INTERIOR GRADIENT BOUNDS FOR NON-UNIFORMLY ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS OF DIVERGENCE FORM

by

L.M. SIMON

A Thesis submitted for the Degree of
Doctor of Philosophy
in the University of Adelaide,
Department of Pure Mathematics,
December, 1971.

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SUMMARY

Let Ω be a domain in R^n. The equation

$$\sum_{i,j=1}^{n} A_{ij}(x,u,\nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j} = B(x,u,\nabla u) \ ,$$

where $B(x,z,p)$, $A_{ij}(x,z,p)$, $i,j=1,\ldots,n$, are given functions of the variables $(x,z,p) \in \Omega\times R\times R^n$, and where

$$0< \sum_{i,j=1}^{n} A_{ij}(x,z,p)\xi_i\xi_j \ , \xi \in R^n-\{0\}, (x,z,p) \in \Omega\times R\times R^n \ ,$$

is called a quasilinear elliptic equation on Ω.

We will here be concerned only with quasilinear elliptic equations which can be written in divergence form - i.e. equations of the form

$$(1) \ \sum_{i=1}^{n}\frac{d}{dx_i}A_i(x,u,\nabla u) \left(\equiv \sum_{i,j=1}^{n} A_{ip_j}(x,u,\nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} A_{iz}(x,u,\nabla u)\frac{\partial u}{\partial x_i} + \sum_{i=1}^{n} A_{ix_i}(x,u,\nabla u)\right) = B(x,u,\nabla u) \ ,$$

where the vector function $A(x,z,p) = (A_1(x,z,p),\ldots,A_n(x,z,p))$ is such that

$$0< \sum_{i,j=1}^{n} A_{ip_j}(x,u,\nabla u) \ \xi_i\xi_j , \xi \in R^n-\{0\}, \ (x,z,p) \in \Omega\times R\times R^n \ .$$

Note that (1) can be written in the integral form

$$\sum_{i=1}^{n} \int_\Omega A_i(x,u,\nabla u) \ \frac{\partial \zeta}{\partial x_i} \ dx = - \int_\Omega B(x,u,\nabla u) \ \zeta \ dx \ , \quad \zeta \in Lip_c(\Omega) \ ,$$

where $Lip_c(\Omega)$ is the set of all Lipschitz functions with compact support contained in Ω.

In case there exist positive continuous functions $c(z)$, $c'(z)$, defined for all $z \in R$ and such that

$$c(z)\ |\xi|^2 \leq \sum_{i,j=1}^{n} A_{ip_j}(x,z,p)\xi_i\xi_j \leq c'(z)\ |\xi|^2 ,$$

for all $\xi \in R^n$ and $(x,z,p) \in \Omega\times R\times R^n$,

then (1) is said to be uniformly elliptic .

If however the maximum and minimum eigenvalues of the matrix $\frac{1}{2}(A_{ip_j}(x,z,p) + A_{jp_i}(x,z,p))$, denoted $\Lambda(x,z,p)$ and $\lambda(x,z,p)$ respectively , are such that the quotient

$$\Lambda(x,z,p)/\ \lambda(x,z,p)$$

is not bounded as $|p| \to \infty$, then (1) is said to be non-uniformly elliptic . An example of a non-uniformly elliptic divergence form equation of classical interest is the minimal surface equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i} / \sqrt{1 + |\nabla u|^2}\right) = 0 .$$

For this example we have

$$A_i(x,z,p) = p_i/ \sqrt{1 + |p|^2}$$

and

$$\lambda(x,z,p) =(1+ |p|^2)^{-3/2},\ \Lambda(x,z,p)=(1+ |p|^2)^{-1/2},$$

so that

$$\Lambda(x,z,p)/\ \lambda(x,z,p) = 1 + |p|^2.$$

This thesis will be concerned with non-uniformly elliptic equations. In particular , we will be concerned with proving the existence of local a-priori interior gradient bounds for sufficiently smooth solutions to equations of the form (1). That is , for suitable equations of the form (1) we will establish the existence of a real-valued function $\Gamma(\rho,M)$, defined for $\rho>0$, $M>0$, such that any sufficiently smooth solution u of (1) with $|u|\leq M$ must satisfy

$$\sup_{\Omega'} |\nabla u| \leq \Gamma(\rho,M)$$

for all subdomains $\Omega' \subset \Omega$ which are such that the distance between Ω' and $\partial\Omega$ is no less than ρ.

It is to be emphasised that $\Gamma(\rho,M)$ does not depend on the particular solution u , but that $\Gamma(\rho,M)$ may not be bounded as $\rho \to 0$ or as $M \to \infty$. In fact it can be arranged that Γ does not depend explicitly on the functions A , B of (1) ; rather it can be arranged that Γ depends on certain structural quantities which are invariant under appropriate changes in A and B .

It is not difficult to demonstrate the importance of such local gradient bounds. For example , many of the results in [1] can be extended to the case when the boundary data is merely continuous (rather than C^2) provided a local gradient bound is known .

Generally speaking , one can think of a local gradient bound as locally reducing a non-uniformly elliptic equation to a uniformly elliptic one , thus making it possible to use the theory of uniformly elliptic equations to study the solutions to non-uniformly elliptic equations .

The gradient bounds obtained in this thesis generalize the work of Bombieri-de Giorgi-Miranda [3] and Ladyzhenskaya-Ural'tseva [4],Part II . The main results appear in Chapter 2 . The techniques used are generally modifications and extensions of those used by the above authors . The main analytic tool is the Sobolev-type inequality derived in Chapter 1 . Chapter 3 consists of extensions of the results of Chapter 2 to other equations , under the assumption that an estimate of Hölder continuity of the solutions is already known . In Chapter 4 refined gradient estimates for a certain sub-class of those equations dealt with in Chapter 2 are obtained .