information time lag. Shapley's stochastic games [2] and Everett's recursive games [3] can both be considered as games in which the players move alternately, and receive information with a time lag of one move. The 'Bomber and Battleship' game described by Isaacs in [1] (p. 355) had been solved by Isaacs and Karlin [4 - 7] in 1954, and independently by Dubins [8]. The only published works on continuous time games with partial information which were in existence in 1965 appear to be Grenander's papers [9 ,10] of 1963 and 1965 respectively, dealing with pursuit games of a stationary character, and a 1964 paper of Oguztoreli which investigates minimax solutions of pursuit games with delayed information [11].

Since 1965 attention has been confined almost solely to the determination of minimax (or maximin) pure strategies, or to games which have pure strategy saddle points.

Differential games with an information time lag were considered by Friedman [12] in 1969 and Ciletti [13,14] in 1970. Friedman showed that the upper (or minimax) value and the lower (or maximin) value of a game with information time lag both converged to the value of the corresponding perfect information game as the time lag approached zero. Ciletti obtained the lower value for a linear-quadratic differential game, and the corresponding minimax and maximising strategies of the players. Again it was observed that as the time lag vanished, the lower value approached the value of the corresponding game with perfect information.

Papers on games with noise corrupted measurements have recently appeared in profusion. With few exceptions, the games treated have been linear-quadratic differential or multistage games with measurements corrupted by additive white noise. The first paper on the subject [15] by Rhodes and Luenberger appeared in 1969, and treated five of the six possible information patterns which may be obtained by allowing the players to have either no information, noise corrupted state measurements or perfect information. The remaining case, in which both players have noise corrupted measurements was later treated in a limited way by the same authors [16]. Willman [17] also examined this case by considering it as the limit of a sequence of multistage games. These and several succeeding papers on the same topic have all claimed to obtain pure strategy solutions of the games which they consider. However, in the case of linear-quadratic differential games with additive white noise corrupting the measurements, the validity of these "solutions" is questioned by a critique in Chapter 4 of this thesis.

Games with no information have as pure strategies, the so-called "open-loop" strategies. Conditions for the existence of solutions to a differential game in open-loop strategies were first examined by Fichefet [18,19] in 1968, but although the main theorems of his first paper are correct, several serious mathematical errors in its early pages invalidate the proofs of these results. Rhodes and Luenberger [15] in 1969, Schmitendorf [20-22], Auslender [23] and Rekasius [24] (1969-70), all gave open-loop saddle point conditions for linear-quadratic differential games. In his paper, Rekasius also obtains conditions for pure strategy saddle points in games where one player has no information and the other has perfect information.
In 1969, Fichefet [25] became the first to obtain mixed strategy solutions of differential games with no information by applying important results of Fleming [26] discovered fifteen years earlier. Fichefet's results, and my own solution of Isaacs' "Princess and Monster" game on the

circle [27] are, as far as I am aware, the only mixed strategy solutions of differential games with no information which have been obtained to date.

3. Notation and Conventions

The Chapters of this thesis are subdivided into sections. Within any section theorems, lemmas, corollaries and examples are numbered consecutively from the beginning of the section, so that a reference to any of these made within the same chapter can be given by specifying two numbers - the section number of the reference and the order of occurrence of the reference within that section. Inter-chapter references are made by prefixing the intra-chapter reference number with the number of the chapter in which the reference occurs. The end of a theorem, lemma, corollary or example is indicated by the sharp symbol, "#", and equations occurring are numbered consecutively from the beginning. Equations occurring elsewhere in the text are numbered in the same way as, but independently of, the theorems etc. All equation reference numbers are parenthesised.

We now give a list of the standard notations which we use and of some symbols and concepts defined within the text.

Set Notation: We use the standard symbols $\cup$, $\cap$, $\subset$, $\in$, $\{x_1, x_2, \ldots, x_n\}$, A-B to denote union, intersection, inclusion (not 'strict'), membership, the set consisting of the elements $x_1, x_2, \ldots x_n$ and the complement of the set B in the set A respectively.

The symbol R always denotes the set of real numbers, and $[-\infty,\infty]$ denotes the set of extended real numbers ($-\infty,\infty$ are 'ideal' numbers with the properties of being less and greater respectively than every real number). The supremum and infimum of a subset A of $[-\infty,\infty]$ are denoted respectively by sup A and inf A.

Functions and Topology: We use the notations $f:A\to B$, $f(W)$ and $\psi_0\varphi$ respectively to denote a function mapping the set A into the set B, the image of the set W under the function f and the composition of the functions ψ and φ. If $f:A\times B\to C$ then for $b \in B$, $f(\cdot,b)$ denotes the function from A to C defined by

$$[f(\cdot,b)](a) = f(a,b) \quad \text{for } a \in A.$$

The supremum and infimum of a function f over a set A are denoted by $\sup_{a\in A} f(a)$ and $\inf_{a\in A} f(a)$ respectively. If the function f attains its extrema on the set A then we shall write $\max_{a\in A} f(a)$ and $\min_{a\in A} f(a)$ for these extrema.

If (X,ρ) is a metric space and $A \subset X$ then the ε-neighbourhood, $\{x \in X;\ \inf_{a\in A} \rho(x,a) < \varepsilon\}$ of A will be denoted by $N(A,\varepsilon)$. If $x \in X$ then we shall write $N(x,\varepsilon)$ for $N(\{x\},\varepsilon)$. The closure of a subset A of a topological space will be denoted by $\overline{A}$.

Linear Spaces and Matrices: All the linear spaces used in this work are assumed to be real linear spaces. R^n, $||a||$ will denote the linear space of n-tuples of real numbers (Euclidean n-space) and the usual Pythagorean length of $a \in R^n$ respectively. The jth component of a vector $a \in R^n$ will be denoted variously by a^j or a_j whatever is the more convenient in the given context. I, M', $\ker(M)$, $||M||$ will denote the unit matrix, the transpose of the matrix M, the kernel (or nullspace) of the matrix M and the norm of the matrix M respectively. The norm we shall use for matrices is defined by

$$||M|| = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |M_{ij}|^2 \right)^{\frac{1}{2}} \text{ for an } m \times n \text{ matrix } M.$$

If X is a linear space, then X^* will denote the algebraic dual of X (i.e. the set of linear real valued functions on X). A real valued function Q on X is a quadratic form if the function $\overline{Q}$ on $X \times X$ defined by

$$\overline{Q}(x,y) = \tfrac{1}{4}\{Q(x+y) - Q(x-y)\} \text{ is bilinear.}$$

$\overline{Q}$ is called the polar bilinear form of Q.

Measure Theory: The Borel sets of a topological space will be taken as the sets in the σ-algebra generated by its open sets. We shall use the abbreviation $\mathcal{B}$-measurable to mean Borel measurable. Lebesgue measure on any Euclidean space will be denoted by μ. (The dimension of the measure μ will be clear in any given context in which it occurs). We shall use the abbreviation a.e. $t \in A$ to mean "almost every $t \in A$." The mean value (expectation) of a random variable Y will be denoted by $E(Y)$.

Normed Linear Spaces and Operators: If A is a subset of a Euclidean space we shall denote by $C_n(A)$ and $\mathcal{L}_n^p(A)$ respectively, the set of continuous functions from A into R^n and the set of $\mathcal{B}$-measurable functions f from A into R^n such that $\int_A \|f(a)\|^p d\mu(a) < \infty$. The norm of a normed linear space will usually be denoted by $\| \|$. The norm of $C_n(A)$ will be taken as the uniform norm and the norm of a function $f \in \mathcal{L}_n^p(A)$ as $\left(\int_A \|f(a)\|^p d\mu(a)\right)^{\frac{1}{p}}$.

We shall only find it necessary to distinguish between two different norms in the case of a function f belonging to both $\mathcal{L}_n^1(A)$ and $\mathcal{L}_n^2(A)$. In this case we write

$$\|f\|_1 = \int_A \|f(a)\| d\mu(a))$$

and

$$\|f\|_2 = \left(\int_A \|f(a)\|^2 d\mu(a)\right)^{\frac{1}{2}}$$

The inner product of $\mathcal{L}_n^2(A)$ will be denoted by $(\, , \,)_n$.

If T is a continuous linear operator mapping a Hilbert space $\mathcal{H}_1$ into another Hilbert space $\mathcal{H}_2$, then we shall denote by $\mathcal{R}(T)$, T^* and $\ker(T)$ the range, the adjoint and the nullspace of T respectively. The subspace orthogonal to a subset V of Hilbert space will be denoted by $V^{\perp}$.

Differentiation: In this thesis "partial derivative" will always mean "partial derivative of the first order". The partial derivative of a function h with respect to its jth real argument will be denoted by $h_{,j}$. The derivative of a function φ on R will be denoted by φ' or $\dot{\varphi}$ depending on the context, the second notation usually being interpreted as differentiation with respect to time. A real variable which has an interpretation as time will normally be the last argument of a function. In this case also we will denote the (partial) derivative of such a function f with respect to its last argument by $\dot{f}$.

Defined Symbols

CHAPTER 2

STRATEGY SPACES FOR DIFFERENTIAL GAMES

1. Introductory Definitions

The definitions which follow are intended to describe the differential games we shall be considering. The concepts of Isaacs elaborated in [1] are taken as a model.

1.1 Definition: Let p,q,n be positive integers, A,B,C be open convex sets in $R^p, R^q, [0,\infty) \times R^n$ respectively and let U,V,θ be closed subsets of A,B,C respectively. Let $f:R^n \times A \times B \to R^n$, $g:R^n \times A \times B \to R$, $h:C \to R$ have bounded, continuous partial derivatives on their domains of definition. The sextuple (f,g,h,U,V,θ) will then be called the dynamics of a regular (n-dimensional) differential game. If θ has the special form $\{T\} \times R^n$, then (f,g,h,U,V,θ) will be called the dynamics of a regular differential game of prescribed duration T. #

The differential game with dynamics (f,g,h,U,V,θ) is played as follows:

The motion of a point z in R^n (called the 'state' of the game) is controlled by two players through the agency of the differential equation

$$\dot{z} = f(z,u,v)$$
$$z(0) = x.$$

The point $x \in R^n$ is the point at which the state begins its motion at time $t=0$. The 'control' u is chosen at each instant from the set U by one player P, and the 'control' v is chosen from the set V by a second player E. The game finishes at the first instant t for which $(t,z(t)) \in \theta$. If the game terminates at time τ then the player E receives a payoff

$$J(x,u,v) = h(z(\tau)) + \int_0^\tau g(z(t),u(t),v(t))dt$$

which he tries to maximise through judicious choice of v, and P tries to minimise by similarly selecting u. We assume that the minimising player P wishes the game to terminate and the maximising player E would prefer non-termination. For non terminating play E therefore receives the payoff $+\infty$. The controls u,v chosen by the players at a given instant will in general depend on the past history of the state of the game. The exact nature of this dependence is determined by the information available to the players when they make their choices. The definitions which follow nominate the sets of functions which we shall allow as strategy spaces. First we introduce some convenient notation.

1.2 <u>Definition</u>: If I is an interval of R, and U a subset of R^p, we shall denote by $\mathcal{B}_U(I)$ the set of all $\mathcal{B}$-measurable functions $u:I \to U$ such that $\int_0^\tau \|u(s)\|ds < \infty$ for every $\tau \in I$; we shall denote by $AC_n(I)$ the set of absolutely continuous functions $x:I \to R^n$; and we shall denote by $\mathcal{A}_U^n$ the set of functions

$u:AC_n[0,\infty) \to \mathcal{B}_U[0,\infty)$ with the property that for every $\tau \in [0,\infty)$ and every $x_1, x_2 \in AC_n[0,\infty)$ with $x_1(t)=x_2(t)$ for $t \in [0,\tau]$, $u(x_1)(t)=u(x_2)(t)$ for $t \in [0,\tau]$. The elements of $\mathcal{A}^n_U$ will be loosely referred to as <u>admissible strategies</u>. #

The property satisfied by an element u of $\mathcal{A}^n_U$ requires that the value of $u(x)$ at the instant t depends only on the values of x prior to the time t and is independent of the future values of x. It is clear that the strategies of a differential game in which the players cannot foretell the future must have this property. However the strategies must be even further restricted if the game is to be well defined.

1.3 <u>Definition</u>: Let $D = (f,g,h,U,V,\theta)$ be the dynamics of a regular n-dimensional differential game. Then

i) the functions of $\mathcal{A}^n_U$ and $\mathcal{A}^n_V$ which are constant on $AC_n[0,\infty)$ will be denoted by $\mathcal{O}^1_D$ and $\mathcal{O}^2_D$ respectively. The elements of $\mathcal{O}^1_D$, $\mathcal{O}^2_D$ will be called <u>open-loop strategies in D</u> of the first and second players respectively.

ii) A pair $(\mathcal{P},\mathcal{Q})$ of sets with $\mathcal{O}^1_D \subset \mathcal{P} \subset \mathcal{A}^n_U$ and $\mathcal{O}^2_D \subset \mathcal{Q} \subset \mathcal{A}^n_V$ will be called a pair of <u>playable strategy sets for D</u> if the differential equation

$$\dot{z}(t) = f(z(t),u(z)(t),v(z)(t)) \qquad \text{a.e. } t \in [0,\infty)$$
$$z(0) = x$$

has a unique absolutely continuous solution $z:[0,\infty) \to R^n$ for any $u \in \mathcal{P}$, $v \in \mathcal{Q}$ and $x \in R^n$.

The elements of such a pair $\mathcal{P},\mathcal{Q}$ will be referred to loosely as <u>playable strategies</u> for D. #

With regard to these definitions we make the following observations.

i) An open-loop strategy of the first (second) player in (f,g,h,U,V,θ) is completely identified by its unique value in $\mathcal{B}_U$ ($\mathcal{B}_V$). In future chapters we shall normally regard the open-loop strategies as elements of $\mathcal{B}_U$ and $\mathcal{B}_V$ and shall often denote these sets by $\mathcal{U},\mathcal{V}$ respectively. (We have used $\mathcal{B}_U$ here to denote $\mathcal{B}_U[0,\infty)$).

ii) Because of the restrictions placed on the dynamics D of a regular differential game it follows from standard existence theorems of differential equations that $(\mathcal{O}_D^1,\mathcal{O}_D^2)$ will be a pair of playable strategy sets for D,

iii) If D is the dynamics of a regular differential game of fixed duration T, then the interval $[0,\infty)$ may be replaced throughout definitions 1.2 and 1.3 by $[0,T]$. Since this is notationally more convenient, and since we shall be concerned almost exclusively with games of fixed duration, this replacement will be implicitly assumed in future chapters.

iv) The reason for the regularity assumptions on the dynamics D of a (regular) differential game are two-fold. First they guarantee the existence and uniqueness of the motion of the state variable under fairly general conditions on the controls u,v. Secondly, they enable us to write down the general necessary condition for the optimality of open-loop strategies in games of prescribed duration. However, for certain special games considered in the next chapter, these assumptions are unnecessary, and we shall not let the restrictiveness of the definitions deter us from calling these games '(non-regular) differential games'.

In differential games with no information, the players make their choice of controls at a given instant with no knowledge of the prior values of the state of the game (except for its initial state, which we regard as one of the 'rules' of the game). Thus, under these circumstances, their pure strategies in an n-dimensional differential game with dynamics (f,g,h,U,V,θ) will be the functions of $\mathcal{A}^n_U$, $\mathcal{A}^n_V$ which are independent of the state history of the game. These are just the functions which constitute the open-loop strategy sets $\mathcal{O}^1_D$, $\mathcal{O}^2_D$ respectively.

In general, any restrictions of a non-stochastic nature on the information available to the players can be taken into account by choosing suitable pairs of playable strategy sets for their sets of pure strategies. Time lags and intermittent interruptions in the state measurements can always be allowed for in this way.

For differential games with perfect information, it is difficult to find pairs of playable strategy sets which are not too restrictive on the players' choices of controls. These difficulties are discussed in the next section.

We conclude our description of a differential game by formalising the concept of payoff.

1.4 Definition: If D is the dynamics of a regular (n-dimensional) differential game and $(\mathcal{P},\mathcal{Q})$ a pair of playable strategy sets for D, then the triple $(D,\mathcal{P},\mathcal{Q})$ will be called a regular (n-dimensional) differential game. In particular $(D,\mathcal{O}^1_D,\mathcal{O}^2_D)$ will be called a regular (n-dimensional) differential game with no information. #

1.5 Definition: Let $((f,g,h,U,V,\theta),\mathcal{P},\mathcal{Q})$ be a regular n-dimensional differential game. Then

i) The unique function $z:R^n \times \mathcal{P} \times \mathcal{Q} \times [0,\infty) \to R^n$ satisfying the equation

$$\dot{z}(x,u,v,t) = f(z(x,u,v,t),u(z(x,u,v,.))(t),v(z(x,u,v,.))(t)) \quad \text{a.e. } t \in [0,\infty)$$

$$z(x,u,v,0) = x$$

for $u \in \mathcal{P}$, $v \in \mathcal{Q}$, $x \in R^n$ will be called the trajectory (or path) function of the game.

ii) If $S(x,u,v)$ denotes the set $\{t \in [0,\infty); (t,z(x,u,v,t)) \in \theta\}$, the extended real valued function

$\tau: R^n \times \mathscr{P} \times \mathscr{Q} \to [0,\infty]$ defined by

$$\tau(x,u,v) = \begin{cases} \inf S(x,u,v) & \text{if } S(x,u,v) \neq \emptyset \\ +\infty & \text{if } S(x,u,v) = \emptyset \end{cases}$$

will be called the <u>termination time</u> of the game. #

1.6 <u>Definition</u>: Let $((f,g,h,U,V,\theta),\mathscr{P},\mathscr{Q})$ be a regular differential game with path function z and termination time τ. The extended real valued function $J: R^n \times \mathscr{P} \times \mathscr{Q} \to (-\infty,\infty]$ defined by

$J(x,u,v) =$

$$\begin{cases} h(\tau(x,u,v),z(x,u,v,\tau(x,u,v))) \\ \quad + \int_0^{\tau(x,u,v)} g(z(x,u,v,s),u(z(x,u,v,.))(s),v(z(x,u,v,.))(s))ds \\ \qquad\qquad\qquad\qquad \text{if } \tau(x,u,v) < \infty \\ +\infty \quad \text{if } \tau(x,u,v) = +\infty \end{cases}$$

will be called the <u>payoff function</u> of the game. #

If $((f,g,h,U,V,\theta),\mathscr{P},\mathscr{Q})$ is a regular differential game with payoff J then the conditions satisfied by f,g,h and $\mathscr{P},\mathscr{Q}$ ensure that $J(x,u,v)$ is a well defined element of $[0,\infty]$ for every $x \in R^n$, and $(u,v) \in \mathscr{P} \times \mathscr{Q}$. Thus for every $x \in R^n$, $(\mathscr{P},\mathscr{Q},-J(x,.,.))$ will constitute a classical two person zero sum game in normal form. (The minus sign is inserted to accord with the convention that the payoff of a classical game goes to the first player who tries to maximise it.)

2. Strategies for Games with Perfect Information

As indicated earlier we shall now discuss the difficulties of defining pairs of playable strategy sets for differential games of perfect information. In such games the players know, and can recall at any instant, the complete prior state history of the game. Thus for an n-dimensional regular differential game with dynamics (f,g,h,U,V,θ) one might expect the pure strategies of players with perfect information to be the sets $\mathcal{A}_U^n$, $\mathcal{A}_V^n$ defined in 1.2. But for arbitrary $u \in \mathcal{A}_U^n$, $v \in \mathcal{A}_V^n$, $x \in R^n$ the differential equation

$$(2.1)\quad \dot{z}(t) = f(z(t),u(z)(t),v(z)(t)) \quad \text{a.e.}\, t \in [0,\infty)$$
$$z(0) = x$$

will not necessarily have a solution z, or may have more than one solution. Ingenious attempts (e.g. in Blaquière et.al.[28]) have been made to allow for the possibility of non-uniqueness and non-existence of the paths of differential games. Although the non-uniqueness aspect can be handled satisfactorily by considering the payoff to be multivalued, attempts to treat non-existence in a similar way have been unconvincing.

The approach used here of forcing the existence and uniqueness of the solutions of (2.1) by restricting the players' choice of strategies to be in an unspecified pair $(\mathcal{P},\mathcal{Q})$ of playable strategy sets is taken from Berkovitz

[29]. The main drawback of this approach is that no completely acceptable pairs of playable strategy sets can be found in all cases. There follows the definition of a pair of possible candidates.

2.1 Definition: Let n,p be positive integers, and U a closed subset of R^p. Then

i) The set of all $u \in \mathcal{A}^n_U$ such that $u(x_1)(t) = u(x_2)(t)$ whenever $x_1, x_2 \in AC_n[0,\infty)$ and $x_1(t) = x_2(t)$ will be denoted by $\mathcal{C}^n_U$. The elements of $\mathcal{C}^n_U$ will be loosely referred to as closed-loop strategies.

ii) The set of all $u \in \mathcal{A}^n_U$ such that for some constant $K > 0$ and all $x_1, x_2 \in AC_n[0,\infty)$, $t \in [0,\infty)$

$$\|u(x_1)(t) - u(x_2)(t)\| \leq K\|x_1(t) - x_2(t)\|$$

will be denoted by $\mathcal{M}^n_U$. The elements of $\mathcal{M}^n_U$ will be loosely referred to as Lipschitz strategies

iii) If $((f,g,h,U,V,\theta),\mathcal{P},\mathcal{Q}) = G$ is a regular n-dimensional differential game, the sets $\mathcal{P} \cap \mathcal{C}^n_U$, $\mathcal{Q} \cap \mathcal{C}^n_V$ will be denoted by $\mathcal{C}^1_G$, $\mathcal{C}^2_G$ respectively. The elements of $\mathcal{C}^1_G$, $\mathcal{C}^2_G$ will be called closed-loop strategies in G of the first and second players respectively. #

Note that if $\mathcal{C}$ denotes the set of all functions $\bar{u}: R^n \times [0,\infty) \to U$ then $\mathcal{C}^n_U$ may be put into a one-to-one correspondence with a subset of $\mathcal{C}$ by means of the relation

$$u(x)(t) = \bar{u}(x(t),t) \quad u \in \mathcal{C}^n_U,\ x \in AC_n[0,\infty),\ t \in [0,\infty)$$

It is more usual in the literature for the elements of $\mathscr{C}$ to be called closed-loop strategies. Isaacs ([1]p.39) has given sound reasons why the players of a differential game may be expected to restrict themselves to closed loop strategies. However, we prefer to retain full generality by allowing the controls to depend on all the past values of the state variables.

It is clear that $\mathscr{M}_U^n \subset \mathscr{C}_U^n$, and it also follows from standard existence and uniqueness theorems of differential equations that $(\mathscr{M}_U^n, \mathscr{M}_V^n)$ is a pair of playable strategy sets for the dynamics (f,g,h,U,V,θ) of a regular n-dimensional differential game. However it seems arbitrary and unrealistic to restrict the players to choose Lipschitz strategies, if only because many otherwise completely acceptable solutions of differential games would be thereby precluded from selection.

Because of the problems outlined above, many ideas similar to Karlin and Isaacs' K-strategy concept ([1] p.38) have appeared in the literature. This approach proceeds by dividing time into discrete intervals, and forcing the players to hold their controls constant over these intervals. Variations on this theme have been examined by Fleming [30,31], Varaiya [32], Varaiya and Lin [33] and Friedman [12], but although many elegant results have been obtained we feel the approach to be aesthetically unpleasant.

To illustrate the difficulties discussed above we consider the following example.

2.2 Example: Isaacs' 'Homicidal Chauffeur' game ([1] pp.11,27) has the 2-dimensional regular dynamics (f,g,h,U,V,θ) with

$$\theta = [0,\infty)\times\{x \in R^2;\ \|x\| \leqslant \ell\}$$

$$U = [-1,1]$$

$$V = R$$

$$f^1(x,y,\varphi,\psi) = -\frac{w_1}{r}y\varphi + w_2 \sin\psi$$

$$f^2(x,y,\varphi,\psi) = \frac{w_1}{r}x\varphi - w_1 + w_2 \cos\psi$$

$$g(x,y,\varphi,\psi) = 1$$

$$h(t,x,y) = 0$$

for $(x,y) \in R^2$, $\varphi \in U$, $\psi \in V$, $t \in [0,\infty)$.

In these equations, ℓ, r, w_1, w_2 are all positive real numbers. We shall also assume that $\ell \geqslant r$.

If $w_1 > w_2$ and $\frac{\ell}{r} > \sqrt{1-\left(\frac{w_1}{w_2}\right)^2} + \arcsin\left(\frac{w_2}{w_1}\right) - 1$

then the first player P can force termination from any point of R^2 ([1] p.237). In a sufficiently small neighbourhood of the positive y-axis the optimal tactics ([1] p.38) $\bar{\varphi},\bar{\psi}$ are given by

$$\bar{\varphi}(x,y) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$$\sin \bar{\psi}(x,y) = \frac{-r(x-r) + y\sqrt{x^2+y^2-2xr}}{(x-r)^2 + y^2}$$

$$\cos \bar{\psi}(x,y) = \frac{yr + (x-r)\sqrt{x^2+y^2-2xr}}{(x-r)^2 + y^2}$$

(see [1] p.193ff.,p 232ff.)

In this game the positive y-axis is a universal surface ([1], Chapter 7) into which optimal paths merge from either side.

Suppose now that a play of the game starts at $t=0$ from a point $(0,y_0)$, with $y_0 > \ell$, that P uses his optimal strategy $\bar{\varphi}$, and that the second player E uses a strategy ψ (not necessarily optimal). We now show that if a solution to the path equations (1.2) exists under these circumstances then $\sin \psi(0,y) = 0$ for $y \in (\ell,y_0]$, and the path never leaves the positive y-axis.

Let $z = (z^1,z^2)$ denote the solution to (1.2) with $f = (f^1,f^2)$ defined as above, $x = (0,y_0)$ and strategies u,v defined by $u(z)(t) = \bar{\varphi}(z(t))$, $v(z)(t) = \psi(z(t))$.

Suppose now that there exists a time t_1 such that $z^1(t_1) \neq 0$, and $(t,z(t)) \notin \theta$ for any $t \in [0,t_1]$ (i.e. t_1 occurs prior to termination).

Let $t^+ = \sup\{t < t_1 ; z^1(s) = 0 \text{ for } s \in [0,t]\}$.

(Note that since $z^1(0) = 0$, and z^1 is continuous then $t^+ \in [0,t_1)$).

Since termination does not occur prior to t_1, and $z^2(0) > 0$, then by continuity

$$z^1(t^+) = 0 \quad \text{and} \quad z^2(t^+) > \ell$$

and there exists $t_2 > t^+$ such that $z^2(t) > \ell$ for $t \in [t^+, t_2]$.

Also, there exists $t_3 \in [t^+, t_2]$ with $z^1(t_3) \neq 0$ (by definition of t^+).

We now contradict the choice of t_3 by showing that $z^1(t_3) = 0$.

Suppose first that $z^1(t_3) > 0$ and put

$$t_4 = \sup\{t < t_3 ; z^1(t) \leq 0\}$$

so that $t_4 \in [t^+, t_3)$, $z^1(t_4) = 0$, and $z^1(t) > 0$ for $t \in (t_4, t_3]$.

Now

$$\dot{z}(t) = -\frac{w_1}{r} z^2(t)\bar{\varphi}(z^1(t), z^2(t)) + w_2 \sin \psi(z^1(t), z^2(t))$$

$$\leq -\frac{w_1}{r} z^2(t) + w_2$$

$$< w_2 - w_1 \quad \text{for} \quad t \in (t_4, t_3]$$

(since $z^1(t) > 0$ for $t \in (t_4, t_3]$, then $\bar{\varphi}(z^1(t), z^2(t)) = 1$. Note also that $z^2(t) \geq \ell \geq r$).

Thus $z^1(t_3) = z^1(t_4) + \int_{t_4}^{t_3} \dot{z}^1(t)dt < 0$.

This contradicts the supposition that $z^1(t_3) > 0$, and it follows that $z^1(t_3) \leq 0$.

By a similar argument it may be shown that $z^1(t_3) \geq 0$, and consequently $z^1(t_3) = 0$ contradicting the choice of t_3. Our original assumption that $z^1(t_1) \neq 0$ for some t_1 prior to termination is therefore untenable, and $z^1(t) = 0$ for all t earlier than termination.

If now $y \in (\ell, y_0]$ then by continuity $z^2(t) = y$ for some t prior to termination. From the path equations (1.2) we have

$$0 = \dot{z}^1(t) = -\frac{w_1}{r} z^2(t)\overline{\varphi}(z^1(t), z^2(t)) + w_2 \sin \psi(z^1(t), z^2(t))$$

$$= w_2 \sin \psi(0,y) \quad (\text{since } z^1(t) = 0).$$

Thus $\sin \psi(0,y) = 0$ as claimed.

In the terminology of Blaquière et.al. [28] the only 'playable' strategy pairs of the form $(\overline{\varphi}, \psi)$ for a play of the game starting on the positive y-axis are those for which $\sin \psi \equiv 0$. (Note that the term 'playable' is used here in a slightly different sense to that of our definition 1.3 ii)). It seems excessively tyrannous to force E to play such a strategy, since this only allows him to run directly towards or directly away from P, but if E plays any other strategy, then the path equations (1.2) have no solution. In this case the trouble is caused by allowing P to react instantaneously to the exit of the state from the positive y-axis, and it seems that similar problems will occur on any surface of the universal type. These problems may be obviated by forcing P to use K-strategies ([1] p.38), but in regions of the playing space removed from singular surfaces, his optimal tactic $\overline{\varphi}$ is perfectly regular and there seems to be no good reason to force him to use K- strategies in these regions. #

3. Universality of Optimal Open-Loop Strategies

A differential game as defined in 1.4 is really a family of games in the classical sense, a separate game being defined for each initial value of the state. Thus although a strategy might be optimal for some initial state, it may not be optimal for another, and therefore not possess the global property of optimality required of strategies which are 'optimal' in the usually accepted sense of Isaacs ([1] Chapters 2 and 4). To avoid confusion we therefore adopt the following definitions.

3.1 Definition: Let $G = (D,\mathcal{P},\mathcal{Q})$ be a regular n-dimensional differential game with payoff J, and let $x \in R^n$. Then:

i) A strategy pair $(u^*,v^*) \in \mathcal{P} \times \mathcal{Q}$ will be called an x-saddle point (or x-solution) of G if it constitutes a saddle point of the (classical) game $(\mathcal{P},\mathcal{Q},-J(x,.,.))$

$(u^*,v^*$ thus satisfy the inequalities

$J(x,u^*,v) \leq J(x,u^*,v^*) \leq J(x,u,v^*)$ for every $(u,v) \in \mathcal{P} \times \mathcal{Q})$; u^*,v^* will be called x-optimal strategies of the first and second players respectively.

ii) For $\varepsilon > 0$, a strategy pair $(u^*,v^*) \in \mathcal{P} \times \mathcal{Q}$ will be called an x-ε-saddle point of G if it constitutes an ε-saddle point of the game $(\mathcal{P},\mathcal{Q},-J(x,.,.))$.

u^*, v^* will be called <u>x-ε-optimal strategies</u> of the first and second players respectively.

A strategy pair $(u^*, v^*) \in \mathcal{P} \times \mathcal{Q}$ will be called a <u>global saddle point</u> (or global solution) or a <u>global ε-saddle point</u> if it constitutes an x-saddle point or an x-ε-saddle point respectively, for every $x \in R^n$. u^*, v^* will then be called <u>globally optimal</u> strategies or <u>globally ε-optimal strategies</u> of the first and second players respectively. #

In most games with large enough strategy sets the existence of an x-saddle point for every x implies the existence of a global saddle point in closed-loop strategies. (A game in which this is not the case is given in [34]). However, if the pure strategies of the players are severely restricted, one cannot normally assert the existence of global solutions. Quite a few games, for instance have x-saddle points in open-loop strategies for every x, but globally optimal open-loop strategies practically never occur.

Rekasius has conjectured [24] that "The existence of a pair of optimal open-loop strategies in a differential game implies also the existence of an optimal pair in which one strategy is open-loop and the other closed-loop, which, in turn, implies the existence of a pair of optimal closed-loop strategies."

If, in the above conjecture we interpret "optimal" as meaning x-optimal throughout then the conjecture can be proved quite simply. A pair of x-optimal open loop strategies are, in fact, also a pair of x-optimal closed loop strategies. This result has important consequences which, up to the present, seem to have been unrecognised. The proof follows.

3.2 <u>Theorem</u>: Let $D = (f,g,h,U,V,\theta)$ be the dynamics of a regular n-dimensional differential game with payoff J and termination time τ. Let $(\mathcal{P}_1,\mathcal{Q}_1)$, $(\mathcal{P}_2,Q_2)$ be pairs of playable strategy sets for D with $\mathcal{P}_1 \subset \mathcal{P}_2$, $\mathcal{Q}_1 \subset \mathcal{Q}_2$. Then if $x \in R^n$, and (u^*,v^*) is an x-saddle point of $(D,\mathcal{P}_1,\mathcal{Q}_1)$ it is also an x-saddle point of $(D,\mathcal{P}_2,\mathcal{Q}_2)$.

<u>Proof</u>: Since $\mathcal{O}_D^1 \subset \mathcal{P}_1$, $\mathcal{O}_D^2 \subset \mathcal{Q}_1$ then by definition 3.1

(1) $J(x,u^*,v) \leqslant J(x,u^*,v^*) \leqslant J(x,u,v^*)$

$$\text{for } (u,v) \in \mathcal{O}_D^1 \times \mathcal{O}_D^2 .$$

Now let $u \in \mathcal{P}_2$, $v \in \mathcal{Q}_2$ and let z_1 denote the solution of the equation

$$\dot{z}_1(t) = f(z_1(t), u^*(z_1)(t), v(z_1)(t)) \quad \text{a.e.} t \in [0,\infty]$$
$$z_1(0) = x$$

and let $v_1 \in \mathcal{O}_D^2$ be the open-loop control defined by

$$v_1(y)(t) = v(z_1)(t) \quad \text{for} \quad y \in AC_n[0,\infty) \quad t \in [0,\infty).$$

Then z_1 also satisfies the equation

$$\dot{z}_1(t) = f(z_1(t), u^*(z_1)(t), v_1(z_1)(t)) \quad \text{a.e.} t \in [0,\infty).$$

Therefore

$$\tau(x,u^*,v) = \tau(x,u^*,v_1), \text{ and}$$

(2) $$J(x,u^*,v) = h(\tau(x,u^*,v),z_1(\tau(x,u^*,v))) + \int_0^{\tau(x,u^*,v)} g(z_1(t),u^*(z_1)(t),v(z_1)(t))dt$$

$$= h(\tau(x,u^*,v_1),z_1(\tau(x,u^*,v_1))) + \int_0^{\tau(x,u^*,v_1)} g(z_1(t),u^*(z_1)(t),v_1(z_1)(t))dt$$

$$= J(x,u^*,v_1) \leq J(x,u^*,v^*) \text{ by (1).}$$

Similarly, if z_2 is the solution of

$$\dot{z}_2(t) = f(z_2(t),u(z_2)(t),v^*(z_2)(t))$$
$$z_2(0) = x$$

and $u_1 \in \mathcal{O}_D^1$ is the open-loop control defined by $u_1(y)(t) = u(z_2)(t)$ for $y \in AC_n[0,\infty)$, $t \in [0,\infty)$ then

(3) $J(x,u,v^*) = J(x,u_1,v^*) \geq J(x,u^*,v^*)$ by (1).

Combining (2) and (3) we have

$$J(x,u^*,v) \leq J(x,u^*,v^*) \leq J(x,u,v^*)$$

and since $(u,v) \in \mathcal{P}_2 \times \mathcal{Q}_2$ were arbitrary, then we have shown that (u^*,v^*) constitutes an x-saddle point of $(D,\mathcal{P}_2,\mathcal{Q}_2)$ as was to be proved. #

Rekasius' conjecture (interpreted in the sense of the remarks prior to 3.2) now follows as a corollary.

3.3 <u>Corollary</u>: Let $G = (D,\mathcal{P},\mathcal{Q})$ be a regular n-dimensional differential game, and $x \in R^n$. Then:

i) If (u^*,v^*) is an x-saddle point of $(D,\mathcal{O}_D^1,\mathcal{O}_D^2)$ then it is an x-saddle point of $(D,\mathcal{O}_D^1,\mathcal{C}_G^2)$ (and also of G).

ii) If (u^*,v^*) is an x-saddle point of $(D,\mathcal{O}^1_D,\mathcal{C}^2_G)$ then it is an x-saddle point of $(D,\mathcal{C}^1_G,\mathcal{C}^2_G)$ (and also of G).

Proof: Since $\mathcal{O}^1_D \subset \mathcal{C}^1_G \subset \mathcal{P}$ and $\mathcal{O}^2_D \subset \mathcal{C}^2_G \subset \mathcal{Q}$, then the result follows from 3.2. #

Thus, if a differential game has x-optimal open-loop strategies for every x then the x-optimal open-loop strategies will also be x-optimal closed-loop strategies, and the values of the game in open-loop and in closed-loop strategies will be the same. The standard techniques (e.g. as given in [1]) for solving differential games in closed-loop strategies may then be used to obtain the value of the game for open-loop strategies. The necessary conditions, given in the next chapter, which must be satisfied by x-optimal open-loop strategies and the corresponding optimal paths are similar to those satisfied by optimal closed-loop strategies, and may be solved by the same methods. Provided that the functions involved are sufficiently 'well behaved' the optimal paths for the two cases will coincide, and the x-optimal open-loop strategies will be generated by the optimal closed-loop strategies along an optimal path through x. (This will be the case, for example, if the necessary conditions have a unique solution). This result has been predicted by a second conjecture of Rekasius in [24].

By a simple proof similar to 3.2 we can assert the more general result that if the differential game $(D,\mathcal{O}_D^1,\mathcal{O}_D^2)$ has an x-saddle point (u^*,v^*), then (u^*,v^*) will constitute an x-saddle point of <u>any</u> differential game with dynamics D in which the players receive partial information that includes the initial value x of the state. (Including, for example, games with noise corrupted measurements). The strategies (u^*,v^*) thus have a universal optimality in the sense that when the first player (second player) plays optimally in a game with dynamics D which starts from x, the second player (first player) can do no better than to play with the strategy $v^*(u^*)$. This conclusion seems to have been overlooked throughout the literature on games with noise corrupted measurements.

CHAPTER 3

GAMES WITH PURE STRATEGY SOLUTIONS

1. A Necessary Condition for the Optimality of Open-Loop Strategies.

Throughout the remainder of the thesis we shall be concerned with differential games of prescribed duration T with no information. As indicated by the remarks following 2-1.3 we shall consider such a game to be the triple $(D,\mathcal{U},\mathcal{V})$, where $D = (f,g,h,U,V,\theta)$ is the dynamics of a regular differential game of prescribed duration T, $\mathcal{U} = \mathcal{B}_U[0,T]$ and $\mathcal{V} = \mathcal{B}_V[0,T]$. The adoption of $(D,\mathcal{U},\mathcal{V})$ rather than $(D,\mathcal{O}_D^1,\mathcal{O}_D^2)$ for this role is made for notational simplicity, and the results obtained in either case are equivalent. The definitions of path and payoff functions of $(D,\mathcal{U},\mathcal{V})$ are obvious from 2-1.5 and 2-1.6, but for the sake of completeness, all these concepts are formalised below.

1.1 Definition: Let $D = (f,g,h,U,V,\theta)$ be the dynamics of a regular n-dimensional differential game of prescribed duration T, $\mathcal{U} = \mathcal{B}_U[0,T]$ and $\mathcal{V} = \mathcal{B}_V[0,T]$. Then:

i) The triple $G = (D,\mathcal{U},\mathcal{V})$ will be called a regular (n-dimensional) blind differential game of prescribed duration T. The elements of $\mathcal{U},\mathcal{V}$ will be loosely referred to as open-loop strategies

ii) The (unique) function $z: R^n \times \mathcal{U} \times \mathcal{V} \times [0,T] \to R^n$

satisfying the differential equation

$$\dot{z}(x,u,v,t) = f(z(x,u,v,t),u(t),v(t)) \quad \text{a.e.}\, t \in [0,T]$$

$$z(x,u,v,0) = x$$

for $x \in R^n$, $(u,v) \in \mathcal{U} \times \mathcal{V}$ will be called the <u>trajectory (or path) function</u> of G.

(iii) The function $J: R^n \times \mathcal{U} \times \mathcal{V} \to R$ defined by

$$J(x,u,v) = h(T,z(x,u,v,T)) + \int_0^T g(z(x,u,v,t),v(t),v(t))dt$$

for $x \in R^n, (u,v) \in \mathcal{U} \times \mathcal{V}$ will be called the <u>payoff function</u> of G. #

The concept of x-saddle points, and x-ε-saddle points for blind differential games are defined exactly as in 2-3.1 and it is unnecessary to repeat the definitions here.

The theorem which follows gives a necessary condition for optimal control of a blind differential game. For piecewise continuous strategies such a condition can be derived easily from the maximum principle of control theory [35]. However, it seems that the extension to the arbitrary $\mathcal{B}$-measurable strategies which we consider here has hitherto been neglected.

1.2 <u>Theorem</u>: Let U,V be compact subsets of R^p, R^q respectively, and $G = ((f,g,h,U,V,\theta),\mathcal{U},\mathcal{V})$ be a regular n-dimensional blind differential game of prescribed duration T. Let $x \in R^n$, and (u^*,v^*) be an x-saddle point of G, and define $\lambda:[0,T] \to R^n$ to be the (unique) solution of the differential equation

$$\dot{\lambda}^j(t) = -\sum_{i=1}^{n} \lambda^i(t) f^i_{,j}(z(x,u^*,v^*,t),u^*(t),v^*(t)) - g_{,j}(z(x,u^*,v^*,t),u^*(t),v^*(t))$$

$$\lambda^j(T) = h_{,j+1}(T,z(x,u^*,v^*,T)).$$

Also define $H:R \times U \times V \to R$ by

$$H(t,u,v) = \sum_{i=1}^{n} \lambda^i(t) f^i(z(x,u^*,v^*,t),u,v)+g(z(x,u^*,v^*,t),u,v)$$

$$\text{for } u \in U,\ v \in V.$$

Then

i) $H(t,u^*(t),v^*(t))$ is constant for a.e. $t \in [0,T]$

ii) $$\sup_{v\in V} H(t,u^*(t),v) = H(t,u^*(t),v^*(t)) = \inf_{u\in U} H(t,u,v^*(t))$$

for a.e. $t \in [0,T]$.

#

This result follows from 5-3.3 as a particular case, and the proof will be deferred to Chapter 5.

2. Conditions for the Existence of Open-Loop Solutions

The problem of finding conditions for the existence of open-loop solutions for differential games seems to have been completely ignored until Fichefet [18,19] published his results in 1968. Since then existence conditions for open-loop solutions of linear-quadratic games have been obtained by various authors, but no games of a more general nature have been considered.

Unfortunately, the conclusions of Fichefet's first paper [18] are invalidated by several mathematical errors, the most serious being the assertion of the weak continuity of a certain class of functionals on a closed convex subset $\mathcal{U}$ of the Hilbert space $\mathcal{L}_p^2[t_0,T]$. The functionals for which weak continuity is asserted include among their number, the square of the norm of $\mathcal{L}_p^2[t_0,T]$, and the set $\mathcal{U}$ contains an infinite set of mutually orthogonal functions. Since $\|(\cdot)\|_2^2$ is not weakly continuous on such a set $\mathcal{U}$, the claims of weak continuity cannot be true in general. The main theorems of the second paper [19] have extremely complicated proofs involving advanced concepts of functional analysis. By making use of a simple lemma on the weak semicontinuity of concave and convex functions, we can provide a relatively concise proof of a general result which includes the main theorems of [18] and [19] as special cases.

2.1 Lemma: If C is a closed convex set in a real Hilbert space $\mathcal{H}$, and f a real valued, lower semicontinuous (resp. upper semicontinuous) and convex (resp. concave) function on C, then f is weakly lower semicontinuous (resp. weakly upper semicontinuous) on C.

Proof: By the convexity and lower semicontinuity of f the set $S = \{x \in C;\ f(x) \leq \alpha\}$ is closed and convex for every real α and is thus weakly closed for each real α.

It follows that f is weakly lower semicontinuous.

The case of a concave function is treated similarly.

#

This result has also been noted independently by Auslender in a recent paper [23]. Note that the lemma holds more generally for quasi-concave and quasi-convex functions (see A2.1 of the Appendix for definitions) as does the following result.

2.2 <u>Theorem</u>: If C_1, C_2 are bounded, closed, convex sets in the real Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ respectively and f a real valued function on $C_1 \times C_2$ such that $f(\cdot, y)$ is lower semicontinuous and convex on C_1 for each $y \in C_2$, and $f(x, \cdot)$ is upper semicontinuous and concave on C_2 for each $x \in C_1$, then

$$\min_{x \in C_1} \max_{y \in C_2} f(x,y) = \max_{y \in C_2} \min_{x \in C_1} f(x,y).$$

<u>Proof</u>: By 2.1 $f(\cdot, y)$ is weakly lower semicontinuous on C_1 for each $y \in C_2$ and $f(x, \cdot)$ is weakly upper semicontinuous on C_2 for each $x \in C_1$, and since C_1, C_2 are both convex and weakly compact (being bounded closed and convex subsets of a Hilbert space) then the theorem follows from Fan's or Sion's minimax theorems (see A2.2 of the Appendix). #

For the concept of weak continuity and the proof of weak compactness of bounded, closed convex subsets of a

Hilbert space we refer the reader to Dunford and Schwartz [36].

Thus if the open loop strategies of a blind differential game form bounded, closed convex subsets of the Hilbert spaces $\mathcal{L}_p^2[0,T]$, $\mathcal{L}_q^2[0,T]$, and the payoff of the game is continuous with respect to the norms of these spaces, and is convex-concave, then we can assert the existence of an open-loop x-saddle point of the game for every x. Conditions under which the strategies and payoff have these properties are established by the following lemmas.

2.3 <u>Lemma</u>: Let U be a non-empty subset of R^p and denote by $\mathcal{U}$ the set of all $u \in \mathcal{L}_p^2[0,T]$ with values in U almost everywhere on $[0,T]$.

Then i) If U is closed, $\mathcal{U}$ is closed

ii) If U is bounded, $\mathcal{U}$ is bounded.

iii) If U is convex, $\mathcal{U}$ is convex.

<u>Proof</u>: i) Let $u_0 \in \overline{\mathcal{U}}$ and let $\{u_r\}_{r=1}^{\infty}$ be a sequence of $\mathcal{U}$ converging to u_0. Let $w_0 \in U$ and for each $r=1,2,..$ define

$$u_r^1(t) = \begin{cases} u_r(t) & \text{if } u_r(t) \in U \\ w_0 & \text{if } u_r(t) \notin U \end{cases}$$

Then for each $r=1,2,...$ $u_r^1 = u_r$ (regarded as elements of $\mathcal{L}_p^2[0,T]$), and $u_r^1(t) \in U$ for every $t \in [0,T]$ and also $\lim\limits_{r\to\infty} u_r^1 = u_0$.

Let $S = \{t \in [0,T];\ u_0(t) \notin U\}$

and $S_r = \{t \in [0,T];\ \inf_{u \in U} \|u - u_0(t)\| \geq \frac{1}{r}\}$

$= \{t \in [0,T];\ u_0(t) \notin N(U, \frac{1}{r})\}$ for $r = 1, 2, \ldots$

Then for each $r = 1, 2 \ldots, S_r$ is measurable and since U is closed

$$S = \bigcup_{r=1}^{\infty} S_r$$

But if r is a positive integer, and $\varepsilon > 0$ then there exists a positive integer n such that

$\|u_0 - u_n\|_2 < \frac{\sqrt{\varepsilon}}{r}$ and therefore

$$\mu(S_r)\frac{1}{r^2} \leq \int_{S_r} \|u_0(t) - u_n(t)\|^2 dt$$

$$\leq \int_0^T \|u_0(t) - u_n(t)\|^2 dt \leq \frac{\varepsilon}{r^2}$$

and $\mu(S_r) \leq \varepsilon$.

Since $\varepsilon > 0$ was arbitrary it follows that

$$\mu(S_r) = 0$$

and then that $\mu(S) = \mu(\bigcup_{r=1}^{\infty} S_r) = 0$.

Consequently $u_0 \in \mathcal{U}$, and $\mathcal{U}$ is closed.

ii) If $\sqrt{L}$ is a bound for U, then for any $u \in \mathcal{U}$, $\int_0^T \|u(t)\|^2 dt \leq LT$ and $\mathcal{U}$ is therefore bounded.

iii) If U is convex, $u_1, u_2 \in \mathcal{U}$ and $\lambda \in [0,1]$ then $\lambda u_1(t) + (1-\lambda)u_2(t) \in U$ a.e. $t \in [0,T]$ and consequently $\lambda u_1 + (1-\lambda)u_2 \in \mathcal{U}$. That is, $\mathcal{U}$ is convex. #

The theorem on which the continuity of the payoff functions of certain differential games is based now follows. Since a similar result is useful in other contexts the theorem is proved with more generality than is immediately required.

2.4 <u>Theorem</u>: Let W be a subset of R^n, A a compact subset of R^p, and $G: A \times W \to R^s$ a bounded $\mathcal{B}$-measurable function such that $G(a,.)$ is continuous on W for every $a \in A$. Let $\mathcal{W}$ denote the set of functions $\{w \in \mathcal{L}_n^2(A);\ w(a) \in W \text{ for a.e. } a \in A\}$ and $P: \mathcal{W} \to \mathcal{L}_s^2(A)$ be defined by

$$P(w)(a) = G(a,w(a)) \qquad w \in \mathcal{W},\ a \in A.$$

Then P is continuous on $\mathcal{W}$ (i.e. with respect to the $\mathcal{L}_n^2(A)$ and $\mathcal{L}_s^2(A)$ norms on $\mathcal{W}$ and $P(\mathcal{W})$ respectively).

<u>Proof</u>: Let $w_0 \in \mathcal{W}$ and $\varepsilon>0$, and let M be a bound for G. Also let S be a countable, dense subset of W (which exists since R^n is separable), and

$$H_r = \{(x,x') \in S \times S;\ \|x-x'\| < \frac{1}{r}\}$$

for each $r=1,2,\ldots$

The sets

$$\mathcal{S}_r = \bigcap_{(x,x') \in H_r} \{a \in A;\ \|G(a,x) - G(a,x')\| \leqslant \frac{\varepsilon}{\sqrt{3\mu(A)}}\}$$

are clearly $\mathcal{B}$-measurable for $r=1,2,\ldots$ (since G is $\mathcal{B}$-measurable), and

$$\left\{a \in A;\ \|G(a,x) - G(a,x')\| \leq \frac{\varepsilon}{3\mu(A)} \text{ for every } x,x' \in W \text{ with } \|x-x'\| < \frac{1}{r}\right\} \subset \mathcal{S}_r$$

We now show that the set on the left hand side of the above expression contains $\mathcal{S}_r$ and must therefore be equal to $\mathcal{S}_r$.

Let $a \in \mathcal{S}_r$. Then

$$\|G(a,x) - G(a,x')\| \leq \frac{\varepsilon}{\sqrt{3\mu(A)}} \quad \text{for } x,x' \in S$$

$$\text{with } \|x-x'\| < \frac{1}{r}.$$

Now if $x_1, x_2 \in W$ and $\|x_1 - x_2\| < \frac{1}{r}$ then there exist sequences $\{x_1^{(r)}\}_{r=1}^{\infty}$, $\{x_2^{(r)}\}_{r=1}^{\infty}$ of S such that $\lim_{r\to\infty} x_1^{(r)} = x_1$ and $\lim_{r\to\infty} x_2^{(r)} = x_2$.

There now exists n_0 such that

$$\|x_1^{(n)} - x_2^{(n)}\| < \frac{1}{r} \quad \text{for every } n \geq n_0$$

which in turn implies

$$\|G(a,x_1^{(n)}) - G(a,x_2^{(n)})\| \leq \frac{\varepsilon}{\sqrt{3\mu(A)}}$$

$$\text{for every } n \geq n_0, \text{ since } x_1^{(n)}, x_2^{(n)} \in S$$

Taking limits as $n \to \infty$ now gives

$$\|G(a,x_1) - G(a,x_2)\| \leq \frac{\varepsilon}{\sqrt{3\mu(A)}}$$

since $G(a,.)$ is continuous on W.

As $x_1, x_2 \in W$ were arbitrary, it follows that $a \in \{\alpha \in A; \|G(\alpha,x)-G(\alpha,x')\| \leq \frac{\varepsilon}{3\mu(A)}$ for $x,x' \in W$ with $\|x-x'\| < \frac{1}{r}\}$ and as $a \in \mathcal{S}_r$ was arbitrary it follows that this set is equal to $\mathcal{S}_r$. Consequently, if $a \in A$ then by the continuity of $G(a,.)$, $a \in \mathcal{S}_r$ for some r, and therefore $\{\mathcal{S}_r\}_{r=1}^{\infty}$ is an increasing sequence of Borel sets with $\bigcup_{r=1}^{\infty} \mathcal{S}_r = A$.

Hence, $\lim_{r\to\infty} \mu(\mathcal{S}_r) = \mu(A)$, and there exists a positive integer q such that

(1) $$\mu(A) - \frac{\varepsilon^2}{12M^2} < \mu(\mathcal{S}_q) \leq \mu(A).$$

Now let $\delta = \frac{\varepsilon}{2qM\sqrt{3}}$ and $w \in \mathcal{W}$ such that $\|w - w_0\|_2 < \delta$.

Let $U = \{a \in A; \ \|w(a)-w_0(a)\| \geq \frac{1}{q}\}$.

Then

$$\frac{\mu(U)}{q^2} \leq \int_U \|w(\alpha) - w_0(\alpha)\|^2 d\alpha$$

$$\leq \int_A \|w(\alpha) - w_0(\alpha)\|^2 d\alpha < \frac{\varepsilon^2}{12M^2q^2}$$

i.e. $\mu(U) < \frac{\varepsilon^2}{12M^2}$.

Thus

$$\|P(w) - P(w_0)\|_2^2 = \int_A \|G(\alpha,w(\alpha)-G(\alpha,w_0(\alpha)\|^2 d\alpha$$

$$\leqslant \int_{\mathcal{S}_q} \|G(\alpha,w(\alpha)) - G(\alpha,w_0(\alpha))\|^2 d\alpha + 4M^2\mu(A-\mathcal{S}_q)$$

$$\leqslant \int_{\mathcal{S}_q - U} \|G(\alpha,w(\alpha))-G(\alpha,w_0(\alpha))\|^2 d\alpha + 4M^2\mu(U) + \frac{\varepsilon^2}{3}$$

(by using (1))

$$\leqslant \mu(A)\,\frac{\varepsilon^2}{3\mu(A)} + \frac{\varepsilon^2}{3} + \frac{\varepsilon^2}{3} = \varepsilon^2$$

$\Big($since $a \in \mathcal{S}_q - U \Rightarrow \|w(a) - w_0(a)\| < \frac{1}{q}$

$\Rightarrow \|G(a, w(a)) - G(a,w_0(a))\| \leqslant \frac{\varepsilon}{\sqrt{3\mu(A)}}\Big)$

and consequently

$$\|P(w) - P(w_0)\|_2 \leqslant \varepsilon.$$

Since $\varepsilon>0$ was arbitrary this proves the continuity of P. #

We now prove the continuity of the payoff function of certain (non-regular) differential games, and thence via 2.1, 2.2 and 2.3 assert the existence of x-saddle points of such games when the payoff is convex-concave.

2.5 <u>Theorem</u>: Let A,B be open convex subsets of R^p, R^q respectively, and U,V compact subsets of A,B respectively. Let $f:[0,T] \times R^n \times A \times B \to R^n$, $f_1:[0,T] \times R^m \times A \to R^m$ and $f_2:[0,T] \times R^r \times B \to R^r$ be $\mathcal{B}$-measurable functions such that

a) For each $t \in [0,T]$, $f(t,\cdot,\cdot,\cdot)$, $f_1(t,\cdot,\cdot)$, $f_2(t,\cdot,\cdot)$

have partial derivatives which are continuous and bounded uniformly with respect to $t \in [0,T]$

b) For each $x \in R^n$, $y \in R^m$, $w \in R^r$, $(u,v) \in U \times V$ the functions $f(.,x,u,v)$, $f_1(.,y,u)$, $f_2(.,w,v)$ are bounded on $[0,T]$.

Let $\mathcal{U},\mathcal{V}$ respectively denote the elements of $\mathcal{L}^2_p[0,T]$, $\mathcal{L}^2_q[0,T]$ with values lying in U,V almost everywhere on $[0,T]$, and for each $x \in R^n$, $y \in R^m$, $w \in R^r$, $(u,v) \in \mathcal{U} \times \mathcal{V}$ let

$z(x,u,v,.):[0,T]\to R^n$, $z_1(y,u,.):[0,T]\to R^m$ and $z_2(w,v,.):[0,T]\to R^r$

be the solutions of the differential equations

$$(1)\ \begin{cases} \dot z(x,u,v,t) = f(t,z(x,u,v,t),\ u(t),v(t)) & \text{a.e.}\ t \in [0,T] \\ z(x,u,v,0) = x \end{cases}$$

$$(2)\ \begin{cases} \dot z_1(y,u,t) = f_1(t,z_1(y,u,t),u(t)) & \text{a.e.}\ t \in [0,T] \\ z_1(y,u,0) = y \end{cases}$$

$$(3)\ \begin{cases} \dot z_2(w,v,t) = f_2(t,z_2(w,v,t),v(t)) & \text{a.e.}\ t \in [0,T] \\ z_2(w,v,0) = w \end{cases}$$

Let $h:R^{n+m+r} \to R$ be continuous and $g:[0,T] \times R^{n+m+r} \times U \times V \to R$ a bounded $\mathcal{B}$-measurable function such that for each $(t,w,v) \in [0,T] \times R^r \times V$ $g(t,.,.,w,.,v)$ is continuous on $R^n \times R^m \times U$, and for each $(t,y,u) \in [0,T] \times R^m \times U$ $g(t,.,y,.,u,.)$ is continuous on $R^n \times R^r \times V$. Let $J:R^{n+m+r} \times \mathcal{U} \times \mathcal{V} \to R$ be defined by

$$J(x,y,w,u,v) = h(z(x,u,v,T),z_1(y,u,T),z_2(w,v,T)) + \\ + \int_0^T g(t,z(x,u,v,t),z_1(y,u,t),z_2(w,v,t),u(t),v(t))dt$$

for $(x,y,w,u,v) \in R^{n+m+r} \times \mathcal{U} \times \mathcal{V}$

Then:

i) For each $(x,y,w) \in R^{n+m+r}$ and $u \in \mathcal{U}$, $J(x,y,w,u,.)$ is continuous on $\mathcal{V}$ (with respect to the norm of $\mathcal{L}^2_q[0,T]$) and for each $(x,y,w) \in R^{n+m+r}$ and $v \in \mathcal{V}$, $J(x,y,w,.,v)$ is continuous on $\mathcal{U}$ (with respect to the norm of $\mathcal{L}^2_p[0,T]$)

ii) If U and V are convex, $(x,y,w) \in R^{n+m+r}$ and $J(x,y,w,.,v)$ is convex on $\mathcal{U}$ for each $v \in \mathcal{V}$ and $J(x,y,w,u,.)$ is concave on $\mathcal{V}$ for each $u \in \mathcal{U}$ then there exists $(u^*,v^*) \in \mathcal{U} \times \mathcal{V}$ such that
$J(x,y,w,u^*,v) \leq J(x,y,w,u^*,v^*) \leq J(x,y,w,u,v^*)$
for every $u,v \in \mathcal{U} \times \mathcal{V}$.

<u>Proof</u>: i) Let $x,y,w \in R^{n+m+r}$ and $u \in \mathcal{U}$ be fixed.

Since h is continuous, then it is clear from the continuity of $z(x,u,.,T)$ and $z_2(w,.,T)$ on $\mathcal{V}$ (see A1.2 iii) of the Appendix) that $h(z(x,u,.,T),z_1(y,u,T), z_2(w,.,T))$ is continuous on $\mathcal{V}$.

Now define $G:[0,T] \times R^{n+r} \times V \to R$ by

$$G(t,\zeta,\zeta_1,v) = g(t,\zeta,z_1(y,u,t),\zeta_1,u(t),v)$$
$$\text{for } (t,\zeta,\zeta_1,v) \in [0,T]\times R^{n+r}\times V.$$

Then G clearly satisfies the conditions of 2.4 (with $A = [0,T]$, $W = R^{n+r}\times V$) and thus the function $P:\mathcal{L}^2_n[0,T] \times \mathcal{L}^2_r[0,T] \times \mathcal{V} \to \mathcal{L}^2[0,T]$ defined by

$$P(\zeta,\zeta_1,v)(t) = G(t,\zeta(t),\zeta_1(t),v(t))$$

$$t \in [0,T],\ \zeta \in \mathcal{L}^2_n[0,T],\ \zeta_1 \in \mathcal{L}^2_r[0,T],\ v \in \mathcal{V}$$

is continuous (by the results of 2.4).

But by A1.2 iii) of the Appendix the function $\phi : \mathcal{V} \to \mathcal{L}^2_n[0,T] \times \mathcal{L}^2_r[0,T] \times \mathcal{V}$ defined by

$$\phi(v)(t) = (z(x,u,v,t),\ z_2(w,v,t),\ v(t)) \quad \text{for} \quad v \in \mathcal{V}, t \in [0,T]$$

is continuous also, and consequently $P_0\phi : \mathcal{V} \to \mathcal{L}^2[0,T]$ is continuous. But

$$J(x,y,w,u,v) = h(z(x,u,v,T),z_1(y,u,T);z_2(w,v,T)) + \int_0^T \{[P_0\phi](v)\}(t)dt \quad \text{for} \quad v \in \mathcal{V}$$

We have already noted the continuity of the first term of the above expression in v. The continuity of the second term follows from the inequality (Schwarz' inequality)

$$\left|\int_0^T \{[P_0\phi](v_1)\}(t)dt - \int_0^T \{[P_0\phi](v_2)\}(t)dt\right|^2 \leq T\int_0^T \left|\{[P_0\phi](v_1)\}(t) - \{[P_0\phi](v_2)\}(t)\right|^2 dt \quad \text{for} \quad v_1, v_2 \in \mathcal{V}$$

and the continuity of $P_0\phi$.

Thus $J(x,y,w,u,.)$ is continuous on $\mathcal{V}$ with respect to the $\mathcal{L}^2_q[0,T]$ norm. In a similar way $J(x,y,w,.,v)$ is continuous on $\mathcal{U}$ with respect to the $\mathcal{L}^2_p[0,T]$ norm, for each $v \in \mathcal{V}$. This proves i).

ii) By 2.3 $\mathcal{U},\mathcal{V}$ are bounded, closed convex sets when U and V are convex. The result ii) then follows from i) and 2.2. (Since continuity implies semicontinuity.) #

As particular examples of games to which the above theorem applies we give the following

<u>2.6 Example</u> A,B,U,V,$\mathcal{U},\mathcal{V}$ are as in 2.5 with U,V convex. C,D,H,F are continuous $m\times m, m\times p, r\times r$ and $r\times q$ matrix functions respectively on $[0,T]$, and for each $x \in R^m$, $y \in R^r$, $u \in \mathcal{U}$, $v \in \mathcal{V}$ the functions $z_1(x,u,.):[0,T] \to R^m$, $z_2(y,v,.):[0,T] \to R^r$ are the respective solutions of the differential equations

$$\begin{cases} \dot{z}_1(x,u,t) = C(t)z_1(x,u,t)+D(t)u(t) & \text{a.e.} t \in [0,T] \\ z_1(x,u,0) = x \end{cases}$$

$$\begin{cases} \dot{z}_2(y,v,t) = H(t)z_2(y,v,t)+F(t)v(t) & \text{a.e.} t \in [0,T] \\ z_2(y,v,0) = y \end{cases}$$

$h:R^m \times R^r \to R$ is a continuous function such that $h(.,\zeta_2)$ is convex on R^m for each $\zeta_2 \in R^r$ and $h(\zeta_1,.)$ is concave on R^r for each $\zeta_1 \in R^m$. $g:[0,T]\times R^m\times A\times R^r\times B\to R$ is a bounded function such that for each $(t,y,v) \in [0,T] \times R^r \times V, g(.,.,.,y,v)$ is continuous on $[0,T] \times R^m \times U$, and $g(t,.,.,y,v)$ is convex on $R^m \times U$, and for each $(t,x,u) \in [0,T] \times R^m \times U$, $g(.,x,u,.,.)$ is dontinuous on $[0,T] \times R^r \times V$, and $g(t,x,u,.,.)$ is concave on $R^r \times V$ (under these circumstances it can be shown that g is $\mathcal{B}$-measurable). The payoff $J: R^{m+r} \times \mathcal{U} \times \mathcal{V} \to R$ is defined by

$$J(x,y,u,v) = h(z_1(x,u,T), z_2(y,v,T)) + \int_0^T g(t, z_1(x,u,t), u(t), z_2(x,v,t), v(t))dt$$
$$\text{for } (x,y) \in R^{m+r}, (u,v) \in \mathcal{U} \times \mathcal{V}.$$

This example satisfies the hypotheses of 2.5 ii) and consequently, for each $(x,y) \in R^{m+r}$, $-J(x,y,\cdot,\cdot)$ has a saddle point. The boundedness assumption on g can in fact be dropped. The other properties of g are strong enough to imply that for any compact subsets S_1, S_2 of R^m, R^r respectively, g is bounded on $[0,T] \times S_1 \times S_2 \times U \times V$. Since the trajectories z_1, z_2 lie in compact sets for any fixed initial state (see A1.2 ii) of the Appendix) it is not difficult to extend 2.5 to cover this case. #

2.7 <u>Example</u>: U, V are compact, convex subsets of R^p, R^q respectively, and $\mathcal{U}, \mathcal{V}$ are defined as in 2.5. A,B,C,Q,S are continuous $n \times n$, $n \times p$, $n \times q$, $p \times p$ and $q \times q$ matrix functions respectively on $[0,T]$, and for each $u \in \mathcal{U}$, $v \in \mathcal{V}$, $x \in R^n$, $z(x,u,v,\cdot):[0,T] \to R^n$ is the solution of the differential equation

$$\begin{cases} \dot{z}(x,u,v,t) = A(t)z(x,u,v,t) + B(t)u(t) + C(t)v(t) \\ z(x,u,v,0) = x. \end{cases}$$

G is an $n \times n$ matrix and the payoff $J: R^n \times \mathcal{U} \times \mathcal{V} \to R$ is defined by

$$J(x,u,v) = z(x,u,v,T)'G^2 z(x,u,v,T) + \int_0^T \{u(t)'Q(t)u(t)+v(t)'S(t)v(t)\}dt$$

for $(x,u,v) \in R^n \times \mathcal{U} \times \mathcal{V}$.

This example satisfies the conditions of 2.5 i), and if Q is positive definite on $[0,T]$ then $J(x,.,v)$ is convex on $\mathcal{U}$ for each $x \in R^n$, $v \in \mathcal{V}$ (see 4-2.1).

If S is negative definite on $[0,T]$ and certain other conditions (given in 4-2.1) are satisfied, then $J(x,u,.)$ is also concave on $\mathcal{V}$ for each $x \in R^n$, $u \in \mathcal{U}$. In this case the function $-J(x,.,.)$ will have a saddle point. #

Finally we note that if U,V are compact, and $((f,g,h,U,V,\theta),\mathcal{U},\mathcal{V})$ is a regular blind differential game, then f,g,h,U,V satisfy the conditions of the first part of 2.5 and consequently its payoff will be continuous in each of its function arguments for any fixed value of the other.

3. <u>A Sufficient Condition for Optimality of Open-Loop Strategies</u>

We now describe a procedure for determining whether an open-loop strategy pair constitutes an x-saddle point of a differential game. The method consists of solving the pair of optimal control problems obtained by fixing the strategy of each of the players in turn at its postulated optimal value, and then optimising the payoff for the player whose strategy is not fixed.

Consider a regular blind differential game of prescribed duration T with strategy sets $\mathcal{U},\mathcal{V}$ and payoff J. Suppose u_0, v_0 are open-loop strategies in the game which are believed to be x-optimal (u_0, v_0 may, for instance, have been generated by optimal closed-loop strategies along a path through x). By standard methods of control theory the maximum $\overline{\nu}$ of $J(x,u_0,.)$ and the minimum $\underline{\nu}$ of $J(x,.,v_0)$ may be obtained. If $\overline{\nu} = \underline{\nu}$ then $J(x,u_0,v_0) = \overline{\nu} = \underline{\nu}$ is the x-value of the game and u_0, v_0 are x-optimal strategies. This procedure may be formulated as the following sufficiency theorem.

3.1 <u>Theorem</u>: Let U,V be compact subsets of R^p, R^q respectively and $D = ((f,g,h,U,V,\theta),\mathcal{U},\mathcal{V})$ a regular n-dimensional blind differential game of prescribed duration T, with payoff function J, and trajectory function z. Let $(u_0,v_0) \in \mathcal{U} \times \mathcal{V}$, and $x_0 \in R^n$, and suppose there exist functions $V^1: R^n \times [0,T] \to R$, $V^2: R^n \times [0,T] \to R$, $\varphi: R^n \times [0,T] \to U$ and $\psi: R^n \times [0,T] \to V$ with the following properties:

i) V^1, V^2 are continuous on $R^n \times [0,T]$ and have continuous partial derivatives on $R^n \times (0,T)$, and $V^1(x,T) = V^2(x,T) = h(T,x)$ for $x \in R^n$.

ii) $u_0(t) = \varphi(z(x_0,u_0,v_0,t),t)$

$v_0(t) = \psi(z(x_0,u_0,v_0,t),t)$ for $t \in [0,T]$

iii) $$0 = \sum_{i=1}^{n} V^1_{,i}(x,t)f^i(x,\varphi(x,t),v_0(t)) + V_{,n+1}(x,t)+g(x,\varphi(x,t),v_0(t))$$

$$= \inf_{u\in U}\{\sum_{i=1}^{n} V^1_{,i}(x,t)f^i(x,u,v_0(t)) + V_{,n+1}(x,t)+g(x,u,v_0(t))\}$$

$$= \sum_{i=1}^{n} V^2_{,i}(x,t)f^i(x,u_0(t),\psi(x,t)) + V_{,n+1}(x,t)+g(x,u_0(t),\psi(x,t))$$

$$= \sup_{v\in V}\{\sum_{i=1}^{n} V^2_{,i}(x,t)f^i(x,u_0(t),v) + V_{,n+1}(x,t)+g(x,u_0(t),v)\}$$

for all $(x,t) \in R^n \times (0,T)$.

Then (u_0,v_0) is an x_0-saddle point of D, and $J(x_0,u_0,v_0) = V^1(x_0,0) = V^2(x_0,0)$.

__Proof__: Substituting $x = z(x_0,u_0,v_0,t)$ in the first equation in iii), and using the first identity of ii) gives

$$0 = \sum_{i=1}^{n} V^1_{,i}(z(x_0,u_0,v_0,t),t)f^i(z(x_0,u_0,v_0,t),u_0(t),v_0(t)) + V^1_{,n+1}(z(x_0,u_0,v_0,t),t) + g(z(x_0,u_0,v_0,t),u_0(t),v_0(t))$$

$$= \sum_{i=1}^{n} V^1_{,i}(z(x_0,u_0,v_0,t),t)\dot{z}(x_0,u_0,v_0,t) + V^1_{,n+1}(z(x_0,u_0,v_0,t),t) + g(z(x_0,u_0,v_0,t),u_0(t),v_0(t)).$$

For any $\varepsilon \in (0,\frac{1}{2}T)$, integration of the above equation over $[\varepsilon,T-\varepsilon]$ gives

$$0 = V^1(z(x_0,u_0,v_0,T-\varepsilon),T-\varepsilon)-V^1(z(x_0,u_0,v_0,\varepsilon),\varepsilon) + \int_{\varepsilon}^{T-\varepsilon} g(z(x_0,u_0,v_0,t),u_0(t)v_0(t))dt$$

and taking limits as $\varepsilon \to 0^+$ we obtain

$$0 = V^1(z(x_0,u_0,v_0,T),T)-V^1(z(x_0,u_0,v_0,0),0)+\int_0^T g(z(x_0,u_0,v_0,t),u_0(t),v_0(t))dt$$

$$= h(T,z(x_0,u_0,v_0,T))-V^1(x_0,0)+\int_0^T g(z(x_0,u_0,v_0,t),u_0(t),v_0(t))dt$$

$$= J(x,u_0,v_0) - V^1(x_0,0) \quad \text{(where we have used i)).}$$

If now $u \in \mathcal{U}$, then substituting $x = z(x_0,u,v_0,t)$ in the second equation of iii) gives

$$0 \leq \sum_{i=1}^{n} V^1_{,i}(z(x_0,u,v_0,t),t)f^i(z(x_0,u,v_0,t),u(t),v_0(t))+ V^1_{,n+1}(z(x_0,u,v_0,t),t) +g(z(x_0,u,v_0,t),u(t),v_0(t)) \quad \text{for } t \in (0,T).$$

Integrating this inequality as above gives

$$0 \leq h(T,z(x_0,u,v_0,T))-V^1(x_0,0)+\int_0^T g(z(x_0,u,v_0,t),u(t),v_0(t))dt$$

$$= J(x,u,v_0) - V^1(x_0,0).$$

Therefore we have

(1) $V^1(x_0,0) = J(x_0,u_0,v_0) \leq J(x_0,u,v_0)$ for any $u \in \mathcal{U}$.

In a similar way from the third and fourth equations in iii) we obtain

(2) $J(x_o, u_o, v) \leq J(x_o, u_o, v_o) = V^2(x_o, 0)$ for any $v \in \mathcal{V}$

(1) and (2) imply that (u_o, v_o) is an x_o-saddle point of D and $J(x_o, u_o, v_o) = V^1(x_o, 0) = V^2(x_o, 0)$ which proves the theorem. #

It should be possible to devise theorems similar to the above for games which are not of prescribed duration. However, in such cases, the possibility of the games not terminating will greatly complicate the formulation and proof of these theorems.

When the above procedure is applied to certain games with dynamics of the type $(f_1+f_2, g_1+g_2, h, U, V, \theta)$, where f_1, g_1 are independent of $v \in V$ and f_2, g_2 are independent of $u \in U$, then it is found that the optimal closed-loop strategies generate open-loop x-saddle points, provided that the initial state x is sufficiently close to the target set θ and that this target set is not deficient in dimension (i.e. θ has dimension n). It would be interesting to know if this is generally true for all such games.

CHAPTER 4

QUADRATIC GAMES

1. Quadratic Games on Hilbert Space

Schmitendorf [21] seems to have been the first to obtain valid necessary and sufficient conditions for linear-quadratic differential games to have open-loop solutions although the problem had been studied earlier by Rhodes and Luenberger [15]. Rekasius [24] has also considered the problem and obtained similar results. The conditions given also provide a means for obtaining the open-loop solutions in the form of a matrix Ricatti equation. By considering the payoff of such a game as a quadratic function on a Hilbert space we obtain necessary and sufficient conditions for the existence of open-loop solutions to the game which have the form of a system of linear algebraic equations. The coefficients in these equations are obtained by solving a linear matrix differential equation and consequently it might be expected that these conditions may be more easily solvable in practice than those involving matrix Ricatti equations. The conditions obtained are also not as stringent as those of Schmitendorf. This seems to be due to the fact that Schmitendorf required his open-loop strategy solutions to generate paths which satisfied a regularity condition while the conditions we give apply to arbitrary square integrable functions.

The approach adopted here seems to have occurred to several workers within a short space of time. Since this work was done, Auslender [23] and Lukes and Russell [37] have published results of a more general nature, which were obtained by a similar approach.

All our results are based on the following general theorem.

1.1 <u>Theorem</u>: Let X,Y be real linear spaces and Q,S real quadratic forms on X,Y respectively, with corresponding polar bilinear forms $\overline{Q},\overline{S}$. Let M be a real bilinear form on $X \times Y$ and $a \in X^{\#}$, $b \in Y^{\#}$, and define the real valued function P on $X \times Y$ by

$$P(u,v) = Q(u) + S(v) + M(u,v) + a(u) + b(v) \quad \text{for } u \in X,\ v \in Y.$$

Then necessary and sufficient conditions that $\alpha \in X$ and $\beta \in Y$ and the non negative real number ε satisfy

$$(1) \quad \begin{cases} P(\alpha,v) \leq P(\alpha,\beta) + \varepsilon & \text{for all } v \in Y \\ P(\alpha,\beta) - \varepsilon \leq P(u,\beta) & \text{for all } u \in X \end{cases}$$

are that Q be non negative, S non positive and

$$(2) \quad \begin{cases} |2\overline{Q}(x,\alpha)+M(x,\beta)+a(x)| \leq 2\sqrt{\varepsilon Q(x)} & \text{for all } x \in X \\ |M(\alpha,y)+2\overline{S}(y,\beta)+b(y)| \leq 2\sqrt{-\varepsilon S(y)} & \text{for all } y \in Y \end{cases}$$

<u>Proof</u>: Using the linearity of $\overline{Q},\overline{S},M$ and a,b we have for any $v \in Y$

(3) $P(\alpha,v) - P(\alpha,\beta) = \overline{S}(v,v)-\overline{S}(\beta,\beta)+M(\alpha,v-\beta)+b(v-\beta)$

$= \overline{S}(v,v)-2\overline{S}(v,\beta)+\overline{S}(\beta,\beta)+2\overline{S}(v-\beta,\beta)+M(\alpha,v-\beta)+b(v-\beta)$

$= S(v-\beta)+2\overline{S}(v-\beta,\beta)+M(\alpha,v-\beta)+b(v-\beta)$

and for any $u \in X$

(4) $P(u,\beta)-P(\alpha,\beta)=Q(u-\alpha)+2\overline{Q}(u-\alpha,\alpha)+M(u-\alpha,\beta)+a(u-\alpha).$

Suppose now that Q and $-S$ are non-negative, and that condition (2) is satisfied. Then from (2) and (3) we have

$$P(\alpha,v) - P(\alpha,\beta) \leqslant S(v-\beta) + 2\sqrt{-\varepsilon S(v-\beta)}$$

$$= -[\sqrt{-S(v-\beta)} - \sqrt{\varepsilon}]^2 + \varepsilon$$

$$\leqslant \varepsilon \quad \text{for any } v \in Y.$$

And in a similar way, (2) and (4) give

$$-\varepsilon \leqslant P(u,\beta) - P(\alpha,\beta) \quad \text{for any } u \in X.$$

Thus the saddle point condition (1) is satisfied.

Conversely if we suppose that (1) is satisfied, then for any real number t and any $y \in Y$ we obtain from (3)

$$S(ty) + 2\overline{S}(ty,\beta) + M(\alpha,ty) + b(ty) \leqslant \varepsilon$$

by substituting $v = \beta + ty$.

Thus

(5) $t^2S(y) + t[2\overline{S}(y,\beta) + M(\alpha,y) + b(y)] \leqslant \varepsilon$

for every real t and every $y \in Y$.

It follows that $S(y) \leqslant 0$ for all $y \in Y$.

Further, if $S(y) = 0$, then (5) gives

$$|2\overline{S}(y,\beta) + M(\alpha,y) + b(y)| = 0 \leqslant 2\varepsilon^{\frac{1}{2}}\sqrt{-S(y)}$$

and if $S(y) \neq 0$, then substitution of

$t = - \dfrac{[2\overline{S}(y,\beta)+M(\alpha,y)+b(y)]}{2S(y)}$ into (5) gives

$$\frac{(2\overline{S}(y,\beta)+M(\alpha,y)+b(y))^2}{-4S(y)} \leqslant \varepsilon$$

and therefore

$$|2\overline{S}(y,\beta) + M(\alpha,y) + b(y)| \leqslant 2\sqrt{-\varepsilon S(y)} \ .$$

Consequently the first part of condition (2) holds.

In a similar way (4) implies that Q is non-negative, and

$$|2\overline{Q}(x,\alpha) + M(x,\beta) + a(x)| \leqslant 2\sqrt{\varepsilon Q(x)} \quad \text{for} \quad x \in X.$$

Thus (2) holds and the theorem is proved. #

As a particular case we have the following

1.2 Corollary: Let $X,Y,Q,S,\overline{Q},\overline{S},a,b$ and P be defined as in 2.1, then a necessary and sufficient condition that $\alpha \in X$ and $\beta \in Y$ satisfy the saddle point condition

$$P(\alpha,v) \leqslant P(\alpha,\beta) \leqslant P(u,\beta) \quad \text{for all} \quad u \in X,\ v \in Y$$

is that Q be non negative, S non positive, and

$$2\overline{Q}(x,\alpha) + M(x,\beta) + a(x) = 0 \quad \text{for all} \quad x \in X$$

$$M(\alpha,y) + 2\overline{S}(y,\beta) + b(y) = 0 \quad \text{for all} \quad y \in Y.$$

Proof: Put $\varepsilon=0$ in 2.1. #

In the case when the linear spaces X,Y of the above theorems are real Hilbert spaces and Q,S,M,a,b are continuous we obtain:

1.3 Corollary: Let X,Y be real Hilbert spaces with inner products $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ respectively.

Let $Q:X \to X$, $S:Y \to Y$ and $M:Y \to X$ be bounded linear operators, $a \in X$, $b \in Y$ and define the function $P:X \times Y \to R$ by

$$P(u,v) = (u,Qu) + \langle v,Sv\rangle + (u,Mv) + (a,u) + \langle b,v\rangle$$
$$\text{for } u \in X,\ v \in Y.$$

Then necessary and sufficient conditions that $\alpha \in X$ and $\beta \in Y$ satisfy

$$P(\alpha,v) \leq P(\alpha,\beta) \leq P(u,\beta) \quad \text{for all } u \in X,\ v \in Y$$

are that Q^*+Q be non negative, S^*+S non positive, and

$$(Q^* + Q)\alpha + M\beta + a = 0$$
$$M^*\alpha + (S^* + S)\beta + b = 0$$

<u>Proof</u>: This result follows immediately from 2.2 by using the facts that

$$(u,(Q+Q^*)u) = 2(u,Qu),\ \langle v,(S+S^*)v\rangle = 2\langle v,Sv\rangle$$

and $(u,Mv) = \langle M^*u,v\rangle$ for every $u \in X$, $v \in Y$ and that if $z \in X$, $w \in Y$ satisfy $\langle y,w\rangle = (x,z) = 0$ for every $x \in X$, $y \in Y$, then $z = w = 0$. #

Although the above corollary gives necessary and sufficient conditions for the existence of a (saddle point) solution to a quadratic game, these conditions are not necessary for the game to have a value since it may have an ε-saddle point for every $\varepsilon>0$ yet not necessarily possess a saddle point. We now establish a condition under which a quadratic game has no value. In the case of games with finite dimensional strategy spaces, this condition is satisfied if and only if the game has no saddle point solution as given by 2.3. For such a game $(X,Y,-P)$ at least one of the following conditions must hold. Either

a) The game $(X,Y,-P)$ has a saddle point in pure strategies.

b) $\sup_{v\in Y} P(u,v) = \infty$ for every $u \in X$

or c) $\inf_{u\in X} P(u,v) = -\infty$ for every $v \in Y$.

(Note that both b) and c) may hold simultaneously).

1.4 Theorem: Let X,Y be real Hilbert spaces with inner products $(\cdot,\cdot)$ and $<\cdot,\cdot>$ respectively, and let $\mathcal{H} = X \times Y$ be the product Hilbert space with inner product $[\cdot,\cdot]$ (defined by $[\{x_1,y_1\},\{x_2,y_2\}] = (x_1,x_2) + <y_1,y_2>$ for $\{x_1,y_1\},\{x_2,y_2\} \in \mathcal{H}$). Let $Q:X \to X$, $S:Y \to Y$ and $M:Y \to X$ be bounded linear operators, $a \in X$, $b \in Y$, and define the linear operator $T:\mathcal{H} \to \mathcal{H}$ and function $P:\mathcal{H} \to R$ by $T\{u,v\} = \{(Q^*+Q)u+Mv, M^*u+(S^*+S)v\}$ $\{u,v\} \in \mathcal{H}$

$$P(u,v) = (u,Qu) + <v,Sv> + (u,Mv) + (a,u) + <v,b>.$$

Suppose further that $\{a,b\} \notin \overline{\mathcal{R}(T)}$. Then either:

(1) $\sup_{v\in Y} P(u,v) = \infty$ for every $u \in X$

or (2) $\inf_{u\in X} P(u,v) = -\infty$ for every $v \in Y$.

Proof: Note that $T:\mathcal{H} \to \mathcal{H}$ is a bounded, self-adjoint linear operator on $\mathcal{H}$, and consequently

$$\mathcal{R}(T)^{\perp} = \overline{\mathcal{R}(T)}^{\perp} = \ker(T).$$

If Q^*+Q is not non negative then the theorem is trivially true. For in this case there exists $u_0 \in X$ such that $(u_0(Q+Q^*)u_0) < 0$. It then follows that $\lim_{\alpha\to\infty} P(\alpha\, u_0, v) = -\infty$ for any $v \in Y$ and thus (2) holds.

Similarly if S^*+S is not non positive then (1) holds.

Suppose then that Q^*+Q is non-negative and S^*+S non-positive and let $\{a,b\} \notin \overline{\mathcal{R}(T)}$.

If $\{x_0, y_0\}$ is the perpendicular projection of $\{a,b\}$ on $\ker(T) = \overline{\mathcal{R}(T)}^{\perp}$ then it follows that $\{x_0, y_0\} \neq 0$ and thus

(3) $[\{x_0,y_0\},\{a,b\}] = [\{x_0,y_0\},\{x_0,y_0\}] \neq 0$

But $T\{x_0,y_0\} = 0 = \{0,0\}$

and therefore

(4) $$(Q^* + Q)x_0 + My_0 = 0$$

(5) $$M^*x_0 + (S^* + S)y_0 = 0$$

taking inner products of x_0 with (4) and subtracting from it the inner product of y_0 with (5) gives

$$(x_0,(Q^* + Q)x_0) - \langle y_0,(S^* + S)y_0\rangle = 0$$

and since $Q^* + Q$, $-(S^* + S)$ are non negative it follows that

$$(x_0,(Q^* + Q)x_0) = \langle y_0,(S^* + S)y_0\rangle = 0.$$

Consequently

$$(Q^* + Q)x_0 = 0 \quad \text{and} \quad (S^* + S)y_0 = 0$$

(since $Q^* + Q$ and $S^* + S$ are semidefinite)

and then from (4) and (5)

$$My_0 = 0 \quad \text{and} \quad M^*x_0 = 0.$$

(6) Thus $P(\alpha\, x_0, v) = \langle v, Sv\rangle + \alpha(a, x_0) + \langle b, v\rangle$ for $v \in Y,\ \alpha \in R$

(7) and $P(u, \beta y_0) = (u, Qu) + (a, u) + \beta(b, y_0)$ for $u \in X,\ \beta \in R.$

However, from (3)

$$[\{x_0, y_0\}, \{a, b\}] = (x_0, a) + \langle y_0, b\rangle \neq 0$$

and thus either $(x_0, a) \neq 0$, in which case (6) gives $\lim_{r\to\infty} P(-r(a, x_0)x_0, v) = -\infty$ for every $v \in Y$ and (2) holds, or otherwise $\langle y_0, b\rangle \neq 0$, in which case (7) gives $\lim_{r\to\infty} P(u, r\langle b, y_0\rangle y_0) = \infty$ for every $u \in X$ and (1) holds. #

If the mapping T of the above theorem has a closed range (hence if, in particular, the spaces X, Y are finite dimensional) then the theorem implies that $(X, Y, -P)$ must satisfy one of the conditions a), b), c) given immediately preceding it. However if X and Y are infinite dimensional then it is possible that $\mathcal{R}(T)$ is not closed, and consequently that $\{a, b\} \in \overline{\mathcal{R}(T)} - \mathcal{R}(T)$. If $\{a, b\} \in \mathcal{R}(T)$ then 1.3 assures us that the game with payoff $-P$ has a saddle point, whilst if $\{a, b\} \in H - \overline{\mathcal{R}(T)}$ then 1.4 implies that one player of the game can make the payoff arbitrarily favourable for any given pure strategy of the other. But neither 1.3 nor 1.4 covers the case when $\{a,b\} \in \overline{\mathcal{R}(T)} - \mathcal{R}(T)$. Examples of quadratic games with payoff $-P$ can be found for which

$$-\infty < \sup_{v \in Y} \inf_{u \in X} P(u,v) < \inf_{u \in X} \sup_{v \in Y} P(u,v) < \infty$$

and consequently do not satisfy any of the conditions a) b) or c). Presumably for some $\{a,b\} \in \overline{\mathcal{R}(T)} - \mathcal{R}(T)$ the game with payoff $-P$ will have an ε - saddle point for every $\varepsilon > 0$, but conditions under which this is the case seem difficult to obtain.

2. Quadratic Differential Games

We now apply the results of the last section to linear quadratic differential games of prescribed duration T.

2.1 Theorem: Let A,B,C,Q,S be continuous $n \times n, n \times p, n \times q, p \times p$ and $q \times q$ matrix functions respectively on $[0,T]$ with Q positive definite and S negative definite. Let $x_0 \in R^n$, $\mathcal{U} = \mathcal{L}_p^2[0,T]$ and $\mathcal{V} = \mathcal{L}_q^2[0,T]$ and for each $u \in \mathcal{U}$, $v \in \mathcal{V}$ let $z(u,v,.) : [0,T] \to R^n$ be the solution of the differential equation

$$(1) \quad \begin{cases} \dot{z}(u,v,t) = A(t)z(t) + B(t)u(t) + C(t)v(t) & \text{a.e. } t \in [0,T] \\ z(u,v,0) = x_0 \end{cases}$$

Let G be an $n \times n$ symmetric matrix and $J : \mathcal{U} \times \mathcal{V} \to R$ be defined by

(2) $$J(u,v)=z(u,v,T)'G^2z(u,v,T)+\int_0^T\{u(t)'Q(t)u(t) + v(t)'S(t)v(t)\}dt$$

for $u \in \mathcal{U}$, $v \in \mathcal{V}$.

Let M denote the fundamental matrix of (1) (so that M satisfies the differential equation $\dot{M}(t) = -M(t)A(t)$, $M(0) = I$) and put

$$\Phi(t) = B(t)'M(t)'(M(T)^{-1})'G$$

$$\Psi(t) = C(t)'M(t)'(M(T)^{-1})'G \quad \text{for} \quad t \in [0,T]$$

$$L = \int_0^T \Phi(t)'Q(t)^{-1}\Phi(t)dt$$

$$P = \int_0^T \Psi(t)'S(t)^{-1}\Psi(t)dt$$

Then a necessary and sufficient condition that $u_0 \in \mathcal{U}$, $v_0 \in \mathcal{V}$ satisfy the inequalities

(3) $J(u_0,v) \leqslant J(u_0,v_0) \leqslant J(u,v_0)$ for every $u \in \mathcal{U}$, $v \in \mathcal{V}$ is that:

i) $I+P$ be non negative

ii) $u_0(t) = Q(t)^{-1}\Phi(t)k$

$v_0(t) = S(t)^{-1}\Psi(t)\ell$ for a.e. $t \in [0,T]$

where $k,\ell \in R^n$ satisfy the equations

$$L\{(I+L)k + P\ell + GM(T)^{-1}x_0\} = 0$$

$$P\{Lk + (I+P)\ell + GM(T)^{-1}x_0\} = 0$$

Proof: Writing the solution to (1) as

$$z(u,v,t) = M(t)^{-1}x_0+M(t)^{-1}\int_0^t \{M(s)B(s)u(s)+M(s)C(s)v(s)\}ds$$

and substituting into the expression for J we obtain

$$z(u,v,T)'G^2z(u,v,T) = x_0'(M(T)^{-1})'G^2M(T)^{-1}x_0 +$$

$$+ \int_0^T\int_0^T u(s)'B(s)'M(s)'(M(T)^{-1})'G^2M(T)^{-1}M(t)B(t)u(t)dsdt$$

$$+ \int_0^T\int_0^T v(s)'C(s)'M(s)'(M(T)^{-1})'G^2M(T)^{-1}M(t)C(t)v(t)dsdt$$

$$+ 2\int_0^T\int_0^T u(s)'B(s)'M(s)'(M(T)^{-1})'G^2M(T)^{-1}M(t)C(t)v(t)dsdt$$

$$+ 2\int_0^T x_0'(M(T)^{-1})'G^2M(t)B(t)u(t)dt$$

$$+ 2\int_0^T x_0'(M(T)^{-1})'G^2M(T)^{-1}M(t)C(t)v(t)dt.$$

Consequently the game is a quadratic game of the type considered in the last section with the payoff being given by

$$J(u,v) = (u,Q_0u)_p + (v,S_0v)_q + (u,M_0v)_p + (a,u)_p + (b,v)_q + c \quad \text{for } u \in \mathcal{U},\ v \in \mathcal{V}$$

where $c = x_0'M(T)'G^2M(T)x_0$ is a constant and

$$(Q_0u)(t) = Q(t)u(t) + \int_0^T \Phi(t)\Phi(s)'u(s)ds$$

$$(S_0v)(t) = S(t)v(t) + \int_0^T \Psi(t)\Psi(s)'v(s)ds$$

$$(M_0v)(t) = 2\int_0^T \Phi(t)\Psi(s)'v(s)ds$$

$$a(t) = 2\Phi(t)GM(T)^{-1}x_0$$

$$b(t) = 2\Psi(t)GM(T)^{-1}x_0$$

Q_0 is clearly non negative, so that by 1.3, $(u_0, v_0) \in \mathcal{U} \times \mathcal{V}$ will satisfy (3) if and only if S_0 is non positive and

$$(4) \quad \begin{cases} 2Q_0u_0 + M_0v_0 = -a \\ M_0^* + 2S_0v_0 = -b \end{cases}$$

Substitution in (4) of the above expressions for Q_0, M_0, S_0 gives the equivalent equations

$$(5) \quad \begin{cases} Q(t)u_0(t) + \int_0^T \Phi(t)\Phi(s)'u_0(s)ds + \int_0^T \Phi(t)\Psi(s)'v_0(s)ds = -\Phi(t)GM(T)^{-1}x_0 \\ \int_0^T \Psi(t)\Phi(s)'u_0(s)ds + \int_0^T \Psi(t)\Psi(s)'v_0(s)ds + S(t)v_0(t) = -\Psi(t)GM(T)^{-1}x_0 \end{cases}$$

for a.e. $t \in [0,T]$

Thus if $(u_0, v_0) \in \mathcal{U} \times \mathcal{V}$ satisfies (3) then since Q, S are positive and negative definite respectively, (5) gives

$$(6) \quad \begin{cases} u_0(t) = Q(t)^{-1}\Phi(t)k \\ v_0(t) = S(t)^{-1}\Psi(t)\ell \end{cases}$$

where $\ell = k = -\{\int_0^T \Phi(s)'u_0(s)ds + \int_0^T \Psi(s)'v_0(s)ds + GM(T)^{-1}x_0\} \in R^n$

Substitution of these expressions for u_0, v_0 back into (5) gives

$$\Phi(t)\{(I+\int_0^T\Phi(s)'Q(s)^{-1}\Phi(s)ds)k + \int_0^T\Psi(s)'S(s)^{-1}\Psi(s)ds\ell+GM(T)^{-1}x_0\} = 0$$

$$\Psi(t)\{(I+\int_0^T\Psi(s)'S(s)^{-1}\Psi(s)ds)\ell + \int_0^T\Phi(s)'Q(s)^{-1}\Phi(s)dsk+GM(T)^{-1}x_0\} = 0$$

for a.e. $t \in [0,T]$.

That is

$$(7) \quad \begin{cases} \Phi(t)\{(I+L)k+P\ell+GM(T)^{-1}x_0\} = 0 \\ \Psi(t)\{Lk+(I+P)\ell+GM(T)^{-1}x_0\} = 0 \end{cases}$$

for all $t \in [0,T]$ (since Φ,Ψ are continuous).

Now if $x \in R^n$ satisfies $\Phi(t)x = 0$ for all $t \in [0,T]$ then

$$Lx = \int_0^T\Phi(t)'Q(t)^{-1}\Phi(t)xdt = 0$$

and conversely, if $Lx = 0$, then

$$0 = x'Lx = \int_0^T(\Phi(t)x)'Q(t)^{-1}\Phi(t)x\ dt$$

and hence $(\Phi(t)x)'Q(t)^{-1}\Phi(t)x = 0$ for all $t \in [0,T]$ (since the integrand of the above expression is continuous and non negative).

But since $Q(t)^{-1}$ is definite this implies that

$$\Phi(t)x = 0 \quad \text{for all} \quad t \in [0,T].$$

Thus $\Phi(t)x = 0$ for all $t \in [0,T]$ if and only if

$Lx = 0$, and in the same way $\Psi(t)y = 0$ for all $t \in [0,T]$ if and only if $Py = 0$.

Consequently (7) is equivalent to

$$(8) \quad \begin{cases} L\{(I+L)k+P\ell+GM(T)^{-1}x_0\} = 0 \\ P\{Lk+(I+P)\ell+GM(T)^{-1}x_0\} = 0 \end{cases}$$

which establishes the necessity of condition ii) in the theorem. Conversely, if S_0 is non positive and condition ii) holds, then by the equivalence of (8) and (7), k and ℓ satisfy (7). Substitution of the expressions (6) for u_0, v_0 into the left hand side of (5) then show that (5) is satisfied and hence that u_0, v_0 satisfy (4).

Thus the theorem will be proved if we can demonstrate the equivalence of the conditions that S_0 is non positive and that $(I+P)$ is non negative. This we do by showing that the following conditions are equivalent

(A) $I+P$ is non negative

(B) S_0 is non positive

(C) $P(I+P)$ is non positive.

We shall show (A) => (B) => (C) => (A).

Suppose first that $I+P$ is non negative, let $v \in \mathcal{V}$ and put

$$(9) \quad \alpha(t) = (S_0 v)(t) = S(t)v(t)+\int_0^T \Psi(t)\Psi(s)'v(s)ds \quad \text{for } t \in [0,T].$$

Then $v(t) = S(t)^{-1}\Psi(t)k + S(t)^{-1}\alpha(t)$

where $k = \int_0^T \Psi(s)'v(s)ds \in R^n$.

Substitution for v in (9) gives

$$(10)\quad \Psi(t)\{(I+P)k+\int_0^T \Psi(s)'S(s)^{-1}\alpha(s)ds\} = 0 \quad \text{for} \quad t \in [0,T]$$

and consequently (from the comments preceding (8))

$$(11)\quad P\{(I+P)k + \bar{a}\} = 0$$

where $\bar{a} = \int_0^T \Psi(s)'S(s)^{-1}\alpha(s)ds.$

Let $k = k_1+k_2$, $\bar{a} = \bar{a}_1+\bar{a}_2$

where $k_1, \bar{a}_1 \in (\ker P)^\perp$ and $k_2, \bar{a}_2 \in \ker P$.

Then $Pk_2 = 0$ and therefore (again using the comments preceding (8)) $\Psi(t)k_2 = 0$ for all $t \in [0,T]$

and $v(t) = S(t)^{-1}\Psi(t)k_1 + S(t)^{-1}\alpha(t)$.

Thus

$$(12)\quad (v,S_0v)_q = \int_0^T v(t)'\alpha(t)dt \quad \text{(using (9))}$$

$$= k_1'\bar{a} + \int_0^T \alpha(t)'S(t)^{-1}\alpha(t)dt$$

$$\leqslant k_1'\bar{a} = k_1'\bar{a}_1$$

(since $S(t)^{-1}$ is negative definite).

But from (11)

$$P(I+P)k + P\bar{a} = 0$$

$$= P(I+P)k_1 + P\bar{a}_1$$

and thence $(I+P)k_1 + \bar{a}_1 = 0$

(since P is symmetric, and $\bar{a}_1, (I+P)k_1 \in \ker(P)^\perp = \mathcal{R}(P)$)

(13) and $k_1'\bar{a}_1 = -k_1'(I+P)k_1 \leqslant 0$ (by hypothesis).

Now from (12) and (13) it immediately follows that S_0 is non positive (since v was chosen arbitrarily).

Thus (A) => (B).

Now suppose that S_0 is non positive, and let $x \in R^n$, $v(t) = S(t)^{-1}\Psi(t)x$ for $t \in [0,T]$

then $0 \geq (v, S_0 v)$

$$= \int_0^T x'\Psi(t)'S(t)^{-1}(\Psi(t)x+\Psi(t)\int_0^T \Psi(s)'S(s)^{-1}\Psi(s)x\,ds)dt$$

$$= \int_0^T x'\Psi(t)'S(t)^{-1}\Psi(t)x\,dt + x'\int_0^T \Psi(t)'S(t)^{-1}\Psi(t)\,dt\,Px$$

$$= (x, (P+P^2)x)$$

and since x was arbitrary this shows that $P+P^2$ is non positive.

Thus (B) => (C).

Now suppose that $P+P^2$ is non positive.

Then for every $x \in R^n$

$$(x, Px) + (Px, Px) \leq 0$$

i.e. $\|Px\|^2 \leq (x, -Px)$

$\leq \|x\|\ \|Px\|$ and hence

(14) $$\|Px\| \leq \|x\|.$$

Therefore $(x, -Px) \leq \|x\|\ \|Px\|$

$\leq \|x\|^2 = (x, x)$ (by (14))

i.e. $0 \leq (x, (I+P)x)$

and since x was arbitrary $I+P$ is non negative. Thus (C) => (A) and the proof is complete. #

In [21] Schmitendorf gave the condition that the matrix $I+P$ occurring in the above theorem be positive definite as being necessary and sufficient for the game to have an open-loop saddle point whose corresponding path satisfied a certain regularity condition. It is not difficult to show that if $I+P$ is positive definite, then the equations appearing in condition ii) of the theorem always have a solution (see 2.2). The converse, however is not true; these equations may be solvable and $I+P$ be non negative without being positive definite. It is therefore seen that our conditions are slightly weaker than Schmitendorf's condition.

The necessary and sufficient conditions of 2.1 provide a method of finding open-loop saddle points of quadratic games which should usually be simpler than directly solving the matrix Ricatti equation usually associated with such saddle points. The procedure is as follows: (we use the notation of 2.1)

Step I. Solve the differential equations

$$(1) \quad \begin{cases} \dot{M}(t) = -M(t)A(t), & \text{for } t \in [0,T] \\ M(0) = I \end{cases}$$

$$(2) \quad \begin{cases} \dot{W}(t) = A(t)W(t) & \text{for } t \in [0,T] \\ W(0) = I \end{cases}$$

(W will be the inverse of M. Instead of solving (2), we can equivalently invert $M(t)$ over the interval $[0,T]$).

Step II. Invert the matrices $Q(t)$, $S(t)$ over the interval $[0,T]$.

Step III. Compute the matrices L,P by performing the required integrations over $[0,T]$.

Step IV. Solve the linear algebraic equations given in condition ii) of 2.1.

To find the optimal strategies it will always be necessary to carry out step II. Steps I, III and IV here replace the commonly given procedure of directly solving a matrix Ricatti equation.

The algebraic equations in 2.1 ii) contain $2n$ 'unknowns'. However, because of the special form of the equations they are equivalent to another set of equations in n 'unknowns' given in the following lemma.

2.2 <u>Lemma</u>: Let P,L be $n\times n$ matrices, and $y \in R^n$. Then in order that there exist $k,\ell \in R^n$ satisfying the equations

$$(1) \quad \begin{cases} L\{(I+L)k+P\ell+y\} = 0 \quad \text{and} \\ P\{Lk+(I+P)\ell+y\} = 0 \end{cases}$$

it is necessary and sufficient that there exists $h \in R^n$ satisfying the equations

$$(2) \quad \begin{cases} L\{(I+L+P)h+y\} = 0 \quad \text{and} \\ P\{(I+L+P)h+y\} = 0. \end{cases}$$

In particular, if

i) $I+L+P$ is non singular,

or ii) L and P are symmetric, L non negative and $I+P$ positive definite, then there exist $k,\ell \in R^n$ satisfying (1).

<u>Proof</u>: The sufficiency of condition (2) is trivial. One merely puts $k=\ell=h$ in (1).

Suppose now that $k,\ell \in R^n$ satisfy (1).

Then put $h = -Lk-P\ell-y$.

Substituting for h in the first equation of (2) gives

$$\begin{aligned} L\{(I+L+P)h+y\} &= -L\{(L+L^2+PL)k+(P+LP+P^2)\ell-(L+P)y\} \\ &= -L\{P\{Lk+(I+P)\ell\}+L\{(I+L)k+P\ell\}-(L+P)y\}] \\ &= 0 \quad \text{by virtue of (1).} \end{aligned}$$

Thus h satisfies the first equation of (2), and by similar substitution of h in the second equation it is found that h satisfies this equation also.

If condition i) is satisfied then there exists $h \in R^n$ such that

$$(I+P+L)h + y = 0 \quad \text{(in fact, } h \text{ is unique)}$$

and h clearly satisfies (2).

If condition ii) is satisfied, then $(I+P+L)$ is positive definite and therefore non singular and satisfies condition i).

This completes the proof of the lemma. #

Finally, we note that in the case of quadratic differential games the mapping T defined in 1.4 can easily be shown to have a closed range. This is done by making use of the special form of the solutions $\{u_0, v_0\}$ to the equation $T\{u_0, v_0\} = \{a,b\}$ which were obtained in 2.1. Thus, a quadratic differential game will have a (finite) value if and only if it has a saddle point. The possibility that such a game has an ε-saddle point for every $\varepsilon>0$ but not a saddle point cannot oocur.

3. Quadratic Games with Noise Corrupted Measurements

We now elaborate on the comments following 2-3.3 and present an informal criticism of the recently proposed methods of solving quadratic games with noisy measurements. This problem was first considered by Rhodes and Luenberger [15] in whose games the players use Kalman filters to estimate the state of the game at a given time from the noisy measurements available. Since then, several authors have adopted similar approaches and many papers developing this line of attack appear in [38].

In the typical quadratic game with noise corrupted measurements, the trajectory z, and payoff function J are given by (1) and (2) of 2.1 with the difference that the first and second player obtain measurements $z_1(t) \in R^i$, $z_2(t) \in R^j$ respectively at the instant t, which are given by

$$(I) \quad \begin{cases} z_1(t) = H_1(t)z(u,v,t) + w_1(t) \\ z_2(t) = H_2(t)z(u,v,t) + w_2(t) \end{cases}$$

and can base their selections of u,v at t on the values $z_1(t)$, $z_2(t)$ and all the previous values of z_1, z_2. In (I), H_1, H_2 are continuous $i \times n$ and $j \times n$ matrix functions respectively, and w_1, w_2 are independent i- and j-dimensional stochastic processes respectively, with zero mean. The strategies of the players will therefore be selected from spaces similar to $\mathcal{A}^{i}_{Rp}$ and $\mathcal{A}^{j}_{Rq}$ defined in 2-1.2.

Our first observation is that if the 'noise' w_m $(m=1,2)$ is white (in the sense that $w_m(t_1), w_m(t_2), \ldots w_m(t_r)$ are independent random variables for any $\{t_k\}_{k=1}^{r}$ satisfying $0 < t_1 < t_2 < \ldots < t_r < T$) and satisfies the reasonable condition $E(\|w_m\|) < \infty$ (it is difficult to see how any process purporting to represent errors of physical measurements in the real world could violate this condition) then the noise can be completely filtered out of $z_m(t)$ to give the value of $H_m(t)z(u,v,t)$. This is done by taking the limit,

$$\lim_{k\to\infty} \frac{1}{k} \sum_{r=1}^{k} z_m(t - \tfrac{1}{r}) = \lim_{k\to\infty} \left\{ \frac{1}{k} \sum_{r=1}^{k} H_m(t - \tfrac{1}{r}) z(u,v,t - \tfrac{1}{r}) + \frac{1}{k} \sum_{r=1}^{k} w_m(t - \tfrac{1}{r}) \right\}$$

as an estimate for $H_m(t)z(u,v,t)$. By continuity, the first sum on the right hand side approaches $H_m(t)z(u,v,t)$, and the second approaches zero with probability 1 by the laws of large numbers. Thus, with probability 1 the estimate is exact. Even if the mth player is only allowed to use a finite number of his measurements he can make an estimate of $H_m(t)z(u,v,t)$ whose error exceeds a given positive number with arbitrarily small probability. Therefore it seems that the use of 'white' noises (in the sense defined above) as a model for the errors in the measurements will not lead to useful results which cannot also be obtained by completely neglecting them.

The second comment we wish to make applies to games with open-loop saddle points and arbitrary noises w_1, w_2. We have asserted in the remarks following 2-3.3 that an x-saddle point in open-loop strategies for such a game is indeed x-optimal in the game with noisy measurements. The proof is similar to that of 2-3.2 and we outline the argument here. Suppose (u^*, v^*) is an open-loop x-saddle point in the quadratic game with payoff J and that the first player uses the open-loop strategy u^*, in the game with noisy measurements starting from x. If ω is the realisation of the noises in the measurements which occurs during the play of the game, then the strategy of the second player (however constituted) will generate an open-loop strategy v_ω along the path, and the payoff will be $J(u^*, v_\omega)$. But this is never greater than $J(u^*, v^*)$ (by the saddle point condition). Thus, by playing u^* the first player ensures that the payoff (and hence its expected value) never exceeds $J(u^*,v^*)$. Similarly, the second player can guarantee a payoff no smaller than $J(u^*,v^*)$ by playing his open-loop strategy v^*. The strategies u^*,v^* are therefore x-optimal in the game with noisy measurements, even though they completely disregard these measurements. The only advantage to be gained by taking account of the measurements is that one's opponent may be penalised even more severely when he departs from his x-optimal strategy. But when the x-optimal open-loop strategies are unique, even this advantage is doubtful. The reason is that even though one's opponent may be using his x-optimal open-loop strategy, random fluctuations in the measurements will cause one's own measurement dependent strategy to deviate randomly from its x-optimal open-loop value, and thereby incur a loss in payoff. Under these circumstances the measurements will only be useful when they guarantee absolutely that one's opponent has departed from his x-optimal open-loop strategy. (This could be the case, for instance, when the noises are bounded).

CHAPTER 5

MIXED STRATEGIES IN THE GENERAL GAME

1. On the Nature of Mixed Strategies

The problem of finding mixed strategy solutions of differential games with partial information is virtually unexplored, a few scattered results only having been obtained by a handful of authors. In this Chapter we show that the mixed strategies of most differential games (according to any reasonable definition of mixed strategy) can be represented in an especially simple way. It is further shown that the regular blind differential game $((f,g,h,U,V,\theta),\mathcal{U},\mathcal{V})$ always has a value if U and V are compact, and that it has saddle points in mixed strategies if the functions f,g have a certain special form. In the last section of the Chapter, a condition which must be satisfied by optimal mixed strategies is derived in the form of a minimax principle.

Fleming [26] has shown that games of a certain type over function spaces have mixed strategy solutions of a simple kind. The games which he considered have a payoff J given by

$$J(\varphi,\psi) = \int_T g(t,\varphi(t),\psi(t))dw(t)$$

where T is a compact metric space, w a finite, non-atomic Borel measure on T and g a continuous function on $T \times A \times B$, A and B being compact subsets of R^p, R^q respectively. The (pure) strategies φ, ψ are $\mathcal{B}$-measurable functions on T with values in A and B respectively. Fleming showed that these games have a mixed strategy solution in which the two players independently select numbers α, β at random from the unit interval $[0,1)$ and then play pure strategies $\varphi_\alpha, \psi_\beta$ which are determined by the numbers so chosen. Fichefet [25] has applied these results to blind differential games with payoffs which can be written in this form. Unfortunately games with such payoffs form a very restricted class, and Fleming's results are not applicable to the vast majority of differential games. We now extend Fleming's result to show that if the strategy spaces of a differential game can be embedded in compact metric spaces (in fact, σ-compactness will do) then <u>all</u> mixed strategies (considered as Borel probability measures with respect to the metric) can be represented in the simple way explained above. The proof follows.

1.1 <u>Theorem</u>: If X is a compact metric space, and F a Borel probability measure on X, then there exists a $\mathcal{B}$-measurable function $x:[0,1) \to X$ which has F as its distribution with respect to Lebesgue measure on $[0,1)$. i.e. $\mu\{\alpha \in [0,1); x(\alpha) \in A\} = F(A)$ for every Borel $A \subset X$.

<u>Proof</u>: We shall begin by constructing a sequence $\{\mathscr{S}_r\}_{r=1}^{\infty}$ of finite, indexed partitions of X with the following properties:

i) For each r, $\mathscr{S}_{r+1}$ is a refinement of $\mathscr{S}_r$

ii) The elements of $\mathscr{S}_r$ are $\mathscr{B}$-measurable for every r

iii) For each r, the elements of $\mathscr{S}_r$ have diameter no greater than $\frac{1}{r}$.

$\{\mathscr{S}_r\}_{r=1}^{\infty}$ is defined as follows.

For each r, the collection $\{N(x,\frac{1}{2r});x\in X\}$ is an open covering of X, and thus has a finite subcovering $\{N(x_1^r,\frac{1}{2r}), N(x_2^r,\frac{1}{2r}),\ldots\ldots,N(x_{k_r}^r,\frac{1}{2r})\}$.

For each r we define

$$P_1^r = N(x_1^r,\tfrac{1}{2r}) \quad \text{and}$$

$$P_i^r = N(x_i^r,\tfrac{1}{2r}) - \bigcup_{j=1}^{i-1} N(x_j^r,\tfrac{1}{2r}) \quad \text{for} \quad i=2,3,\ldots,k_r$$

and then S_i^r is defined inductively by

$$S_i^1 = P_i^1 \qquad i=1,2,\ldots,k_1$$

and for $i=1,2,\ldots,k_1k_2\ldots k_r$

$$S_i^r = S_j^{r-1} \cap P_{i-(j-1)k_r}^r,$$ where j is the (unique) integer of $\{1,2,\ldots\ldots,k_1k_2\ldots k_{r-1}\}$ such that $i\in\{(j-1)k_r+1,(j-1)k_r+2,\ldots,jk_r\}$.

Now put $\mathscr{S}_r = \{S_1^r,S_2^r,\ldots,S_{j_r}^r\}$ where $j_r=k_1k_2\ldots k_r$. By induction $\{\mathscr{S}_r\}_{r=1}^{\infty}$ has properties i)-iii). We now define a corresponding sequence $\{\mathscr{G}_r\}_{r=1}^{\infty}$ of finite indexed partitions of $[0,1)$ with the properties i) and ii).

Put $I_1^r = [0, F(S_1^r))$ for each r and $I_j^r = [F(\bigcup_{i=1}^{j-1} S_i^r), F(\bigcup_{i=1}^{j} S_i^r))$ for $j = 2, \ldots, j_r$ and each $r=1,2,\ldots$

$\{\mathscr{I}_r\}_{r=1}^{\infty}$ is then a sequence of partitions of $[0,1)$ with properties i) and ii), and $\mu(I_j^r) = F(S_j^r)$ for each r and $j=1,2,\ldots j_r$ (since $\{S_j^r\}_{j=1}^{j_r}$ forms a partition of X).

Also if $m > n$ and I_j^m is a non empty element of $\mathscr{I}_m$ with $I_j^m \subset I_k^n$, then S_j^m is a non empty element of $\mathscr{S}_m$ with $S_j^m \subset S_k^n$.

(The proof of this by induction is straightforward, but tedious, and is omitted.)

Now for each $\alpha \in [0,1)$ and $r=1,2,\ldots$ let $I_{j(r,\alpha)}^r$ be the (unique) interval of $\mathscr{I}_r$ which contains α. Then $\{I_{j(r,\alpha)}^r\}_{r=1}^{\infty}$ is a decreasing sequence of non empty intervals and therefore $\{\overline{S_{j(r,\alpha)}^r}\}_{r=1}^{\infty}$ is a decreasing sequence of non empty compact subsets of X. Therefore $\bigcap_{r=1}^{\infty} \overline{S_{j(r,\alpha)}^r}$ is non empty, and since $\overline{S_{j(r,\alpha)}^r}$ has diameter no greater than $\frac{1}{r}$ then it follows that there is <u>exactly one</u> element of X in $\bigcap_{j=1}^{\infty} \overline{S_{j(r,\alpha)}^r}$. We denote this element by $x(\alpha)$.

It will now be shown that the function $x:[0,1) \to X$ has the required properties.

Let A be a closed subset of X, and for each

$n=1,2,\ldots$ let K_n denote the set of integers $i \leq j_n$ such that

$$\overline{S_i^n} \cap A \neq \emptyset$$

Let $C_n = \bigcup_{i \in K_n} S_i^n$

and $J_n = \bigcup_{i \in K_n} I_i^n$ for each $n=1,2\ldots$

Then $F(C_n) = \sum_{i \in K_n} F(S_i^n) = \sum_{i \in K_n} \mu(I_i^n) = \mu(J_n)$, and $\{C_n\}_{n=1}^{\infty}, \{J_n\}_{n=1}^{\infty}$ are decreasing sequences of $\mathscr{B}$-measurable sets with $A = \bigcap_{n=1}^{\infty} C_n$ (since each point of C_n lies at a distance from A of at most $\frac{1}{n}$).

Also, if $x(\alpha) \in A$, then $\overline{S_{j(r,\alpha)}^r} \cap A \neq \emptyset$ for every r (since $x(\alpha) \in \overline{S_{j(r,\alpha)}^r} \cap A$) and therefore $j(r,\alpha) \in K_r$ and $\alpha \in I_{j(r,\alpha)}^r \subset J_r$.

Thus $x^{-1}(A) \subset J_r$ for every r.

Conversely, if $\alpha \in J_r$ for every r, then $j(r,\alpha) \in K_r$ (since the interval $I_{j(r,\alpha)}^r$ of $\mathscr{I}_r$ in which α lies is unique, it must be one of the intervals whose uinion constitutes J_r) and $\overline{S_{j(r,\alpha)}^r} \cap A \neq \emptyset$ for every r.

Therefore $\bigcap_{r=1}^{\infty} \overline{S_{j(r,\alpha)}^r} \cap A \neq \emptyset$ and consequently $x(\alpha) \in A$. Thus we have shown that

$$x^{-1}(A) = \bigcap_{r=1}^{\infty} J_r, \text{ and is } \mathscr{B}\text{-measurable.}$$

Furthermore,

$$\mu(x^{-1}(A)) = \lim_{r\to\infty} \mu(J_r) = \lim_{r\to\infty} F(C_r) = F(A).$$

Since A was an arbitrary closed subset of X, and a Borel measure on a metric space is completely determined by its value on closed sets (e.g. see Billingsley [39]), then it follows that x is $\mathcal{B}$-measurable, and

$$\mu(x^{-1}(B)) = F(B) \quad \text{for all Borel}$$

subsets B of X. #

It is seen that by allowing the partitions $\mathcal{S}_r, \mathcal{J}_r$ occurring in the above proof to be countably infinite, the theorem could be proved for σ-compact spaces. However, since the strategy spaces we shall be dealing with are in general compact or precompact we shall not require the generalization of 1.1 to σ-compact spaces.

As a corollary to 1.1 we have:

1.2 <u>Corollary</u>: If $\mathcal{U}$ is a bounded Borel subset of a separable Hilbert space $\mathcal{H}$, and F a Borel probability measure on $\mathcal{U}$, then there exists a $\mathcal{B}$-measurable function $x_0 : [0,1) \to \mathcal{U}$ such that $\mu\{\alpha \in [0,1); x_0(\alpha) \in A\} = F(A)$ for every Borel subset A of $\mathcal{U}$.

<u>Proof</u>: We first observe that the Borel sets generated by the weak topology of $\mathcal{H}$ coincide with the Borel sets generated by the norm topology of $\mathcal{H}$. For if $N(x,\varepsilon)$ is any open sphere of $\mathcal{H}$, then $\overline{N(x,\varepsilon + \frac{1}{r})}$ are weakly closed subsets of $\mathcal{H}$ with $\bigcap_{r=1}^{\infty} \overline{N(x,\varepsilon + \frac{1}{r})} = N(x,\varepsilon)$, and since $\mathcal{H}$ is separable then the Borel sets of $\mathcal{H}$ (generated by the norm topology) are generated by the open spheres of $\mathcal{H}$.

Thus, if we let $\overline{c}(\mathcal{U})$ denote the closed convex hull of $\mathcal{U}$, then $\overline{c}(\mathcal{U})$ is bounded, convex and closed, and therefore is compact and metrizable (since $\mathcal{H}$ is separable) with respect to the weak topology of $\mathcal{H}$.

Therefore, if we define the Borel measure F_1 on $\overline{c}(\mathcal{U})$ by

$$F_1(A) = F(A \cap \mathcal{U}) \quad \text{for } A \text{ a Borel subset of } \overline{c}(\mathcal{U})$$

then there exists a $\mathcal{B}$-measurable function $x:[0,1) \to \overline{c}(\mathcal{U})$ with $F_1(A) = \mu(x^{-1}(A))$ for every Borel subset A of $\overline{c}(\mathcal{U})$. Now since $F_1(\overline{c}(\mathcal{U})-\mathcal{U}) = 0$, then $x^{-1}(\overline{c}(\mathcal{U})-\mathcal{U})$ has Lebesgue measure zero, and if we let $u_0 \in \mathcal{U}$ and define $x_0:[0,1)\to\mathcal{U}$ by

$$x_0(\alpha) = \begin{cases} x(\alpha) & \alpha \in x^{-1}(\mathcal{U}) \\ u_0 & \alpha \in x^{-1}(\overline{c}(\mathcal{U})-\mathcal{U}) \end{cases}$$

then x_0 is a $\mathcal{B}$-measurable function with the required property. #

The spaces of pure strategies for the differential games which we consider are all bounded subsets of a separable Hilbert space. Therefore, if the players of such games are allowed to select as mixed strategies any Borel measures on the strategy spaces, then by 1.2 these strategies can be represented in the simple way described at the beginning of this section.

In our games, the pure strategies consist of functions on an interval $[0,T]$ with values in some subset of a Euclidean space. The lemma which follows shows that the mixed strategies for these games can be regarded as

$\mathcal{B}$-measurable functions on $[0,1) \times [0,T]$, with values in the given subset of Euclidean space.

1.3 Lemma: i) If $z \in \mathcal{L}^2_p([0,1) \times [0,T])$, and $u:[0,1) \to \mathcal{L}^2_p[0,T]$ is defined by

$$u(\alpha) = z(\alpha,.) \quad \text{for} \quad \alpha \in [0,1), \quad \text{then}$$

u is $\mathcal{B}$-measurable.

ii) If $u:[0,1) \to \mathcal{L}^2_p[0,T]$ is $\mathcal{B}$-measurable, and $\int_0^1 \|u(\alpha)\|_2^2 d\alpha < \infty$, then there exists a function

$$z \in \mathcal{L}^2_p([0,1) \times [0,T]) \quad \text{such that}$$

$$z(\alpha,.) = u(\alpha) \quad \text{for almost every} \quad \alpha \in [0,1).$$

iii) If $u:[0,1) \to C_p[0,T]$ is $\mathcal{B}$-measurable, then the function $z:[0,1) \times [0,T] \to R^p$ defined by

$$z(\alpha,t) = u(\alpha)(t) \quad \text{for} \quad (\alpha,t) \in [0,1) \times [0,T]$$

is $\mathcal{B}$-measurable.

Proof: i) If $y \in \mathcal{L}^2_p[0,T]$, then the real valued function ψ on $[0,1)$ given by

$$\psi(\alpha) = \int_0^T \|z(\alpha,t) - y(t)\|^2 dt = \int_0^T \|u(\alpha)(t) - y(t)\|^2 dt$$

is $\mathcal{B}$-measurable, and therefore the set

(1) $\{\alpha \in [0,1);\ u(\alpha) \in N(y,\varepsilon)\} = \{\alpha \in [0,1); \psi(\alpha) < \varepsilon^2\}$ is $\mathcal{B}$-measurable.

But since $\mathcal{L}^2_p[0,T]$ is separable, its Borel sets are generated by its open spheres and (1) therefore implies that u is $\mathcal{B}$-measurable.

ii) Since $\mathcal{L}_p^2[0,T]$ is separable, let $\{x_r\}_{r=1}^{\infty}$ be a countable, dense sequence in $\mathcal{L}_p^2[0,T]$ and for each $i,r=1,2,\ldots$ put

$$P_1^r = N(x_1,\frac{1}{r})$$

$$P_i^r = N(x_i,\frac{1}{r}) - \bigcup_{j=1}^{i-1} N(x_j,\frac{1}{r}).$$

Then each P_i^r is a Borel subset of $\mathcal{L}_p^2[0,T]$ of diameter no greater than $\frac{2}{r}$, and $\{P_i^r\}_{i=1}^{\infty}$ is a partition of $\mathcal{L}_p^2[0,T]$ for each r.

Now for every $r=1,2,\ldots$, $u^{-1}(P_i^r)$ is $\mathcal{B}$-measurable for all i, and $\{u^{-1}(P_i^r)\}_{i=1}^{\infty}$ is a partition of $[0,1)$. We therefore define

$$x^r(\alpha) = x_i \quad \text{for} \quad \alpha \in u^{-1}(P_i^r)$$

and $z^r(\alpha,t) = x^r(\alpha)(t)$ for $(\alpha,t) \in [0,1) \times [0,T]$.
$x^r:[0,1) \to \mathcal{L}_p^2[0,T]$ is then $\mathcal{B}$-measurable for every r, and if S is a Borel subset of R^p

$$\{(\alpha,t) \in [0,1) \times [0,T];\ z^r(\alpha,t) \in S\} =$$

$$\bigcup_{i=1}^{\infty} u^{-1}(P_i^r) \times \{t \in [0,T];\ x_i(t) \in S\} \quad \text{is}$$

$\mathcal{B}$-measurable. Consequently z^r is measurable for each r.

Also for every $\alpha \in [0,1)$ and every $r=1,2,\ldots$

$\|u(\alpha) - x^r(\alpha)\|_2 \leq \frac{1}{r}$, and therefore

$$\int_0^1 \|x^r(\alpha)\|_2^2 d\alpha \leq \int_0^1 \|u(\alpha)\|_2^2 d\alpha + \frac{2}{r}\int_0^1 \|u(\alpha)\|_2 + \frac{1}{r^2} < \infty$$

(since $\int_0^1 \|u(\alpha)\|_2 d\alpha \leq 1.\left(\int_0^1 \|u(\alpha)\|_2^2 d\alpha\right)^{\frac{1}{2}}$ by Schwarz' inequality)

i.e. $$\int_0^1 \left\{ \int_0^T \|x^r(\alpha)(t)\|^2 dt \right\} d\alpha = \int_{[0,1)\times[0,T]} \|z^r(\alpha,t)\|^2 d\mu(\alpha,t) < \infty$$

and $z^r \in \mathcal{L}_p^2([0,1) \times [0,T])$ for each r.

Also

$$\|x^r(\alpha) - x^n(\alpha)\|_2^2 = \int_0^T \|z^r(\alpha,t) - z^n(\alpha,t)\|^2 dt$$

$$\leq \left(\frac{1}{r} + \frac{1}{n}\right)^2 \quad \text{for every } \alpha \in [0,1) \text{ and}$$

$r,n=1,2,\ldots$. Thus $\{z^r\}_{r=1}^{\infty}$ is a Cauchy sequence of $\mathcal{L}_p^2([0,1) \times [0,T])$ and therefore converges to an element $z \in \mathcal{L}_p^2([0,1) \times [0,T])$. By i) the function $\varphi:[0,1) \to \mathcal{L}_p^2[0,T]$ defined by

$$\varphi(\alpha)(t) = z(\alpha,t) \quad \text{for} \quad (\alpha,t) \in [0,1) \times [0,T]$$

is $\mathcal{B}$-measurable, and consequently the sets M_r, $r=1,2,\ldots$ given by

$$M_r = \{\alpha \in [0,1); \|z(\alpha,.) - u(\alpha)\|_2 > \frac{4}{r}\}$$

are $\mathcal{B}$-measurable, and if $\alpha \in M_r$ then

$$\|z(\alpha,.) - z^r(\alpha,.)\|_2 > \frac{3}{r}$$

$$\left(\text{since } \|z^r(\alpha,.) - u(\alpha)\|_2 \leq \frac{1}{r}\right)$$

and therefore $\|z(\alpha,.) - z^s(\alpha,.)\|_2 > \frac{1}{r}$ for $s > \frac{1}{r}$

$$\left(\text{since } \|z^s(\alpha,.) - z^r(\alpha,.)\|_2 \leq \frac{1}{r} + \frac{1}{s}\right).$$

Hence

$$(2) \quad \frac{1}{r^2}\mu(M_r) \leq \int_0^1 \int_0^T \|z(\alpha,t) - z^s(\alpha,t)\|^2 dt d\alpha = \|z - z^s\|_2^2 \quad \text{for } s \geq r.$$

But for any $\varepsilon > 0$ there exists $s \geq r$ such that $\|z - z^s\|_2 < \frac{\varepsilon}{r^2}$, and consequently by (2), $\mu(M_r) \leq \varepsilon$. Therefore $\mu(M_r) = 0$.

But $\{\alpha \in [0,1); \|z(\alpha,.) - u(\alpha)\|_2 \neq 0\} = \bigcup_{r=1}^{\infty} M_r$, and therefore $z(\alpha,.) = u(\alpha)$ except if $\alpha \in \bigcup_{r=1}^{\infty} M_r$ which is a set of measure zero. This proves ii).

iii) Again, since $C_p[0,T]$ is separable, we construct functions x^r, z^r in the same way as was done in i). In this case, however the sequence $\{z^r\}_{r=1}^{\infty}$ converges uniformly on $[0,1) \times [0,T]$ to z which is consequently $\mathcal{B}$-measurable. #

2. Existence Theorems

In this section we investigate the existence of mixed strategy solutions to blind differential games. The proofs of existence theorems were carried out by constructing topologies on the strategy spaces in such a way that they were compact, and the payoff was continuous. In the most general case the best that could be done was to provide topologies with respect to which the payoff was continuous and the strategy spaces precompact. Although this guarantees a value, and ε-optimal strategies for every $\varepsilon > 0$, we cannot assert the existence of a saddle point. In so-called pursuit games, in which the players have individual control over separate state variables, the strategy spaces are compact with respect to the constructed topologies, and the games possess saddle points in mixed strategies. The topologies are defined as follows

2.1 Definition: Let U,V be compact subsets of R^p, R^q respectively, and $D = ((f,g,h,U,V,\theta),\mathcal{U},\mathcal{V})$ be a regular n-dimensional blind differential game of prescribed duration T with payoff J and path function z. Then:

i) We define the metric functions $\rho_D^1 : R^n \times \mathcal{U} \times \mathcal{U} \to R$, $\rho_D^2 : R^n \times \mathcal{V} \times \mathcal{V} \to R$ by

$$\rho_D^1(x,u,u') = \sup_{v \in \mathcal{V}} \{\|z(x,u,v,T) - z(x,u',v,T)\|^2 +$$

$$+\left|\int_0^T (g(z(x,u,v,t),u(t),v(t)) - g(z(x,u',v,t),u'(t),v(t)))dt\right|^2\}^{\frac{1}{2}}$$

for $u,u' \in \mathcal{U}$, $x \in R^n$

$$\rho_D^2(x,v,v') = \sup_{u \in \mathcal{U}} \{\|z(x,u,v,T) - z(x,u,v',T)\|^2 +$$

$$+\left|\int_0^T (g(z(x,u,v,t),u(t),v(t)) - g(z(x,u,v',t),u(t),v'(t)))dt\right|^2\}^{\frac{1}{2}}$$

for $v,v' \in \mathcal{V}$, $x \in R^n$.

ii) The modified strategy spaces $\mathcal{U}_D^x$, $\mathcal{V}_D^x$ are defined for each $x \in R^n$ to be the equivalence classes

$$\mathcal{U}_D^x = \{\{u \in \mathcal{U}; \rho_D^1(x,u,u')=0\};\ u' \in \mathcal{U}\}$$

$$\mathcal{V}_D^x = \{\{v \in \mathcal{V}; \rho_D^2(x,v,v')=0\};\ v' \in \mathcal{V}\}$$

These spaces will be considered as metric spaces with the metrics $d_D^1(x,\cdot,\cdot)$, $d_D^2(x,\cdot,\cdot)$ given by

$$d_D^1(x,u,u') = \rho_D^1(x,u_1,u_2) \quad \text{where} \quad u_1 \in u \in \mathcal{U}_D^x, u_2 \in u' \in \mathcal{U}_D^x$$

$$d_D^2(x,v,v') = \rho_D^2(x,v_1,v_2) \quad \text{where} \quad v_1 \in v \in \mathcal{V}_D^x, v_2 \in v' \in \mathcal{V}_D^x.$$

iii) The modified payoff $J_D(x,\cdot,\cdot):\mathcal{U}_D^x \times \mathcal{V}_D^x \to R$ is defined for each $x \in R^n$ by

$J_D(x,u,v) = J(x,u_1,v_1)$ where $u_1 \in u \in \mathcal{U}_D^x$, and $v_1 \in v \in \mathcal{V}_D^x$. #

We note here that since U,V are compact in the above definition then by A1.2 ii) of the Appendix ρ_D^1,ρ_D^2 are well defined real valued functions. Also if $\rho_D^1(x,u_1,u_1') = 0 = \rho_D^1(x,u_2,u_2')$ then

$$\rho_D^1(x,u_1,u_2) = \rho_D^1(x,u_1',u_2') \text{ and}$$

consequently d_D^1 is a well defined metric, as is d_D^2. Finally, if $\rho_D^1(x,u_1,u_2) = 0 = \rho_D^2(x,v_1,v_2)$ then $J(x,u_1,v_1) = J(x,u_2,v_2)$ and therefore J_D is also well defined.

From the point of view of the first player, selection of a strategy from $\mathcal{U}_D^x$ is equivalent to the selection of one from $\mathcal{U}$, since if $u \in \mathcal{U}_D^x$ and $u_1 \in u$, $u_2 \in u$ then $J(x,u_1,v) = J(x,u_2,v)$ for every $v \in \mathcal{V}$. Similarly, for the second player, all the strategies in an equivalence class $v \in \mathcal{V}_D^x$ give him exactly the same payoff, and selection of any one of these is equivalent to the selection of v itself.

The compactness and continuity properties of the strategy spaces and payoff function are now established in the following theorems.

2.2 Theorem: Let U, V be compact subsets of R^p, R^q respectively and $D = ((f,g,h,U,V,\theta),\mathcal{U},\mathcal{V})$ a regular n-dimensional blind differential game of prescribed duration T with payoff J and path function z, and in which $g(x,u,v) = 0$ for all $x \in R^n$, $(u,v) \in U \times V$ (i.e. the payoff of the game is "terminal"). Then for each $x \in R^n$, the spaces $\mathcal{U}^x_D$, $\mathcal{V}^x_D$ are precompact (i.e. every sequence of the space has a Cauchy subsequence) with respect to the metrics $d^1_D(x,.,.)$, $d^2_D(x,.,.)$ respectively, and $J_D(x,.,.)$ is uniformly continuous on $\mathcal{U}^x_D \times \mathcal{V}^x_D$ with respect to the product topology.

Proof: Let $x \in R^n$ be fixed, and let M be a bound for the function $\|z(x,.,.,T)\|$ (see A1.2 ii) of the Appendix for the justification of this).

Now let $\{u_r\}_{r=1}^{\infty}$ be a sequence of the space $\mathcal{U}^x_D$, and for each $r=1,2,...$ let $u'_r \in \mathcal{U}$ be an element of u_r.

By A1.2 iii) of the Appendix there exists a constant K such that for $r=1,2,...$, and $v,v' \in \mathcal{V}$

$$\|z(x,u'_r,v,T) - z(x,u'_r,v',T)\| \leqslant K\,\|v-v'\|_2.$$

The sequence $\{z(x,u_r,.,T\}_{r=1}^{\infty}$ is therefore a uniformly bounded and equicontinuous sequence of functions from a separable space (namely, $\mathcal{V}$ with the $\mathcal{L}^2_q([0,T])$ norm) into a compact space (namely, the closed sphere of R^n with radius M). By the Arzelà-Ascoli theorem (see Dunford and Schwartz [36] p.266) this sequence has a subsequence $\{z(x,u_{n_r},\cdot,T)\}_{r=1}^{\infty}$ which satisfies the Cauchy condition uniformly over $\mathcal{V}$. The sequence $\{u_{n_r}\}_{r=1}^{\infty}$ is then a Cauchy sequence of $\mathcal{U}_D^x$ with respect to the metric $d_D^1(x,\cdot,\cdot)$. Thus, $\mathcal{U}_D^x$ is precompact, and in exactly the same way it may be shown that $\mathcal{V}_D^x$ is precompact.

Now let $\varepsilon > 0$. Then since h has continuous bounded partial derivatives, there exists $\delta > 0$ such that

$|h(x_1) - h(x_2)| < \frac{\varepsilon}{2}$ for all $x_1, x_2 \in R^{n+1}$ with $\|x_1 - x_2\| < \delta$.

Therefore, if $u_1, u_2 \in \mathcal{U}$ and

$$\rho_D^1(x,u_1,u_2) = \sup_{v\in\mathcal{V}} \|z(x,u_1,v,T) - z(x,u_2,v,T)\| < \delta$$

Then

$$|J(x,u_1,v) - J(x,u_2,v)| = |h(T,z(x,u_1,v,T)) - h(T,z(x,u_2,v,T))| \leqslant \frac{\varepsilon}{2} \text{ for every } v \in \mathcal{V},$$

and similarly if $v_1, v_2 \in \mathcal{V}$ and $\rho_D^2(x,v_1,v_2) < \delta$, then

$$|J(x,u,v_1) - J(x,u,v_2)| \leqslant \frac{\varepsilon}{2} \text{ for every } u \in \mathcal{U}.$$

It follows that if $(u_0, v_0) \in \mathcal{U}_D^x \times \mathcal{V}_D^x$ and $(u_1, v_1) \in \mathcal{U}_D^x \times \mathcal{V}_D^x$ with $d_D^1(x,u_0,u_1) < \delta$, $d_D^2(x,v_0,v_1) < \delta$ then

$$|J_D(x,u_0,v_0) - J_D(x,u_1,v_1)|$$
$$\leqslant |J(x,u_0',v_0') - J(x,u_1',v_0')| + |J(x,u_1',v_0') - J(x,u_1',v_1')|$$

where $u_0' \in u_0$, $v_0' \in v_0$, $u_1' \in v_1$, $v_1' \in v_1$, and

since $\rho_D^1(x,u_0',u_1') = d_D^1(x,u_0,u_1) < \delta$,

$$\rho_D^2(x,v_0',v_1') = d_D^2(x,v_0,v_1) < \delta$$

then it follows that

$$|J_D(x,u_0,v_0) - J_D(x,u_1,v_1)| \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary this proves the uniform continuity of $J_D(x,\cdot,\cdot)$ on $\mathcal{U}_D^x \times \mathcal{V}_D^x$. #

Games with non terminal payoff can always be reduced to ones which have a terminal payoff by adding an extra state variable. By doing this we can prove the precompactness of the spaces $\mathcal{U}_D^x$, $\mathcal{V}_D^x$ for arbitrary regular blind differential games D in which the strategies have values lying in compact sets.

2.3 Corollary: Let U,V be compact subsets of R^p, R^q respectively, and $D = ((f,g,h,U,V,\theta),\mathcal{U},\mathcal{V})$ a regular n-dimensional blind differential game of prescribed duration T with payoff J. Then the spaces $\mathcal{U}_D^x$, $\mathcal{V}_D^x$ are precompact for every $x \in R^n$, and $J_D(x,\cdot,\cdot)$ is uniformly continuous on $\mathcal{U}_D^x \times \mathcal{V}_D^x$.

Proof: We define a regular n+1-dimensional differential game which has the strategy spaces $\mathcal{U},\mathcal{V}$ and the payoff function J.

If $f:R^n \times A \times B \to R^n$ and $g:R^n \times A \times B \to R$ with A,B open convex subsets of R^p, R^q respectively, containing U and V, then we define $f_1:R^{n+1} \times A \times B \to R^{n+1}$, and $h_1:R^{n+2} \to R$ by

$$f_1^j\ (x,u,v) = f^j(x^1,x^2..x^n,u,v) \text{ for } x \in R^{n+1}, u,v \in U\times V, \quad j=1,2..n$$

$$f_1^{n+1}(x,u,v) = g\ (x^1,x^2,..x^n,u,v) \text{ for } x \in R^{n+1}, \text{ and } u,v \in U\times V.$$

$h_1(y) = h(y^1, y^2, \ldots y^{n+1}) + y^{n+2}$ for $y \in R^{n+2}$

and we define $\theta_1 = \{T\} \times R^{n+1}$.

Then $D_1 = ((f_1, 0, h_1, U, V, \theta_1), \mathcal{U}, \mathcal{V})$ is a regular n+1-dimensional blind differential game of prescribed duration T with payoff J_1 given by

(1) $$J_1(x,u,v) = J(x^1, x^2 \ldots x^n, u, v) + x^{n+1}$$
$$\text{for } x \in R^{n+1}, (u,v) \in \mathcal{U} \times \mathcal{V}.$$

Furthermore, for $x \in R^n$

(2) $$\mathcal{U}_D^x = \mathcal{U}_{D_1}^{(x,0)} \text{ and } \mathcal{V}_D^x = \mathcal{V}_{D_1}^{(x,0)}.$$

Since D_1 satisfies the conditions of 2.2, then by (1) and (2) $\mathcal{U}_D^x$, $\mathcal{V}_D^x$ are precompact, and $J(x,.,.)$ is uniformly continuous on $\mathcal{U}_D^x \times \mathcal{V}_D^x$. #

We are now in a position to assert the existence of a value for a blind differential game of prescribed duration T. The assertion follows.

2.4 <u>Theorem</u>: Let U, V be compact subsets of R^p, R^q respectively, and $D = ((f,g,h,U,V,\theta), \mathcal{U}, \mathcal{V})$ a regular n-dimensional blind differential game of prescribed duration T with payoff J. Let $\mathcal{F}, \mathcal{G}$ denote the set of all Borel probability measures on $\mathcal{U}, \mathcal{V}$ respectively and for each $x \in R^n$ define $\mathcal{J}_x : \mathcal{F} \times \mathcal{G} \to R$ by

$$\mathcal{J}_x(F,G) = \int_{\mathcal{U} \times \mathcal{V}} J(x,u,v) \, d(F \times G)(u,v)$$

Then $$\inf_{F \in \mathcal{F}} \sup_{G \in \mathcal{G}} \mathcal{J}_x(F,G) = \sup_{G \in \mathcal{G}} \inf_{F \in \mathcal{F}} \mathcal{J}_x(F,G)$$
for each $x \in R^n$.

(Consequently by the results of 1.2 and 1.3, we can assert that if $X \in R^n$ then there exists a value V, and for each $\varepsilon > 0$ there exist functions $u \in \mathcal{L}_p^2([0,1)\times[0,T])$, $v \in \mathcal{L}_q^2([0,1)\times[0,T])$ such that

$$\sup_{v\in\mathcal{V}} \int_0^1 J(x,u(\alpha,.),v)d\alpha \leqslant V + \varepsilon$$

and

$$\inf_{u\in\mathcal{U}} \int_0^1 J(x,u,v(\beta,.))d\beta \geqslant V - \varepsilon \;)$$

Proof: Let $x \in R^n$ be fixed, and let $\mathcal{F}_1, \mathcal{G}_1$ denote the sets of Borel probability measures on $\mathcal{U}_D^x$ and $\mathcal{V}_D^x$ respectively. Define $\mathcal{J}_1 : \mathcal{F}_1 \times \mathcal{G}_1 \to R$ by

$$\mathcal{J}_1(F,G) = \int_{\mathcal{U}_D^x\times\mathcal{V}_D^x} J_D(x,u,v)d(F\times G)(u,v)$$

By the result 2.3, $J_D(x,.,.)$ is uniformly continuous on $\mathcal{U}_D^x \times \mathcal{V}_D^x$ with respect to the product topology of the metrics $d_D^1(x,.,.)$ and $d_D^2(x,.,.)$, and it therefore follows from A2.4 of the Appendix that

$$\inf_{F\in\mathcal{F}_1} \sup_{G\in\mathcal{G}_1} \mathcal{J}_1(F,G) = \sup_{G\in\mathcal{G}_1} \inf_{F\in\mathcal{F}_1} \mathcal{J}_1(F,G).$$

Furthermore, if we put $V = \inf_{F\in\mathcal{F}_1} \sup_{G\in\mathcal{G}_1} \mathcal{J}_1(F,G)$ and let $\varepsilon > 0$, then from the proof of A2.4 it can be seen that there exist atomic measures $F_0 \in \mathcal{F}_1$, $G_0 \in \mathcal{G}_1$, each with a finite number of atoms, such that

$$\int_{\mathcal{U}_D^x} J_D(x,u,v)dF_0(u) < V+\varepsilon \quad \text{for every} \quad v \in \mathcal{V}_D^x$$

and $V-\varepsilon < \int_{\mathcal{V}_D^x} J_D(x,u,v)dG_0(v)$ for every $u \in \mathcal{U}_D^x$.

Now let $u_1, u_2, \ldots u_s \in \mathcal{U}_D^x$ and $v_1, v_2, \ldots v_r \in \mathcal{V}_D^x$ be the atoms of F_0, G_0 respectively, and let $u_i' \in u_i$ for $i=1,2\ldots s$, and $v_j' \in v_j$ for $j=1,2,\ldots,r$. Let $F_1 \in \mathcal{F}$, $G_1 \in \mathcal{G}$ be the atomic measures with atoms u_i' $i=1,2,\ldots s$, and v_j' $j=1,2,\ldots r$ respectively, and

$$F_1(\{u_i'\}) = F_0(\{u_i\}), \quad G_1(\{v_j'\}) = G_0(\{v_j\}).$$

We then have

$$\int_{\mathcal{U}} J(x,u,v)dF_1(u) < V+\varepsilon \quad \text{for every} \quad v \in \mathcal{V}$$

(since $\int_{\mathcal{U}} J(x,u,v')\, dF_1(u) = \int_{\mathcal{U}_D^x} J_D(x,u,v)dF_0(u)$ for $v' \in v \in \mathcal{V}_D^x$)

and similarly

$$\int_{\mathcal{V}} J(x,u,v)dG_1(v) > V-\varepsilon \quad \text{for every} \quad u \in \mathcal{U}$$

it then follows (as in A2.4) that

$$\inf_{F\in\mathcal{F}} \sup_{G\in\mathcal{G}} \mathcal{J}_x(F,G) = \sup_{G\in\mathcal{G}} \inf_{F\in\mathcal{F}} \mathcal{J}_x(F,G)$$

which was to be proved. #

We now consider the case of so-called pursuit games in which each player has complete control over a separate part of the state variable. We first require a lemma concerning the compactness of the set of solutions of a certain differential equation.

2.5 Lemma: Let A be an open convex subset of R^p and U a compact subset of A. Let $\varphi: R^n \times A \to R^n$ have continuous, bounded partial derivatives, and let $\varphi(x,U)$ be convex for every $x \in R^n$. For each $x \in R^n$ and $u \in \mathcal{B}_U[0,T]$ let $y(x,u,.)$ denote the solution of the equation,

$$\dot{y}(x,u,t) = \varphi(y(x,u,t),u(t))$$

$$y(x,u,0) = x$$

and set $\mathcal{S}_x = \{y(x,u,.);\ u \in \mathcal{B}_U[0,T]\}$. Then for each $x \in R^n$, $\mathcal{S}_x$ is a compact subset of $C_n[0,T]$ with respect to the uniform norm.

<u>Proof</u>: By A1.2 ii) $\mathcal{S}_x$ is uniformly bounded, and consequently there exist a constant K such that $\|\varphi(z(t),u)\| < K$ for every $z \in \mathcal{S}_x$ and $u \in U$ (as U is compact, and φ continuous.)

It then follows that $\|\dot{z}(t)\| \leqslant K$ for every $z \in \mathcal{S}_x$ and thence that $\mathcal{S}_x$ is equicontinuous. By the Arzelà-Ascoli theorem ([36] p.266) it remains only to show that $\mathcal{S}_x$ is closed. Let y be an element of the closure of $\mathcal{S}_x$ with respect to the uniform norm, and let $\{y_r\}_{r=1}^{\infty}$ be a sequence of $\mathcal{S}_x$ converging uniformly on $[0,T]$ to y.

Then we must show that $y = y(x,u,.)$ for some $u \in \mathcal{B}_U[0,T]$.

Since $\|\dot{y}_r(t)\| \leq K$ for every r and $t \in [0,T]$ then $\|y_r(t_1)-y_r(t_2)\| \leq K|t_1-t_2|$ for every r, and $t_1,t_2 \in [0,T]$. It then follows that

(1) $\|y(t_1)-y(t_2)\| \leq K|t_1-t_2|$ for $t_1,t_2 \in [0,T]$ and hence that y is absolutely continuous, and $\|\dot{y}(t)\| \leq K$ for a.e. $t \in [0,T]$ (see A1.1 of the Appendix).

Let $t_0 \in [0,T]$ be such that y is differentiable at t_0, and then choose $\varepsilon > 0$. Since φ has bounded partial derivatives it is uniformly continuous, and there exists $\delta > 0$ such that

$$\|\varphi(y_1,u) - \varphi(y_2,u)\| < \frac{\varepsilon}{2} \text{ for } y_1,y_2 \in R^n \text{ with}$$

$\|y_1-y_2\| < \delta$ and $u \in U$. Also, there exists a positive integer m such that

$$\|y-y_r\| < \delta \quad \text{for} \quad r \geq m.$$

Thus if $r \geq m$, and $|t-t_0| < \frac{\delta}{K}$, then

$$\begin{aligned}\|\varphi(y_r(t),u_r(t))-\varphi(y(t_0),u_r(t))\| \leq \\ &\|\varphi(y_r(t),u_r(t)) - \varphi(y(t),u_r(t))\| \\ &+ \|\varphi(y(t),u_r(t))-\varphi(y(t_0),u_r(t))\| \\ &< \varepsilon\end{aligned}$$

(since $\|y(t)-y(t_0)\| < \delta$ by (1), and $\|y_r(t)-y(t)\| \leq \|y-y_r\| < \delta$ for all $t \in [0,T]$).

Therefore, for $r \geq m$ and $|t-t_0| < \frac{\delta}{K}$, $\varphi(y_r(t),u_r(t))$ lies in the closed ε-neighbourhood $\overline{N(\varphi(y(t_0),U),\varepsilon)}$ of the set $\varphi(y(t_0),U)$. Since

$\overline{N(\varphi(y(t_o),U),\varepsilon)}$ is convex and closed then

$$\frac{y_r(t_o+\rho)-y_r(t_o)}{\rho} = \frac{\int_{t_o}^{t_o+\rho}\varphi(y_r(s),u_r(s))ds}{\rho} \in \overline{N(\varphi(y(t_o),U),\varepsilon)}$$

for $r \geq m$, and $|\rho| < \frac{\delta}{K}$. Taking limits as $r \to \infty$ we obtain

$$\frac{y(t_o+\rho)-y(t_o)}{\rho} \in \overline{N(\varphi(y(t_o),U),\varepsilon)}$$

and then taking limits as $\rho \to 0$

$$\dot{y}(t_o) \in \overline{N(\varphi(y(t_o),U),\varepsilon)} .$$

Since $\varepsilon > 0$ was arbitrary, then

$$\dot{y}(t_o) \in \bigcap_{\varepsilon>0} \overline{N(\varphi(y(t_o),U),\varepsilon)} = \overline{\varphi(y(t_o),U)}$$
$$= \varphi(y(t_o),U)$$

(since φ has bounded partial derivatives, and U is compact, then $\varphi(y(t_o),U)$ is closed).

Thus, if we put

$S = \{(z,w) \in R^{2n};\ z-\varphi(w,u) = 0$ for some $u \in U\}$

then we have $(\dot{y}(t),y(t)) \in S$ for almost every $t \in [0,T]$ (viz. those t of $[0,T]$ at which y is differentiable).

Also by A3.4 S is $\mathcal{B}$-measurable, and the function $\psi:S \to U$ defined by

$$\psi(z,w) = \text{lex.sup}\ \{u \in U; z-\varphi(w,u) = 0\}$$

is $\mathcal{B}$-measurable on S. Consequently, the function u defined almost everywhere on $[0,T]$ by

$$u(t) = \psi(\dot{y}(t), y(t)) \text{ is } \mathcal{B}\text{-measurable}$$

and $\dot{y}(t) = \varphi(y(t), u(t))$ for a.e. $t \in [0,T]$.

Also, we have $y(0) = \lim_{r\to\infty} y_r(0) = x$, and hence by defining $u(t)$ to be an arbitrary element of U for those $t \in [0,T]$ at which it is not already defined, we get $u \in \mathcal{B}_U[0,T]$, and $y = y(x,u,.)$. #

We now show that so-called pursuit games in which the player have separate control over independent state variables and the payoff is terminal, have a saddle point in mixed strategies.

2.6 <u>Theorem</u>: Let A,B be open convex subsets of R^p, R^q respectively and U,V compact sub sets of A,B respectively. Let $f_1: R^n \times A \to R^n$ and $f_2: R^m \times A \to R^m$ have continuous bounded partial derivatives, and for each $x \in R^n$, $y \in R^m$ suppose that $f_1(x,U)$, $f_2(y,V)$ are convex. Set $\mathcal{U} = \mathcal{B}_U[0,T]$ and $\mathcal{V} = \mathcal{B}_V[0,T]$, and for each $u \in \mathcal{U}$, $v \in \mathcal{V}$ $x \in R^n, y \in R^m$ let $z_1(x,u,.):[0,T] \to R^n$, $z_2(y,v,.):[0,T] \to R^m$ be the solutions of the respective differential equations

$$\begin{cases} \dot{z}_1(x,u,t) = f_1(z_1(x,u,t),u(t)) & \text{a.e. } t \in [0,T] \\ z_1(x,u,0) = x \end{cases}$$

$$\begin{cases} \dot{z}_2(y,v,t) = f_2(z_2(y,v,t),v(t)) & \text{a.e. } t \in [0,T] \\ z_2(y,v,0) = y \end{cases}$$

Let $h: R^{n+m} \to R$ be continuous and $J: R^{n+m} \times \mathcal{U} \times \mathcal{V} \to R$ be defined by

$J(x,y,u,v)=h(z_1(x,u,T),z_2(y,v,T))$ for $(x,y,u,v)\in R^{n+m}\times\mathcal{U}\times\mathcal{V}$.

Then for each $z\in R^{n+m}$, there exist $\mathcal{B}$-measurable functions $u_0:[0,1)\times[0,T]\to U$ and $v_0:[0,1)\times[0,T]\to V$ such that

$$\int_0^1 J(z,u_0(\alpha,\cdot),v)d\alpha \leq \int_0^1\int_0^1 J(z,u_0(\alpha,\cdot),v_0(\beta,\cdot))d\alpha d\beta$$

$$\leq \int_0^1 J(z,u,v_0(\beta,\cdot))d\beta$$

for every $u\in\mathcal{U}$, $v\in\mathcal{V}$.

__Proof:__ Let $z=(x,y)\in R^n\times R^m$ be fixed, and put

$$\mathcal{S}^1=\{z_1(x,u,\cdot);\ u\in\mathcal{U}\}$$

$$\mathcal{S}^2=\{z_2(y,v,\cdot);\ v\in\mathcal{V}\}$$

Then by 2.5 $\mathcal{S}^1$ and $\mathcal{S}^2$ are compact with respect to the uniform norm, and if

$J_1:\mathcal{S}^1\times\mathcal{S}^2\to R$ is defined by

$J_1(z_1,z_2)=h(z_1(T),z_2(T))$

then J_1 is clearly continuous on $\mathcal{S}^1\times\mathcal{S}^2$. Therefore, by A2.3 there exist Borel measures F_0,G_0 on $\mathcal{S}^1$ and $\mathcal{S}^2$ respectively, such that

$$\int_{\mathcal{S}^1} J_1(z_1,\zeta_2)dF_0(z_1)\leq \int_{\mathcal{S}^1\times\mathcal{S}^2} J_1(z_1,z_2)d(F_0\times G_0)(z_1,z_2)$$

$$\leq \int_{\mathcal{S}^2} J_1(\zeta_1,z_2)dG_0(z_2)$$

for every $\zeta_1\in\mathcal{S}^1$ and $\zeta_2\in\mathcal{S}^2$.

Now by 1.1 and 1.3 iii) there exist $\mathcal{B}$-measurable functions $\zeta_1:[0,1)\times[0,T]\to R^n$, $\zeta_2:[0,1)\times[0,T]\to R^m$

with

$\mu(\{\alpha \in [0,1); \zeta_1(\alpha,.) \in A\}) = F_0(A)$

$\mu(\{\beta \in [0,1); \zeta_2(\beta,.) \in B\}) = G_0(B)$ for any Borel subsets A,B of $\mathcal{S}^1, \mathcal{S}^2$ respectively. Then we have

$$(1) \quad \int_0^1 J_1(\zeta_1(\alpha,.), z_2)\, d\alpha \leq \int_0^1\int_0^1 J_1(\zeta_1(\alpha,.), \zeta_2(\beta,.))\, d\alpha\, d\beta$$

$$\leq \int_0^1 J_1(z_1, \zeta_2(\beta,.))\, d\beta$$

for $z_1 \in \mathcal{S}^1$, $z_2 \in \mathcal{S}^2$

(Note that $\zeta_1(\alpha,.) \in \mathcal{S}^1$ and $\zeta_2(\alpha,.) \in \mathcal{S}^2$ for almost every $\alpha \in [0,1)$).

If now

$S_1 = \{(\eta_1,\xi_1) \in R^{2n};\ \eta_1 - f_1(\xi_1,u) = 0$ for some $u \in U\}$

$S_2 = \{(\eta_2,\xi_2) \in R^{2m};\ \eta_1 - f_2(\xi_2,v) = 0$ for some $v \in V\}$,

and $\psi_1 : S_1 \to U$, $\psi_2 : S_2 \to V$ are defined by

$\psi_1(\eta_1,\xi_1) = \text{lex.sup}\{u \in U; \eta_1 - f_1(\xi_1,u) = 0\}$ for $\eta_1,\xi_1 \in S_1$

and

$\psi_2(\eta_2,\xi_2) = \text{lex.sup}\{v \in V; \eta_2 - f_2(\xi_2,v) = 0\}$ for $\eta_2,\xi_2 \in S_2$

and u_0, v_0 are further given by

$u_0(\alpha,t) = \psi_1(\dot{\zeta}_1(\alpha,t), \zeta_1(\alpha,t))$

$v_0(\alpha,t) = \psi_2(\dot{\zeta}_2(\alpha,t), \zeta_2(\alpha,t))$ a.e $(\alpha,t) \in [0,1) \times [0,T]$

then by A3.4 and the properties of ζ_1 and ζ_2, u_0, v_0 are $\mathcal{B}$-measurable functions defined almost everywhere on $[0,1) \times [0,T]$. By suitably defining u_0, v_0 on the remainder of $[0,1) \times [0,T]$ we obtain functions which are everywhere defined, and are such that for a.e. $\alpha \in [0,1)$

$$(2) \quad \begin{cases} \zeta_1(\alpha,t) = z_1(x,u_0(\alpha,.),t) \\ \zeta_2(\alpha,t) = z_2(y,v_0(\alpha,.),t) \quad \text{for every} \quad t \in [0,T] \end{cases}$$

But since $J(x,y,u,v) = J_1(z_1(x,u,.),z_2(y,v,.))$ for $u \in \mathcal{U}$ and $v \in \mathcal{V}$, then it follows from (1) and (2) that u_0, v_0 are the required functions. #

The above results, in asserting the existence of mixed strategy values and saddle points for various differential games with no information, have extended those of Fichefet [25]. The Theorem 2.4 shows that differential games of a fairly comprehensive class have mixed strategy ε-saddle points, and by the conclusion of 1.2, any mixed strategy in these games can be represented in the simple way first proposed by Fleming [26]. The result immediately above establishes the existence of mixed strategy saddle points for games of a more restricted nature.

Games with no information which are not of fixed duration may also be expected to have similar simple representations for their mixed strategies, but these games are complicated by the discontinuity of the payoff and by the fact that the player who desires termination may not be able to attin it with probability 1. In certain special cases (such as the "Princess and Monster" game on a compact set in R^n ([1]p 349)) the payoff is upper or lower semi-continuous, the strategy spaces are compact, and one player (the "Pursuer") desires and can force termination of the game with probability 1. In such cases these games will also have values in mixed strategies, one player having an optimal strategy and the other having an ε-optimal strategy for every positive ε.

3. Necessary Conditions for Optimality

In this section we derive a minimax principle which a mixed strategy solution of a regular blind differential game must satisfy. Although the principle is easily derived in an informal way, its proof is rendered difficult by the general nature of the strategies allowed (arbitrary $\mathcal{B}$-measurable functions). Again we find it easier to deal first with games in which payoff is terminal and then treat the general game by adding an extra state variable. The proof of the principle proceeds by way of a preliminary minimum principle.

3.1 Lemma: Let A be an open convex subset of R^p, U a compact subset of A and $\mathcal{U} = \mathcal{B}_U[0,T]$. Let $f:R^n \times A \times [0,T] \times [0,1) \to R^n$ be a $\mathcal{B}$-measurable function such that:

i) For each $(t,\alpha) \in [0,T] \times [0,1)$, $f(\cdot,\cdot,t,\alpha)$ has partial derivatives which are continuous and bounded uniformly with respect to $(t,\alpha) \in [0,T] \times [0,1)$.

ii) For each $(x,u) \in R^n \times U$, $f(x,u,\cdot,\cdot)$ is bounded.

Let $z:R^n \times \mathcal{U} \times [0,1) \times [0,T] \to R^n$ be the (unique) solution of the differential equation

$$(1)\quad \begin{cases} \dot{z}(x,u,\alpha,t) = f(z(x,u,\alpha,t),u(t),t,\alpha) & \text{a.e. } t \in [0,T] \\ z(x,u,\alpha,0) = x & \text{for } (x,u,\alpha) \in R^n \times \mathcal{U} \times [0,1) \end{cases}$$

Let $h:R^n \to R$ have continuous partial derivatives and define the functional $G:R^n \times \mathcal{U} \to R$ by

$$G(x,u) = \int_0^1 h(z(x,u,\alpha,T))d\alpha.$$

Suppose that $x_0 \in R^n$ and that $u_0 \in \mathcal{U}$ minimises $G(x_0,.)$ over $\mathcal{U}$. If now $\lambda:[0,1) \times [0,T] \to R^n$ is the solution of the differential equation

$$(2)\quad \begin{cases} \dot{\lambda}_j(\alpha,t) = -\sum_{i=1}^{n} f^i_{,j}(z(x_0,u_0,\alpha,t),u_0(t),t,\alpha)\lambda_i(\alpha,t) \\ \qquad\qquad\qquad\qquad \text{a.e. } t \in [0,T] \\ \lambda_j(\alpha,T) = h_{,j}(z(x_0,u_0,\alpha,T)) \quad \text{for } \alpha \in [0,1) \end{cases}$$

and $H:[0,T] \times U \to R$ is defined by

$$H(t,u) = \int_0^1 \sum_{i=1}^{n} \lambda_i(\alpha,t)f^i(z(\alpha_0,u_0,\alpha,t),u,t,\alpha)d\alpha$$

then for almost every $t \in [0,T]$

$$H(t,u_0(t)) = \inf_{u \in \mathcal{U}} H(t,u).$$

<u>Proof</u>: Let $z_0:[0,1) \times [0,T] \to R^n$ be defined by

$$z_0(\alpha,t) = z(x_0,u_0,\alpha,t) \quad \text{for } (\alpha,t) \in [0,1) \times [0,T]$$

and let F denote the $\mathcal{B}$-measurable matrix function defined on $[0,1) \times [0,T]$ by

$F_{jk}(\alpha,t)=f^j_{,k}(z_0(\alpha,t),u_0(t),t)$ for $(\alpha,t) \in [0,1) \times [0,T]$.

Also, denote by M the solution of the matrix differential equation

$$\dot{M}(\alpha,t) = -M(\alpha,t)F(\alpha,t) \qquad \text{a.e.}\ t \in [0,T]$$
$$M(\alpha,0) = I.$$

We also define the gradient $\nabla h: R^n \to R^n$ of h by

$$\nabla h_j(z) = h_{,j}(z) \quad \text{for} \quad z \in R^n.$$

The equations (2) may now be written

$$(3) \quad \begin{cases} \dot{\lambda}(\alpha,t) = -F(\alpha,t)'\lambda(\alpha,t) & \text{a.e.}\ t \in [0,T] \\ \lambda(\alpha,T) = \nabla h(z_0(\alpha,T)) & \text{for} \quad \alpha \in [0,1) \end{cases}$$

which have the solution

$$(4) \quad \lambda(\alpha,t) = (M(\alpha,T)^{-1}M(\alpha,t))'\nabla h(z_0(\alpha,T)) \qquad (\alpha,t) \in [0,1) \times [0,T]$$

as can be verified by substitution.

Now for each $u \in U$ let D_u denote the set of all $t \in (0,T)$ such that:

i) $$\lim_{\delta t \to 0} \frac{\int_t^{t+\delta t} \int_0^1 \nabla h(z_0(\alpha,T))'M(\alpha,T)^{-1}M(\alpha,s)f(z_0(\alpha,s),u,s,\alpha)\,d\alpha\,ds}{\delta t}$$
$$= \int_0^1 \nabla h(z_0(\alpha,T))'M(\alpha,T)^{-1}M(\alpha,t)f(z_0(\alpha,t),u,t,\alpha)\,d\alpha$$

ii) $$\lim_{\delta t \to 0} \frac{1}{\delta t} \int_t^{t+\delta t} \|u-u_0(s)\|\,ds = \|u-u_0(t)\|.$$

Since the integrands in the above expressions are bounded $\mathcal{B}$-measurable functions (see A1.2) then by well-known results of measure theory (see A1.1) $\mu(D_u) = T$ for every $u \in U$. We also let D be the set of $t \in [0,T]$ for which i) holds when u is replaced by the function u_0. Then $\mu(D) = T$ also.

Now let $u \in U$ be fixed and $t_0 \in D_u \cap D$.

If $u = u_0(t_0)$, then trivially

(5) $$H(t_0, u_0(t_0)) \leq H(t_0, u).$$

Therefore suppose that $u \neq u_0(t_0)$ and for each $\varepsilon \in (0, T-t_0]$ define $u_\varepsilon \in \mathcal{U}$ by

$$u_\varepsilon(t) = \begin{cases} u_0(t) & \text{for } t \in [0,t_0) \cup [t_0+\varepsilon, T] \\ u & \text{for } t \in [t_0, t_0+\varepsilon) \end{cases}$$

and put $z_\varepsilon(\alpha,t) = z(x_0, u_\varepsilon, \alpha, t)$ for $(\alpha,t) \in [0,1) \times [0,T]$

Also, let $\delta_\varepsilon : [0,1) \times [0,T] \to R^n$ be the solution of the differential equation

(6) $$\begin{cases} \dot{\delta}_\varepsilon(\alpha,t) = F(\alpha,t)\delta_\varepsilon(\alpha,t) + f(z_\varepsilon(\alpha,t), u_\varepsilon(t), t, \alpha) - \\ \qquad\qquad f(z_\varepsilon(\alpha,t), u_0(t), t, \alpha) \\ \delta_\varepsilon(\alpha,0) = 0 \qquad \text{for a.e. } t \in [0,T] \text{ and } \alpha \in [0,1) \end{cases}$$

So that

(7) $$\delta_\varepsilon(\alpha,t) = M(\alpha,t)^{-1} \int_0^t M(\alpha,s)\{f(z_\varepsilon(\alpha,s), u_\varepsilon(s), s, \alpha) - f(z_\varepsilon(\alpha,s), u_0(s), s, \alpha)\}ds$$

for $(\alpha,t) \in [0,1) \times [0,T]$.

Also, since

$$\lim_{\varepsilon \to 0} \frac{\|u_\varepsilon - u_0\|_1}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\int_0^T \|u_\varepsilon(t) - u_0(t)\| dt}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\int_{t_0}^{t_0+\varepsilon} \|u - u_0(t)\| dt}{\varepsilon}$$

$$= \|u - u_0(t_0)\| \neq 0 \quad \text{(by ii)).}$$

then $\|u_\varepsilon - u_0\|_1 \neq 0$ if ε is sufficiently small, and therefore by A1.3 of the Appendix

$$(8) \quad \lim_{\varepsilon\to 0} \frac{\|z_\varepsilon(\alpha,t)-z_0(\alpha,t)-\delta_\varepsilon(\alpha,t)\|}{\varepsilon} =$$

$$\lim_{\varepsilon\to 0} \frac{\|z_\varepsilon(\alpha,t)-z_0(\alpha,t)-\delta_\varepsilon(\alpha,t)\|}{\|u_\varepsilon - u_0\|_1} \cdot \frac{\|u_\varepsilon - u_0\|_1}{\varepsilon}$$

$$= 0 \quad \text{uniformly on } [0,1) \times [0,T].$$

Now by the minimising property of u_0, we have

$$0 \leq \int_0^1 \{h(z_\varepsilon(\alpha,T))-h(z_0(\alpha,T))\}d\alpha \text{ for every } \varepsilon \in (0,T-t_0]$$

$$= \int_0^1\int_0^1 \nabla h(z_0(\alpha,T)+r(z_\varepsilon(\alpha,T)-z_0(\alpha,T)))'(z_\varepsilon(\alpha,T)-z_0(\alpha,T))drd\alpha$$

$$= \int_0^1 I_\varepsilon(\alpha)'\delta_\varepsilon(\alpha,T)d\alpha + \int_0^1 I_\varepsilon(\alpha)'(z_\varepsilon(\alpha,T)-z_0(\alpha,T)-\delta_\varepsilon(\alpha,T))d\alpha.$$

where $I_\varepsilon(\alpha) = \int_0^1 \nabla h(z_0(\alpha,T)+r(z_\varepsilon(\alpha,T)-z_0(\alpha,T)))dr.$

And since $z_\varepsilon(\alpha,t) \to z_0(\alpha,t)$ uniformly on $[0,1) \times [0,T]$ as $\varepsilon\to 0$ (see A1.2 iii)) then $I_\varepsilon(\alpha) \to \nabla h(z_0(\alpha,T))$ uniformly on $[0,1)$ as $\varepsilon\to 0$. Thus

$$0 \leq \lim_{\varepsilon\to 0} \frac{\int_0^1 \{h(z_\varepsilon(\alpha,T))-h(z_0(\alpha,T))\}d\alpha}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0} \frac{\int_0^1 \nabla h(z_0(\alpha,T))'\delta_\varepsilon(\alpha,T)d\alpha}{\varepsilon}$$

$$+ \lim_{\varepsilon\to 0} \frac{\int_0^1 I_\varepsilon(\alpha)'(z_\varepsilon(\alpha,T)-z_0(\alpha,T)-\delta_\varepsilon(\alpha,T))d\alpha}{\varepsilon}$$

$$+ \lim_{\varepsilon\to 0} \int_0^1 (I_\varepsilon(\alpha)-\nabla h(z_0(\alpha,T))'\frac{\delta_\varepsilon(\alpha,T)}{\varepsilon}\, d\alpha \;.$$

Since $I_\varepsilon(.)$ and $\frac{\delta_\varepsilon(.,T)}{\varepsilon}$ are bounded uniformly with respect to $\varepsilon \in (0,T-t_0]$, then in view of (8) and the fact that $I_\varepsilon(\alpha) \to \nabla h(z_0(\alpha,T))$ uniformly on $[0,1)$ as $\varepsilon \to 0$, the second and third terms of the above expression are zero. Thus

$$(9) \qquad 0 \leqslant \lim_{\varepsilon\to 0} \frac{\int_0^1 \nabla h(z_0(\alpha,T))'\delta_\varepsilon(\alpha,T)d\alpha}{\varepsilon}$$

By substituting for δ_ε from (7) we obtain

$$\int_0^1 \nabla h(z_0(\alpha,T))'\delta_\varepsilon(\alpha,T)d\alpha =$$

$$= \int_0^1 \nabla h(z_0(\alpha,T))'M(\alpha,T)^{-1}\int_0^T M(\alpha,s)\{f(z_\varepsilon(\alpha,s),u_\varepsilon(s),s,\alpha)- f(z_\varepsilon(\alpha,s),u_0(s),s,\alpha)\}dsd\alpha$$

$$= \int_{t_0}^{t_0+\varepsilon}\int_0^1 \nabla h(z_0(\alpha,T))'M(\alpha,T)^{-1}M(\alpha,s)\{f(z_\varepsilon(\alpha,s),u_\varepsilon(s),s,\alpha) - f(z_\varepsilon(\alpha,s),u_0(s),s,\alpha)\}d\alpha ds$$

$$= \int_{t_0}^{t_0+\varepsilon}\int_0^1 \lambda(\alpha,s)'\{f(z_\varepsilon(\alpha,s),u,s,\alpha)-f(z_\varepsilon(\alpha,s),u_0(s),s,\alpha)\}d\alpha ds$$

where we have substituted λ from (4).

Therefore

$$\lim_{\varepsilon\to 0} \frac{\int_0^1 \nabla h(z_0(\alpha,T))'\delta_\varepsilon(\alpha,T)d\alpha}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0} \frac{\int_{t_0}^{t_0+\varepsilon}\int_0^1 \lambda(\alpha,s)'\{f(z_0(\alpha,s),u,s,\alpha) - f(z_\varepsilon(\alpha,s),u_0(s),s,\alpha)\}d\alpha ds}{\varepsilon}$$

$$+ \lim_{\varepsilon\to 0} \frac{\int_{t_0}^{t_0+\varepsilon}\int_0^1 \lambda(\alpha,s)'\{f(z_\varepsilon(\alpha,s),u,s,\alpha)-f(z_0(\alpha,s),u,s,\alpha)\}d\alpha ds}{\varepsilon}$$

$$+ \lim_{\varepsilon\to 0} \frac{\int_{t_0}^{t_0+\varepsilon}\int_0^1 \lambda(\alpha,s)'\{f(z_0(\alpha,s),u_0(s),s,\alpha) - f(z_\varepsilon(\alpha,s),u_0(s),s,\alpha)\}d\alpha ds}{\varepsilon}$$

But since $z_\varepsilon(\alpha,s) \to z_0(\alpha,s)$ uniformly on $[0,1) \times [0,T]$ as $\varepsilon\to 0$, and λ is bounded, then the second and third terms of the above expression are zero. Since $t_0 \in D_u \cup D$, then the first term is just

$$\int_0^1 \lambda(\alpha,t_0)'\{f(z_0(\alpha,t_0),u,t_0,\alpha)-f(z_0(\alpha,t_0),u_0(t_0),t_0,\alpha)\}d\alpha = H(t_0,u)-H(t_0,u(t_0)) \quad \text{(see i) above}$$

and the subsequent comments). Consequently by (9) we have

$$(10)\quad 0 \leq \lim_{\varepsilon\to 0} \frac{\int_0^1 \nabla h(z_0(\alpha,T))'\,\delta_\varepsilon(\alpha,T)\,d\alpha}{\varepsilon} = H(t_0,u)-H(t_0,u(t_0))$$

Now let $\{u_r\}_{r=1}^{\infty}$ be a (countable) dense sequence in U and let $t\in[0,T]$. Then $H(t,u_0(t)) = \inf_{u\in U} H(t,u)$ if and only if $H(t,u_0(t)) \leq H(t,u_r)$ for every $r=1,2,\ldots$ (since $H(t,.)$ is continuous on U).

In particular if $t \in \bigcap_{r=1}^{\infty} D_{u_r} \cap D$, then by (10)

$$H(t,u_0(t)) = \inf_{u\in U} H(t,u).$$

But since $\mu(\bigcap_{r=1}^{\infty} D_{u_r} \cap D) = T$ then the lemma is proved. #

The minimax principle for games with terminal payoff now follows.

3.1 <u>Theorem</u>: Let U,V be compact subsets of R^p,R^q respectively, and $D=((f,0,h,U,V,\theta),\mathcal{U},\mathcal{V})$ a regular n-dimensional blind differential game of prescribed duration T with payoff J and path function z.

Suppose that $u_0:[0,1)\times[0,T]\to U$ and $v_0:[0,1)\times[0,T]\to V$ are $\mathcal{B}$-measurable functions, and $x_0\in R^n$ such that

$$\int_0^1 J(x_0,u_0(\alpha,.),v)\,d\alpha \leq \int_0^1\int_0^1 J(x_0,u_0(\alpha,.),v_0(\beta,.))\,d\alpha\,d\beta \leq \int_0^1 J(x_0,u,v(\beta,.))\,d\beta$$

for all $u\in\mathcal{U}$ and $v\in\mathcal{V}$.

Let $\lambda:[0,1)\times[0,1)\times[0,T]\to R^n$ be the solution of the differential equation

$$\dot\lambda_j(\alpha,\beta,t) = -\sum_{i=1}^{n} f^i_{,j}(z(x_0,u_0(\alpha,.),v_0(\beta,.),t),u_0(\alpha,t),\\ v_0(\beta,t))\lambda_i(\alpha,\beta,t)$$

$$\lambda_j(\alpha,\beta,T) = h_{,j+1}(T,z(x_0,u_0(\alpha,.),v_0(\beta,.),T))$$

for a.e. $t\in[0,T]$ and $(\alpha,\beta)\in[0,1)^2$

and define $H:[0,1)\times[0,1)\times[0,T]\times U\times V\to R$ by

$$H(\alpha,\beta,t,u,v) = \sum_{i=1}^{n}\lambda_i(\alpha,\beta,t)f^i(z(x_0,u_0(\alpha,.),v_0(\beta,.),t),u,v).$$

Then:

i) For almost every $(\alpha,\beta,t)\in[0,1)\times[0,1)\times[0,T]$

(a) $$\int_0^1 H(\alpha,\beta,t,u_0(\alpha,t),v_0(\beta,t))d\beta=\inf_{u\in U}\int_0^1 H(\alpha,\beta,t,u,v_0(\beta,t))d\beta$$

and

(b) $$\sup_{v\in V}\int_0^1 H(\alpha,\beta,t,u_0(\alpha,t),v)d\alpha=\int_0^1 H(\alpha,\beta,t,u_0(\alpha,t),v_0(\beta,t))d\alpha$$

and

ii) $\int_0^1\int_0^1 H(\alpha,\beta,t,u_0(\alpha,t),v_0(\beta,t))d\alpha d\beta$ has the same value

for almost every $t\in[0,T]$.

Proof: We first show that under the hypotheses of the theorem

(1) $$\int_0^1 J(x_0,u_0(\alpha,.),v_0(\beta,.))d\beta = \inf_{u\in\mathcal{U}}\int_0^1 J(x_0,u,v_0(\beta,.))d\beta$$

for almost every $\alpha\in[0,1)$.

Let M be the set of all $\alpha \in [0,1)$ such that (1) holds. Then M is $\mathcal{B}$-measurable, and if $\{u_r\}_{r=1}^{\infty}$ is a (countable) dense sequence of $\mathcal{U}$ (with respect to the $\mathcal{L}_p^2[0,T]$ norm) and M_r denotes the set of $\alpha \in [0,1)$ such that

(2) $$\int_0^T J(x_0,u_0(\alpha,.),v_0(\beta,.))d\beta = \inf_{r=1,2,\dots} \int_0^1 J(x_0,u_r,v_0(\beta,.))d\beta$$

then $M = \bigcap_{r=1}^{\infty} M_r$

(since $J(x_0,.,v_0(\beta,.))$ is continuous on $\mathcal{U}$ for every $\beta \in [0,1)$ by 3-2.5 ii)).

Also if $\mu(M_r) < 1$ for some $r = 1,2,\dots$ then there exists $\varepsilon > 0$ and a set Z of positive Lebesgue measure such that

$$\int_0^1 J(x_0,u_r,v_0(\beta,.))d\beta \leq \int_0^1 J(x_0,u_0(\alpha,.),v_0(\beta,.))d\beta - \varepsilon \quad \text{for } \alpha \in Z.$$

Therefore, by putting

$$u_1(\alpha,t) = \begin{cases} u_r(t) & \text{for } \alpha \in Z \\ u_0(\alpha,t) & \text{for } \alpha \notin Z \end{cases}$$

we obtain a $\mathcal{B}$-measurable function $u_1:[0,1) \times [0,T] \to U$ with

$$\int_0^1\int_0^1 J(x_0,u_1(\alpha,.),v_0(\beta,.)d\alpha d\beta \leq \int_0^1\int_0^1 J(x_0,u_0(\alpha,.),v_0(\beta,.))d\alpha d\beta - \varepsilon\mu(Z).$$

But since we must have

$$\int_0^1\int_0^1 J(x_0,u_0(\alpha,.),v_0(\beta,.)d\alpha d\beta \leq \int_0^1 J(x_0,u_1(\alpha,.),v_0(\beta,.))d\beta$$

for every α by hypothesis, this is a contradiction. Therefore $\mu(M_r) = 1$ for every $r=1,2...$ and consequently $\mu(M) = \mu(\bigcap_{r=1}^{\infty} M_r) = 1$.

Thus for $\alpha \in M$, the function $u_0(\alpha,.)$ minimises the functional $G:\mathcal{U} \to R$ defined by

$$G(u) = \int_0^1 J(x_0,u,v(\beta,.))d\beta \quad \text{for} \quad u \in \mathcal{U} \quad \text{and the}$$

inequality i) a) then follows for almost every $t \in [0,T]$ by the results of 3.1.

Therefore i) a) holds for almost every $(\alpha,t) \in [0,1) \times [0,T]$, and in a similar way i b) holds for almost every $(\beta,t) \in [0,1) \times [0,T]$. It now remains only to prove ii).

For brevity, put

$$\mathcal{H}(t) = \int_0^1\int_0^1 H(\alpha,\beta,t,u_0(\alpha,t),v_0(\beta,t))d\alpha d\beta \quad \text{for} \quad t \in [0,T]$$

$$z(\alpha,\beta,t) = z(x_0,u_0(\alpha,.),v_0(\beta,.),t) \quad \text{for} \quad (\alpha,\beta,t) \in [0,1) \times [0,1) \times [0,T]$$

$$\varphi(t_1,t_2,t_3) = \int_0^1\int_0^1 \sum_{i=1}^{n} \lambda_i(\alpha,\beta,t_1)f^i(z(\alpha,\beta,t_1),u_0(\alpha,t_2),v_0(\beta,t_3))d\alpha d\beta$$

Then $\mathcal{H}(t_1) - \mathcal{H}(t_2) = \varphi(t_1,t_1,t_1) - \varphi(t_2,t_2,t_2)$

$$= \varphi(t_1,t_1,t_2) - \varphi(t_2,t_1,t_2)$$
$$+ \varphi(t_2,t_1,t_2) - \varphi(t_2,t_2,t_2)$$
$$+ \varphi(t_1,t_1,t_1) - \varphi(t_1,t_1,t_2).$$

But if $t_1 \in [0,T]$ then from i) (a) we have for a.e.$(\alpha,t) \in [0,1) \times [0,T]$

$$\int_0^1 H(\alpha,\beta,t,u_0(\alpha,t),v_0(\beta,t))d\beta \leq \int_0^1 H(\alpha,\beta,t,u(\alpha,t_1),v_0(\beta,t))d\beta$$

and then by integration with respect to α over $[0,1)$

$$0 \leq \varphi(t,t_1,t) - \varphi(t,t,t) \quad \text{for a.e.} t \in [0,T].$$

Similarly from i) (b) we obtain for $t_2 \in [0,T]$

$$0 \leq \varphi(t,t,t) - \varphi(t,t,t_2) \quad \text{for a.e.} t \in [0.T].$$

Substituting these inequalities into the above expression for $\mathcal{H}(t_1) - \mathcal{H}(t_2)$ we find

(3) $\mathcal{H}(t_1) - \mathcal{H}(t_2) \geq \varphi(t_1,t_1,t_2) - \varphi(t_2,t_1,t_2)$

for a.e.$(t_1,t_2) \in [0,T]^2$.

But since $\dot{\lambda}$ and $\dot{z}$ are both bounded on $[0,1) \times [0,1) \times [0,T]$, and f has bounded continuous partial derivatives, then for every $(t_1,t_2) \in [0,T]^2$, $\varphi(\cdot,t_1,t_2)$ is differentiable almost everywhere on $[0,T]$, and for $t_1 > t_2$

(4) $\varphi(t_1,t_1,t_2)-\varphi(t_2,t_1,t_2) = \int_{t_2}^{t_1} \varphi_{,1}(t,t_1,t_2)dt$

$$= \int_{t_2}^{t_1}\int_0^1\int_0^1 \sum_{i=1}^{n}\{\dot{\lambda}_i(\alpha,\beta,t)f^i(z(\alpha,\beta,t),u_0(\alpha,t_1),v_0(\beta,t_2))$$

$$+\sum_{j=1}^{n}\lambda_i(\alpha,\beta,t)f^i_{,j}(z(\alpha,\beta,t),u_0(\alpha,t_1),v_0(\beta,t_2))\dot{z}^j(\alpha,\beta,t)\}d\alpha d\beta dt$$

$$=\int_{t_2}^{t_1}\int_0^1\int_0^1\{-\sum_{i=1}^{n}\sum_{j=1}^{n}[\lambda_j(\alpha,\beta,t)f^j_{;i}(z(\alpha,\beta,t),u_0(\alpha,t),v_0(\beta,t)).$$
$$f^i(z(\alpha,\beta,t),u_0(\alpha,t_1),v_0(\beta,t_2))$$
$$-\lambda_i(\alpha,\beta,t)f^i_{;j}(z(\alpha,\beta,t),u_0(\alpha,t_1),v_0(\beta,t_2))f^j(z(\alpha,\beta,t),$$
$$u_0(\alpha,t),v_0(\beta,t))]\}d\alpha d\beta dt.$$

(By substitution from the differential equations satisfied by z and λ).

Since the integrand in the above expression is bounded and $\mathcal{B}$-measurable on $[0,1) \times (0,1) \times [0,T]$, then there exists a constant $L_1 > 0$ such that $|\varphi(t_1,t_1,t_2)-\varphi(t_2,t_1,t_2)|<L_1|t_1-t_2|$ for all $t_1,t_2 \in [0,T]$ with $t_1 > t_2$. In exactly the same way it follows that there exists a constant $L_2 > 0$ such that $|\varphi(t_2,t_1,t_2)-\varphi(t_1,t_1,t_2)|<L_2|t_2-t_1|$ for all $t_1,t_2 \in [0,T]$ with $t_2 > t_1$. It then follows from (3) that

(5) $|\mathcal{H}(t_1)-\mathcal{H}(t_2)|\leqslant\max(L_1,L_2)|t_1-t_2|$

for almost every $(t_1,t_2) \in [0,T]^2$

and under these circumstances there exists a unique continuous function $\mathcal{H}_1:[0,T] \to R$ such that

(6) $\mathcal{H}_1(t) = \mathcal{H}(t)$ for a.e. $t \in [0,T]$.

From the inequality (5) and A1.1 iii) it follows that $\mathcal{H}_1$ is differentiable almost everywhere on $[0,T]$ and

(7) $\mathcal{H}_1(t_1)-\mathcal{H}_1(t_2) = \int_{t_2}^{t_1} \dot{\mathcal{H}}(t)dt$ for $t_1,t_2 \in [0,T]$.

We now show that $\dot{\mathcal{H}}(t) = 0$ for almost every $t \in [0,T]$.

Let $S = \{(t_1,t_2) \in [0,T]^2; \ t_1 > t_2\}$. Then from (3) we have

$$(9) \quad \frac{\mathcal{H}_1(t_1)-\mathcal{H}_1(t_2)}{t_1 - t_2} \geq \frac{\varphi(t_1,t_1,t_2)-\varphi(t_2,t_1,t_2)}{t_1 - t_2}$$

for a.e. $(t_1,t_2) \in S$.

Also, by 1.3 i) the function $w:[0,T] \to \mathcal{L}^2_q[0,1)$ defined by $[w(t)](\beta) = v_0(\beta,t)$ for $t \in [0,T]$, $\beta \in [0,1)$ is $\mathcal{B}$-measurable, and therefore by Lusin's theorem (see A3.1 in the Appendix) for each $r=1,2,\ldots$ there exists a $\mathcal{B}$-measurable subset A_r of $[0,T]$ with measure $\mu(A_r) \geq T - \frac{1}{r}$ such that w is continuous (with respect to the $\mathcal{L}^2_q[0,1)$ norm) on A_r. By elimination of at most a countable number of elements of A_r we can obtain a set A_r' such that for any $t \in A_r'$ there is an increasing sequence $\{t_s\}_{s=1}^{\infty}$ of A_r with $\lim_{s\to\infty} t_s = t$ and $\lim_{s\to\infty} w(t_s) = w(t)$.

If we then put

$$A = \bigcup_{r=1}^{\infty} A_r$$

$$A' = \bigcup_{r=1}^{\infty} A_r'$$

it follows that $\mu(A) = \mu(A') = T$ and that for any $t \in A'$ there exists an increasing sequence $\{t_s\}_{s=1}^{\infty}$ of A such that $\lim_{s\to\infty} t_s = t$, and

(10) $$\lim_{s\to\infty} \int_0^1 \|v_0(\beta,t) - v_0(\beta,t_s)\|^2 d\beta = 0$$

$$\text{(i.e. } \lim_{s\to\infty} w(t_s) = w(t))$$

Now (4) can be written as

(11) $$\varphi(t_1,t_1,t_2)-\varphi(t_2,t_1,t_2) = \int_{t_2}^{t_1}\int_0^1\int_0^1 \{\psi(\alpha,\beta,t)' f(z(\alpha,\beta,t),u_0(\alpha,t_1),v_0(\beta,t_2))$$

$$+\lambda(\alpha,\beta,t)' F(\alpha,\beta,t,t_1,t_2)\xi(\alpha,\beta,t)\}d\alpha d\beta dt$$

$$\text{for } t_1 > t_2$$

where F is a matrix function and ψ,ξ are vector functions given by

$$F_{ij}(\alpha,\beta,t) = f^j_{;i}(z(\alpha,\beta,t),u_0(\alpha,t_1),v_0(\beta,t_2))$$

$$\psi^i(\alpha,\beta,t) = -\sum_{j=1}^{n}\lambda_j(\alpha,\beta,t)f^j_{;i}(z(\alpha,\beta,t),u_0(\alpha,t),v_0(\beta,t))$$

$$\xi^j(\alpha,\beta,t) = f^j(z(\alpha,\beta,t),u_0(\alpha,t),v_0(\beta,t))$$

and if we put

$$T_1(t_1,t_2) = \int_{t_2}^{t_1}\int_0^1\int_0^1 \psi(\alpha,\beta,t)' f(z(\alpha,\beta,t),u_0(\alpha,t_1),v_0(\beta,t_2))\, d\alpha d\beta dt$$

$$T_2(t_1,t_2) = \int_{t_2}^{t_1}\int_0^1\int_0^1 \lambda(\alpha,\beta,t)' F(\alpha,\beta,t,t_1,t_2)\xi(\alpha,\beta,t)\, d\alpha d\beta dt.$$

Then (11) becomes

(12) $$\varphi(t_1,t_1,t_2) - \varphi(t_2,t_1,t_2) = T_1(t_1,t_2) + T_2(t_1,t_2),$$

and

$$(13)\quad \frac{T_1(t_1,t_2)}{t_1-t_2} - \int_0^1\int_0^1 \psi(\alpha,\beta,t_1)'f(z(\alpha,\beta,t_1),u_o(\alpha,t_1),v_o(\beta,t_1))d\alpha d\beta dt$$

$$= (t_1-t_2)^{-1}\Big\{\int_{t_2}^{t_1}\int_0^1\int_0^1 [\psi(\alpha,\beta,t)-\psi(\alpha,\beta,t_1)]'f(z(\alpha,\beta,t_1),u_o(\alpha,t_1),v_o(\beta,t_1))d\alpha d\beta dt$$

$$+ \int_{t_2}^{t_1}\int_0^1\int_0^1 \psi(\alpha,\beta,t)[f(z(\alpha,\beta,t),u_o(\alpha,t_1),v_o(\beta,t_2))-f(z(\alpha,\beta,t_1),u_o(\alpha,t_1),v_o(\beta,t_1))]d\alpha d\beta dt\Big\}.$$

Now for almost every $(\alpha,\beta,t_1)\in[0,1]\times[0,1]\times[0,T]$ $\lim_{r\to\infty} r\int_{t_1-\frac{1}{r}}^{t_1}\psi(\alpha,\beta,t)dt = \psi(\alpha,\beta,t_1)$ (which can be shown by using A1.1 and the boundedness and measurability of ψ) and it then follows by interchanging the integrals in the first term of the right hand side of (13) and using the Lebesgue dominated convergence theorem, that

$$\lim_{r\to\infty} r\int_{t_1-\frac{1}{r}}^{t_1}\int_0^1\int_0^1 [\psi(\alpha,\beta,t)-\psi(\alpha,\beta,t_1)]'f(z(\alpha,\beta,t_1),u_o(\alpha,t_1),v_o(\beta,t_1))d\alpha d\beta dt = 0$$
$$\text{for a.e. } t_1\in[0,T],$$

and this further implies that

$$(14)\quad \lim_{\delta\to 0^+} \delta^{-1}\int_{t_1-\delta}^{t_1}\int_0^1\int_0^1 [\psi(\alpha,\beta,t)-\psi(\alpha,\beta,t_1)]'f(z(\alpha,\beta,t_1),u_o(\alpha,t_1),v_o(\beta,t_1))d\alpha d\beta dt = 0$$
$$\text{for a.e. } t\in[0,T].$$

Thus, if we let B denote the set of all $t_1\in[0,T]$ such that (14) holds, then we have $\mu(B) = T$.

Now let G be a bound for ψ on $[0,1) \times [0,1) \times [0,T]$, and K a bound for the partial derivatives of f. Then the second term on the right hand side of (13) is dominated by

$$(15) \quad (t_1-t_2)^{-1}\int_{t_2}^{t_1}\int_0^1\int_0^1 G(nK\|z(\alpha,\beta,t)-z(\alpha,\beta,t_1)\| + qK\|v_0(\beta,t_1)-v_0(\beta,t_2)\|)\,d\alpha d\beta dt$$

$$\leqslant nGK \sup_{t\in[t_2,t_1]} \int_0^1\int_0^1 \|z(\alpha,\beta,t)-z(\alpha,\beta,t_1)\|\,d\alpha d\beta + qGK\left(\int_0^1 \|v_0(\beta,t_1)-v_0(\beta,t_2)\|^2 d\beta\right)^{\frac{1}{2}}.$$

(where we have used Schwarz' inequality on the second term). Now from A1.2, z, and hence $f(z(\cdot,\cdot,\cdot),u_0(\cdot,\cdot,\cdot),u_0(\cdot,\cdot))$ is bounded on $[0,1) \times [0,1) \times [0,T]$ and there exists a constant L such that $\|z(\alpha,\beta,t)-z(\alpha,\beta,t_1)\| \leqslant L|t-t_1|$ for $\alpha,\beta \in [0,1)$ and $t_1,t \in [0,T]$. Thus if $t_1 \in B \cap A'$ then there exists an increasing sequence t_s of A such that (10) holds and $\lim_{s\to\infty} t_s=t_1$. It then follows from (10), (13), (14) and (15), that

$$(16) \quad \lim_{s\to\infty} \frac{T_1(t_1,t_s)}{t_1 - t_s} = \int_0^1\int_0^1 \psi(\alpha,\beta,t_1)'f(z(\alpha,\beta,t_1), u_0(\alpha,t_1),v_0(\beta,t_1))\,d\alpha d\beta$$

$$= \int_0^1\int_0^1 \Big\{-\sum_{i=1}^n \sum_{j=1}^n \lambda_j(\alpha,\beta,t_1) f^j_{;i}(z(\alpha,\beta,t_1),u_0(\alpha,t_1),v_0(\beta,t_1)) f^i(z(\alpha,\beta,t_1),u_0(\alpha,t_1),v_0(\beta,t_1))\Big\}\,d\alpha d\beta.$$

In exactly the same way we can define a subset C of $[0,T]$ with $\mu(C)=T$, such that for any

$t_1 \in C \cap B \cap A'$ there exists an increasing sequence $\{t_s\}_{s=1}^{\infty}$ of A with limit t_1 such that (16) holds, and also

$$(17)\quad \lim_{s\to\infty} \frac{T_2(t_1,t_s)}{t_1 - t_s} = \int_0^1\int_0^1 \lambda(\alpha,\beta,t_1)'F(\alpha,\beta,t_1,t_1,t_1)\,\xi(\alpha,\beta,t_1)\,d\alpha d\beta$$

$$= \int_0^1\int_0^1 \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_j(\alpha,\beta,t_1)f^j_{;i}(z(\alpha,\beta,t_1),u_0(\alpha,t_1),v_0(\beta,t_1))\,f^i(z(\alpha,\beta,t_1),u_0(\alpha,t_1),v_0(\beta,t_1))\,d\alpha d\beta.$$

Consequently, from (9) and (12)

$$\liminf_{s\to\infty} \frac{H_1(t_1)-H(t_s)}{t_1 - t_s} \geq \lim_{s\to\infty} \frac{T_1(t_1,t_s)}{t_1 - t_s} + \lim_{s\to\infty} \frac{T_2(t_1,t_s)}{t_1 - t_s} = 0$$

Thus if H_1 is differentiable at t_1 and $t_1 \in C \cap B \cap A'$ then $\dot{H}_1(t_1) \geq 0$. This means that $\dot{H}(t) \geq 0$ for a.e. $t \in [0,T]$. In a similar way we may show that $\dot{H}(t) \leq 0$ for a.e. $t \in [0,T]$, and consequently that $\dot{H}(t) = 0$ for a.e. $t \in [0,T]$. The conclusion ii) then follows from (6) and (7). #

As a corollary we can state the minimax principle for arbitrary regular blind differential games of prescribed duration.

3.3 <u>Corollary</u>: Let U,V be compact subsets of R^p,R^q respectively, and $D = ((f,g,h,U,V,\theta),\mathcal{U},\mathcal{V})$ a regular blind differential game of prescribed duration T with payoff J and path function z. Suppose that $u_0:[0,1) \times [0,T] \to U$ and $v_0:[0,1) \times [0,T] \to V$ are

$\mathcal{B}$-measurable functions and $x_0 \in R^n$ such that

$$\int_0^1 J(x_0,u(\alpha,.),v)d\alpha \leq \int_0^1\int_0^1 J(x_0,u_0(\alpha,.),v_0(\beta,.))d\alpha d\beta$$
$$\leq \int_0^1 J(x_0,u,v_0(\beta,.))d\beta$$

for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$.

Let $\lambda:[0,1) \times [0,1) \times [0,T] \to R^n$ be the solution of the differential equation

$$\begin{cases} \dot{\lambda}_j(\alpha,\beta,t) = -\sum_{i=1}^{n} f^i_{,j}(z(x_0,u_0(\alpha,.),v_0(\beta,.),t), \\ \qquad u_0(\alpha,t),v_0(\beta,t))\lambda_i(\alpha,\beta,t) \\ \qquad -g_{,j}(z(x_0,u_0(\alpha,.),v_0(\beta,.),t),u_0(\alpha,t),v_0(\beta,t)) \\ \lambda_j(\alpha,\beta,T) = h_{,j+1}(T,z(x_0,u_0(\alpha,.),v_0(\beta,.),T)) \end{cases}$$

for a.e. $t \in [0,T]$ and $(\alpha,\beta) \in [0,1)^2$

and define $H:[0,1) \times [0,1) \times [0,T] \times U \times V \to R$ by

$$H(\alpha,\beta,t,u,v) = \sum_{i=1}^{n} \lambda_i(\alpha,\beta,t)f^i(z(x_0,u_0(\alpha,.),v_0(\beta,.),t),u,v)$$
$$+ g(z(x_0,u_0(\alpha,.),v_0(\beta,.),t),u,v)$$

Then:

i) For almost every $(\alpha,\beta,t) \in [0,1) \times [0,1) \times [0,T]$

a) $\displaystyle\int_0^1 H(\alpha,\beta,t,u_0(\alpha,t),v_0(\beta,t))d\beta = \inf_{u\in U}\int_0^1 H(\alpha,\beta,t,u,v_0(\beta,t))d\beta$

and

b) $\displaystyle\sup_{v\in V}\int_0^1 H(\alpha,\beta,t,u_0(\alpha,t),v)d\alpha = \int_0^1 H(\alpha,\beta,t,u_0(\alpha,t),v_0(\beta,t))d\alpha$

and

ii) $\int_0^1\int_0^1 H(\alpha,\beta,t,u_o(\alpha,t),v_o(\beta,t))d\alpha d\beta$ has the same value for almost every $t \in [0,T]$.

Proof: Define the regular n+1-dimensional blind differential game D_1 as in the proof of 2.3. D_1 has the form $((f_1,0,h_1,U,V,\theta_1),\mathcal{U},\mathcal{V})$ and the results of 3.2 can be applied to it to give the theorem. #

We note here that the result 3-1.2 giving necessary conditions for the optimality of pure strategies in a blind differential game of prescribed duration follows immediately from 3.3 as a special case.

The equations involved in the minimax principles of 3.2 and 3.3 provide a means of obtaining the mixed strategy solutions of a blind differential game. However, the solution of these equations will in general be hampered by two main difficulties. First, the form of the equations is of such a nature as to render their solution much more difficult than the corresponding equations for pure strategy solutions. Secondly, we observe that if $u_o:[0,1) \times [0,T] \to U$, $v_o:[0,1) \times [0,T] \to V$ are any solutions of the minimax equations, and ℓ_1,ℓ_2 are measure preserving transformations on $[0,1)$, then the functions $u_o(\ell_1(\cdot),\cdot)$, $v_o(\ell_2(\cdot),\cdot)$ will also satisfy the equations, so that if one solution exists then an infinite class of solutions will exist. This second difficulty might conceivably be ameliorated by placing further restrictions on the class of functions allowed as mixed strategies. If this could be done in such a way that any Borel measure on the strategy spaces is represented by exactly one function from the class allowed as mixed strategies, then presumably the minimax equations would in general have only a finite number of solutions subject to the given restrictions.

Finally we observe that the minimax equations of 3.3 may also be of help in obtaining the value and ε-optimal strategies of a game which has no saddle point. If the strategy spaces of a regular blind differential game satisfying the conditions of 3.3 are $\mathcal{U}$ and $\mathcal{V}$, and $\mathcal{U}\times\mathcal{V}$ is provided with a topology which makes it compact, and the payoff of the game continuous, then any sequence $\{(u_r, v_r)\}_{r=1}^{\infty}$ of <u>mixed</u> strategy pairs will have a subsequence which converges in a certain sense (which we shall not define here) to another mixed strategy pair (u^*, v^*). We conjecture that if (u_r, v_r) is a $\frac{1}{r}$ - optimal pair for each r, then (u^*, v^*) will satisfy the minimax equations of 3.3. Having solved these equations for (u^*, v^*) it should then be possible to reconstruct an ε-optimal strategy pair.

CHAPTER 6

CONCLUSIONS

1. Objectives

In this chapter we briefly review the aims of the research described in the previous chapters, and discuss the limitations and possible extensions of this work.

In Chapter 1 we stated that our objectives were "to describe the nature of strategies, both pure and mixed, which may be used "in differential games with no information, and "to find conditions under which they have solutions and to provide methods of determining these solutions when they do exist." We have, in fact, restricted our attention to games of prescribed duration whose 'dynamics' have a certain regularity property, so that the analysis might be carried out with complete rigor.

It has long been realised that the pure strategies of differential games with no information are just the so-called 'open-loop' strategies. In Chapters 3 and 4 we have extended the work of several authors by providing existence conditions for open-loop strategy solutions of games of different kinds and some necessary and sufficient conditions for the optimality of open-loop strategies. The necessary condition given in Chapter 3 provides a method for determining solutions of games in open-loop strategies if such solutions exist, and the sufficiency condition of the same Chapter provides a means of proving the optimality of these solutions.

The way in which mixed strategy spaces of differential games with no information should be defined has, to the present, been much less clear (e.g. see the footnote in [26] p.583). Chapter 5 has proposed the use of Borel measures (with respect to suitably defined topologies) for the role of mixed strategies, and shown that in most practical circumstances these can be represented as functions on the real unit interval with values in the pure strategy spaces. It was also shown in Chapter 5 that games of a reasonably comprehensive class have values in mixed strategies and that a more restricted type of game possess mixed strategy saddle points. The necessary conditions for the optimality of mixed strategies which were given in Chapter 5, also provide a basis for determining the value and optimal (or ε-optimal) mixed strategies, although it is doubtful whether the present form of these conditions can be solved in any practical circumstances.

2. Limitations

The major limitation of this work is the restrictive nature of the games considered. We shall discuss the three aspects of differential games on which we have placed most restriction. Namely:

i) The dynamics ii) The termination, and
iii) The information structure.

The restrictions on the dynamics were imposed in order to ensure the existence and uniqueness of a trajectory and payoff for any given pair of pure strategies. Although these conditions can be relaxed slightly (e.g. see Friedman [12]), some such will always be necessary if the desirable properties of existence and uniqueness are required. The formulation of the necessary conditions of Chapters 3 and 5 for arbitrary measurable functions also forced the nature of these conditions to be more restrictive than would otherwise have been necessary.

The confinement of our attention to games of prescribed duration was done for similar reasons. When games of a more general nature are considered, problems of whether or not termination will occur complicate the analysis, and discontinuities appear in the payoff function, so that the proof of existence theorems is rendered difficult. Although we feel these problems could be overcome, at least in part, their treatment here would have made this thesis much longer than was desirable.

The third restriction, that on the information structure, is by far the most serious, since most practically occurring games do not have an information structure of this kind. This drawback is partly offset by two considerations. First, as pointed out at the end of Chapter 2, pure strategy solutions of a game with no information constitute solutions of any game with the same dynamics, whatever its information structure. Secondly, we believe that a complete understanding of the nature of these games, which are the most simple kind of differential game with partial information, will indicate the way in which more general games are likely to be solved. This theme is taken up in the concluding section.

3. Finale

We now briefly indicate the way in which we think the results of this thesis will be of most use.

The first observation we make is a reiteration of the possible use of the necessary conditions of Chapter 5 as a starting point for solving differential games with no information. However, much work needs to be done before these conditions are likely to be solved in any realistic cases.

Second, and more important, we believe that the simple representation of mixed strategies given in Chapter 5 should be applicable to most differential games with partial information, and that similar necessary conditions for optimality of mixed strategies could easily be derived. The application of these ideas to differential games with an information lag, with noise corrupted measurements, and with intermittent interruptions in their information should yield interesting results.

APPENDIX

For the sake of completeness we collect here some of the standard and semistandard results used in the thesis and whose occurrence in the body of the work would be too destructive of continuity. Standard results are quoted without proof and a reference given, while proofs which follow frequently used lines of reasoning are merely sketched. The few special results of the Appendix whose occurrence is peculiar to this work are given complete proofs.

A1 DIFFERENTIABILITY THEOREMS AND DIFFERENTIAL EQUATIONS

In this section, the differentiability properties of absolutely continuous functions are listed, and some of the existence theorems and properties of solutions of differential equations used in the text are established.

A1.1 Theorem: If $\varphi:[0,T] \to R^n$ is absolutely continuous on $[0,T]$ then φ is differentiable almost everywhere on $[0,T]$, and

$$\int_a^b \varphi'(s)ds = \varphi(b) - \varphi(a) \quad \text{for} \quad a,b \in [0,T]$$

In particular

i) If $\psi:[0,T] \to R^n$ is integrable over $[0,T]$ then

$$\lim_{\delta t \to 0} \frac{\int_t^{t+\delta t} \psi(s)ds}{\delta t} = \psi(t) \quad \text{for a.e.} \ t \in [0,T]$$

ii) If there exists a constant K such that

$$\|\varphi(t_1)-\varphi(t_2)\| \leqslant K|t_1-t_2| \text{ for } t_1, t_2 \in [0,T]$$

then φ is differentiable almost everywhere on $[0,T]$ and

$$\int_a^b \varphi'(s)ds = \varphi(b) - \varphi(a) \quad \text{for} \quad a,b \in [0,T]$$

iii) If $\varphi:[0,T] \to R^n$ is absolutely continuous then $\|\varphi(.)\|$ is differentiable almost everywhere on $[0,T]$

$$\int_a^b \frac{d}{dt}\|\varphi(t)\|dt = \|\varphi(b)\|-\|\varphi(a)\| \quad \text{for} \quad a,b \in [0,T]$$

and
$$\frac{d}{dt}\|\varphi(t)\| \leqslant \|\dot{\varphi}(t)\|.$$

<u>Proof</u>: The definition of absolutely continuous scalar functions is given in Dunford and Schwartz ([36]p.242).

Absolutely continuous vector functions are defined in an exactly similar way. All the statements of the above theorem except (iii) follow immediately from standard results of measure theory (e.g. see [36] p.210ff.)

If $\varphi:[0,T] \to R^n$ is absolutely continuous and $t_i \in [0,T]$ for $i = 1,2...,m$ are such that $t_1 < t_2 < ... < t_m$, then

$$\sum_{i=1}^{m-1} \big|\, \|\varphi(t_{i+1})\| - \|\varphi(t_i)\| \,\big| \leqslant \sum_{i=1}^{m} \|\varphi(t_{i+1}) - \varphi(t_i)\|$$

and hence $\|\varphi(.)\|$ is absolutely continuous.

Since $\big|\, \|\varphi(t+\delta t)\| - \|\varphi(t)\| \,\big| \leqslant \|\varphi(t+\delta t) - \varphi(t)\|$

for $t, t+\delta t \in [0,T]$ then

$$\lim_{\delta t \to 0} \left| \frac{\|\varphi(t+\delta t)\| - \|\varphi(t)\|}{\delta t} \right| \leq \lim_{\delta t \to 0} \frac{\|\varphi(t+\delta t) - \varphi(t)\|}{|\delta t|}$$

for a.e. $t \in [0,T]$

and (iii) then follows from the previous statements. #

The following result justifies the assertion made in Chapter 2 that the open-loop strategies of a regular differential game form playable strategy sets, and is used several times elsewhere in the text (e.g. in Chapters 3 and 5). The playability of the sets of Lipschitz strategies may be established in a similar manner.

A1.2 <u>Theorem</u>: Let A be an open, convex subset of R^s, and W a compact subset of A.

Let $f: R^n \times A \times [0,T] \times [0,1) \to R^n$ be a $\mathcal{B}$-measurable function such that:

a) For each $(t,\alpha) \in [0,T] \times [0,1)$, $f(\cdot,\cdot,t,\alpha)$ has partial derivatives which are continuous and bounded uniformly with respect to $(t,\alpha) \in [0,T] \times [0,1)$

b) For each $(x,w) \in R^n \times W$, $f(x,w,\cdot,\cdot)$ is bounded.

Then for each $x \in R^n$, $w \in \mathcal{B}_W[0,T]$, $\alpha \in [0,1)$ there exists a unique absolutely continuous function $z(x,w,\alpha,\cdot):[0,T] \to R^n$ satisfying the differential equation.

$$(1) \quad \begin{cases} \dot{z}(x,w,\alpha,t) = f(z(x,w,\alpha,t),w(t),t,\alpha) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t \in [0,T] \\ z(x,w,\alpha,0) = x \end{cases}$$

This function has the following properties

i) For each $x \in R^n$, $w \in \mathcal{B}_W[0,T]$ the function $z(x,w,.,.)$ is $\mathcal{B}$-measurable on $[0,1) \times [0,T]$

ii) For each $x \in R^n$ the function $z(x,.,.,.)$ is bounded on $\mathcal{B}_W[0,T] \times [0,1) \times [0,T]$

iii) There exist constants K_1, K_2 such that

$$\|z(x,w_1,\alpha,t)-z(x,w_2,\alpha,t)\| \leq K_1\int_0^T\|w_1(y)-w_2(y)\|dy$$

and

$$\|z(x,w_1,\alpha,t)-z(x,w_2,\alpha,t)\| \leq K_2\left\{\int_0^T\|w_1(y)-w_2(y)\|^2dy\right\}^{\frac{1}{2}}$$

for every $(x,t,\alpha) \in R^n \times [0,T] \times [0,1)$ and every $w_1, w_2 \in \mathcal{B}_W[0,T]$.

<u>Proof</u>: We denote $\mathcal{B}_W[0,T]$ throughout by $\mathcal{W}$.

Let Y be a compact subset of R^n. Then we shall show that f is bounded on $Y \times W \times [0,T] \times [0,1)$.

Let $y_0 \in Y$ and $w_0 \in W$. Then for any $(y,w,t,\alpha) \in Y \times W \times [0,T] \times [0,1)$

$$f(y,w,t,\alpha) =$$

$$f(y_0,w_0,t,\alpha) +$$

$$+\sum_{j=1}^{n}\left\{\int_0^1 f_{,j}(y_0+r(y-y_0),w_0+r(w-w_0),t,\alpha)dr(y^j-y_0^j)\right\}$$

$$+\sum_{j=1}^{s}\left\{\int_0^1 f_{,n+j}(y_0+r(y-y_0),w_0+r(w-w_0),t,\alpha)dr(w^j-w_0^j)\right\}$$

Now let K be a (uniform) bound for the partial derivatives of f, and L be a bound for $f(y_o,w_o,\cdot,\cdot)$. Then the above equation gives

$$\|f(y,w,t,\alpha)\| \leqslant L + nK\|y-y_o\| + sK\|w-w_o\|$$
$$\text{for } (y,w,t,\alpha) \in Y \times W \times [0,T] \times [0,1)$$

and since Y,W were compact it follows that f is bounded on $Y \times W \times [0,T] \times [0,1)$.

Thus the Picard iterates

$z_r : R^n \times \mathcal{W} \times [0,1) \times [0,T] \to R^n$ given by

$$z_o(x,w,\alpha,t) = x$$
$$z_{r+1}(x,w,\alpha,t) = x + \int_0^t f(z_{r-1}(x,w,\alpha,y),w(y),y,\alpha)dy$$

for $(x,w,\alpha,t) \in R^n \times \mathcal{W} \times [0,1) \times [0,T]$ and $r = 0,1,2,\ldots,$ can be shown to be well defined by induction, and to satisfy the inequality

$$(2)\quad \|z_{r+1}(x,w,\alpha,t)-z_r(x,w,\alpha,t)\| \leqslant \frac{(nKt)^r}{r!}\int_0^t \|f(x,w(y),y,\alpha)\|dy$$
$$\leqslant T\,A(x)\,\frac{(nKt)^r}{r!}$$

for $r=0,1,2,\ldots$ and every $(x,w,\alpha,t) \in R^n \times \mathcal{W} \times [0,1)\times[0,T]$; where $A(x)$ is a bound for $\|f(x,\cdot,\cdot,\cdot)\|$ on $W \times [0,T] \times [0,1)$.

It follows that for each $x \in R^n, \{z_r(x,\cdot,\cdot,\cdot)\}_{r=1}^{\infty}$ converges uniformly on $\mathcal{W} \times [0,1) \times [0,T]$ to a function z_x which satisfies

(3) $z_x(w,\alpha,t) = x + \int_0^t f(z_x(w,\alpha,y),w(y),y,\alpha)dy$

for $(w,\alpha,t) \in \mathcal{W} \times [0,1) \times [0,T]$.

Thus $z_x(w,\alpha,0) = x$ and differentiation of (3) establishes the existence of the solution to (1).

To prove the uniqueness of this solution we fix $(x,w,\alpha) \in R^n \times \mathcal{W} \times [0,1)$ and let

$$z_1(t) = z_x(w,\alpha,t) \quad \text{for} \quad t \in [0,T].$$

Now suppose that $z_2:[0,T] \to R^n$ is absolutely continuous and also satisfies (1).

Let $M = \int_0^T \|f(z_2(y),w(y),y,\alpha)\|dy$.

Then we have

$$z_2(t) = x + \int_0^t f(z_2(y),w(y),y,\alpha)dy$$

$$\text{for} \quad t \in [0,T]$$

and hence

$$\|z_2(t) - x\| \leq M \quad \text{for} \quad t \in [0,T].$$

It can now be established by induction that z_2, z_r satisfy the inequalities

(4) $\|z_2(t) - z_r(x,w,\alpha,t)\| \leq M \dfrac{(nKt)^r}{r!}$ for $t \in [0,T]$

By taking limits in (4) as $r \to \infty$ we obtain immediately $z_2(t) = z_1(t)$ for $t \in [0,T]$.

Uniqueness of the solution of (1) is consequently established.

Property i) follows immediately since for each $(x,w,) \in R^n \times \mathcal{W}$ the rth iterate $z_r(x,w,.,.)$ is $\mathcal{B}$-measurable on $[0,1) \times [0,T]$ and the sequence $\{z_r(x,w,.,.)\}_{r=1}^{\infty}$ converges uniformly on $[0,1) \times [0,T]$ to $z(x,w,.,.)$.

Property ii) can be proved by establishing the inequality

$$\|z_{r+1}(x,w,\alpha,t) - x\| \leq TA(x)\sum_{j=0}^{r}\frac{(nKt)^j}{j!}$$

$$\text{for } (x,w,\alpha,t) \in R^n \times \mathcal{W} \times [0,1) \times [0,T]$$

and $r = 0,1,2,\ldots$ by induction. Taking limits as $r \to \infty$ gives $\|z(x,w,\alpha,t) - x\| \leq TA(x)e^{Knt}$

$$\text{for } (x,w,\alpha,t) \in R^n \times \mathcal{W} \times [0,1) \times [0,T]$$

Property iii) can be proved by establishing a similar inequality. Namely

$$\|z_r(x,w_1,\alpha,t)-z_r(x,w_2,\alpha,t)\| \leq sK\int_0^T\|w_1(y)-w_2(y)\|dy.\sum_{j=0}^{r-1}\frac{(nKt)^j}{j!}$$

$$\text{for } (x,\alpha,t) \in R^n \times [0,1) \times [0,T];$$

$w_1,w_2 \in \mathcal{W}$ and $r = 1,2,\ldots$. Again taking limits yields

$$\|z(x,w_1,\alpha,t)-z(x,w_2,\alpha,t)\| \leq sKe^{Kt}\int_0^T\|w_1(y)-w_2(y)\|dy$$

$$\leq sKe^{KT}\int_0^T\|w_1(y)-w_2(y)\|dy$$

$$\text{for } (x,\alpha,t) \in R^n \times [0,1) \times [0,T]; w_1,w_2 \in \mathcal{W}$$

Schwarz' inequality now gives

$$\|z(x,w_1,\alpha,t) - z(x,w_2,\alpha,t)\| \leq sKe^{KT}T\left(\int_0^T \|w_1(y)-w_2(y)\|^2 dy\right)^{\frac{1}{2}}$$

#

In the proof of 5-3.1 we use a further property of the solutions of equation (1) of A1.2. This property is now established.

A1.3 <u>Lemma</u>: Let A,W,f be as in A1.2, let $w_0 \in \mathscr{B}_W[0,T]$ and $x_0 \in R^n$. For each $(w,\alpha)\in\mathscr{B}_W[0,T]\times[0,1)$ let $z(w,\alpha,.)$ denote the (unique) absolutely continuous function satisfying the differential equation

$$(1)\quad \begin{cases} \dot z(w,\alpha,t) = f(z(w,\alpha,t),w(t),t,\alpha) & \text{a.e. } t\in[0,T] \\ z(w,\alpha,0) = x_0, & \end{cases}$$

let $z_1(\alpha,t) = z(w_0,\alpha,t)$ for $t\in[0,T]$ and let $\delta z(w,\alpha,.)$ denote the (unique) absolutely continuous solution of the equation

$$(2)\quad \begin{cases} \delta\dot z(w,\alpha,t) = \sum_{i=1}^{n} f_{,i}(z_1(\alpha,t),w_0(t),t,\alpha)\,\delta z^i(w,\alpha,t) \\ \qquad + f(z(w,\alpha,t),w(t),t,\alpha) - f(z(w,\alpha,t),w_0(t),t,\alpha) \\ \qquad\qquad \text{a.e. } t\in[0,T] \\ \delta z(w,\alpha,0) = 0. \end{cases}$$

Then

$$\frac{\|z(w,\alpha,t) - z_1(\alpha,t) - \delta z(w,\alpha,t)\|}{\|w - w_0\|_1} \to 0 \text{ uniformly}$$

on $[0,1)\times[0,T]$ as $\|w - w_0\|_1 \to 0$.

<u>Proof</u>: Let A,B be the n×n matrix functions defined on $[0,1) \times [0,T]$ and $\mathcal{W} \times [0,1) \times [0,T]$ respectively by [1]

$$A_{ij}(\alpha,t) = f^i_{,j}(z_1(\alpha,t),w_0(t),t,\alpha) \quad \text{for } i,j = 1,2,\dots n$$
$$\text{and } (\alpha,t) \in [0,1) \times [0,T]$$

$$B_{ij}(w,\alpha,t) = \int_0^1 f^i_{,j}(z_1(\alpha,t)+r(z(w,\alpha,t)-z_1(\alpha,t)),w_0(t),t,\alpha)dr$$
$$\text{for } i,j=1,2,\dots n \text{ and } (w,\alpha,t) \in \mathcal{W} \times [0,1) \times [0,T]$$

and let $b:\mathcal{W} \times [0,1) \times [0,T] \to R^n$ be given by

$$b(w,\alpha,t) = f(z(w,\alpha,t),w(t),t,\alpha) - f(z(w,\alpha,t),w_0(t),t,\alpha)$$
$$\text{for } (w,\alpha,t) \in \mathcal{W} \times [0,1) \times [0,T].$$

Then equation (2) may be written

$$(3) \quad \delta\dot{z}(w,\alpha,t) = A(\alpha,t)\delta z(w,\alpha,t) + b(w,\alpha,t)$$

And if $dz(w,\alpha,t) = z(w,\alpha,t) - z_1(\alpha,t)$, then we have

$$\begin{aligned} d\dot{z}(w,\alpha,t) &= f(z(w,\alpha,t),w(t),t,\alpha) - f(z_1(\alpha,t),w_0(t),t,\alpha) \\ &= f(z(w,\alpha,t),w_0(t),t,\alpha) - f(z_1(\alpha,t),w_0(t),t,\alpha) + \\ &+ f(z(w,\alpha,t),w(t),t,\alpha) - f(z(w,\alpha,t),w_0(t),t,\alpha) \\ &= B(w,\alpha,t)(z(w,\alpha,t) - z_1(\alpha,t)) + b(w,\alpha,t) \end{aligned}$$
$$\text{for } (w,\alpha,t) \in \mathcal{W} \times [0,1) \times [0,T]$$

That is

$$(4) \quad d\dot{z}(w,\alpha,t) = B(w,\alpha,t)dz(w,\alpha,t) + b(w,\alpha,t)$$

Now if K is a bound for the partial derivatives of f then

---

[1] In this proof we have again used $\mathcal{W}$ to denote $\mathcal{B}_W[0,T]$

$\|A(\alpha,t)\| \leq nK$

$\|B(w,\alpha,t)\| \leq nK$

and

$\|b(w,\alpha,t)\| \leq sK\|w(t) - w_o(t)\|$

$$\text{for} \quad (w,\alpha,t) \in \mathcal{W} \times [0,1) \times [0,T].$$

Substitution of the inequalities for B and b in (4) gives

(5) $$\|d\dot{z}(w,\alpha,t)\| \leq nK\|dz(w,\alpha,t)\| + sK\|w(t) - w_o(t)\|$$
$$\text{for} \quad (w,\alpha,t) \in \mathcal{W} \times [0,1) \times [0,T].$$

By A1.1 iii) $\|dz(w,\alpha,.)\|$ is absolutely continuous and $\frac{d}{dt}\|dz(w,\alpha,t)\| \leq \|d\dot{z}(w,\alpha,t)\|$.

(5) may therefore be integrated to give

$$\|dz(w,\alpha,t)\| \leq sK\int_0^t e^{nK(t-y)}\|w(y)-w_o(y)\|dy$$

$$\leq L\|w-w_o\|_1 \quad \text{for} \quad (w,\alpha,t) \in \mathcal{W} \times [0,1)\times[0,T]$$

$$\text{where} \quad L = e^{nKT}sK$$

(3) and (4) now give

$$\begin{aligned}\|\delta\dot{z}(w,\alpha,t) - d\dot{z}(w,\alpha,t)\| &= \|(A(\alpha,t) - B(w,\alpha,t))dz(w,\alpha,t) \\ &\quad + A(\alpha,t)(\delta z(w,\alpha,t) - dz(w,\alpha,t)\| \\ &\leq \|A(\alpha,t) - B(w,\alpha,t)\| \, \|dz(w,\alpha,t)\| \\ &\quad + \|A(\alpha,t)\| \, \|\delta z(w,\alpha,t)-dz(w,\alpha,t)\| \\ &\quad \text{for} \quad (w,\alpha,t) \in \mathcal{W} \times [0,1) \times [0,T].\end{aligned}$$

Again this inequality can be integrated to give

$$\|\delta z(w,\alpha,t)-dz(w,\alpha,t)\| \leq L\|w-w_0\|_1 \int_0^T e^{nK(t-y)}\|A(\alpha,y)-B(w,\alpha,y)\|dy$$

and thus

$$\frac{\|z(w,\alpha,t)-z_1(\alpha,t)-\delta z(w,\alpha,t)\|}{\|w - w_0\|_1} \leq L_1\int_0^T \|A(\alpha,y)-B(w,\alpha,y)\|dy$$

for $(w,\alpha,t) \in \mathcal{W} \times [0,1) \times [0,T]$

where $L_1 = Le^{nKT}$.

Now if $\varepsilon > 0$, then by the uniform continuity of $f_{,j}$ with respect to its first n arguments, and the property A1.2 iii) satisfied by z then there exists $\delta > 0$ such that

$$\|w-w_0\|_1 < \delta \Rightarrow \|A(\alpha,y) - B(w,\alpha,y)\| < \varepsilon/L_1 T$$

for every $(\alpha,y) \in [0,1) \times [0,T]$.

Then $0 < \|w-w_0\|_1 < \delta$

$$\Rightarrow \frac{\|z_1(w,\alpha,t) - z_1(\alpha,t) - \delta z(w,\alpha,t)\|}{\|w - w_0\|_1} < \varepsilon$$

for $(\alpha,y) \in [0,1) \times [0,T]$.

This proves the Lemma. #

A2 MINIMAX THEOREMS

The minimax theorems used in the text are all special results derived from the theorems of Fan [40], and Sion [41] both of which are stated below. Fan's theorem requires the concepts of convexity and concavity of a function on spaces without a linear structure, and Sion's requires the concepts of quasi-convexity and quasi-concavity.

A2.1 <u>Definition</u>: a) If M,N are sets and f a real valued function on $M \times N$ then f is said to be:

i) <u>convex</u> on M if for every $m_1, m_2 \in M$ and $\lambda \in [0,1]$ there exists $m \in M$ such that

$$\lambda f(m_1,n) + (1-\lambda)f(m_2,n) \geq f(m,n) \quad \text{for every} \quad n \in N$$

ii) <u>concave</u> on N if for every $n_1, n_2 \in N$ and $\lambda \in [0,1]$ there exists $n \in N$ such that

$$\lambda f(m,n_1) + (1-\lambda)f(m,n_2) \leq f(m,n) \quad \text{for every} \quad m \in M$$

b) If X is a real linear space M a subset of X and f a real valued function on M then f is said to be <u>quasi-convex</u> (quasi-concave) on M if for every real α the set

$$\{x \in M; \ f(x) \leq \alpha\} \quad (\text{the set } \{x \in M; \ f(x) \geq \alpha\})$$

is convex. #

The minimax theorems now follow.

A2.2 <u>Theorem:</u> i) (Fan's minimax theorem). Let X,Y be compact Hausdorff spaces and f a real valued function on $X \times Y$ such that for every $x \in X$, $f(x,.)$ is upper semicontinuous on Y and for every $y \in Y$, $f(.,y)$ is lower semicontinuous on X. Then if f is convex on X and concave on Y

$$\min_{x \in X} \max_{y \in Y} f(x,y) = \max_{y \in Y} \min_{x \in X} f(x,y)$$

ii) (Sion's minimax theorem). Let M,N be compact convex subsets of the topological linear spaces X,Y respectively, and let f be a real valued function on

$M \times N$ such that for every $x \in M$, $f(x,.)$ is upper semicontinuous and quasi-concave on N and for every $y \in N$, $f(.,y)$ is lower semicontinuous and quasi-convex on M. Then

$$\min_{x\in M}\max_{y\in N} f(x,y) = \max_{y\in N}\min_{x\in M} f(x,y)$$

#

Fan's theorem may be used to prove the existence of saddle points in mixed strategies of a game with a continuous payoff on compact metric spaces. The proof follows.

A2.3 <u>Theorem</u>: Let X,Y be compact metric spaces and f a real valued continuous function on $X\times Y$. Let $\mathcal{X},\mathcal{Y}$ denote the set of all Borel probability measures on X,Y respectively, and let $\mathcal{F}:\mathcal{X} \times \mathcal{Y} \to R$ be the function defined by

$$\mathcal{F}(\nu_1,\nu_2) = \int_{X\times Y} f(x,y)d(\nu_1\times\nu_2)(x,y)$$

$$\text{for } (\nu_1,\nu_2) \in \mathcal{X} \times \mathcal{Y}$$

Then

$$\min_{\nu_1\in\mathcal{X}}\max_{\nu_2\in\mathcal{Y}} \mathcal{F}(\nu_1,\nu_2) = \max_{\nu_2\in\mathcal{Y}}\min_{\nu_1\in\mathcal{X}} \mathcal{F}(\nu_1,\nu_2).$$

<u>Proof</u>: With respect to their Prohorov (see Billingsley [39]) topologies, $\mathcal{X}$ and $\mathcal{Y}$ are compact Hausdorff spaces, and since f is (uniformly) continuous on $\mathcal{X} \times \mathcal{Y}$ then $\int_Y f(.,y)d\nu_2(y)$ is continuous on X for every $\nu_2 \in \mathcal{Y}$, and $\mathcal{F}(.,\nu_2)$ is therefore continuous on $\mathcal{X}$ (with respect to the

Prohorov topology) for every $\nu_2 \in \mathcal{Y}$.

Similarly $\mathcal{F}(\nu_1,.)$ is continuous on $\mathcal{Y}$ for every $\nu_1 \in \mathcal{X}$.

It is not difficult to show that for each $\nu_1 \in \mathcal{X}$, $\nu_2 \in \mathcal{Y}$, $\mathcal{F}(\nu_1,.)$ is concave on $\mathcal{Y}$ and $\mathcal{F}(.,\nu_2)$ is convex on $\mathcal{X}$.

The result now follows from Fan's minimax theorem A2.2 i). #

A2.4 <u>Theorem</u>: Let X,Y be precompact metric spaces with metrics ρ_1,ρ_2 respectively and f a real valued function which is uniformly continuous on $X \times Y$.

(Under these circumstances, if $\bar{X},\bar{Y}$ denote the completions of X,Y respectively, then the function f has a unique continuous extension to $\bar{X} \times \bar{Y}$.)

Let $\mathcal{X},\mathcal{Y}$ denote the set of Borel probability measures on X,Y respectively, and let $\mathcal{F}:\mathcal{X} \times \mathcal{Y} \to R$ be the function defined by

$$\mathcal{F}(\nu_1,\nu_2) = \int_{X \times Y} f(x,y)\, d(\nu_1 \times \nu_2)(x,y) \quad \text{for} \quad (\nu_1,\nu_2) \in \mathcal{X} \times \mathcal{Y}$$

Then

$$\inf_{\nu_1 \in \mathcal{X}} \sup_{\nu_2 \in \mathcal{Y}} \mathcal{F}(\nu_1,\nu_2) = \sup_{\nu_2 \in \mathcal{Y}} \inf_{\nu_1 \in \mathcal{X}} \mathcal{F}(\nu_1,\nu_2).$$

<u>Proof</u>: Let $\bar{X},\bar{Y}$ denote the completions of X,Y respectively, and $\bar{f}$ the (unique) continuous extension of f to $\bar{X} \times \bar{Y}$. Let $\bar{\mathcal{X}},\bar{\mathcal{Y}}$ denote the sets of Borel probability

measures on $\overline{X}, \overline{Y}$ respectively and define the function $\overline{\mathcal{F}}: \overline{\mathfrak{X}} \times \overline{\mathcal{Y}} \to R$ by

$$\overline{\mathcal{F}}(\nu_1, \nu_2) = \int_{\overline{X} \times \overline{Y}} \overline{f}(x,y) d(\nu_1 \times \nu_2)(x,y) \quad \text{for } \nu_1, \nu_2 \in \overline{\mathfrak{X}} \times \overline{\mathcal{Y}}.$$

Then $\overline{X}, \overline{Y}$ are compact and $\overline{f}$ is continuous on $\overline{X} \times \overline{Y}$, so that by A2.3 there exist $\nu_1^* \in \overline{\mathfrak{X}}$, $\nu_2^* \in \overline{\mathcal{Y}}$ such that

(1) $$\overline{\mathcal{F}}(\nu_1^*, \nu_2) \leq \overline{\mathcal{F}}(\nu_1^*, \nu_2^*) \leq \overline{\mathcal{F}}(\nu_1, \nu_2^*)$$
for every $(\nu_1, \nu_2) \in \overline{\mathfrak{X}} \times \overline{\mathcal{Y}}$.

Let $\overline{\mathcal{F}}(\nu_1^*, \nu_2^*) = V$, and $\varepsilon > 0$. Then there exists a finite partition $\mathcal{S} = \{S_1, S_2, \ldots S_r\}$ of $\overline{X}$ consisting of Borel sets S_j; $j = 1,2,\ldots r$, with non-empty interiors, such that for each j

$$|\overline{f}(x,y) - \overline{f}(x',y)| \leq \varepsilon \quad \text{for } x', x \in S_j \text{ and } y \in \overline{Y}$$

Since each S_j has non-empty interior there exists an element $x_j \in S_j \cap X$.

Let $\nu_1 \in \mathcal{F}$ be the atomic measure with atoms x_j, and $\nu_1(\{x_j\}) = \nu_1^*(S_j)$ for $j = 1,2,\ldots r$.

Then for every $y \in Y$

$$\left|\int_X f(x,y) d\nu_1(x) - \int_{\overline{X}} \overline{f}(x,y) d\nu_1^*(x)\right|$$

$$= \left|\sum_{j=1}^{r} (f(x_j,y)\nu_1^*(S_j) - \int_{\overline{X}} \overline{f}(x,y) d\nu_1^*(x)\right|$$

$$\leq \sum_{j=1}^{r} \int_{S_j} |\overline{f}(x_j,y) - \overline{f}(x,y)| d\nu_1^*(x) \leq \varepsilon$$

Thus

$$(3) \quad \int_X f(x,y)d\nu_1(x) \leq \overline{\int_X} \bar{f}(x,y)d\nu_1^*(x) + \varepsilon \leq V + \varepsilon \quad \text{for } y \in Y \text{ by}$$

virtue of (1).

Therefore $\mathcal{F}(\nu_1, \nu_2) \leq V + \varepsilon$ for every $\nu_2 \in \mathcal{Y}$ (by integrating (3) with respect to ν_2).

Consequently

$$\sup_{\nu_2 \in \mathcal{Y}} \mathcal{F}(\nu_1, \nu_2) \leq V + \varepsilon,$$

and

$$\inf_{\nu \in \mathcal{X}} \sup_{\nu_2 \in \mathcal{Y}} \mathcal{F}(\nu, \nu_2) \leq V + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary then

$$(4) \qquad \inf_{\nu \in \mathcal{X}} \sup_{\nu_2 \in \mathcal{Y}} \mathcal{F}(\nu, \nu_2) \leq V$$

Similarly, it may be shown that

$$(5) \qquad \sup_{\nu \in \mathcal{Y}} \inf_{\nu' \in \mathcal{X}} \mathcal{F}(\nu', \nu) \geq V$$

But since $\sup_{\nu_2 \in \mathcal{Y}} \inf_{\nu_1 \in \mathcal{X}} \mathcal{F}(\nu_1, \nu_2) \leq \inf_{\nu_1 \in \mathcal{X}} \sup_{\nu_2 \in \mathcal{Y}} \mathcal{F}(\nu_1 \nu_2)$,

then the result follows from (4) and (5). #

A3 LUSIN'S THEOREM AND MEASURABILITY.

In the proof of 5-3.2 we used Lusin's theorem concerning the continuity of a measurable function on a closed subset of its domain. In the standard texts on measure theory (e.g. [42]) this is usually shown only for functions with values in finite dimensional Euclidean spaces. Since we require the theorem for functions with values in separable Hilbert spaces we give a proof of the theorem here.

A3.1 <u>Theorem</u> (Lusin) Let $\mathcal{H}$ be a real, separable Hilbert space, A a Borel subset of R^n and $u:A \to \mathcal{H}$ a $\mathcal{B}$-measurable function. Then for any $\varepsilon > 0$ there exists a closed subset B of A with $\mu(B) \geq \mu(A) - \varepsilon$ such that u is continuous on B.

<u>Proof</u>: Let $\{x_r\}_{r=1}^{\infty}$ be the enumeration of a countable dense subset of $\mathcal{H}$, and

$$\mathcal{S} = \{N(x_i, \tfrac{1}{r});\quad i = 1,2,\ldots;\ r = 1,2,\ldots\} \cup \{\emptyset\}$$

Then $\mathcal{S}$ is a countable collection of open sets, and any open set in $\mathcal{H}$ can be expressed as the union of a subcollection of $\mathcal{S}$. (Namely, the subcollection consisting of those sets of $\mathcal{S}$ which are contained in the given open set).

If $\mathcal{S}_1 = \{\mathcal{H}-N(x_i, \tfrac{1}{r});\ i = 1,2,\ldots;\ r = 1,2,\ldots\} \cup \{\mathcal{H}\}$ then $\mathcal{S}_1$ is a countable collection of closed sets, and any closed set of $\mathcal{H}$ may be expressed as the intersection of a subcollection of $\mathcal{S}_1$ (namely, that subcollection consisting of the sets in $\mathcal{S}_1$ which contain the given closed set).

Let $\{C_r\}_{r=1}^{\infty}$ be an enumeration of $\mathcal{S}_1$ and let $\varepsilon > 0$.

For each r, $u^{-1}(C_r)$ and $A-u^{-1}(C_r)$ are measurable and therefore contain closed subsets $B_r^{(1)}$, $B_r^{(2)}$ respectively with

$$\mu(B_r^{(1)}) \geq \mu(u^{-1}(C_r)) - \frac{\varepsilon}{2^{r+1}}$$
$$\mu(B_r^{(2)}) \geq \mu(A-u^{-1}(C_r)) - \frac{\varepsilon}{2^{r+1}}$$

(this regularity property of Lebesgue measure μ is established, for example, by Billingsley in [39] (p.7)).

Now put $B_r = B_r^{(1)} \cup B_r^{(2)}$. Then B_r is a closed subset of A with $\mu(B_r) \geq \mu(A) - \frac{\varepsilon}{2^r}$, and $B_r \cap u^{-1}(C_r) = B_r^{(1)}$ is closed.

Hence if $B = \bigcap_{r=1}^{\infty} B_r$, then B is a closed subset of A with $\mu(B) \geq \mu(A) - \varepsilon$, and for each r

$$B \cap u^{-1}(C_r) = B \cap B_r \cap u^{-1}(C_r) \text{ (since } B \subset B_r)$$
is closed.

If now C is any closed subset of $\mathcal{H}$ and $J = \{r; C \subset C_r\}$

Then $$C = \bigcap_{r\in J} C_r$$

and $$B \cap u^{-1}(C) = B \cap \bigcap_{r\in J} u^{-1}(C_r)$$

$$= \bigcap_{r\in J} (B \cap u^{-1}(C_r)) \text{ is closed.}$$

Consequently u is continuous on B, and the theorem is proved. #

In the proof of 5-.2.6 we used the concept of lexicographic supremum (lex.sup.) to define a unique function satisfying a given property and which was required to be $\mathcal{B}$-measurable. The notion of lexicographic supremum is defined below and the required measurability is then established.

A3.2 Definition: i) If $x_1, x_2 \in R^n$ with $x_1 \neq x_2$, if j is the smallest positive integer i such that $x_1^i \neq x_2^i$ and if $x_1^j < x_2^j$ then we write $x_1 \lhd x_2$.

If either $x_1 \lhd x_2$ or $x_1 = x_2$ then we say x_2 dominates x_1 lexicographically, and write $x_1 \unlhd x_2$.

ii) If $S \subset R^n$, $x \in R^n$ and $y \unlhd x$ for every $y \in S$ then x is called a lexicographic upper bound (abbreviated lex.u.b.) of S.

iii) If $S \subset R^n$ and $x \in R^n$ is a lex.u.b. of S such that $x \unlhd y$ for any y which is also a lex.u.b. of S then x is called a lexicographic supremum (abbreviated lex.sup) of S, and is denoted by lex.sup S (such an x must be unique). #

The relation $\unlhd$ is transitive, reflexive and anti-symmetric on R^n, and any two elements are comparable with respect to $\unlhd$ which therefore constitutes a total ordering of R^n. The uniqueness of a lex.sup of a set follows from the antisymmetry of $\unlhd$, but it is not true in general that an arbitrary subset of R^n possesses a lex.sup.. In the case of a compact set S, lex.sup S always exists and lies in S.

A3.3 <u>Lemma</u>: Let S be a non-empty compact subset of R^n. Then S has a lex.sup x with $x \in S$.

<u>Proof</u>: The proof is by induction on n.

For subsets of R, the lex.sup is just the ordinary supremum and the lemma is thus true.

Suppose the result is true for $n=m$, and let S be a non-empty compact subset of R^{m+1}.

Let $S_1 = \{x \in R^m; (x,y) \in S \text{ for some } y \in R\}$.

Then S_1 is a non-empty compact subset of R^m and has a lex.sup α by the induction hypothesis. Furthermore $\alpha \in S_1$.

The set $S_2 = \{y \in R; (\alpha,y) \in S\}$ is thus a non-empty compact subset of R and contains a supremum k. It follows that $(\alpha,k) \in S$.

If $x \in S$ then $(x^1,x^2,\ldots x^m) \in S_1$ and therefore $(x^1,x^2,\ldots x^m) \trianglelefteq \alpha$.

Thus, if $(x^1,x^2,\ldots x^m) \neq \alpha$, then $x \trianglelefteq (\alpha,k)$.

If, on the other hand, $(x^1,x^2,\ldots,x^m) = \alpha$, then $x^{m+1} \leqslant k$, and hence $x \trianglelefteq (\alpha,k)$.

In either case, therefore, $x \trianglelefteq (\alpha,k)$.

Since $x \in S$ was arbitrary, and $(\alpha,k) \in S$ then it follows that $(\alpha,k) = \text{lex.sup } S$, which proves the lemma. #

A3.4 <u>Lemma</u>: Let U be a compact subset of R^p, $\varphi: R^q \times U \to R^n$ a $\mathcal{B}$-measurable function such that $\varphi(x,.)$

is continuous on U for each $x \in R^q$, and let

$S = \{x; \varphi(x,u) = 0$ for some $u \in U\}$.

Then S is $\mathcal{B}$-measurable, and the function $\psi: S \to U$ defined by

$$\psi(x) = \text{lex.sup}\ \{u \in U; \varphi(x,u) = 0\}$$

is $\mathcal{B}$-measurable on S.

<u>Proof</u>: Note first that ψ is well defined, since $\{u \in U; \varphi(x,u) = 0\}$ is compact for each $x \in R^q$.

Now let D be a countable dense subset of U (which exists since U is separable).

Then $x \in S$ if and only if for every positive integer r there exists $u \in D$ such that $\|\varphi(x,u)\| < \frac{1}{r}$.

Therefore

$$S = \bigcup_{r=1}^{\infty} \bigcap_{u \in D} \{x; \|\varphi(x,u)\| < \frac{1}{r}\}$$

is $\mathcal{B}$-measurable, which was the first assertion of the Lemma.

The remainder of the proof proceeds by induction on the components of ψ.

If $\alpha \in R$, then it can be shown that $\psi^1(x) < \alpha$ if and only if there exists a positive integer r such that $\|\varphi(x,u)\| \geq \frac{1}{r}$ for every $u \in D$ with $u^1 \geq \alpha - \frac{1}{r}$.

Consequently, if $D_r = \{u \in D;\ u^1 \geq \alpha - \frac{1}{r}\}$, then

$$\{x; \psi^1(x) < \alpha\} = \bigcup_{r=1}^{\infty} \bigcap_{u \in D_r} \{x \in S; \|\varphi(x,u)\| \geq \frac{1}{r}\}$$

is $\mathcal{B}$-measurable.

Since α was arbitrary it follows that ψ^1 is $\mathcal{B}$-measurable on S.

If now $\psi^1, \psi^2, \ldots, \psi^j$ are $\mathcal{B}$-measurable on S, and $\alpha \in R$, it can be shown that $\psi^{j+1}(x) < \alpha$ if and only if there exists a positive integer r such that for every $u \in D$ with $u^{j+1} \geqslant \alpha - \frac{1}{r}$ either $\|\varphi(x,u)\| \geqslant \frac{1}{r}$ or $\psi^i(x) - u^i \geqslant \frac{1}{r}$ for some $i = 1,2,\ldots,j$.

Hence, if $D_r' = \{u \in D;\ u^{j+1} \geqslant \alpha - \frac{1}{r}\}$, then

$$\{x \in S; \psi^{j+1}(x) < \alpha\} = \bigcup_{r=1}^{\infty} \bigcap_{u \in D_r'} \Big[\{x; \|\varphi(x,u)\| \geqslant \tfrac{1}{r}\} \cup \bigcup_{i=1}^{j} \{x; \psi^i(x) - u^i \geqslant \tfrac{1}{r}\}\Big]$$

is $\mathcal{B}$-measurable. Again since α was arbitrary, then ψ^{j+1} is $\mathcal{B}$-measurable, and it then follows by induction that ψ is $\mathcal{B}$-measurable. #

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