5/11/81

OVERFLOW TRAFFIC IN TELEPHONY

by

RONDA MAY POTTER M.Sc.(Adelaide)

Thesis submitted for the Degree of

Doctor of Philosophy

in the University of Adelaide,

Department of Applied Mathematics.

November, 1979.

degree awarded April/81

ERRATA

CHAPTERS 2,3,4

p. 13, 2.3: Equation (2.3.12) is valid for $n < N$.

p. 15, 2.3.1: Read $\Gamma(s/\mu+r)/\Gamma(s/\mu)$ for $\Gamma(s+r)/\Gamma(s)$.

p. 17, 2.3.1: Read 'relationship' for 'new relationship'.

p. 40, 3.4: Read $\alpha_{(n-1)\pi}$ for $\alpha_{(n-1)\pi}{}^{\pi}$

p. 43, 3.4: Read W for w.

p. 43, 3.4: Read $\frac{\mu^2}{2}\psi''(\xi)$ for $\frac{\mu^2}{2} - \psi''(\xi)$ in equation (3.4.17).

p. 60, 4.4: Read $\leqslant$ for $\geqslant$ in (4.4.10).

CHAPTER 5

p. 72, 5.2: Read $(n+1)$ for (n) in equation (5.2.14).

p. 92, 5.5: Read $0 \leqslant m < n$ for $0 \leqslant m \leqslant n$.

p. 92, 5.5: Read $0 \leqslant k < n$ for $0 \leqslant k \leqslant n$.

p. 96, 5.5: $\bar{R}_{N,L.}$ is the product of the lost calls to carried calls with the secondary group size.

p. 96, 5.5: Read] for)]

p. 96, 5.5: Equation (5.5.36) should be

$$\bar{R}_{N,L.} = L\left[\frac{(N+L)!}{N!} - 1 + M_N \sum_{r=1}^{L} \frac{(N+L)!}{(N+L+1-r)!} A^{-r}\right]^{-1}$$

CHAPTER 6

p. 102, 6.2: Read (2.3.17) for (2.2.17).

p. 102, 6.2: Omit last sentence.

p. 104, 6.3: $\chi(s)$ is the Laplace-Stieltjes transform of any renewal stream.

p. 108, 6.4.1: Read $z_{(1)}$ for $z_{(1)}-1$ in last line.

p. 109, 6.4: The result (6.4.7) implies that successive overflows have increasing peakedness whilst remaining smooth. Once a stage occurs where the overflow traffic becomes rough it remains rough for all successive stages whilst the increasing property of the peakedness need not be true.

p. 112, 6.5: Read $j=1$ for $j=0$ (product index).

p. 112, 6.5: Read (5.3.19) for (5.3.20).

p. 130, 7.4: Read f_{N+1} for f_N in last sentence.

p. 131, 7.4: Read f_{N+1} for f_N in equation (7.4.8).

p. 132, 7.4.1: Read $(N+1)f_N$ for $(N+1)F_N$ in equation (7.4.1.1).

p. 132, 7.4.1: Equations (7.4.16) and (7.4.17) hold only for negative exponential input streams and for α_3 consistent with the values of α_1 and α_2.

CONTENTS

OVERFLOW TRAFFIC IN TELEPHONY

R.M. POTTER

SUMMARY

This thesis is concerned with an analysis of the Kosten and Brockmeyer overflow systems with renewal input.

Two models of teletraffic overflow systems are included. The first, or 'group' model, considers an overflow system as either a $G/M/\infty$ or $G/M/L$ queueing system with an overflow stream as input. This overflow stream is produced by offering a renewal input stream to a finite primary group of trunks. The second, or atomic model, considers sequential overflow streams from individual trunks.

The atomic model is used to study such characteristics of an overflow stream as its peakedness and coefficient of variation. However properties of the overflow traffic and the Laplace-Stieltjes transforms of the interoverflow distribution, developed by the first approach, are used to prove the overflow traffic factorial moment theorem.

A key feature of this thesis is the classification of traffic by its 'weakness', a new concept to telephony.

Explicit formulae for all offered and carried overflow traffic moments are derived in terms of finite differences of the overflow traffic's weakness, or equivalently, Laplace-Stieltjes transforms of the input renewal stream. The finite difference version is inverted to provide insight into the effect of specifying a finite number of overflow traffic moments on dimensioning teletraffic overflow systems.

A new dimensioning procedure, called the Equivalent Non Random Method is developed in the final chapter.

SIGNED STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief, the thesis contains no material previously published or written by any other person, except where due reference is made in the text of the thesis.

Ronda May Potter.

ACKNOWLEDGEMENTS

I happily acknowledge my gratitude to my supervisor, Dr. C.E.M. Pearce for his guidance, help and criticism throughout the work on this interesting topic.

I warmly thank Professor R.B. Potts for his interest in my work and his help and supervision during Dr. Pearce's absence.

I would like to take this opportunity to thank Dr. R. Harris, Professor A. Myskja, and Mr. D. Sutton for their interest and comments on my research.

I am grateful to Rob, Kate and Janey for their encouragement and understanding.

Finally, I would like to thank Ms. E. Henderson for her excellent typing.

CHAPTER I

INTRODUCTION

The work contained in this thesis is a discussion of overflow traffic resulting from offering a general renewal input stream to a finite fully-available primary group. The concepts of 'overflow stream' and 'overflow traffic' are analysed using queueing and renewal theory. "Some Formulae Old and New for Overflow Traffic in Telephony" by Pearce and Potter [25] is based on definitions and results found in Chapters II through IV. These three initial chapters contain a unified methodology for finding

(i) the Laplace-Stieltjes transform of the interoverflow event distribution, see Chapter II,

(ii) the binomial moments of offered and carried traffic, see Chapters III and IV respectively.

The distinction between continuous time and the imbedded Markov chain steady state occupancy distributions corresponding to any given renewal input stream is emphasized throughout these early chapters. The expressions derived for the overflow stream and the binomial traffic moments are shown to be equivalent to early results of Takács and Cohen, see Syski [34]. The general inter-overflow distribution formula simplifies for random input traffic. In this case our simplified general result verifies that a hyperexponential distribution characterises the overflow stream, see Khinchine [18].

The basic queueing systems, $G/M/\infty$ and $G/M/L$ are needed to analyse 'offered' and 'carried' traffic. Theory relevant to these concepts are reviewed in detail in Chapters III and IV.

The determination of explicit formulae for all overflow traffic moments from a renewal input stream is given in Chapter V. This work is summarised in "Explicit Formular for All Overflow Traffic Moments of Kosten and Brockmeyer Systems with Renewal Input", see Potter [26]. The 'group' interpretation of these systems is to consider the overflow stream from a finite primary group as the input renewal stream for either the $G/M/\infty$ or the $G/M/L$ queueing systems. The factorial overflow traffic moment theorem is proved for any general renewal input stream, thus extending its known region of applicability, see Nightingale [21].

The classification of renewal streams by the 'weakness', that is the reciprocal of their intensity, is a feature of this work. A comprehensive list of explicit formulae for all offered and carried overflow traffic moments and related functions (defined at arrival instants and in continuous time) concludes Chapter V. These formulae are expressed in terms of

(i) the divided difference of the overflow traffic weakness

and

(ii) the Laplace-Stieltjes transform of the input stream.

By simplifying the general overflow traffic binomial moment formulae for the case of random input traffic, the resulting expressions are shown equivalent to those derived by Schehrer, [32].

In Chapter VI an alternate view of overflow systems is considered. This 'atomic' approach, based on sequential overflows from individual trunks is described and its power demonstrated by deriving characteristics of the peakedness and coefficient of variation of overflow traffic from a renewal input stream. Curves of overflow traffic peakedness versus either the primary group size or the input streams weakness are included to illustrate salient features common to a class of renewal input streams, namely a set of Erlang input streams of different phase. The intuitive result that 'overflow traffic arising from random input traffic is rough' is verified as a special case of the more general peakedness result valid for all renewal input streams. Pearce [23] extended this work by analysing the effect of both heavy and light input traffic on the peakedness of offered and carried overflow traffic.

The final chapter, a detailed version of Potter [27], contains a consideration of the effect on an overflow system when a finite number of overflow traffic moments are given specified values. The inversion of the factorial overflow traffic moment formula is fundamental to this work, resulting in an expression relating the overflow traffic weakness from an increased primary group with the number of prescribed overflow traffic moments. Using results derived in Chapter V, a list of overflow quantities fixed by the given number of specified overflow traffic moment values is established. Two overflow traffic moments, an assumption common to most teletraffic dimensioning techniques, are shown to specify the marginal occupancy, independently of the form of the renewal input stream. Additional restrictions arising when three overflow traffic moments are given set values are also considered. If the input traffic is random, exact dimensioning formulae are obtained in terms of three overflow traffic moments.

A dimensioning technique, based on a recurrence expression derived in Chapter V for the overflow traffic variance involving the marginal occupancy and the mean overflow traffic, is developed. This Equivalent Non Random (E.N.R.) Method is dependent on charts produced by superimposing two families of curves. One family corresponding to constant input weakness values whereas the other corresponds to constant primary group size values. Examples of such charts for different Erlang input streams are included and their common features discussed.

Throughout this work the holding time distribution is assumed to be negative exponential with parameter μ.

CHAPTER II

THE RENEWAL OVERFLOW STREAM AND ITS PROPERTIES

CHAPTER II

2.1 Introduction

When a renewal stream is offered to a finite set of trunks and forced to overflow, the overflow stream is also renewal, as shown by Takács [38] and Descloux [12]. The invariance of the renewal property will be verified in the subsequent analysis of the inter-arrival time distribution of the overflow stream.

Initially the notation and terminology needed to analyse the input and overflow streams will be discussed.

A queueing theory approach to this overflow stream will be used to derive the Laplace-Stieltjes transform of the inter overflow distribution. The same approach, thus providing a unified methodology, is used in Chapters 3 and 4 when analysing the steady state occupancy distributions on a set of trunks (finite or infinite).

The equivalence of the author's expression for the Laplace-Stieltjes transform of the overflow stream with that of Takács [37], is established. An identity resulting from this equivalence provides insight into the structure of the o.f. stream.

In the last section of this chapter relevant quantities, such as the weakness of an overflow stream, the n^{th} divided difference of the weakness, the probability of congestion and their associated props are introduced. Such quantities are essential in the subsequent derivation of explicit formulae for all the overflow traffic moments.

2.2 Notation and Terminology

Consider a telephone exchange at which calls arrive at time instants $\tau_1 \leqslant \tau_2 \leqslant \ldots \leqslant \tau_n \leqslant \ldots$ where $X_n = \tau_n - \tau_{n-1}$ (n=1,2,...,

$:\tau_0=0)$ are independently and identically distributed random variables with distribution function $F(t)$:

$$p(X_n \leq t) = F(t) \quad \text{for all} \quad n=1,2,\ldots \quad ,$$

that is, calls arrive in a renewal stream with interevent time distribution $F(t)$.

The mean interevent time, m:

$$m = E[X_n] = \int_0^\infty t dF(t) \quad \text{for all} \quad n=1,2,\ldots \ . \tag{2.2.1}$$

Let $\phi(s)$ be the Laplace-Stieltjes transform of $F(t)$,

$$\phi(s) = \int_0^\infty e^{-sx} dF(x)$$

An alternate expression for the expected time between successive events is given by

$$m = - \phi'(0) \tag{2.2.2}$$

If this renewal stream, F, is offered to a finite set of N trunks with negative exponential service distribution, parameter μ, the weakness, f_0, of the input stream:

$$f_0 = - \mu\phi'(0) \tag{2.2.3}$$

Hence, using (2.2.2),

$$f_0 = + \mu m \tag{2.2.4}$$

Teletraffic engineers use the term, intensity, I_0, to describe the ratio of the arrival rate of calls to the service rate, thus

$$I_0 = \frac{1}{m\mu} \tag{2.2.5}$$

Thus the weakness of the renewal stream, F, is the reciprocal of its intensity.

Let $\bar{\pi}_N$ be the probability that an arriving call finds all N trunks occupied. The expected number of calls between consecutive instants at which arriving calls find all N trunks busy is given by

$$E[\nu] = \frac{1}{\bar{\pi}_N} \quad , \tag{2.2.6}$$

where ν is the number of interevent times occurring between successive overflow instants.

Suppose t_i and t_{i+1} are consecutive instants at which arriving calls finds all N trunks busy. The interevent times $x_{i+1} = t_{i+1} - t_i$ are independent, identically distributed random variables with distribution G:

$$p(x_{i+1} \leq t) = G(t) \quad , \quad i=1,2,\ldots \; ,$$

that is, calls overflow in a renewal stream with interevent time distribution G(t). Let M_N be the mean interevent time of G, thus

$$M_N = - \psi_N'(0) \tag{2.2.7}$$

where $\psi_N(s)$ is the Laplace-Steiltjes transform of G(t). If $x_i = X_1 + \ldots + X_\nu$ where ν is the number of interevent times occurring between successive instants when calls find all N trunks busy, (ν is a random variable), then by Wald's Theorem, see Takács [36],

$$E[x_i] = E[\nu]\; E[X_1] \tag{2.2.8}$$

providing $E[\nu] < \infty$ and the event ν=n and subsequent time intervals $X_{n+1}, X_{n+2} \ldots$ are independent. Equation (2.2.5) can be expressed as

$$M_N = \frac{1}{\bar{\pi}_N} m \tag{2.2.9}$$

The weakness, f_N, of the overflow stream:

$$f_N = -\mu\psi_N{}'(0) \tag{2.2.10}$$

hence, using equation (2.2.7),

$$f_N = \mu M_N \tag{2.2.11}$$

The intensity, I_N, of the overflow stream:

$$I_N = \frac{1}{\mu M_N} \tag{2.2.12}$$

Substituting for m and M_N in terms of f_0 and f_N, given by equations (2.2.4) and (2.2.11), equation (2.2.9) becomes

$$\bar{\pi}_N = \frac{f_0}{f_N} \tag{2.2.13}$$

Thus the congestion probability is the ratio of the weakness of the input stream to the weakness of the overflow stream. A similar interpretation of this loss probability is used by Descloux [12], page 331/1 and Pearce and Potter [25], equation (15).

I assume throughout this thesis that the holding time of each trunk has a negative exponential distribution, parameter μ.

2.3 Analysis of the Overflow Stream, G

Let $f_n(t)$ be defined as the distribution function for the time from an epoch where a call joins a group of N trunks to find n of them occupied till the instant of the first subsequent overflow.

Suppose a call at τ_i finds n trunks busy and that the first subsequent overflow occurs at time $\tau_i + t$. Let $y = \tau_{i+1} - \tau_i$ where $0 \leq y \leq t$. If a call at τ_{i+1} finds j, where $0 \leq j \leq n+1$, trunks busy, then $(n+1-j)$ calls must have finished in time y whilst the remaining j calls did not. The time to the first subsequent overflow from τ_{i+1} now becomes $(t-y)$, hence

$$f_n(t) = \sum_{j=0}^{n+1} \binom{n+1}{j} \int_0^t (1-e^{-\mu y})^{n+1-j} e^{-j\mu y} f_j(t-y)\,dF(y), \quad 0 \leq n < N. \quad (2.3.1)$$

Since the overflow is instantaneous when a call arrives to find all N trunks busy, let

$$f_N(t) = \delta(t-0) \quad (2.3.2)$$

where δ is the Dirac Delta measure defined by

$$\int_{-\infty}^{\infty} s(t)\delta(t-a)\,dt = s(a) \quad \text{for any generalized function } s(\cdot).$$

The condition of equation (2.3.2) puts a physical boundary on equation (2.3.1).

Taking the Laplace-Stieltjes transform of equation (2.3.1) gives

$$f_n^*(s) = \sum_{j=0}^{n+1} \int_0^{\infty} e^{-st} \binom{n+1}{j} (1-e^{-\mu t})^{n+1-j} e^{-j\mu t}\,dF(t) f_j^*(s), \quad 0 \leq n < N \quad (2.3.3)$$

The solution to these equations (2.3.3) is the same as that of the unrestricted set for which $0 \leq n < \infty$ with the imposed supplementary boundary condition

$$f_N^*(s) \equiv 1. \quad (2.3.4)$$

By taking factorial generating functions, the extended set of equations becomes

$$f^*(z) = \int_0^\infty e^{-st} \frac{d}{dz}\left[f^*(ze^{-\mu t})\exp\{z(1-e^{-\mu t})\}\right]dF(t) \qquad (2.3.5)$$

Proof of equation (2.3.5)

The factorial generating function, $f^*(z)$ of $f_n^*(s)$ is given by

$$f^*(z) = \sum_{n=0}^{\infty} \frac{f_n^*(s)z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n+1} \int_0^\infty e^{-st} \binom{n+1}{j} (1-e^{-\mu t})^{n+1-j} e^{-j\mu t} \frac{z^n}{n!} dF(t) f_j^*(s) \text{ from (2.3.3)}$$

$$= f_0^*(s)\int_0^\infty e^{-st} (1-e^{-\mu t})\exp(z(1-e^{-\mu t}))dF(t)$$

$$+ \sum_{j=1}^{\infty} f_j^*(s) \sum_{r=0}^{\infty} \int_0^\infty e^{-st}\binom{r+j}{j} (1-e^{-\mu t})^r e^{-j\mu t} \frac{z^{r+j-1}}{(r+j-1)!} dF(t)$$

$$= \sum_{j=0}^{\infty} \frac{f_j^*(s)}{j!} z^j e^{-j\mu t} \int_0^\infty e^{-st} (1-e^{-\mu t})\exp\{z(1-e^{-\mu t})\}dF(t)$$

$$+ \sum_{j=1}^{\infty} \frac{f_j^*(s)}{j!} z^j \left(\frac{j}{z}\right) e^{-\overline{j-1}\mu t} e^{-\mu t} \int_0^\infty e^{-st}\exp\{z(1-e^{-\mu t})\}dF(t)$$

$$= \int_0^\infty e^{-st} \frac{d}{dz}\left[f^*(ze^{-\mu t})\exp\{z(1-e^{-\mu t})\}\right]dF(t)$$

By substituting $k(z) = f^*(z)e^{-z}$, equation (2.3.5) can be rewritten as

$$k(z) = \int_0^\infty e^{-st} \left(\frac{d}{dz} + 1\right)k(ze^{-\mu t})dF(t) \qquad (2.3.6)$$

which on expanding

$$k(z) = \sum_{n=0}^{\infty} \frac{k_n(s)}{n!} z^n$$

gives

$$k(z) = \int_0^{\infty} e^{-st} \left(\sum_{n=0}^{\infty} \frac{k_n(s)e^{-n\mu t}}{n!} z^n + \sum_{n=0}^{\infty} \frac{k_{n+1}(s)}{n!} z^n e^{-n\mu t} e^{-\mu t} \right)^{**} dF(t)$$

Equating coefficients of $\frac{z^n}{n!}$ gives

$$k_n(s) = \int_0^{\infty} e^{-st} (k_n(s)e^{-n\mu t} + k_{n+1}(s)e^{-(n+1)\mu t})dF(t) \; . \qquad (2.3.7)$$

Since $\phi(s) = \int_0^{\infty} e^{-st} dF(t)$, equation (2.3.7) can be rewritten as

$$k_{n+1}(s) = k_n(s) \frac{1-\phi(s+n\mu)}{\phi(s+\overline{n+1}\mu)} \qquad (2.3.8)$$

$$\therefore \quad k_n(s) = k_0(s) \prod_{j=1}^{n} \frac{1-\phi(s+\overline{j-1}\mu)}{\phi(s+j\mu)} \qquad (2.3.9)$$

Since $k(z) = e^{-z} f^*(z)$

$$f_n^*(s) = \sum_{r=0}^{n} \binom{n}{r} k_r(s) \qquad (2.3.10)$$

$$= k_0(s) \sum_{r=0}^{n} \binom{n}{r} \prod_{j=1}^{r} \frac{1-\phi(s+\overline{j-1}\mu)}{\phi(s+j\mu)} \quad \text{by (2.3.10)}$$

** $$\frac{d}{dz} k(z) = \sum_{n=0}^{\infty} \frac{k_{n+1}(s) z^n}{n!}$$

$$\frac{d}{dz} k(ze^{-\mu t}) = \sum_{n=0}^{\infty} k_{n+1}(s) e^{-\overline{n+1}\mu t} \frac{z^n}{n!}$$

But the physical boundary condition (2.3.4) gives

$$f_N{}^*(s) \equiv 1 = k_0(s) \sum_{r=0}^{N} \binom{N}{r} \prod_{j=1}^{r} \frac{1-\phi(s+\overline{j-1}\mu)}{\phi(s+j\mu)} \qquad (2.3.11)$$

Thus $f_n{}^*(s)$ can be divided by $f_N{}^*(s)$ to give

$$f_n{}^*(s) = \frac{f_n{}^*(s)}{f_N{}^*(s)} = \frac{k_0(s) \sum\limits_{r=0}^{n} \binom{n}{r} \prod\limits_{j=1}^{r} \frac{1-\phi(s+\overline{j-1}\mu)}{\phi(s+j\mu)}}{k_0(s) \sum\limits_{r=0}^{N} \binom{n}{r} \prod\limits_{j=0}^{r} \frac{1-\phi(s+\overline{j-1}\mu)}{\phi(s+j\mu)}} \qquad (2.3.12)$$

The $k_0(s)$ cancel enabling it to be given any value w.l.o.g. by expressing $f_n{}^*(s)$ by equation (2.3.12). We take $k_0(s) \equiv 1$ in the expression for $k_r(s)$ hence

$$k_r(s) = \begin{cases} \prod\limits_{j=1}^{r} \frac{1-\phi(s+\overline{j-1}\mu)}{\phi(s+j\mu)} & , \quad r \geqslant 1 \\ 1 & , \quad r = 0 \end{cases} \qquad (2.3.13)$$

One property of $k_r(s)$ useful for later work is

$$k_n(0) = \delta_{n0} \qquad (2.3.14)$$

Proof of (2.3.14)

When $s = 0$, $f_n{}^*(0) = \int_0^\infty df_n(t) = 1$ and the corresponding functions $f^*(z)$ and $k(z)$ become

$$f^*(z) = \sum_{n=0}^{\infty} \frac{f_n{}^*(0)z^n}{n!} = e^z$$

$$k(z) = e^{-z} f^*(z) = 1$$

However we have defined $k_0(s) \equiv 1$ for all s,

$$\therefore \quad k_0(0) = 1 \; .$$

But $k(z) = \sum_{n=0}^{\infty} \frac{k_n(s) z^n}{n!}$

$$\therefore \quad k_0(0) + \sum_{n=1}^{\infty} \frac{k_n(0) z^n}{n!} = 1$$

thus

$$k_n(0) = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \; . \end{cases}$$

By setting $n = N - 1$ in equation (2.3.12)

$$f_{N-1}{}^*(s) = \frac{\sum_{r=0}^{N-1} \binom{N-1}{r} k_r(s)}{\sum_{r=0}^{N} \binom{N}{r} k_r(s)} \tag{2.3.15}$$

$G(t)$, the distribution function of the time separating two consecutive epochs at which an arriving call finds all N trunks occupied, satisfies,

$$G(t) = \sum_{j=0}^{N} \binom{N}{j} \int_0^t (1-e^{-\mu y})^{N-j} e^{-j\mu y} f_j(t-y)\,dF(y) \tag{2.3.16}$$

Equation (2.3.16) is the same as equation (2.3.1) with $n = N - 1$. This result is reasonable physically. After the arrival of one more call, the full set of channels and that with $(N-1)$ calls become indistinguishable (except for the overflowing call).

Thus the Laplace-Stieltjes transform, $\psi_N(s)$ of the overflow stream G, satisfies equation (2.3.15), giving

$$\psi_N(s) = \frac{\sum_{r=0}^{N-1} \binom{N-1}{r} k_r(s)}{\sum_{r=0}^{N} \binom{N}{r} k_r(s)} \tag{2.3.17}$$

where $k_r(s)$ is defined by equation (2.3.13).

Takács [36] obtained the following expression for $\psi_N(s)$,

$$\psi_N(s) = \frac{\sum_{r=0}^{N} \binom{N}{r} \ell_r(s)}{\sum_{r=0}^{N+1} \binom{N+1}{r} \ell_r(s)} \tag{2.3.18}$$

where

$$\ell_r(s) = \begin{cases} \prod_{j=1}^{r} \dfrac{1-\phi(s+\overline{j-1}\mu)}{\phi(s+\overline{j-1}\mu)} & , \quad r \geqslant 1 \\ 1 & , \quad r = 0 \end{cases} \tag{2.3.19}$$

2.3.1 Overflow Stream Distribution Resulting from a Negative Exponential Input Stream

If $F(t) = 1-e^{-\lambda t}$, then $\phi(s) = \dfrac{\lambda}{\lambda+s}$.

Substituting for $\phi(s)$ in the expression (2.3.19) for $\ell_r(s)$, gives

$$\ell_r(s) = \begin{cases} A^{-r} \dfrac{\Gamma(s+r)}{\Gamma(s)} & , \quad r \geqslant 1 \\ 1 & , \quad r = 0 \end{cases}$$

where $A = \dfrac{\lambda}{\mu}$.

If we assume $\mu = 1$, then $\ell_r(s)$ can be rewritten as

$$\ell_r(s) = \begin{cases} \lambda^{-r} (-1)^r \binom{-s}{r} r! & , \quad r \geqslant 1 \\ 1 & , \quad r = 0 \end{cases} \tag{2.3.1.1}$$

Substituting this expression (2.3.1.1) for $\ell_r(s)$ into equation (2.3.18) for $\psi_N(s)$, gives

$$\psi_N(s) = \frac{\sum_{r=0}^{N} \binom{N}{r} \lambda^{-r} (-1)^r \binom{-s}{r} r!}{\sum_{r=0}^{N+1} \binom{N+1}{r} \lambda^{-r} (-1)^r \binom{-s}{r} r!}$$

$$= \frac{C_N(-s;\lambda)}{C_{N+1}(-s;\lambda)} \tag{2.3.1.2}$$

where $C_N(-s;\lambda)$ is the Charlier Polynomial, see Bateman manuscript [1], section 10.25, defined by

$$C_N(x;\lambda) = \sum_{r=0}^{N} \binom{N}{r}(-1)^r \binom{x}{r} r!\lambda^{-r} \quad , \quad x \quad \text{real} \tag{2.3.1.3}$$

with generating function $g(x,z:\lambda)$ satisfying

$$g(x,z;\lambda) = \sum_{N=0}^{\infty} C_N(x;\lambda)\frac{z^N}{N!}$$

$$= e^z (1-\frac{z}{\lambda})^x \quad , \text{ for } \quad x \quad \text{real and} \quad |z| < 1. \tag{2.3.1.4}$$

Hence,

$$C_0(x;\lambda) = 1 \tag{2.3.1.5}$$

$$C_1(x;\lambda) = 1 - \frac{x}{\lambda} \tag{2.3.1.6}$$

$$C_2(x;\lambda) = \frac{x^2-(2\lambda+1)x+\lambda^2}{\lambda^2} \tag{2.3.1.7}$$

Since $C_N(x;\lambda)$, x positive has N distinct real roots $\varepsilon(0,\infty)$, see Szëgo [35], Theorem 3.3.1, page 44, the ratio of $C_N(x;\lambda)$ to $C_{N+1}(x;\lambda)$ can be expressed by $(N+1)$ partial fractions. If $\xi_{N+1,1}, \xi_{N+1,2}, \ldots, \xi_{N+1,N+1}$ are the $(N+1)$ real, positive zeros of $C_{N+1}(x;\lambda)$ then

$$\frac{C_N(x;\lambda)}{C_{N+1}(x;\lambda)} = \sum_{r=1}^{N+1} \frac{\beta_r}{(x-\xi_{N+1,r})} \quad , \quad \xi_{N+1,r} > 0 \tag{2.3.1.8}$$

and

$$\beta_r = \frac{C_N(\xi_{N+1,r};\lambda)}{C_{N+1}'(\xi_{N+1,r};\lambda)} \tag{2.3.1.9}$$

Equation (2.3.1.2) can be rewritten as

$$\psi_N(s) = \sum_{r=1}^{N+1} \frac{\beta_r}{(-s-\xi_{N+1,r})}$$

$$= - \sum_{r=1}^{N+1} \frac{\beta_r}{s+\xi_{N+1,r}} \qquad (2.3.1.10)$$

where $\xi_{N+1,r} > 0$ for all N and r and β_r satisfies equation (2.3.1.9).

If the inter overflow stream has distribution and density functions $G_N(t)$, $g_N(t)$ respectively, then inverting equation (2.3.1.10) gives

$$g_N(t) = - \sum_{r=1}^{N+1} \beta_r e^{-\xi_{N+1,r} t} \qquad (2.3.1.11)$$

But $\int_0^\infty g_N(t)dt = 1$, $\therefore \sum_{r=1}^{N+1} \frac{-\beta_r}{+\xi_{N+1,r}} = 1$

or equivalently

$$\sum_{r=1}^{N+1} \frac{C_N(\xi_{N+1,r};\lambda)}{C_{N+1}'(\xi_{N+1,r};\lambda)} \frac{1}{\xi_{N+1,r}} = -1 \qquad (2.3.1.12)$$

Equation (2.3.1.12) is a new relationship connecting the zeros of the extended Charlier polynomials. The distribution function $G_N(t)$ corresponding to (2.3.1.11), is

$$G_N(t) = 1 - \sum_{r=1}^{N+1} \frac{-\beta_r}{\xi_{N+1,r}} e^{-\xi_{N+1,r} t} \qquad (2.3.1.13)$$

which agrees with equation (33.1) of Khinchine [18] p. (94), but gives explicit expressions for his constants a_{Nr} in terms of the zeros of Charlier polynomials, i.e.

$$a_{Nr} = \frac{-\beta_r}{\xi_{N+1,r}} \qquad (2.3.1.14)$$

where β_r and $\{\xi_{N+1,r}\}$ satisfy equations (2.3.1.10) and (2.3.19) respectively. In telephony, a convex combination of negative exponential distributions as in equation (2.3.1.13) is known as a hyper exponential distribution.

Equation (2.3.1.13) can be rewritten as

$$G_N(t) = \sum_{r=1}^{N+1} \frac{-\beta_r}{\xi_{N+1,r}} [1 - e^{-\xi_{N+1,r}t}] \qquad (2.3.1.15)$$

Equivalently, Palm's function $\phi_N(t)$, the complementary distribution of $G_N(t)$, satisfies

$$\phi_N(t) = \sum_{r=1}^{N+1} \frac{-\beta_r}{\xi_{N+1,r}} e^{-\xi_{N+1,r}t} \qquad (2.3.1.16)$$

To demonstrate this methodology for determining the $G_N(t)$ we consider the simple case when $N = 1$.

$$G_1(t) = \sum_{r=1}^{2} \frac{-\beta_r}{\xi_{2,r}} [1-e^{-\xi_{2,r}t}] \qquad (2.3.1.17)$$

where $\{\xi_{2,r}\}$ are the zeros of $C_2(x;\lambda)$ defined by equation (2.3.1.7) and

$$\beta_r = \frac{C_1(\xi_{2,r})}{C_2'(\xi_{2,r})}$$

by equation (2.3.1.10).

Now the zeros $\xi_{2,1}$, and $\xi_{2,2}$ satisfy

$$\xi_{2,1} = \frac{2\lambda+1 - \sqrt{4\lambda+1}}{2} \qquad (2.3.1.18)$$

$$\xi_{2,2} = \frac{2\lambda+1 + \sqrt{4\lambda+1}}{2} \qquad (2.3.1.19)$$

giving

$$C_2'(\xi_{2,1};\lambda) = -\frac{\sqrt{4\lambda+1}}{\lambda^2}$$

$$C_2'(\xi_{2,2};\lambda) = \frac{\sqrt{4\lambda+1}}{\lambda^2}$$

Hence

$$\frac{-\beta_1}{\xi_{2,1}} = \frac{+(1-\frac{\xi_{2,1}}{\lambda})}{\xi_{2,1}} \cdot \frac{\lambda^2}{\sqrt{4\lambda+1}}$$

$$= \frac{-\lambda+\xi_{2,2}}{\sqrt{4\lambda+1}} \qquad \text{since} \quad \xi_{2,1}\xi_{2,2} = \lambda^2$$

$$= \frac{1}{2} + \frac{1}{2\sqrt{4\lambda+1}} \tag{2.3.1.20}$$

and

$$\frac{-\beta_2}{\xi_{2,2}} = \frac{-(1-\frac{\xi_{2,2}}{\lambda})}{\xi_{2,2}} \frac{\lambda^2}{\sqrt{4\lambda+1}}$$

$$= \frac{-(\xi_{2,1}-\lambda)}{\sqrt{4\lambda+1}} \qquad \text{since} \quad \xi_{2,2}\xi_{2,1} = \lambda^2$$

$$= \frac{-\frac{1}{2} + \frac{\sqrt{4\lambda+1}}{2}}{\sqrt{4\lambda+1}}$$

$$= +\frac{1}{2} - \frac{1}{2\sqrt{4\lambda+1}} \tag{2.3.1.21}$$

Thus $$G_1(t) = (\frac{1}{2} + \frac{1}{2\sqrt{4\lambda+1}})[1 - \exp[\frac{2\lambda+1-\sqrt{4\lambda+1}}{2}]^t]$$

$$+ [\frac{1}{2} - \frac{1}{2\sqrt{4\lambda+1}}][1 - \exp[\frac{2\lambda+1+\sqrt{4\lambda+1}}{2}]^t] \tag{2.3.1.22}$$

This equation (2.3.1.21) for $G_1(t)$ is equivalent to the following expression for Palm's $\phi_1(t)$ given on page 95 of Khintchine [18],

$$\phi_1(t) = (\frac{1}{2} + \frac{1}{2\sqrt{4\lambda+1}})\exp[\frac{2\lambda+1-\sqrt{4\lambda+1}}{2}]$$

$$+ [\frac{1}{2} - \frac{1}{2\sqrt{4\lambda+1}}]\exp[\frac{2\lambda+1+\sqrt{4\lambda+1}}{2}]$$

Kuczura [13], approximates the overflow distribution from any negative exp. input stream by an interrupted Poisson Process, which is a Process which is alternately turned on for an exponentially distributed time and then turned off for another (independent) exponentially distributed time. If the interrupted Poisson Process approximates the overflow stream, then the interevent time distribution, $A(t)$, between successive overflow instants (see Kuczura [13], page 444, equation 16), satisfies

$$A(t) = k_1[1 - e^{-r_1 t}] + k_2[1 - e^{-r_2 t}] \qquad (2.3.1.23)$$

where

$$r_1 = \frac{1}{2}[\lambda + \omega + \nu + \sqrt{(\lambda+\omega+\nu)^2 - 4\lambda\omega}] \qquad (2.3.1.24)$$

$$r_2 = \frac{1}{2}[\lambda + \omega + \nu - \sqrt{(\lambda+\omega+\nu)^2 - 4\lambda\omega}] \qquad (2.3.1.25)$$

$$k_1 = \frac{\lambda - r_2}{r_1 - r_2} \qquad (2.3.1.26)$$

$$k_2 = 1 - k_1 \qquad (2.3.1.27)$$

$\frac{1}{\lambda}$ = mean interarrival time of the input stream F,

$\frac{1}{\nu}$ = mean on time of the random switch

$\frac{1}{\omega}$ = mean off time of the random switch.

and μ is assumed to be unity.

Note that the form of Kuczura's $A(t)$, is identical to equation (2.3.1.17), which gives the interoverflow distribution from a single primary trunk. The following relationships hold between the various parameters if $A(t)$ is equated with $G_1(t)$,

$$r_1 = \xi_{2,2} \; ; \quad r_2 = \xi_{2,1} \tag{2.3.1.28}$$

$$k_1 = \frac{-\beta_2}{\xi_{2,2}} \; ; \quad k_2 = \frac{-\beta_1}{\xi_{2,1}} \tag{2.3.1.29}$$

Equations (2.3.1.24) and (2.3.1.19) are identical when

$$\omega = \lambda, \quad \nu = 1 , \tag{2.3.1.30}$$

that is $r_1 = \xi_{2,2}$, $r_2 = \xi_{2,1}$.

Equation (2.3.1.26) for k_1 can be rewritten using equation (2.3.1.28) as follows

$$k_1 = \frac{\lambda - \xi_{2,1}}{\xi_{2,2} - \xi_{2,1}}$$

$$= \frac{-\beta_2}{\xi_{2,2}} \qquad \text{by (2.3.1.20)}$$

which checks with the expression for k_1 in (2.3.1.29). Hence the overflow stream from an individual trunk can be considered to be an interrupted Poisson Process with a unit mean 'on time' of the random switch and a mean 'off time' of the random switch being the same as the mean interarrival time of the input stream.

2.4 Equivalence of Formulae for the Overflow Stream, G.

Equations (2.3.17) and (2.3.18) are equivalent if

$$\sum_{r=0}^{N+1} \binom{N+1}{r} \ell_r(s) = [\phi(s)]^{-1} \sum_{r=0}^{N} \binom{N}{r} k_r(s) \tag{2.4.1}$$

Proof of identity (2.4.1)

Step 1.

$$D_{N+1}(s) - D_N(s) = \frac{1-\phi(s)}{\phi(s)} D_N(s+\mu) , \tag{2.4.2}$$

where

$$D_N(s) = \sum_{r=0}^{N} \binom{N}{r} \ell_r(s) \tag{2.4.3}$$

Proof of Step 1.

$$\ell_r(s+\mu) = \prod_{j=1}^{r} \frac{1-\phi(s+j\mu)}{\phi(s+j\mu)}$$

$$= \frac{\phi(s)}{1-\phi(s)}\, \ell_{r+1}(s) \qquad (2.4.4)$$

$$D_{N+1}(s) - D_N(s) = \sum_{r=0}^{N+1} \binom{N+1}{r} \ell_r(s) - \sum_{r=0}^{N} \binom{N}{r} \ell_r(s)$$

$$= \sum_{r=1}^{N+1} \binom{N}{r-1} \ell_r(s)$$

$$= \sum_{r=0}^{N} \binom{N}{r} \ell_{r+1}(s)$$

$$= \frac{1-\phi(s)}{\phi(s)} \sum_{r=0}^{N} \binom{N}{r} \ell_r(s+\mu) \quad \text{by (2.4.4)}$$

$$= \frac{1-\phi(s)}{\phi(s)}\, D_N(s+\mu)$$

Step 2.

$$k_r(s) = \frac{\phi(s)}{\phi(s+r\mu)}\, \ell_r(s) \qquad (2.4.5)$$

Proof of (2.4.5) follows from definitions of $k_r(s)$ and $\ell_r(s)$.

Step 3.

$$\frac{1-\phi(s)}{\phi(s)}\, \ell_r(s+\mu) = \frac{k_r(s)}{\phi(s)} - \ell_r(s) \qquad (2.4.6)$$

Proof of (2.4.6)

$$\ell_r(s+\mu) = \frac{\phi(s)}{1-\phi(s)} \frac{1-\phi(s+r\mu)}{\phi(s+r\mu)} \prod_{j=0}^{r-1} \frac{1-\phi(s+j\mu)}{\phi(s+j\mu)}$$

$$= \frac{\phi(s)}{1-\phi(s)} \frac{1-\phi(s+r\mu)}{\phi(s+r\mu)}\, \ell_r(s)$$

$$\therefore \frac{1-\phi(s)}{\phi(s)}\, \ell_r(s+\mu) = \frac{\ell_r(s)}{\phi(s+r\mu)} - \ell_r(s)$$

$$= \frac{k_r(s)}{\phi(s)} - \ell_r(s) \quad \text{by } (2.4.5)$$

Multiplying equation (2.4.6) through by $\binom{N}{r}$ and summing from $r=0$ to N gives

$$\frac{1-\phi(s)}{\phi(s)} D_N(s+\mu) = \frac{1}{\phi(s)} \sum_{r=0}^{N} \binom{N}{r} k_r(s) - D_N(s) \qquad (2.4.7)$$

$$\therefore \quad D_{N+1}(s) - D_N(s) = \frac{1}{\phi(s)} \sum_{r=0}^{N} \binom{N}{r} k_r(s) - D_N(s) \quad \text{by } (2.4.2)$$

$$\therefore \quad D_{N+1}(s) = \frac{1}{\phi(s)} \sum_{r=0}^{N} \binom{N}{r} k_r(s), \quad \text{proving } (2.4.1).$$

A consequence of equation (2.4.1) with $s=\mu$ is

$$\sum_{r=0}^{N+1} \binom{N+1}{r} \ell_r(\mu) = [\phi(\mu)]^{-1} \sum_{r=0}^{N} \binom{N}{r} k_r(\mu) \qquad (2.4.8)$$

This equation (2.4.8) can be proved using congestion probabilities as will be demonstrated in Chapter 4.

2.5 Properties of the Overflow Stream

In this section, we will prove properties of the overflow stream and comment on their relevance to further results, proved in subsequent chapters.

Property 1.

$$\psi_N'(0) = \frac{\phi'(0)}{\phi(\mu)} \sum_{r=0}^{N-1} \binom{N-1}{r} k_r(\mu) \qquad (2.5.1)$$

Proof of (2.5.1)

Differentiating equation (2.3.17) and putting $s=0$ gives

$$\psi_N'(0) = \frac{\sum_{r=0}^{N-1} \binom{N-1}{r} k_r'(0) \sum_{r=0}^{N} \binom{N}{r} k_r(0) - \sum_{r=0}^{N-1} \binom{N-1}{r} k_r(0) \sum_{r=0}^{N} \binom{N}{r} k_r'(0)}{\left[\sum_{r=0}^{N} \binom{N}{r} k(0)\right]^2}$$

but by equation (2.3.14), $k_r(0) = \delta_{r0}$

$$\therefore \quad \psi_N'(0) = \sum_{r=0}^{N-1} \binom{N-1}{r} k_r'(0) - \sum_{r=0}^{N} \binom{N}{r} k_r'(0) \tag{2.5.2}$$

Differentiating equation (2.3.13) and putting s=0 gives

$$\therefore \; k_r'(0) = k_0'(0) \prod_{j=1}^{r} \frac{1-\phi(0+\overline{j-1}\mu)}{\phi(j\mu)} + k_0(0)\frac{d}{ds}\left[\prod_{j=1}^{r} \frac{1-\phi(s+\overline{j-1}\mu)}{\phi(s+j\mu)}\right]_{s=0}$$

$$= \frac{d}{ds}\left[\frac{[1-\phi(s)][1-\phi(s+\mu)]\ldots[1-\phi(s+\overline{r-1}\mu)]}{\phi(s+\mu)\phi(s+2\mu)\ldots\phi(s+r\mu)}\right]_{s=0}$$

$$= \frac{d}{ds}\left[\frac{1-\phi(s)}{\phi(s+\mu)}\right]_{s=0} \prod_{j=2}^{r} \frac{1-\phi(\overline{j-1}\mu)}{\phi(j\mu)} + \frac{d}{ds}\left[\frac{1-\phi(s+\mu)}{\phi(s+2\mu)}\right]_{s=0} \frac{1-\phi(0)}{\phi(\mu)} \prod_{j=3}^{r} \frac{1-\phi(\overline{j-1}\mu)}{\phi(j\mu)}$$

$$+ \text{ terms of form } \frac{1-\phi(0)}{\phi(\mu)} \times \ldots$$

Since $\phi(0) = 1$, $\displaystyle k_r'(0) = \frac{-\phi'(0)}{\phi(\mu)} \prod_{j=2}^{r} \frac{1-\phi(\overline{j-1}\mu)}{\phi(j\mu)}$

$$= -\frac{\phi'(0)}{\phi(\mu)} k_{r-1}(\mu) \tag{2.5.3}$$

Substituting equation (2.5.3) into (2.5.2) gives

$$\psi_N'(0) = -\frac{\phi'(0)}{\phi(\mu)}\left[\sum_{r=1}^{N-1} \binom{N-1}{r} k_{r-1}(\mu) + k_0'(0) - \sum_{r=1}^{N} \binom{N}{r} k_{r-1}(\mu) - k_0'(0)\right]$$

$$= -\frac{\phi'(0)}{\phi(\mu)}\left[\sum_{r=1}^{N-1}\left[\binom{N-1}{r} - \binom{N}{r}\right]k_{r-1}(\mu) - k_{N-1}(\mu)\right]$$

$$= \frac{\phi'(0)}{\phi(\mu)} \sum_{r=0}^{N-1} \binom{N-1}{r} k_r(\mu) \; .$$

Using equation (2.4.8) gives Takács' [36] alternate representation, for $\psi_N'(0)$,

$$\psi_N'(0) = \phi'(0) \sum_{r=0}^{N} \binom{N}{r} \ell_r(\mu) \tag{2.5.4}$$

A consequence of equation (2.5.1) is the following expression for the weakness, f_N, of the overflow stream G,

$$f_N = \frac{f_0}{\phi(\mu)} \sum_{r=0}^{N-1} \binom{N-1}{r} k_r(\mu) \tag{2.5.5}$$

where f_N and f_0 are defined by equations (2.2.10) and (2.2.3) respectively.

Alternately the weakness can be expressed as

$$f_N = f_0 \sum_{r=0}^{N} \binom{N}{r} \ell_r(\mu) \quad \text{using equation (2.4.8).} \tag{2.5.6}$$

The intensity, I_N, of the overflow stream satisfies

$$I_N = I_0 \left[\sum_{r=0}^{N} \binom{N}{r} \ell_r(\mu) \right]^{-1} , \tag{2.5.7}$$

since I_N and I_0 are the reciprocals of f_N and f_0 respectively.

Property 2.

The nth divided difference of the overflow streams' weakness, $f(N,N+1,N+2,\ldots,N+n)$ satisfies

$$f(N,N+1,N+2,\ldots,N+n) = \frac{f_0}{n!} \sum_{r=0}^{N} \binom{N}{r} \ell_{n+r}(\mu) \tag{2.5.8}$$

where

$$f(N,N+1,N+2,\ldots,N+n) \equiv \frac{\Delta^n}{n!} f_N \tag{2.5.9}$$

and Δ is the forward difference operator defined in Appendix 1.

Proof of equation (2.5.8)

Equation (2.5.8) is proved if

$$\Delta^n f_N = f_0 \sum_{r=0}^{N} \binom{N}{r} \ell_{n+r}(\mu) \tag{2.5.10}$$

Proof of (2.5.10) follows by Mathematical Induction.

When $n=1$, $\Delta f_N = f_{N+1} - f_N$

$$= f_0 \left[\sum_{r=0}^{N+1} \binom{N+1}{r} \ell_r(\mu) - \sum_{r=0}^{N} \binom{N}{r} \ell_r(\mu) \right] \quad \text{from (2.5.6)}$$

$$= f_0 \sum_{r=1}^{N+1} \binom{N}{r-1} \ell_r(\mu)$$

$$= f_0 \sum_{r=0}^{N} \binom{N}{r} \ell_{r+1}(\mu)$$

Assume

$$\Delta^s f_N = f_0 \sum_{r=0}^{N} \binom{N}{r} \ell_{s+r}(\mu) \tag{2.5.11}$$

$$\therefore \quad \Delta^{s+1} f_N = \Delta^s f_{N+1} - \Delta^s f_N$$

$$= f_0 \left[\sum_{r=0}^{N+1} \binom{N+1}{r} \ell_{s+r}(\mu) - \sum_{r=0}^{N} \binom{N}{r} \ell_{s+r}(\mu) \right]$$

$$= f_0 \sum_{r=0}^{N} \binom{N}{r} \ell_{s+r+1}(\mu)$$

thus proving equation (2.5.10) by Mathematical Induction.

Property 2 of the overflow stream is basic to the derivation of explicit formulae for the overflow traffic moments given in Chapter 5.

An alternate expression for $\Delta^n f_N$ is

$$\Delta^n f_N = f_0 \; \ell_n(\mu) \sum_{r=0}^{N} \binom{N}{r} \ell_r(\overline{n+1}\mu) \tag{2.5.12}$$

Proof of (2.5.12) follows if

$$\ell_{n+r}(\mu) = \ell_n(\mu)\ell_r(\overline{n+1}\mu) \tag{2.5.13}$$

Now

$$\ell_{n+r}(\mu) = \prod_{j=1}^{n+r} \frac{1-\phi(j\mu)}{\phi(j\mu)}$$

$$= \prod_{j=1}^{n} \frac{1-\phi(j\mu)}{\phi(j\mu)} \prod_{j=n+1}^{n+r} \frac{1-\phi(j\mu)}{\phi(j\mu)}$$

$$= \ell_n(\mu) \prod_{j=1}^{r} \frac{1-\phi(\overline{n+j}\mu)}{\phi(\overline{n+j}\mu)}$$

$$= \ell_n(\mu)\ell_r(\overline{n+1}\mu), \quad \text{proving (2.5.13).}$$

Property 3.

The Laplace-Stieltjes transform of the overflow stream satisfies the following recurrence relation,

$$\psi_N(s+\mu) = \psi_N(s)\left[1 - \frac{1}{\psi_N(s)}\right]\left[1 - \frac{1}{\psi_{N+1}(s)}\right]^{-1} \tag{2.5.14}$$

Proof of (2.5.14)

Step 1. $$k_r(s+\mu) = \frac{\phi(s+\mu)}{1-\phi(s)} k_{r+1}(s) . \tag{2.5.15}$$

Since $$k_r(s+\mu) = \prod_{j=1}^{r} \frac{1-\phi(s+j\mu)}{\phi(s+\overline{j+1}\mu)}$$

$$= \frac{\phi(s+\mu)}{1-\phi(s)} \prod_{j=1}^{r+1} \frac{1-\phi(s+\overline{j-1}\mu)}{\phi(s+j\mu)}$$

thus proving equation (2.5.15).

Step 2.
$$\frac{\psi_N(s)}{1-\psi_N(s)} = \frac{\sum_{r=0}^{N-1}\binom{N-1}{r}k_r(s)}{\sum_{r=0}^{N-1}\binom{N-1}{r}k_{r+1}(s)} \tag{2.5.16}$$

Proof of (2.5.16)

$$\frac{\psi_N(s)}{1-\psi_N(s)} = \frac{\sum_{r=0}^{N-1}\binom{N-1}{r}k_r(s)}{\sum_{r=0}^{N}\binom{N}{r}k_r(s) - \sum_{r=0}^{N-1}\binom{N-1}{r}k_r(s)} \quad \text{using (2.3.17)}$$

$$= \frac{\sum_{r=0}^{N-1}\binom{N-1}{r}k_r(s)}{\sum_{r=1}^{N}\binom{N-1}{r-1}k_r(s)}$$

$$= \frac{\sum_{r=0}^{N-1}\binom{N-1}{r}k_r(s)}{\sum_{r=0}^{N-1}\binom{N-1}{r}k_{r+1}(s)} ,$$

thus proving equation (2.5.16).

Now

$$\frac{\psi_N(s)}{1-\psi_N(s)}\ \frac{1-\psi_{N+1}(s)}{\psi_{N+1}(s)} = \frac{\sum_{r=0}^{N-1}\binom{N-1}{r}k_r(s)\ \sum_{r=0}^{N}\binom{N}{r}k_{r+1}(s)}{\sum_{r=0}^{N-1}\binom{N-1}{r}k_{r+1}(s)\ \sum_{r=0}^{N}\binom{N}{r}k_r(s)}$$

$$= \psi_N(s)\ \frac{\sum_{r=0}^{N}\binom{N}{r}k_{r+1}(s)}{\sum_{r=0}^{N-1}\binom{N-1}{r}k_{r+1}(s)}$$

$$= \psi_N(s)\ \frac{\sum_{r=0}^{N}\binom{N}{r}k_r(s+\mu)}{\sum_{r=0}^{N-1}\binom{N-1}{r}k_r(s+\mu)} \quad \text{by (2.5.15)}$$

$$= \psi_N(s)\,[\psi_N(s+\mu)]^{-1} \tag{2.5.17}$$

Rearranging equation (2.5.17), proves (2.5.14).

Equation (2.5.14) is a rearrangement of the recurrence relation established by Takács [37], equation (8), page 136,

$$\psi_N(s) = \frac{\psi_{N-1}(s+\mu)}{1-\psi_{N-1}(s) + \psi_{N-1}(s+\mu)} \tag{2.5.18}$$

However it is the form given in equation (2.5.14) which is necessary to prove the factorial moment theorem for overflow traffic, given in Chapter 5.

Property 4.

The probability of blocking or the congestion probability $\bar{\pi}_N$, on the primary group of N trunks, satisfies equation (2.2.13), that is,

$$\bar{\pi}_N = \frac{f_0}{f_N} \quad . \tag{2.5.19}$$

Property 5.

The ratio of f_N to f_{N+1} is a measure of $\psi_N(s)$ at $s=\mu$.

$$\frac{f_N}{f_{N+1}} = \psi_N(\mu) \tag{2.5.20}$$

Proof of (2.5.20)

$$\frac{f_N}{f_{N+1}} = \frac{\sum_{r=0}^{N} \binom{N}{r} \ell_r(\mu)}{\sum_{r=0}^{N+1} \binom{N+1}{r} \ell_r(\mu)} \qquad \text{by equation (2.5.6)}$$

$$= \psi_N(\mu) \qquad \text{by equation (2.3.18)}$$

The second order moments of the overflow traffic will be shown in Chapters 3 and 4 to depend on $\psi_N(\mu)$, hence equation (2.5.20) enables such moments to be expressed in terms of the weakness of the overflow stream from N and $N+1$ trunks. Equation (2.5.20) is basis of the formulation, given in Chapter 7, of the Equivalent Non Random Method for dimensioning overflow systems. An alternate expression for $\psi_N(\mu)$, is

$$\psi_N(\mu) = \frac{I_{N+1}}{I_N} \qquad (2.5.21)$$

since the intensity of the stream is the reciprocal of its weakness.

<u>Property 6.</u>

$$\psi_N(2\mu) = \frac{f_N - f_{N+1}}{f_{N+1} - f_{N+2}} \; . \qquad (2.5.22)$$

Proof of (2.5.22)

$$\psi_N(2\mu) = \psi_N(\mu)\left[1 - \frac{1}{\psi_N(\mu)}\right]\left[1 - \frac{1}{\psi_{N+1}(\mu)}\right]^{-1} \quad \text{by (2.5.14)}$$

$$= \frac{f_N}{f_{N+1}}\left[1 - \frac{f_{N+1}}{f_N}\right]\left[1 - \frac{f_{N+2}}{f_{N+1}}\right]^{-1} \quad \text{by (2.5.17)}$$

$$= \frac{f_N - f_{N+1}}{f_{N+1} - f_{N+2}} \quad \text{thus proving (2.5.22) .}$$

A generalization of (2.5.22), viz.

$$\psi_N(\overline{n+1\mu}) = \frac{\Delta^n f_N}{\Delta^n f_{N+1}} \qquad (2.5.23)$$

is a consequence of the factorial moment theorem of overflow traffic to be proved in Chapter 5. However equation (2.5.23) can be proved directly using equations (2.5.12) and (2.5.13).

Proof of (2.5.23).

Since

$$\psi_N(\overline{n+1\mu}) = \frac{\sum_{r=0}^{N} \binom{N}{r} \ell_r(\overline{n+1\mu})}{\sum_{r=0}^{N+1} \binom{N+1}{r} \ell_r(\overline{n+1\mu})} \qquad \text{by (2.3.18) with } s=\overline{n+1\mu}$$

$$= \frac{\sum_{r=0}^{N} \binom{N}{r} \ell_{n+r}(\mu)}{\sum_{r=0}^{N+1} \binom{N+1}{r} \ell_{n+r}(\mu)} \qquad \text{by (2.5.13)}$$

$$= \frac{\Delta^n f_N}{\Delta^n f_{N+1}} \qquad \text{by (2.5.12)} .$$

CHAPTER III

OFFERED TRAFFIC DISTRIBUTIONS

3.1 Introduction

3.2 Imbedded Markov Chain Occupancy Distribution

3.3 Continuous Time Occupancy Distribution

3.4 Properties of the Offered Traffic Moments

3.5 Application of Traffic Formulae for Specified Input Streams.

CHAPTER III

3.1 Introduction

Suppose the arrival instants of calls at an infinite group of trunks form a renewal stream, with an interevent time distribution $G(t)$ and corresponding Laplace-Stieltjes transform $\psi(s)$. The steady state occupancy distribution, π, of the calls at arrival instants is called the imbedded Markov Chain distribution of the offered traffic. The steady state continuous time occupancy distribution, q, is known as the offered traffic distribution. The mean, variance and other central moments of the offered traffic are determined from the q distribution. The call congestion, discussed in the previous chapter, depends on the imbedded Markov Chain occupancy distribution for a finite group.

Expressions for the binomial moments and the occupancy probabilities of both the π and q offered traffic distributions will be derived using the approach of Pearce and Potter [25]. The technique consists of the following four stages.

1. Determine (using the fundamental ergodic theorem) a system of equations satisfied by the relevant occupancy distributions;

2. obtain an integral equation satisfied by the relevant occupancy generating functions;

3. obtain binomial moments, using successive differentiation of the relevant generating functions;

4. recover (if necessary) the occupancy probabilities.

The same methodology is used in the next chapter when analysing the carried traffic's imbedded Markov Chain and continuous time distributions.

The literature discussing the occupancy distributions for the $G/M/\infty$ queueing system is diverse, see Takács [36], Palm [22] and Syski [34]. However, to demonstrate the simplicity of the above fore mentioned technique and to obtain suitable, convenient forms for the offered traffic moments, a complete analysis of this $G/M/\infty$ system will be given.

The final section of this chapter contains proofs and comments on key results related to the offered traffic and the input stream. For example, the intuitive result that the mean offered traffic is the reciprocal of the weakness of the input stream is verified and the existence of an inverse relation between the ψ^s and the factorial traffic moments is established. Such results are fundamental to the factorial overflow traffic theorem of Chapter 5. Expressions for statistical quantities such as the peakedness and coefficient of variation of the offered traffic are derived for any renewal input stream.

Simplified formulae are obtained for the following specific input distributions; (i) Erlang distribution of order k, (ii) deterministic distribution and (iii) negative exponential distribution.

3.2 Imbedded Markov Chain Occupancy Distribution

Let $N(t)$ be the number of calls in progress at time t. For the renewal input stream F, $N(t)$ is not a Markov Process unless the arrivals form a Poisson stream, that is $F(t)$ is negative exponential. As in the previous chapter, suppose arrivals occur at time instants, τ_k, $k=1,2,\ldots$ and $N(\tau_{k-0})$ is known, then until the next call arrives at τ_{k+1} the number of calls in progress is a simple death process with death rate μ per trunk. No additional knowledge of $N(t)$ for $t < \tau_k$ is of prognostic relevance to $N(t)$ for $t > \tau_k$ when $N(\tau_{k-0})$ is known. Thus the $N(\tau_{k-0})$ form a

Markov Chain imbedded in the non Markov Process $N(t)$. The π occupancy distribution is the steady state distribution of the $N(\tau_{k-0})$ whereas the q occupancy distribution is the steady state distribution of the $N(t)$.

Let π_j be the steady state probability that an arriving customer finds j busy trunks, thus

$$\pi_j = \lim_{k\to\infty} p(N(\tau_{k-0}) = j) \ . \tag{3.2.1}$$

Let q_j be the steady state probability that at time t, j trunks are busy, thus

$$q_j = \lim_{t\to\infty} p(N(t) = j) \ . \tag{3.2.2}$$

Step 1. <u>Derivation of π_j system of equations</u>

An arriving call could find j trunks busy, if the previous call had found m trunks busy and $\overline{m-j-1}$ calls terminate during this interarrival time, when $m=j-1,j,\ldots,$ thus

$$\pi_j = \sum_{m=j-1}^{\infty} \pi_m \int_0^{\infty} \binom{m+1}{j} (1-e^{-\mu x})^{m-j+1} e^{-j\mu x} dG(x), \tag{3.2.3}$$

$$j=1,2,\ldots$$

Step 2. Generating function $\pi(z)$

$$\pi(z) = \int_0^{\infty} w(1+\overline{z-1}e^{-\mu x}) dG(x) \tag{3.2.4}$$

where $\qquad w(z) = z\pi(z).$

Proof:

Let $\pi(z) = \sum_{j=0}^{\infty} \pi_j z^j$, then equation (3.2.3) becomes

$$\pi(z) = \pi_0 + \sum_{j=1}^{\infty} \sum_{m=j-1}^{\infty} \pi_m \int_0^{\infty} \binom{m+1}{j} e^{-j\mu x} (1-e^{-\mu x})^{m+1-j} z^j dG(x), \quad j=1,2,\ldots$$

$$= \pi_0 + \sum_{m=0}^{\infty} \pi_m \int_0^{\infty} (1-e^{-\mu x}+ze^{-\mu x})^{m+1} dG(x) - \sum_{m=0}^{\infty} \pi_m \int_0^{\infty} (1-e^{-\mu x})^{m+1} dG(x)$$

$$= \pi_0 + \int_0^{\infty} (1-e^{-\mu x}+ze^{-\mu x})\pi(1-e^{-\mu x}+ze^{-\mu x})dG(x)$$

$$- \int_0^{\infty} (1-e^{-\mu x})\pi(1-e^{-\mu x})dG(x).$$

But $\pi(1) = 1$

$$\therefore \quad 1 = \pi_0 + 1 - \int_0^{\infty} (1-e^{-\mu x})\pi(1-e^{-\mu x})dG(x)$$

$$\therefore \quad \pi(z) = \int_0^{\infty} (1-e^{-\mu x}+ze^{-\mu x})\pi(1-e^{-\mu x}+ze^{-\mu x})dG(x) \qquad (3.2.5)$$

$$= \int_0^{\infty} w(1+\overline{z-1}e^{-\mu x})dG(x).$$

Step 3.

$$\beta_{n\pi} \equiv \frac{\pi^{(n)}(1)}{n!} = h_n(\mu) \qquad (3.2.6)$$

where

$$h_n(\mu) = \begin{cases} \prod_{j=1}^{n} \dfrac{\psi(j\mu)}{1-\psi(j\mu)} & , \quad n \geq 1 \\ 1 & , \quad n = 0 \end{cases} \qquad (3.2.7)$$

Proof of (3.2.6)

Taking the n^{th} derivative of equation (3.2.5) and applying Leibnitz rule gives,

$$\pi^{(n)}(1) = \int_0^{\infty} (1-e^{-\mu x}+ze^{-\mu x})\Big]_{z=1} \pi^{(n)}(1-e^{-\mu x}+ze^{-\mu x})\Big]_{z=1} dG(x)$$

$$+ \int_0^\infty \binom{n}{1} e^{-\mu x} \pi^{(n-1)} (1-e^{-\mu x}+ze^{-\mu x}) \Big]_{z=1} dG(x)$$

$$= \int_0^\infty \pi^{(n)}(1)e^{-n\mu x} dG(x) + \int_0^\infty ne^{-n\mu x}\pi^{(n-1)}(1)dG(x)$$

$$\therefore \quad \pi^{(n)}(1) = n\,\pi^{(n-1)}(1)\,\frac{\psi(n\mu)}{1-\psi(n\mu)} \tag{3.2.8}$$

$$\therefore \quad \pi^{(n)}(1) = \pi(1)n! \prod_{j=1}^{n} \frac{\psi(j\mu)}{1-\psi(j\mu)} , \tag{3.2.9}$$

thus proving equation (3.2.6).

Step 4.

$$\pi_k = \sum_{n=k}^{\infty} \binom{n}{k} (-1)^{n-k} h_n(\mu) . \tag{3.2.10}$$

Proof of (3.2.10).

The binomial moments $\beta_{n\pi}$ of the π distribution:

$$\beta_{n\pi} \equiv \sum_{k=n}^{\infty} \binom{k}{n} \pi_k , \tag{3.2.11}$$

hence equation (3.2.10) is the inverse of equation (3.2.11).

3.3 Continuous Time Occupancy Distribution

The q distribution is accessible from the π distribution.

Step 1.

$$q_j = K \sum_{m=0}^{\infty} \pi_{j-1+m} \int_0^\infty\!\!\int_0^t \binom{j+m}{m} (1-e^{-\mu x})^m e^{-j\mu x} dx dG(t), \; j \geqslant 1 \tag{3.3.1}$$

Proof of equation (3.3.1)

Consider an arbitrary instant in steady state. Let T be the length of the interarrival interval in which this point lies. The probability that this given instant falls in an interval of prescribed length is proportional both to the frequency of intervals of that length and to the length itself, that is

$$p(t < T < t+dt) = K\, t\, dG(t)$$

where $dG(t) = \text{prob}(t <$ length of arbitrary selected interval $\leqslant t+dt)$

and K is an arbitrary constant.

But

$$\int_0^\infty p(t < T < t+dt) = 1$$

$$\therefore\ K^{-1} = \int_0^\infty t\, dG(t) \qquad (3.3.2)$$

$$\therefore\ K = M^{-1} \qquad (3.3.3)$$

where M is the mean of $G(t)$.

Now $p(j$ trunks busy at an arbitrary instant/instant lies in an interval of length $t)$

$= \frac{1}{t}\int_0^t p\,(j$ trunks busy at time x after beginning of interval$)\, dx$

$\therefore q_j = p(j$ trunks busy at arbitrary instant in steady state$)$

$= \int_0^\infty p\,(j$ trunks busy at arbitrary instant/interval has length $T = t)\ p(t < T \leqslant t+dt)$

$= K\int_0^\infty \frac{1}{t}\int_0^t p\,(j$ trunks busy at time x after beginning of interval$)\, dx\ t\, dG(t)$

$= K\int_0^\infty \int_0^t p\,(j$ trunks busy at time x after beginning of interval$)\, dx\, dG(t)$

$$= K \sum_{m=0}^{\infty} \pi_{j-1+m} \int_0^\infty\int_0^t \binom{j+m}{m}(1-e^{-\mu x})^m e^{-j\mu x}\, dx\, dG(t),\ j \geqslant 1\ .$$

Step 2.

$$q(z) = 1 + f^{-1} \sum_{n=1}^{\infty} (z-1)^n h_{n-1}(\mu)/n \tag{3.3.4}$$

where f is the weakness of the input stream G.

Proof of (3.3.4)

Using equation (3.3.1),

$$q(z) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \pi_{j-1+m} K\int_0^{\infty} dG(t) \int_0^t \binom{j+m}{m} e^{-j\mu x} (1-e^{-\mu x})^m z^j\, dx$$

$$= K\int_0^{\infty} dG(t) \int_0^t \sum_{m=0}^{\infty} \sum_{k=m-1}^{\infty} \pi_k \binom{k+1}{m} (ze^{-\mu x})^{k-m+1} (1-e^{-\mu x})^m\, dx$$

where $k=j-1+m$

$$= K\int_0^{\infty} dG(t) \int_0^t w(1+\overline{z-1}e^{-\mu x})\, dx \tag{3.3.5}$$

Let $w(1+y) = \sum_{n=0}^{\infty} w_n \frac{y^n}{n!}$.

For $n \geqslant 1$, $w_n = w^{(n)}(1+y)\Big]_{y=0} = \frac{d^n}{dy^n}(1+y)\pi(1+y)\Big]_{y=0}$

$= \pi^{(n)}(1) + n\pi^{(n-1)}(1)$ by Leibnitz Theorem.

$$= n\frac{\pi^{(n-1)}(1)}{1-\psi(n\mu)} \quad \text{by (3.2.8)} \tag{3.3.6}$$

and $w_0=1$.

Substituting for w, equation (3.3.5) becomes

$$q(z) = K\int_0^{\infty} dG(t) \int_0^t \sum_{n=0}^{\infty} w_n \frac{(z-1)^n}{n!} e^{-n\mu x}\, dx$$

$$= K\int_0^{\infty} dG(t) \left[t + \sum_{n=1}^{\infty} w_n \frac{(z-1)^n}{n!} \frac{(1-e^{-n\mu t})}{n\mu} \right]$$

$$= K\int_0^\infty t dG(t) + K \sum_{n=1}^{\infty} w_n \frac{(z-1)^n}{n!} \frac{1-\psi(n\mu)}{n\mu}$$

$$= 1 + K \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!} \frac{\pi^{(n-1)}(1)}{\mu} \quad \text{by equation (3.3.6)}$$

$$= 1 + \frac{K}{\mu} \sum_{n=1}^{\infty} \frac{(z-1)^n}{n} h_{n-1}(\mu) \quad \text{by equation (3.2.6)}$$

But $f^{-1} = K/\mu$ by equation (2.2.4), thus proving (3.3.4).

Step 3.

$$\beta_{nq} \equiv \frac{q^{(n)}(1)}{n!} = \frac{h_{n-1}(\mu)}{nf} \tag{3.3.7}$$

Proof of (3.3.7)

Since $q(z) = \sum_{n=0}^{\infty} \frac{q^{(n)}(1)}{n!} (z-1)^n$ and $q(z)$ satisfies (3.3.4), equation (3.3.7) is obtained by equating coefficients of $(z-1)^n$.

Step 4.

$$g_k = f^{-1} \sum_{n=k}^{\infty} \binom{n}{k} (-1)^{n-k} \frac{h_{n-1}(\mu)}{n} \tag{3.3.8}$$

$$q_0 = 1 - f^{-1} \sum_{n=1}^{\infty} \frac{h_{n-1}(\mu)}{n} \tag{3.3.9}$$

Equations (3.3.8) and (3.3.9) follow from the definition of the binomial moments β_{nq}.

3.4 Properties of the Offered Traffic Moments

Property 1. The factorial moments of the π and q distributions satisfy

$$\alpha_{(n-1)\pi} = f\,\alpha_{nq} \quad , \quad n \geqslant 1 \tag{3.4.1}$$

where
$$\begin{cases} \alpha_{n\pi} \equiv \pi^{(n)}(1) \\ \alpha_{nq} \equiv q^{(n)}(1) \end{cases}$$

Proof of (3.4.1)

$$q^{(n)}(1) = \frac{n!}{nf} h_{n-1}(\mu) \quad \text{by equation (3.3.7)}$$

$$= \frac{(n-1)!}{f} h_{n-1}(\mu)$$

$$= \frac{\pi^{(n-1)}(1)}{f} \quad \text{by equation (3.2.6)} \qquad (3.4.2)$$

thus proving (3.4.1).

Property 2.

$$\left[1 + n \frac{\alpha_{nq}}{\alpha_{\overline{n+1\,q}}}\right]^{-1} = \psi(n\mu) \qquad (3.4.3)$$

Proof of (3.4.3)

Since $q^{(n)}(1) = \frac{(n-1)!}{f} h_{n-1}(\mu)$ by (3.3.7),

$$\frac{q^{(n+1)}(1)}{q^{(n)}(1)} = n \frac{h_n(\mu)}{h_{n-1}(\mu)} \qquad (3.4.4)$$

$$= n \frac{\psi(n\mu)}{1-\psi(n\mu)} \quad \text{from (3.2.7)}$$

$$\therefore \quad \frac{1}{\psi(n\mu)} - 1 = n \frac{q^{(n)}(1)}{q^{(n+1)}(1)} \qquad (3.4.5)$$

thus proving (3.4.3).

This property relating $\psi(n\mu)$ with the factorial moments of the offered traffic is basic to the derivation of explicit formulae for the overflow traffic moments for any renewal input stream.

Equation (3.4.4) relates the π and q factorial moments in the following way,

$$\frac{q^{(n+1)}(1)}{q^{(n)}(1)} = n \frac{\pi^{(n)}(1)}{\pi^{(n-1)}(1)} \qquad (3.4.6)$$

Property 3. The mean offered traffic, M, is given by α_{1q} where

$$\alpha_{1q} = \beta_{1q} = f^{-1} \tag{3.4.7}$$

Equation (3.4.7) follows from equation (3.3.7) with $n=1$. Implicit in this result is the equivalence of a renewal streams' intensity and the mean offered traffic produced by this stream. This is intuitively obvious but is important when considering the offered traffic moments of an overflow renewal stream.

Property 4. The variance V of the offered traffic satisfies

$$V = f^{-1}[1 - f^{-1} + f^{*-1}] \tag{3.4.8}$$

where f^{-1} is the mean offered traffic and

$$f^* = (\alpha_{1\pi})^{-1}, \tag{3.4.9}$$

that is, f^* is the mean offered traffic for the π distribution.

Proof of (3.4.8).

$$V = q^{(2)}(1) + q^{(1)}(1) - [q^{(1)}(1)]^2 \tag{3.4.10}$$

$$= q^{(1)}(1)[1-q^{(1)}(1) + \pi^{(1)}(1)] \tag{3.4.11}$$

by equation (3.4.2), for $n = 2$

thus result follows by (3.4.7) and (3.4.9).

Since $\pi^{(1)}(1) = h_1(\mu)$ by equation (3.2.6)

$$= \frac{\psi(\mu)}{1-\psi(\mu)}$$ from (3.2.7) with $n=1$,

then

$$V = M(1-M+h_1) \tag{3.4.12}$$

or alternatively

$$V = f^{-2}\left[\frac{f}{1-\psi(\mu)} - 1\right] \tag{3.4.13}$$

Property 5. The peakedness, z, and the coefficient of variation, w, of the offered traffic satisfy

$$z \equiv \frac{V}{M} = f^{-1}\left[\frac{f}{1-\psi(\mu)} - 1\right] \qquad (3.4.14)$$

$$w \equiv \frac{V}{M^2} = \frac{f}{1-\psi(\mu)} - 1 \qquad (3.4.15)$$

The equations (3.4.14), (3.4.15) follow directly from equation (3.4.13).

Property 6. The peakedness of the offered traffic is greater than or equal to ½. The minimum value is only achieved for a deterministic stream of very low weakness. Variation in the weakness of the renewal stream is equivalent to leaving the interevent times fixed and varying the parameter μ of the holding times, see Pearce [23]. A large value of μ corresponds to high traffic weaknesses (or equivalently low intensities), and small values of μ correspond to low weaknesses. To prove property 6, we consider z as a function of μ with λ fixed.

By the mean value theorem

$$\psi(\mu) = 1 + \mu\psi'(0) + \frac{\mu^2}{2}\psi''(\xi) \qquad (3.4.16)$$

for μ small for some $\xi \in (0,\mu)$. Equation (3.4.14) can be rewritten as

$$z = (1-\psi(\mu))^{-1} - [-\mu\psi'(0)]^{-1}$$

$$= [-\mu\psi'(0) - \frac{\mu^2}{2}\psi''(\xi)]^{-1} - [-\mu\psi'(0)]^{-1} \quad \text{by (3.4.16)}$$

$$= \frac{\psi''(\xi)}{2\psi'(0)\,[\psi'(0) + \frac{\mu^2}{2}\psi''(\xi)]} \qquad (3.4.17)$$

$$\therefore \lim_{\mu\to 0} z(\mu) = \frac{\psi''(0)}{2[\psi'(0)]^2} \tag{3.4.18}$$

If the input stream G has mean, m, and variance, v, then the coefficient of variation, w, of the interevent times satisfies

$$w \equiv \frac{v}{m^2} = \frac{\psi''(0) - [\psi'(0)]^2}{[\psi'(0)]^2} \tag{3.4.19}$$

$$\text{i.e.} \quad \frac{\psi''(0)}{\psi'(0)} = 1 + w \tag{3.4.20}$$

Therefore equation (3.4.18) becomes

$$\lim_{\mu\to 0} z(\mu) = \frac{1+w}{2} \tag{3.4.21}$$

$$\text{i.e.} \qquad \lim_{\mu\to 0} z(\mu) \geqslant \tfrac{1}{2} \quad \text{since} \quad w \geqslant 0 \tag{3.4.22}$$

This limiting value of ½ occurs for a stream with low weakness and $w=0$, that is, a deterministic stream of low weakness.

Kuczura, [14] p. 1315, in analysing the variability of a traffic stream uses Jensen's inequality , to show that for a fixed mean interarrival time, m, a stream's peakedness defined by equation (3.4.14) attains a minimum when $G(t)$ is the one point distribution defined by

$$G(t) = \begin{cases} 0 , & t < m \\ 1 , & m \leqslant t \end{cases} \tag{3.4.23}$$,

$$\text{with} \qquad \psi(\mu) = e^{-f_0} \quad \text{where} \quad f_0 = m\mu \tag{3.4.24}$$.

On substituting equation (3.4 .24) for $\psi(\mu)$ into equation (3.4.14) for z, he showed that the peakedness for this distribution can vary from ½ (when the intensity = ∞, or equivalently the weakness = 0) to unity 1 when the intensity = 0 or equivalently the weakness = ∞).

Property 7. The coefficient of variation, W, is a monotonic increasing function of μ.

Equation (3.4.15) can be rewritten as

$$W = \frac{-\mu\psi'(0)}{1-\psi(\mu)} - 1 \tag{3.4.25}$$

$$\therefore \quad \frac{dW}{d\mu} = \frac{-\psi'(0)}{(1-\psi(\mu))}\left[1 + \frac{\mu\psi'(\mu)}{1-\psi(\mu)}\right] \tag{3.4.26}$$

$$= \frac{-\psi'(0)}{1-\psi(\mu)} \frac{\mu^2\psi''(\xi)}{2(1-\psi(\mu))} \qquad \text{by (3.4.16)}$$

$$> 0 . \tag{3.4.27}$$

Hence W is strictly monotone increasing with μ. From equation (3.4.25) it follows as $\mu \to 0$, $W \to 0$ and as $\mu \to \infty$, $W \to \infty$, hence the coefficient of variation of the offered traffic ranges from 0 to ∞ as the stream's weakness ranges from 0 to ∞.

3.5 Application of Traffic Formulae for Specified Input Streams.

(i) Erlang Distribution order k input stream.

A k-phase Erlang input stream, characterised by an interarrival distribution G_{ER}, satisfies

$$G_{ER}(x) = 1 - \sum_{j=0}^{k-1} \frac{(\lambda_1 x)^j}{j!} e^{-\lambda_1 x} , \quad k \text{ integer}, \lambda_1 > 0 , \tag{3.5.1}$$

with a corresponding Laplace-Stieltjes transform ψ_{ER}, given by

$$\psi_{ER}(s) = \left(\frac{\lambda_1}{\lambda_1+s}\right)^k \tag{3.5.2}$$

This distribution arises quite naturally in the consideration of input processes. Suppose, that a device distributes requests for service to two groups of trunks on an alternating basis. Then the input to each group of trunks is Erlangian with $k=2$ phases.

The Erlang Distribution is the discrete case of the well known mathematical gamma distribution,

$$\Gamma(x) = \frac{(\lambda x)^{p-1}}{\Gamma(p)} \lambda e^{-\lambda x} \quad (p > 0,\ \lambda > 0,\ x \geqslant 0)\ . \qquad (3.5.3)$$

The weakness, f, of this stream is given by

$$f_{ER} = -\mu\psi'_{ER}(0) \quad \text{from } (2.2.3)$$

$$= \frac{+\mu k}{\lambda_1} \qquad (3.5.4)$$

$$= kA^{-1} \qquad (3.5.5)$$

where $$A = \frac{\lambda_1}{\mu}\ . \qquad (3.5.6)$$

The intensity, or equivalently the mean traffic M_{ER}, is the reciprocal of f_{ER}, thus

$$M_{ER} = \frac{\lambda_1}{k\mu} \qquad (3.5.7)$$

$$= \frac{A}{k}\ .$$

If the mean interarrival time of an Erlang distribution order k is assumed identical to that of a negative exponential stream, parameter λ then the intensity, of both streams is the same,

i.e. $$M_{ER} = A \qquad (3.5.8)$$

where $$A = \lambda/\mu \qquad (3.5.9)$$

and $$\lambda_1 = k\lambda \,. \qquad (3.5.10)$$

Now the variance of the traffic produced by the Erlang input stream satisfies equation (3.4.12) giving

$$V_{ER} = M_{ER}\left(-M_{ER} + \frac{1}{1-\psi_{ER}(\mu)}\right) \qquad (3.5.11)$$

$$= A\left[-A + \frac{1}{1-\left[\frac{k\lambda}{k\lambda+\mu}\right]^{k}}\right] \quad \text{using (3.5.8) and (3.5.10)}$$

$$= A\left[-A + \frac{1}{1-\left[1+\frac{1}{kA}\right]^{-k}}\right] . \qquad (3.5.12)$$

The peakness and coefficient of variation of the traffic produced by the Erlang input stream satisfies equations (3.4.14), (3.4.15) respectively, hence

$$z_{ER} = -A + \frac{1}{1-\left(1+\frac{1}{kA}\right)^{-k}} \qquad (3.5.13)$$

$$W_{ER} = -1 + \frac{A^{-1}}{1-\left(1+\frac{1}{kA}\right)^{-k}} \qquad (3.5.14)$$

The Erlang distribution provides a model for a range of input processes characterised by complete randomness when $k=1$ and no randomness when $k=\infty$.

(ii) Negative Exponential input stream .

A negative exponential stream with parameter λ, is an Erlang distribution of order 1. Hence by letting $k=1$ in equations

(3.3.8), (3.5.12), (3.5.13), and (3.5.14),

$$M_{EX} = A \tag{3.5.15}$$

$$V_{EX} = A \tag{3.5.16}$$

$$z_{EX} = 1 \tag{3.5.17}$$

$$W_{EX} = A^{-1} \tag{3.5.18}$$

(iii) Deterministic input stream.

A deterministic stream with constant interarrival time equal to mean interarrival time of the negative exponential distribution, parameter λ, corresponds to the limiting case of an Erlang Distribution of infinite order. Hence by taking the limit as k tends to infinity in equations (3.5.8), (3.5.12), (3.5.13) and (3.5.14)

$$M_{DET} = A \tag{3.5.19}$$

$$V_{DET} = \lim_{k \to \infty} A\left[-A + \frac{1}{1-[1+\frac{1}{kA}]^{k}}\right]$$

$$= A\left[-A + \frac{1}{1-e^{-1/A}}\right] \tag{3.5.20}$$

$$z_{DET} = -A + \frac{1}{1-e^{-1/A}} \tag{3.5.21}$$

$$W_{DET} = -1 + \frac{A^{-1}}{1-e^{-1/A}} \tag{3.5.22}$$

An alternate method for obtaining these expressions for V_{DET}, z_{DET}, W_{DET} is to simplify equations (3.4.13), (3.4.14) and (3.4.15) with

$$f = A^{-1} \tag{3.5.23}$$

and

$$\psi_{DET}(s) = e^{-\frac{s}{\lambda}} . \tag{3.5.24}$$

CHAPTER IV

CARRIED TRAFFIC DISTRIBUTIONS

CHAPTER IV

CARRIED TRAFFIC DISTRIBUTIONS

4.1 Introduction

To establish notation and certain properties of the carried traffic of the stream G, the techniques used will parallel those of the previous chapter which corresponds to the limiting case $L = \infty$.

Key relations between the $\bar{\pi}$ and $\bar{q}$ binomial moments will be stressed as well as expressions for the time congestion, call congestion and the first two central moments of the carried traffic. One significant expression resulting from the analysis of this chapter is a relation linking the offered traffic moments with the carried traffic moments. This expression is fundamental in the derivation of explicit formulae for the carried overflow traffic moments, given in Chapter 5.

The chapter concludes with a few comments on the possible divergence between the continuous time and the imbedded Markov Chain distributions for the traffic on a primary set of trunks.

4.2 Imbedded Markov Chain Carried Traffic Distribution

As in section 3.2, define $\bar{\pi}_j$ and $\bar{q}_j$ by equations (3.2.1) and (3.2.2) where the 'bar' signifies that the set of trunks being offered the renewal stream G is finite of size L.

Step 1. Derivation of $\bar{\pi}_j$ systems of equations.

Analogously to equation (3.2.3), the equations satisfied by the $\bar{\pi}_j$ are

$$\bar{\pi}_j = \sum_{m=j-1}^{L-1} \bar{\pi}_m \int_0^{\infty} \binom{m+1}{j} e^{-j\mu x} (1-e^{-\mu x})^{m+1-j} \, dG(x)$$

$$+ \bar{\pi}_L \int_0^{\infty} \binom{L}{j} e^{-j\mu x} (1-e^{-\mu x})^{L-j} \, dG(x) \quad , \quad 0 < j \leqslant L \qquad (4.2.1)$$

The extra term included in equation (4.2.1) which is not in equation (3.2.3) is the expression for the probability of an arriving call finding all L channels occupied, that is the situation when an overflowing call is produced or when congestion occurs.

Step 2. <u>Generating Function $\bar{\pi}(z)$.</u>

$$\bar{\pi}(z) = \int_0^{\infty} w(1+\overline{z-1}e^{-\mu x}) dG(x) - \bar{\pi}_L \int_0^{\infty} (z-1) e^{-\mu x} (1+\overline{z-1}e^{-\mu x})^L dG(x) \qquad (4.2.2)$$

where $\bar{\pi}_L$ is the congestion probability defined in Chapter 2 by equation (2.2.6).

Equation (3.2.3) is the limiting case of equation (4.2.2) when $L \to \infty$.

Proof of equation (4.2.2)

$$\bar{\pi}(z) = \sum_{j=0}^{L} \bar{\pi}_j z^j$$

$$= \bar{\pi}_0 + \sum_{j=1}^{L} \sum_{m=j-1}^{L-1} \bar{\pi}_m \int_0^{\infty} \binom{m+1}{j} e^{-j\mu x} (1-e^{-\mu x})^{m+1-j} z^j \, dG(x)$$

$$+ \sum_{j=1}^{L} \bar{\pi}_L \int_0^{\infty} \binom{L}{j} e^{-j\mu x} (1-e^{-\mu x})^{L-j} z^j \, dG(x)$$

$$= \bar{\pi}_0 + \sum_{m=0}^{L-1} \bar{\pi}_m \int_0^{\infty} [e^{-\mu x}(z-1)+1]^{m+1} dG(x) + \bar{\pi}_L \int_0^{\infty} [e^{-\mu x}(z-1)+1]^L dG(x)$$

$$- \int_0^{\infty} [\sum_{m=0}^{L-1} \bar{\pi}_m (1-e^{-\mu x})^{m+1} + \bar{\pi}_L (1-e^{-\mu x})^L] dG(x)$$

$$= \bar{\pi}_0 + \int_0^{\infty} \bar{w}(\overline{1+z-1e^{-\mu x}})dG(x) + \bar{\pi}_L \int_0^{\infty} (\overline{1+z-1e^{-\mu x}})^L [1-(\overline{1+z-1e^{-\mu x}})]dG(x)$$

$$- \int_0^{\infty} \bar{w}(1-e^{-\mu x})dG(x) - \bar{\pi}_L \int_0^{\infty} (1-1-e^{-\mu x})(1-e^{-\mu x})^L dG(x)$$

but $$\bar{\pi}(1) = 1 = \bar{\pi}_0 + 1 - \int_0^{\infty} \bar{w}(1-e^{-\mu x})dG(x) + \bar{\pi}_L \int_0^{\infty} e^{-\mu x}(1-e^{-\mu x})^L dG(x)$$

thus proving (4.2.2).

Step 3.

$$\bar{\beta}_{n\pi} \equiv \frac{\bar{\pi}^{(n)}(1)}{n!}$$

$$= h_n(\mu)[1 - \bar{\pi}_L \sum_{k=0}^{n-1} \binom{L}{k} h_k^{-1}(\mu)], \quad 0 < n \leqslant L \tag{4.2.3}$$

where $h_n(\mu)$ is defined by (3.2.7).

Proof of equation (4.2.3)

Differentiation of equation (4.2.2) n times, via Leibnitz Rule, gives

$$\bar{\pi}^{(n)}(z) = \int_0^{\infty} [(\overline{1+z-1e^{-\mu x}})\bar{\pi}^{(n)}(\overline{1+z-1e^{-\mu x}}) + n\bar{\pi}^{(n-1)}(\overline{1+z-1e^{-\mu x}})e^{-\mu x}]dG(x)$$

$$- \bar{\pi}_L \int_0^{\infty} [(z-1)e^{-\mu x}([\overline{1+z-1e^{-\mu x}}]^L)^{(n)} + ne^{-\mu x}[(\overline{1+z-1e^{-\mu x}})^L]^{(n-1)}]dG(x)$$

$$\therefore \quad \frac{\bar{\pi}^{(n)}(1)}{n!} = \int_0^{\infty} [e^{-n\mu x} \frac{\bar{\pi}^{(n)}(1)}{n!} + \frac{\bar{\pi}^{(n-1)}(1)}{(n-1)!} e^{-n\mu x}]dG(x)$$

$$- \bar{\pi}_L \int_0^{\infty} \binom{L}{n-1} e^{-n\mu x} dG(x)$$

$$\therefore \quad \frac{\bar{\pi}^{(n)}(1)}{n!}[1 - \psi(n\mu)] = \left[\frac{\bar{\pi}^{(n-1)}(1)}{(n-1)!} - \bar{\pi}_L \binom{L}{n-1}\right]\psi(n\mu) \tag{4.2.4}$$

Syski [34] p. 258 notes that this recurrence relation was originally obtained by Cohen. Takács [36] used an integral equation approach to derive (4.2.4). Now from equation (4.2.4),

$$\frac{\bar{\pi}^{(n)}(1)}{n!} = \prod_{k=1}^{n} \frac{\psi(k\mu)}{1-\psi(k\mu)} - \bar{\pi}_L \sum_{k=0}^{n-1} \binom{L}{k} \frac{h_n(\mu)}{h_k(\mu)}, \tag{4.2.5}$$

thus proving (4.2.3).

Step 4.

$$\bar{\pi}_k = \sum_{n=k}^{L} \binom{n}{k} (-1)^{n-k} \bar{\beta}_{n\pi} \tag{4.2.6}$$

Equation (4.2.6) is the inversion formula needed to recover the occupancy probabilities from the binomial moments.

4.3 Continuous Time Carried Traffic Distribution

The $\bar{q}$ distribution is accessible from the $\bar{\pi}$ distribution in the same way as the q distribution was from the π distribution.

Step 1.

$$\bar{q}_j = \sum_{m=j-1}^{L-1} K\, \bar{\pi}_m \int_0^\infty dG(t) \int_0^t \binom{m+1}{j} e^{-j\mu x} (1-e^{-\mu x})^{m+1-j}\, dx$$

$$+ \bar{\pi}_L\, K \int_0^\infty dG(t) \int_0^t \binom{L}{j} e^{-j\mu x} (1-e^{-\mu x})^{L-j}\, dx \tag{4.3.1}$$

where K is defined by (3.3.2).

The analysis used to obtain equation (4.3.1) from (4.2.1) is the same as that used to obtain (3.3.1) from (3.2.3).

Step 2.

$$\bar{q}(z) = 1+f^{-1} \sum_{n=1}^{L} \frac{(z-1)^n}{n} \frac{\bar{\pi}^{(n)}(1)}{n!} \frac{1-\psi(n\mu)}{\psi(n\mu)} \tag{4.3.2}$$

Proof of (4.3.2)

$$\bar{q}(z) = \bar{\pi}_0 + \sum_{j=1}^{L} \sum_{m=j-1}^{L-1} K\,\bar{\pi}_m \int_0^\infty dG(t) \int_0^t \binom{m+1}{j} e^{-j\mu x} (1-e^{-\mu x})^{m+1-j} z^j \, dx$$

$$+ K\,\bar{\pi}_L \sum_{j=1}^{L} \int_0^\infty dG(t) \int_0^t \binom{L}{j} e^{-j\mu x} (1-e^{-\mu x})^{L-j} z^j \, dx$$

$$= \bar{\pi}_0 + K \sum_{m=0}^{L-1} \bar{\pi}_m \int_0^\infty dG(t) \int_0^t [(1+\overline{z-1}e^{-\mu x})^{m+1} - (1-e^{-\mu x})^{m+1}] dx$$

$$+ K\,\bar{\pi}_L \int_0^\infty dG(t) \int_0^t [(1+\overline{z-1}e^{-\mu x})^{L} - (1-e^{-\mu x})^{L}] dx$$

$$= \bar{\pi}_0 + K \int_0^\infty dG(t) \bar{w}(1+\overline{z-1}e^{-\mu x}) dx - \sum_{m=0}^{L-1} \bar{\pi}_m K \int_0^\infty dG(t) \int_0^t (1-e^{-\mu x})^{m+1} dx$$

$$- K\,\bar{\pi}_L \int_0^\infty dG(t) \int_0^t (1+\overline{z-1}e^{-\mu x})^{L} (1+\overline{z-1}e^{-\mu x} - 1) dx$$

$$- K\,\bar{\pi}_L \int_0^\infty dG(t) \int_0^t (1-e^{-\mu x})^{L} dx \; .$$

But $\bar{q}(1) = 1$

$$\therefore \qquad \bar{\pi}_0 = K \sum_{m=0}^{L-1} \bar{\pi}_m \int_0^\infty dG(t) \int_0^t (1-e^{-\mu x})^{m+1} dx$$

$$+ K\,\bar{\pi}_L \int_0^\infty dG(t) \int_0^t (1-e^{-\mu x})^{L} dx$$

thus

$$\bar{q}(z) = K \int_0^\infty dG(t) \int_0^t \bar{w}(1+\overline{z-1}e^{-\mu x}) dx$$

$$- K\,\bar{\pi}_L \int_0^\infty dG(t) \int_0^t (z-1) e^{-\mu x} (1+\overline{z-1}e^{-\mu x})^{L} dx \qquad (4.3.3)$$

Let $\bar{w}(1+y) = \sum_{n=0}^{L} \bar{w}_n \dfrac{y^n}{n!}$ as in section 3.3, then

$$\bar{w}_n = \begin{cases} n\,\dfrac{\bar{\pi}^{(n-1)}(1)}{1-\psi(n\mu)} & , \quad 1 \leqslant n \leqslant L \\ 1 & , \quad n = 0 \end{cases} \tag{4.3.4}$$

Substituting for $\bar{w}$ in (4.3.3), gives

$$\bar{q}(z) = K\int_0^{\infty} dG(t) \int_0^{t} \Big[\sum_{n=0}^{L} \frac{(z-1)^n}{n!} e^{-n\mu x}\, \bar{\pi}^{(n)}(1)$$

$$+ \sum_{n=0}^{L} \frac{(z-1)^{n+1}}{n!} e^{-(n+1)\mu x}\, \bar{\pi}^{(n)}(1)$$

$$- \bar{\pi}_L (z-1)e^{-\mu x} \sum_{n=0}^{L} \binom{L}{n}(z-1)^n e^{-n\mu x} \Big] dx$$

$$= K\int_0^{\infty} dG(t) \Big[\sum_{n=1}^{L} \frac{-(z-1)^n}{n\mu} e^{-n\mu x} \frac{\bar{\pi}^{(n)}(1)}{n!} + x$$

$$+ \sum_{n=1}^{L} \left[\frac{\bar{\pi}^{(n)}(1)}{(n-1)!} - \bar{\pi}_L \binom{L}{n-1} \right] \frac{(z-1)^n e^{-n\mu x}}{-n\mu} \Big]_0^t$$

but $\dfrac{\bar{\pi}^{(n-1)}(1)}{(n-1)!} - \bar{\pi}_L \dbinom{L}{n-1} = \dfrac{1-\psi(n\mu)}{\psi(n\mu)} \dfrac{\bar{\pi}^{(n)}(1)}{n!}$ by (4.2.4)

$$\therefore \qquad \bar{q}(z) = 1 + \frac{K}{\mu} \sum_{n=1}^{L} \frac{(z-1)^n}{n} \frac{\bar{\pi}^{(n)}(1)}{n!} \left(\frac{1-\psi(n\mu)}{\psi(n\mu)}\right)$$

Step 3.

$$\bar{\beta}_{nq} \equiv \frac{\bar{q}^{(n)}(1)}{n!}$$

$$= f^{-1} \frac{h_{n-1}}{n} \left[1 - \bar{\pi}_L \sum_{k=0}^{n-1} \binom{L}{k} h_k^{-1}(\mu) \right] \tag{4.3.5}$$

Proof of (4.3.5)

Differentiation of (4.3.2) n times gives

$$\bar{q}^{(n)}(1) = f^{-1} \left[\frac{\bar{\pi}^{(n)}(1)}{n} \frac{1-\psi(n\mu)}{\psi(n\mu)} \right] \tag{4.3.6}$$

But $\frac{\bar{\pi}^{(n)}(1)}{n!} = h_n(\mu)[1 - \bar{\pi}_L \sum_{k=0}^{n-1} \binom{L}{k} h_k^{-1}(\mu)]$ by (4.2.5), thus on substituting for $\frac{\bar{\pi}^{(n)}(1)}{n!}$ in equation (4.3.6) gives

$$\bar{q}^{(n)}(1) = (n-1)!f^{-1}h_n(\mu)\left[\frac{1-\psi(n\mu)}{\psi(n\mu)}\right]\left[1 - \bar{\pi}_L \sum_{k=0}^{n-1} \binom{L}{k} h_k^{-1}(\mu)\right]$$

$$= f^{-1}(n-1)!h_{n-1}(\mu)\left[1 - \bar{\pi}_L \sum_{k=0}^{n-1} \binom{L}{k} h_k^{-1}(\mu)\right] \tag{4.3.7}$$

thus proving (4.3.5).

Step 4.

$$\bar{q}_k = \sum_{n=k}^{L} \binom{n}{k} (-1)^{n-k} \bar{\beta}_{nq} \ . \tag{4.3.8}$$

4.4 Properties of the Carried Traffic

Property 1.

The following relationship is satisfied by the binomial moments of the carried traffic,

$$n\ f\ \bar{\beta}_{nq} = [\bar{\beta}_{(n-1)\pi} - \bar{\pi}_L \binom{L}{n-1}] \ . \tag{4.4.1}$$

Proof of (4.4.1)

Substituting for $\bar{\pi}^{(n)}(1)\ \frac{1-\psi(n\mu)}{\psi(n\mu)}$ using (4.2.4), equation (4.3.6) becomes

$$f\ \bar{q}^{(n)}(1) = \bar{\pi}^{(n-1)}(1) - \bar{\pi}_L (n-1)! \binom{L}{n-1} \tag{4.4.2}$$

Dividing by $n!$ gives the required result.

Property 2. The time congestion, $\bar{q}_L \equiv \frac{\bar{q}^{(L)}(1)}{L!}$ satisfies equation (4.3.5) with $n=L$, that is

$$Lf\ \bar{q}_L = h_{L-1}(\mu)[1 - \bar{\pi}_L \sum_{k=0}^{L-1} \binom{L}{k} h_k^{-1}(\mu)] \ . \qquad (4.4.4)$$

However the call congestion probability $\bar{\pi}_L$ is related to the time congestion probability $\bar{q}_L$ by equation (4.3.6), with $n=L$, that is

$$\bar{q}_L \equiv \frac{\bar{q}^{(L)}(1)}{L!} = f^{-1} \frac{\bar{\pi}^{(L)}(1)}{L.L!} \frac{1-\psi(n\mu)}{\psi(n\mu)} \ ,$$

giving

$$Lf\ \bar{q}_L = \bar{\pi}_L \frac{1-\psi(L\mu)}{\psi(L\mu)} \qquad (4.4.5)$$

Rewriting equation (4.4.5) gives

$$\frac{1}{\psi(L\mu)} = 1 + Lf \frac{\bar{q}_L}{\bar{\pi}_L} \qquad (4.4.6)$$

One interpretation of equation (4.4.6) is that the Laplace-Stieltjes transform of any renewal input stream at $s=L\mu$ gives a measure of the rates of the congestion probabilities produced on a set of L trunks. This approach is used later in the thesis.

Property 3

Mean Carried Traffic, $\bar{M}$, as given by equation (4.3.7) with $n=1$, thus

$$\bar{M} = f^{-1}[1-\bar{\pi}_L] \qquad (4.4.7)$$

This expression for $\bar{M}$ could be obtained using equation (2.2.13), arising from the theory of section (2.2) in the following way.

If $\bar{\pi}_L$ is the congestion probability of the input stream with weakness f, then the weakness, f_L of the overflow stream resulting from this congestion satisfies (2.2.13), thus

$$f_L = \frac{f}{\bar{\pi}_L} .$$

But the average traffic carried on the L trunks, $\bar{M}$, is the difference between the intensity of the input stream and the overflow stream, that is

$$\bar{M} = \frac{1}{f} - \frac{1}{f_L} \tag{4.4.8}$$

$$= \frac{1}{f}[1 - \bar{\pi}_L]$$

<u>Property 4.</u>

The variance of the carried traffic, $\bar{V}$, satisfies

$$\bar{V} = \bar{M}[1 - \bar{M} + h_1 - \frac{L\,\bar{\pi}_L}{1-\bar{\pi}_L}] \tag{4.4.9}$$

Proof of (4.4.9)

$$\bar{V} = \bar{q}^{(2)}(1) + \bar{q}^{(1)}(1) - [\bar{q}^{(1)}(1)]^2$$

$$= \frac{1}{f}[\bar{\pi}^{(1)}(1) - \bar{\pi}_L L] + \bar{M} - \bar{M}^2 \quad \text{using (4.4.2) with } n=2.$$

But $\bar{\pi}^{(1)}(1) = h_1(\mu)[1-\bar{\pi}_L]$ from (4.2.5) with $n=1$ and $\frac{1}{f} = \frac{\bar{M}}{1-\bar{\pi}_L}$ from (4.4.8), hence

$$\bar{V} = \frac{\bar{M}}{1-\bar{\pi}_L}[(1-\bar{\pi}_L)h_1 - L\bar{\pi}_L] + \bar{M} - \bar{M}^2$$

which simplifies to give equation (4.4.9).

Equation (4.4.9) is a generalization of Wallstrom's [40] equation (3.1.35) page 208.

These expressions for $\bar{M}$, and $\bar{V}$ as well as those for M and V given by equations (3.4.7) and (3.4.13) are the basis of a possible study examining the errors introduced when the Kosten System is approximated by the Brockmeyer System.

Property 5

If $\bar{R}_L$ denotes the proportion of lost calls to carried calls on the set of L trunks, that is

$$\bar{R}_L \equiv \frac{L\,\bar{\pi}_L}{1-\bar{\pi}_L} \quad ,$$

then the following conditions are satisfied;

(i) $\bar{R}_L \geqslant h_1(\mu)$ with equality only when $L=1$, (4.4.10)

(ii) $\{R_L\}$ is decreasing as L increases , (4.4.11)

(iii) $R_L \to 0$ as $L \to \infty$. (4.4.12)

Proof: Putting $n=L$ in equation (4.2.3) gives

$$\bar{\pi}_L \equiv \frac{\bar{\pi}^{(L)}(1)}{L!} = [\sum_{k=0}^{L} \binom{L}{k} h_k^{-1}(\mu)]^{-1} \tag{4.4.13}$$

$$\therefore \quad \frac{1-\bar{\pi}_L}{L\,\bar{\pi}_L} = \frac{\sum_{k=0}^{L} \binom{L}{k} h_k^{-1}(\mu) - 1}{L}$$

$$= \frac{1}{L} \sum_{k=1}^{L} \binom{L}{k} h_k^{-1}(\mu)$$

$$= h_1^{-1}(\mu) + \frac{1}{L} \sum_{k=2}^{L} \binom{L}{k} h_k^{-1}(\mu) \tag{4.4.14}$$

Since the h_k^s are positive, condition (4.4.14) holds. This condition (4.4.14) implies from equation (4.4.9) for the variance of the carried traffic, that

$$\bar{V} \geqslant \bar{M}[1-\bar{M}] \quad \text{with equality only when } L=1. \tag{4.4.15}$$

Consider $L_1 > L_2$, then

$$\frac{1-\bar{\pi}_{L_1}}{L_1\ \bar{\pi}_{L_1}} = h_1^{-1}(\mu) + \frac{1}{L_1}\sum_{k=2}^{L_1}\binom{L_1}{k}h_k^{-1}(\mu) \quad \text{by (4.4.14)}$$

$$> h_1^{-1}(\mu) + \frac{1}{L_1}\sum_{k=2}^{L_2}\binom{L_1}{k}h_k^{-1}(\mu) \quad \text{since } L_1 > L_2$$

$$> h_1^{-1}(\mu) + \frac{1}{L_2}\sum_{k=2}^{L_2}\binom{L_2}{k}h_k^{-1}(\mu) \quad \text{since } L_1 > L_2 \text{ in each term in the sum}$$

$$= \frac{1-\bar{\pi}_{L_2}}{L_2\ \bar{\pi}_{L_2}} ,$$

hence (4.4.15) is proved.

Since $\bar{V} \to V$ and $\bar{M} \to M$ as $L \to \infty$, considering equation (4.4.9) in the limiting case proves (4.4.12).

4.5 A relationship between Carried Traffic Moments and Offered Traffic Moments.

Equation (4.3.6) relating $\bar{q}^{(n)}(1)$ to $\bar{\pi}^{(n)}(1)$ implies that the ratio of $\bar{q}^{(n)}(1)$ to $\bar{\pi}^{(n)}(1)$ is independent of L even though their individual expressions given by equations (4.2.3) and (4.3.5), depend on L. Since this quotient is independent of L its value can be obtained by letting $L \to \infty$ and is $\dfrac{q^{(n)}(1)}{\pi^{(n)}(1)}$, which by equation (3.4.2) equals $q^{(n)}(1)/fq^{(n+1)}(1)$, thus

$$\frac{\bar{q}^{(n)}(1)}{\bar{\pi}^{(n)}(1)} = \frac{q^{(n)}(1)}{fq^{(n+1)}(1)} , \quad 0 < n < L . \tag{4.5.1}$$

This equation (4.5.1) is a key relationship between the carried and offered traffic moments. It is fundamental to the derivation of explicit formulae for carried overflow traffic.

Equation (4.4.2) relates the $\bar{q}^{(n)}(1)$ with $\bar{\pi}^{(n-1)}(1)$ and the probability of loss, $\bar{\pi}_L$, enabling equation (4.5.1) to be expressed in the following ways.

(i)
$$\frac{\bar{\pi}^{(n-1)}(1) - (n-1)!\bar{\pi}_L\binom{L}{n-1}}{\bar{\pi}^{(n)}(1)} = \frac{q^{(n)}(1)}{q^{(n+1)}(1)} \tag{4.5.2}$$

or equivalently

(ii)
$$\frac{\bar{q}^{(n)}(1)}{f\bar{q}^{(n+1)}(1) + n!\bar{\pi}_L\binom{L}{n}} = \frac{1}{f}\frac{q^{(n)}(1)}{q^{(n+1)}(1)} \tag{4.5.3}$$

Equation (4.5.2) can be expressed in terms of

(i) the factorial moments $\bar{\alpha}$, α giving

$$\bar{\alpha}_{n\pi} = \frac{\alpha_{(n+1)q}}{\alpha_{nq}}\left[\bar{\alpha}_{(n-1)\pi} - (n-1)!\bar{\pi}_L\binom{L}{n-1}\right] \tag{4.5.4}$$

(ii) the binomial moments $\bar{\beta}$, β

$$\bar{\beta}_{n\pi} = \frac{(n+1)\beta_{(n+1)q}}{n\,\beta_{nq}}\left[\bar{\beta}_{(n-1)\pi} - \bar{\pi}_L\binom{L}{n-1}\right]. \tag{4.5.5}$$

Similarly equation (4.5.3) can be expressed in terms of

(i) the factorial moments giving

$$\bar{\alpha}_{(n+1)q} = \frac{1}{f}\left[f\,\bar{\alpha}_{nq}\frac{\alpha_{(n+1)q}}{\alpha_{nq}} - n!\bar{\pi}_L\binom{L}{n}\right] \tag{4.5.6}$$

(ii) the binomial moments giving

$$\bar{\beta}_{(n+1)q} = \frac{1}{f}\left[f\,\bar{\beta}_{nq}\frac{\beta_{(n+1)q}}{\beta_{nq}} - \frac{\bar{\pi}_L}{(n+1)}\binom{L}{n}\right] \tag{4.5.7}$$

4.6 Possible Divergence Between the Continuous Time and the Imbedded Markov Chain Occupancy Distribution

The formulation of the q and $\bar{q}$ distributions from the π and $\bar{\pi}$ distributions demonstrates that the continuous time distributions differ from the corresponding imbedded markov chain distributions. The possible extreme divergence which can occur for different input streams has been investigated by Pearce [24], Beneš [2]. The two distributions are identical when the input stream is negative exponential due to the memoryless property of the Poisson Process, thus the relevance of the possible divergence and its effect on the traffic moments for congestion systems has not until recently been investigated. Since the overflow stream G, produced when an input stream F is offered to a primary set of trunks, is not negative exponential, the possible divergence of the q and π or the $\bar{q}$ and $\bar{\pi}$ distributions of the overflow traffic could be significant.

Pearce [24] discusses the extreme divergence in relation to the fundamental paradox of renewal theory. He constructs renewal streams for which the following paradoxes hold.

If $\varepsilon, \varepsilon'$ are two arbitrary constants,

(i) A renewal stream, representing arriving calls to a set of L trunks exists for which both

$$\left.\begin{array}{ll} & \bar{\pi}_0(L) < \varepsilon \\ \text{and} & \bar{q}_0(L) > 1-\varepsilon' \end{array}\right\} \text{hold simultaneously,} \quad 0 < L \leqslant \infty .$$

Thus, the arriving calls emptiness probability can be arbitrarily close to zero which the observers emptiness probability is arbitrarily close to unity, regardless of the number of trunks in the finite set.

(ii) If $\varepsilon, \varepsilon'$ are any two arbitrary constants, then a renewal stream exists for which

$$\bar{\pi}_L > 1-\varepsilon \quad \text{and} \quad \bar{q}_L < \varepsilon' \quad \text{hold simultaneously.}$$

Thus, for this input stream the probability of overflow is all but certain for an arriving call even when the set of trunks is full only an arbitrarily small proportion of the time.

Pearce also discusses the influence these paradoxes could exert on the underdesign of teletraffic networks.

CHAPTER V

OVERFLOW TRAFFIC - GROUP APPROACH

CHAPTER V

OVERFLOW TRAFFIC - GROUP APPROACH

5.1 Introduction

An overflow stream resulting from a renewal input stream has been shown in Chapter II to retain its renewal property, with only the precise functional form of the interevent time distribution of calls changing. This invariant renewal property of overflow streams is basic to the 'group' approach for analysing overflow systems with renewal input.

We consider an overflow system as comprising two trunk groups with negative exponential services, parameter μ. The first or primary group is finite whereas the second or secondary group can be either finite or infinite. A renewal input stream F, is offered to a primary group of N trunks and the resulting overflow stream G, (determined in Chapter II), is then offered to the secondary group. When the secondary group is infinite the overflow system is called a Kosten system with renewal input, (w,r,i), otherwise a Brockmeyer system w.r.i occurs. The steady state continuous time occupancy distribution of the Kosten and Brockmeyer systems w.r.i determine the 'offered' and 'carried' overflow traffic distributions.

The approach or 'group method' used for determining overflow traffic moments of both systems, is to consider the overflow stream G from a finite primary group as input for either a $G/M/\infty$ or $G/M/L$ queueing system. Many formulae and properties relevant to moments of steady state occupancy distributions on either an infinite or finite group were determined in Chapters III and IV. All such results are therefore valid for a particular input stream which has this interevent distribution G characterising an overflow stream from the primary group. When the input stream for the $G/M/\infty$ or $G/M/L$ queueing system is being considered in this special way, all functions corresponding to this stream G are subscripted by N. Although Cohen, see Syski [34] page (416), mentioned that this approach could be used to analyse overflow systems, no detailed analysis eventuated. If however the complicated expression for the ψ^s corresponding to G are inserted, for example, into the expressions for the q moments, the mathematics becomes extremely complicated and messy and yields no explicit formulae directly.

The link which enables the overflow traffic moments to be found explicitly is a theorem relating the overflow traffic moments with the size of the primary group. Two perspectives of an overflow system provide interesting yet different proofs of this overflow traffic moment theorem. Both are described in this chapter.

Nightingale [21] and Freeman [16] mention that the overflow traffic moments from a negative exponential input stream satisfy this theorem, however its wider application to general renewal input streams is new to telephony.

The overflow traffic moments are expressed explicitly in terms of either

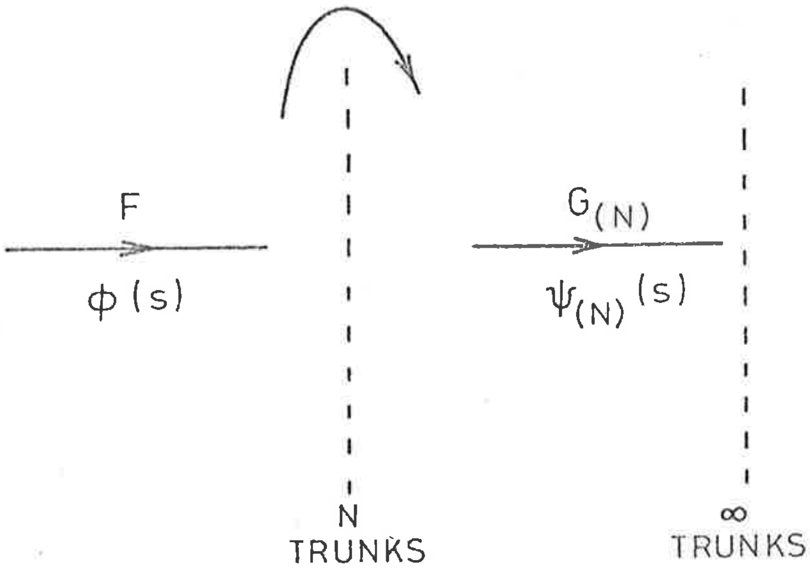
(i) finite differences of the overflow stream's weakness

or

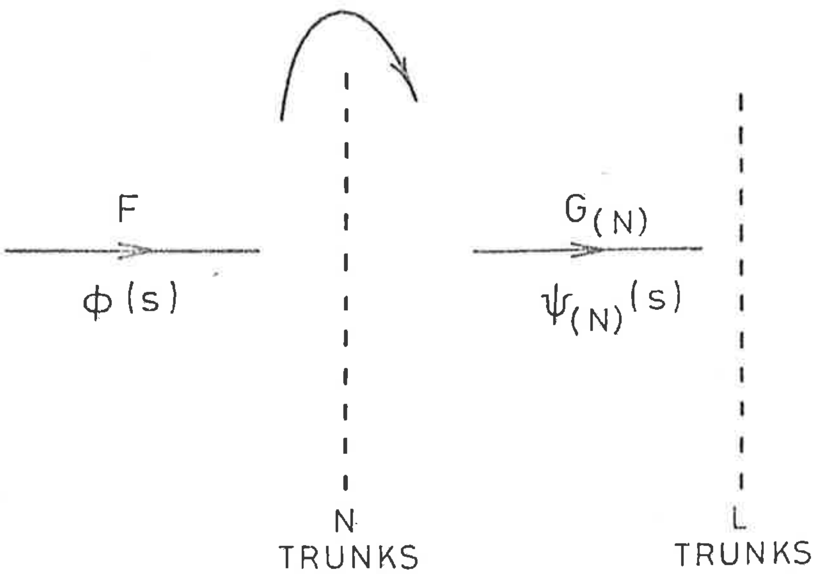
(ii) Laplace-Stieltjes transforms corresponding to the input stream.

Explicit expressions for such statistical quantities as peakedness, marginal occupancy and coefficient of variation which teletraffic engineers use to describe characteristics of overflow systems, are also derived.

KOSTEN SYSTEM WITH RENEWAL INPUT



BROCKMEYER SYSTEM WITH RENEWAL INPUT



Specific formulae, obtained by simplifying these general renewal input results for a negative exponential input stream are compared with results derived by Wallstrom [40], Schehrer [32], Mina [19].

Schehrer [32], by means of a joint probability approach derives expressions for all moments of the Kosten and Brockmeyer overflow systems with negative exponential input streams, whereas the earlier work of Wallstrom [40] again using a joint probability approach contains explicit expressions for only the first two moments. Mina [19] studies properties of the peakedness of carried overflow traffic by defining each of the offered input traffic, the carried primary and secondary traffic and the traffic overflowing the secondary group by a random variable, again for the negative exponential input stream. Possible bounds on the congestion probability for the secondary group for any renewal input stream were found by Holtzman [10].

The derivation of explicit overflow traffic moment formulae is summarised in Potter [26].

5.2 The Overflow Traffic Factorial Moment Theorem

The factorial overflow traffic moment theorem can be expressed mathematically as

$$\frac{1}{q_{N+1}^{(n)}(1)} = \frac{1}{q_N^{(n)}(1)} + \frac{n}{q_N^{(n+1)}(1)} \quad , \; n \geqslant 1 \qquad (5.2.1)$$

where $q_N^{(n)}(1)$ is the n^{th} factorial moment of the overflow traffic from a primary group of N trunks. These factorial moments satisfy equation (3.3.7), hence

$$nf_N q_N^{(n)}(1) = \begin{cases} \prod_{j=1}^{n-1} \dfrac{\psi_N(j\mu)}{1-\psi_N(j\mu)} & , \quad n > 1 \\ 1 & , \quad n = 1 \end{cases} , \tag{5.2.2}$$

where (i) $\psi_N(s)$ is the Laplace-Stieltjes transform of the overflow stream G, determined by equation (2.3.17),

(ii) f_N is the weakness of the overflow stream given by equation (2.5.6).

If the n^{th} factorial moment of the overflow traffic is represented by $\alpha_{nq}(N)$, equation (5.2.1) becomes

$$\alpha_{nq}^{-1}(N+1) = \alpha_{nq}^{-1}(N) + n\,\alpha_{(n+1)q}^{-1}(N) . \tag{5.2.3}$$

Proof 1. Direct Approach.

The overflow stream G, from a primary group of N trunks satisfies equation (2.5.14) with $s=n-1$, hence

$$\psi_N(n\mu) = \psi_N(\overline{n-1}\mu)\left[1 - \frac{1}{\psi_N(\overline{n-1}\mu)}\right]\left[1 - \frac{1}{\psi_{N+1}(\overline{n-1}\mu)}\right]^{-1} \tag{5.2.4}$$

$$\therefore \quad \psi_N^{-1}(n\mu) = \psi_N^{-1}(\overline{n-1}\mu)\left[1 - \frac{1}{\psi_N(\overline{n-1}\mu)}\right]^{-1}\left[1 - \frac{1}{\psi_{N+1}(\overline{n-1}\mu)}\right] \tag{5.2.5}$$

But equation (3.4.3) related the factorial moments of the overflow traffic to the overflow stream, by

$$\psi_N^{-1}(n\mu) = 1 + n\,\frac{q_N^{(n)}(1)}{q_N^{(n+1)}(1)} . \tag{5.2.6}$$

Equation (5.2.5) on substituting for ψ in terms of the q^s by using equation (5.2.6) becomes

$$1 + n \frac{q_N^{(n)}(1)}{q_N^{(n+1)}(1)} = \left[1 + (n-1) \frac{q_N^{(n-1)}(1)}{q_N^{(n)}(1)}\right] \frac{q_N^{(n)}(1)}{q_N^{(n-1)}(1)} \frac{q_{N+1}^{(n-1)}(1)}{q_{N+1}^{(n)}(1)} \quad (5.2.7)$$

which on rearranging can be expressed as

$$\left[1 + n \frac{q_N^{(n)}(1)}{q_N^{(n+1)}(1)}\right] \frac{q_{N+1}^{(n)}(1)}{q_N^{(n)}(1)} = \left[1 + (n-1) \frac{q_N^{(n-1)}(1)}{q_N^{(n)}(1)}\right] \frac{q_{N+1}^{(n-1)}(1)}{q_N^{(n-1)}(1)} \quad (5.2.8)$$

Since

$$\left[1 + (n-1) \frac{q_N^{(n-1)}(1)}{q_N^{(n)}(1)}\right] \frac{q_{N+1}^{(n-1)}(1)}{q_N^{(n-1)}(1)} = 1 \quad \text{when} \quad n = 1,$$

equation (5.2.8) becomes

$$\left[1 + n \frac{q_N^{(n)}(1)}{q_N^{(n+1)}(1)}\right] \frac{q_{N+1}^{(n)}(1)}{q_N^{(n)}(1)} = 1 \quad \text{for all} \quad n \geqslant 1 \quad (5.2.9)$$

The statement of the theorem by equation (5.2.1) is a rearrangement of equation (5.2.9).

Proof 2. Divided Difference Approach.

Equation (2.5.23) relates the overflow stream with the divided difference of its weakness, f_N, giving

$$\psi_N^{-1}(n\mu) = \frac{\Delta^{n-1} f_{N+1}}{\Delta^{n-1} f_N} \quad (5.2.10)$$

where $\Delta^n f_N = f_0 \sum_{r=0}^{N} \binom{N}{r} \ell_{n+r}(\mu)$ by equation (2.5.10). Thus the overflow traffic moments can be expressed in terms of the weakness of the overflow stream by equating equations (5.2.6) and (5.2.10), to give

$$\frac{\Delta^{n-1} f_{N+1}}{\Delta^{n-1} f_N} = 1 + n \frac{q_N^{(n)}(1)}{q_N^{(n+1)}(1)} \tag{5.2.11}$$

$$\therefore \quad \frac{\Delta^{n-1} f_{N+1} - \Delta^{n-1} f_N}{\Delta^{n-1} f_N} = n \frac{q_N^{(n)}(1)}{q_N^{(n+1)}(1)}$$

$$\therefore \quad \frac{\Delta^{n} f_N}{\Delta^{n-1} f_N} = n \frac{q_N^{(n)}(1)}{q_N^{(n+1)}(1)}$$

i.e. $$(\Delta^n f_N) q_N^{(n+1)}(1)/n! = (\Delta^{n-1} f_N) q_N^{(n)}(1)/(n-1)! \tag{5.2.12}$$

Now $$q_N^{(1)}(1) = \frac{1}{f_N} \quad \text{by equation (3.4.7)}, \tag{5.2.13}$$

hence $$(\Delta^{n-1} f_N) q_N^{(n)}(1)/(n-1)! = 1 \quad \text{when} \quad n = 1.$$

Thus by equation (5.2.12),

$$\Delta^n f_N \, q_N^{(n)}(1)/n! = 1 \quad \text{for all} \quad n \geqslant 1 \tag{5.2.14}$$

or alternatively

$$q_N^{(n+1)}(1) = \left[\frac{\Delta^n f_N}{n!}\right]^{-1} . \tag{5.2.15}$$

Rewriting equation (5.2.11) using equation (5.2.15) gives

$$\frac{q_N^{(n)}(1)}{q_{N+1}^{(n)}(1)} = 1 + n \frac{q_N^{(n)}(1)}{q_N^{(n+1)}(1)}$$

which on rearranging gives equation (5.2.1).

The result, (5.2.15) incorporated as part of proof 2, can be considered as a lemma to the theorem when proved by the approach adopted in proof 1.

Proof of (5.2.15).

A mathematical induction proof is used with the theorem being applied at the inductive step.

Since $q_N^{(1)}(1) = [f_N]^{-1}$ by equation (3.4.7), equation (5.2.15) holds for $n = 1$.

Assume $$q_N^{(m)}(1) = \left[\frac{\Delta^{m-1} f_N}{(m-1)!}\right]^{-1}, \tag{5.2.16}$$

then equation (2.5.1) can be expressed as

$$[q_N^{(m+1)}(1)]^{-1} = \frac{1}{m}[[q_{N+1}^{(m)}(1)]^{-1} - [q_N^{(m)}(1)]^{-1}]$$

$$\therefore \quad [q_N^{(m+1)}(1)]^{-1} = \frac{1}{m!}[\Delta^{m-1} f_{N+1} - \Delta^{m-1} f_N] \quad \text{by} \quad (5.2.16)$$

$$= \frac{1}{m!}\Delta^m f_N$$

thus proving (5.2.15) for all $n \geq 1$ by Principle of Mathematical Induction.

5.3 Explicit Overflow Traffic Moment Formulae

Equation (5.2.15) which expresses the overflow traffics' factorial moments as a function of the weakness of the overflow stream, enables explicit formulae for all the overflow traffic moments to be found. The formulae can be given either in terms of the divided difference of the overflow stream's weakness which corresponds to a recurrence expression, or directly in terms of the Laplace-Stieltjes transform of the input stream.

5.3.1 Factorial Moments

The divided difference operator Δ is related to the forward difference operator E, by

$$\Delta^n = (E-1)^n = \sum_{r=0}^{n} \binom{n}{r} E^r (-1)^{n-r} \qquad (5.3.0)$$

Equation (5.2.15) can be rewritten as

$$[q_N^{(n+1)}(1)]^{-1} = \frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} f_{N+r} \quad , \quad n \geq 0 \qquad (5.3.1)$$

giving

$$\alpha_{n+1,q}(N) = n! \left[\sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} f_{N+r} \right]^{-1} . \qquad (5.3.2)$$

The overflow traffic factorial moments corresponding to the π occupancy distribution are obtained from the $\alpha_{nq}(N)$ by equation (3.4.1),

$$\alpha_{n\pi}(N) = f_N \, \alpha_{(n+1)q}(N) \qquad (5.3.3)$$

where f_N satisfies equation (2.5.6).

Using equation (5.3.2), equation (5.3.3) becomes

$$\alpha_{n\pi}(N) = n! \, f_N [\sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} f_{N+r}]^{-1} . \qquad (5.3.4)$$

Explicit formulae for $\alpha_{n+1,q}(N)$ and $\alpha_{n\pi}(N)$ in terms of the input stream's Laplace-Stieltjes transform are obtained by substituting for $\Delta^n f_N$ by using equation (2.5.10), giving

$$\alpha_{n+1,q}(N) = n! [f_0 \sum_{r=0}^{N} \binom{N}{r} \ell_{n+r}(\mu)]^{-1} \qquad (5.3.2a)$$

and

$$\alpha_{n\pi}(N) = n! \sum_{r=0}^{N} \binom{N}{r} \ell_r(\mu) [\sum_{r=0}^{N} \binom{N}{r} \ell_{n+r}(\mu)]^{-1} \qquad (5.3.4a)$$

where $f_0 = -\mu\phi'(0)$ by equation (2.2.3) and $\ell_r(\mu)$ is defined by equation (2.3.19).

5.3.2 Binomial Moments

The binomial moments β_n of any distribution are related to the corresponding factorial moments by

$$n!\beta_n = \alpha_n \quad . \tag{5.3.5}$$

Hence the binomial moments of the overflow traffic corresponding to

(i) equations (5.3.2) and (5.3.2a) are

$$\beta_{n+1,q}(N) = \frac{1}{(n+1)} \left[\sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} f_{N+r} \right]^{-1} \tag{5.3.6}$$

or

$$\beta_{n+1,q}(N) = \frac{1}{(n+1)} \left[f_0 \sum_{r=0}^{N} \binom{N}{r} \ell_{n+r}(\mu) \right]^{-1} \tag{5.3.6a}$$

(ii) equations (5.3.4) and (5.3.4a) are

$$\beta_{n,\pi}(N) = f_N \left[\sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} f_{N+r} \right]^{-1} \tag{5.3.7}$$

or

$$\beta_{n,\pi}(N) = \sum_{r=0}^{N} \binom{N}{r} \ell_r(\mu) \left[\sum_{r=0}^{N} \binom{N}{r} \ell_{n+r}(\mu) \right]^{-1} \quad . \tag{5.3.7a}$$

5.3.3 Ordinary Moments

The ordinary moments θ_n of any distribution are related to the corresponding factorial moments by

$$\theta_n = \sum_{k=1}^{n} \sigma_{n,k}\, \alpha_k \tag{5.3.8}$$

where

$$\sigma_{n,k} = \frac{1}{k!} \sum_{i=1}^{k} (-1)^{i+k} \binom{k}{i} i^n \quad , \tag{5.3.9}$$

that is, the $\sigma_{n,k}$ are Stirling numbers of the second kind.

Hence the ordinary moments of the overflow traffic corresponding to

(i) equations (5.3.2) and (5.3.2a) are

$$\theta_{n+1,q}(N) = \sum_{k=1}^{n+1} \sigma_{n+1,k}(k-1)!\Big[\sum_{s=0}^{k-1} \binom{k-1}{s}(-1)^{k-1-s} f_{N+s}\Big]^{-1} \tag{5.3.10}$$

or

$$\theta_{n+1,q}(N) = \sum_{k=1}^{n+1} \sigma_{n+1,k}(k-1)!\Big[f_0 \sum_{s=0}^{N} \binom{N}{s} \ell_{k-1+s}(\mu)\Big]^{-1} \tag{5.3.10a}$$

(ii) equations (5.3.4) and (5.3.4a) are

$$\theta_{n,\pi}(N) = \sum_{k=1}^{n} \sigma_{n,k}\, k!\, f_N \Big[\sum_{s=0}^{k} \binom{k}{s}(-1)^{k-s} f_{N+s}\Big]^{-1} \tag{5.3.11}$$

or

$$\theta_{n,\pi}(N) = \sum_{k=1}^{n} \sigma_{n,k}\, k! \sum_{s=0}^{N} \binom{N}{s} \ell_s(\mu) \Big[\sum_{s=0}^{N} \binom{N}{s} \ell_{k+s}(\mu)\Big]^{-1} . \tag{5.3.11a}$$

5.3.4 Central Moments

The central moments c_n of any distribution are related to the corresponding ordinary moments by

$$c_r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \theta_{r-i}\, \theta_1^r \tag{5.3.12}$$

Hence the central moments of the overflow traffic corresponding to

(i) equations (5.3.2) and (5.3.2a) are

$$c_{n+1,q}(N) = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} \theta_{n+1-i,q}(N)\theta_1^{n+1}$$

$$= \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} \sum_{k=1}^{n+1-i} \sigma_{n+1-i,k}(k-1)!\left[\sum_{s=0}^{k-1} \binom{k-1}{s}(-1)^{k-1-s} f_{N+s}\right]^{-1}\theta_1^{n+1} \tag{5.3.13}$$

or

$$c_{n+1,q}(N) = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} \sum_{k=1}^{n+1-i} \sigma_{n+1,i,k}(k-1)!\theta_1^{n+1}\left[f_0 \sum_{s=0}^{N} \binom{N}{s} \ell_{k-1+s}(\mu)\right]^{-1} \tag{5.3.13a}$$

(ii) equations (5.3.4) and (5.3.4a) are

$$c_{n,\pi}(N) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \theta_{n-i,\pi}(N)\theta_1^n$$

$$= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \sum_{k=1}^{n-i} \sigma_{n-i,k} k! f_N \left[\sum_{s=0}^{k} \binom{k}{s}(-1)^{k-s} f_{N+s}\right]^{-1} \tag{5.3.14}$$

or

$$c_{n,\pi}(N) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \sum_{k=1}^{n-i} \sigma_{n-i,k} k! \sum_{s=0}^{N} \binom{N}{s} \ell_s(\mu)\left[\sum_{s=0}^{N} \binom{N}{s} \ell_{k+s}(\mu)\right]^{-1} \tag{5.3.14a}$$

5.3.5 Special Formulae

(i) Mean overflow traffic, M_N, is given by equation (3.4.7) with $n = 1$, hence

$$M_N \equiv q_N^{(1)}(1) = \frac{1}{f_N} . \tag{5.3.15}$$

(ii) Variance of the overflow traffic, V_N, satisfies

$$V_N = q_N^{(2)}(1) + q_N^{(1)}(1) - [q_N^{(1)}(1)]^2$$

$$= \frac{1}{f_{N+1}-f_N} + \frac{1}{f_N} - [\frac{1}{f_N}]^2 \quad \text{by} \quad (5.2.15)$$

$$= M_N[\frac{M_N}{M_N-M_{N+1}} - M_N] \quad \text{by} \quad (5.3.15) \tag{5.3.16}$$

$$= M_N{}^2[\frac{1}{H_N} - 1] \tag{5.3.17}$$

where H_N is defined to be the marginal occupancy of the overflow system, see Pratt [28], defined by

$$H_N \equiv M_N - M_{N+1} \quad . \tag{5.3.18}$$

The marginal occupancy is the decrease in the overflow traffic when the size of the primary group is increased by one.

The expression (5.3.16) can be obtained directly from equation (3.4.12) as follows

$$V_N = M_N[1 - M_N + \frac{\psi_N(\mu)}{1-\psi_N(\mu)}] \quad \text{by } (3.4.12)$$

but

$$\psi_N(\mu) = \frac{f_N}{f_{N+1}} \quad \text{by } (2.5.20)$$

$$= \frac{M_{N+1}}{M_N} \quad \text{by } (5.3.15)$$

which on substituting into (3.4.12) gives (5.3.16).

(iii) Peakedness of the overflow traffic, Z_N, is defined to be its variance to mean ratio, thus

$$Z_N \equiv \frac{V_N}{M_N}$$

$$= 1 - M_N + \frac{\psi_N(\mu)}{1-\psi_N(\mu)} \quad \text{by (3.4.12)} \qquad (5.3.19)$$

or alternatively $\quad Z_N = M_N[\frac{1}{H_N} - 1]$ by (5.3.16) .

The peakedness is a quantity which is used in telephony as a measure of the 'roughness' or 'smoothness' of the traffic. Traffic is said to be rough if its peakedness is greater than unity and is said to be smooth if its peakedness is less than unity.

(iv) Coefficient of variation of the overflow traffic, W_N, is defined by

$$W_N \equiv \frac{V_N}{M_N^{\ 2}}$$

$$= \frac{1}{H_N} - 1 \quad \text{by (5.3.20).}$$

The physical interpretation of the marginal occupancy implies that it must have a value lying between zero and one . The corresponding value of W_N can therefore range between zero and infinity.

Chapter 6 contains a study of the nature and properties of the peakedness and coefficient of variation of overflow traffic.

5.4 Carried Overflow Traffic Moments

When the overflow stream G from a primary group of N trunks is input to the G/M/L queueing system, the steady state $\bar{q}$ occupancy distribution on the secondary group can be found using formulae derived in Chapter 4. This distribution will be called the carried overflow traffic distribution or alternatively the distribution of the overflow traffic carried on the secondary group.

Equation (4.5.1) can be subscripted by N to denote that this Brockmeyer overflow system with renewal input is being considered and reads,

$$f_N \bar{q}_N^{(n)}(1) = \bar{\pi}_N^{(n)}(1) \frac{q_N^{(n)}(1)}{q_N^{(n+1)}(1)} \qquad 1 \leqslant n < L \tag{5.4.1}$$

where f_N satisfies (2.5.6) and the overflow traffic moments $q_N^{(n)}(1)$ satisfy (5.2.15).

Equation (5.4.1) can be rewritten in terms of finite differences of the overflow stream's weakness, as

$$f_N \bar{q}_N^{(n)}(1) = \bar{\pi}_N^{(n)}(1) \frac{\Delta^n f_N}{n\Delta^{n-1} f_N}, \quad 1 \leqslant n < L . \tag{5.4.2}$$

The mean carried overflow traffic, $\bar{M}_{N,L}$ is given by $\bar{q}_N^{(1)}(1)$ and satisfies equation (4.4.8), hence

$$\bar{M}_{N,L} \equiv \bar{q}_N^{(1)}(1)$$

$$= M_N[1 - \bar{\pi}_L] \tag{5.4.3}$$

where M_N, the mean overflow traffic, satisfies (5.3.15) and $\bar{\pi}_L$ is the probability of congestion on the secondary group. Substitution of $\bar{q}_N^{(1)}(1)$ given by (5.4.3) into equation (5.4.2) with $n = 1$, gives the following expression for $\bar{\pi}_N^{(1)}(1)$,

$$\bar{\pi}_N^{(1)}(1) = (1 - \bar{\pi}_L) \frac{f_N}{\Delta f_N} . \tag{5.4.4}$$

But $\bar{\pi}_N^{(2)}(1)$ is related to $\bar{\pi}_N^{(1)}(1)$ by equation (4.5.4) with $n = 2$, that is

$$\bar{\pi}_N^{(2)}(1) = [\bar{\pi}_N^{(1)}(1) - \bar{\pi}_L\binom{L}{1}]\,\frac{\alpha_{3q}(N)}{\alpha_{2q}(N)} . \qquad (5.4.5)$$

An explicit expression for the $\bar{\pi}_N^{(n)}(1)$ is given by

$$\bar{\pi}_N^{(n)}(1) = n!\,F(N,L,n)\,[\Delta^n f_N]^{-1}, \quad 1 \leqslant n < L \qquad (5.4.6)$$

where

$$F(N,L,n) = \bar{\pi}_N^{(1)}(1)\Delta f_N - \bar{\pi}_L \sum_{\ell=1}^{n-1} \binom{L}{\ell}\Delta^\ell f_N . \qquad (5.4.7)$$

Proof of (5.4.6)

Equation (5.4.5) is identical to equation (5.4.6) when $n = 2$. Suppose $\bar{\pi}_N^{(n-1)}(1) = (n-1)!\,F(N,L,n-1)\,[\Delta^{n-1} f_N]^{-1}$, then substituting this expression for $\bar{\pi}_N^{(n-1)}(1)$ into (4.5.4) gives

$$\bar{\pi}_N^{(n)}(1) = \frac{n\Delta^{n-1} f_N}{\Delta^n f_N}\,[(n-1)!\,F(N,L,n-1)\,[\Delta^{n-1} f_N]^{-1} - (n-1)!\,\bar{\pi}_L\binom{L}{n-1}]$$

$$= n!\,F(N,L,n-1)\,[\Delta^n f_N]^{-1} - [\Delta^n f_N]^{-1}\{n!\,\bar{\pi}_L\binom{L}{n-1}\Delta^{n-1} f_N\}$$

but $\quad F(N,L,n-1) - \bar{\pi}_L\binom{L}{n-1}\Delta^{n-1} f_N = F(N,L,n)$

$$\bar{\pi}_N^{(n)}(1) = n!\,F(N,L,n)\,[\Delta^n f_N]^{-1} ,$$

thus proving (5.4.6) by Principle of Mathematical Induction.

An explicit expression for the $\bar{q}_N^{(n)}(1)$ is

$$\bar{q}_N^{(n)}(1) = n!\,F(N,L,n)\,[n f_N \Delta^{n-1} f_N]^{-1}, \quad 1 \leqslant n \leqslant L . \qquad (5.4.8)$$

Proof of (5.4.8) follows by substituting for $\bar{\pi}_N^{(n)}(1)$, given by equation (5.4.6), into equation (5.4.2).

Equation (5.4.8) is a generalization of Schehrer's [32] equation (72) established in the case of a negative exponential input stream.

All the expressions for all moments of the overflow traffic which are contained in the following section of this chapter will be in terms of f_N and the finite difference $\Delta^n f_N$. These equations can be rewritten using relations (5.3.0) or (2.5.10) for $\Delta^n f_N$.

5.4.1 Factorial Moments

Equations (5.4.6), (5.4.8) giving the $\bar{\pi}$ and $\bar{q}$ factorial moments can be rewritten as

$$\bar{\alpha}_{n\pi}(N) = n!F(N,L,n)[\Delta^n f_N]^{-1}, \quad 1 \leqslant n \leqslant L \tag{5.4.9}$$

$$\bar{\alpha}_{nq}(N) = n!F(N,L,n)[nf_N\Delta^{n-1} f_N]^{-1}, \quad 1 \leqslant n \leqslant L \tag{5.4.10}$$

5.4.2 Binomial Moments

Using equation (5.3.5), the binomial carried overflow traffic moments corresponding to equations (5.4.9) and (5.4.10) are

$$\bar{\beta}_{n\pi}(N) = F(N,L,n)[\Delta^n f_N]^{-1}, \quad 1 \leqslant n \leqslant L \tag{5.4.11}$$

and

$$\bar{\beta}_{nq}(N) = F(N,L,n)[nf_N\Delta^{n-1} f_N]^{-1}, \quad 1 \leqslant n \leqslant L \ . \tag{5.4.12}$$

The probability of congestion, $\bar{\pi}_L$ on the secondary group of L trunks is given by $\bar{\beta}_{L\pi}(N)$, that is

$$\bar{\pi}_L = \bar{\beta}_{L\pi}(N) = F(N,L,L)[\Delta^L f_N]^{-1} \tag{5.4.13}$$

where

$$F(N,L,L) = \bar{\pi}_N^{(1)}(1)\Delta f_N - \bar{\pi}_L \sum_{\ell=1}^{L-1} \binom{L}{\ell}\Delta^\ell f_N$$

$$\therefore \qquad \bar{\pi}_L = [-\bar{\pi}_L \sum_{\ell=1}^{L-1} \binom{L}{\ell}\Delta^\ell f_N + \bar{\pi}_N^{(1)}(1)\Delta f_N][\Delta^L f_N]^{-1}$$

$$\therefore \qquad \bar{\pi}_L [\Delta^L f_N + \sum_{\ell=1}^{L-1} \binom{L}{\ell}\Delta^\ell f_N] = \bar{\pi}_N^{(1)}(1)\Delta f_N$$

$$= (1 - \bar{\pi}_L) f_N \quad \text{by (5.4.4)}$$

$$\therefore \qquad \bar{\pi}_L [\sum_{\ell=0}^{L} \binom{L}{\ell}\Delta^\ell f_N] = f_N \tag{5.4.15}$$

or alternatively

$$\bar{\pi}_L = f_N [\sum_{\ell=0}^{L} \binom{L}{\ell}\Delta^\ell f_N] \quad . \tag{5.4.15}$$

But inverting equation (5.3.0), gives

$$\sum_{\ell=0}^{L} \binom{L}{\ell}\Delta^\ell = E^L \tag{5.4.16}$$

therefore equation (5.4.15) becomes

$$\bar{\pi}_L = f_N [f_{N+L}]^{-1} \quad . \tag{5.4.17}$$

This proves that the probability of congestion on the secondary group is the ratio of the weakness of the overflow stream from the primary group to the weakness of the overflow stream arising if both the primary and secondary group were combined. Equation (5.4.17) is a generalization of Wallstrom's [40] equation (3.1.40) on page 208.

The time congestion, $\bar{q}_L$ on the secondary group of L trunks is given by $\bar{\beta}_{Lq}(N)$, thus

$$\bar{q}_L = \frac{\bar{\pi}_L}{Lf_N} \frac{\Delta^L f_N}{\Delta^{L-1} f_N} \quad \text{by equation (5.4.2)} \tag{5.4.18}$$

$$= \Delta^L f_N [L\, f_{N+L} \Delta^{L-1} f_N] \quad \text{by equation (5.4.17).} \tag{5.4.19}$$

5.4.3 Ordinary Moments

Using equations (5.3.8) and (5.3.9), the ordinary moments of overflow traffic corresponding to equations (5.4.9) and (5.4.10) are

$$\bar{\theta}_{n\pi}(N) = \sum_{k=1}^{n} \sigma_{n,k} k! F(N,L,k)(\Delta^k f_N)^{-1}, \quad 1 \leqslant n \leqslant L \tag{5.4.20}$$

and

$$\bar{\theta}_{nq}(N) = \sum_{k=1}^{n} \sigma_{n,k} k! F(N,L,k)[kf_N \Delta^{k-1} f_N]^{-1}, \quad 1 \leqslant n \leqslant L \tag{5.4.21}$$

where $\sigma_{n,k}$ satisfies equation (5.3.9).

5.4.4 Central Moments

Using equation (5.3.12) the central moments of overflow traffic corresponding to equations (5.4.9) and (5.4.10) are

$$\bar{c}_{n\pi}(N) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \bar{\theta}^n_{1\pi} \sum_{k=1}^{n-i} \sigma_{n-i,k} k! F(N,L,k)(\Delta^k f_N)^{-1}, \quad 1 \leqslant n \leqslant L \tag{5.4.22}$$

and

$$\bar{c}_{nq}(N) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \bar{\theta}^n_{1q} \sum_{k=1}^{n-i} \sigma_{n-i,k} k! F(N,L,k)[kf_N \Delta^{k-1} f_N]^{-1}, \quad 1 \leqslant n \leqslant L \tag{5.4.23}$$

where $\sigma_{n,k}$ satisfies equation (5.3.9).

Alternate expressions using equations (5.3.0) and (2.5.10) for all moments, are listed in Appendix II .

5.4.5 Special Formulae

(a) The mean traffic carried on the secondary group, $\bar{M}_{N,L}$ satisfies equation (5.4.3), that is

$$\bar{M}_{N,L} = (1 - \bar{\pi}_L)M_N$$

$$= (1 - \frac{M_{N+L}}{M_N})M_N \quad \text{by equation (5.4.17)}$$

$$= M_N - M_{N+L} \quad . \qquad (5.4.24)$$

This agrees with intuitive reasoning which says the average traffic carried on the finite secondary group is the difference between the intensities of the input stream and the overflow stream from a combined set of (N+L) trunks. Equation (5.4.24) is a generalization of Wallstrom's [42] equation (3.1.34), page 208 and Schehrer's [33] equation (62).

(b) The ratio of the lost calls to carried calls on the secondary group, $\bar{R}_{N,L}$ is given by

$$\bar{R}_{N,L} = \frac{L\ \bar{\pi}_L}{1-\bar{\pi}_L}$$

$$= \frac{L\ f_N}{f_{N+L} - f_N} \quad \text{by (5.4.17)} \ . \qquad (5.4.25)$$

$\bar{R}_{N,L}$ satisfies equations (4.4.14), (4.4.15) and (4.4.16) given in Chapter IV.

(c) The variance of the carried overflow traffic, $\bar{V}_{N,L}$, satisfies equation (4.4.9), hence

$$\bar{V}_{N,L} = \bar{M}_{N,L}\left[1 - \bar{M}_{N,L} + h_1 - \frac{L\,\bar{\pi}_L}{1-\bar{\pi}_L}\right] \tag{5.4.26}$$

where (i) $\bar{M}_{N,L}$ satisfies equation (5.4.22)

(ii) $\bar{\pi}_L$ satisfies equation (5.4.16)

(iii) $h_1 = \dfrac{\psi_N(\mu)}{1-\psi_N(\mu)}$

$$= \left[\frac{f_{N+1}}{f_N} - 1\right]^{-1} \quad \text{by equation (2.5.20)} \tag{5.4.27}$$

$$= \frac{M_{N+1}}{M_N - M_{N+1}} \quad \text{by equation (5.3.15)} \tag{5.4.28}$$

Equation (5.4.25) can be written in terms of the weakness functions f_N, f_{N+1} and f_{N+L} giving

$$\bar{V}_{N,L} = \left(\frac{1}{f_N} - \frac{1}{f_{N+L}}\right)\left(\frac{f_{N+L} - f_N - Lf_N f_{N+L}}{f_{N+L}(f_{N+L} - f_N)} - \frac{f_{N+1} - f_N - f_N f_{N+1}}{f_N(f_{N+1} - f_N)}\right). \tag{5.4.29}$$

Proof of equation (5.4.29)

$$\bar{M}_{N,L} = \frac{1}{f_N} - \frac{1}{f_{N+L}} \quad \text{from equation (5.4.23)} \tag{5.4.30}$$

$$1+h_1(\mu) = 1 + \frac{\psi_N(\mu)}{1-\psi_N(\mu)} \quad \text{by (3.2.7)}$$

$$= \frac{1}{1-\psi_N(\mu)}$$

$$= \frac{f_{N+1}}{f_N - f_{N+1}} \quad \text{by (2.5.20)} \tag{5.4.31}$$

Substitution for $\bar{M}_{N,L}$, $1+h_1(\mu)$, $\bar{R}_{N,L}$ given by equations (5.4.30), (5.4.31) and (5.4.25) respectively, gives the required result.

Equation (5.4.26) is a generalization of Wallstrom's [40] equation (3.1.35), page 208.

(d) The peakedness of the carried overflow traffic, $\bar{z}_{N,\,L.}$ is given by a rearrangement of equation (5.4.26),

$$\bar{z}_{N,\,L.} = 1 - \bar{M}_{N,\,L.} + h_1 - \frac{L\ \bar{\pi}_L}{1-\bar{\pi}_L} \quad . \tag{5.4.32}$$

Equation (5.4.32) is a generalization of Mina's [19] equation (1).

Note that the quantities $\bar{M}_{N,\,L.}$, $\bar{V}_{N,\,L.}$ and $\bar{z}_{N,\,L.}$ are functions of M_N, M_{N+1} and M_{N+L} when N and L are given. If therefore the functional dependence of the mean overflow traffic on N is known, these quantities can be calculated from equations (5.4.24), (5.4.29) and (5.4.32).

5.5 Formulae Applicable to the Kosten and Brockmeyer Systems with Negative Exponential Input

(i) $F(t) = 1 - e^{-\lambda t}$ with $\phi(s) = \dfrac{\lambda}{\lambda+s}$. (5.5.1)

(ii) $f = -\mu\phi'(0)$ from 2.2.3)

$= \dfrac{\mu}{\lambda}$

$= A^{-1}$ where A is defined as the intensity of the input stream. (5.5.2)

(iii) $\ell r(\mu) = \prod_{j=1}^{r} \frac{1-\phi(j\mu)}{\phi(j\mu)}$ from (2.3.19)

$$= \prod_{j=1}^{r} \frac{j\mu}{\lambda}$$

$$= r!A^{-r} \tag{5.5.3}$$

(iv) $f_N = f_0 \sum_{r=0}^{N} \binom{N}{r} \ell r(\mu)$ from (2.5.6)

$$= A^{-1} \sum_{r=0}^{N} \binom{N}{r} r!A^{-r}$$

$$= N!A^{-(N+1)} \sum_{r=0}^{N} \frac{A^r}{r!} \tag{5.5.4}$$

$M_N = f_N^{-1}$ from (5.3.15)

$$= \frac{A^{N+1}}{N!} \Big/ \sum_{r=0}^{N} \frac{A^r}{r!} \tag{5.5.5}$$

Equation (5.5.5) is the well known expression for the mean overflow traffic, see Wallstrom [40], equation (3.1.11), page 205, and is usually written in terms of the Erlang Loss function, $E_N(A)$ as

$$M_N = A\, E_N(A)$$

where $$E_N(A) \equiv \frac{A^N}{N!} \Big/ \sum_{r=0}^{N} \frac{A^r}{r!} \; .$$

(v) Recurrence relation satisfied by f_N.

$f_{N+1} = f_0 \sum_{r=0}^{N+1} \binom{N+1}{r} \ell r(\mu)$ from (2.5.6)

$$= \frac{N+1}{A} A^{-(N+1)} N! \sum_{r=0}^{N} \frac{A^r}{r!} + \frac{1}{A}$$

$$= \frac{N+1}{A} f_N + \frac{1}{A} \; . \tag{5.5.6}$$

Equation (5.5.6) is a rearrangement of the following well known recurrence relation satisfied by M_N,

$$M_{N+1} = \frac{A\,M_N}{N+1+M_N} \quad . \tag{5.5.7}$$

(vi) $$\psi_N(\mu) = \frac{f_N}{f_{N+1}} \quad \text{by (2.5.20)}$$

$$= \frac{A}{N+1+M_N} \quad \text{by (5.5.7)}. \tag{5.5.8}$$

(vii) $$f_{N+k} = \frac{(N+k)!}{N!} A^{-k} f_N + \sum_{r=1}^{k} \frac{(N+k)!}{(N+k+1-r)!} A^{-r} \tag{5.5.9}$$

The proof of equation (5.5.9) follows by mathematical induction for which equation (5.5.6) is the induction step.

The following proof is based on equation (5.5.4) with N=N+k.

$$f_{N+k} = \frac{(N+k)!}{A^{N+k+1}} \sum_{r=0}^{N+k} \frac{A^r}{r!} \quad \text{by (5.5.4)}$$

$$= \frac{(N+k)!}{N!} A^{-k} \frac{N!}{A^{N+1}} \left[\sum_{r=0}^{N} \frac{A^r}{r!} + \sum_{r=N+1}^{N+k} \frac{A^r}{r!} \right]$$

$$= \frac{(N+k)!}{N!} A^{-k} f_N + (N+k)!A^{-(k+N+1)} \sum_{r=1}^{k} \frac{A^{N+r}}{(N+r)!}$$

$$= \frac{(N+k)!}{N!} A^{-k} f_N + (N+k)! \sum_{r=1}^{k} \frac{A^{r-k-1}}{(N+r)!}$$

$$= \frac{(N+k)!}{N!} A^{-k} f_N + (N+k)! \sum_{s=1}^{k} \frac{A^{-s}}{(N+k+1-s)!} \quad \text{where } s=k+1-r$$

(viii) $$\Delta^n f_N = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f_{N+k} \quad \text{by (5.3.1)}$$

$$= (-1)^n f_N + \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} \left[\frac{(N+k)!}{N!} A^{-k} f_N + (N+k)! \sum_{s=1}^{k} \frac{A^{-s}}{(N+k+1-s)!} \right]$$

by (5.5.9) (5.5.10)

(ix) $$q^{(n+1)}(1) = \left[\frac{\Delta^n f_N}{n!} \right]^{-1} \quad \text{by (5.2.15)}$$

$$= n! \left[(-1)^n f_N + \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} \left[\frac{(N+k)!}{N!} A^{-k} f_N + (N+k)! \sum_{s=1}^{k} \frac{A^{-s}}{(N+k+1-s)!} \right] \right]^{-1}$$

by (5.5.10) (5.5.11)

Equation (5.5.11) can be rewritten in terms of A, N and M_N to give

$$q^{(n+1)}(1) = n! A^n M_N \left[(-A)^n + \sum_{k=1}^{n} \binom{n}{k} (-A)^{n-k} \left[\frac{(N+k)!}{N!} + M_N \sum_{s=1}^{k} \frac{(N+k)!}{(N+k+1-s)!} A^{k-s} \right] \right]^{-1} \tag{5.5.12}$$

Schehrer [32], equation (43) gives the following expression for $q^{(n+1)}(1)$,

$$q^{(n+1)}(1) = n! A^n M_N \left[n! \sum_{w=0}^{N} \frac{(-1)^w}{w!} \binom{N+n-w}{n-w} \left[A^w - M_N \sum_{k=0}^{w-1} \binom{N}{k} k! A^{w-1-k} \right] \right]^{-1} \tag{5.5.13}$$

(x) <u>Equivalence of Schehrer's and Potter's higher order moments.</u>

The denominator of Schehrer's equation (5.5.13) can be written as $X + Y$ where

$$X = n! \sum_{w=0}^{N} \frac{(-1)^w}{w!} \binom{N+n-w}{n-w} A^w \tag{5.5.14}$$

and $$Y = -n!M_N \sum_{w=1}^{N} \binom{N+n-w}{n-w} \frac{(-1)^w}{w!} \sum_{k=0}^{w-1} \binom{N}{k} k! A^{w-1-k} \; . \qquad (5.5.15)$$

The denominator of equation (5.5.12) can be written as $X_1 + Y_1$ where

$$X_1 = (-A)^n + \sum_{k=1}^{n} \binom{n}{k} (-A)^{n-k} \frac{(N+k)!}{N!} \qquad (5.5.16)$$

and $$Y_1 = M_N \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} A^n (N+k)! \sum_{s=1}^{k} \frac{A^{-s}}{(N+k+1-s)!} \; . \qquad (5.5.17)$$

<u>Step 1</u> $$X = X_1 \qquad (5.5.18)$$

Proof of (5.5.18)

$$X = n!\{\binom{N+n}{n} + \sum_{w=1}^{n} \frac{(-1)^w}{w!} \binom{N+n-w}{n-w} A^w\} \; .$$

Let $k = n - w$

$$\therefore \quad X = \frac{(N+n)!}{N!} + n! \sum_{k=0}^{n-1} \frac{(-1)^{n-k}}{(n-k)!} \binom{N+k}{k} A^{n-k}$$

$$= \frac{(N+n)!}{N!} + \sum_{k=0}^{n-1} \frac{(-1)^{n-k}}{N!} (N+k)! \binom{n}{k} A^{n-k}$$

$$= \sum_{k=0}^{n} (-1)^{n-k} \frac{(N+k)!}{N!} \binom{n}{k} A^{n-k}$$

$$= X_1 \; .$$

<u>Step 2</u> $$Y = -A^n M_N \sum_{s=0}^{n-1} \binom{n}{s} (-1)^{n-s} (N+s)! \sum_{k=s+1}^{n} \frac{A^{-k}}{(N+s+1-k)!} \; . \qquad (5.5.19)$$

Proof of (5.5.19)

Let $w = n - s$ in (5.5.15)

$$\therefore \quad Y = -n!M_N \sum_{s=0}^{n-1} \frac{(-1)^{n-s}}{(n-s)!} \binom{N+s}{s} \sum_{k=0}^{n-(s+1)} \binom{N}{k} k! A^{n-k-(s+1)}$$

$$= -A^n M_N \sum_{s=0}^{n-1} \frac{(-1)^{n-s}}{(n-s)!} \frac{n!}{s!} \frac{(N+s)!}{N!} \sum_{k=0}^{n-(s+1)} \frac{N!}{(N-k)!} A^{-k-(s+1)}$$

$$= -A^n M_N \sum_{s=0}^{n-1} (-1)^{n-s} \binom{n}{s} (N+s)! \sum_{k=s+1}^{n} \frac{A^{-k}}{(N+s+1-k)!}$$

$$= +A^n M_N Z$$

where $$Z = -\sum_{s=0}^{n-1} (-1)^{n-s} \binom{n}{s} (N+s)! \sum_{k=s+1}^{n} \frac{A^{-k}}{(N+s+1-k)!} \; . \qquad (5.5.20)$$

Step 3 $$Z = Y_1/M_N A^n \qquad (5.5.21)$$

Proof: Equation (5.5.21) is proved if the coefficient of A^{-m} in Z = coefficient of A^{-m} in $Y_1/M_N A^n$ for $1 \leqslant m \leqslant n$.

Let Y_m be the coefficient of A^{-m} in $Y_1/M_N A^n$ and let Z_m be the coefficient of A^{-m} in Z.

$$Z_m = -\sum_{r=0}^{m-1} \frac{\binom{n}{r}(N+r)!}{(N+r-m-1)!} (-1)^{n-r}$$ from equation (5.5.20) where $1 \leqslant m \leqslant n$ (5.5.22)

$$Y_m = \sum_{r=m}^{n} \frac{\binom{n}{r}(N+r)!}{(N+r-m-1)!} (-1)^{n-r}$$ from equation (5.5.19). (5.5.23)

To show $Y_m - Z_m = 0$. (5.5.24)

$$Y_m - Z_m = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} \frac{(N+r)!}{(N+r-m-1)!} \; . \qquad (5.5.25)$$

Equation (5.5.24) follows from the following theorem:

$$\sum_{r=0}^{n} P_k(r) \binom{n}{r} (-1)^{n-r} = 0 \quad \text{for} \quad 0 \leqslant k \leqslant n$$

where $P_k(r)$ is any polynomial in r of degree k. (5.5.26)

Proof of (5.5.26)

Any polynomial in r of degree k can be rewritten in the following form

$$P_k(r) = B_k \prod_{j=0}^{k-1} (r-j) + B_{k-1} \prod_{j=0}^{k-2} (r-j) + \ldots + B_0 \qquad (5.5.27)$$

$$\therefore \sum_{r=0}^{n} P_k(r) \binom{n}{r} (-1)^{n-r} t^r = B_k t^k \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} \prod_{j=0}^{k-1} (r-j) t^{r-k}$$

$$+ B_{k-1} t^{k-1} \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} \prod_{j=0}^{k-2} (r-j) t^{r-(k-1)}$$

$$+ B_0 \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} t^r$$

$$= B_k t^k f^{(k)}(t) + B_{k-1} t^{k-1} f^{(k-1)}(t) + \ldots + B_0 f(t) \qquad (5.5.28)$$

where $$f(t) = (t-1)^n$$

$$\therefore \quad f^{(k)}(1) = 0 \quad \text{for} \quad 0 \leqslant k \leqslant n \, .$$

Hence

$$\sum_{r=0}^{n} P_k(r) \binom{n}{r} (-1)^{n-r} = B_k f^{(k)}(1) + B_{k-1} f^{(k-1)}(1) + \ldots + B_0 f(1)$$

$$= 0 \, . \qquad \text{q.e.d.}$$

Equation (5.5.25) can be written as

$$Y_m - Z_m = \sum_{r=0}^{n} P_{m-1}(r) \binom{n}{r} (-1)^{n-r}$$

since $\dfrac{(N+r)!}{(N+r-m-1)!}$ is a polynomial in r of degree $(m-1)$ for fixed values of N.

Hence $Y_m - Z_m = 0$ by (5.5.26) since $1 \leqslant m \leqslant n$.

Therefore equation (5.5.21) is proved.

$$\therefore \qquad Y = Y_1$$

$$\therefore \qquad n!A^n M_N [X+Y]^{-1} = n!A^n M_N [X_1+Y_1]^{-1} ,$$

thus proving the equivalence of Schehrer's and Potter's expressions.

The following particular formulae are calculated from equation (5.5.11) for n=1,2,3,4 respectively,

(a) n=1,
$$q_N^{(2)}(1) = \frac{A\ M_N}{N+1+M_N-A} . \qquad (5.5.29)$$

This expression is well known in telephony and when substituted into equation (3.4.10) gives the following well known expression for the variance of overflow traffic,

$$V_N = M_N \left[1 - M_N + \frac{A}{N+1+M_N-A}\right] . \qquad (5.5.30)$$

(b) n=2,
$$q_N^{(3)}(1) = 2!A^2 M_N [(N+2)M_N - AM_N + (N+1)(N+2)$$
$$-2(N+1)A+A^2]^{-1} . \qquad (5.5.31)$$

Equation (5.5.31) is a rearrangement of Schehrer's [32] equation (44).

(c) n=3,
$$q_N^{(4)}(1) = 3!A^3 M_N [(N+3)(N+2)M_N - (2N+3)AM_N + A^2 M_N$$
$$+(N+1)(N+2)(N+3)-3(N+1)(N+2)A+3(N+1)A^2-A^3]^{-1} . (5.5.32)$$

Equation (5.5.32) is a rearrangement of Schehrer's [32] equation (45a).

(d) $n=4$, $q_N^{(5)}(1) = 4!A^4 M_N[(N+4)(N+3)(N+2)M_N-(N+3)(3N+4)AM_N$

$$+(3N+4)A^2M_N-A^3M_N+(N+4)(N+3)(N+2)(N+1)$$

$$-4(N+1)(N+2)(N+3)(N+4)A+6(N+1)(N+2)A^2$$

$$-4(N+1)A^3+A^4]^{-1} . \tag{5.5.33}$$

Equation (5.5.33) is a rearrangement of Schehrer's [32] equation (46b).

(xi) Probability of Loss on the Secondary Group

$$\bar{\pi}_L = \frac{f_N}{f_{N+L}} \qquad \text{by equation (5.4.17)}$$

$$= f_N[\frac{(N+L)!}{N!} A^{-L} f_N + \sum_{r=1}^{L} \frac{(N+L)!}{(N+L+1-r)!} A^{-r}]^{-1} \qquad \text{by (5.5.9)}$$

$$= [\frac{(N+L)!}{N!} A^{-L} + M_N \sum_{r=1}^{L} \frac{(N+L)!}{(N+L+1-r)!} A^{-r}]^{-1} . \tag{5.5.34}$$

(xii) Mean traffic carried on the secondary group

$$\bar{M}_{N,L.} = M_N[1-\bar{\pi}_L] \qquad \text{from (5.4.3)}$$

$$= M_N - [f_{N+L}]^{-1} \qquad \text{from (5.4.17)}$$

$$= M_N - M_{N+L} \qquad \text{from (5.3.15)}$$

$$= M_N - \left[\frac{(N+L)!}{N!} A^{-L} f_N + \sum_{r=1}^{L} \frac{(N+L)!}{(N+L+1-r)!} A^{-r}\right]^{-1} \quad \text{from (5.5.9)}$$

$$= M_N\left[1 - \left\{\frac{(N+L)!}{N!} A^{-L} + M_N \sum_{r=1}^{L} \frac{(N+L)!}{(N+L+1-r)!} A^{-r}\right\}^{-1}\right] . \quad (5.5.35)$$

(xiii) The ratio of lost calls to carried calls on the secondary group.

$$\bar{R}_{N,L.} = \frac{L\,f_N}{f_{N+L} - f_N}$$

$$= Lf_N\left[f_N\left(\frac{(N+L)!}{N!} A^{-L} - 1\right) + \sum_{r=1}^{L} \frac{(N+L)!}{(N+L+1-r)!} A^{-r}\right)\Bigg]^{-1} \text{ by (5.5.9)}$$

which simplifies to

$$= \left[(N+L)! - L + LN!A^{L} M_N \sum_{r=1}^{L} \frac{(N+L)!}{(N+L+1-r)!} A^{-r}\right]^{-1} . \quad (5.5.36)$$

CHAPTER VI

OVERFLOW TRAFFIC - ATOMIC APPROACH

6.1 Introduction

6.2 Formulation of $\psi_N(s)$

6.3 Coefficient of Variation of Overflow Traffic

6.4 Peakedness of Overflow Traffic

6.5 Features of Overflow Peakedness Charts

CHAPTER VI

OVERFLOW TRAFFIC - ATOMIC APPROACH

6.1 Introduction

In the foregoing 'group' analysis, an overflow system is considered as comprising of two groups, namely the primary and secondary group. An alternate analysis, based on individual trunks and sequential overflows is discussed in this chapter. By analogy with other areas of mathematics, this approach is called an 'atomic' analysis of overflow systems.

The overflow system is considered as a sequence of individual trunks each being offered the overflow stream from the preceding trunk. This situation is illustrated in Figure 6.1. The overflow stream from a primary group of N trunks is interpreted as the N^{th} overflow stream from the N^{th} individual trunk which has been offered the $(N-1)^{st}$ overflow stream from the $(N-1)^{st}$ trunk. In general, if the Brockmeyer system with renewal input is being analysed, the overflow stream from a finite secondary group of L trunks, becomes the stream overflowing the $(N+L)^{th}$ individual trunk when offered the $(N+L-1)^{st}$ overflow stream from the $(N+L-1)^{st}$ trunk.

This 'atomic' approach is valid only when the services are memoryless. For more general service distributions, the 'group' approach could be extended, however the overflow traffic moments for the corresponding queueing systems become extremely complicated and intractable.

To demonstrate the methodology used for an 'atomic' analysis of an overflow system, the Laplace-Stieltjes transform $\psi_N(s)$ of the overflow stream G, from a finite primary group, is rederived. The three phases of the atomic approach are illustrated in figure 6.1. New properties of the overflow traffic's peakedness and coefficient of variation are then established. A significant consequence of this study on peakedness is a simple proof of the well established practical teletraffic result, "The overflow traffic arising from randomly offered traffic is rough". Wilkinson [41], intuitively gives reasons why such a result must necessarily hold.

This study of the overflow traffic's peakedness is extended by producing peakedness charts for different Erlang input streams.

Notation

The inter event time distribution of the overflow from the i^{th} individual trunk when offered the overflow stream from the $(i-1)^{st}$ individual trunk is denoted by $G_{(i)}(t)$ with $\psi_{(i)}(s)$ as its Laplace-Stieltjes transform.

6.2 Derivation of $\psi_N(s)$ by the atomic approach

6.2.1 Phase I

The overflow stream from the first trunk when offered the input stream F satisfies

$$\psi_{(1)}(s) = \frac{k_0(s)}{k_0(s)+k_1(s)} \qquad (6.2.1)$$

where

$$k_0(s) = 1 \qquad (6.2.2)$$

$$k_1(s) = \frac{1-\phi(s)}{\phi(s+\mu)} \quad . \qquad (6.2.3)$$

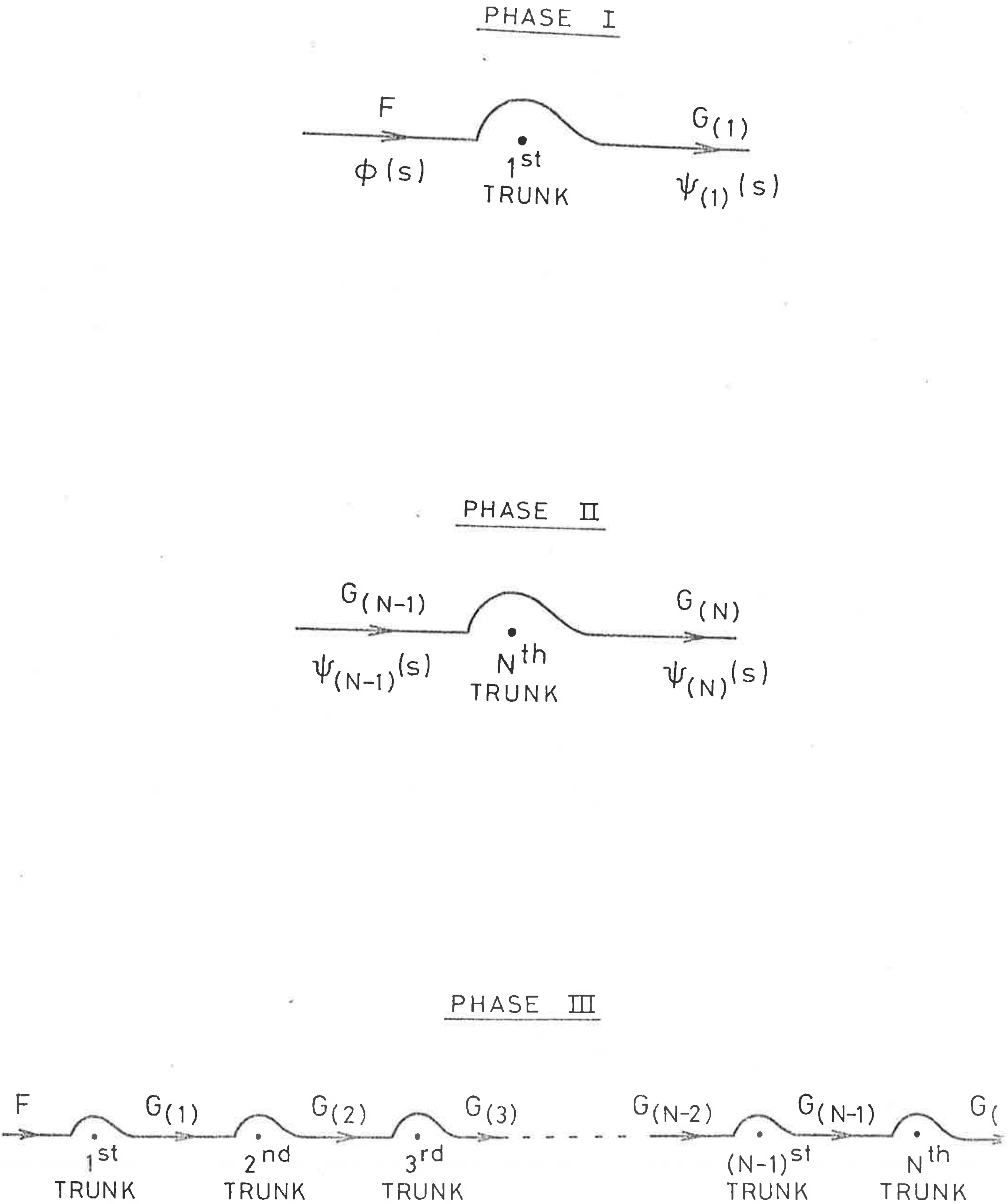

PHASE I
F
G(1)
φ(s)
1st TRUNK
ψ(1)(s)
PHASE II
G(N-1)
G(N)
ψ(N-1)(s)
Nth TRUNK
ψ(N)(s)
PHASE III
F
G(1)
G(2)
G(3)
G(N-2)
G(N-1)
1st TRUNK
2nd TRUNK
3rd TRUNK
(N-1)st TRUNK
Nth TRUNK

FIGURE 6.1

Proof of (6.2.1)

If $f_0(t)$ is the distribution function for the time separating an instant when a call arrives to find the trunk already occupied and the instant of the first subsequent overflow, then

$$f_0(t) = \int_0^t (1-e^{-\mu y}) f_0(t-y)\,dF(y) + \int_0^t e^{-\mu y} f_1(t-y)\,dF(y) \qquad (6.2.4)$$

where we interpret $f_1(t) = \delta(t-0)$.

Taking the Laplace-Stieltjes transform of equation (6.2.4), gives

$$f_0{*}(s) = \int_0^\infty e^{-st}(1-e^{-\mu t}) f_0{*}(s)\,dF(t) + \int_0^\infty e^{-st} e^{-\mu t} f_1{*}(s)\,dF(t).$$

We impose the boundary condition $f_1{*}(s) \equiv 1$ then

$$f_0{*}(s)[1 - \phi(s) + \phi(s+\mu)] = \phi(s+\mu) \quad . \qquad (6.2.5)$$

But $\psi_{(1)}(s) = f_0{*}(s)$

$$= 1/[1 + \frac{1-\phi(s)}{\phi(s+\mu)}] \quad \text{by equation (6.2.5)} \qquad (6.2.6)$$

thus proving equation (6.2.1).

6.2.2 Phase II

The overflow stream from the N^{th} individual trunk when offered the overflow stream from the $(N-1)^{st}$ trunk satisfies

$$\psi_{(N)}(s) = 1/\left[1 + \frac{1-\psi_{(N-1)}(s)}{\psi_{(N-1)}(s+\mu)}\right] . \qquad (6.2.7)$$

Proof of (6.2.7)

If the input stream F and overflow stream $G_{(1)}$ are replaced by $G_{(N-1)}$ and $G_{(N)}$ the situation depicted in Phase I is identical with that of Phase II, hence equation (6.2.6) holds when $\psi_{(1)}(s)$ and $\phi(s)$ are replaced by $\psi_{(N)}(s)$ and $\psi_{(N-1)}(s)$ respectively.

6.2.3 Phase III

The overflow stream $G_{(N)}$ from the N^{th} individual trunk satisfies

$$\psi_{(N)}(s) = \frac{\sum_{r=0}^{N-1} \binom{N-1}{r} k_r(s)}{\sum_{r=0}^{N} \binom{N}{r} k_r(s)} \tag{6.2.8}$$

where

$$k_r(s) = \begin{cases} 1 & , \quad r=0 \\ \prod_{j=1}^{r} \dfrac{1-\phi(s+\overline{j-1}\mu)}{\phi(s+j\mu)} & , \quad r \geqslant 1 \end{cases} \tag{6.2.9}$$

Proof: Equation (6.2.8) simplifies to equation (6.2.1) when $N=1$.

If equation (6.2.8) is valid for $\psi_{(N-1)}(s)$, then

$$\psi_{(N)}(s) = 1\Big/\left[1 + \frac{1-\psi_{(N-1)}(s)}{\psi_{(N-1)}(s+\mu)}\right] \quad \text{by equation (6.2.7)}$$

Now $k_{r+1}(s) = \dfrac{1-\phi(s)}{\phi(s+\mu)} k_r(s+\mu)$ by equation (2.5.15)

$$\therefore \quad \psi_{(N-1)}(s+\mu) = \frac{\sum_{r=0}^{N-2} \binom{N-2}{r} k_{r+1}(s)}{\sum_{r=0}^{N-1} \binom{N-1}{r} k_{r+1}(s)} \tag{6.2.10}$$

$$1-\psi_{(N-1)}(s) = 1 - \frac{\sum_{r=0}^{N-2} \binom{N-2}{r} k_r(s)}{\sum_{r=0}^{N-1} \binom{N-1}{r} k_r(s)} \quad \text{by equation (6.2.8) with } N=N-1$$

$$= \frac{\sum_{r=1}^{N-1} \binom{N-2}{r-1} k_r(s)}{\sum_{r=0}^{N-1} \binom{N-1}{r} k_r(s)} \tag{6.2.11}$$

$$\therefore \qquad \frac{1-\psi_{(N-1)}(s)}{\psi_{(N-1)}(s+\mu)} = \frac{\sum_{r=0}^{N-1} \binom{N-1}{r} k_{r+1}(s)}{\sum_{r=0}^{N-1} \binom{N-1}{r} k_r(s)} \tag{6.2.12}$$

Substituting equation (6.2.12) into the expression for $\psi_{(N)}(s)$ given by equation (6.2.7) shows $\psi_{(N)}(s)$ satisfies equation (6.2.8).

Since the stream overflowing the N^{th} trunk in sequence is identical to the overflow stream from a primary group,

$$\psi_{(N)}(s) = \psi_N(s) \tag{6.2.13}$$

hence, equation (6.2.8) and (2.2.17) are equivalent.

The inductive form of the proof of equation (6.2.8) is essentially equivalent to the conditional probability argument of Takács [36] . He applied an integral equation technique to obtain the solution. He was unaware of our physical 'atomic' interpretation of the system and his argument appears outwardly dissimilar.

6.3 The Coefficient of Variation of the Overflow Traffic from N Trunks

Notation:

Let $f_{(N)}$, $M_{(N)}$, $V_{(N)}$, $W_{(N)}$, $z_{(N)}$ denote the weakness, mean, variance, coefficient of variation, peakedness of the overflow traffic resulting from the overflow stream $G_{(N)}$ of section 6.2. We assume these quantities are defined for $N=0$ by defining

$$G_{(0)} = F . \tag{6.3.1}$$

6.3.1 Phase I

The overflow traffic from the first trunk satisfies

$$W_{(1)} > W_{(0)} \quad . \tag{6.3.2}$$

Proof of expression (6.3.2)

The input stream F satisfies equation (3.4.12), giving

$$V_{(0)} = M_{(0)}\left[1 - M_{(0)} + \frac{\phi(\mu)}{1-\phi(\mu)}\right] \tag{6.3.3}$$

where $M_{(0)} = \frac{1}{f_{(0)}} = -\frac{1}{\mu\phi'(0)}$ by equation (2.2.10).

Since $W_{(0)} = \frac{V_{(0)}}{[M_{(0)}]^2}$, equation (6.3.3) becomes

$$W_{(0)} = f_{(0)}\left[1 - \frac{\phi(\mu)}{1-\phi(\mu)}\right] - 1 \tag{6.3.4}$$

$$= -\mu\phi'(0)\left[1 - \frac{\phi(\mu)}{1-\phi(\mu)}\right] - 1. \tag{6.3.5}$$

But equation (6.3.5) is valid for any renewal stream and in particular holds for the stream $G_{(1)}$ overflowing from the first trunk, thus

$$W_{(1)} = -\mu\psi_{(1)}'(0)[1 - \psi_{(1)}(\mu)]^{-1} - 1 \quad . \tag{6.3.6}$$

But $\psi_{(1)}(\mu) = \frac{\phi(2\mu)}{\phi(2\mu)+1-\phi(\mu)}$ by equation (6.2.6) (6.3.7)

and $\psi_{(1)}'(0) = \frac{\phi'(0)}{\phi(\mu)}$ by equation (2.5.1) with $N=1$, (6.3.8)

therefore the expression for $W_{(1)}$ given by equation (6.3.6) can be simplified to

$$W_{(1)} = -1 - \frac{\mu\phi'(0)}{1-\phi(\mu)} \frac{\phi(2\mu)+1-\phi(\mu)}{\phi(\mu)} \tag{6.3.9}$$

Holtzman [10] notes that for any renewal stream,

$$\chi(j\mu) \geqslant [\chi(\mu)]^j \quad , \quad j \geqslant 1 \tag{6.3.10}$$

and Winsten [42] shows that strict inequality holds in the case when $j=2$ unless the underlying distribution is degenerate, that is

$$\phi(2\mu) > [\phi(\mu)]^2 \quad \text{since } F \text{ is not degenerate.} \tag{6.3.11}$$

Now $[1 - \phi(\mu)]^2 > 0$ since $\phi(\mu) \neq 1$

$$\therefore \qquad [\phi(\mu)]^2 - \phi(\mu) + 1 > \phi(\mu)$$

$$\therefore \qquad \frac{[\phi(\mu)]^2-\phi(\mu)+1}{\phi(\mu)} > 1 \quad . \tag{6.3.12}$$

Hence using the inequality, (6.3.11) gives

$$\frac{\phi(2\mu)-\phi(\mu)+1}{\phi(\mu)} > 1 \quad . \tag{6.3.13}$$

Thus the latter term in the product of $W_{(1)}$ is greater than unity, proving the inequality (6.3.2).

6.3.2 Phase II

The overflow traffic from the N^{th} individual trunk when offered the overflow stream from the $(N-1)^{st}$ trunk satisfies

$$W_{(N)} > W_{(N-1)} \quad . \tag{6.3.14}$$

Proof of expression (6.3.14)

The argument used in the proof of phase I is valid for any renewal input stream and its corresponding renewal overflow stream from one trunk. Hence equations (6.3.5), (6.3.6) and (6.3.9) hold when F and $G_{(1)}$ are replaced by $G_{(N-1)}$ and $G_{(N)}$, giving

$$W_{(N-1)} = -\mu\psi_{(N-1)}'(0)\,[1 - \psi_{(N-1)}(\mu)]^{-1} - 1 \qquad (6.3.15)$$

$$W_{(N)} = -\mu\psi_{(N)}'(0)\,[1 - \psi_{(N)}(\mu)]^{-1} - 1 \qquad (6.3.16)$$

and

$$W_{(N)} = -1 - \mu\psi_{(N-1)}'(0)\left[1 - \psi_{(N-1)}(\mu)\right]^{-1}\left[\frac{\psi_{(N-1)}(2\mu)+1-\psi_{(N-1)}(\mu)}{\psi_{(N-1)}(\mu)}\right] \qquad (6.3.17)$$

Hence the inequality (6.3.14) follows since

$$\psi_{(N-1)}(2\mu) + 1 - \psi_{(N-1)}(\mu) > \psi_{(N-1)}(\mu) \quad \text{by equation (6.3.12)}.$$

This result given in Phase II, proves that the coefficient of variation of the overflow traffic is an increasing function of N.

6.3.3 Phase III

The coefficient of variation of the overflow traffic from N trunks satisfies

$$W_N \equiv W_{(N)} > W_{(0)} \quad . \qquad (6.3.18)$$

This result follows by induction on (6.3.12), having proved (6.3.1).

The increasing property of $W_{(0)}$ with increasing weakness of the input stream was proved by equation (3.4.25).

If the identical analysis of varying μ for a fixed λ is now applied to $W_{(N)}$ given by equation (6.3.16), the derivative of $W_{(N)}$ with respect to μ satisfies

$$\frac{dW_{(N)}}{d\mu} = - \frac{\psi_{(N)}{}'(0)}{1-\psi_{(N)}(\mu)} \frac{\mu^2 \psi''_{(N)}(\xi)}{2(1-\psi_{(N)}(\mu))} \quad \text{where } \xi \in (0,\mu) \ . \qquad (6.3.19)$$

Hence $W_{(N)} \to 0$ as $\mu \to 0$ and $W_{(N)} \to \infty$ as $\mu \to \infty$; that is, the coefficient of variation of the overflow traffic ranges from 0 to ∞ as the input weakness ranges from 0 to ∞.

When the input stream is Erlang order k, defined by equation (3.5.1) and satisfying equation (3.5.10),

$$W_{(0)} = W_{ER}$$

$$= -1 + \frac{A^{-1}}{1-(1+\frac{1}{kA})^{-k}} \quad \text{by equation (3.5.14).}$$

Hence the coefficient of variation of the overflow traffic from N trunks produced by this input stream satisfies

$$W_N > -1 + \frac{A^{-1}}{1-(1+\frac{1}{kA})^{-k}} \quad \text{by (6.3.18)} \ . \qquad (6.3.20)$$

If $k=1$, that is the input stream is negative exponential, the inequality (6.3.20) reduces to

$$W_N > A^{-1} \ , \qquad (6.3.21)$$

whereas if $k=\infty$, that is the input stream is deterministic, the inequality reduces to

$$W_N > -1 + \frac{A^{-1}}{1-e^{-1/A}} \tag{6.3.22}$$

For Telecom Australia [39] p. 31, the peakedness z defined by (3.4.14) is accepted as the standard for measuring variability of a stream. The standard being $z = 1$ for the negative exponential input stream and traffic being called smooth if $z < 1$ or rough if $z > 1$. However Kuczura [15], calls a stream smooth if $W < 1$ and peaked $W > 1$ with the standard still being the negative exponential stream for which $W = 1$, although he mentions "the same dichotomy is effected by the inequalities $z < 1$ and $1 < z$."

6.4 The Peakedness of the Overflow Traffic from N Trunks

Telephony classifies the various streams of an overflow system by means of the peakedness of the traffic produced by that stream.

If the peakedness of the traffic produced by a renewal stream is greater than or less than unity, the stream is called rough or smooth respectively. The label $\left\{\begin{matrix}\text{rough}\\\text{smooth}\end{matrix}\right.$ implies that the traffic is $\left\{\begin{matrix}\text{rougher}\\\text{smoother}\end{matrix}\right.$ than pure chance traffic for which the variance equals the mean.

6.4.1 Phase I

The overflow traffic from the first trunk satisfies

$$z_{(1)} > \min(1, z_{(0)}) \,. \tag{6.4.1}$$

Proof of (6.4.1)

Since $z_{(0)} = \dfrac{V_{(0)}}{M_{(0)}}$, equation (6.3.3) becomes

$$z_{(0)} = 1 + \frac{1}{\mu\phi'(0)} + \frac{\phi(\mu)}{1-\phi(\mu)} \tag{6.4.2}$$

This equation (6.4.2) is true for any renewal stream, and in particular holds for the stream $G_{(1)}$ overflowing from the first trunk, thus

$$z_{(1)} = 1 + \frac{1}{\mu\psi_{(1)}'(0)} + \frac{\psi_{(1)}(\mu)}{1-\psi_{(1)}(\mu)} \qquad (6.4.3)$$

$$\therefore \quad z_{(1)} - 1 = \frac{1}{\mu\psi_{(1)}'(0)} + \frac{\psi_{(1)}(\mu)}{1-\psi_{(1)}(\mu)} \qquad (6.4.4)$$

$$= \frac{\phi(\mu)}{\mu\phi'(0)} + \frac{\phi(2\mu)}{1-\phi(\mu)} \quad \text{by equations (6.3.7) and (6.3.8).} \qquad (6.4.5)$$

But the stream F satisfies (6.3.11),

$$\therefore \quad z_{(1)} - 1 > \phi(\mu)[z_{(0)} - 1] \ . \qquad (6.4.6)$$

The condition expressed by (6.4.6) means that

if $z_{(0)} - 1 > 0$ then $z_{(1)} - 1 > 0$

but

if $z_{(0)} - 1 < 0$ then $z_{(1)} - 1 > z_{(0)} - 1$,

or equivalently

$$z_{(1)} - 1 > \min(1, z_{(0)}) \ .$$

6.4.2 Phase II

The overflow traffic from the N^{th} trunk satisfies

$$z_{(N)} > \min(1, z_{(N-1)}) \ . \qquad (6.4.7)$$

Proof of (6.4.7)

Equations (6.4.2), (6.4.4) and (6.4.5) hold for any renewal input stream and its corresponding overflow stream from 1 trunk. In particular if F and $G_{(1)}$ are replaced by $G_{(N-1)}$ and $G_{(N)}$

these equations can be rewritten as

$$z_{(N-1)} = 1 + \frac{1}{\mu\psi_{(N-1)}'(0)} + \frac{\psi_{(N-1)}(\mu)}{1-\psi_{(N-1)}(\mu)} \tag{6.4.8}$$

$$z_{(N)} - 1 = \frac{1}{\mu\psi_{(N)}'(0)} + \frac{\psi_{(N)}(\mu)}{1-\psi_{(N)}(\mu)} \tag{6.4.9}$$

$$= \frac{\psi_{(N-1)}(\mu)}{\mu\psi_{(N-1)}'(0)} + \frac{\psi_{(N-1)}(2\mu)}{1-\psi_{(N-1)}(\mu)} \;. \tag{6.4.10}$$

But inequality (6.3.12) holds for any non degenerate renewal stream,

$$\therefore \qquad z_{(N)} - 1 > \psi_{(N-1)}(\mu)\,[z_{(N-1)} - 1] \tag{6.4.11}$$

that is $\qquad z_{(N)} > \min(1, z_{(N-1)})$.

This result (6.4.7) implies that if the input stream is smooth then the peakedness of subsequent individual overflows increases until the overflow traffic becomes rough; once this occurs all subsequent overflows remain rough but whether the increasing property of the peakedness is maintained is not necessarily true. Tables obtained from computer calculations for the overflow traffic peakedness for different Erlang input streams imply that a unique max value, greater than unity exists.

6.4.3 Phase III

The overflow traffic from a set of N trunks satisfies

$$z_N > \min(1, z_{(0)}) \tag{6.4.12}$$

This result is a consequence of induction on (6.4.7) since (6.4.1) is valid.

When the input stream is Erlang order k, defined by equation (3.5.1) and satisfying equation (3.5.10),

$$z_{(0)} = z_{ER}$$

$$= -A + \frac{1}{1-(1+\frac{1}{kA})^{-k}} \quad \text{by equation (3.5.13).}$$

Hence the peakedness of the overflow traffic from N trunks produced by this input stream satisfies

$$z_N > \min(1, -A + \frac{1}{1-(1+\frac{1}{kA})^{-k}} \quad \text{by (6.5.12)} . \qquad (6.4.13)$$

If $k=1$, that is the input stream is negative exponential, the inequality (6.4.13) reduces to

$$z_N > \min(1,1) = 1 . \qquad (6.4.14)$$

This proves the well known practical result that the overflow traffic corresponding to a negative exponential input stream is rough.

If $k=\infty$, that is the input stream is deterministic, the inequality (6.4.13) reduces to

$$z_N > \min(1, -A + \frac{1}{1-e^{-\frac{1}{A}}}) . \qquad (6.4.15)$$

The peakedness of the input stream has a lower bound of $\frac{1}{2}$ which occurs for a deterministic distribution of very high intensity, as shown by equation (3.4.22), hence

$$z_N > \frac{1}{2} \text{ by (6.5.12)} . \tag{6.4.16}$$

This confirms the analysis of section 7.2 of Pearce and Potter [25].

Table (6.5.1) gives the peakedness of the input traffic corresponding to Erlang distributions of order $k=1,3,6,10,\infty$ for the indicated values of A.

Equations (3.5.13) and (3.5.21) were used to calculate these values.

		MEAN INPUT TRAFFIC (A)						
		5	7	9	11	13	15	(19)
	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	3	.6812	.6774	.6748	.6735	.6726	.6715	
Erlang	6	.5993	.5948	.5923	.5907	.5897	.5889	
Phase	10	.5666	.5617	.5591	.5586	.5565	.5560	
k	∞	.5318	.5121	.5099	.5076	.5067	.5058	.5048

TABLE 6.5.1

6.5 Features of Overflow Peakedness from Computed Charts

If an input stream, F, is Erlang order k, $k=1,2,\ldots,$ with its average interevent time identical to that of a negative exponential distribution parameter λ, then

(i) $f_0 = A^{-1}$ from equation (3.5.23) (6.5.1)

where $A = \lambda/\mu$ from equation (3.5.9).

(ii)

$$\phi(s) = \begin{cases} \left(\dfrac{k\lambda}{k\lambda+s}\right)^k, & k \text{ finite, from equations (3.5.2) and (3.5.10)} \\ e^{-s/\lambda}, & k \text{ infinite, from equation (3.5.24)} \end{cases} \tag{6.5.2}$$

(iii)

$$\phi(j\mu) = \begin{cases} (A/[A+j/k])^k, & k \text{ finite} \\ e^{-jA^{-1}}, & k \text{ infinite .} \end{cases} \tag{6.5.3}$$

Substituting for $\phi(j\mu)$, defined by equation (6.5.3), into equation (2.3.19) gives

(iv)

$$\ell_r(\mu) = \begin{cases} \prod_{j=0}^{r} \dfrac{1-\left(\dfrac{A}{A+j/k}\right)^k}{\left(\dfrac{A}{A+j/k}\right)^k}, & k \text{ finite, } r \geqslant 1 \\ \prod_{j=0}^{r} \left(e^{-jA^{-1}} - 1\right), & k \text{ infinite, } r \geqslant 1 \\ 1, & r = 1 \text{ all } k \end{cases} \tag{6.5.4}$$

The overflow weakness, f_N, corresponding to the Erlang input stream is obtained by substituting for $\ell_r(\mu)$, given by equation (6.5.4) into equation (2.5.6).

The mean overflow traffic, M_N which is the reciprocal of the overflow stream's weakness was calculated for Erlang input streams of order $1,3,6,10,\infty$.

The variance V_N, and the peakedness z_N, of the overflow traffic were found by substituting calculated values of M_N and M_{N+1} into equation (5.3.16) and (5.3.20) respectively, for these particular Erlang input streams.

Families of curves were produced from these results. Figures 6.5.1 to 6.5.5 corresponding to the different Erlang streams illustrate the dependence of the overflow peakedness $z_N(A)$, on the primary group size N, when the input stream has a fixed intensity value, A.

The effect of the phase of the Erlang stream, or equivalently, the effect of smoothing the input traffic on the overflow traffic's peakedness is demonstrated in Figure 6.5.6.

Families of overflow traffic peakedness curves, corresponding to fixed primary group size are given in Figures 6.5.7 to 6.5.10 for the various Erlang input streams.

The effect of the phase of the Erlang stream being offered to a primary group of fixed size, on the overflow peakedness is depicted in Figure 6.5.11.

One striking feature of all the graphs is their common shape.

The curves of Figures 6.5.1 to 6.5.6 suggest that for all input streams, E_k of constant intensity A, there exists a unique maximum value for the overflow traffic peakedness at $N=N^*$ say. In the case of $A=9$ Erlangs illustrated in Figure 6.5.6, all the input streams $E_1, E_3, E_6, E_{10}, E_\infty$ have their max value occurring at $N^*=12$, but the value of $z_{12}(9)$ ranges from 2.14 to 1.52 corresponding to $k=1$ or ∞.

Similarly the curves illustrated in Figures 6.5.7 to 6.5.10 suggest that there exists a unique maximum value of the overflow peakedness at $A=A^*$ for all Erlang input streams offered to a primary group of fixed size. Figure 6.5.11 shows that the maximum value of $z_9(A)$ ranges from 1.97 for a negative exponential input stream to 1.41 when the input stream is deterministic, but the maximum value for all $E_1, E_3, E_{10}, E_\infty$ occurs at $A^*=7.5$.

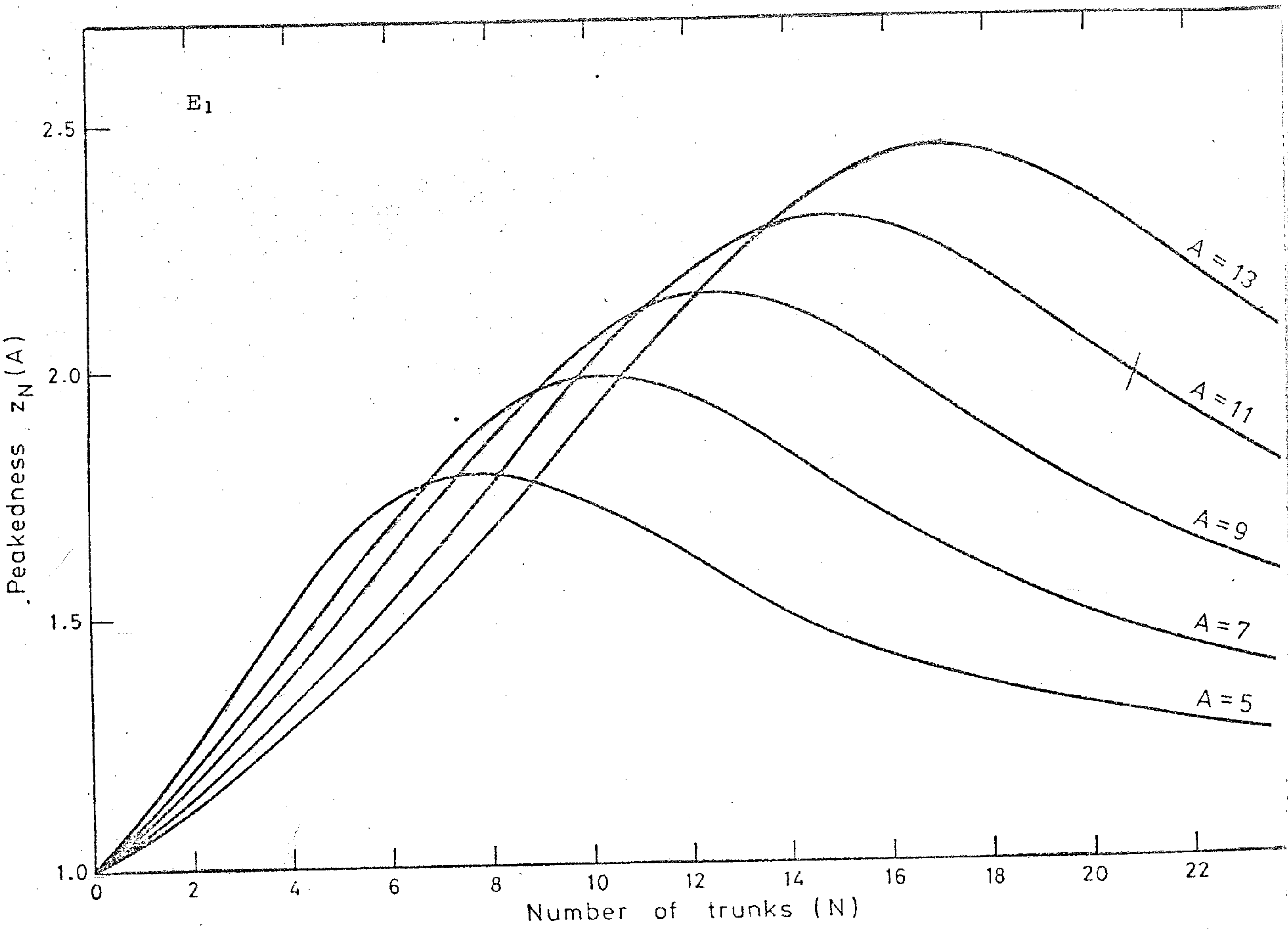


FIGURE 6.5.1

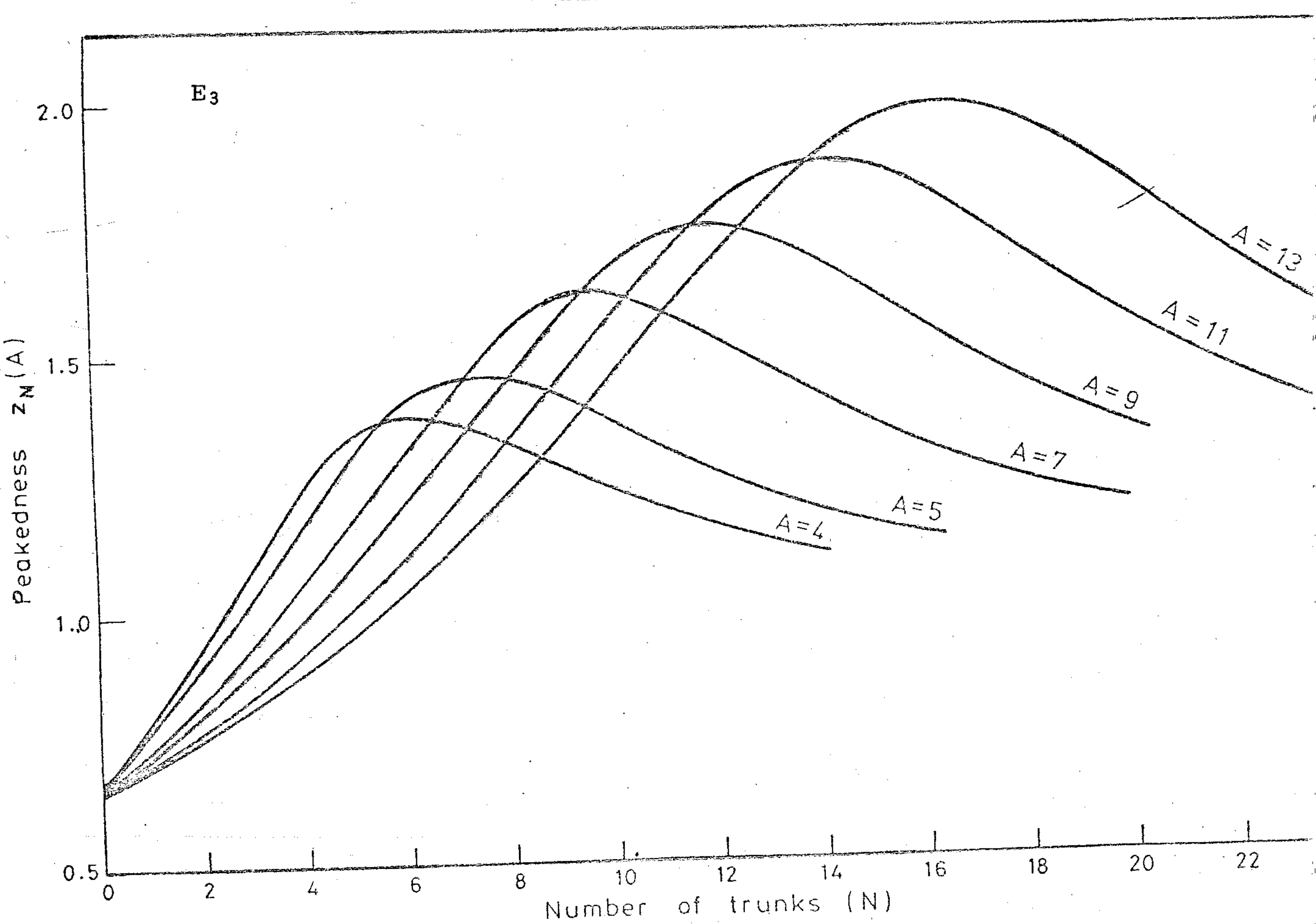


FIGURE 6.5.2

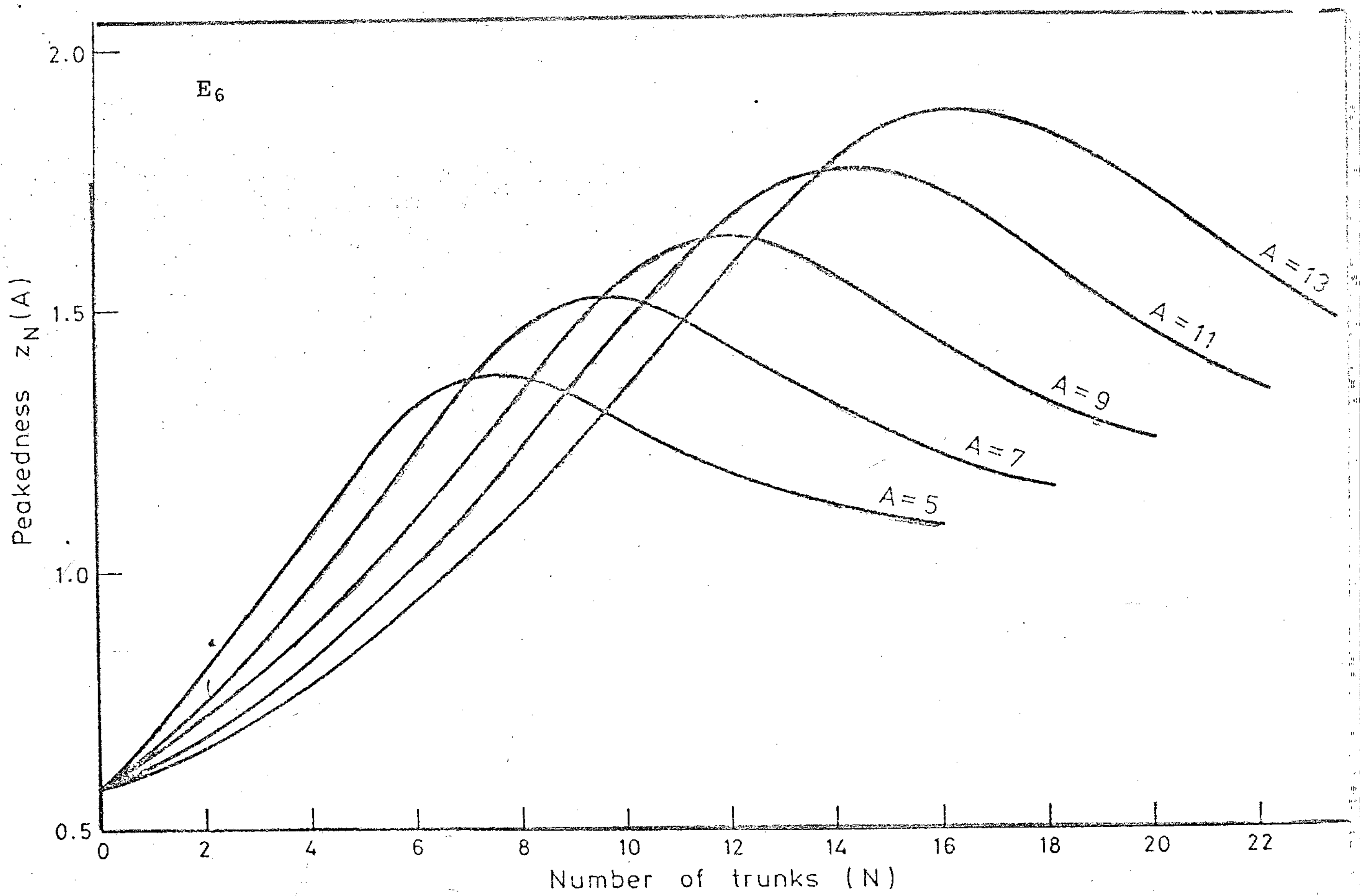


FIGURE 6.5.3

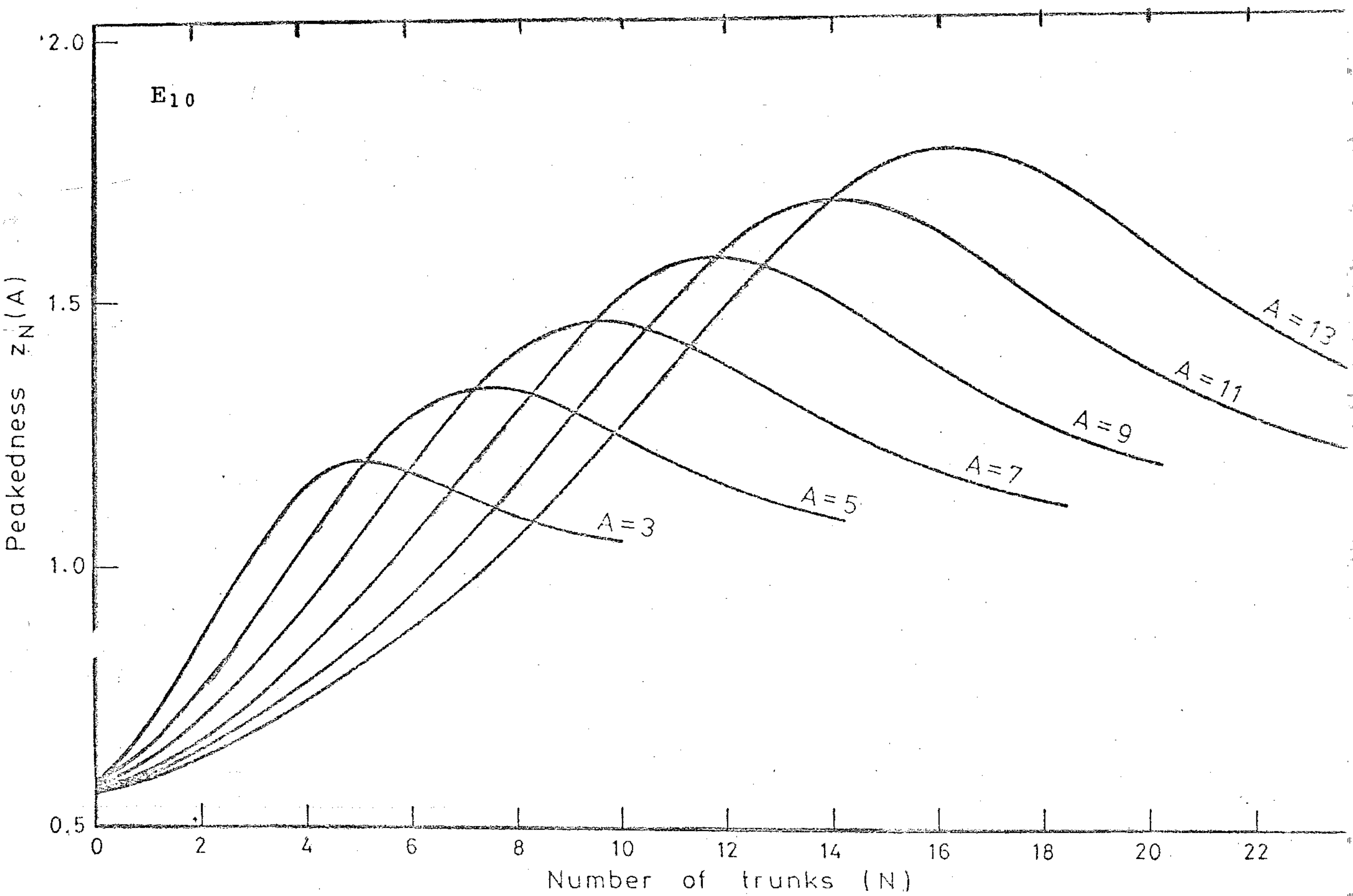
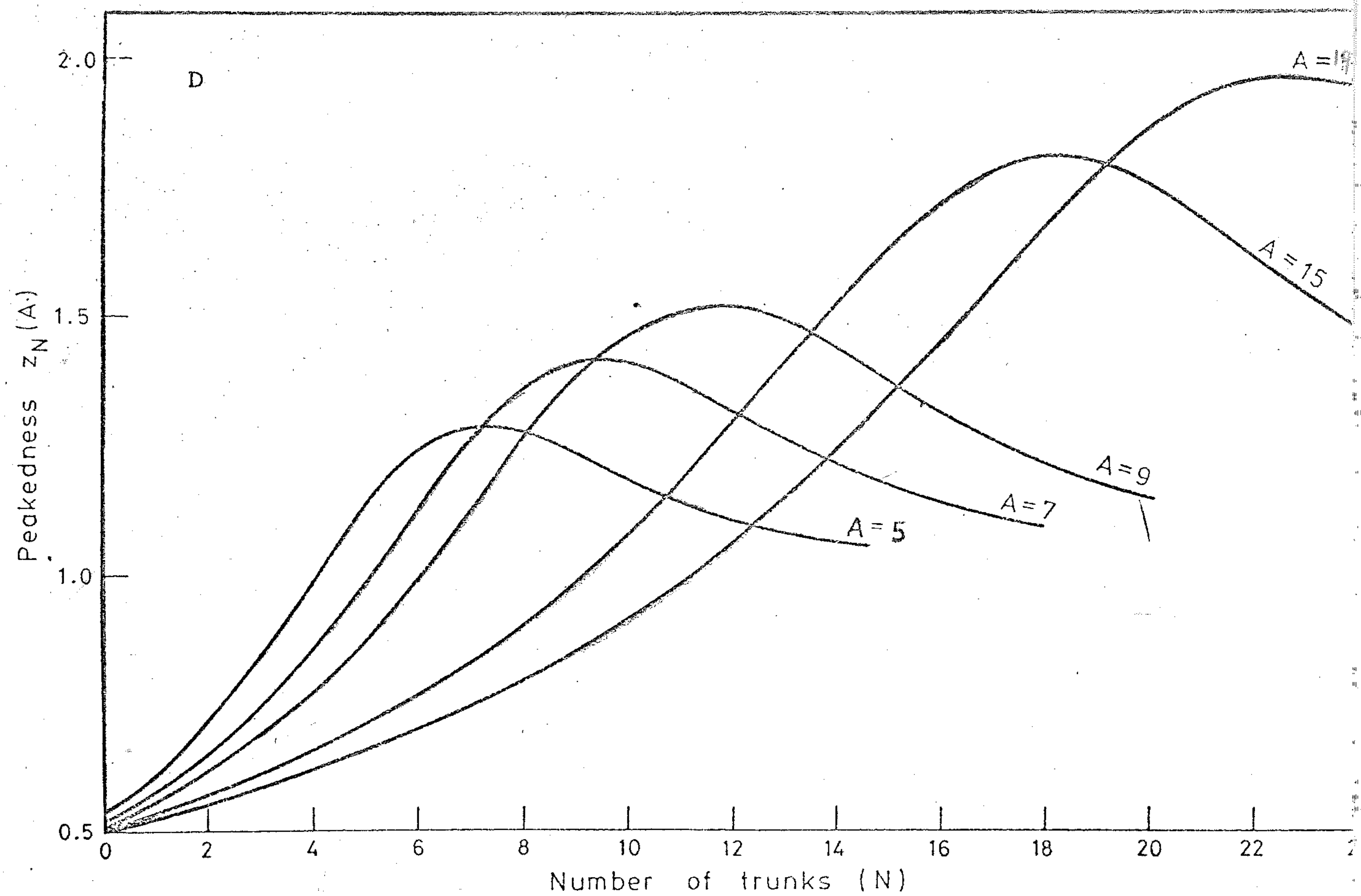


FIGURE 6.5.4

FIGURE 6.5.5



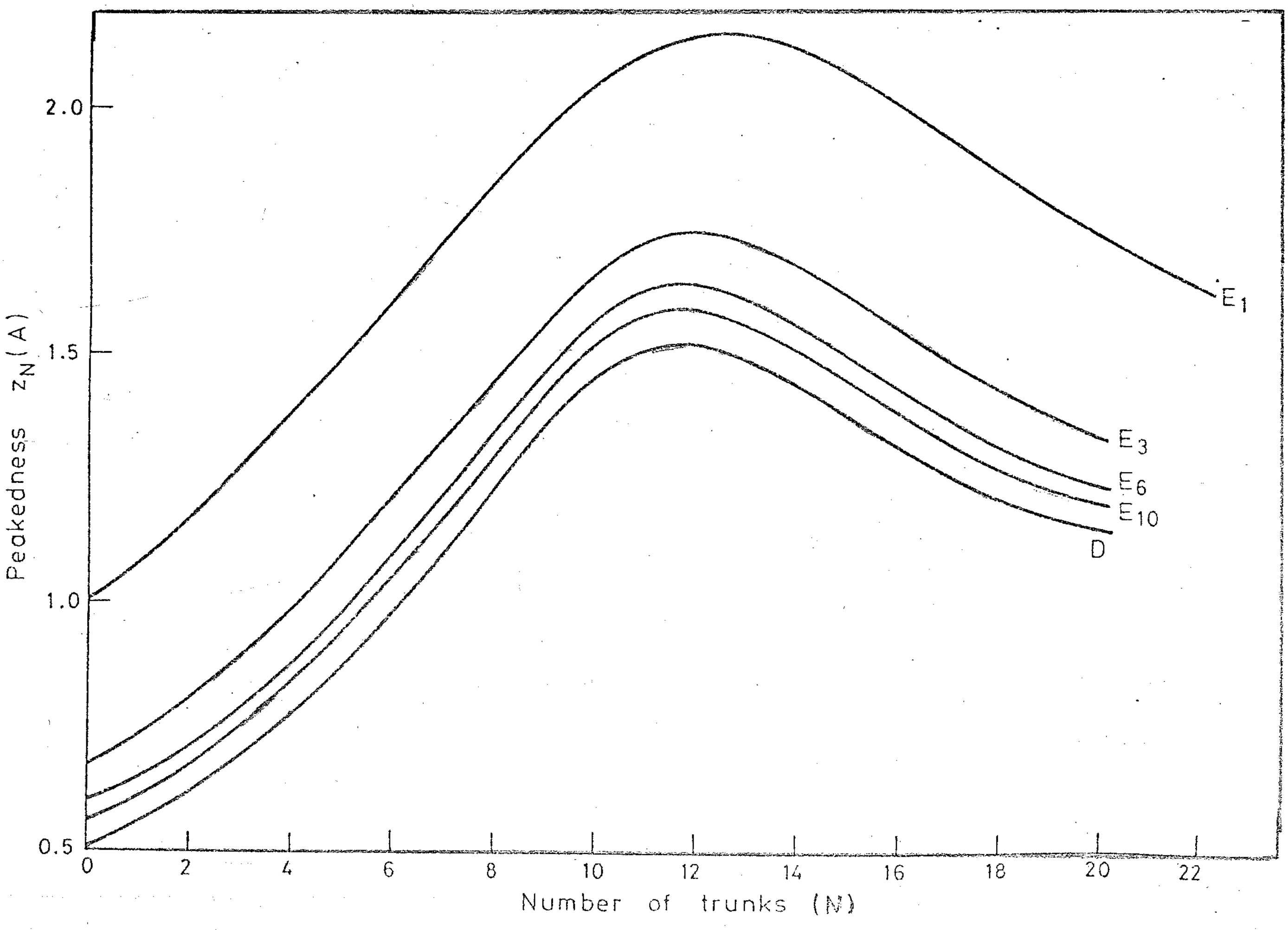


FIGURE 6.5.6

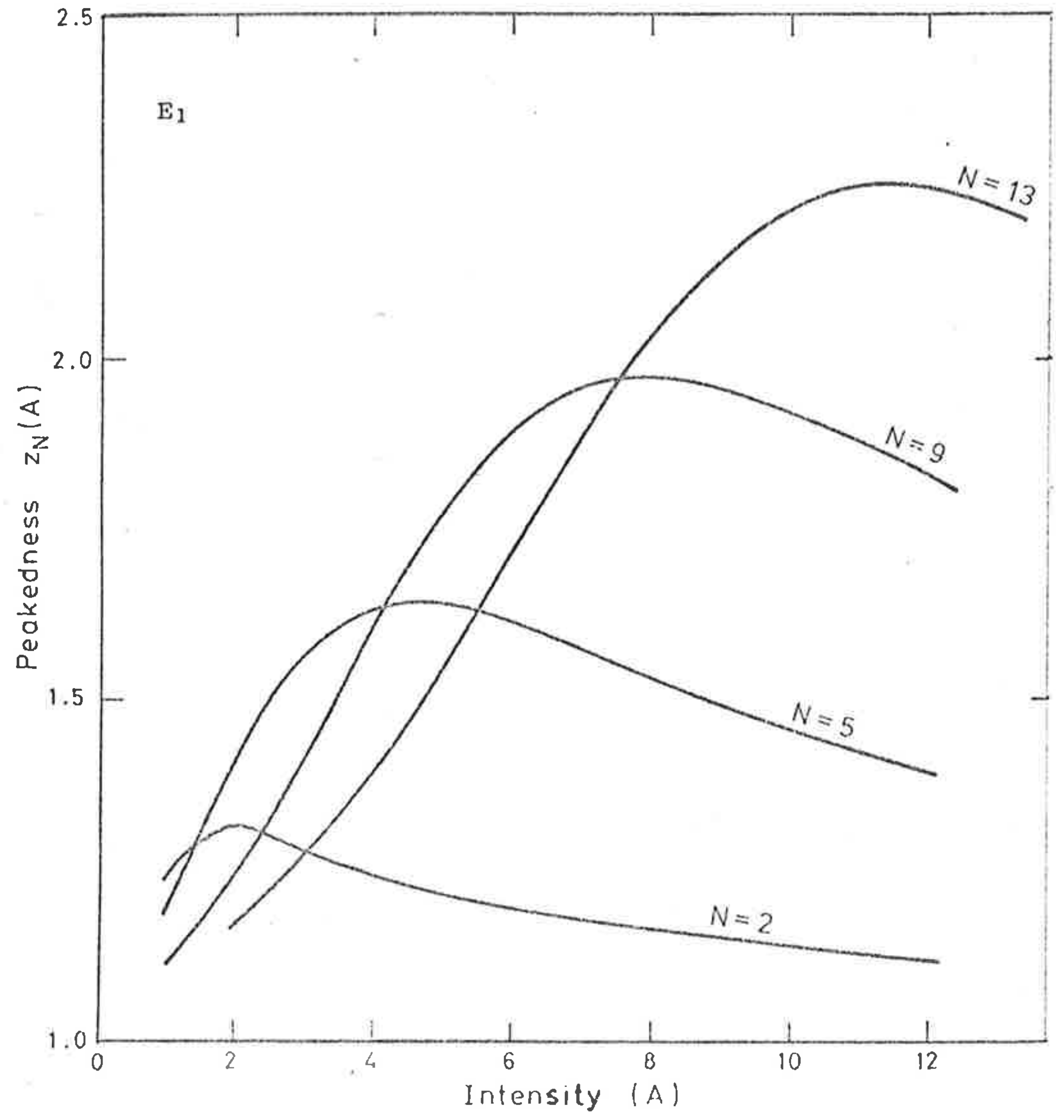

E1
N = 13
N = 9
N = 5
N = 2
2.5
2.0
1.5
1.0
0
2
4
6
8
10
12
Peakedness $z_N(A)$
Intensity (A)

FIGURE 6.5.7

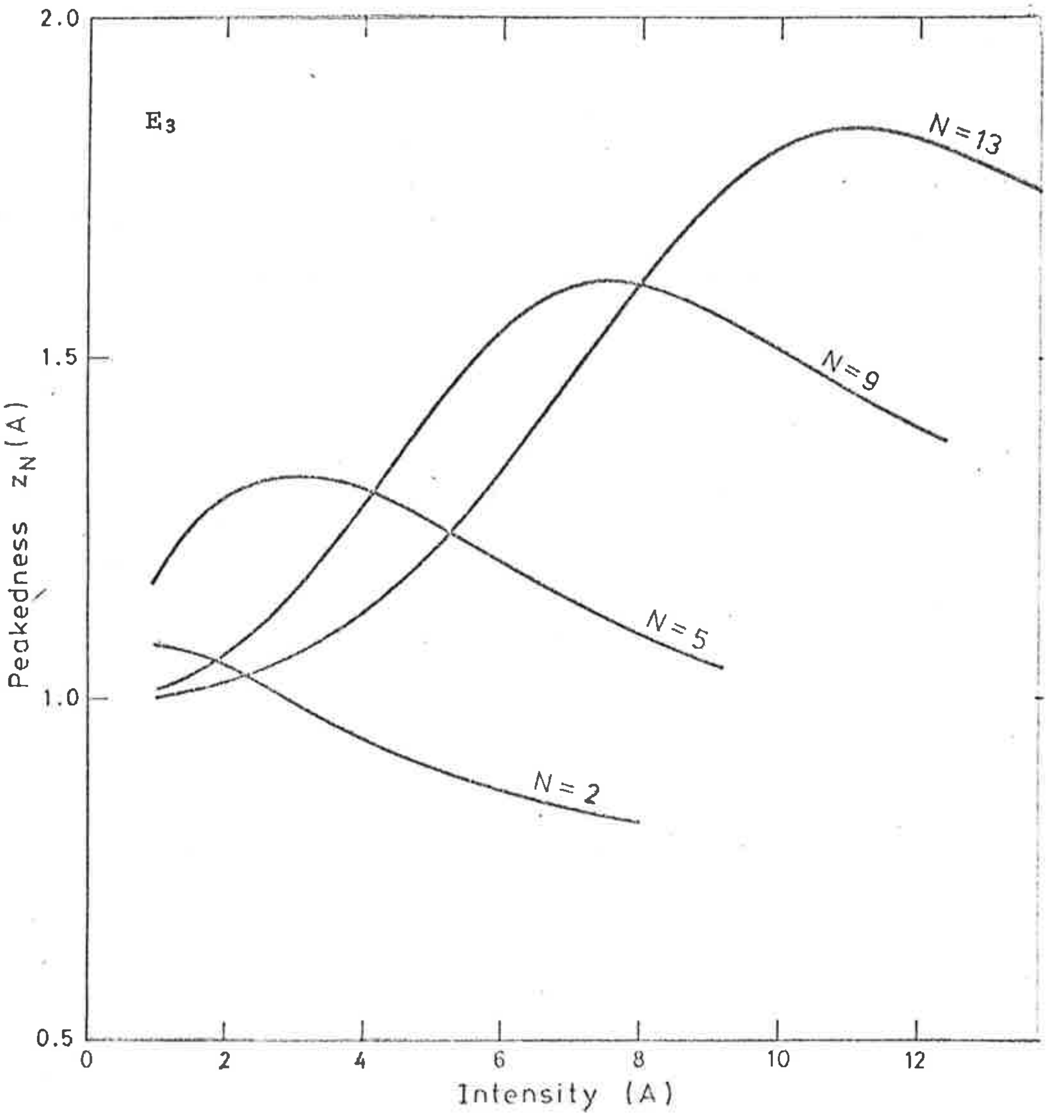

E3
N = 13
N = 9
N = 5
N = 2
2.0
1.5
1.0
0.5
Peakedness $z_N(A)$
0
2
4
6
8
10
12
Intensity (A)

FIGURE 6.5.8

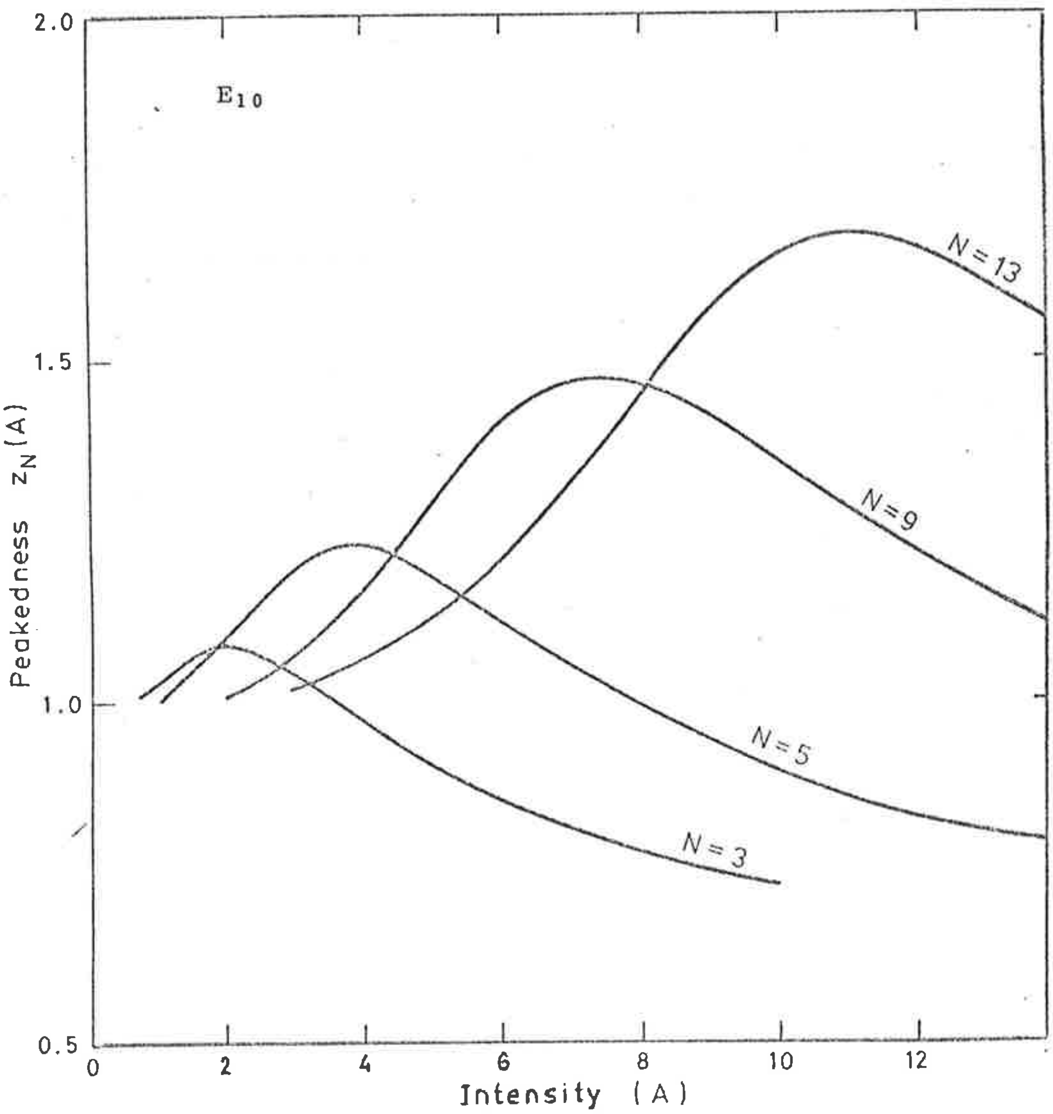

2.0
1.5
1.0
0.5
E_{10}
Peakedness $z_N(A)$
N = 13
N = 9
N = 5
N = 3
0
2
4
6
8
10
12
Intensity (A)

FIGURE 6.5.9

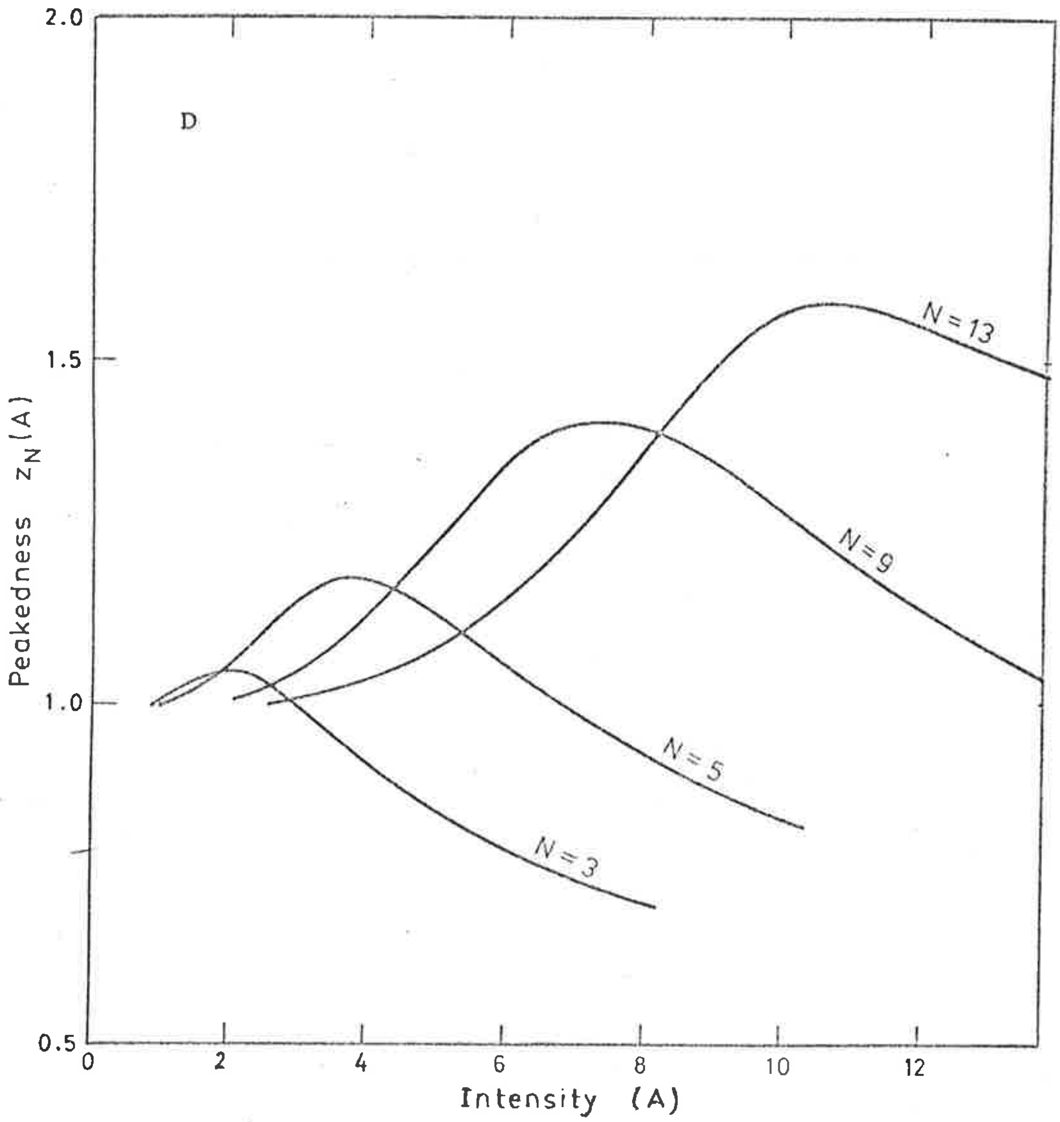

2.0
D
N = 13
1.5
N = 9
Peakedness $z_N(A)$
1.0
N = 5
N = 3
0.5
0
2
4
6
8
10
12
Intensity (A)

FIGURE 6.5.10

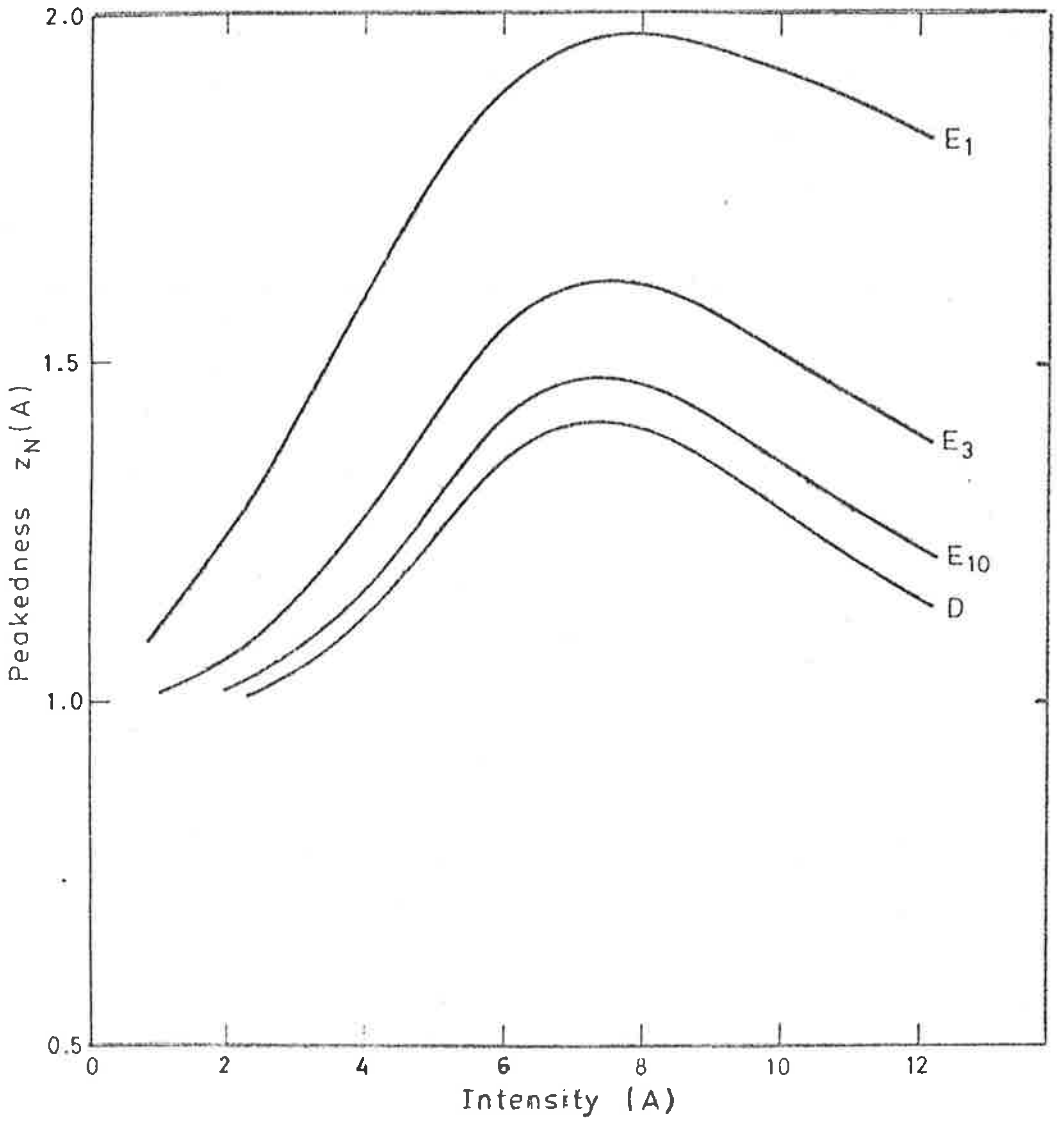


FIGURE 6.5.11

CHAPTER VII

THE E.N.R. METHOD AND RESTRICTIONS IMPOSED ON RENEWAL OVERFLOW SYSTEMS BY DIMENSIONING

CHAPTER VII

THE E.N.R. METHOD AND RESTRICTIONS IMPOSED ON RENEWAL OVERFLOW SYSTEMS BY DIMENSIONING

7.1 Introduction

"What is the effect on any general overflow system, (one with any general renewal input), when a finite number of the overflow moments are fixed?" Any restriction imposed on the system by these moment values would necessarily be independent of the particular distribution function chosen to model the interarrival times of the input stream. Holtzman [10] studied some invariant features of a general overflow system produced when values of the mean and variance of overflow traffic are given specified values. However, most dimensioning procedures, ranging from the early work of Erlang [5] through to the recent work of Bretschneider [3], Nightingale [21], Schehrer [32] and Rubas [31], have assumed a particular input stream. Any such procedures must necessarily incorporate properties peculiar to the chosen input distribution.

The explicit moment formulae, derived in Chapter 5, are used to both study the posed question and provide basic formulae needed to establish a more general dimensioning method. This chapter is divided into two sections. In the first section, we consider those characteristics of general overflow systems which are made invariant by specifying values for a finite number of overflow traffic moments. In the second section, we develop a dimensioning method, called the Equivalent Non Random (E.N.R.) method which is applicable for any renewable input stream.

al

The inversion of explicit formulae for the factorial overflow traffic moments is basic to any study on the invariant features produced when dimensioning an overflow system by a finite number of moments. One inversion expression gives the weakness, f_{N+r}, of the overflow stream from an increased primary group (the extra number of trunks is determined by the number of specified overflow traffic moments), as a simple function of the fixed overflow traffic moments. Many quantities applicable to overflow systems were shown in Chapter 5 to be functions of only these f_{N+r} s, hence all these quantities are invariant whenever the f_{N+r} are. The particular effect on an overflow system caused by specifying two or three overflow traffic moments is discussed in detail. With the added restriction imposed by assuming a negative exponential input stream, the inversion formulae provide closed expressions for the input stream's weakness, A^{-1}, and the corresponding primary group size, N, involving only three overflow traffic values. Nightingale [21] in his ER-W dimensioning model, recognised that three moment values were needed to provide an exact dimensioning model if the input traffic was assumed random.

The E.N.R. method is based on formulae for the first two overflow traffic moments for any Erlang input stream, phase k, $k=1,2,\ldots,\infty$. Using computed values for the overflow traffic's mean and variance, dimensioning charts, similar to the well known Wormald Charts, (see Nightingale [21], page (49)), are produced. Two families of curves are superimposed to produce a E.N.R. chart. Examples of these charts are given for various input streams ranging from random to smooth. Features common to all charts are discussed.

7.2 Inversion of the Overflow Traffic Factorial Moment Formula

Since only the q (or $\bar{q}$) steady state occupancy distributions on the secondary group need be considered in this chapter, the following notation for the factorial moments of the overflow traffic will be used,

$$\alpha_n(N) \equiv \alpha_{n,q}(N) \; . \tag{7.2.1}$$

Now the overflow traffic factorial moments were shown to satisfy

$$\alpha_{n+1}(N) = n!\left[\sum_{r=0}^{n} \binom{n}{r}(-1)^{n-r} f_{N+r}\right]^{-1} \quad \text{from equation (5.3.2)}$$

or equivalently

$$\alpha_{n+1}(N) = n!\left[f_0 \sum_{r=0}^{N} \binom{N}{r} \ell_{n+r}(\mu)\right]^{-1} \quad \text{from equation (5.3.2a).}$$

When the finite difference representation of $\alpha_{n+1}(N)$ given by equation (5.3.2) is inverted, then

$$f_{N+r} = \sum_{n=0}^{r} \binom{r}{n} n! \; \alpha_{n+1}^{-1}(N) \; , \tag{7.2.2}$$

whereas equation (5.3.2a) on inversion becomes

$$\ell_{n+r}(\mu) = f_0^{-1} \sum_{N=0}^{r} \binom{r}{N}(-1)^{r-N} n! \; \alpha_{n+1}^{-1}(N) \; . \tag{7.2.3}$$

The implications of equation (7.2.2) on dimensioning general overflow systems are investigated in this section.

If a finite number of overflow moments, say $\alpha_1, \alpha_2, \ldots, \alpha_{r+1}$ are specified, then all of $f_{N+1}, f_{N+2}, \ldots, f_{N+r}$ can be determined from equation (7.2.2). Hence any of the overflow system's quantities dependent only on any of the $f_{N+1}, f_{N+2}, \ldots, f_{N+r}$ are necessarily made identical for all the input streams by the chosen values of the r overflow traffic moments.

Overflow quantities which are functions of $f_N, f_{N+1}, \ldots, f_{N+r}$ are listed below.

(i) Marginal occupancy for $(N+r-1)$ primary trunks, given by equation (5.3.18) with $N=N+r-1$, satisfies

$$H_{N+r-1} = f_{N+r-1}^{-1} - f_{N+r}^{-1}, \quad r=1,2,\ldots \quad . \tag{7.2.4}$$

(ii) Laplace-Stieltjes transform at $s=\mu$ of the overflow distribution from $(N+r-1)$ primary trunks, given by equation (2.5.20) with $N=N+r-1$, satisfies

$$\psi_{N+r-1}(\mu) = \frac{f_{N+r-1}}{f_{N+r}}, \quad r=1,2,\ldots \quad . \tag{7.2.5}$$

(iii) Laplace-Stieltjes transform at $s=r\mu$ of the overflow distribution from N primary trunks, given by equation (2.5.23), satisfies

$$\psi_N(r\mu) = \frac{\Delta^{r-1} f_N}{\Delta^{r-1} f_{N+1}}, \quad r=1,2,\ldots \quad . \tag{7.2.6}$$

(iv) Probability of loss on a secondary group of r trunks, given by equation (5.4.17) with $L=r$, satisfies

$$\bar{\pi}_r = \frac{f_N}{f_{N+r}}, \quad r=1,2,\ldots \quad . \tag{7.2.7}$$

(v) Mean carried overflow traffic on a secondary group of r trunks, given by equation (5.4.24) with $L=r$, satisfies

$$\bar{M}_{N,r} = f_{N+r-1}^{-1} - f_{N+r}^{-1} \quad , \quad r=1,2,\ldots \quad . \tag{7.2.8}$$

(vi) Ratio of lost calls to carried calls on a secondary group of r trunks, given by equation (5.4.25) with $L=r$, satisfies

$$\bar{R}_{N,r} = \frac{r\, f_N}{f_{N+r} - f_N} \quad , \quad r=1,2,\ldots \quad . \tag{7.2.9}$$

(vii) Variance of carried overflow traffic on a secondary group of r trunks, given by equation (5.4.29) with $L=r$, satisfies

$$\bar{V}_{N,r} = [f_N^{-1} - f_{N+r}^{-1}]\left(\frac{f_{N+r} - f_N - r f_N f_{N+r}}{f_{N+r}(f_{N+r} - f_N)} - \frac{f_{N+1} - f_N - f_N f_{N+1}}{f_N(f_{N+1} - f_N)}\right) . \tag{7.2.10}$$

The values of any overflow traffic quantity listed above is the same for all renewal input streams once values are assigned to r overflow traffic moments. Hence dimensioning procedures involving r overflow traffic moments restrict the structure of a general overflow system by predetermining values (in terms of $\alpha_1, \alpha_2, \ldots, \alpha_r$) for all these listed quantities.

7.3 Specification of Two Overflow Traffic Moments

Most practical dimensioning procedures used in telephony involve two overflow traffic moments which are usually the mean and variance. Letting $r=1$, equation (7.2.2) becomes

$$f_{N+1} = \alpha_1^{-1}(N) + \alpha_2^{-1}(N) . \tag{7.3.1}$$

Thus any overflow quantity which is a function of f_N and f_{N+1} is invariant of the form of the input stream once values are specified for α_1 and α_2. Putting $r=1$, in equations (7.2.4) to (7.2.10) gives the following specified quantities,

(i) $$H_N = f_N^{-1} - f_{N+1}^{-1} \qquad (7.3.2)$$

(ii) $$\psi_N(\mu) = \frac{f_N}{f_{N+1}} \qquad (7.3.3)$$

(iii) $$\bar{\pi}_1 = \frac{f_N}{f_{N+1}} \qquad (7.3.4)$$

(iv) $$\bar{M}_{N,1.} = f_N^{-1} - f_{N+1}^{-1} \qquad (7.3.5)$$

(v) $$\bar{R}_{N,1.} = \frac{f_N}{f_{N+1}-f_N} \qquad (7.3.6)$$

(vi) $$\bar{V}_{N,1.} = (f_N^{-1} - f_{N+1}^{-1})\left(\frac{f_{N+1}-f_N-f_Nf_{N+1}}{f_{N+1}(f_{N+1}-f_N)} - \frac{f_{N+1}-f_N-f_Nf_{N+1}}{f_N(f_{N+1}-f_N)}\right) \qquad (7.3.7)$$

which simplifies to

$$\bar{V}_{N,1.} = (f_N^{-1} - f_{N+1}^{-1})[1 - f_N^{-1} + f_{N+1}^{-1}] \quad . \qquad (7.3.8)$$

The marginal occupancy, H_N is used in telephony to describe the decrease in overflow traffic when the primary group size is increased by one. The substitution of f_{N+1} given by equation (7.3.1) into equation (7.3.2), gives

$$H_N = \alpha_1 - [\alpha_1^{-1} + \alpha_2^{-1}]^{-1} \quad . \qquad (7.3.9)$$

Equations (7.3.2) and (7.3.5) imply

$$H_N \equiv \bar{M}_{N,1.} \quad . \qquad (7.3.10)$$

This result is apparent when the overflow system is viewed from an 'atomic' perspective, since the traffic carried on an additional primary trunk is the same as that carried on a single trunk (secondary group) when offered the N^{th} overflow stream from the N previous individual trunks.

Equations (7.3.3) and (7.3.4) imply

$$\psi_N(\mu) = \bar{\pi}_1 \quad . \tag{7.3.11}$$

Equation (7.3.11) means that $\psi_N(\mu)$ can be interpreted as the probability of loss when an overflow stream from the primary group is offered to a single secondary trunk group. Holtzman [10] recognised that $\psi_N(\mu)$ was independent of the input stream for any dimensioning procedure based on two overflow traffic moments.

The implication of equation (7.3.8) is that not only the mean but also the variance of the overflow carried traffic on a single secondary trunk is invariant for any general overflow system once values are assigned to α_1 and α_2.

7.4 Specification of Three Overflow Traffic Moments

Some dimensioning procedures, see Nightingale [21] and Freeman [16], involve three overflow traffic moments. Methods for measuring or estimating three overflow traffic moments are not usually considered practical by teletraffic engineers. However, this section shows that a great deal of additional information concerning the structure of the overflow system is provided by assigning a value to α_3.

If $\alpha_1, \alpha_2, \alpha_3$ are given specified values, then equation (7.2.2) with $r=2$, becomes

$$f_{N+2} = \alpha_1^{-1} + 2\alpha_2^{-1} + 2\alpha_3^{-1} \quad . \tag{7.4.1}$$

Therefore, all quantities which are functions of not only f_N and f_{N+1}, see Section 7.3, but also of f_N, f_{N+1} and f_{N+2} are completely determined by α_1, α_2, and α_3 regardless of the form of the input stream.

The following quantities are functions of f_{N+1} and f_{N+2}.

(i) $H_{N+1} = f_{N+1}^{-1} - f_{N+2}^{-1}$ from equation (7.2.4) with $r=2$ (7.4.2)

or equivalently

$\bar{M}_{N+1,1} = f_{N+1}^{-1} - f_{N+2}^{-1}$ from equation (7.2.8) with $N=N+1$ and $r=1$ (7.4.3)

(ii) $\psi_{N+1}(\mu) = \dfrac{f_{N+1}}{f_{N+2}}$ from equation (7.2.5) with $r=2$ (7.4.4)

(iii) $\bar{R}_{N+1,1} = \dfrac{f_{N+1}}{f_{N+2} - f_{N+1}}$ from equation (7.2.9) with $N=N+1$ and $r=1$ (7.4.5)

(iv) $\bar{V}_{N+1,1} = (f_{N+1}^{-1} - f_{N+2}^{-1})(1 - f_{N+1}^{-1} + f_{N+2}^{-1})$ from equation (7.3.8) with $N=N+1$. (7.4.6)

Therefore the marginal occupancy for a primary group of $(N+1)$ trunks, the Laplace-Stieltjes transform at $(s=\mu)$ of the corresponding overflow stream and the variance of overflow traffic carried on a single secondary trunk are fixed for any input stream once values are specified for α_1, α_2 and α_3 .

The following quantities determined by substituting $r=2$ into equations (7.2.4) to (7.2.9) are functions of f_N and f_{N+2}

(i) $$\bar{\pi}_2 = \frac{f_N}{f_{N+2}} \tag{7.4.7}$$

(ii) $$\bar{M}_{N,2} = f_N^{-1} - f_{N+2}^{-1} \tag{7.4.8}$$

(iii) $$\bar{R}_{N,2} = \frac{f_N}{f_{N+2}-f_N} \; . \tag{7.4.9}$$

The following quantities are functions of f_N, f_{N+1} and f_{N+2}

(i) $$\psi_N(2\mu) = \frac{\Delta f_N}{\Delta f_{N+1}} \quad \text{from equation (7.2.6) with } r=2 \tag{7.4.10}$$

(ii) $$\bar{V}_{N,2} = (f_N^{-1}-f_{N+2}^{-1})\left(\frac{f_{N+2}-f_N-2f_Nf_{N+2}}{f_{N+2}(f_{N+2}-f_N)} - \frac{f_{N+1}-f_N-f_Nf_{N+1}}{f_N(f_{N+1}-f_N)}\right) \tag{7.4.11}$$

from equation (7.2.10) with $r=2$.

Equations (7.3.3) and (7.4.10) imply that the overflow streams Laplace-Stieltjes transform at $s=\mu$ and 2μ is invariant of the input stream once three overflow moment values are given. Similarly, equations (7.4.8) and (7.4.11) indicate that the mean and variance of the overflow traffic carried on a secondary group of two trunks, are independent of the form of the input stream when three overflow traffic moments are specified.

7.4.1 Dimensioning an Overflow System with Negative Exponential Input using Three Overflow Traffic Moments

It was shown in section 7.3 that f_{N+1} and f_{N+2} for any renewal input stream must satisfy equations (7.3.1) and (7.4.1) when α_1, α_2 and α_3 are given particular values.

If the input stream is now taken as being negative exponential, the resulting overflow stream's weakness from N trunks must also satisfy the recurrence relation given by equation (5.5.6), that is

f_N

$$Af_{N+1} = (N+1)F_N + 1 \,. \qquad (7.4.1.1)$$

Putting $N=N+1$, equation (7.4.1.1) becomes

$$Af_{N+2} = (N+2)f_{N+1} + 1 \,. \qquad (7.4.1.2)$$

Substituting for f_{N+1} and f_{N+2} in terms of α_1, α_2 and α_3 from equations (7.3.1) and (7.3.2), equations (7.4.1.1) and (7.4.1.2) can be expressed as

$$A = \frac{(N+1)\alpha_1^{-1}+1}{\alpha_1^{-1} + \alpha_2^{-1}} \qquad (7.4.1.3)$$

and

$$A = \frac{(N+2)(\alpha_1^{-1}+\alpha_2^{-1})+1}{\alpha_1^{-1}+2\alpha_2^{-1}+2\alpha_3^{-1}} \qquad (7.4.1.4)$$

$$\therefore \quad \frac{(N+1)\alpha_1^{-1}+1}{\alpha_1^{-1}+\alpha_2^{-1}} = \frac{(N+2)(\alpha_1^{-1}+\alpha_2^{-1})+1}{\alpha_1^{-1}+2\alpha_2^{-1}+2\alpha_3^{-1}} \,. \qquad (7.4.1.5)$$

Equation (7.4.1.5) a linear equation in N, simplifies to

$$N = \frac{(\alpha_1^{-1}+\alpha_2^{-1})^2 - (\alpha_2^{-1}+2\alpha_3^{-1})}{2\alpha_1^{-1}\alpha_3^{-1} - \alpha_2^{-2}} - 1 \,. \qquad (7.4.1.6)$$

Substituting for N using equation (7.4.1.6), equation (7.4.1.3) simplifies to

$$A = \frac{\alpha_1^{-1}(\alpha_1^{-1}+\alpha_2^{-1}) - \alpha_2^{-1}}{2\alpha_1^{-1}\alpha_3^{-1} - \alpha_2^{-2}} \,. \qquad (7.4.1.7)$$

Equation (7.4.1.7) is equivalent to Nightingale [21], page 46, equation (26).

A simple expression for the third factorial moment of the overflow traffic involving only the first two moments, A and N is obtained by subtracting equation (7.4.1.7) from (7.4.1.6), giving

$$N+1 - A = \frac{(\alpha_1^{-1}+\alpha_2^{-1})^2-(\alpha_2^{-1}+2\alpha_3^{-1})-\alpha_1^{-1}(\alpha_1^{-1}+\alpha_2^{-1})+\alpha_2^{-1}}{2\alpha_1^{-1}\alpha_3^{-1} - \alpha_2^{-2}} . \tag{7.4.1.8}$$

Equation (7.4.1.8) simplifies to

$$N+1 - A = \frac{\alpha_2^{-1}(\alpha_1^{-1}+\alpha_2^{-1}) - 2\alpha_3^{-1}}{2\alpha_1^{-1}\alpha_3^{-1} - \alpha_2^{-2}} . \tag{7.4.1.9}$$

Rearrangement gives

$$\alpha_3^{-1} = \frac{1}{2}\,\frac{\alpha_2^{-2}[N+2-A]+\alpha_1^{-1}\alpha_2^{-1}}{1+\alpha_1^{-1}(N+1-A)} . \tag{7.4.1.10}$$

The closed form expressions for A and N, given by equations (7.4.1.6) and (7.4.1.7) can also be obtained by solving the explicit expressions (5.5.29) and (5.5.31) which give α_2 and α_3 as functions of A, N and α_1. The existence of such closed form expressions implies that an exact dimensioning procedure exists when the input stream is assumed to be negative exponential and three overflow traffic moments have known values.

7.5 Equivalent Non Random Method

The Equivalent Non Random (E.N.R.) method is a departure from conventional dimensioning procedures for smooth overflow systems, for which the peakedness of the overflow traffic < 1. The approach of Bretschneider [3] and Nightingale [21] for dimensioning such overflow systems is to assume a negative exponential input stream, (although it necessarily must be smooth by equation (6.4.12), and extend the range of the equivalent primary group size to include negative real values. The E.N.R. method assumes the input stream to be Erlang with sufficiently large phase for the specified smoothness of the overflow traffic. The particular Erlang distributions considered in this section are E_3, E_6, E_{10}, and E_∞ although the procedure can be applied to any given renewal stream. The intensity of the assumed Erlang stream is made identical to that of a negative exponential stream, parameter λ, see section 6.5.

The mean overflow traffic, M_N, produced from any Erlang input stream satisfies equation (5.3.2a) with $n=0$, giving

$$M_N = \alpha_1 = [f_0 \sum_{r=0}^{N} \binom{N}{r} \ell_r(\mu)]^{-1} \tag{7.5.1}$$

where f_0 and $\ell_r(\mu)$ satisfy equations (6.5.1) and (6.5.4) respectively.

The variance, V_N, corresponding to this overflow traffic is determined from equation (5.3.16), that is

$$V_N = M_N^2 [\frac{1}{M_N - M_{N+1}} - 1] . \tag{7.5.2}$$

Values of M_N and V_N corresponding to a given input stream are computed for ranges of values of both A and N. The tabulated results are then used to produce dimensioning charts of M_N versus V_N. Each specified input stream has a corresponding chart. Figures 7.5.1 to 7.5.4 are examples of these E.N.R. charts for E_1, E_3, E_{10}, D input streams. Two families of curves, illustrated in figures 7.5.5 and 7.5.6, are superimposed to produce these charts. The family of curves depicted in figure 7.5.5 corresponds to increasing the input stream's intensity for fixed N values whereas figure 7.5.6 illustrates the effect of increasing the primary group size for fixed A values.

Once an E.N.R. chart has been produced for any particular renewal input stream, the values of A and N which correspond to the specified values of M_N and V_N can be determined from the chart.

Wilkinson pioneered the dimensioning of overflow systems by charts, see Wilkinson [41] p.448-451 . The original 'Wilkinson' chart was produced from equations (5.5.5) and (5.5.30) for the mean and variance of the overflow traffic from a negative exponential input stream. Using the 'Wormald' chart, Bretschneider [3] and Nightingale [21] extended Wilkinson's concept of a dimensioning chart for a negative exponential input stream, by permitting negative N values. The assumption of an Erlang input stream of sufficiently large phase is basic to the E.N.R. method. This guarantees a positive number of equivalent primary trunks, irrespective of the degree of smoothness specified by the given M_N and V_N values.

7.5.1 Features of the E.N.R. Charts

(i) All the charts have a common shape. The invariance of the marginal occupancy for a mean overflow traffic of 7.7 is illustrated in figure (7.5.7) when F is E_3. This property has been shown in section 7.4 to hold for any renewal input stream.

(ii) As the Erlang phase k increases, the peakedness of the overflow traffic for a heavy input stream, decreases. This property was demonstrated in section 6.4. Table (7.5.1) gives the peakedness, taken as the reciprocal of the slope of the curve corresponding to one trunk, for different input streams.

Phase k	1	2	3	6	10	∞
Overflow peakedness	1.0098	.7569	.6714	.58706	.5533	.5015

Table (7.5.1)

(iii) A linearity condition relating the mean and variance of the overflow traffic is shown in Figure (7.5.8), and can be expressed by

$$M_{1+k}(A+k) - M_1(A) = s_A[V_{1+k}(A+k) - V_1(A)] \; . \qquad (7.5.1)$$

The slope of such lines is a function of the input intensity A. Table (7.5.2) shows the decrease in slope with increasing input intensity calculated for $F = E_{10}$. Such tables can be calculated for every E.N.R. chart.

A	8	9	10	11	12	13	14
s_A	.1459	.12811	.11402	.10266	.0923	.08544	.0739

Table (7.5.2)

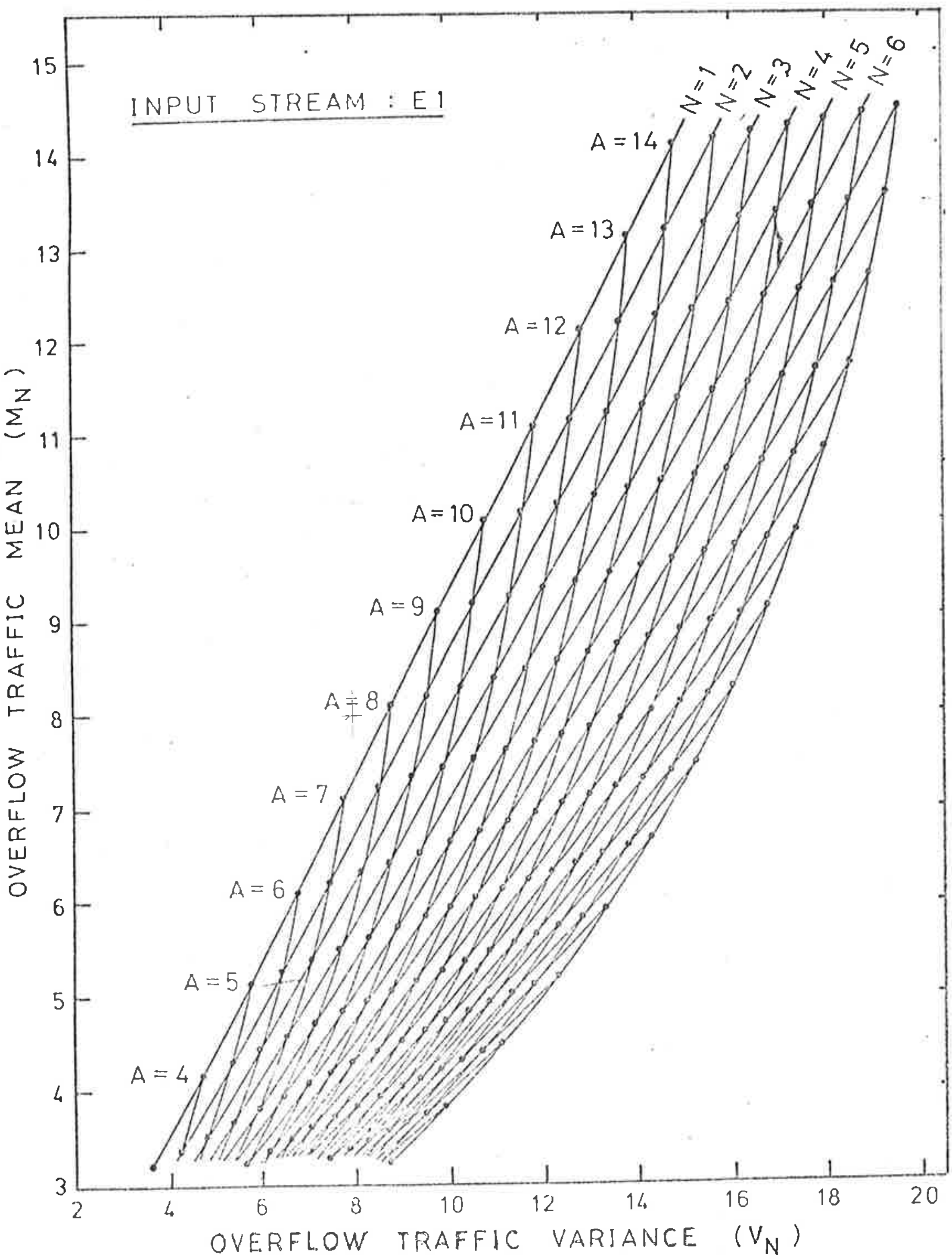
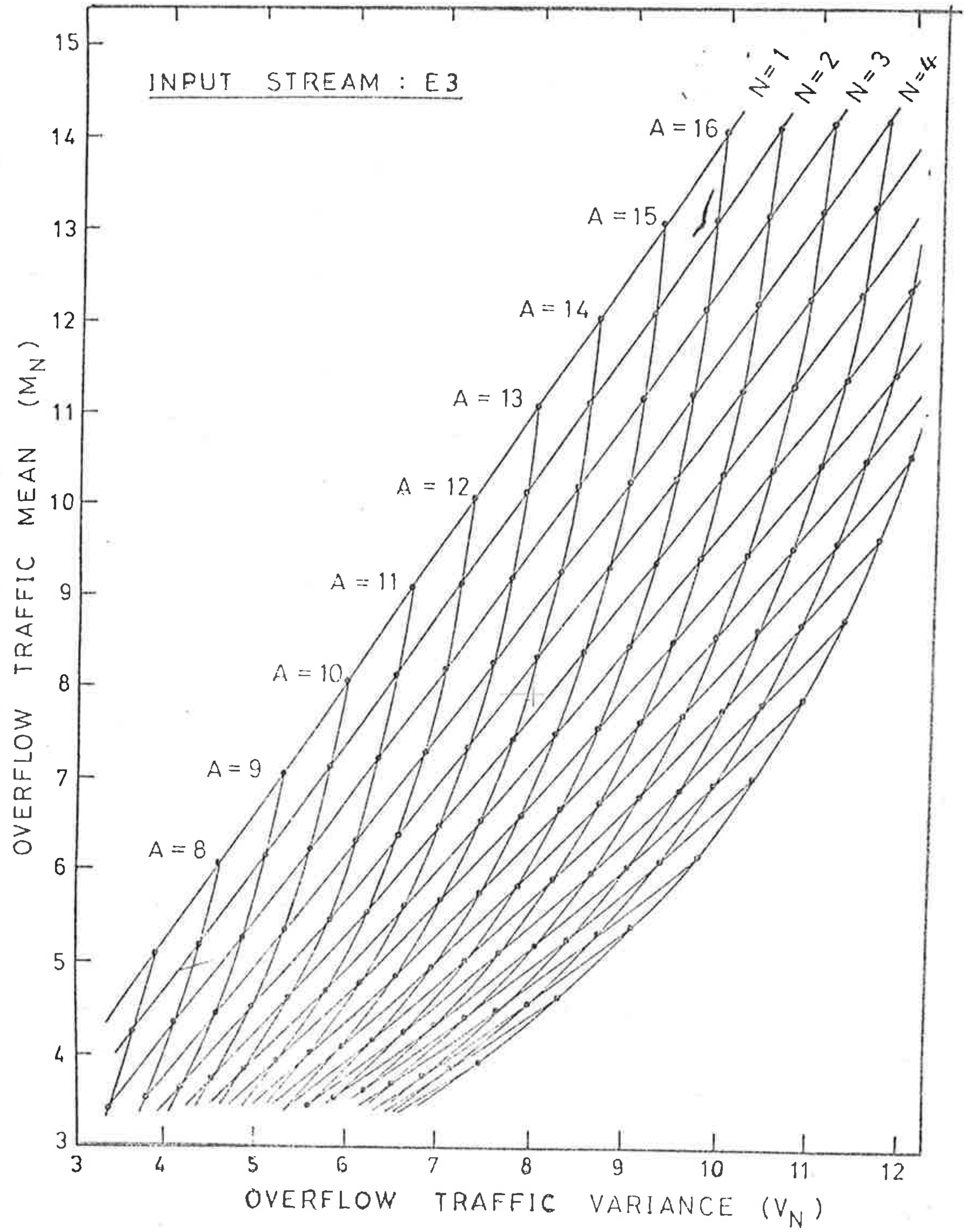

INPUT STREAM : E1
OVERFLOW TRAFFIC MEAN (M_N)
OVERFLOW TRAFFIC VARIANCE (V_N)
A = 4
A = 5
A = 6
A = 7
A = 8
A = 9
A = 10
A = 11
A = 12
A = 13
A = 14
N = 1
N = 2
N = 3
N = 4
N = 5
N = 6
3
4
5
6
7
8
9
10
11
12
13
14
15
2
4
6
8
10
12
14
16
18
20

FIGURE 7.5.1



INPUT STREAM : E3
N=1
N=2
N=3
N=4
A = 16
A = 15
A = 14
A = 13
A = 12
A = 11
A = 10
A = 9
A = 8
OVERFLOW TRAFFIC MEAN (M_N)
OVERFLOW TRAFFIC VARIANCE (V_N)
3
4
5
6
7
8
9
10
11
12
13
14
15

FIGURE 7.5.2

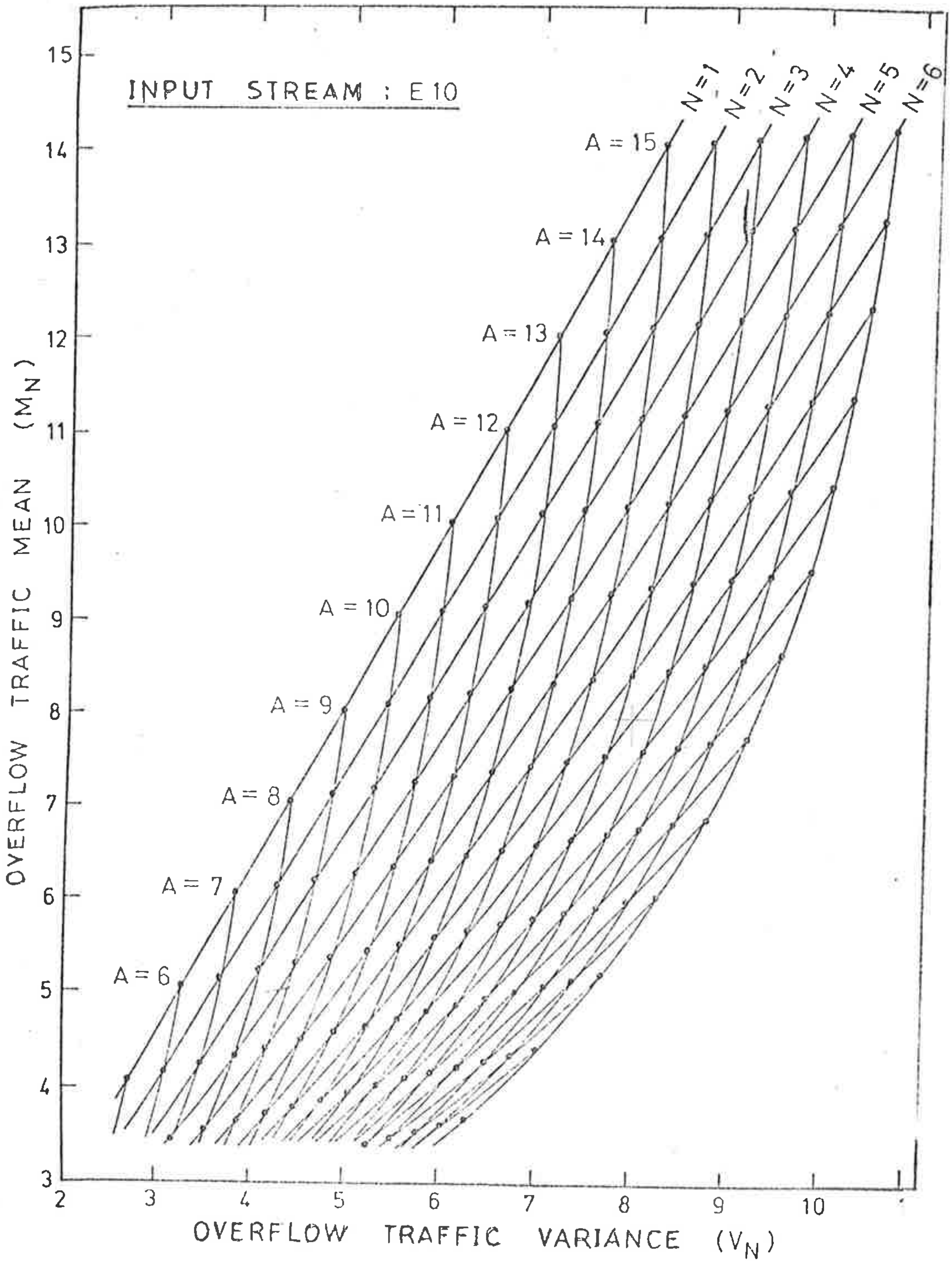

INPUT STREAM : E10
N=1
N=2
N=3
N=4
N=5
N=6
A = 15
A = 14
A = 13
A = 12
A = 11
A = 10
A = 9
A = 8
A = 7
A = 6
OVERFLOW TRAFFIC MEAN (M_N)
OVERFLOW TRAFFIC VARIANCE (V_N)
3
4
5
6
7
8
9
10
11
12
13
14
15
2
3
4
5
6
7
8
9
10

FIGURE 7.5.3

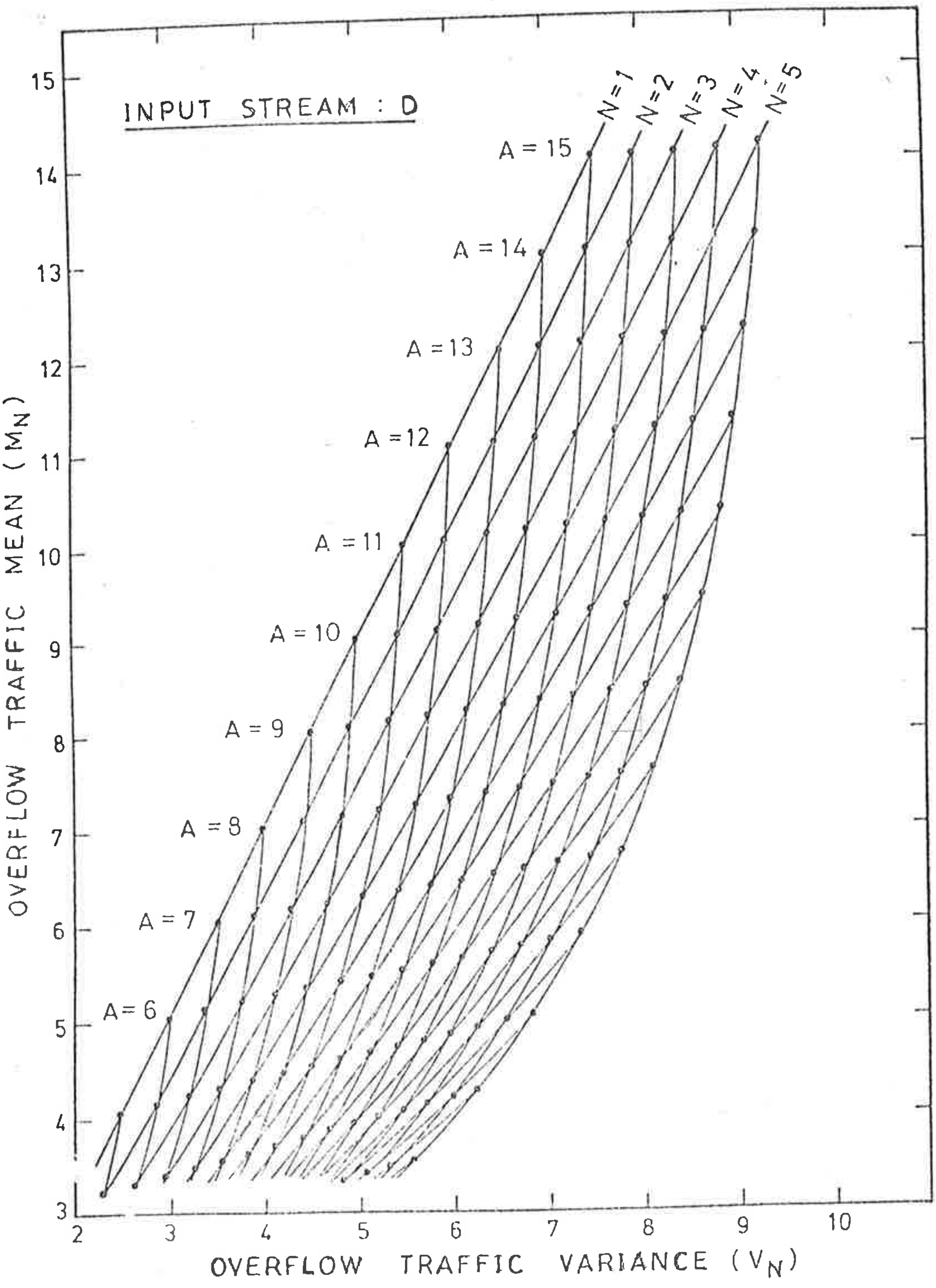

INPUT STREAM : D
N=1
N=2
N=3
N=4
N=5
A = 15
A = 14
A = 13
A = 12
A = 11
A = 10
A = 9
A = 8
A = 7
A = 6
15
14
13
12
11
10
9
8
7
6
5
4
3
2
3
4
5
6
7
8
9
10
OVERFLOW TRAFFIC MEAN (M_N)
OVERFLOW TRAFFIC VARIANCE (V_N)

FIGURE 7.5.4

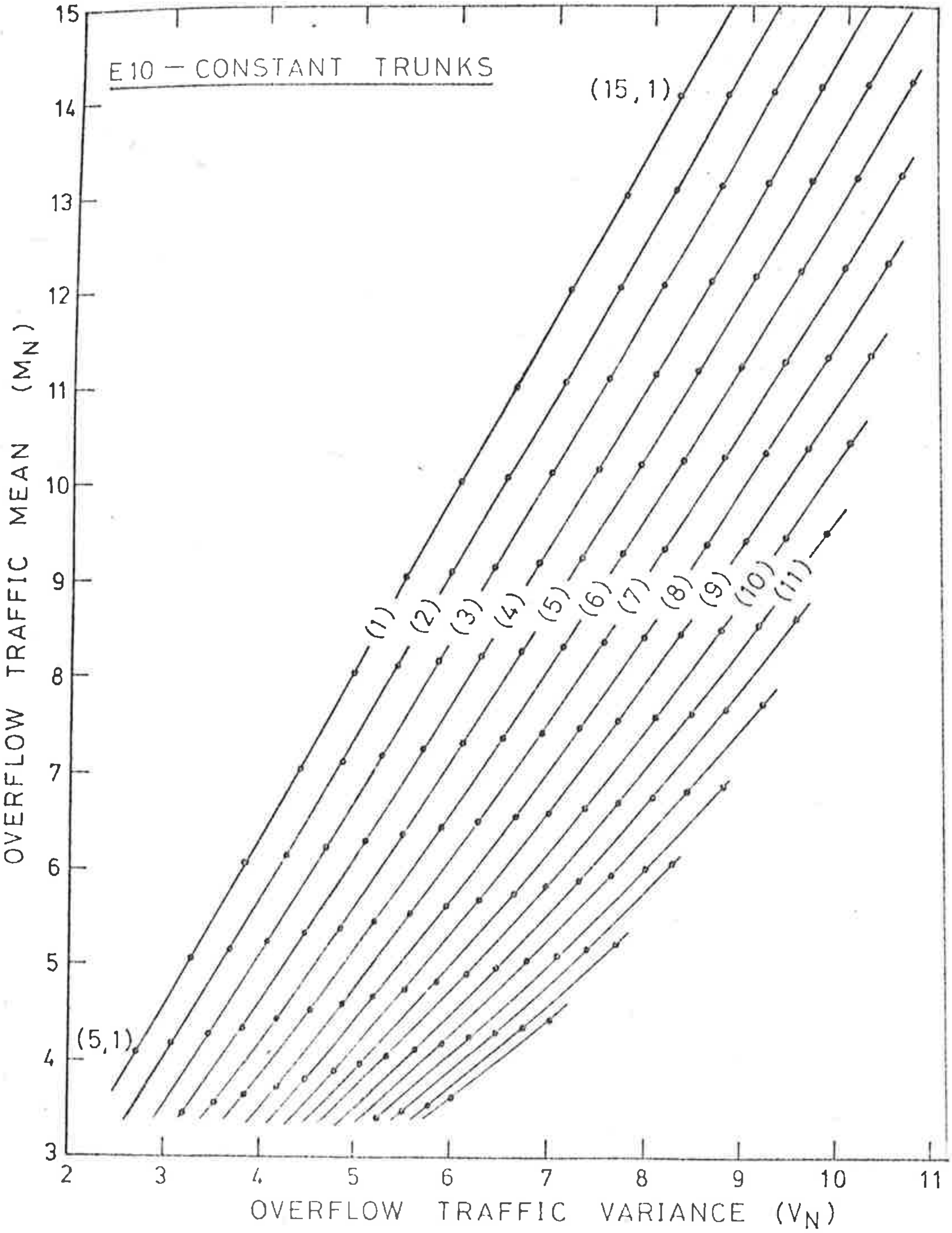

E10 — CONSTANT TRUNKS
(15,1)
(5,1)
(1)
(2)
(3)
(4)
(5)
(6)
(7)
(8)
(9)
(10)
(11)
OVERFLOW TRAFFIC MEAN (M_N)
OVERFLOW TRAFFIC VARIANCE (V_N)
15
14
13
12
11
10
9
8
7
6
5
4
3
2
3
4
5
6
7
8
9
10
11

FIGURE 7.5.5

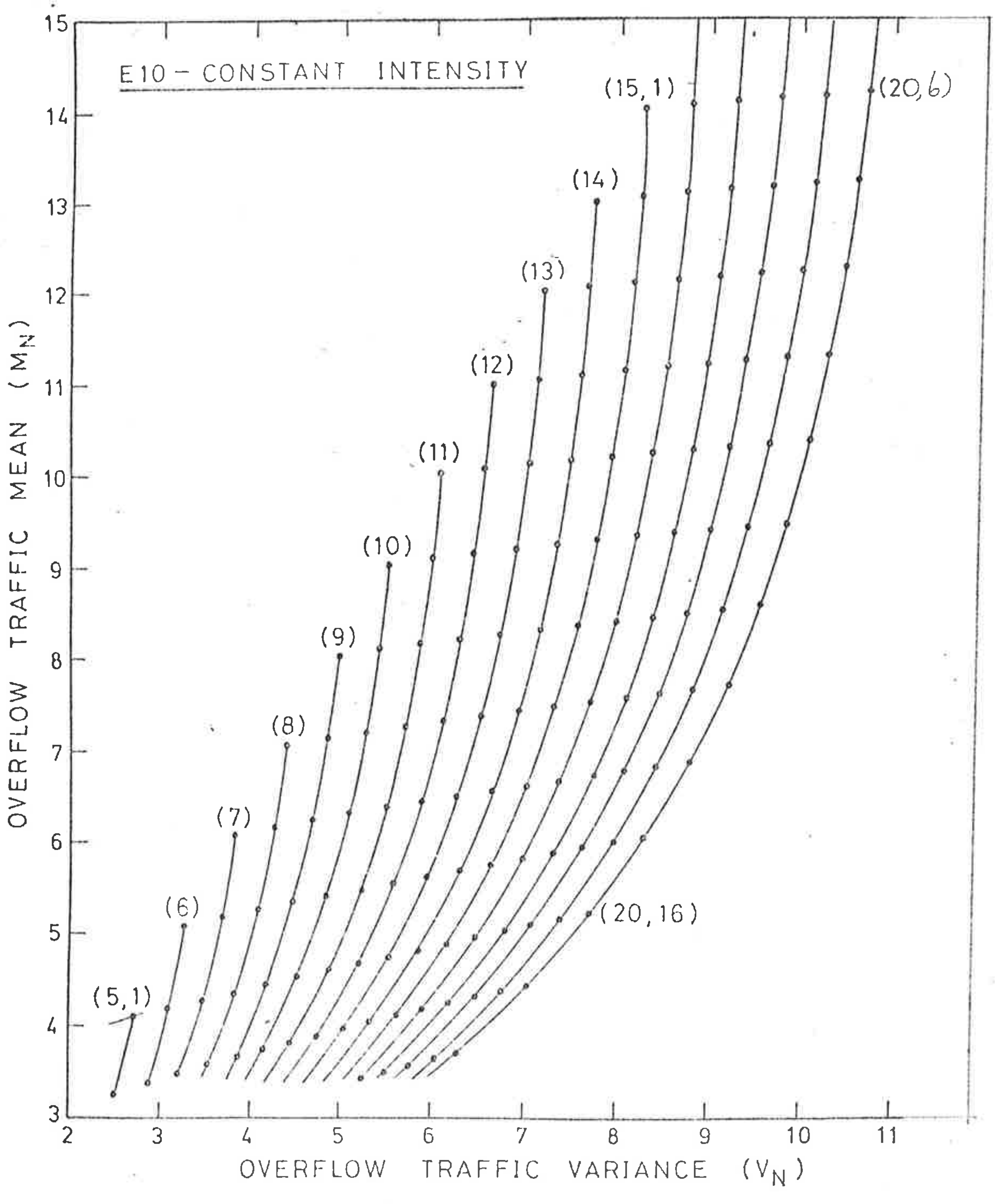

E10 - CONSTANT INTENSITY
OVERFLOW TRAFFIC MEAN (M_N)
OVERFLOW TRAFFIC VARIANCE (V_N)
(5,1)
(6)
(7)
(8)
(9)
(10)
(11)
(12)
(13)
(14)
(15,1)
(20,6)
(20,16)
3
4
5
6
7
8
9
10
11
12
13
14
15
2
3
4
5
6
7
8
9
10
11

FIGURE 7.5.6

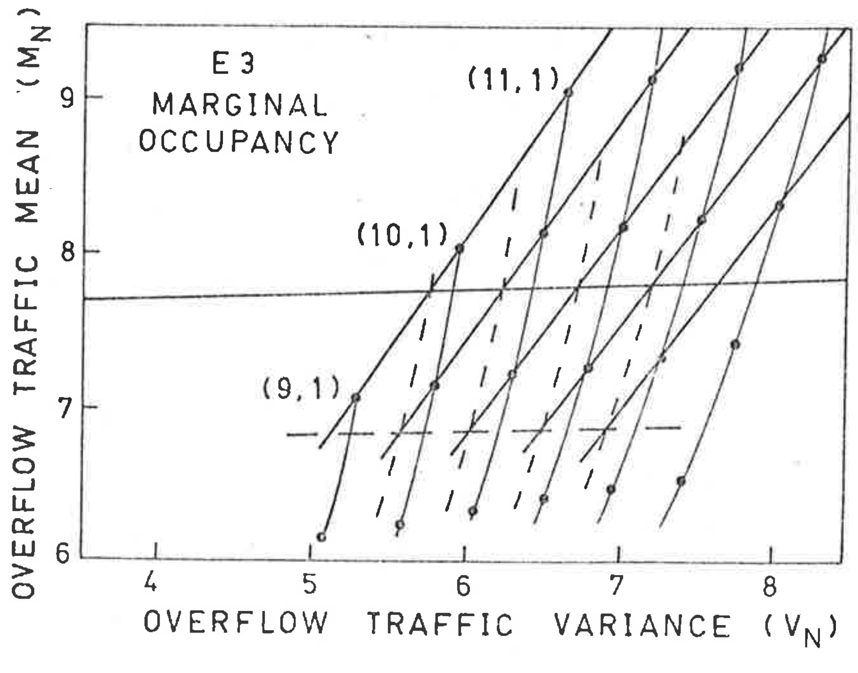

E 3
MARGINAL
OCCUPANCY
(11, 1)
(10, 1)
(9, 1)
9
8
7
6
4
5
6
7
8
OVERFLOW TRAFFIC MEAN (M_N)
OVERFLOW TRAFFIC VARIANCE (V_N)

FIGURE 7.5.7

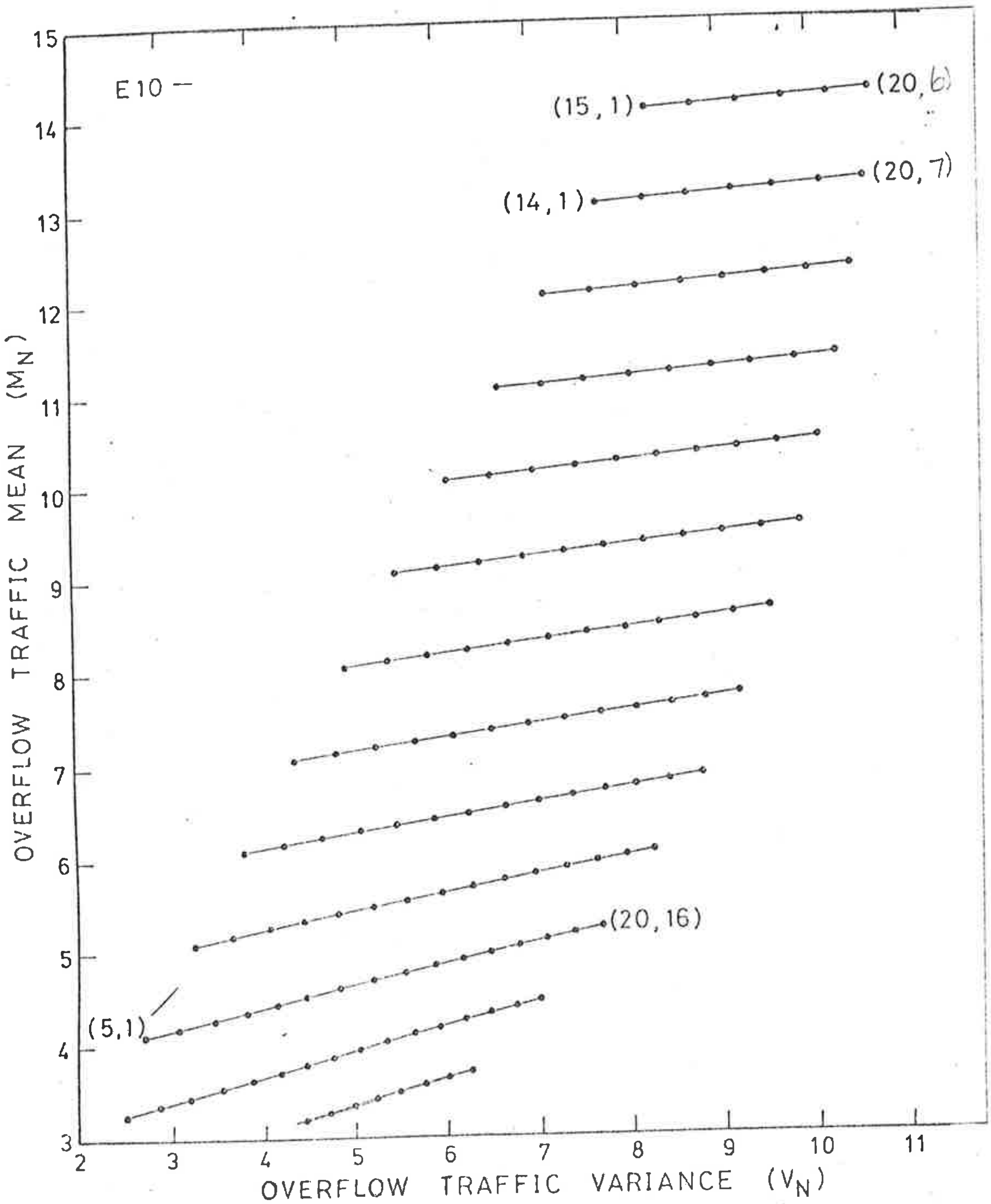

E10 —
(15, 1)
(20, 6)
(14, 1)
(20, 7)
(20, 16)
(5,1)
OVERFLOW TRAFFIC MEAN (M_N)
OVERFLOW TRAFFIC VARIANCE (V_N)
3
4
5
6
7
8
9
10
11
12
13
14
15
2
3
4
5
6
7
8
9
10
11

FIGURE 7.5.8

CHAPTER VIII

CONCLUSIONS

A new intensity measure for a renewal stream, that of weakness, was used extensively as a basic descriptor for many overflow traffic quantities throughout this thesis. Expressions for the renewal overflow stream's Laplace-Stieltjes transform and the binomial moments of the steady state q, $\bar{q}$, π, $\bar{\pi}$, occupancy distributions for a group of trunks were obtained by means of a unified queueing methodology.

The 'group' and 'atomic' views of renewal overflow systems enabled features of overflow traffic to be examined. The 'group' approach, with the overflow traffic factorial moment theorem, resulted in explicit formulae for all offered and carried overflow traffic moments as well as the related statistical quantities of peakedness and coefficient of variation. These formulae were expressed either as functions of divided differences of the overflow stream's weakness or by equivalent functions of the input stream's Laplace-Stieltjes transform.

Properties of the peakedness and the coefficient of variation of overflow traffic were examined by means of the 'atomic' approach.

Features of graphs produced for various Erlang input streams, provided insight into possible characteristics of the peakedness of overflow traffic. One avenue which might be pursued in future research is that of an analytic study on the existence, uniqueness and value of a maximum overflow traffic peakedness.

The inversion of the explicit overflow traffic moment formulae was fundamental to the examination of the effect on general overflow systems when a finite number of overflow traffic moments have specified values. One consequence for common dimensioning models, which are based on two overflow traffic moments, was that the marginal occupancy is completely determined by the two moment values irrespective of the form of the input stream. An exact dimensioning procedure, using three overflow traffic moments, was established for random input traffic. The existence and simplicity of such formulae question the accepted convention of basing dimensioning procedures on two moments. A study on the implications and permitted ranges for overflow traffic moment values on the possible form of an input stream is a possible extension of this work.

The basis and formulation of the E.N.R. dimensioning procedure was discussed. To illustrate E.N.R. dimensioning charts, various Erlang streams were chosen to typify possible smooth input streams. Common features of shape and linearity as well as properties of peakedness and marginal occupancy were characteristics of these charts. Future analytic research on underlying mathematical structures evidenced by these charts might provide insight into their common shape and linearity conditions.

APPENDIX I

FINITE DIFFERENCE CALCULUS

It is appropriate to introduce the following notation. Let E be an operator, used in Finite Difference Calculus, see Wylie [43], p. 132 to advance the argument of a function by one, i.e.

$$E^k f_N = f_{N+k} \qquad \text{(I.1)}$$

where

$$(E-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} E^k \ . \qquad \text{(I.2)}$$

The forward difference operator Δ, is defined by

$$\Delta^n = (E-1)^n \qquad \text{(I.3)}$$

and the n^{th} divided difference function $X(N)$, denoted by $X(N,N+1,\ldots,N+n)$ satisfies

$$X(N,N+1,\ldots,N+n) = \frac{\Delta^n}{n!} X(N) \ . \qquad \text{(I.4)}$$

APPENDIX II

MOMENTS OF OVERFLOW CARRIED TRAFFIC

Carried Overflow Moments in Terms of the ℓ_r^s

$$\bar{\alpha}_{n\pi}(N) = n!F(N,L,n)\left[f_0 \sum_{r=0}^{N} \binom{N}{r} \ell_{n+r}(\mu)\right]^{-1} \quad (II.1)$$

$$\bar{\alpha}_{nq}(N) = n!F(N,L,n)\left[nf_0^2 \sum_{r=0}^{N} \binom{N}{r} \ell_r(\mu) \sum_{r=0}^{N} \binom{N}{r} \ell_{n-1+r}(\mu)\right]^{-1} \quad (II.2)$$

$$\bar{\beta}_{n\pi}(N) = F(N,L,n)\left[f_0 \sum_{r=0}^{N} \binom{N}{r} \ell_{n+r}(\mu)\right]^{-1} \quad (II.3)$$

$$\bar{\beta}_{nq}(N) = F(N,L,n)\left[nf_0^2 \sum_{r=0}^{N} \binom{N}{r} \ell_r(\mu) \sum_{r=0}^{N} \binom{N}{r} \ell_{n-1+r}(\mu)\right]^{-1} \quad (II.4)$$

$$\bar{\theta}_{n\pi}(N) = \sum_{k=1}^{n} \sigma_{n,k} k!F(N,L,k)\left[f_0 \sum_{r=0}^{N} \binom{N}{r} \ell_{k+r}(\mu)\right]^{-1} \quad (II.5)$$

$$\bar{\theta}_{nq}(N) = \sum_{k=1}^{n} \sigma_{n,k} k!F(N,L,k)\left[kf_0^2 \sum_{r=0}^{N} \binom{N}{r} \ell_r(\mu) \sum_{r=0}^{N} \binom{N}{r} \ell_{k-1+r}(\mu)\right]^{-1} \quad (II.6)$$

$$\bar{C}_{n\pi}(N) = \sum_{i=0}^{n} (-1)^i \bar{\theta}_{1\pi}^{n} \sum_{k=1}^{n-i} \sigma_{n-i,k} k!F(N,L,k)\left[f_0 \sum_{r=0}^{N} \binom{N}{r} \ell_{k+r}(\mu)\right]^{-1} \quad (II.7)$$

$$\bar{C}_{nq}(N) = \sum_{i=0}^{n} (-1)^i \bar{\theta}_{1q}^{n} \sum_{k=1}^{n-i} \sigma_{n-i,k} k!F(N,L,k)\left[kf_0^2 \sum_{r=0}^{N} \binom{N}{r} \ell_r(\mu) \sum_{r=0}^{N} \binom{N}{r} \ell_{k-1+r}(\mu)\right]^{-} \quad (II.8)$$

where

$$F(N,L,n) = f_N - \bar{\pi}_L \sum_{s=0}^{L-1} \binom{L}{s} \sum_{r=0}^{N} \binom{N}{r} \ell_{s+r}(\mu) \quad (II.9)$$

and

$$f_N = f_0 \sum_{r=0}^{N} \binom{N}{r} \ell_r(\mu) \quad (II.10)$$

and

$$\bar{\pi}_L = \sum_{r=0}^{N} \binom{N}{r} \ell_r(\mu) \Big/ \sum_{r=0}^{N+L} \binom{N+L}{N} \ell_r(\mu) \quad (II.11)$$

and

$$f_0 = -\mu\phi'(0).$$

REFERENCES

[1] Bateman Manuscript Project, "Higher Transcendental Functions", California Institute of Technology.

[2] Beneš, V.E., "On Trunks with Negative Exponential Holding Times Serving a Renewal Process", B.S.T.J., 38, 1959, pp. 211-215.

[3] Bretschneider, G., "Extension of the E.R. Method to Smooth Traffics", I.T.C. 7, (Stockholm), paper 411, pp. 1-9.

[4] Brockmeyer, E., "Det Simple Overflow Problem i Telefontia Ficteorien", Teleteknik, 5, No. 4, 1954, pp. 361-374.

[5] Brockmeyer, E., Halstrom, H.L., Jensen, A., "The Life and Works of A.K. Erlang", Acta Polytechnica Scandinavica "Mathematics and Computing Machinery Series, No. 6", Copenhagen, 1960.

[6] Chastang, J., "Contributions to Studies of Overflow Traffic", Electrical Communication, 38, No. 1, 1963, pp. 120-129.

[7] Cohen, J.W., "The Full Availability Group of Trunks with an Arbitrary Distribution of the Inter-Arrival Times and the Negative Exponential Holding Time Distribution", Simon Stevin Wis-en Natuurkimdig Tijdschrift, 26, 1957, pp. 169-181.

[8] Cooper, R.B., "Introduction to Queueing Theory", Toronto, Macmillan.

[9] Harris, R.J., Rubas, J., "Calculation of Overflow Traffic Moments for Full and Limited Availability Systems with Binomially-Distributed Offered Traffic", A.T.R., 8, No. 2, pp. 42-45.

[10] Holtzman, J.M., "The Accuracy of the Equivalent Random Method with Renewal Inputs", 7th I.T.C., (Stockholm), 1973, paper 414.

[11] Heffes, H. , Holtzman, J.M., "Peakedness of Traffic Carried by a Finite Trunk Group with Renewal Input", B.S.T.J., 52, 1973, pp. 1617-1642.

[12] Descloux, A., "On Markovian Servers with Recurrent Input", 7th I.T.C. (Stockholm) 1973, paper 331.

[13] Kuczura, A., "The Interrupted Poisson Process as an Overflow Process", B.S.T.J., 52, No. 3, 1973, pp. 437-449.

[14] Kuczura, A., Neal, S.R., "The Accuracy of Call Congestion Measurements for Loss Systems with Renewal Input", B.S.T.J., 51, No. 10, 1972, pp. 2197-2208.

[15] Kuczura, A., "Queues with Mixed Renewal and Poisson Inputs", B.S.T.J., 51, No. 6, 1972, pp. 1305-1326.

[16] Freeman, A.H., "The Accuracy of Overflow Traffic Models", A.T.R., 11, No. 1, 1977, pp. 98-101.

[17] Jensen, E., "On Arrival Processes for Overflow Calls", 7th I.T.C., (Stockholm), 1973, paper 416.

[18] Khintchine, A.Y., "Mathematical Methods in the Theory of Queueing", Griffins Statistical Monographs and Courses, No. 7, Edited by Stewart, A.

[19] Mina, R.R., "The Rigorous Calculation of the Blocking Probability and its Application in Traffic Measurement", 8th I.T.C. (Melbourne), 1976, paper 263.

[20] Neal, S.R., "The Equivalent Group Method for Estimating the Capacity of Partial Access Service Systems which Carry Overflow Traffic", B.S.T.J., 51, No. 3, 1972, pp. 777-783.

[21] Nightingale, D.T., "Computations with Smooth Traffics and the Wormald Chart", A.T.R., 11, No. 1, 1977, pp. 45-51.

[22] Palm, C., "Research on Telephone Traffic Carried by Full Availability Groups", Tele, 1, 1957.

[23] Pearce, C.E.M., "On the Peakedness of Primary and Secondary Processes", A.T.R., 12, No. 2, 1978, pp. 18-24.

[24] Pearce, C.E.M., "Teletraffic Theory and the Renewal Paradox", OR' 78, North-Holland Publishing Company (1979), 1092-1103.

[25] Pearce, C.E.M., Potter, R.M., "Some Formulae Old and New for Overflow Traffic in Telephony", A.T.R., 11, No. 1, 1977, pp. 92-97.

[26] Potter, R.M., "Explicit Formulae for All Overflow Traffic Moments of Kosten and Brockmeyer Systems with Renewal Input", A.T.R., 13, No. 2, 1979, pp.

[27] Potter, R.M., "The Equivalent Non Random Method and Restrictions Imposed on Renewal Overflow Systems by the Specification of a Finite Number of Overflow Traffic Moments", 9th I.T.C. (Torremolinos), 1979, paper 247.

[28] Pratt, C.W., "The Concept of Marginal Overflow in Alternate Routing", A.T.R., 1 and 2, 1967.

[29] Riordan, J., "Derivation of Moments of Overflow Traffic", in "Theories for Toll Traffic Engineering in the U.S.A.", by R.I. Wilkinson, B.S.T.J., 35, No. 2, 1956, pp. 87-91.

[30] Rapp, Y., "Application of Wilkinson's Equivalent Random Theory to Numerical Calculation", Ericsson Technics, No. 1, 1964, pp. 113-130.

[31] Rubas, J., "Dimensioning of Alternative Routing Networks Offered Smooth Traffic", A.T.R., 11, No. 1, 1977, pp. 38-44.

[32] Schehrer, R., "On Higher Order Moments of Overflow Traffic behind Groups of Full Access", 8th I.T.C., Melbourne, 1976, paper 422.

[33] Schehrer, R., "On the Exact Calculation of Overflow Systems", 6th I.T.C., (Munich), 1970, paper 147.

[34] Syski, R., "Congestion Theory in Telephone Systems", Edinburgh and London, Oliver and Boyd.

[35] Szegö, G., "Orthogonal Polynomials", New York, 1939.

[36] Takács, L., "Introduction to the Theory of Queues", Oxford University Press, New York, 1962.

[37] Takács, L., "On the Limiting Distribution of the Number of Coincidences Concerning Telephone Traffic", Annals of Math. Stats., 30, 1959, pp. 134-141.

[38] Takács, L., "On Fluctuation Problems in the Theory of Queues", Advanced Applied Probability, 8, 1976, pp. 548-583.

[39] Telecom Australia, "A Course in Teletraffic Engineering", Planning Services Branch, 1978, Melbourne.

[40] Wallstrom, B., "Congestion Studies in Telephone Systems with Overflow Facilities", Ericsson Technics, 22, No. 3, 1966, pp. 202-237.

[41] Wilkinson, R.I., "Theories for Toll Traffic Engineering in the U.S.A.", B.S.T.J., 35, No. 2, 1956.

[42] Winsten, C.B., "Geometric Distributions in the Theory of Queues", J. Roy. Statist. Soc. Vol. 21, 1959, pp. 1-35.

[43] Wylie, C.R., "Advanced Engineering Mathematics", McGraw-Hill, New York, Toronto, London, pp. 130-139.