GENERALISATIONS OF MINKOWSKI'S THEOREM IN THE PLANE

by

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For my late friend Simeon,

"G'Bye Boo".

The results presented in this thesis are my own except where due acknowledgement is given in the text.

John R. Arkinstall

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CONTENTS

ABSTRACT

Several Generalisations of Minkowski's Convex Body Theorem, in the Euclidean Plane, are obtained. In the first, the perimeter of an 0-symmetric convex set containing no interior lattice points besides 0 is used as an additional parameter in bounding the area of the set. Secondly, the number of chords of a convex set bisected by 0 is introduced as a measure of symmetry of the set about 0. With this measure of symmetry, a simultaneous generalisation of both Minkowski's Theorem and Ehrhart's Theorem is obtained. Similar generalisations are shown for a number of related symmetry conditions. Finally, the same symmetry measure is used to extend the Minkowski-Van der Corput Theorem in the plane, for sets containing at most four interior lattice points.

0. PRELIMINARIES

0.1 Introduction

A great deal of work has been done, by various authors, in the search for extensions of Minkowski's convex body theorem, and in the general area of convex bodies and lattice points. The only results of other authors which appear in this thesis are those for which I have particular application. These results form the principal content of Chapter 1. I have made no attempt to give a general survey of all known results which extend Minkowski's theorem. To justify this omission I first note that a number of good surveys have recently been published, for example:

J. Hammer: Unsolved Problems Concerning Lattice Points, Research Notes in Mathematics. No. 15, London, Pitman, 1977.

Peter M. Gruber: Geometry of Numbers, Contributions to Geometry: Proceedings of the Geometry Symposium in Siegen 1978, Birkhaüser Verlag Basel 1979 pp. 186-224.

I secondly note that the type of result I have found of most use often has little directly to do with Minkowski's Convex Body Theorem, but is simply a result of the Euclidean plane. To attempt to supply an exhaustive list of recent results in so general an area would clearly be an enormous task largely irrelevant to the exposition of this thesis.

In Chapter 2, we consider the following type of set. Let K be a convex body in the Euclidean plane which is centrally symmetric about 0, the only point of the integer lattice in its interior. Minkowski's Theorem asserts that the area of K, $A(K)$ is at most four. We consider the perimeter $P = P(K)$ of such a set K as an additional parameter. A tighter bound than four is deduced for A(K), in terms of P for those values of P up to a certain value. The result is best possible, as for each value of P we exhibit a set whose area is this tighter bound.

This work was motivated by a conjecture of my supervisor, Dr. P. Scott [11], and I received a good deal of initial assistance from him. The principal results of Chapter 2 have been published in the following article, with P. Scott as co-author

An Isoperimetric Problem with Lattice Point Constraints,
Journal of the Australian Math. Soc. Ser. A, 27(1979), 27-36.

Simultaneous with the publication of this work, H.T. Croft [4], published an article which also confirmed Scott's conjecture [11]. The articles by myself and Croft are similar in many respects. However, neither yields a method which can be applied to the same problem for a general 2-dimensional lattice, nor in higher dimensions. For a general 2-dimensional lattice, I have been able by geometrical methods, as opposed to the analytic methods of Chapter 2, to obtain partial results, alluded to in Conjecture 2.10.1. However, even the simplest 3-dimensional problem, with a cubic lattice in E^3, presents an intractable problem, as little is known of the surfaces involved in extreme cases, other than that they have constant mean curvature.

In Chapter 3, we introduce a measure of symmetry of a convex set K which contains 0 as an interior point. We let s(K) be the number of chords of K through 0 which are bisected by 0. Heuristically, a large value of s(K) indicates that either K is roughly centrally symmetric (if the chords involved are evenly spread about) or that K has a roughly symmetric section of boundary (if the chords involved are clustered). We show that if 0 is the only lattice point in the interior of K, and if $s(K) \geqslant 2$, then the area of K is bounded. Specifically, if $s(K) = 2$ or $s(K) \geqslant 4$, then $A(K) \leqslant 4$, while if $s(K) = 3$ then $A(K) \leqslant 4\frac{1}{2}$.

The principal results of Chapter 3 appear in print in the following article:

Minimal Conditions for Minkowski's Theorem in the Plane I, Bulletin of the Australian Mathematical Society (22) 1980, 259-274.

In Chapter 4, we introduce a number of measures of symmetry which we show are related to $s(K)$. These measures of symmetry are the number of chords of K through 0 which partition evenly the area of K, the perimeter of K, the number of chords through 0 midway between parallel supporting lines of K (midchords), the number of reflection axes of K through 0, and the number of rotation symmetries of K. For each of these measures, results similar to the results of Chapter 3 are proved, although with perimeter symmetry our proof requires an extra condition. Most results from this chapter have appeared in part II of the article above, in the same volume, pages 275-284.

In Chapter 5, we generalise the Minkowski-Van der Corput Theorem. In the plane this result states that a convex set K which is centrally symmetric about 0 and which contains in its interior t (necessarily t is even) integer lattice points besides 0, has area at most $2(t+2)$. We again use the measure of symmetry $s(K)$ for a convex set K containing 0 as an interior point and t other points of the integer lattice besides 0 in its interior, where $t \leqslant 3$. We show that if $s(K) = 2$ or $s(K) \geqslant 4$ then $A(K) \leqslant 2(t+2)$ and that if $s(K) = 3$ then

$A(K) \leqslant 2(t+2) + (2(t+1))^{-1}$.

The methods employed in this chapter form a natural extension of methods developed in Chapter 3. Unfortunately, many of the results needed in this extension are proved in a manner which is less than elegant. While the arguments in this section on paper appear to be no more than endless casesplitting and argument by contradiction, when translated to sketches of the lattice polygons concerned they are very simple. It is my belief that the restriction $t \leqslant 3$ is spurious to the main result of this chapter. Confirmation of this belief will no doubt require a significant improvement upon the methods I have used in section 5.5.

In each of Chapters 3, 4 and 5, the proof techniques used employ a number of results which are peculiar to the plane and not to higher dimensional spaces. For this reason, I have made no effort to suggest higher dimensional analogues for either the symmetry measure $s(K)$ or for the results involving this measure of symmetry. That is not to say that I disbelieve that such analogues exist. However, given the historical resistance to proof of some conjectures, for example the conjectured extensions of Erharts Theorem to higher dimensions, I consider it imprudent to propose conjectures in this area.

0.2 Basic Definitions

In order to avoid any possible confusion of ideas, we give in this section some basic definitions, and an indication of the significance of some of the properties of the terms defined. All the following terms are set in Euclidean n-space, E^n.

A convex set is a subset of E^n which contains all points of all segments with endpoints belonging to the set.

We say a set K is centrally symmetric about the point A if whenever $X \in K$ then $2A - X \in K$. In particular K is symmetric about the origin 0, or is 0-symmetric, exactly when $X \in K$ implies $-X \in K$.

The integer lattice Λ_0 is the set of points of E^n all of whose coordinates are integers. In particular Λ_0 is a discrete subgroup of the additive group E^n, and we define a general lattice Λ in E^n to be any such discrete subgroup of E^n, with rank n. It is readily shown that any lattice Λ is an image of Λ_0 under a nonsingular linear transformation, and that the determinants of all such transforms from Λ_0 to Λ have the same modulus.

This is called the determinant of the lattice Λ, $\det(\Lambda)$, and clearly measures the volume (area) of a fundamental cell of the lattice Λ, the image of the unit volume with vertex 0 and adjacent edges of unit length along the coordinate axes. As the affine transformations of E^n form a group, many results in this thesis which can be expressed in terms invariant under the group can then be applied to any lattice.

In particular $A(K)/\det(\Lambda)$ and the ratio in which a line segment is divided by a point are invariant when an affine transformation is applied to map a set K and lattice Λ to a new set and lattice. This result is employed often in the text, particularly throughout Chapters 3, 4 and 5, to claim that results hold generally, for all planar lattices.

A second important group of transformations is the group of integral unimodular transformations, affine transformations whose determinant is ± 1, whose coefficients are integral, and whose translation component is integral. This group has the important property of being the automorphism group of the integer lattice Λ_0. In many instances, when proving results about the integer lattice Λ_0, we employ the group of integral unimodular transformations to make simplifying assumptions about the properties being demonstrated. This is particularly so in section 5.5.

Many further terms, and particularly geometric properties of convex bodies in Euclidean spaces, are employed in the text. We introduce these terms in the body of the text as needed.

CHAPTER 1

Minkowski's Theorem and its Generalisations

1.1 Statements of Minkowski's Theorem

The following six results are each statements of the Fundamental Theorem of Minkowski, (1896) [9], the last three as generalised by Van der Corput, (1935) [3]. We give these six forms for ease of later reference.

Theorem 1.1.1 (Minkowski): A bounded 0-symmetric convex body in E^n with volume $V(K) > 2^n \det(\Lambda)$ contains at least one point $u \neq 0$ of the lattice Λ.

Theorem 1.1.2 (Minkowski): A compact 0-symmetric convex body in E^n with volume $V(K) \geqslant 2^n \det(\Lambda)$ contains at least one point $u \neq 0$ of the lattice Λ.

Theorem 1.1.3 (Minkowski): A compact 0-symmetric convex body in E^n containing no nonzero point of the lattice Λ in its interior, has volume $V(K) \leqslant 2^n \det(\Lambda)$.

Theorem 1.1.4 (Minkowski-van der Corput): Let K be a bounded 0-symmetric convex body in E^n of volume $V(K) > 2^n k.\det(\Lambda)$, where k is a positive integer. Then K contains at least k pairs of points $\pm u^i \neq 0$ of the lattice Λ.

Theorem 1.1.5 (Minkowski-van der Corput): Let K be a compact 0-symmetric convex body in E^n with volume $V(K) \geqslant 2^n k \det(\Lambda)$, where k is a positive integer. Then K contains at least k pairs of points $\pm u^i \neq 0$ of the lattice Λ.

Theorem 1.1.6 (Minkowski-van der Corput): Let m be a positive integer. Let K be a bounded 0-symmetric convex body containing just m points of the lattice Λ (including 0). Then $V(K) \leqslant 2^n\left(\frac{m+1}{2}\right)\det(\Lambda)$.

Theorems 1.1.1, 1.1.2 and 1.1.3 are simply special cases of Theorems 1.1.4, 1.1.5 and 1.1.6 corresponding to $k=m=1$. Theorem 1.1.6 is simply the contrapositive statement to Theorem 1.1.4 taking $m = 2k-1$. A proof of Theorems 1.1.4 and 1.1.5 can be found in Lekkerkerker [8] p. 44.

1.2 Generalisations in E^2 by Ehrhart and Scott

We now give some generalisations to Minkowski's Theorem, due to Ehrhart and Scott.

Theorem 1.2.1 (Ehrhart, [6]): A bounded convex body in E^2 with area $A(K) > 4.5 \det(\Lambda)$ and with centre of gravity 0, contains at least two points of the lattice besides 0.

Since a later result, Theorem 3.1.2, generalises Ehrhart's result, and employs a lemma due to Ehrhart used by him in the proof of Theorem 1.2.1, we sketch Ehrharts proof. We give firstly results needed for this proof.

Theorem 1.2.2 (Winternitz): If a bounded convex figure in E^2 is divided into two parts by a line ℓ through its centre of gravity G, then the ratio of the areas of the two parts always lies between the bounds 4/5 and 5/4.

Proof. A proof of this result can be found in Yaglom and Boltyanski [17], section 3.10.

Lemma 1.2.3 (Ehrhart): A bounded convex set K has at least three chords through its centre of gravity G which are bisected by G.

Proof. See Ehrhart [6].

The following result of Ehrhart [6] is used in section 3.3, and forms an essential part of the proof given there. Since the proof given in [6] is brief, and takes as read several subcases, we give a proof. This gives us an opportunity to introduce the function $d(\theta)$, much use of which is made in Chapters 3, 4 and 5.

Lemma 1.2.4 (Ehrhart): Let K be a bounded convex set in E^2 which has at least three chords bisected by a common point 0. Then K has three such chords which can be labelled successively $\overrightarrow{AB}$, $\overrightarrow{A'B'}$ and $\overrightarrow{A''B''}$ so that the semitangents to the right at the endpoints of $\overrightarrow{AB}$, to the left at the endpoints of $\overrightarrow{A'B'}$ and to the right at the endpoints of $\overrightarrow{A''B''}$, meet or are parallel.

Proof. If all the chords of K through 0 have 0 as midpoint, then K is 0-symmetric, and so all the respective pairs of semitangents either side of any chord through 0 meet or are parallel. The result is then, in this case, trivially true. Hence, with no loss of generality, we choose an axis $\overrightarrow{0X}$ such that the chord $K \cap \overleftrightarrow{0X}$ is not bisected by 0. We define the function $d(\theta)$, for $\theta \in [0,\pi]$, to be the difference of squared lengths $|B0|^2 - |0A|^2$, where the chord of K, $\overrightarrow{A0B} = c(\theta)$ makes an angle θ with $\overrightarrow{0X}$. Indeed we choose to orient $\overrightarrow{0X}$ in such a way that $d(0) < 0$. The function $d(\theta)$ is continuous on $[0,\pi]$, and clearly $d(0) + d(\pi) = 0$. The function $d(\theta)$ has at least three zeros on $(0,\pi)$, as $d(0) \neq 0$ and since $d(\theta) = 0$ if and only if $c(\theta)$ has midpoint 0.

If $d(\theta) = 0$ for all θ in the interval $(\theta_1,\theta_2) \subset (0,\pi)$, then the convex hull K' of the chords $c(\theta)$, $\theta \in (\theta_1,\theta_2)$ is a centrally symmetric set. By our first remark above, all the semitangent pairs (to K') at the endpoints of $c(\theta)$, $\theta \in (\theta_1,\theta_2)$ meet or are parallel. As a semitangent at these points to K' is also a semitangent to K, the result is proved in this case also. We therefore assume that in any subinterval of $(0,\pi)$ there is a value of θ for which $d(\theta) \neq 0$.

As $d(\theta)$ is continuous on $[0,\pi]$, and $d(0) < 0$, $d(\pi) > 0$, the value

$$\theta_1 = \sup\{x/d(\theta) \leqslant 0 \quad \forall\theta \in (0,x)\}$$

is a zero of $d(\theta)$, distinct from both 0 and π. Further, $d(\theta) > 0$ for all values of θ sufficiently close to but greater than θ_1, and so $d(\theta)$ is increasing at θ_1.

If $d(\theta) \geqslant 0$ for all $\theta \in (\theta_1,\pi)$, then the remaining zeros of $d(\theta)$, there are at least two, all correspond to extrema of $d(\theta)$. Otherwise, we let

$$\theta_2 = \sup\{x/d(\theta) \geqslant 0, \ \forall\theta \in (\theta_1,x)\}$$

which exists since $d(\theta) < 0$ for some $\theta \in (\theta_1,\pi)$, and which is clearly less than π as $d(\pi) > 0$. Clearly $d(\theta)$ is decreasing at θ_2. Similarly,

$$\theta_3 = \sup\{x/d(\theta) \leqslant 0, \ \forall\theta \in (\theta_2,x)\}$$

exists, is distinct from both θ_2 and π, and $d(\theta)$ is increasing at θ_3.

Hence we have shown that either $d(\theta)$ has three zeros at which $d(\theta)$ is successively increasing, decreasing and increasing, or $d(\theta)$

has, besides θ_1, a zero at which $d(\theta)$ is increasing, at least two zeros at which $d(\theta)$ is an extremum. We now show that the behaviour of $d(\theta)$ at its zeros determines the properties of the semitangents at the endpoints of $c(\theta)$.

We first suppose that θ_4 is a zero of $d(\theta)$ at which $d(\theta)$ is a local extremum. Let $\bar{K}$ denote the reflection of K in O. Since $d(\theta)$ is an extremum at θ_4, at the endpoints of $c(\theta_4)$ the boundaries of K and $\bar{K}$ touch internally, and so K and $\bar{K}$ share a supporting line ℓ at one of these endpoints. The lines ℓ and $\bar{\ell}$, the reflection of ℓ in O, are parallel supporting lines to K at the endpoints of $c(\theta_4)$. Therefore the semitangents to K at the endpoints of $c(\theta_4)$, on either side of $c(\theta_4)$, either meet or are parallel.

We next suppose that θ_5 is a zero of $d(\theta)$ at which $d(\theta)$ is increasing. In this case the boundaries of K and $\bar{K}$ cross at the endpoints of $c(\theta_5)$. If θ^- is just smaller than θ_5, $d(\theta)$ is nonpositive and so the endpoint B^- of $c(\theta)$ is no further from O than endpoint A^-. If θ^+ is just greater than θ_5, the opposite occurs. As the intersection of sets K and $\bar{K}$ is convex at B, the angle formed by the left semitangent t_1 to $\bar{K}$ at B and the right semitangent t_2 to K at B is no larger than π. We note that left and right are as seen from the direction of the chord $\overrightarrow{AOB} = c(\theta_5)$. Hence $\bar{t}_1$, the reflection of t_1 in O, is the right semitangent to K at A, and $\bar{t}_1$ meets or is parallel to t_2. Similarly, a zero θ_6 of $d(\theta)$ at which $d(\theta)$ is decreasing corresponds to a chord $c(\theta_6)$ at whose endpoints the semitangents to the left meet or are parallel.

If θ_1 θ_2 and θ_3 are zeros of $d(\theta)$ at which $d(\theta)$ is successively increasing, decreasing and increasing, we deduce the result by the above. Otherwise, we have shown that $d(\theta)$ is increasing at θ_1 and an extremum at at least two other zeros. Since the semitangents to either side at these other zeros meet or are parallel, we have no difficulty labelling the chords as required by the statement of the lemma.

We are now in a position to prove Ehrharts result, Theorem 1.2.1. Since this result appears in Ehrhart [6], we give a sketch only.

<u>Proof of Theorem 1.2.1:</u> By lemmas 1.2.3 and 1.2.4, K has three chords c_1, c_2, c_3 bisected by 0 the centre of gravity of K. The semitangents to the right of c_1 and c_3 at their endpoints, and to the left of c_2 meet or are parallel. Let s_i be that open part of K cut off by chord c_i to the side of c_i to which the semitangents are drawn. By Winternitz Theorem, and noting that $A(K) > 9/2 \det(\Lambda)$, we have that

$$A(s_i) \geqslant 4/9\ A(K) > 2 \det(\Lambda) \qquad i=1,2,3.$$

Let $\bar{s}_i$ denote the reflection of s_i in 0. The set $s_i \cup \bar{s}_i \cup c_i$ is a convex figure, centrally symmetric about 0, with area greater than $4 \det(\Lambda)$. By Theorem 1.1.1, it therefore contains a pair of lattice points $\pm u_i \neq 0$ in its interior, and so either both $\pm u_i$ belong to c_i or one of them, $u_i \in s_i$. If this latter condition holds for $i=1,2,$ and 3, we note that the three regions s_i, by lemma 1.2.4 have no point in common. In order for each set to contain a lattice point u_i, K must then contain at least two points of Λ. In any event then, K contains two points of Λ, and the result is proved.

We now state a result due to Scott [15], which we will employ later in much the same way the above proof employs Winternitz' Theorem.

Theorem 1.2.5 (Scott): Let A_1 A_2 and A_3 be three open, convex sets in E^2 satisfying $\bigcap_{i=1}^{3} A_i = \phi$. Then

$$A\left(\bigcup_{i=1}^{3} A_i\right) \geqslant 9/5 \min_j A(A_j)$$

and this result is best possible, and achieved only if $\bigcup_{i=1}^{3} A_i$ is a triangle, and each A_i is a trapezium bounded by the sides of the triangle and a chord through the centre of gravity of the triangle.

Proof: See Scott [15].

We now prove a result of Scott, two corollaries of which are of particular use in later chapters.

Definition 1.2.6: Let K be a closed convex body in E^n and π an intersecting hyperplane. We say that the section $K \cap \pi$ is *visible* if there is some point P exterior to π (perhaps "at infinity") from which every boundary point of $K \cap \pi$ (relative to π) is visible. We call such a point P a *point of visibility*. Further, if π partitions E^n into the two closed halfspaces π^+ and π^-, and if $V(\pi^+ \cap K) \geqslant V(\pi^- \cap K)$, then π^+ is *a heavy side of* π (relative to K).

Theorem 1.2.7 (Scott): Let K be a closed convex body in E^n containing the origin 0 and let π be a hyperplane through 0 such that $\pi \cap K$ is symmetric about 0, and $\pi \cap K$ is visible from a heavy side of π. Then if $V(K) \geqslant 2^n$, K contains a nonzero point of the integral lattice, on the heavy side of π.

Proof (Scott [13]: Let π partition K into the sets K_1 and K_2, and suppose that a point of visibility V lies on the K_1 side of π, which is a heavy side of π. Let $\bar{K}_1$ denote the reflection of K_1 in 0 and set $K^* = K_1 \cup \bar{K}_1$. By construction K^* is symmetric in 0, and $V(K^*) \geqslant 2^n$.

Let B be a boundary point of K^*. To show that K^* is convex it suffices to show that for each such point B there is a hyperplane σ which supports K^* locally (that is, if $N(B,\varepsilon)$ is a spherical ε-neighbourhood of $B(\varepsilon > 0)$, then σ supports $K^* \cap N(B,\varepsilon)$.) If $B \notin \pi$ such a hyperplane clearly exists, since K_1 and $\bar{K}_1$ are convex. Suppose then that $B \in \pi$. Let V' denote the reflection of V in 0, and let C be the double cone having common base $K \cap \pi$ and vertices V and V'. Each half of C is convex and since VV' meets $C \cap \pi$ in 0, C is itself convex. Hence at B there exists a hyperplane of support to C. Since $K^* \subseteq C$, this hyperplane also supports K^*, as required. Therefore K^* is convex.

Now, by Minkowski's Theorem 1.1.2, K^* contains a nonzero point of the integral lattice. Using the symmetry of K^*, we deduce that K_1 contains a nonzero point of the integral lattice.

Theorem 1.2.8: Let K be a closed convex body in E^n containing the origin 0 and let π be a hyperplane through 0 such that $\pi \cap K$ is symmetric about 0, and $\pi \cap K$ is visible from both sides of π. Then if $V(K) \geqslant 2^{n-1}(t+1)$, K contains at least t points of the integral lattice besides 0.

Proof: Since $\pi \cap K$ is visible from both sides of π, we can apply the method of proof of 1.2.7 to both sides K_1 and K_2 of K. As in 1.2.7, both the sets $K_1^* = K_1 \cup \bar{K}_1$ and $K_2^* = K_2 \cup \bar{K}_2$ are convex and 0-symmetric. We denote by v_1 and v_2 the volumes $V(K_1)$ and $V(K_2)$ respectively, so that $v_1+v_2 \geqslant 2^{n-1}(t+1)$. Let k_i be the integer $[2^{1-n}v_i]$. Since $V(K_i^*) = 2v_i \geqslant 2^n k_i$, by Theorem 1.1.5, K_i^* contains at least k_i pairs of lattice points $\pm u^j$, besides 0, for i=1,2. Noting the possibility that K_1^* and K_2^* may share lattice points on π, we thus deduce that K has at least k_1+k_2 lattice points [illegible], besides 0. Since $v_1+v_2 \geqslant 2^{n-1}(t+1)$, the sum $k_1+k_2 > t-1$, as the sum of the fractional parts of $2^{1-n}v_i$, i=1,2 is strictly less than 2. As k_1+k_2 is an integer, we deduce that $k_1+k_2 \geqslant t$, and so the Theorem is proved.

Theorem 1.2.9 (Scott, [12]): Let K be a closed convex set in the plane which contains 0 as an interior point. Suppose that there is a chord A0B of K which has midpoint 0, and which partitions K into two disjoint regions of equal area. Then if $A(K) \geqslant 4$, K contains a nonzero point of the integral lattice.

Proof: The chord A0B with midpoint 0 is the intersection of K with a hyperplane (line) and is symmetric about 0. As supporting lines to K drawn at each of the endpoints of A0B must meet or be parallel, the section A0B is visible from some point. As K is partitioned into two regions of equal area, it is visible from the heavy side of A0B. Therefore we can apply Theorem 1.2.5 to deduce that K contains a nonzero point of the integral lattice.

Theorem 1.2.10: Let K be a closed convex set in the plane which contains the origin 0 as an interior point. Suppose that there is a chord A0B which has midpoint 0, and that K has parallel supporting lines at A and B. Then if $A(K) \geqslant 2(t+1)$, K contains t nonzero points of the integral lattice.

Proof: As K has parallel supporting lines at A and B, the symmetric section AOB is visible from both sides of AOB. Hence the result follows from Theorem 1.2.8.

1.3 A boundedness result from Minkowski's Theorem

We now give a result of H. Cohn who employs Minkowski's Theorem to deduce a bound on any convex set K which contains a neighbourhood of the origin, but no other lattice points.

Theorem 1.3.1 (Cohn [2]): Let K be a convex body in E^d containing in its interior, the hypersphere $\sum_d(r)$ of radius r and centre at the origin $(r < 1)$. If K fails to contain in its interior those lattice points other than the origin in a second sphere $\sum_d(c_d/r^{d-1})$, then it will contain no lattice points at all except the origin. In fact K will then lie entirely within the second sphere.

Proof: See Cohn [2].

In our particular use of Theorem 1.3.1, the precise radius of the outer sphere is of no importance. We deduce a corollary to Theorem 1.3.1 suited to our needs.

Theorem 1.3.2: Let K be a convex body in E^d containing in its interior the hypersphere $\sum_d(r)$ of radius r and centre at the origin. If K contains in its interior only those lattice points from a finite set $S = \{\lambda_0, \lambda_1, \ldots, \lambda_t\}$, then K lies completely within a second sphere whose radius is c_d/r^{d-1}, where c_d is a constant dependant on S and d.

Proof: Since S is a finite set, we can choose a sublattice of Λ_0, say $p\Lambda_0$, similar to Λ_0 such that none of the points $\lambda_1,\ldots,\lambda_t$ belong to $p\Lambda_0$. We can then apply Theorem 1.3.1 to this lattice and the set K, and deduce the above result.

1.4 Lattice polygon results

In later work we will need the following simple yet powerful result due to G. Pick.

Theorem 1.4.1 (G. Pick [10]): Let K be a convex polygon in the plane whose vertices lie at points of the integer lattice Λ_0. If K has b points of Λ_0 on its boundary, and i points of Λ_0 in its interior, then the area of K is given by

$$A(K) = i + \frac{1}{2}b - 1.$$

Proof: See Pick [10] or Honsburger [7], page 27.

We shall also use the following related result of P. Scott.

Theorem 1.4.2 (P. Scott): Let K be a convex polygon in the plane whose vertices lie at points of the integer lattice, and such that $i \geqslant 1$. Then

$$b \leqslant 2i + 7$$

and the equality holds if and only if K is an integral unimodular transform of the triangle with vertices (0,0) (3,0) and (0,3).

Proof: See Scott [14].

1.5 Blaschke Selection Theorem

In later sections we will use the following result.

Theorem 1.5.1 (Selection Theorem of Blaschke): Let H_k, $(k=1,2,\ldots)$ be a sequence of closed bounded convex sets contained in a fixed cube K. Then there exists a subsequence H_{k_r} converging to a convex set H in the Hausdorff metric.

Proof: See Lekkerkerker [8], page 12.

1.6 Results on the perimeter of a convex set.

Theorem 1.6.1 (Classical Isoperimetric Problem): Of all convex compact sets of a given perimeter, a circular disc is the unique set of largest area.

Proof: See Yaglom and Boltyanski [17], p. 51, or Honsberger [7], p. 67.

Theorem 1.6.2 (Besicovitch, Singmaster, Soupouris): Let S be a convex compact set in E^2, with perimeter $P(S)$, inradius r_0 and let P_0 be the perimeter of the union of all incircles of S. Let $\mathcal{T}_p$ denote the collection of all compact subsets of S, with perimeter equal to $P \leqslant P(S)$.

Then $\mathcal{T}_p$ contains a set of largest area, T, and

(a) If $0 < P \leqslant 2\pi r_0$, T is a closed disc of radius $P/2\pi$.

(b) If $2\pi r_0 < P \leqslant P_0$, T is a rectangle with semicircular ends of radius r_0.

(c) If $P_0 < P \leqslant P(S)$, T is the union of all closed discs of some radius r which lie in S, for some r, $0 \leqslant r < r_0$.

Proof: See Singmaster and Soupouris [16], or Besicovitch [1], Variant III.

Finally in this section we discuss Steiner Symmetrisation of a compact convex set in E^2. Let S be a compact convex set in E^2, and let ℓ be a given line in E^2. We construct a new set which is symmetric about ℓ as follows. For each line p orthogonal to ℓ, replace the segment $p \cap S$ by a congruent segment of p having its midpoint on ℓ. Let S' be the union of these translated segments. We refer to S' as the Steiner symmetrand of S about ℓ, and refer to the process as Steiner Symmetrisation.

Theorem 1.6.3: Steiner symmetrisation preserves convexity, area and central symmetry of sets. It does not increase perimeter, and actually decreases it unless S is already symmetric in a line parallel to ℓ.

Proof: (See Eggleston [5], p. 90).

CHAPTER 2

Minkowski's Theorem with Perimeter Constraints

2.1 Introduction and Statement of Results

The results of this chapter, in particular section 2.9, have been also obtained by Croft [4].

Let S be a bounded convex set in E^2 having area $A(S)$ and perimeter $P(S)$. Let $\mathcal{S}$ denote the set of all such sets which are symmetric in the origin 0, and which contain no non-zero point of the integral lattice. Write

$$A_p(\mathcal{S}) = \max_{\substack{S \in \mathcal{S} \\ P(S)=P}} A(S)$$

The existence of $A_p(\mathcal{S})$ is assured by a simple application of the Blaschke selection theorem, Theorem 1.5.1.

Theorem 2.1.1

(a) If $0 \leqslant P \leqslant 2\pi$, then $A_p(\mathcal{S}) = P^2/4\pi$.

(b) If $2\pi \leqslant P \leqslant 4+\pi$, then $A_p(\mathcal{S}) = 4 - (8-P)^2/4(4-\pi)$.

(c) If $4+\pi \leqslant P \leqslant 2\sqrt{2}(4+\pi)/(1+\sqrt{3})$, then $A_p(\mathcal{S})$

$$= 4 - [8(4+\pi)-P^2]/4(4+\pi).$$

(d) If $2\sqrt{2}(4+\pi)/(1+\sqrt{3}) \leqslant P \leqslant 2\sqrt{2}(1+\sqrt{3})$, then $A_p(\mathcal{S})$

$$= 4 - [2\sqrt{2}(1+\sqrt{3})-P]^2/4(2\sqrt{3}-\pi).$$

(e) If $2\sqrt{2}(1+\sqrt{3}) \leqslant P$, then $A_p(\mathcal{S}) = 4$.

The result in (a) is a restatement of the classical isoperimetric result, Theorem 1.6.1, since for $0 \leqslant P \leqslant 2\pi$, the circular disc having centre 0 and perimeter P is a member of $\mathcal{S}$. Minkowski's Theorem, Theorem 1.1.3 gives the upper bound 4 for $A_p(\mathcal{S})$. The result in (e) gives those values of P for which this bound is attained.

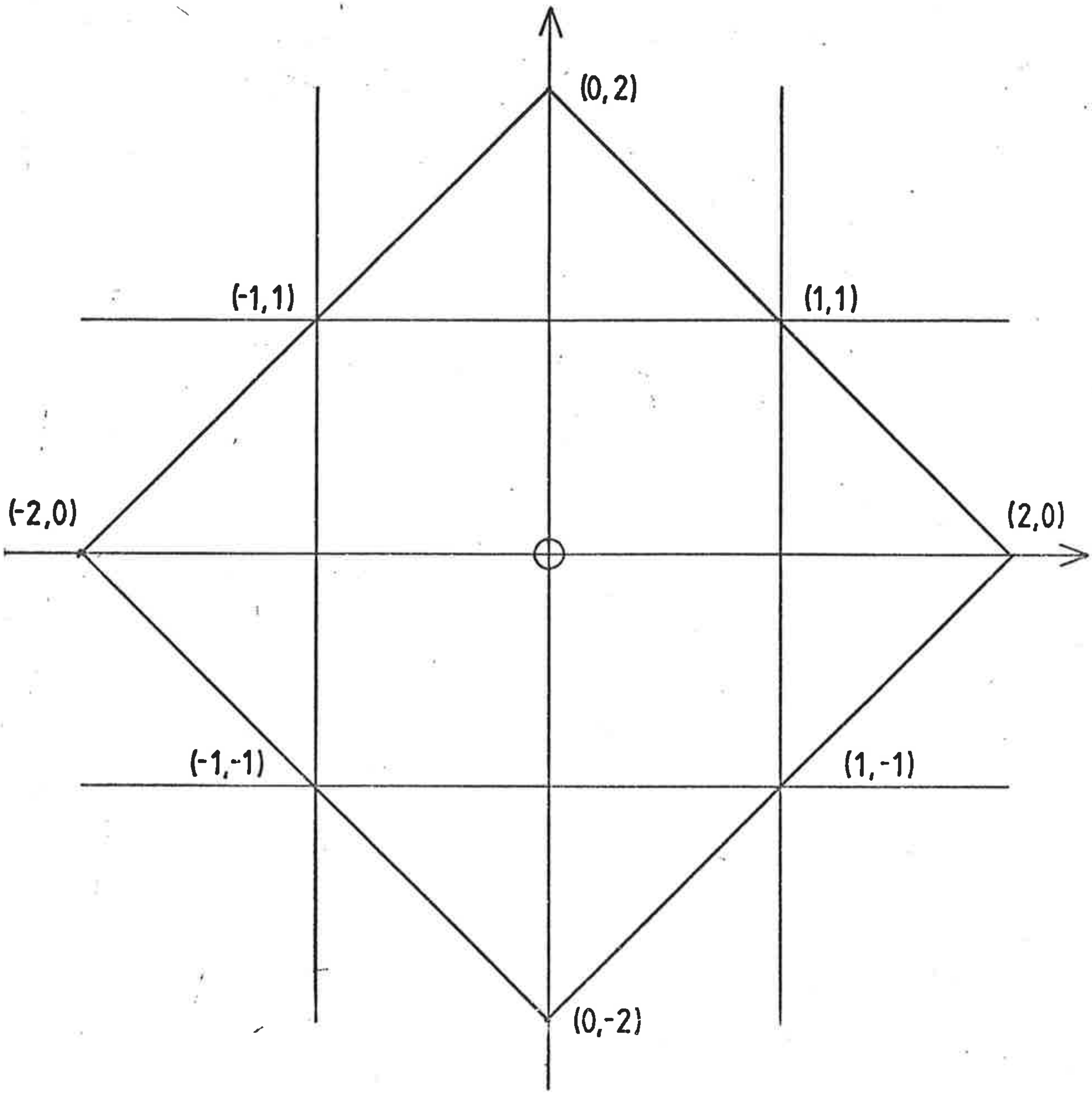


FIGURE 2.1

2.2 An Application of Steiner Symmetrisation.

The proof of Theorem 2.1.1 can be greatly simplified using Steiner symmetrization, as described in §1.6.

For $S \in \mathcal{S}$, we use the notation $d(S)$, ∂S for the diameter and boundary of S respectively. Let U be the convex hull of the points $\pm(1,\pm 1)$, and let Q be the convex hull of $\pm(2,0)$, $\pm(0,2)$. The polygon Q is shown in Figure 2.1, opposite.

Lemma 2.2.1 Let $S \in \mathcal{S}$ be a set for which $d(S) < 4$. Then there exists a set $S' \in \mathcal{S}$ which satisfies the following

(a) $S' \subseteq Q$.

(b) S' is symmetric in the lines $y = \pm x$.

(c) $A(S') = A(S)$, $P(S') \leqslant P(S)$ and $P(S') = P(S)$ only if $S' = S$.

(d) $S' \subset H$, where $H \in \mathcal{S}$ is a hexagon, and $H = Q \cap L$ where L is a parallelogram, symmetric in the lines $y = \pm x$, with sides passing through points $\pm(1,0)$, $\pm(0,1)$.

Proof We first show that S extends beyond Q in at most two quadrants of the plane. Let q be a point of $S{\sim}Q$ in the first quadrant. By reflecting S in the line $y = x$ if necessary, we may assume that q lies in the second octant of the plane. Now since q lies in the half-plane $y \geqslant 1$, and $(0,1)$ is not interior to S, it follows by convexity that any point r of $S{\sim}Q$ in the second quadrant must lie in the half-plane $y \leqslant 1$, and so in the fourth octant of the plane. Denoting the point $(-1,1)$ by r', we see that

$$A(\Delta 0qr) > A(\Delta 0qr') > 1.$$

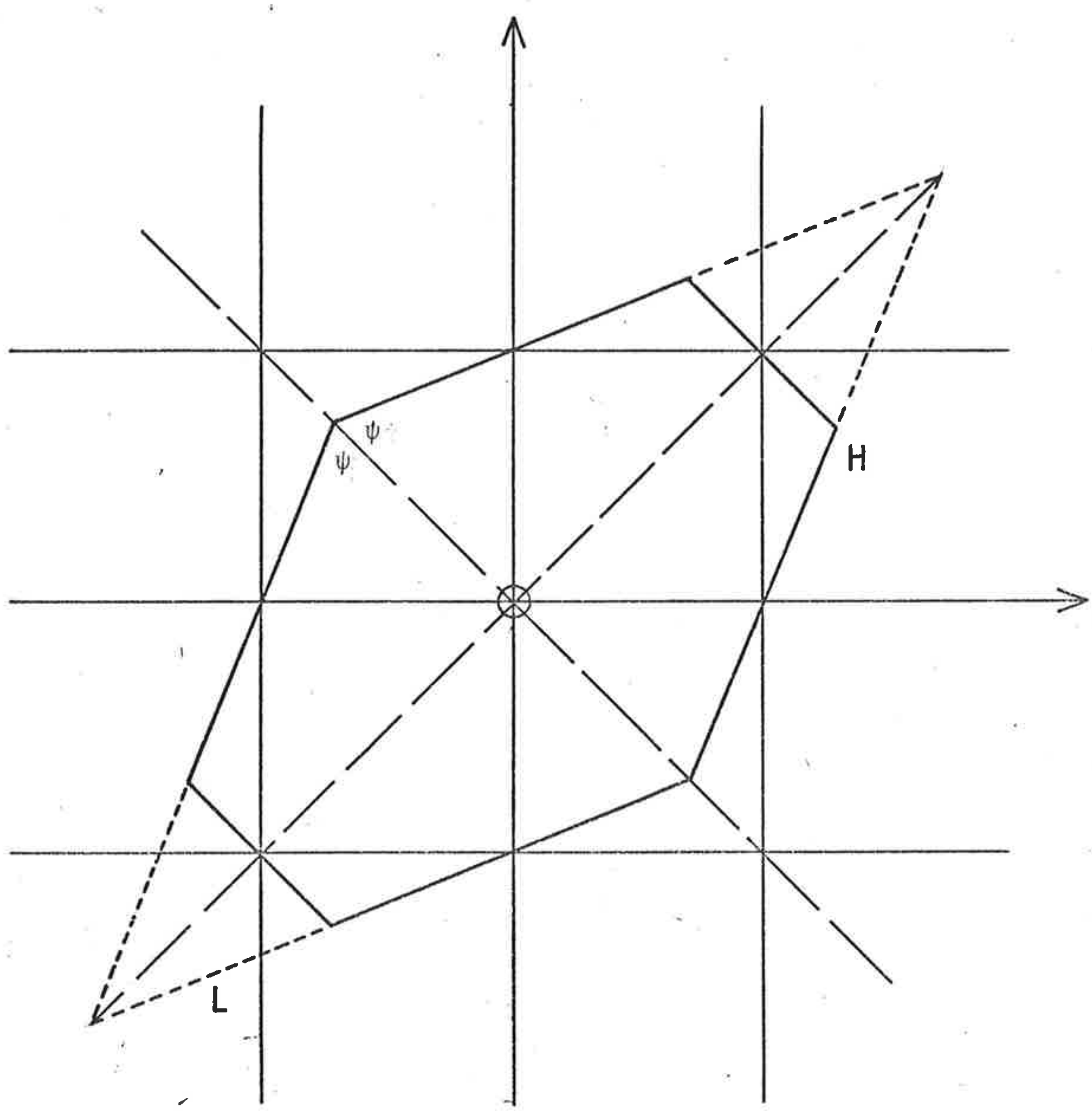


FIGURE 2.2

Hence the parallelogram contained in S and having vertices $\pm q$, $\pm r$ has area greater than 4. This contradicts the bound on $A(S)$ given by Minkowski's theorem, Theorem 1.1.3.

By reflecting S in the y-axis if necessary, we may assume then that S does not extend beyond Q in the second and fourth quadrants of the plane. In fact, since $d(S) < 4$, S lies in the strip $|x-y| \leqslant 2$.

We now obtain a new set S' by symmetrizing S in the line $y = -x$. The lines $y = x+1$, $y = x$, $y = x-1$ intercept the interior of S (and so the interior of S') in segments of length at most $\sqrt{2}$, $2\sqrt{2}$, $\sqrt{2}$ respectively. Since S lies in the strip $|x-y| \leqslant 2$ it follows that S' contains no non-zero lattice points in its interior.

By Theorem 1.6.3, $S' \in \mathcal{S}$, $A(S') = A(S)$, and $P(S') \leqslant P(S)$ with equality only if $S' = S$. Since S' is both centrally symmetric in 0 and symmetric in $y = -x$, it is symmetric in $y = x$. This symmetry of S together with its convexity and the fact that $\pm(1,1)$ are not interior to S', enables us to deduce that $S' \subset Q$. Further, S' does not contain $\pm(1,0)$ or $\pm(0,1)$ as interior points; hence S' is contained in a parallelogram whose sides pass through $\pm(1,0)$, $\pm(0,1)$, and whose diagonals lie along the lines $y = \pm x$. Finally, $S' \subseteq Q \cap L = H$, and $H \in \mathcal{S}$. This completes the proof of Lemma 2.2.1.

We note that by reflecting S' in the y-axis, if necessary, we may assume that ∂H has non-negative slope at $(0,1)$. Let the angles common to the hexagon H and the parallelogram L, of Lemma 2.2.1, measure 2ψ. We can then denote these figures explicitly by $H(\psi)$, $L(\psi)$ respectively, where $\frac{\pi}{4} \leqslant \psi \leqslant \frac{\pi}{2}$. These polygons are shown in Figure 2.2, opposite.

2.3 Hexagons which have area 4.

We now establish part (e) of Theorem 2.1.1.

Let $S \in \mathcal{S}$ be a set with $A(S) = 4$. We show that $P(S) \geqslant 2\sqrt{2}(1 + \sqrt{3})$ (≈ 7.73). Let us assume that $P(S) < 8$. Then $d(S) < 4$, and by Lemma 2.2.1, there exists a set $S' \in \mathcal{S}$ where $A(S') = 4$, $P(S') \leqslant P(S)$ and S' is contained in a hexagon $H(\psi)$. In fact since $H(\psi)$ has area 4, $S' = H(\psi)$. It is easily verified that the perimeter of $H(\psi)$ is $P(H(\psi)) = 2\sqrt{2}(2 \operatorname{cosec} \psi + 1 - \cot \psi)$.

As $\frac{d}{d\psi}(P(H(\psi))) = 2\sqrt{2}(1 - 2 \cos \psi)(\sin \psi)^{-2}$, the function $P(H(\psi))$ assumes its minimal value of $2\sqrt{2}(1 + \sqrt{3})$ when $\psi = \frac{\pi}{3}$; $H(\frac{\pi}{3})$ is an equiangular hexagon. Therefore $P(S) \geqslant P(S') \geqslant 2\sqrt{2}(1 + \sqrt{3})$, as required.

It remains to be shown that for each value of P in this range, there is a set $S \in \mathcal{S}$ having perimeter P and area 4. For $P \geqslant 8$ we obtain S by suitably shearing the square U parallel to the x-axis. Also, over the interval $\frac{\pi}{4} \leqslant \psi \leqslant \frac{\pi}{3}$, the perimeter of $H(\psi)$ is a continuous function of ψ. Since $H\left(\frac{\pi}{4}\right) = U$, we deduce that for each value of P, $2\sqrt{2}(1+\sqrt{3}) \leqslant P \leqslant 8$, there is a hexagon having perimeter P and area 4. We conclude that for $2\sqrt{2}(1+\sqrt{3}) \leqslant P$, $A_p(\mathcal{S}) = 4$.

2.4 Extreme sets are symmetric in $y = \pm x$.

Let $S \in \mathcal{S}$ be a set for which $A(S) = A_p(\mathcal{S})$ where $P(S) = P < 2\sqrt{2}(1+\sqrt{3})$. We show that S must satisfy the conditions of the set S' in Lemma 2.2.1.

Since $d(S) < 4$, by Lemma 2.2.1 there exists a set $S' \in \mathcal{S}$ for which $A(S') = A(S)$, $P(S') \leqslant P(S)$, and such that S' is contained in a hexagon $H(\psi) = H$. Now define $S(\lambda) = \lambda H + (1-\lambda)S'$ $(0 \leqslant \lambda \leqslant 1)$. Then $S' \subset S(\lambda) \subset H$, and since the perimeter of $S(\lambda)$

is a continuous function of λ, there exists a set $S^* = S(\lambda^*)$ $(0 \leqslant \lambda^* \leqslant 1)$ for which $P(S^*) = P(S)$. If $\lambda > 0$, $A(S^*) > A(S') = A(S)$, contradicting our choice of S. Hence $\lambda = 0$, and $S = S'$, as required.

Since $H(\psi)$ contains no non-zero lattice points in its interior, S is the set of largest area and perimeter P contained in $H(\psi)$. We use a result of Besicovitch, Theorem 1.6.2 which characterises S, and shows that $S \in \mathcal{S}$.

2.5 Extreme sets are rounded polygons.

Let $S \in \mathcal{S}$ be a set for which $A(S) = A_p(\mathcal{S})$. By §2.4, S is contained in a polygon $H(\psi)$, for some ψ. By the definition of $A_p(\mathcal{S})$, S is the largest set of perimeter P contained in the convex set $H(\psi)$. Hence, by Theorem 1.6.2, S is the union of all discs of some particular radius, contained in $H(\psi)$, or, more simply, is $H(\psi)$ with rounded corners. Since the two sides of $H(\psi)$, $\partial Q \cap \partial H(\psi)$, are not longer than the four equal sides of $H(\psi)$, this rounding can occur in one of only two ways. Either each corner is rounded individually (as when the circular arcs have small radius), or each of the short sides is rounded off completely by a single circular arc. In the latter case, $L(\psi)$ has effectively been rounded at each corner, using arcs of some common radius.

We formulate these cases together. A polygon G is rounded at each vertex by arcs of fixed radius r, to form the rounded polygon G_r. We assume that r is small enough for each side of G to contribute a line segment to G_r. We number the vertices of G, and let $2\psi_i$ be the internal angle at the ith vertex. In rounding the corners of G, two segments of length $r \cot \psi_i$ are lost from the boundary of G at the ith vertex of G, and replaced by circular segments. Hence

$$A(G_r) = A(G) - r^2\left(\sum_i \cot\psi_i - \pi\right) = A(G) - r^2 k(G) \quad \text{say} \qquad (1)$$

$$P(G_r) = P(G) - 2r\,k(G). \qquad (2)$$

Solving equation (2) for r, and substituting in equation (1) gives

$$A(G_r) = A(G) - [P(G)-P(G_r)]^2[4k(G)]^{-1}\ . \qquad (3)$$

When G_r is obtained from G by rounding, equation (3) enables us to find the maximum value of $A(G_r)$ when $P(G_r)$ is a given constant. Our problem is complicated by the fact that $G = H(\psi)$ or $G = L(\psi)$ are themselves dependent on a parameter ψ. To find the maximum value of $A(G_r)$ we need to examine the derivative $A(G_r)$ with respect to ψ:

$$\frac{dA(G_r)}{d\psi} = \frac{dA(G)}{d\psi} - \frac{2[P(G)-P(G_r)]}{4k(G)}\frac{dP(G)}{d\psi} + \frac{[P(G)-P(G_r)]^2}{4[k(G)]^2}\frac{dk(G)}{d\psi} \qquad (4)$$

2.6 Rounded hexagon or parallelogram?

Let ψ be given. According to Besicovitch's construction, there will be a certain value $P_0(\psi)$ of the perimeter for which $H_r(\psi)$ and $L_r(\psi)$ coincide. For $P < P_0(\psi)$, it will not be possible to construct $H_r(\psi)$, there being insufficient perimeter to reach the sides of $H(\psi)$ through $\pm(1,1)$. For $P > P_0(\psi)$, $L_r(\psi)$ will contain $\pm(1,1)$ as interior points, and will thus no longer be a member of $\mathcal{S}$.

We show that

$$P_0(\psi) = P(G) - k(G)\frac{\sqrt{2}(1 - \tan\psi)}{1 - \sec\psi}$$

where $G = L(\psi)$.

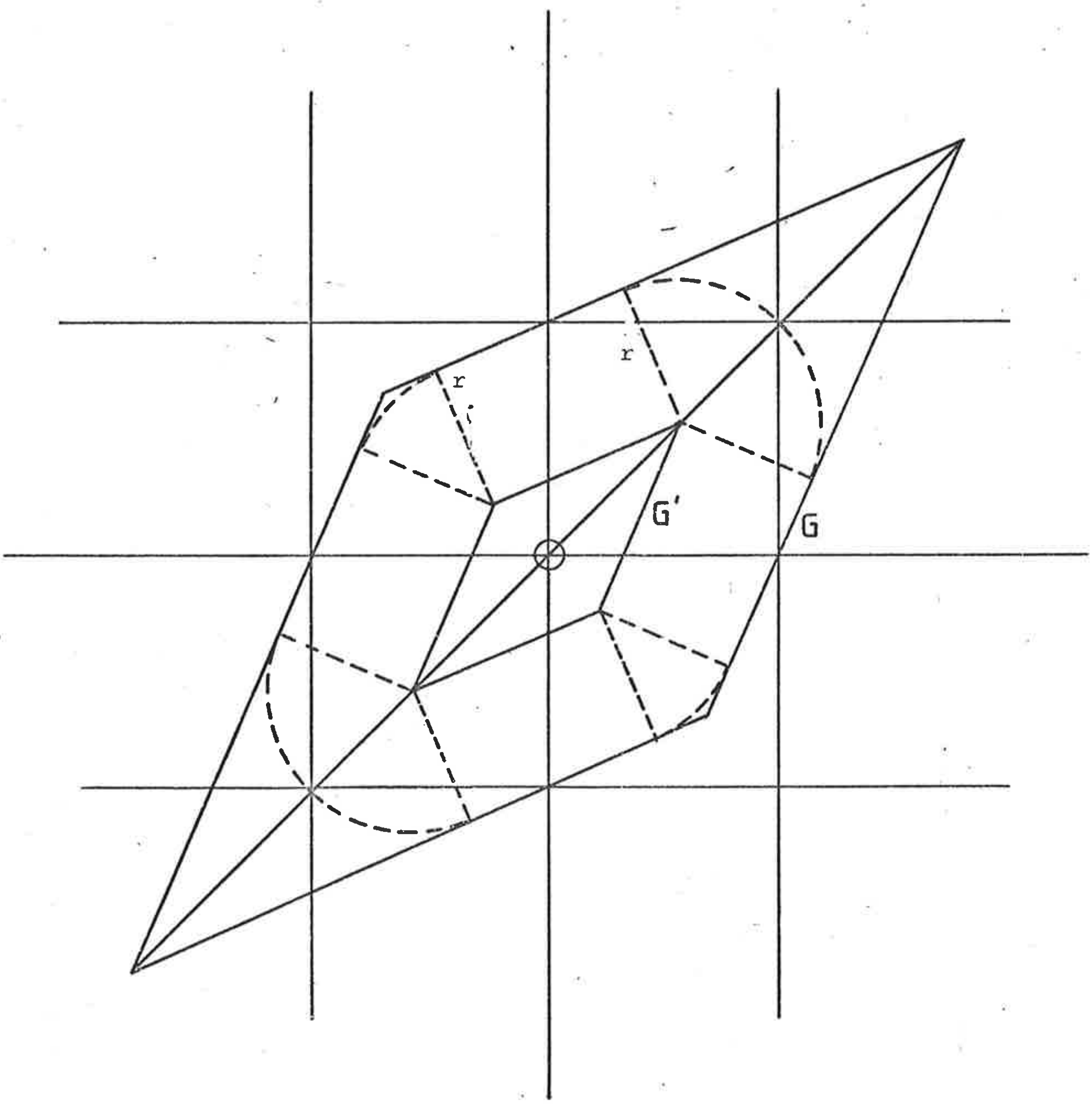


FIGURE 2.3

Let G_r be the rounded parallelogram with perimeter $P_0(\psi)$. Then the points $\pm(1,1)$ lie on the boundary ∂G_r, the diameter of G_r lies along the line $y = x$, and $d(G_r) = 2\sqrt{2}$. We can also calculate $d(G_r)$ by using the construction shown in figure 2.3, opposite.

We note that the parallelogram G has diameter $2\sin\left(\frac{3\pi}{4} - \psi\right).\sec\psi$, and that the similar parallelogram G' is obtained from G by an enlargement of scale factor $1 - r\sec\left(\psi - \frac{\pi}{4}\right)$.
Now, since the vertices of G' are the centres of the circular boundary arcs of $G_r(\psi)$,

$$d(G_r) = 2r + 2\sin\left(\frac{3\pi}{4} - \psi\right).\sec\psi.[1 - r\sec(\psi - \frac{\pi}{4})]$$

$$= 2r(1 - \sec\psi) + \sqrt{2}(1 + \tan\psi). \qquad (5)$$

Equating this to $2\sqrt{2}$, solving for r, and substituting in (2) gives, for $\psi \in [\frac{\pi}{4}, \frac{\pi}{2})$ and $G = L(\psi)$, that

$$P_0(\psi) = P(G) - k(G).\sqrt{2}(1 - \tan\psi).(1 - \sec\psi)^{-1}.$$

Both the functions $P(L(\psi))$ and $k(L(\psi))$ are continuous for $\psi \in [\frac{\pi}{4}, \frac{\pi}{2}]$. We thus deduce that the limiting value of $P_0(\psi)$, as ψ approaches $\frac{\pi}{2}$, is $\sqrt{2}(2 + \pi)$. As this equals the value of $P_0(\psi)$ given by the definition of $P_0(\psi)$, $P_0(\psi)$ is a continuous function on $[\frac{\pi}{4}, \frac{\pi}{2}]$.

We next show that $P_0(\frac{\pi}{2})$ is the least value taken by $P_0(\psi)$, for $\psi \in [\frac{\pi}{4}, \frac{\pi}{2}]$. It is easily calculated that

$$P_0(\psi) - P_0(\tfrac{\pi}{2}) = P(G) - k(G)\sqrt{2}(1 - \tan\psi)(1 - \sec\psi)^{-1} - \sqrt{2}(2+\pi)$$

$$= \sqrt{2}(1 - \cos\psi)^{-1}[2(2-\sin\psi-\cos\psi)\sin\psi^{-1} + \pi(\sin\psi-\cos\psi)]$$

$$- \sqrt{2}(2+\pi)$$

$$= \sqrt{2}(1 - \sin\psi)(1 - \cos\psi)^{-1}[2(2 - \cos\psi)\sin\psi^{-1} - \pi]$$

$$= 0 \text{ iff } \sin\psi = 1 \text{ or } 2(2 - \cos\psi) = \pi\sin\psi.$$

Thus $P_0(\psi) = P_0(\frac{\pi}{2})$ only if $\psi = \frac{\pi}{2}$ or if $2(2-\cos\psi) = \pi\sin\psi$. We examine this later condition as a quadratic equation in $\cos\psi = c$,

$$(2(2-c))^2 = \pi^2(1-c^2) .$$

This equation has discriminant $(\frac{\pi}{2})^2((\frac{\pi}{2})^2-3)$, which is negative.

Hence, by the continuity of $P_0(\psi)$, $P_0(\psi) > P_0(\frac{\pi}{2})$, for all $\psi \in (\frac{\pi}{4}, \frac{\pi}{2})$. We note that $P_0(\frac{\pi}{3}) = 2\sqrt{2}(4+\pi)/(1+\sqrt{3})$.

2.7 Determination of rounded polygon $H(\psi)$ or $L(\psi)$ from P.

We apply the results of §2.5 to our problem.

Case 1. Suppose $G = H(\psi)$. By straightforward calculation we find that

$$A(G) = 4$$

$$P(G) = 2\sqrt{2}(2 \operatorname{cosec}\psi + 1 - \cot\psi)$$

$$\begin{aligned} k(G) &= 2\cot\psi + 4\tan\frac{\psi}{2} - \pi \\ &= \tfrac{1}{2}\sqrt{2}\, P(G) - (2+\pi). \\ &> 0, \quad \text{since } P(G) \geqslant 2\sqrt{2}(1+\sqrt{3}) \end{aligned}$$

Substituting in equation (4) and setting $P(G_r) = P$,

$$\frac{dA(G_r)}{d\psi} = \left[\frac{P(G)-P}{2\sqrt{2}\,k(G)}\right]\left[-\sqrt{2}\,\frac{dP(G)}{d\psi} + \left(\frac{P(G)-P}{\sqrt{2}\,k(G)}\right)\frac{dk(G)}{d\psi}\right]$$

$$= \left[\frac{P(G)-P}{2\sqrt{2}\,k(G)}\right]\frac{dP(G)}{d\psi}\left[-\sqrt{2} + \left(\frac{P(G)-P}{2k(G)}\right)\right]$$

$$= 2r \,.\, (1-2\cos\psi)\operatorname{cosec}^2\psi \,.\, (r-\sqrt{2}).$$

Clearly $r - \sqrt{2} < 0$, since circles of radius $\sqrt{2}$ can not be contained in G. Since we are assuming $2\pi \leqslant P \leqslant 2\sqrt{2}(1+\sqrt{3})$, the

first term is non-negative for all ψ, and zero only when $\psi = \frac{\pi}{3}$.
Finally, $1 - 2\cos\psi$ $(\frac{\pi}{4} \leqslant \psi \leqslant \frac{\pi}{2})$ is an increasing function of ψ with a zero at $\psi = \frac{\pi}{3}$. Hence for any value of P, $A(G_r)$ assumes its maximal value when and only when $\psi = \frac{\pi}{3}$.

Case 2. Suppose now that $G = L(\psi)$. Setting $a = \tan\psi + \cot\psi$ we find that

$$A(G) = a + 2$$

$$P(G) = 2\sqrt{2}(\operatorname{cosec}\psi + \sec\psi)$$

$$= 2\sqrt{2}\, a^{\frac{1}{2}}(a+2)^{\frac{1}{2}}$$

$$k(G) = 2a - \pi .$$

Hence from equation (4), setting $P(G_r) = P$,

$$\frac{dA(G_r)}{d\psi} = \frac{da}{d\psi}\left[1 - \left\{\frac{P(G)-P}{2k(G)}\right\}\frac{2\sqrt{2}(a+1)}{a^{\frac{1}{2}}(a+2)^{\frac{1}{2}}} + 2\left\{\frac{P(G)-P}{2k(G)}\right\}^2\right]$$

Solving this quadratic expression in $r = \frac{P(G)-P}{2k(G)}$, and simplifying gives

$$\frac{dA(G_r)}{d\psi} = \frac{da}{d\psi}(r - (a+2)^{\frac{1}{2}}(2a)^{-\frac{1}{2}})(r - a^{\frac{1}{2}}(2(a+2))^{-\frac{1}{2}}).2$$

$$= \frac{da}{d\psi}\frac{1}{2[k(G)]^2}[P - \sqrt{2}(a+2)^{\frac{1}{2}}a^{-\frac{1}{2}}\pi][P - \sqrt{2}\,a^{\frac{1}{2}}(a+2)^{-\frac{1}{2}}(4+\pi)].$$

Now $\frac{da}{d\psi} = \operatorname{cosec}^2\psi(\tan^2\psi-1) \geqslant 0$ for $\frac{\pi}{4} \leqslant \psi < \frac{\pi}{2}$, with equality only if $\psi = \frac{\pi}{4}$. The term $\frac{1}{2[k(G)]^2}$ is obviously positive. A simple calculation shows that $\sqrt{2}(a+2)^{\frac{1}{2}}a^{-\frac{1}{2}} \leqslant 2$ with equality only for $\psi = \frac{\pi}{4}$. Since $P \geqslant 2\pi$, the term $[P - \sqrt{2}(a+2)^{\frac{1}{2}}a^{-\frac{1}{2}}\pi]$ is non-negative, and zero only when $P = 2\pi$ and $\psi = \frac{\pi}{4}$. Finally, for $\frac{\pi}{4} \leqslant \psi < \frac{\pi}{2}$, $\sqrt{2}\,a^{\frac{1}{2}}(a+2)^{-\frac{1}{2}}$ is an increasing function of ψ, assuming its minimum value of 1 when $\psi = \frac{\pi}{4}$. Hence, if $P \leqslant 4+\pi$,

$$P - \sqrt{2}a^{1/2}(a+2)^{-1/2}(4+\pi) \leqslant P - (4+\pi) \leqslant 0.$$

If $P \geqslant 4 + \pi$, then $P - \sqrt{2}a^{1/2}(a+2)^{-1/2}(4+\pi)$ is a decreasing function of ψ, with a zero at $\psi = T$, corresponding to the root of

$$P = \sqrt{2}\, a^{1/2}(a+2)^{-1/2}(4+\pi). \tag{6}$$

Thus for $2\pi \leqslant P < 4+\pi$, $A(G_r)$ assumes its maximum value at $\psi = \frac{\pi}{4}$. For $4 + \pi \leqslant P \leqslant 2\sqrt{2}(1+\sqrt{3})$, $A(G_r)$ assumes its maximum value when $\psi = T$.

We can now establish part (b) of the theorem. Suppose $2\pi \leqslant P \leqslant 4+\pi < \sqrt{2}(2+\pi)$. By section 2.6, G must be a parallelogram. Hence $A(G_r)$ assumes its maximum when $G = L(\frac{\pi}{4})$, and

$$A_p(S) = A(L_r(\tfrac{\pi}{4})) = 4 - (8 - P)^2/4(4 - \pi).$$

2.8 Determination of $A_p(S)$ for $P > 4 + \pi$

Suppose that $4+\pi \leqslant P \leqslant 2\sqrt{2}(1+\sqrt{3})$. From §2.7 we know that the function $A(G_r(\psi))$ assumes its maximum when $G = L(T)$ or when $G = H(\frac{\pi}{3})$. However, we do not yet know whether the rounded polygons G_r can be formed for P in the given range.

Let $G = L(T)$. For $P \geqslant 4+\pi$, the angle T is determined by (6):

$$P = \sqrt{2}(4+\pi)a^{1/2}(a+2)^{-1/2} = \sqrt{2}(4+\pi)(\sin T + \cos T)^{-1}.$$

From (2), the radius by which $L_r(T)$ is rounded is given by

$$r = (P(G) - P)[2k(G)]^{-1}.$$

Substituting for $P(G)$, $k(G)$, and then for P we obtain

$$r = P/(8+2\pi) = [\sqrt{2}(\sin T + \cos T)]^{-1}. \tag{7}$$

From (5), the diameter of G_r is given by

$$d(G_r) = 2r(1 - \sec T) + \sqrt{2}(1 + \tan T)$$

$$= \sqrt{2}(1 + 2\sin T)(\sin T + \cos T)^{-1} .$$

If $d(G_r) > 2\sqrt{2}$, $L_r(T)$ contains $\pm(1,1)$ as interior points. We find that for $\frac{\pi}{4} \leqslant T \leqslant \frac{\pi}{2}$, $d(G_r) > 2\sqrt{2}$ if and only if $T > \frac{\pi}{3}$. From (7), this occurs precisely when $P > 2\sqrt{2}(4+\pi)/(1+\sqrt{3})$.

The area of $L_r(T)$ can be found from (3) by substituting in the value of P from (6). A short calculation gives

$$A(L_r(T)) = 2 + P^2/4(4+\pi) .$$

Again, directly from (3) we have

$$A(H_r(\tfrac{\pi}{3})) = 4 - [2\sqrt{2}(1+\sqrt{3}) - P]^2/4(2\sqrt{3}-\pi)$$

Graphically, these two area functions of P are represented by parabolas, the first upright, and the second inverted. A lengthy but elementary calculation shows that these parabolas touch when $P = 2\sqrt{2}(4+\pi)/(1+\sqrt{3})$. It follows that $A(L_r(T)) \geqslant A(H_r(\frac{\pi}{3}))$, with equality only when $P = 2\sqrt{2}(4+\pi)/(1+\sqrt{3})$. Since we have just shown that the construction of $L_r(T)$ is valid for $P \leqslant 2\sqrt{2}(4+\pi)/(1+\sqrt{3})$ we have established part (c) of the theorem.

Suppose now that $2\sqrt{2}(4+\pi)/(1+\sqrt{3}) \leqslant P \leqslant 2\sqrt{2}(1+\sqrt{3})$. As we note in §6, $P_0(\frac{\pi}{3}) = 2\sqrt{2}(4+\pi)/(1+\sqrt{3})$, so $H_r(\frac{\pi}{3})$ can be constructed for P in this range. However, $L_r(T)$ is now disallowed, as $\pm(1,1)$ are interior points. Substituting the value of r given by (2) in expression (5), we see that the diameter $d(L(\psi))$ is a continuous function of ψ for $\frac{\pi}{4} \leqslant \psi < \frac{\pi}{2}$. It follows that for values of ψ lying in a certain open interval about T, the rounded parallelogram

$L_r(\psi)$ having perimeter P will contain $\pm(1,1)$ as interior points. Also, the function $A(L_r(\psi))$ is a continuous function of ψ with a single (maximum) turning point at $\psi = T$. Hence the maximum allowable value of $A(L_r(\psi))$ will be assumed for a rounded parallelogram $L_r(\psi)$ which has $\pm(1,1)$ as boundary points. But now $L_r(\psi)$ coincides with $H_r(\psi)$, and of all such rounded hexagons of perimeter P, $H_r(\frac{\pi}{3})$ has the largest area. Since $H_r(\frac{\pi}{3})$ can be constructed, we have established part (d) of Theorem 2.1.1.

This completes the proof of Theorem 2.1.1.

2.9 A related problem.

Consider now the problem of finding

$$h(\beta) = \max_{S \in \mathcal{S}} \frac{A(S)}{[P(S)]^{\beta}}$$

where $0 \leqslant \beta \leqslant 2$. The existence of such a maximum is guaranteed by Blaschke's selection theorem, Theorem 1.5.1. When $\beta = 0$, the solution of this problem is given by Minkowski's Theorem 1.1.3. When $\beta = 2$, we have the classical unconstrained isoperimetric problem, Theorem 1.6.1. This problem has also been solved, in a slightly more general context, in Croft [4].

For given β, let $S^* \in \mathcal{S}$ be the set for which $h(\beta)$ is attained. S^* must have the largest area of all sets of perimeter $P(S^*)$. Hence S^* must be one of the sets S for which $A(S) = A_p(\mathcal{S})$ in the above theorem. In particular, by calculating the ratios $A_p(\mathcal{S})/P$ we are able to confirm a conjecture of Scott [11].

<u>Theorem 2.9.1</u> The value of $h(1)$ is $2(2+\sqrt{\pi})^{-1}$. This is attained when S is a square with rounded corners, $L_r(\frac{\pi}{4})$, whose radius $r = 2(2+\sqrt{\pi})^{-1}$.

Proof. In order to verify this result, by our comment above, we need only verify that, as a function of P, $A_p(S)/P$ has a single local maximum at $P = 4\sqrt{\pi}$.

For $P \in (0,2\pi]$, $A_p(S)/P = P/4\pi \leqslant \frac{1}{2}$, and clearly $A_p(S)/P$ is an increasing function.

For $P \in (2\pi,4+\pi]$, $A_p(S)/P = [4 - (8-P)^2/4(4-\pi)]/P$. The derivative $D_p(A_p(S)/P) = (16\pi-P^2)/4(4-\pi)P^2$, and so $A_p(S)/P$ has a local maximum when $P = 4\sqrt{\pi} = 7.08 \in (2\pi,4+\pi)$.

For $P\in(4+\pi,2\sqrt{2}(4+\pi)/(1+\sqrt{3}))$, $A_p(S)/P = (8(4+\pi)-P^2)/4(4+\pi)P$, and $D_p(A_p(S)/P) = (P^2+8(4+\pi))/(-4(4+\pi)P^2) < 0$ for all P. $A_p(S)/P$ is therefore a decreasing function for P in this range.

For $P \in (2\sqrt{2}(4+\pi)/(1+\sqrt{3}),\ 2\sqrt{2}(1+\sqrt{3}))$,

$$A_p(S)/P = [4-(2\sqrt{2}(1+\sqrt{3})-P)^2/4(2\sqrt{3}-\pi)]/P\ , \quad \text{and}$$

$$D_p(A_p(S)/P) = [16(2-\sqrt{3}+\pi)-P^2]/4(2\sqrt{3}-\pi)P^2, \quad \text{and so}$$

the formula specifying $A_p(S)/P$ in this range has a local maximum when $P = 4\sqrt{2-\sqrt{3}+\pi} = 7.385$. However $7.385 < 7.393 = 2\sqrt{2}(4+\pi)/(1+\sqrt{3})$. Hence $A_p(S)/P$ is decreasing for all P in this range.

For $P > 2\sqrt{2}(1+\sqrt{3})$, $A_p(S)/P = 4/P < 4/2\sqrt{2}(1+\sqrt{3})$ and is clearly a decreasing function of P.

Hence $A_p(S)/P$ has a unique local and global maximum, when $P = 4\sqrt{\pi}$, on $(0,\infty)$. It is readily confirmed that $h(1) = 2(2+\sqrt{\pi})^{-1}$, $r = 2(2+\sqrt{\pi})^{-1}$, and that S is $L_r(\frac{\pi}{4})$ a square with rounded corners.

2.10 Conjectured analogue of Theorem 2.1.1 for a general planar lattice.

Let Λ be a lattice in E^2 with generators a and b, such that $\|a\| \leqslant \|b\| \leqslant \|c\|$ for all $c \in \Lambda$, $c \neq a$, and such that θ, the angle from a to b, $\theta \in [\frac{\pi}{3}, \frac{\pi}{2}]$. We let $S(P)$ denote the set of all compact convex 0-symmetric sets, which contain no non-zero lattice points in their interiors, and which have perimeter $P(S) = P$. By a simple application of the Blaschke selection theorem, Theorem 1.5.1, it can be shown that $\mathcal{S}(P)$ contains at least one set of largest area, for each value of P. Let $\mathcal{S}_*(P)$ be such a set.

Conjecture 2.10.1 Suppose $\|a\| \neq \|b\|$ and that $\theta \neq \frac{\pi}{2}$. Then there are numbers P_1 P_2 and P_3 with the following properties.

$S_*(P)$ is uniquely determined, and for the indicated range of P, $S_*(P)$ is one of the following.

(a) If $0 < P \leqslant 2\pi\|a\|$, $S_*(P)$ is a closed disc.

(b) If $2\pi\|a\| < P \leqslant P_1$, $S_*(P)$ is the convex hull of two discs whose line of centres is orthogonal to a, and whose radii are $\|a\|$.

(c) If $P_1 < P \leqslant P_2$, $S_*(P)$ is the convex hull of four discs, the straight line segments of whose boundary are bisected by the points $\pm a$, $\pm b$.

(d) If $P_2 < P \leqslant P_3$, $S_*(P)$ is the convex hull of six discs, the straight line segments of whose boundary are bisected by the points $\pm a$, $\pm b$, and $\pm(a-b)$, and whose sides when extended form an equiangular hexagon.

(e) If $P > P_3$, $S_*(P)$ is one of the well known critical polygons whose area is $4\|a\|\|b\|\sin\theta = 4\det(\Lambda)$, and so which satisfy the inequality of Minkowski's Theorem with equality.

If $\|a\| = \|b\|$ or $\theta = \frac{\pi}{2}$, $S_*(P)$ is not uniquely determined by the above properties, and indeed for $\|a\| = \|b\|$, $S_*(P)$ is not unique. If $\theta \neq \frac{\pi}{2}$, any set S which satisfies one of conditions (a), (b), (c) or (d) and which belongs to $\mathcal{S}(P)$, is $S_*(P)$ for $P = P(S)$.

Although I cannot prove this result, I have proved enough to suggest that the result is true. The difficulty lies in identifying $S_*(P)$ when it has four lattice points on its boundary.

CHAPTER 3

MINKOWSKI'S THEOREM WITH RELAXED SYMMETRY CONDITIONS

3.1 Introduction

Let Λ be a lattice in the plane having determinant $\det(\Lambda)$. We say the set K is *admissible* if it is a closed convex set in the plane with 0 the only point of Λ in its interior. Minkowski's fundamental theorem, Theorem 1.1.3 asserts that if an admissible set K is centrally symmetric about 0, then its area $A(K)$ is no greater than $4 \det(\Lambda)$. We call a chord of K which is bisected by 0, a *chord of symmetry* of K. Minkowski's hypothesis requires that all chords through 0 be chords of symmetry. We say a chord of symmetry of K is *extremal* if K has parallel supporting lines at its endpoints. Theorem 1.2.8 states that $A(K) \leqslant 4 \det(\Lambda)$ if K has an extremal chord of symmetry.

Let $s(K)$ denote the number of chords of symmetry of an admissible set K. We show

Theorem 3.1.1 If $s(K)$ is even or infinite, $A(K) \leqslant 4 \det(\Lambda)$.

Theorem 3.1.2 If $s(K) > 1$, $A(K) \leqslant 4.5 \det(\Lambda)$.

Theorem 3.1.3 If $s(K) > 3$, $A(K) \leqslant 4 \det(\Lambda)$.

Theorem 3.1.2 is analogous to Theorem 1.2.1 which states that $A(K) \leqslant 4.5 \det(\Lambda)$, whenever K satisfies the more restrictive condition that 0 is the centre of gravity of K. It will be sufficient to establish the theorems when Λ is the integral lattice Λ_0, since $s(K)$ and $A(K)/\det(\Lambda)$ are invariant under a linear transformation of K and Λ. As $\det(\Lambda_0) = 1$, we delete all further reference to $\det(\Lambda)$ in this chapter.

We show that each of the above theorems gives the best possible bound on $A(K)$. Let E_1 be the convex hull of the points $(-1,2)$, $(2,-1)$ and $(-1, 1)$. This triangle E_1 is admissible, and has only three chords of symmetry, passing through $(1,0)$, $(0,1)$ and $(-1,-1)$ respectively. As $A(E_1) = 4.5$ and $s(E_1) = 3$, Theorem 3.1.2 is best possible.

For Theorems 3.1.1 and 3.1.3 we argue as follows. Let U be the convex hull of the four points $(\pm 1,\pm 1)$. Let $C \subseteq U$ be the union of half a regular $2m$-gon and a semicircle on a common diameter with midpoint 0. Then $s(C) = m > 1$. The set $K(\varepsilon) = (1-\varepsilon)U + \varepsilon C$, $0 < \varepsilon < 1$, is an admissible set and since U is centrally symmetric about 0, $s(K(\varepsilon)) = s(C) = m$. Since for sufficiently small ε, $A(K(\varepsilon))$ can be arbitrarily close to $A(U) = 4$, Theorems 3.1.1 and 3.1.3 give the best possible bound for finite $s(K)$. Taking $\varepsilon = 0$, we obtain the square U with $A(U) = 4$ and $s(U)$ infinite.

Finally, let E_2 be the convex hull of the three points $(\pm t,-1)$ and $(0,1/t)$ where $t > 1$. The triangle E_2 is admissible, and has only one chord of symmetry parallel to the x-axis. As $A(E_2) > t$ and $s(E_2) = 1$, we can deduce no upper bound on $A(K)$ from the information that $s(K) = 1$.

3.2 Proof of Theorem 3.1.1

If all the chords of K which pass through 0 are chords of symmetry of K, Minkowski's Theorem 1.1.2 gives the desired result. We may thus assume there is a chord of K, P_0OP_0' which is not a chord of symmetry of K. We denote each chord POP' of

K by the label $c(\theta)$, where θ is the angle P_0OP measured in an anti-clockwise direction and $\theta \in [0,\pi]$. The continuous function $d(\theta) = (PO)^2 - (OP')^2$ has a zero when and only when $c(\theta)$ is a chord of symmetry of K, and by our choice of P_0OP_0', $d(0) = -d(\pi) \neq 0$. By applying the intermediate value theorem to $d(\theta)$, we deduce that $d(\theta)$ has at least one zero on $[0,\pi]$. If $s(K)$ is even, then $d(\theta)$ has an even number of zeros, and so at least one zero θ_*, occurs at an extremum of $d(\theta)$. Let $\bar{K}$ denote the reflection of K in the origin 0, and let P_*OP_*' be the chord $c(\theta_*)$. If $d(\theta)$ has a local maximum at θ_*, then in a neighbourhood of P_* the boundary of K is contained in $\bar{K}$, and so a supporting line s to K at P_*', together with a parallel line s' through P_*, form a pair of parallel supporting lines to K at the endpoints of $c(\theta_*)$, a chord of symmetry of K. We argue similarly if $d(\theta)$ has a local minimum at θ_*. In either case, $c(\theta_*)$ is an extremal chord of symmetry of K, and so by Theorem 1.2.8 we deduce that $A(K) \leqslant 4$.

If $s(K)$ is infinite, using the above notation, there is a point A on the boundary of K, which is an accumulation point of the infinite set I of endpoints P of chords of symmetry. As the boundary of K is continuous, A is itself an endpoint of a chord of symmetry AOA'. If we consider a sequence (P_n) in I which has limit A, the limit l of the sequence of lines (P_nA) is a supporting line to K at A. Similarly the sequence $(P_n'A')$ has as its limit a line l' which is a supporting line to K at A'. Since, for each n, the line P_nA is parallel to the line $P_n'A'$, it follows that l and l' are parallel. Again, Theorem 1.2.8 gives that $A(K) \leqslant 4$, and the proof is complete.

<u>Corollary 3.2.1</u> If $d(\theta)$ has a zero at which $d(\theta)$ is an extremum, then $c(\theta)$ is an extremal chord of symmetry, and $A(K) \leqslant 4$.

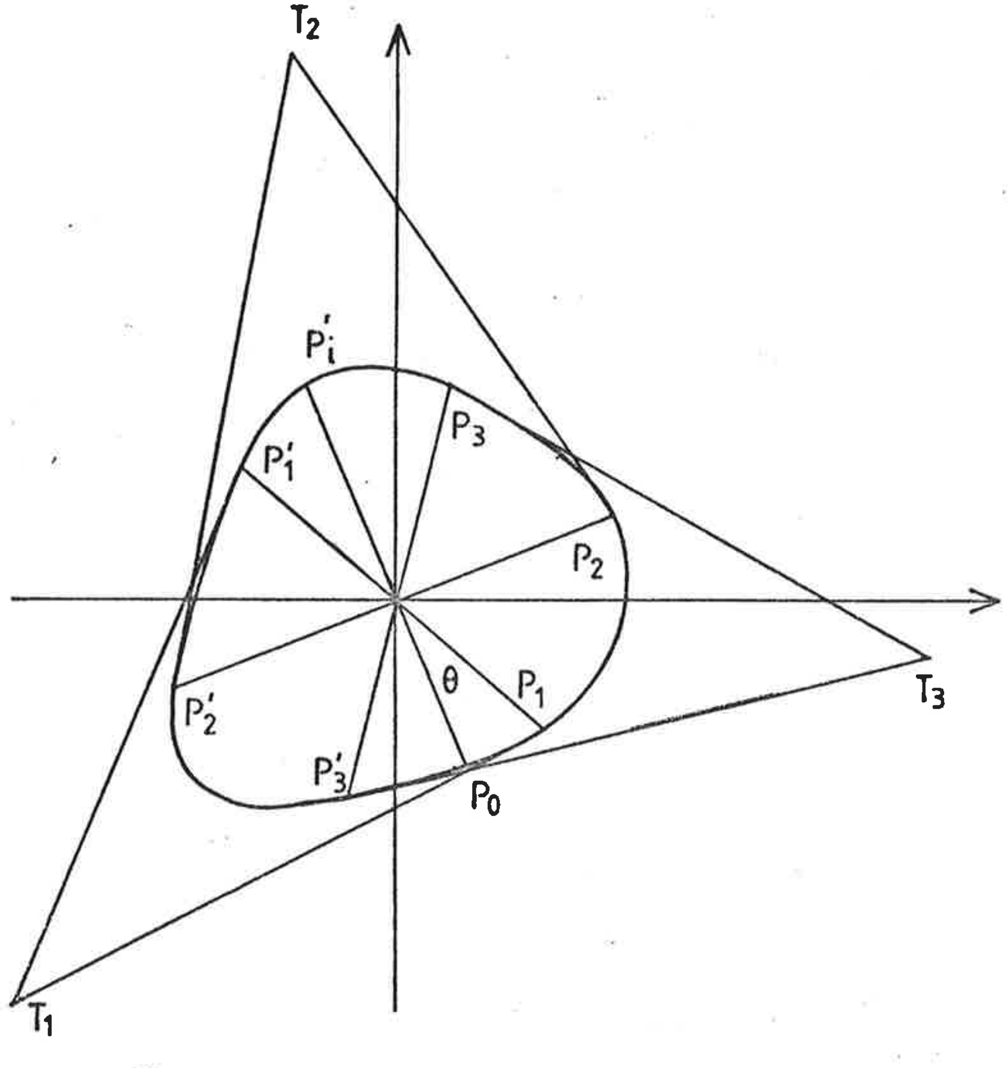


FIGURE 3.1

3.3 Proof of Theorem 3.1.2

By Theorem 3.1.1 and Corollary 3.2.1, $A(K) \leq 4$ if either $s(K) = 2$, or if any zero of $d(\theta)$ occurs at a point where $d(\theta)$ is also an extremum. Suppose then that $s(K) \geq 3$. By lemma 1.2.4 K has three chords of symmetry $P_1'P_1$, $P_2'P_2$ and $P_3'P_3$ labelled successively such that the semitangents to the right at the end-points of $\vec{P_1'P_1}$, to the left at the endpoints of $\vec{P_2'P_2}$ and to the right of $\vec{P_3'P_3}$ meet or are parallel. Since we already have that $A(K) \leq 4$ if any such pair is parallel, we suppose that T_i is the point of intersection of the semitangents indicated at P_i and P_i'. This situation is illustrated in Figure 3.1, opposite.

Let K_i denote the intersection of K with the closed triangular region $P_i P_i' T_i$ $(1 \leq i \leq 3)$, and let $\bar{K}_i$ be the reflection of K_i in 0. By Minkowski's Theorem 1.1.1, as $K_i \cup \bar{K}_i$ is an admissible set, centrally symmetric about 0,

$$A(K_i \cup \bar{K}_i) = 2A(K_i) \leq 4.$$

It is easily seen that $\bigcup_{i=1}^{3} (K \sim K_i) = \bigcup_{i=1}^{3} K_i = K$, and hence that $\bigcap_{i=1}^{3} (K \sim K_i) = \phi$. Let X_i be the interior of $K \sim K_i$. The three sets X_i are open convex sets in E^2 with no point in common. Hence by Theorem 1.2.5 we have that

$$A(K) \geq 9/5 \min_{i=1,2,3} A(K \sim K_i). \tag{2}$$

However, as $A(K_i) \leq 2$ $(1 \leq i \leq 3)$, we also have

$$A(K) \leq 2 + \min_{i=1,2,3} A(K \sim K_i). \tag{3}$$

From (2) and (3) we deduce that $A(K) \leq 4.5$; for equality to be attained here, we required equality in both (2) and (3).

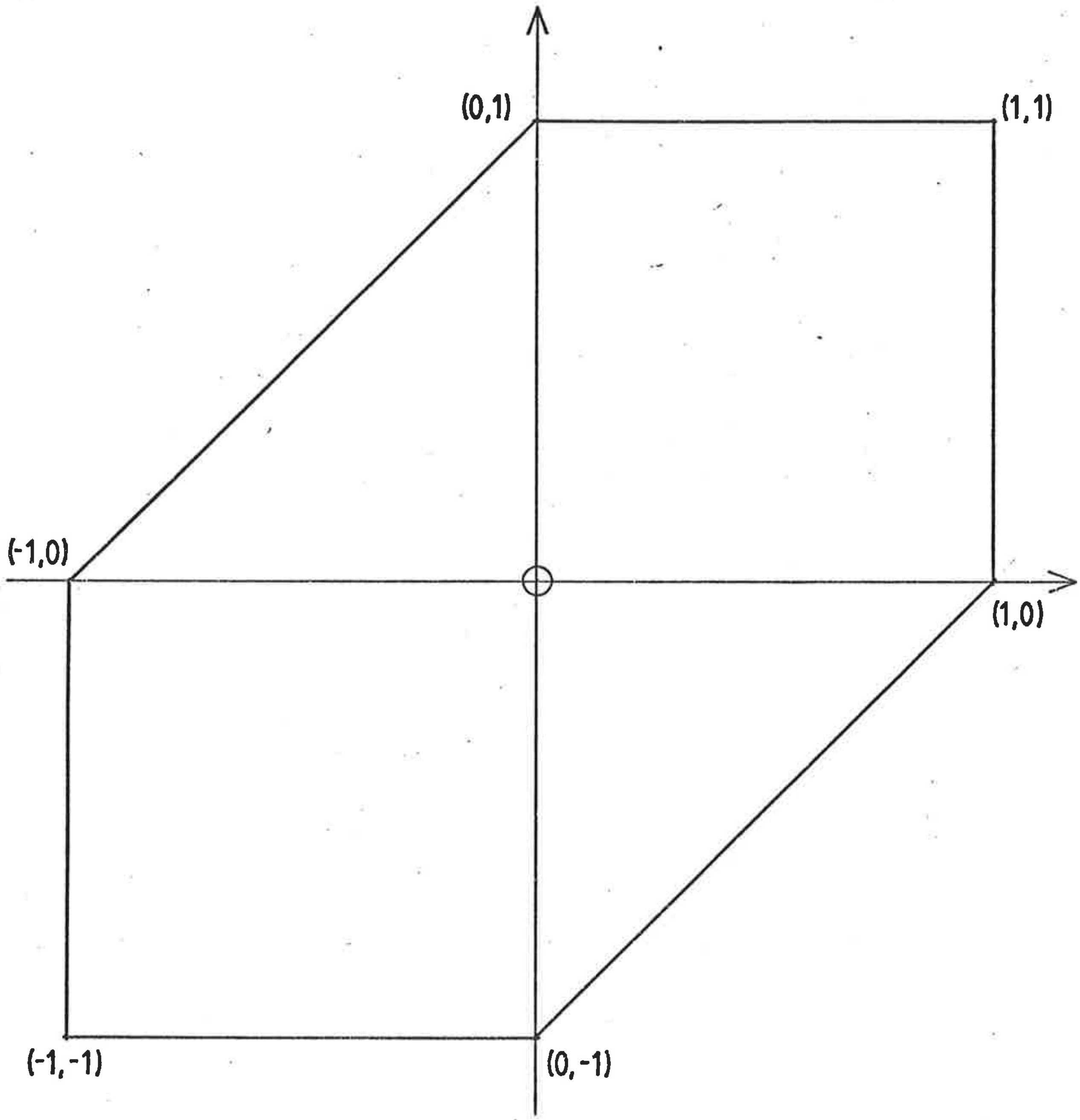


FIGURE 3.2

By Theorem 1.4.1 equality in (2) is possible only when K is a triangle T with centre of gravity 0, and with three chords of symmetry, each parallel to an edge of T. The set E_1 of Section 3.1 is such a triangle.

3.4 Polygons and lattice points

We shall need the following preliminary results. A *lattice polygon* P is a simple polygon which has each vertex at a lattice point. Let v,b,c denote the number of lattice points which are vertices, boundary points, interior points respectively of P, and let A denote the area of P. From Picks' Theorem 1.5.1, it is known that for lattice Λ_0 and lattice polygon P that, $A = \frac{1}{2}b + c - 1$.

Lemma 3.4.1: Let P be a convex lattice polygon.

(i) If $v = 4$, and P has no pair of parallel edges, then $c \geqslant 1$.

(ii) If $v \geqslant 5$, then $c \geqslant 1$.

(iii) If $v = 6$, and $c = 1$, then P is equivalent under an integral unimodular transformation to the centrally symmetric hexagon H_0 illustrated opposite, in Figure 3.2.

(iv) If $v > 6$, $c \geqslant 2$.

Proof Let $P_1 = V_1V_2V_3V_4$ be a convex lattice quadrilateral having no pair of parallel sides and let $P_2 = V_1V_2V_3V_4V_5$ be a convex lattice pentagon. In each case, since it is not possible for the sum of every pair of adjacent angles to be less than or equal π, we may assume that $\angle V_2 + \angle V_3 > \pi$. Suppose too that

V_4 is no further than V_1 from the line V_2V_3. Then, in each case, the vertex X of the parallelogram $V_2V_3V_4X$ is a lattice point which is interior to the polygon P_i. As any lattice polygon with $v > 5$ contains as a subset a lattice pentagon, we have proved (i) and (ii).

Let L be the unique interior lattice point of the convex lattice hexagon $H = V_1V_2V_3V_4V_5V_6$. If L does not lie on V_1V_4, then one of $LV_4V_5V_6V_1$ or $LV_1V_2V_3V_4$ is a proper convex pentagon, and we deduce from (ii) that $c \geqslant 2$. Hence L lies on each of the diagonals V_1V_4, V_2V_5, V_3V_6. In fact, since the lattice points on a line are regularly spaced, the uniqueness of L implies that L is the common midpoint of the diagonals. From (i) we deduce that the diagonals are parallel to the edges of H, and so H is the image of H_0 under some integral unimodular transformation.

Finally, let P be a convex lattice polygon with $v \geqslant 6$ and $c = 1$. Choose 5 vertices of P. By (iii) each remaining vertex of P lies at the unique lattice point which completes a centrally symmetric hexagon about L. Thus $v = 6$. Hence if $v > 6$ we must have $c \geqslant 2$.

<u>Lemma 3.4.2</u> Let H_0, with vertices $V_1V_2V_3V_4V_5V_6$ be the convex lattice hexagon of Figure 3.2. Let X, with vertices $T_1T_2\ldots T_n$ be a convex polygon, with $H_0 \subseteq X$. Let the vertices V_2 and V_3 of H_0 lie on the edge T_2T_3 of X, and let the edges T_1T_2 and T_3T_4 of X have V_1 and V_4 as midpoints respectively. Then T_1, V_5, V_6 and T_4 are collinear. In particular, if each edge of X contains a vertex of H_0, X must be a quadrilateral.

Proof The diagonal V_1V_4 of H_0 is parallel to, and midway between the edges V_2V_3 and V_5V_6. Hence T_1 and T_4 lie at the same distance from V_1V_4 as T_2 and T_3 respectively, that is, on the line V_5V_6.

Lemma 3.4.3 Let M be a polygon having n edges, and having a lattice point at the midpoint of each edge.

(i) If n is odd, each vertex of M is a lattice point.

(ii) If n is even, the lattice polygons with vertices at the alternate midpoints have the same centroid as M.

(iii) If $n = 4m + 2$ $(m \in Z)$, and the midpoints of opposite edges occur in pairs which are symmetric about 0, then M is centrally symmetric about 0.

Proof Let V_i and m_i be the vertices of M and the midpoints of the sides of M respectively, for $i \in I = \{1,2,\ldots,n\}$. With the convention $V_0 = V_n$, we write this as

$$\tfrac{1}{2}(V_{i-1}+V_i) = m_i \in \Lambda, \quad i \in I. \tag{4}$$

When n is odd, equations (4) give

$$v_n = m_1 - m_2 + m_3 \ldots + m_n .$$

Thus $v_n \in \Lambda$, and from (4) we deduce that $v_i \in \Lambda$ for all $i \in I$. When n is even, summing the alternate equations in (4) gives

$$\tfrac{1}{2}(v_1 + \ldots + v_n) = m_2 + m_4 + \ldots + m_n = m_1 + m_3 + \ldots + m_{n-1} . \tag{5}$$

We obtain (ii) simply by dividing (5) by $n/2$.

Now suppose $n = 4m + 2$, and $m_j = -m_{j+2m+1}$, for $j \in J = \{1,2,\ldots,2m+1\}$. Adding the expressions in (4) for m_j and m_{j+2m+1}, we get

$$\tfrac{1}{2}(v_{j-1}+v_j+v_{j+2m}+v_{j+2m+1}) = 0, \quad j \in J. \tag{6}$$

Taking the alternating sum of these equations gives

$$\tfrac{1}{2}(v_{4m+2}+v_{2m+1}) = 0 ;$$

that is $v_{4m+2} = -v_{2m+1}$. Using equations (6), it now follows that M is symmetric in 0.

3.5 An analytical result

In the proof of Theorem 3.1.3 we will use two continuous transformations and a limiting process to reduce the number of admissible sets under consideration. We prove

Lemma 3.5.1 Let (K_i) be a sequence of admissible convex sets such that $K_i \to K$ in the Hausdorff metric. Suppose that for all i, $s(K_i) = k$, an odd number, and also that K_i has no extremal chord of symmetry. Then either

(i) $s(K) = k$, and K has no extremal chord of symmetry,

or (ii) $A(K) \leqslant 4$.

Proof Let $d(\theta)$ be defined for the set K as in Section 3.2, with θ measured from a direction such that $d(0) = -d(\pi) \neq 0$. For all i, we define the function $d_i(\theta)$ corresponding to the set K_i in the same manner, with θ measured from the same direction used for the set K. As no chord of K_i is extremal, by Corollary 3.2.1 we have that no zero of $d_i(\theta)$ is an extremum of $d_i(\theta)$. By elementary analysis, we choose an infinite subsequence (K_i), $i \in G \subseteq N$, such that the zeros of $d_i(\theta)$, $i \in G$, on the interval $[0,\pi)$, form k convergent subsequences. Since we lose no generality by doing so, we assume that the sequence (K_i) is this subsequence (K_i), $i \in G$. We now show that unless $A(K) \leqslant 4$, then the limits $\theta_1, \theta_2, \ldots, \theta_k$ of the sequences of zeros of the functions $d_i(\theta)$ are all the zeros of $d(\theta)$, and are distinct.

In Section 3.2 we noted that each function $d_i(\theta)$ is continuous in $[0,\pi)$. Since (K_i) converges to K in the Hausdorff metric, the functions $d_i(\theta)$ converge to $d(\theta)$ uniformly. Suppose $d(\theta_1) \neq 0$ where θ_1 is the limit of a convergent sequence of zeros of $d_i(\theta)$. In a neighbourhood of θ_1, $|d(\theta)| \geqslant \frac{1}{2}|d(\theta_1)|$, by the continuity of $d(\theta)$. Hence, by the uniform convergence of $d_i(\theta)$ to $d(\theta)$, there exists a number N, so that for $i \geqslant N$, $|d_i(\theta)| > 0$ for θ in this neighbourhood of θ_1. Therefore no zeros of $d_i(\theta)$ lie in this neighbourhood of θ_1, for $i \geqslant N$. By contradiction we have proved that limits of convergent sequences of zeros of $d_i(\theta)$ are zeros for $d(\theta)$.

Suppose θ_{k+1} is a zero of $d(\theta)$ on $[0,\pi)$, not one of the limits above. If θ_{k+1} is an accumulation point of the set of zeros of $d(\theta)$, then K has an infinite number of chords of symmetry, and Theorem 3.1.1 then implies $A(K) \leqslant 4$. If θ_{k+1} is an extremum of $d(\theta)$, then Corollary 3.2.1 implies that $A(K) \leqslant 4$. We may therefore assume that θ_{k+1} lies between two neighbourhoods $[\theta_-, \theta_{k+1})$ and $(\theta_{k+1}, \theta_+]$ in which $d(\theta)$ is non zero and of opposite sign. By the uniform convergence of $d_i(\theta)$ to $d(\theta)$, there exists a number N such that for $i \geqslant N$, $d_i(\theta)$ is non zero and of opposite sign in the neighbourhoods $[\theta_-,(\theta_-+\theta_{k+1})/2]$ and $[(\theta_{k+1}+\theta_+)/2,\theta_+]$. As each $d_i(\theta)$ is a continuous function, for $i \geqslant N$, $d_i(\theta)$ has a zero in the interval $((\theta_-+\theta_{k+1})/2,(\theta_{k+1}+\theta_+)/2)$. By again choosing a subsequence of the sequence of functions $(d_i(\theta))$, we can assert that there is a sequence of zeros of the functions $d_i(\theta)$ lying in the above interval, which converge. As shown in the previous paragraph, this sequence converges to a zero of $d(\theta)$ which lies in the above interval, and so must converge to θ_{k+1}. As this contradicts our choice of θ_{k+1}, we deduce that $\theta_1,\theta_2,\ldots,\theta_k$ are the only zeros of $d(\theta)$.

Finally we show that $\theta_1,\ldots,\theta_k$ are distinct. Suppose to the contrary that $\theta_1 = \theta_2$. Without loss of generality we assume that θ_1 and θ_2 are the limits of $(\theta_1(n))$ and $(\theta_2(n))$ respectively, where $\theta_1(n) < \theta_2(n)$ are zeros of $d_n(\theta)$. For given n, and $j \in \{1,2\}$, the zero of $d_n(\theta)$ at $\theta_j(n)$ corresponds to a chord of symmetry $P_{j,n}OP'_{j,n}$ of K_n. As n becomes large, $P_{1,n}$ and $P_{2,n}$ approach P, on the chord of symmetry POP' of K corresponding to the zero of $d(\theta)$ at $\theta_1 = \theta_2$. By again taking a suitable subsequence of K_i (if necessary) and renaming this (K_i), we assume that the limit as n becomes large of the angle of orientation of the chord $P_{1,n}P_{2,n}$ exists. We call this limit φ.

We show that the line l through P, with angle of orientation φ, is a supporting line to K at P. For suppose to the contrary that l is a proper chord of K which meets the boundary of K at Q. By taking n sufficiently large to make $P_{1,n}$ and $P_{2,n}$ very close to P, and the angle of orientation of the chord $P_{1,n}P_{2,n}$ very close to φ, we may assume that the line $P_{1,n}P_{2,n}$

passes as close as we choose to Q. By convexity K_n is bounded by the line $P_{1,n}P_{2,n}$ outside the segment $P_{1,n}P_{2,n}$. Thus, taking values of n approaching infinity, and since K_n converges to K, K is bounded by the limit of the line $P_{1,n}P_{2,n}$. As this limit is l, we contradict our choice of l as a proper chord of K. Similarly, we can construct a supporting line to K at P_* parallel to l. By Theorem 1.2.8, we then deduce that $A(K) \leqslant 4$.

3.6 Proof of Theorem 3.1.3

We now prove Theorem 3.1.3. From Theorem 3.1.1 we may assume s(K) is odd, and so at least five. As 0 is an interior point of the convex set K, by Cohn's Theorem 1.3.1, K is separated from all non-zero lattice points by a finite number of lines. Hence K is contained in a polygon $\mathcal{K}$, bounded by these lines and supporting lines to K at the endpoints of its chords of symmetry. We may clearly choose $\mathcal{K}$ so that each edge of $\mathcal{K}$ contains in its relative interior at least one endpoint of a chord of symmetry or a lattice point. As $\mathcal{K}$ has no smaller area and no fewer lines of symmetry than K, we take K to be the polygon $\mathcal{K}$. Should an edge of K contain an endpoint of each of three or more chords of symmetry, by the convexity of K the other endpoints lie on a parallel edge. From Theorem 1.2.8, we then know that $A(K) \leqslant 4$. We therefore may assume that the polygon K has at least five edges.

We now modify the polygon K so that each edge of K contains at least one lattice point in its relative interior. If an edge E of K contains no lattice point in its relative interior, we form edge E(t), parallel to E and distance t further than E away from 0. Denote by $K(t)$ the polygon obtained from K by including this new edge and extensions of the edges of K adjacent to it. We continue to increase t until one of three things happens:

(a) the length of $E(t)$ becomes zero, or

(b) $E(t)$ has a lattice point on it, in its relative interior, or

(c) $s(K(t)) \neq s(K)$.

One of these things must happen, since as K contains a disc about 0, each modified set $K(t)$ also contains the same disc, and, by Theorem 1.3.1 such sets are uniformly bounded.

Suppose (c) occurs either with or before either (a) or (b). This can occur in two ways. Firstly, we suppose that $K(t^*)$ is the first modified set for which $s(K) \neq s(K(t^*))$. As neither (a) or (b) are true for $t < t^*$, each of the sets $K_i = K(t^*.(1-1/i))$ is admissible, and since (c) is false also for $t < t^*$, $s(K_i) = s(K)$ is odd for each set. As the sequence (K_i) converges to $K(t^*)$ in the Hausdorff metric, we can deduce from lemma 3.5.1 that $A(K(t^*)) \leqslant 4$.

Secondly, we suppose that $s(K) = s(K(t^*))$ for all $t \in [0,t^*]$, but that this equality holds on no longer interval. By our assumption that (c) occurs with or before (a) or (b), and noting that the transformation is such that (a) or (b) must occur for some first modified set, we may assume that neither (a) nor (b) occur for $t < t^* + \delta$, for some $\delta > 0$. Thus we can choose a sequence $K_i = K(t)$, $t \in (t^*, t^*+\delta.1/i)$, of admissible sets, such that (K_i) converges to $K(t^*)$ in the Hausdorff metric. We assume that $s(K_i)$ is odd and finite for each i, since otherwise we could deduce by Theorem 3.1.1 that $A(K_i)$, and so $A(K(t^*))$, is no greater than 4. In fact, as only three edges of $K(t)$ are modified by the transformation, the value of $s(K_i)$ differs from $s(K(t^*))$ by at most six, for we showed in an earlier argument that $A(K(t)) \leq 4$ if any edge of $K(t)$ contains the endpoints of more than two chords of symmetry. Hence we can choose a subsequence of K_i, which we relabel (K_i), so that each $s(K_i)$ is odd, constant, and not equal to $s(K(t^*))$. By applying Lemma 3.5.1 to this sequence (K_i), we deduce that $A(K(t^*)) \leq 4$.

We may therefore assume that K has at least five edges, each of which contains a lattice point in its relative interior. By Lemma 3.4.1(ii) we deduce that 0 is an interior point of the lattice polygon Y whose vertices are those lattice points on the boundary of K.

We now further modify K so that each edge E of K either contains two lattice points in its relative interior, or one lattice point at its midpoint. We show that such a modification increases the area $A(K)$, does not decrease $s(K)$ and does not remove from the boundary of K any of the lattice points on it. Suppose L, not the midpoint of edge E of K, is the only lattice point in the relative interior of E. We replace E by another edge $E(\psi)$, through L, at angle ψ to E. As L is not the midpoint of E, we may orient ψ so that for (small) positive ψ, the set $K(\psi)$ bounded by $E(\psi)$ and the extensions of edges of K other than E, has area greater than $A(K)$. As the boundary of K is continuous, and since $E(\psi)$ pivots about L which is not its midpoint, $A(K(\psi))$ increases continuously with ψ until one of three things happens:

(d) L is the midpoint of $E(\psi)$, or

(e) $E(\psi)$ contains two lattice points in its relative interior, or

(f) $s(K(\psi)) \neq s(K)$.

No lattice points are lost from the boundary of K in this modification of K. For, were L_1 to be lost at angle ψ, as L is not the mid-point of $E(\psi)$, the lattice point $2L-L_1$ lies in the relative interior of $E(\psi)$. Hence the convex hull of those lattice points on the boundary of K, $Y \subseteq K(\psi)$. As 0 is an interior point of Y, we have by Cohn's Theorem 1.3.1 that the sets $K(\psi)$ are uniformly bounded. Hence, for sufficiently large ψ, (d) or (e) must occur. Suppose (f) occurs either with or before either (d) or (e).

This can occur in two ways. Firstly suppose $K(\psi^*)$ is the first modified set for which $s(K(\psi^*)) \neq s(K)$. The sequence of admissible sets $K_i = K(\psi^*(1-1/i))$ converges to $K(\psi^*)$ in the Hausdorff metric, and $s(K_i) = s(K)$ is odd for each set K_i in the sequence. By applying Lemma 3.5.1 to the sequence (K_i) we deduce that $A(K(\psi^*)) \leqslant 4$.

Secondly, suppose that $s(K) = s(K(\psi))$ for all $\psi \in [0,\psi^*]$, but this equality holds on no longer interval. We can apply the same argument used above for $K(t)$, noting carefully that only three edges of K are modified by this transformation on any sufficiently small open interval. As above we can deduce that $A(K(\psi^*)) \leqslant 4$ in this case.

In performing the above modifications to the polygon K, there is a possibility that an infinite sequence of modifications may occur. For example, a sequence of edge turning modifications may successively upset the previously set lattice point midpoints of edges. However, all the modifications of K contain Y, and so contain a fixed disc about 0. By Cohn's Theorem 1.3.1, all such modifications of K are bounded by a uniform bound.

We may thus apply Blaschke's Selection Theorem 1.7.1 to any such sequence of modifications of K, (K_i), and obtain a limiting figure K^*, which can no longer be modified. However, $s(K_i) = s(K)$ is odd for each of these admissible sets K_i, and the sequence K_i converges to K^* in the Hausdorff metric. Hence by Lemma 3.5.1 we deduce that unless $s(K^*) = s(K)$ then $A(K) \leqslant 4$.

We therefore may assume $s(K^*) = s(K)$, and since K^* is immutable by the above modifications it must be a convex polygon, with at least five edges, each edge containing at least one lattice point in its relative interior. Further, if any edge E of K^* contains only one lattice point L in its relative interior, L is the midpoint of E. We call such an edge E a *lattice midpoint edge*. Since $s(K^*) = s(K) \geqslant 5$, by Theorem 3.1.2 we deduce that $A(K^*) \leqslant 4.5$.

Denote by Z the convex lattice polygon formed by the convex hull of the set of lattice points which lie on the boundary of K^*, but are not vertices of K^*. As 0 is the only possible lattice point in the interior of $Z \subseteq K^*$, we deduce from Lemma 3.4.1 (iv) that K^* has either,

(g) five edges, one containing two lattice points in its relative interior, the other four edges each lattice midpoint edges, or

(h) five edges, each lattice midpoint edges, or

(i) six edges, each lattice midpoint edges.

We first show that cases (g) and (h) cannot arise. In case (g), by Lemma 3.4.1(iii), Z, having six lattice points on its boundary, is a centrally symmetric hexagon H which is an affine transform of H_0, the hexagon shown in Figure 3.2. The incidences of K^* with H listed in (g), become, when transformed by the inverse of this affine transformation, incidences of the image of K^* with H_0. Indeed, these incidences are exactly those needed for Lemma 3.4.2. Hence the image of K^*, and hence K^*, is a quadrilateral. Hence case (g) cannot occur. In case (h), by Lemma 3.4.3(i), each vertex of K^* is also a lattice point, and so K^* has ten boundary lattice points. By Pick's Theorem 1.5.1, we deduce that $A(K^*) = 5$. However, since this contradicts the bound on $A(K^*)$ given above by Theorem 3.1.2, case (h) cannot arise.

Finally, in case (i), by Lemma 3.4.1(iii), as Z has six vertices and only one interior lattice point, it is a centrally symmetric hexagon. Thus the lattice points on the boundary of K^* number 6, and are symmetrically located about 0. By Lemma 3.4.3(iii) then, K^* is itself centrally symmetric about 0. By Minkowski's Theorem 1.1.3, we deduce that $A(K^*) \leqslant 4$.

3.7 Boundary Lattice Points

We now give a simple corollary to Theorem 3.1.2, attained by using lemma 3.4.1 to describe the configuration of lattice points on the boundary of a set.

Corollary 3.7.1 Let K be a strictly convex compact set in E^2 which contains in its interior only one lattice point 0, and which has more than four lattice points on its boundary. Then K contains either 5 or 6 lattice points on its boundary, and has area no more than $4.5\det(\Lambda)$.

<u>Proof</u> Let Z denote the closed convex hull of $K \cap \Lambda$. As K is strictly convex and has more than four lattice points on its boundary, Z is a convex lattice polygon with at least five vertices. Since $Z \subseteq K$ and as K contains only one point 0 of Λ in its interior, we deduce from lemma 3.4.1 that $0 \in Z$, and that Z has at most 6 vertices. Indeed if Z has six vertices it is centrally symmetric about 0, and so the three common chords of Z and K passing through the vertices of Z and 0 are each chords of symmetry of K. By Theorem 3.1.2 therefore, as $s(K) \geqslant 3$, $A(K) \leqslant 4.5\det(\Lambda)$.

Suppose now that Z has only five vertices. We claim that Z and K share two common chords whose endpoints are vertices of Z, and which are bisected by 0. The conclusion of the theorem then follows, as $s(K) \geqslant 2$ and so by Theorem 3.1.2, $A(K) \leqslant 4.5\det(\Lambda)$. Now Z has no non-zero lattice point in its interior. Also, Z has no lattice point in the relative interior of any edge, since the strict convexity of K would then place such a lattice point in the interior of K. We label Z cyclically as $A_1A_2A_3A_4A_5$, and note that since Z has 5 vertices, at least one vertex, say A_1, is the only vertex of Z on the line A_10. The line A_10 must at the boundary of Z between vertices A_3 and A_4 since otherwise either $A_10A_3A_4A_5$ or $A_1A_2A_3A_40$ is a convex lattice 5-gon which by lemma 3.4.1 must contain a second interior lattice point in K. Thus both $A_1A_2A_3O$ and $A_1OA_4A_5$ are convex lattice 4-gons with no point of Λ in their interiors. By lemma 3.4.1 each of these 4-gons has a pair of parallel edges. As these edges are either internal to Z or on the boundary of Z, they contain no points of Λ in their relative interior. Each 4-gon is therefore a parallelogram and so A_1O, A_2A_3 and A_4A_5 are adjacent parallel lattice lines. Let $\bar{A}_1 \neq A_1$ be the lattice point on A_1O adjacent to O. Since there are no points of the lattice between the lines A_2A_3 and A_4A_5 other than those on A_1O, the lattice hexagon $H = A_1A_2A_3\bar{A}_1A_4A_5$ contains only the lattice point O in its interior. By lemma 3.4.1(iii), this hexagon H is therefore an integral unimodular transform of H_0. Thus A_2OA_4 and A_3OA_5 are chords of H, and so chords of symmetry of K as required above.

3.8 Incentre, Circumcentre Results

We now deduce an analogue of Theorem 1.2.1, which stated that an admissible set with centre of gravity O has area at most $4.5\det(\Lambda)$. We show

Theorem 3.8.1 Let K be an admissible set. Then $A(K) \leqslant 4.5\det(\Lambda)$ if either of the following conditions hold

(i) the smallest homothet of any given symmetric convex set K_1 which contains K is centred at O.

(ii) a largest homothet of any given symmetric convex set K_2 which is contained by K is centred at O.

In particular, when K_1 or K_2 is a circle, we can restate the theorem as,

<u>Corollary 3.8.1</u> Let K be an admissible set. Then $A(K) \leqslant 4.5\det(\Lambda)$ if either of the following conditions hold

(i) 0 is the circumcentre of K.

(ii) 0 is an incentre of K.

<u>Proof of Theorem 3.8.1</u> By Theorems 1.2.1 and 3.1.2, the result will follow if we can show that under either of the conditions (i) or (ii), K has either an extremal chord of symmetry, or that $s(K) \geqslant 2$. We show that this is so in the following two lemmas. With these lemmas, then, Theorem 3.8.1 is proved.

<u>Lemma 3.8.2</u> Let K be a convex set in the plane which contains 0 as an interior point. We suppose that the smallest homothet of a given symmetric convex set K_1 which contains K is centred at 0. Then K has either an extremal chord of symmetry or has $s(K) \geqslant 3$.

<u>Proof</u> Without loss of generality, we suppose that K_1 is a symmetric convex set containing K, centred at 0, no smaller homothet of which (about any centre) contains K. In order to discuss the chords of K through 0, we first show that the boundaries of K and K_1, ∂K and ∂K_1, meet in at least three points forming a triangle containing 0, or that K and K_1 share an extremal chord of symmetry.

Plainly $\partial K \cap \partial K_1$ is at least one point, for otherwise a smaller homothet of K_1 centred at 0 will still contain K. If $\partial K \cap \partial K_1$ is just one point P, we translate K_1 a small distance away from P, orthogonal to a supporting line of K

at P, to the set K_1' which still contains K and whose boundary does not meet ∂K. A smaller homothet of K_1' say K_1'' will therefore contain K, and since the product of a homothet and translation is a homothet with different centre, K_1'' is a smaller homothet of K_1 which contains K. By our choice of K_1 we deduce a contradiction. Therefore $\partial K \cap \partial K_1$ is at least two points.

We now suppose that P and P' are points of $\partial K \cap \partial K_1$ which are collinear with 0. As K_1 is 0-symmetric, the chord POP' of K is a chord of symmetry of K. As K_1 is 0-symmetric, K_1 has a pair of parallel supporting lines at P and P', which since $K \subseteq K_1$ are also supporting lines to K at P and P'. Thus K has an extremal chord of symmetry.

We next suppose that $\partial K \cap \partial K_1$ has no pair of points collinear with 0. We show that in this case there are three points P_1 P_2 and P_3 of $\partial K \cap \partial K_1$ such that 0 is interior to the triangle $P_1P_2P_3$. Supposing to the contrary that 0 is not interior to any such triangle, we let P_1P_2 be the common chord of K and K_1 which passes closest to 0. Hence the open arc of the boundary of K between P_1 and P_2, to the side of P_1P_2 containing 0, contains no point of ∂K_1 in its convex hull. Since K_1 is centrally symmetric about 0, the semitangents to K at P_1 and P_2, to the side of P_1P_2 containing 0, are either parallel or meet at a point Q to the side of P_1P_2 not containing 0. Hence we can translate K_1 to a set K_1' in the direction $\vec{0Q}$ by a suitably small distance, so that $\partial K_1' \cap \partial K = \phi$. That this intersection is empty is assured by our choice of P_1 and P_2 and by the fact that there are no points of ∂K_1 on the arc of ∂K from P_1 to P_2 which encloses 0. Since a smaller homothet of K_1', which is a smaller homothet of K_1, thus contains K, we deduce a contradiction against the choice of K_1 above. Hence there are points $P_1P_2P_3$ of $\partial K \cap \partial K_1$ which form a triangle strictly containing 0.

The three chords $P_j OP_j'$, $j=1,2,3$, of K have the property that $|OP_j| > |OP_j'|$, since $K \subset K_1$, and since $\partial K \cap \partial K_1$ has no pair of points collinear with 0. Since 0 is an interior point of the triangle $P_1P_2P_3$, the six endpoints of these chords are given successively around ∂K by $P_1P_3'P_2P_1'P_3P_2'$. Since the boundary of K is continuous, between each of P_1 and P_3', P_3' and P_2 and P_2 and P_1' lies the endpoint of a chord of symmetry of K. Hence $s(K) \geqslant 3$, and the lemma is proved.

<u>Lemma 3.8.3</u> Let K be a convex set in the plane which contains 0 as an interior point. We suppose that a largest homothet of a given symmetric convex set K_2 which is contained by K is centred at 0. Then K has either an extremal chord of symmetry or has $s(K) \geqslant 2$.

<u>Proof</u> Without loss of generality, we suppose that K_2 is a symmetric convex set contained in K, centred at 0 no smaller homothet of which (about any centre) is contained in K. As in Lemma 3.8.2, we can easily show that $\partial K \cap \partial K_2$ consists of at least two points.

We now suppose that points P and P' collinear with 0 belong to $\partial K \cap \partial K_2$. As K_2 is 0-symmetric, the chord POP' is a chord of symmetry of K. A supporting line at P or P'

to K_2 need not be a supporting line to K; this may happen when P or P' is a corner point of K_2. As we need only consider the case when K has no extremal chord of symmetry, we suppose that K has supporting lines ℓ and ℓ' at P and P' respectively, which meet in a point Q. Indeed we choose these lines to be the semitangents to K at P and P' to the side of POP' away from Q. If $\partial K \cap \partial K_2$ has no points on the side of POP' away from Q, we can translate K_2 in the direction $\vec{QO}$ by a sufficiently small distance that its translate K_2' has no points in common with ∂K. By our choice of ℓ and ℓ', neither P nor P' lies on $\partial K_2'$. Since a larger homothet of K_2', and so of K_2, is therefore contained in K we derive a contradiction against our choice of K_2. There is thus at least one point P_1 of $\partial K \cap \partial K_2$ on the side of POP' away from Q.

Since ℓ and ℓ' meet, there are points R_1 and R_2, arbitrarily close to P and P', on the boundary of K, to the side of POP' containing Q, with the following property. The chord R_iOR_i' of K is divided by O such that $|R_iO| < |OR_i'|$, for $i=1,2$. Because $K_2 \subset K$, the chord P_1OP_1' of K has the property that $|P_1'O| \geq |OP_1|$. As the boundary of the convex set K is continuous, between R_1 and P_1', (or possibly at P_1') there is an endpoint P_2 of a chord of symmetry P_2OP_2' of K. That is $|P_2O| = |OP_2'|$. As POP' is also a chord of symmetry of K, $s(K) \geq 2$, and the result follows.

We now can suppose that $\partial K \cap \partial K_2$ has no pair of points collinear with O. As in lemma 3.8.2, we can easily show that there are three points $P_1P_2P_3$ of $\partial K \cap \partial K_2$, such that O is an interior point of the triangle $P_1P_2P_3$. The three chords

$P_i OP_i'$, $i=1,2,3$ of K on which these points lie have the property that $|OP_i| < |OP_i'|$. As O lies in the interior of the triangle $P_1P_2P_3$, the endpoints of these chords are ordered cyclically $P_1P_3'P_2P_1'P_3P_2'$ about ∂K. By the continuity of the boundary of K there is between each of P_1 and P_3', P_3' and P_2 and P_2 and P_1' the endpoint of a chord of symmetry of K. Thus $s(K) \geqslant 3$ in this case, and the proof of the lemma is completed.

CHAPTER 4

Minkowski's Theorem with Other Measures of Symmetry

4.1 Introduction

Let Λ be a lattice in the plane having determinant $\det(\Lambda)$. As in Chapter 3 we say that the set K is *admissible* if it is a closed convex set in the plane with 0 the only point of Λ in its interior.

Definition 4.1.1 We define a *chord of areal symmetry* to be a chord through 0 which splits K into two regions of equal area. A chord through 0 equidistant from two parallel lines of support of K we call a *midchord of symmetry*, while we say that if the set K is invariant under reflection in a chord through 0, then that chord is a *chord of reflective symmetry*. A *chord of perimeter symmetry* passes through 0 and partitions the boundary of K into two arcs of equal length. A diametral chord of perimeter symmetry is such a chord at whose endpoints K has lines of support perpendicular to the chord. We denote the number of each of these types of chords by $a(K)$, $m(K)$, $r(K)$, $p(K)$ and $dp(K)$ respectively. Additionally, if an admissible set K is invariant under a rotation centred at 0 of $2\pi/n$ radians, we let $t(K) = n$, where n is the largest such integer, or ∞ if K is fixed under a rotation by an irrational multiple of π, (in which case K is simply a circular disc).

Definition 4.1.2 We say that an integer valued function (possibly infinite) $f(K)$, defined on the set of admissible sets, is an *M-function*, when

(i) $f(K) > 1$ and $f(K) \neq 3$ imply that $A(K) \leqslant 4 \det(\Lambda)$,

and (ii) $f(K) = 3$ implies that $A(K) \leqslant 4.5 \det(\Lambda)$.

In Chapter 3, we showed that $s(K)$ is an *M-function*.

We show

Theorem 4.1.1 The function $a(K)$ is an *M-function*.

Theorem 4.1.2 The function $m(K)$ is an *M-function*.

Theorem 4.1.3 The function $r(K)$ is an *M-function*, and $r(K) = 3$ only if adjacent chords of reflective symmetry form angles of $\pi/3$.

Theorem 4.1.4 The function $t(K)$ is an *M-function*.

We show that Theorems 4.1.1 and 4.1.2 give the best possible bounds on $A(K)$ for all Λ and all values of $a(K)$ and $m(K)$. Each of Theorems 4.1.3, 4.1.4 gives the best possible bound on $A(K)$ for at least one lattice when $r(K)$, $t(K)$ equals 2,3,4 or 6. Under a linear transformation of K and Λ, the values of $A(K)/\det(\Lambda)$, $a(K)$ and $m(K)$ are invariant. It suffices to show Theorems 4.1.1 and 4.1.2 are best possible with respect to any one lattice.

In the hexagonal lattice generated by $a = (2,0)$ and $b = (1,\sqrt{3})$, the equilateral triangle T, with vertices $v_1 = -a + 2b$, $v_2 = -b + 2a$ and $v_3 = -b - a$ has area $4.5\det(\Lambda)$. Each chord through the origin 0 and a vertex of T is simultaneously a chord of areal symmetry, a midchord of symmetry, a chord of reflective symmetry and a diametral chord of perimeter symmetry. Also, as T is invariant under a rotation of $2\pi/3$ about 0,

$t(T) = 3$. Hence the bound of $4.5\det(\Lambda)$ can be attained in each of the above theorems. In the same lattice, the regular hexagon H with vertices $\pm 2/3v_1$, $\pm 2/3v_2$, $\pm 2/3v_3$, has area $4\det(\Lambda)$, and $r(H) = t(H) = dp(H) = 6$, $a(H) = m(H) = \infty$. We obtain two further critical examples for Theorems 4.1.3, 4.1.4 in the integer lattice Λ_0. Let U be the square with vertices $(\pm 1, \pm 1)$, and let R be the rectangle with vertices $\pm(3/2,-1/2)$, $\pm(1/2,-3/2)$. Only the four chords of U parallel to and midway between the coordinate axes are chords of reflective symmetry for U, while only the chords midway between the axes are chords of reflective symmetry for R. As $A(U) = A(R) = 4 = 4\det(\Lambda_0)$ we have shown theorem 4.1.3 to be best possible, in four instances. Since $t(U) = 4$ and $t(R) = 2$, we have also shown theorem 4.1.4 to be best possible, in these same four instances.

The argument that Theorem 4.1.1 is best possible when $a(K)$ is finite is a little more complicated. Let $\text{int}\, U$ denote the interior of U, and let $C \subseteq \text{int}\, U$ be a closed, convex, proper $2n$-gon $(n \geqslant 2)$ which is centrally symmetric about 0. Further, let the edges of C be labelled $e_1, e_2, \ldots, e_{2n}$ in a clockwise direction, and let e_1 be parallel to the x-axis and bisected by the positive y-axis. For given small $\eta > 0$, we may assume that $A(C) \geqslant A(U) - \eta$. We now modify to produce a set C' for which $C \subseteq C' \subseteq U$ and $a(C') = n$.

First suppose that n is even. To edge e_1, add a small scalene triangle τ for which the area lying in the halfplane $x \leqslant 0$ exceeds the area lying in $x \geqslant 0$ by $\varepsilon/2$ $(\varepsilon > 0)$. To the edge e_{n+1}, add the mirror image of τ in the x-axis. To the edges

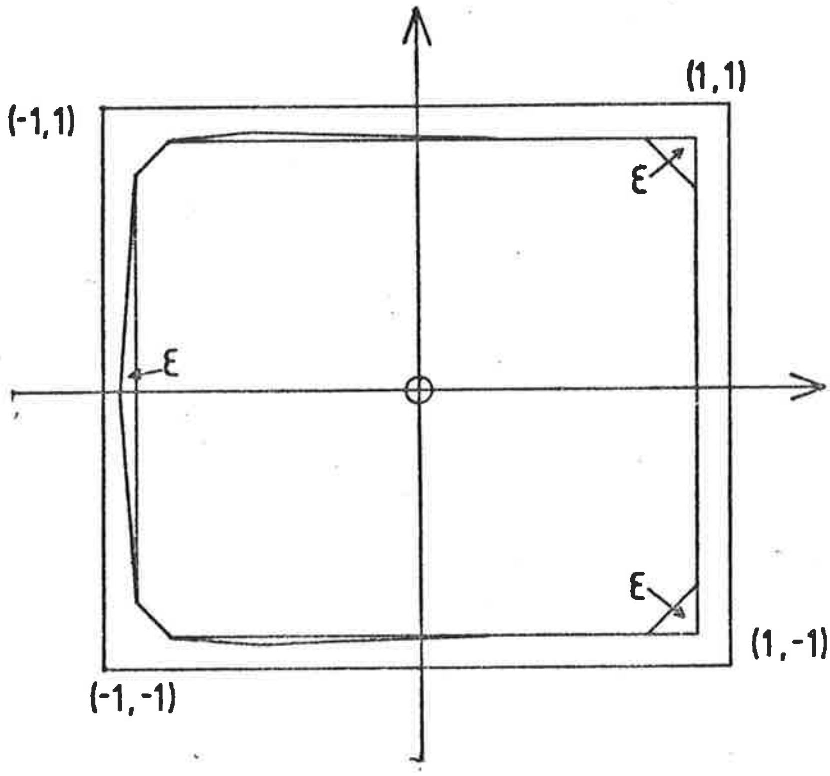


FIGURE 4.1

$$e_2, e_4, \ldots, e_n, e_{n+3}, e_{n+5}, \ldots, e_{2n-1}$$

add a small region of area ε. (Figure 4.1, Opposite, shows the case $n = 4$.) By choosing ε sufficiently small we can ensure that the resulting set C' is still convex, $C' \subseteq U$, and $A(U) - \eta < A(C') < 4 = A(U)$. Also, C' has precisely n chords of areal symmetry, namely, the chord along the y-axis, and the $n/2 + (n/2-1)$ chords which bisect the added regions on edges other than e_1 and e_{n+1}.

For odd n, we need not isolate a pair of edges e_1 and e_{n+1} for special treatment as in the above case for even n. Instead, add to each of the edges $e_2, e_4, \ldots, e_{2n}$ a region of area $\varepsilon > 0$, sufficiently small to ensure that the resulting set $C' \subset U$ is still convex. The chords of areal symmetry of C' are precisely the n chords passing through 0 which bisect the area of one of these added regions.

Finally, we argue that Theorem 4.1.2 is best possible when $m(K)$ is finite. Let $C \subseteq \text{int } U$ be specified as above. We now modify C to produce a set C'' for which $C \subseteq C'' \subseteq U$ and $m(C'') = n$. To each of the edges $e_1, \ldots, e_n$ we add a region of small positive area, so that the resulting set C'' is convex, $C'' \subseteq \text{int } U$, and so that each vertex of C is a boundary point of C''. We choose the added regions, so that C'' has a unique supporting line at each of the $n-1$ vertices incident with $e_2, \ldots, e_{n-1}$, and a unique pair of parallel supporting lines at the pair of opposite vertices incident with e_1 and e_n. This latter pair of parallel supporting lines, together with the $n-1$ unique pairs of parallel supporting lines at the other pairs of opposite vertices, lie parallel to n midchords of symmetry through 0.

We show that there are no other midchords of symmetry. In each pair of parallel supporting lines of C", one line must meet one of the edges $e_{n+1},\ldots,e_{2n}$, and hence be incident with a vertex of C". Therefore any pair of parallel supporting lines equidistant from 0 must meet C" at a pair of opposite vertices of C", and so must be one of the n unique such pairs counted above. As $A(C") > A(C) \geqslant A(U) - \eta = 4\det(\Lambda) - \eta$, the bound of Theorem 4.1.2 cannot be improved.

The example chosen above suggests perhaps that $A(K) = 4\det(\Lambda)$ is impossible when $m(K)$ is finite and not one or three. As a counter-example, the trapezium Z with vertices (-1,2), (2,-1), (-1,0) and (0,-1) is admissible with respect to Λ_0, but $A(Z) = 4\det(\Lambda_0)$, and $m(Z) = 2$.

4.2 Area Symmetry

Proof of Theorem 4.1.1 Theorems 1.2.7 and 1.2.8 show that an admissible set K has area $A(K) \leqslant 4\det(\Lambda)$ whenever

(1) a chord of symmetry of K is also a chord of areal symmetry of K, or

(2) K has parallel supporting lines at the endpoints of a chord of symmetry (that is, K has an *extremal* chord of symmetry, section 3.1).

Let $A_R(\theta)$ be the area of that part of an admissible set K to the right of a directed chord $c(\theta) = \overrightarrow{P'OP}$ of K, which makes an angle $\angle POX = \theta$ $(0 \leqslant \theta \leqslant \pi)$ with the positive x-axis. Similarly, let $A_L(\theta)$ be the area of that part of K to the left of $c(\theta)$. We show that $A_R(\theta)$ is a differentiable function of θ. The areas of the two almost triangular regions of K which lie between the chords $c(\theta)$ and $c(\theta+d\theta)$ are well approximated by $\frac{1}{2}(OP)^2 d\theta$ and $\frac{1}{2}(OP')^2 d\theta$. Then

$$\frac{d}{d\theta}(A_R(\theta)) = \lim_{d\theta\to 0} \frac{\frac{1}{2}(OP)^2 d\theta - \frac{1}{2}(OP')^2 d\theta}{d\theta} = \tfrac{1}{2}[(OP)^2 - (OP')^2].$$

The function $d_A(\theta) = A_R(\theta) - A_L(\theta) = 2A_R(\theta) - A(K)$ is therefore differentiable with respect to θ, and has derivative

$$d_A'(\theta) = 2A_R'(\theta) = (OP)^2 - (OP')^2 .$$

Hence we can identify a zero of $d_A(\theta)$ with a chord of areal symmetry of K, and a zero of its derivative $d_A'(\theta)$ with a chord of symmetry of K.

We now show that simplifying assumptions about the function $d_A(\theta)$ can be made. Suppose that at some angle θ_0, $d_A'(\theta_0) = 0$ but $d_A(\theta_0)$ is not an extremum. Without loss of generality assume that $d_A'(\theta) \geqslant 0$ for θ in a neighbourhood $N(\theta_0, \varepsilon)$ about θ_0. Hence for $\theta \in N(\theta_0, \varepsilon)$, $d(\theta) = P'OP$ has the property that the segment OP is no shorter than OP'. Therefore, a supporting line to K at the endpoint P_0 of $c(\theta_0) = P_0'OP_0$, together with a parallel line through P_0' form a pair of parallel supporting lines at the endpoints of $c(\theta_0)$, a chord of symmetry of K. Theorem 1.2.8 states that $A(K) \leqslant 4$ in this case.

If $c(\theta_1)$ is both a chord of areal symmetry of K and a chord of symmetry of K, Theorem 1.2.7 gives that $A(K) \leqslant 4$. A particular case of this occurs when $a(K)$ is infinite, as the infinite set of zeros of $d_A(\theta)$ must have an accumulation point $\theta_1 \in [0, \pi)$. By continuity θ_1 is also a zero of $d_A(\theta)$, and indeed must also be a zero of its derivative $d_A'(\theta)$.

We may therefore assume that $d'_A(\theta) = 0$ only at extrema of $d'_A(\theta)$, and only when $d_A(\theta) \neq 0$. By applying Rolle's theorem to $d_A(\theta)$, we deduce that $d'_A(\theta)$ has an odd, or infinite, number of zeros between each pair of successive zeros of $d_A(\theta)$. The number of pairs of successive zeros of $d_A(\theta)$ equals $a(K)$, for we identify the last and first zero on $[0,\pi)$ as such a pair. The value of $s(K)$ is therefore either infinite, or $a(K) + 2t$, where t is a nonnegative integer. Theorem 4.1.1 now follows as a simple corollary of the result proved in Chapter 3, that $s(K)$ is an *M-function*.

4.3 Midchord Symmetry

Proof of Theorem 4.1.2 Let K be an admissible set having $m(K)$ midchords of symmetry. Hence K is a subset of the $2(m(K))$-gon, P, centrally symmetric about 0, formed from those pairs of parallel supports of K parallel to the midchords. When $m(K)$ is infinite, we simply regard some of the sides of this polygon P to have lengths equal to or approaching zero. Since the sides of P are all lines of support of K, K contains a point on each side of P. Let S_1 and S_2 be two opposite sides of of P, containing points R_1 and R_2 of K respectively. Let R_3 and R_4 be points such that R_1R_3 and R_2R_4 are chords of K through 0. The midpoint of the chord $R_1 0 R_3$ lies no further from S_1 than S_2, since $R_3 \in K$ and S_2 bounds K, and so the distance $|R_1 0| \geq |0 R_3|$. Similarly the midpoint of $R_1 0 R_4$ lies no further from S_2 than S_1, and so $|R_4 0| \leq |0 R_2|$. Indeed, if equality holds in either of the above expressions, S_1 and S_2 are parallel supports of K at the endpoints of a chord of symmetry of K, and we can deduce that

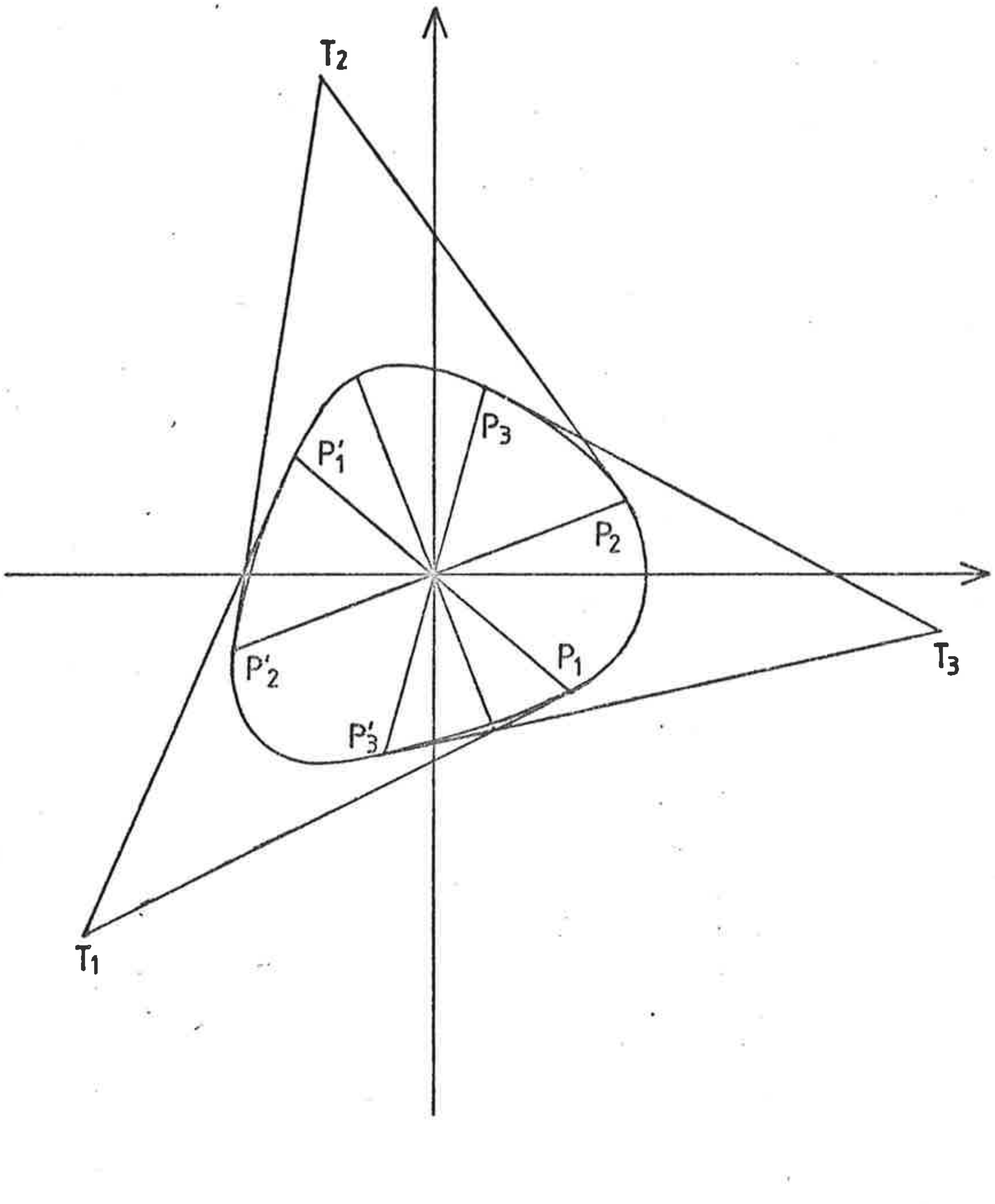


FIGURE 4.2

$A(K) \leqslant 4\det(\Lambda)$ by Scott's Theorem 1.2.8. Otherwise, by the continuity of the boundary of K, there must exist a chord of K, R_5OR_6 between the chords R_1OR_3 and R_4OR_2, such that $|R_5O| = |OR_6|$. Hence we may assume K has at least one chord of symmetry associated exclusively with each of its midchords, and so $s(K) \geqslant m(K)$. Since $s(K)$ is an *M-function*, by Theorems 3.1.1, 3.1.2 and 3.1.3, Theorem 4.1.2 now follows with the exception of the case $m(K) = 2$ and $s(K) = 3$. We demonstrate that this situation can never arise, by showing that if an admissible set K has $s(K) = 3$ and $A(K) > 4\det(\Lambda)$, then $m(K) > 2$.

Suppose K is an admissible set, with $s(K) = 3$ and $A(K) > 4\det(\Lambda)$. Since $A(K) > 4\det(\Lambda)$, we may assume without loss of generality that the supporting lines at the endpoints of the three chords of symmetry meet as in Figure 4.2. This same assumption is justified in section 3.3 in the proof of Theorem 3.1.2, from which we adopt the notation P_iOP_i', T_i, $i \in \{1,2,3\}$ for the chords of symmetry and the intersections of the supporting lines at their endpoints. The chord midway between the two parallel supporting lines of K, parallel to T_1P_1, in Figure 4.2 lies to the same side of O as P_1', since no support parallel to T_1P_1 passes through P_1'. Similarly the chord midway between two parallel supporting lines of K, parallel to T_1P_1' lies to the same side of O as P_1. Hence, as the signed distance between O and a chord of K midway between parallel supports of K at angle θ with the x-axis is a continuous function of θ, there is a midchord of symmetry of K, parallel to T_1X, where X lies on the chord P_1OP_1'. Similarly, midchords of symmetry of K are generated parallel to T_2Y and T_3Z, where

$Y \in P_2OP_2'$ and $Z \in P_3OP_3'$. That the directions of T_1X, T_2Y, and T_3Z are distinct is easily confirmed from the configuration shown in Figure 4.2, opposite. Hence $m(K) \geq 3$, and the proof is complete.

4.4 Reflection/Rotation Symmetry

Proofs of Theorems 4.1.3 and 4.1.4 Every chord of reflective symmetry is also clearly a chord of areal symmetry of K. Hence, by Theorem 4.1.1, when $r(K) > 3$, or when $r(K)$ is infinite, we deduce that $A(K) \leq 4\det(\Lambda)$.

If $r(K) = 2$, the two chords of reflective symmetry must form an angle of $\pi/2$, for otherwise the reflection of one in the other forms a third distinct chord of reflective symmetry. In this case any further chords of areal symmetry give rise to an even number of chords of areal symmetry, by reflection in the two perpendicular chords of reflective symmetry. Hence the number of chords of areal symmetry is even or infinite, and so Theorem 4.1.1 implies that $A(K) \leq 4\det(\Lambda)$. If $r(K) = 3$, the chords must form angles of $\pi/3$, else their images in each other form further chords of reflective symmetry. By Theorem 4.1.1 we thus complete the proof of Theorem 4.1.3.

The set K is centrally symmetric about 0 if $t(K)$ is a multiple of 2, for then K is invariant under a halfturn about 0. All admissible sets have at least one chord of symmetry as we noted in Chapter 3. The images of any such chord under the $t(K)$ rotations belonging to the symmetry group of K are also chords of symmetry of K. Hence the number of chords of symmetry, $s(K)$ is no less than $t(K)$. By Theorems 3.1.1, 3.1.2 and 3.1.3 we thus complete the proof of Theorem 4.1.4.

4.5 Perimeter Symmetry

Let K be an admissible set with $dp(K) > 1$. Let $c(\alpha)$ and $c(\alpha+\beta)$ be two diametral chords of perimeter symmetry of K, separated by an angle $\beta \in (0,\pi)$. As in previous arguments, we denote by $\bar{K}$ the reflection of K in 0, and describe in polar form by functions $a_1(\theta)$ and $a_2(\theta)$, $\theta \in [\alpha,\alpha+\beta]$ the arcs of K and $\bar{K}$ respectively between $c(\alpha)$ and $c(\alpha+\beta)$. Since these chords are chords of perimeter symmetry of K the arcs $a_1(\theta)$ and $a_2(\theta)$ have the same length. As the chords are also diametral the arcs have supporting lines at their endpoints orthogonal to $c(\alpha)$ and $c(\alpha+\beta)$ respectively. We first show

Lemma 4.5.1 For arcs $a_1(\theta)$ and $a_2(\theta)$ as described above, there is at least one angle $\gamma \in (\alpha,\alpha+\beta)$ such that $a_1(\gamma) = a_2(\gamma)$.

Proof With no loss of generality, rotating the coordinate axes if necessary, we assume that $\alpha = 0$. To the contrary we suppose there is no such angle γ. By the symmetry of properties of $a_1(\theta)$ and $a_2(\theta)$, we assume without loss of generality that $a_1(\theta) < a_2(\theta)$ for all $\theta \in (0,\beta)$. We reflect the two arcs about axis $c(\beta)$, and so extend their definition to $\theta \in [0,2\beta]$. Hence $a_i(2\beta-\delta) = a_i(\delta)$ for any $\delta \in [0,\beta]$ and $i=1,2$. Since $c(\beta)$ is a diametral chord, the extended arc $a_i(\theta)$ still has a supporting line at the endpoint of $c(\beta)$, namely that orthogonal to this chord.

The perimeter of a convex set is a strictly increasing function with respect to set inclusion. A proof of this fact can be found in Yaglom and Bottyanski [17], p. 15. Our argument is easy in the following case. We suppose that β divides π, and we extend

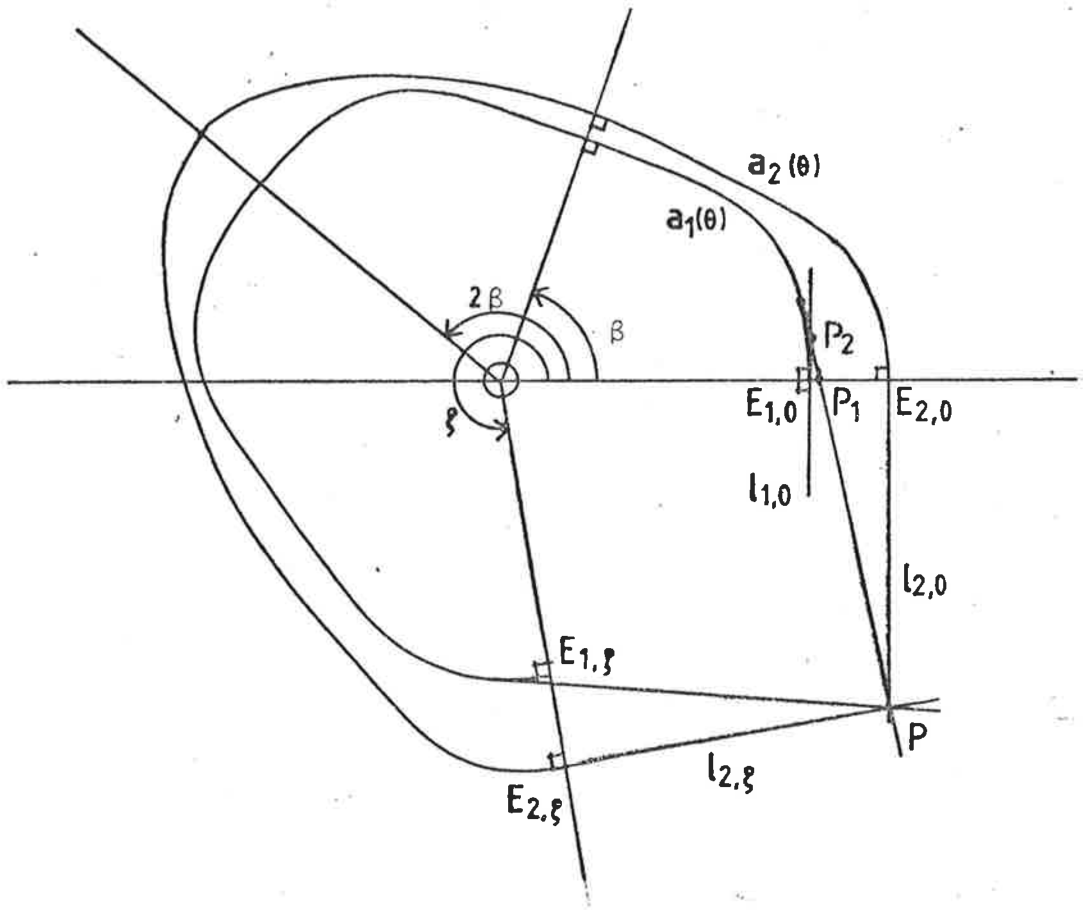


FIGURE 4.3

our definitions of $a_i(\theta)$, $i=1,2$, to $\theta \in [0,2\pi]$ by making each $a_i(\theta)$ periodic with period 2β. Since $a_i(2\beta) = a_i(0)$ the curve so represented is closed for $i=1,2$, and since the same number of units of the original arcs $a_i(\theta)$ are used in each instance, both of these curves have the same length. Finally, since $a_i(\theta)$ bounds a convex set for $\theta \in (0,\beta)$, namely K or $\bar{K}$, at each point in this range a supporting line can be drawn to the curve $a_i(\theta)$. By reflecting in $c(\beta)$ and/or then a rotation by a multiple of 2β, this set of supporting lines is transformed to a set of supporting lines for the closed arc $a_i(\theta)$, which therefore bounds a convex set, L_i. Since $a_1(\theta) < a_2(\theta)$ for $\theta \in (0,\beta)$, it follows that $L_1 \subset L_2$. However as the perimeters of L_1 and L_2 are equal, we have contradicted the first statement of this paragraph, and so proved the result in this simple case.

Suppose now that π is not an integral multiple of B. The following construction is illustrated in Figure 4.3, opposite. We can as above extend the definition of $a_i(\theta)$, so that $a_i(\theta)$ is defined for $\theta \in (0,\xi)$, where $\xi > \pi$, and so that the curves $a_i(\theta)$, of equal length, have supporting lines at all points, and at their endpoints $E_{i,0}$ and $E_{i,\xi}$ have supporting lines $\ell_{i,0}$ and $\ell_{i,\xi}$ orthogonal to the chords $c(0)$ and $c(\xi)$ respectively, for $i=1,2$. Let P denote the point of intersection of lines $\ell_{2,0}$ and $\ell_{2,\xi}$, and let the set L_i be the convex hull of P and the points of curve $a_i(\theta)$, $\theta \in (0,\xi)$, $i=1,2$. By our construction each set L_i is symmetric about the line OP. We now show that the perimeter of L_1 is no less than that of L_2, which will contradict the fact that L_1 is contained in L_2. Since $c(0)$ is a diametral chord of K, $\ell_{2,0}$ is orthogonal to $c(0)$ and so the length

$E_{2,0}P$ is no greater than the length from P to the point P_1, the intersection of $c(0)$ and the boundary of L_1. Indeed, this length is strictly less unless $P_1 = E_{2,0}$. The length of the boundary of L_1 from $c(0)$ to $c(\xi/2)$ is made up from some of the arc $a_1(\theta)$ and a segment (of length 0 only if $P_1=E_{2,0}$) of the supporting line of the curve $a_1(\theta)$ through P_1. Let P_2 be the intersection of this supporting line and the line $\ell_{1,0}$. The length P_2P_1 is no less than length $P_2E_{1,0}$, since $\ell_{1,0}$ is orthogonal to $c(0)$. Therefore the length of the boundary of L_1 from $c(0)$ to $c(\xi/2)$ is no less than the length of boundary of L_1, with $P_2E_{1,0}$ replacing P_2P_1. This second length is however, no less than the length of $a_1(\theta)$ between $c(0)$ and $c(\xi/2)$, since the convex hull of $a_1(\theta)$, $\theta \in (0,\xi/2)$, and the point 0 is contained in the convex hull of 0 with the above curve. By symmetry then the perimeter of L_2 is no less than that of L_1, and by construction $L_1 \subset L_2$. A contradiction thus exists unless $L_1 = L_2$ in which case the point P_1 is $E_{1,0}$ and so $a_1(\theta) = a_2(\theta)$ for $\theta \in (0,\xi)$, in contradiction to the above.

We are now in a position to prove that $dp(\theta)$ is almost an M-function. We can prove

Theorem 4.5.2 Let K be an admissible set having $dp(K)$ diametral chords of perimeter symmetry. Then, if $dp(K) > 1$, $A(K) \leqslant 4.5\det(\Lambda)$ and if $dp(K) > 3$, $A(K) \leqslant 4.\det(\Lambda)$.

Proof By lemma 4.5.1 there lies a chord of symmetry of K strictly between any two diametral chords of perimeter symmetry of K. Hence $s(K) \geqslant dp(K)$. The result follows as a trivial consequence of Theorems 3.1.1, 3.1.2 and 3.1.3. □

It is interesting to note that Theorem 4.5.2 gives the best possible bound on $A(K)$ for at least one lattice when $dp(K) = 2,3,4$ and 6. In section 4.1 we noted examples of this when $dp(K) = 3,4$ and 6. We denote by T' the convex set formed by modifying the equilateral triangle T of section 4.1 by removing from the triangle at each of two of its vertices, an equilateral triangle of side length ε, and at the third vertex a right angled triangle, with base angle $\pi/3$ and base length $\varepsilon/(3-\sqrt{3})$. The perimeter of T' is 3ε less than the perimeter of T, as it is reduced by ε at each corner. The set T' is admissible, and for small ε has just three chords of perimeter symmetry, as had T. The two chords of T', lying on the medians of T, which have an endpoint on the side of one of the two small equilateral triangles removed from T, are clearly diametral chords of perimeter symmetry. The third chord of perimeter symmetry of T' lies close to the third median line of T, but is not diametral since the altitude of the removed right triangle, an edge of T', lies at approximately $\pi/3$ to this chord and contains in its interior one endpoint of the chord. Hence $dp(T') = 2$, and since $A(T')$ is arbitrarily close to $4.5\det(\Lambda)$, the result 4.5.2 is again best possible in this case.

Our results on diametral chords of perimeter symmetry follow from Lemma 4.5.1. It is not true that between any two chords of perimeter symmetry of an admissible set there is always a chord of symmetry of the set. However, I have yet to find a set which contradicts the following conjecture.

<u>Conjecture 4.5.3</u> The function $p(K)$ is an M-function.

CHAPTER 5

The Minkowski-Vander Corput Theorem with Relaxed Symmetry Conditions

5.1 Introduction and Statement of Results

Definition 5.1.1 Let Λ be a lattice in the plane with determinant $\det(\Lambda)$. We say that a closed convex set in the plane, which contains 0 as an interior point, is *t-admissible* if K contains at most t lattice points besides 0 in its interior.

Definition 5.1.2 A *chord of symmetry* of K is a chord of K through 0 bisected by 0. An *extremal chord of symmetry* of K has, in addition, parallel supporting lines at its endpoints.

In this chapter, we let $s(K)$ denote the number of chords of symmetry of a t-admissible set K. We establish the following results, which extend theorems 3.1.1, 3.1.2 and 3.1.3.

Theorem 5.1.1 If $s(K)$ is even or infinite, $A(K) \leqslant 2(t+2)\det(\Lambda)$.

Theorem 5.1.2 If $s(K) > 1$ and $t \leqslant 3$ then $A(K) \leqslant (2(t+2)+(2(t+1))^{-1})\det\Lambda$.

Theorem 5.1.3 If $s(K) > 3$ and $t \leqslant 3$ then $A(K) \leqslant 2(t+2)\det(\Lambda)$.

As in Chapter 3, since $s(K)$ and $A(K)/\det(\Lambda)$ are invariant under a linear transformation of K and Λ, we may suppose that Λ is the integral lattice Λ_0. Since $\det(\Lambda_0) = 1$ we delete all further reference to $\det(\Lambda)$ in this chapter.

We show that each of the above Theorems gives the best possible bound on $A(K)$. Let T_1 be the convex hull of points $(-1,-1)$, $(-1,4)$ and $(1.5,-1)$. This triangle is 1-admissible and has only three chords of symmetry, passing through $(1,0)$, $(.5,1)$ and $(1,-1)$ respectively. As $A(T_1) = 6\frac{1}{4}$, Theorem 5.1.2 is best possible for $t=1$. The triangle T_2 with vertices $(-1,-1)$ $(-1,6)$ and $(1\frac{1}{3},-1)$ similarly has $s(T_2) = 3$ and area $8\frac{1}{6}$, while T_3 with vertices $(-1,-1)$, $(-1,8)$ and $(1\frac{1}{4},-1)$ is best possible for $t=3$.

Examples of sets demonstrating that Theorems 5.1.1 and 5.1.3 are best possible can be constructed similarly to those constructed in section 3.1 by approximating rectangles of the appropriate extreme area with sides parallel to the principal lattice directions, and with width 2.

5.2 Proof of Theorem 5.1.1

This proof is in essence identical to the proof of theorem 3.1.1. If all the chords of K which pass through 0 are chords of symmetry of K, then K is 0-symmetric. As K is t-admissible and 0-symmetric it follows that t is even, and that K does not contain $t/2+1$ pairs of lattice points besides 0. Now the contrapositive statement of Theorem 1.1.5 implies that $A(K) \leqslant 4(t/2+1) = 2(t+2)$.

We may thus assume there is a chord of K, P_0OP_0' which is not a chord of symmetry of K. We denote each chord POP' by the label $c(\theta)$, where θ is the angle P_0OP, $\theta \in [0,\pi]$. We let $d(\theta) = |PO|^2-|OP'|^2$, and note that $d(0) = -d(\pi) \neq 0$, while $d(\theta)$ has a zero exactly when $c(\theta)$ is a chord of symmetry of K.

If $s(K)$ is even, $d(\theta)$ has a zero θ_* corresponding to a local extremum of $d(\theta)$, and so, as in the proof of 3.1.1, we can deduce that $c(\theta_*)$ is an extremal chord of symmetry of K. Since K does not contain $t+1$ lattice points besides 0, the contrapositive statement of Theorem 1.2.8 implies that $A(K) \leqslant 2((t+1)+1) = 2(t+1)$. If $s(K)$ is infinite, the argument given in the proof of 3.1.1 shows that K has parallel supporting lines at the endpoints of any chord of symmetry $c(\theta)$ corresponding to a point of accumulation of the zeros of $d(\theta)$. By again using the contrapositive statement of Theorem 1.2.8, we deduce that $A(K) \leqslant 2(t+2)$.

Corollary 5.2.1 If $d(\theta)$ has a zero at which $d(\theta)$ is an extremum, then $c(\theta)$ is an extremal chord of symmetry, and $A(K) \leqslant 2(t+2)$.

5.3 Preliminary Results Needed in the Proof of Theorems 5.1.2 and 5.1.3

Unfortunately the result of Scott, Theorem 1.2.5 which led to a rapid proof of Theorem 3.1.2, is of no help in proving Theorem 5.1.2.

Consequently we shall prove this result using the machinery of the proof of Theorem 3.1.3, that is by transforming the set and then using the properties of lattice polygons. It is necessary to restate and extend several of the lemmas used in Chapter 3 to suit our needs in this chapter. We prove

Lemma 5.3.1 Let (K_i) be a sequence of admissible convex sets such that $K_i \to K$ in the Hausdorff metric. Suppose that for all i, $s(K_i) = k$, an odd number, and also that K_i has no extremal

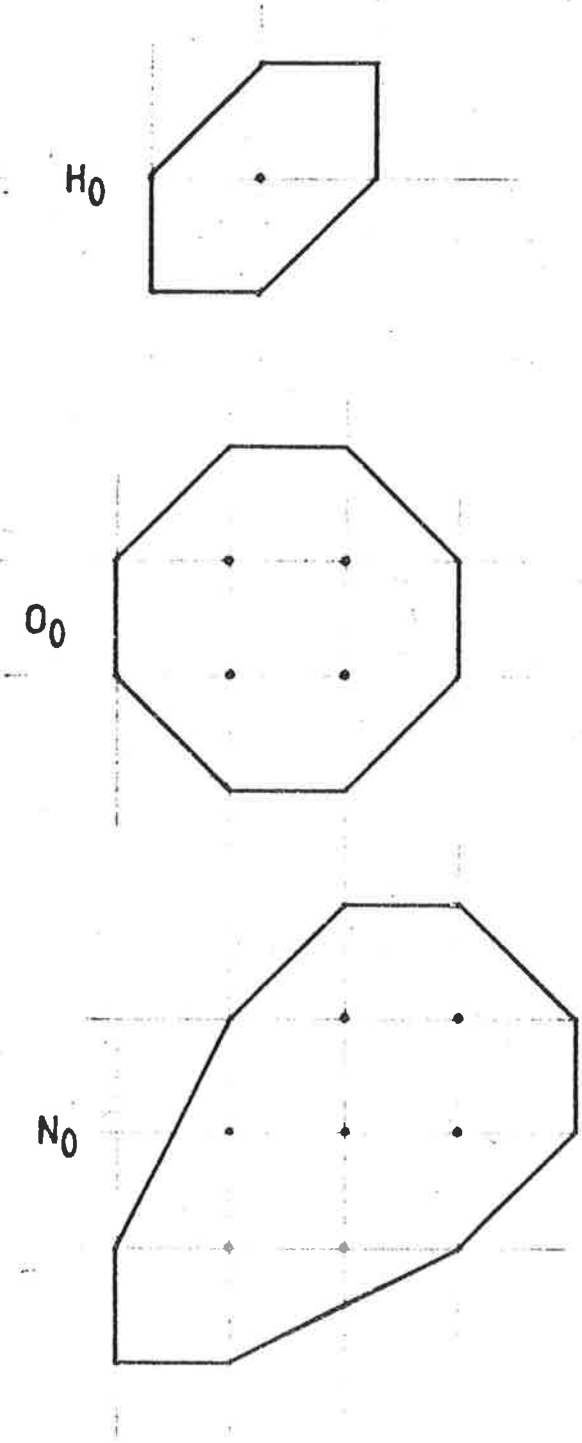


FIGURE 5.1

chord of symmetry. Then either

(i) $s(K) = k$, and K has no extremal chord of symmetry,

or (ii) $A(K) \leqslant 2(t+2)$.

<u>Proof</u> The proof of lemma 3.5.1 applies. It is necessary only to replace the references to results 3.1.1 and 3.2.1 with 5.1.1 and 5.2.1 respectively, and to employ result 1.2.8 in the case of t-admissible sets.

<u>Lemma 5.3.2</u> Let P be a convex lattice polygon, and let v and c denote the numbers of vertices and interior lattice points of P respectively.

(i) If $v = 4$, and P has no pair of parallel edges, then $c \geqslant 1$.

(ii) If $v = 5$, then $c \geqslant 1$.

(iii) If $v = 6$ and $c = 1$, then P is equivalent under an integral unimodular transformation to the centrally symmetric hexagon H_0 illustrated in Figure 5.1 (opposite).

(iv) If $v > 6$, then $c \geqslant 2$.

(v) If $v = 5$, and P has an edge containing two lattice points in its relative interior, then $c \geqslant 2$.

(vi) If $v = 6$ and all the interior points of P are collinear on line ℓ, then ℓ passes through two vertices of P, and is parallel to the two sides of P not incident with these vertices.

(vii) If $v \geqslant 7$, then $c \geqslant 4$ and $c = 4$ only if the lattice points interior to P form a parallelogram.

(viii) If $v = 8$ and $c = 4$, then P is an integral unimodular transform of O_0, the symmetric octahedron shown in Figure 5.1. If

$v = 8$ and $c > 4$, then $c \geqslant 6$.

(ix) If $v = 9$ then $c \geqslant 7$, and $c = 7$ only if P is an integral unimodular transform of N_0, the 9-gon shown in Figure 5.1.

Proof The statements (i), (ii), (iii) and (iv) have been proved in lemma 3.4.1.

(v) Let $P_5 = V_1V_2V_3V_4I_1I_2V_5$ denote a convex lattice 5-gon with lattice points I_1 and I_2 on edge V_4V_5. By result (ii), P_5 contains in its interior a lattice point L. Since at least one of $V_2V_3V_4I_1L$ or $V_1V_2LI_2V_5$ is a convex lattice 5-gon, result (ii) asserts that P_5 contains a second interior lattice point, proving (v).

(vi) Let P_6 be a convex lattice 6-gon, and let ℓ be the line containing all interior lattice points of P_6. Let L_1, L_2 be interior points of P_6. We claim that P_6 has exactly two vertices on ℓ and two on either side. For otherwise P_6 would properly contain a convex lattice 5-gon with vertices L_1L_2 and three vertices of P_6 to one side of ℓ; by result (ii) this would contradict that all interior lattice points of P lie on ℓ. Since the previous argument will be used repeatedly throughout the proof of this lemma, we give it the name "Pentagon argument". A lattice quadrilateral Q given by the intersection of P_6 with a closed halfplane bounded by ℓ contains no lattice points in its interior, and so by result (i) has a pair of parallel edges. The same may be said of the lattice quadrilateral Q' contained in Q, which is given by replacing the vertices of Q on ℓ by L_1 and L_2. Therefore Q has an edge parallel to ℓ. Applying this argument to the halfplanes both sides of ℓ, we deduce that the two edges of P_6, not incident with ℓ, are parallel to ℓ.

We have so proved (vi).

(vii) Let P_7 be a convex lattice polygon with $v \geqslant 7$. By (iv), P_7 contains at least two interior lattice points L_1 and L_2. If the line $\ell = L_1L_2$ meets the boundary of P_7 at two vertices of P_7, they together with the three vertices of P_7 which must be to one side of ℓ form a convex lattice 5-gon with both L_1 and L_2 in the relative interior of one edge. By result (v) this pentagon and so P_7 contains two further lattice points. If the line ℓ meets the boundary of P_7 in just one vertex of P_7, say V, we can form a lattice 6-gon, not a transform of H_0, from edge L_1L_2V and four vertices of P_7 to one side of ℓ; unless $v = 7$ and there are three vertices of P_7 to each side of ℓ. In this case we can then form a lattice 5-gon on each side of ℓ with edge L_1L_2 and each set of three vertices. In either case we can use results (ii) or (iii) to show that P_7 contains two further interior lattice points. Finally, if ℓ does not meet P_7 in any vertex, we can either construct 5-gons to either side of ℓ as above, or we can construct, from edge L_1L_2 and the 5 vertices of P_7 which otherwise must lie to one side of ℓ, a lattice 7-gon. Since result (iv) shows that this 7-gon contains two interior lattice points, we can so deduce that P_7 contains at least four interior lattice points.

Let us assume that P_7 has $c = 4$. If three of the interior lattice points of P_7 lie on line ℓ, the above argument shows that P_7 has, in addition to the lattice points on ℓ, at least two further interior lattice points. Hence if $c = 4$, P_7 has no such 3 collinear interior lattice points.

Again, let P_7 as above with $v \geqslant 7$ have $c = 4$. We now show that the convex hull of these four interior lattice points of P_7 is a quadrilateral. By the above, no three of these points are collinear. The only other possibility is that the four lattice points form a triangle T (say $L_1L_2L_3$) with an interior point L_4. Each exterior open halfplane bounded by an edge (line) of T contains at most two vertices of P_7, for otherwise we could apply the pentagon argument to three such vertices and the two vertices of T on that edge. Since the union of the three open halfspaces to the sides of T and away from T is the whole plane minus the closed triangle T, we deduce that P_7 has at most six vertices, contradicting $v \geqslant 7$.

Hence P_7 contains in its interior a lattice 4-gon $L_1L_2L_3L_4$, and no other lattice points. By result (i) this 4-gon has a pair of parallel sides, and since lattice points are regularly spaced on parallel lattice lines we deduce that it is a parallelogram.

(viii) Let P_8 be a convex lattice 8-gon with $c = 4$, and let $L_1L_2L_3L_4$ be the interior lattice points forming a parallelogram by (vii). No vertex of P_8 lies in the open strip between parallel lines L_1L_2 and L_3L_4, since otherwise we could apply the pentagon argument to it and points $L_1L_2L_3$ and L_4. There can be no three vertices to one side of the closed strip between L_1L_2 and L_3L_4 either, for if there were three to the side of L_1L_2 say, then we could apply the pentagon argument to these three vertices and L_1 and L_2. Hence four vertices of P_8 lie on the lines L_1L_2 and L_3L_4, and as $c = 4$, they are adjacent to points $L_1L_2L_3L_4$ on these lines. We apply this argument also to the strip between parallel lines L_1L_4 and L_2L_3. Therefore all the vertices of P_8 lie on the edges of parallelogram $L_1L_2L_3L_4$. Hence, under an integral unimodular transformation which maps $L_1L_2L_3L_4$ to a square in standard position, P_8 is mapped to the symmetric octahedron O_0.

Now let P_8 be a convex lattice 8-gon with $c \geqslant 5$. We may assume that N, the convex lattice polygon formed as the convex hull of the lattice points interior to P_8, is not a 5-gon, since then (ii) would imply that $c \geqslant 6$ and hence the result. We also assume that $L_1L_2L_3L_4L_5$ are the only lattice points interior to P_8. Now, using the same argument which we applied to triangle T in (vii), each exterior open halfplane of N contains at most 2 vertices of N. If N is a line segment or a triangle we can thus deduce that $v \leqslant 6$, a contradiction. Hence N is a quadrilateral. We distinguish two cases, when L_5 is interior to quadrilateral $N = L_1L_2L_3L_4$ and when L_5 lies on an edge, say L_1L_2 of N.

We suppose first that L_5 is interior to N. By the pentagon argument, there are at most two vertices of P_8 in the exterior open halfplane bounded by side L_1L_2 of N, not containing L_5. The same is true of L_3L_4. Hence in the closed wedge W formed by these two edges L_1L_2 and L_3L_4, P_8 has at least four vertices. However, to either side of N in this wedge there is at most one vertex of P_8, since we can apply the pentagon argument to L_5, L_i, L_j and any two vertices in the intersection of W with the exterior open halfplane bounded by $L_i L_j$ not containing L_5, where the pair (i,j) is either $(1,4)$ or $(2,3)$. Hence the closed wedge between edges L_1L_2 and L_3L_4 contains both at least four and at most two vertices of P_8, a clear contradiction. It follows that $c \geqslant 6$.

We suppose now that L_5 lies on the edge L_1L_2 of N. By result (i) applied to both quadrilaterals $L_1L_2L_3L_4$ and $L_2L_3L_4L_5$, we deduce that L_1L_2 is parallel to L_3L_4. We now apply the argument given above in (viii) to deduce that P_8 has four vertices on lines L_1L_2 and L_3L_4 and its other four vertices outside the closed strip between these lines. Hence the exterior open halfplanes bounded by L_1L_2 and L_3L_4 each contain two vertices of P_8. No vertices other than those lying on L_1L_2 and L_3L_4 can lie outside the closed wedge W formed by lines L_1L_4 and L_2L_3, for we can otherwise apply the pentagon argument to $L_i L_j$ and three vertices of P_8 in the open exterior halfplane bounded by $L_i L_j$ not containing L_5 if $(i,j) = (1,4)$ or $(2,3)$. Hence the two vertices of P_8 in the open exterior halfplane bounded by L_3L_4 not containing L_5 must lie in W, and so form with $L_3L_4L_5$ a pentagon. Applying the pentagon argument to this pentagon, we obtain a contradiction. Hence (viii) is proved.

(ix) Let P_9 be a convex lattice polygon with $v = 9$. Let O_1 and O_2 be two different convex lattice 8-gons whose vertices are vertices of P_9. Since O_1 and O_2 share 7 vertices, they share 4 vertices which are consecutive on the boundary of P_9. The convex lattice 8-gon O_0 can be uniquely completed from any four of its consecutive vertices. If both O_1 and O_2 are transforms of O_0, they can therefore also be uniquely completed from any four of their consecutive vertices. Hence, as O_1 and O_2 share four consecutive vertices, $O_1 = O_2$ in contradiction to our choice. Hence at least one of O_1 and O_2, say O_1 is not a transform of O_0, and so by results (vii) and (viii), we deduce that $c \geqslant 6$.

We first show that P_9 may be assumed to contain in its interior two lattice points such that all other lattice points interior to P_9 lie strictly to one side of the line through them. To the contrary, we suppose that N, the polygonal convex hull of all interior lattice points of P_9 has at least three lattice points on each edge. By the pentagon argument, we know that P_9 has at most two vertices in the open halfplane not containing N bounded by any edge of N. As $v = 9$, N has therefore at least five edges, and so $c \geqslant 10$. We may therefore assume that L_1 and L_2 are two interior lattice points of P_9, with all other interior lattice points of P_9 in the open halfplane H_1 bounded by the line $\ell = L_1L_2$.

By the pentagon argument P_9 has at most two vertices in the open complementary halfplane of H_1 and of course at most two vertices on ℓ. Hence P_9 has at least five vertices in H_1. If P_9 has 6 or more vertices in H_1, they together with L_1L_2 and any vertices on ℓ form either a convex lattice 8-gon, not unimodular equivalent to O_0 since it contains 3 lattice points on ℓ, or a convex lattice 9-gon. In either case result (viii) or our first comment in this proof of (ix) shows that P_9 contains an additional 6 lattice points in its interior, giving the result $c > 7$.

We can therefore assume that P_9, with vertices labelled cyclically has vertices $V_1V_2 \ldots V_9$, with V_9, L_1, L_2, V_6 lying in this order on ℓ. We first argue that $c \geqslant 7$. To the contrary, suppose $c = 6$ and so that P_9 has only four interior lattice points in H_1. If the edge V_7V_8 of P_9 contains in its relative interior any lattice points, we modify P_9 by taking V_7 to be the closest such lattice point to V_8. Since all 6 interior points of P_9 belong to the closure of H_1, both v and c are unaltered in this transformation. We can therefore without loss of generality assume that there are no lattice points in the relative interior of edge V_7V_8. Applying result (i) to both the quadrilaterals $V_6V_7V_8V_9$ and $V_6V_7V_8V_1$, we deduce that edge V_7V_8 is parallel to ℓ. Since L_1 and L_2 are adjacent lattice points on ℓ, and V_7,V_8 adjacent on a line parallel to ℓ, the quadrilateral $L_1L_2V_7V_8$ is a parallelogram.

We show that V_2 lies in the closed halfplane H_2, bounded by L_1V_8, and containing L_2. For if V_2 is outside H_2, $V_1V_2L_1V_8V_9$ is a convex lattice 5-gon which by (ii) contains an interior lattice point L_3, which being interior to P_9 lies in H_1. But then $L_1L_3V_2V_3V_4V_5V_6$ is a convex lattice 7-gon, which by (vi) contains four additional interior lattice points, contrary to our assumption that $c = 6$. A symmetric argument shows that V_4 lies in the closed halfplane H_3 bounded by L_2V_7 containing L_1. We complete the contradiction by simply noting that V_3 must lie in the interior of the strip $H_2 \cap H_3$ bounded by L_1V_8 and V_2V_7; however there are no lattice points in this strip. Hence $c \geqslant 7$.

We now suppose that $c = 7$, and that $V_9L_1L_2$ and V_6 are collinear in this order on ℓ, as was the case prior to our assumption that $c = 6$ above. We note that the description of P_9 so far given is ambiguous, in that the same configuration is attained if the points so far labelled are relabelled, interchanging L_1,L_2 and interchanging V_j and V_k whenever $j+k \equiv 6 \pmod 9$.

We first show that with one of these labellings, there is a chord of P_9 passing through L_2 and a second interior lattice point L_3 of P_9, with $L_1V_7V_8$ and V_9 to one side of it, and with all the other vertices and interior lattice points of P_9 to the other. To the contrary, we assume that with neither labelling there is such a chord. There are therefore no lattice points in the open triangles $V_1L_2V_9$ and $V_5V_6L_1$. We also claim that there is no lattice point V on the line segment between L_1 and V_5. For if there was such a lattice point V closest to L_1, we could produce a lattice point U in the triangle $V_5V_6L_1$ by adding L_2-L_1 to V. By supposition then, U is on the edge V_5V_6. However then the octagon $UL_2V_9V_1V_2V_3V_4V_5$ is not a transform of O_0, since L_1 is interior to the edge L_2V_9, and so by (viii) contains at least 6 lattice points in its interior, making $c \geqslant 8$. We have by a similar argument, applied to the alternative labelling of P_9, that there are no lattice points on the line segment L_2V_1.

Hence neither of the parallelograms with sides V_5V_6 and V_6L_1 or with sides L_2V_9 and V_9V_1 contain any lattice points except those on the sides parallel to ℓ, and so the line V_1V_5 is parallel to ℓ. Were there two or fewer lattice points interior to P_9 on the line segment V_1V_5, there would be insufficient lattice lines parallel to V_5V_6 intersecting P_9 to carry each of V_2V_3 and V_4 on a separate such line, as is required for them to be vertices of P_9. Were there more than three lattice points between V_1 and V_5, by result (v) the lattice 5-gon $V_1V_2V_3V_4V_5$ contains at least two further lattice points, giving $c \geqslant 8$. There are thus exactly 3 lattice points interior to the segment V_1V_5. Because three interior lattice points of P_9 lie on V_1V_5 and two on V_9V_6, the edges V_8V_9 and V_6V_7 of P_9 are not parallel. Therefore, by (i), the quadrilateral $V_6V_7V_8V_9$, which contains no interior lattice points of P_9 in its interior, has a pair of parallel edges, namely V_6V_9 and V_7V_8. Hence when the lines

V_5V_6 and V_1V_9 are extended, they meet the line V_7V_8 in two lattice points, with only one lattice point between them. In order that V_6 and V_9 be vertices of the strictly convex set P_9, it is necessary then for V_7 and V_8 to both be this single lattice point, contradicting the fact that $v = 9$. By this contradiction, we have shown that we may assume that there is a chord of P_9, passing through L_2 and a second interior lattice point L_3 of P_9, with $L_1V_7V_8$ and V_9 to one side of it, and with all the other vertices and interior lattice points of P_9 to the other.

The convex lattice 8-gon $L_2L_3V_1V_2V_3V_4V_5V_6$ contains only four interior lattice points, and so is a transform of O_0. We suppose this 8-gon to be O_0, in standard position. We now claim that V_6L_2 is a side of O_0 parallel to the sides of M, the convex hull of the four interior lattice points of O_0. The side V_6L_2 of O_0, when extended, contains successively L_1 and then V_9. Since when an edge of O_0 not parallel to an edge of M is extended, the second lattice point encountered is collinear with an edge of O_0. As the vertices V_9, V_1 and V_2 of P_9 are not collinear, we deduce the above claim. Indeed, not only are V_9 and L_1 fixed in relation to O_0, up to an isometry of O_0 which is automatically an integral unimodular transformation anyway, but V_7 and V_8 must lie on an adjacent lattice line parallel to ℓ. This follows by applying result (i) to the lattice quadrilateral $V_6V_7V_8V_9$ which contains no lattice points in its interior, and by noting that since V_1V_2 and V_5V_6 are parallel sides of O_0, the lines V_8V_9 and V_6V_7 are not parallel. As there are only two lattice points on this lattice line parallel to ℓ between the lines V_5V_6 and V_1V_9, the points V_7 and V_8 are uniquely specified. This then is the unique configuration N_0. We have thus proved (ix).

This completes the proof of lemma 5.3.2.

With a computer program running a simple search through "likely" convex lattice polygons, I have found polygons with the following values of v and c.

Conjecture 5.3.2 (Extension to lemma 5.3.2) The following table gives the least number c of interior lattice points of a convex lattice polygon having v-vertices.

v	10	12	14	16	18	20	22
c	10	19	34	52	79	112	154
v	24	26	28	30	32	34	36
c	199	262	332	416	508	616	732

Lemma 5.3.3 If K is a centrally symmetric t-admissible set, with $s(K) \geqslant 2$, then K has an extremal chord of symmetry.

Proof. If K is centred at 0, any chord of K is extremal. We suppose then that 0', not 0, is the centre of K. Note there is no need for 0' to be a lattice point. We also suppose that $P_i OP_i'$, for i=1,2, are distinct chords of symmetry of K. Clearly, no chord of (0-) symmetry lies on the line 00'. We can thus assume without loss of generality that P_1 and P_2 lie to the same side of 00'. We let Q_i and Q_i' be the reflections of P_i and P_i' in $0'$, respectively, for $i=1,2$. The line segments P_1P_2 and $Q_1'Q_2'$ are distinct, parallel and lie to the same side of OO'. Since K is convex, $P_1P_2Q_1'$ and Q_2' are therefore collinear. Hence P_1P_2 and Q_1Q_2 are parallel lines of support of K, and so P_1Q_1 is an extremal chord of symmetry.

Lemma 5.3.4 (i) Let n be an odd integer greater than 1, and suppose that P is a convex n-gon in the plane, all of whose edges have a lattice point as midpoint.

If P contains c $(c \geqslant 0)$ lattice points in its interior, then

$$A(P) \leqslant 2c+2 .$$

(ii) Let R be any convex lattice polygon, other than a triangle T containing just one interior lattice point and whose sides each contain 4 lattice points.

If R has c, $c \geqslant 1$, lattice points in its interior, then

$$A(R) \leqslant 2c+2.$$

Proof. We first prove (ii). Let b denote the number of lattice points on the boundary of R. By Theorem 1.4.2, Scott's result on convex lattice polygons, since $R \neq T$ and $c \geqslant 1$,

$$b \leqslant 2c+6. \qquad (1)$$

By Theorem 1.4.1, the area of R is given by

$$A(R) = \tfrac{1}{2}b + c - 1. \qquad (2)$$

From (1) and (2) result (ii) is trivially deduced.

By applying lemma 3.4.3(i) to P, we deduce that P is a lattice polygon with an odd number of lattice points on all sides.

Hence (i) follows from (ii), unless $c = 0$. By lemma 5.3.2(ii),(iv) we deduce that if $c = 0$ then $n = 3$, and that P has just six lattice points on its boundary. Result (i) then follows from equation (2).

5.4 A reduction to t-admissible polygons of a special type.

In this section we generalise the results of Section 3.5, and so prove

Theorem 5.4.1 It is sufficient to prove Theorems 5.1.2 and 5.1.3 for a restricted class of t-admissible sets, which have the following properties

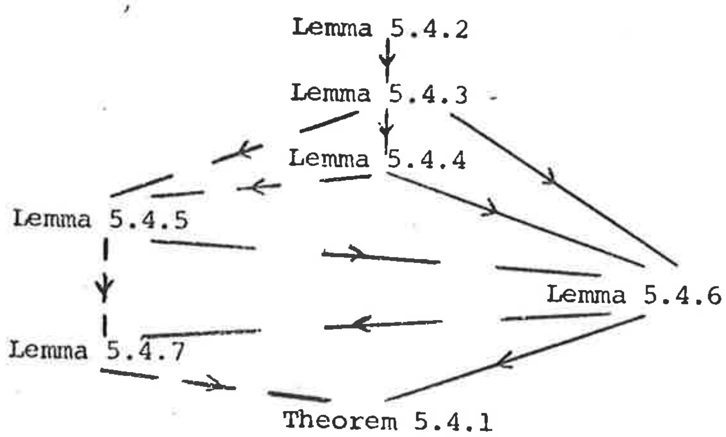
(i) K is a polygon

(ii) Each edge of K is a lattice midpoint edge of K, or a multiple lattice point edge.

(iii) If K is a triangle, it has no lattice midpoint edge, and if K is a quadrilateral, it has at most two lattice midpoint edges.

We shall prove this result by a series of lemmas.
Throughout this chapter we assume that K is a t-admissible set with $s(K) > 1$. To prove Theorems 5.1.2 and 5.1.3 we must show that K is bounded by one of two bounds, the lesser of which is $2(t+2)\det(\Lambda)$. Each of the following lemmas proves that either $A(K) \leqslant 2(t+2)\det(\Lambda)$, or that $A(K)$ is less than the area of a t-admissible set with specific properties. In this manner, we can successively assume that the set K has the properties above.

The following lemmas are interdependent. By lemmas 5.4.2 and 5.4.3 we assume that K is a polygon, each edge of which has a lattice point in its relative interior. Lemma 5.4.4 is a service lemma, detailing how an individual edge of K can be modified. This result is used, together with a boundedness result proved via lemma 5.4.3, in lemma 5.4.6. This essentially completes the proof of Theorem 5.4.1, except that to prove the boundedness result, we need to deal separately with triangular and quadrilateral K. We do this with lemmas 5.4.5 and 5.4.7. The following diagram illustrates the interdependence of the lemmas. The broken arrows indicate the parts of the proof which deal with triangles and quadrilaterals.



<u>Lemma 5.4.2</u> We may assume that K is a convex polygon, with at least s(K) edges.

<u>Proof.</u> From Theorem 5.1.1 we may assume that s(K) is odd and finite. As O is an interior point of the t-admissible set K, by Theorem 1.3.2, K is bounded. Hence K is separated from all the lattice points not in its interior by a finite number of lines. We can therefore construct a polygon K about K which is

bounded by these lines and by the supporting lines to K at the endpoints of its chords of symmetry. We may clearly take $\mathcal{K}$ so that each edge of $\mathcal{K}$ contains in its relative interior at least one endpoint of a chord of symmetry, or a lattice point. We note that $\mathcal{K}$ has no smaller area, no fewer lines of symmetry than K and contains no lattice points in its interior other than those contained in the interior of K. Hence we take K to be the t-admissible polygon $\mathcal{K}$. Should an edge of K contain an endpoint of each of three or more chords of symmetry, by convexity the other endpoints of these chords lie on a parallel edge of K. By Theorem 1.2.8, we then know that $A(K) \leqslant 2(t+2)$. Hence K has at least $s(K)$ edges.

Lemma 5.4.3 We may also assume that each edge of K has a lattice point in its relative interior.

Proof. We let E be an edge of the polygon K, and suppose that E has no lattice point in its relative interior. We modify K, by replacing E with an edge $E(r)$, parallel to E and distance r further away from O than E. Denote by $K(r)$ the polygon obtained from K by including $E(r)$ in place of E, and extending the edges of K adjacent to E. We continue to increase r until one of three things happen.

(a) the length of $E(r)$ becomes zero, or

(b) there is a lattice point in the relative interior of $E(r)$, or

(c) $s(K(r)) \neq s(K)$.

One of these things must happen, for as K contains a disc $\mathcal{D}$ about O, $K(r) \supset K \supset \mathcal{D}$, and by Theorem 1.3.2, $K(r)$ is bounded by a uniform bound for all r.

In section 3.6 we showed that should (c) occur either with or before either (a) or (b), when $r = r^*$, then $A(K(r^*)) \leqslant 4$. This generalises with no complication to the present case, by using results 5.1.1 and 5.3.1 in place of results 3.1.1 and 3.5.1. We can therefore deduce that should (c) occur either with or before either (a) or (b), that then $A(K(r^*)) \leqslant 2(t+2)$.

We note that the set $K(r)$ is still t-admissible, since condition (b) occurs prior to any additional lattice point becoming interior to $K(r)$. By applying the above modification to any edge of K which has no lattice point in its relative interior, successively until there are no such edges left, we produce a modified set K^*, which is t-admissible. Since if (c) occurs, then $A(K) \leqslant 2(t+2)$ as required by Theorems 5.1.2 and 5.1.3, we may assume that $s(K^*) = s(K)$. As K^* has no lesser area than K, we take K to be the t-admissible polygon K^*, all of whose edges have property (b), as required by lemma 5.4.3.

<u>Lemma 5.4.4</u> Let K be as above. We suppose that K has an edge E, which has a single lattice point in its relative interior, but which is not a lattice midpoint edge. Then K may be modified, increasing its area and E replaced by either a multiple lattice point edge or a lattice midpoint edge.

<u>Proof</u>. We suppose that L, not the midpoint of edge E of K, is the only lattice point in the relative interior of E. We let E_+ and E_- be edges of K adjacent to E. We let $K(\psi)$ be the set formed by replacing E with an edge $E(\psi)$, through L, making

an angle ψ with E, and by extending E_+ or E_- to meet $E(\psi)$. As L is not the midpoint of E, we may orient ψ so that for (small) positive ψ, the set $K(\psi)$ has area greater than A(K). As the boundary of K is continuous, and since $E(\psi)$ pivots about L which is not its continuously varying midpoint, $A(K(\psi))$ increases continuously with ψ until one of three things happen.

(d) L is the midpoint of $E(\psi)$, or

(e) the relative interior of $E(\psi)$ contains two or more lattice points, or

(f) $s(K(\psi)) \neq s(K)$.

No lattice points are lost from the boundary of K in this modification of K. For, were L_1 to be lost at angle ψ, as L is not the midpoint of $E(\psi)$, the lattice point $2L-L_1$ lies in the relative interior of $E(\psi)$. Hence all those lattice points on the boundary of K lie on the boundary of $K(\psi)$, although possibly at vertices of $K(\psi)$. Also, all those lattice points besides O which belong to the interior of K continue to belong to $K(\psi)$, although they may now lie on $E(\psi)$, if ψ is an angle such as given by (e). Hence $K(\psi)$ is also a t-admissible set, provided O is an interior point of $K(\psi)$.

We next show that O is still an interior point of $K(\psi)$. To the contrary we suppose that O is not interior to $K(\psi_*)$, and so the edge $E(\psi_*)$ contains O in its relative interior, as well as L. We consider the set $K(\psi)$ as ψ approaches ψ_*, and show that condition (f) occurs before $\psi = \psi_*$. Since O is an interior point of K, there is a neighbourhood of O such that no boundary points of $K(\psi)$ other than those of $E(\psi)$ may lie in this neighbourhood. Now the directions of the edges E_+ and E_- adjacent to $E(\psi)$ are not altered by varying ψ. Further, as each of E_+ and E_- contains a lattice point in its relative interior, the lengths of E_+ and E_- are not reduced to zero by the modification. Hence, when ψ is such that $E(\psi)$ is sufficiently close to O, the sets $K(\psi)$ and its reflection in O, $\overline{K(\psi)}$ can intersect only in the arbitrarily thin figure J bounded by three segments E_-, $E(\psi)$ and E_+, and their three reflections in O. The lattice point O is not the midpoint of $E(\psi_*)$, since, as ψ approaches ψ_*, the edge $E(\psi)$ becomes closer to the interior point O of $K(\psi)$, and therefore O is adjacent to the shorter of the two segments into which L divides $E(\psi)$. Hence, with O not the midpoint of $E(\psi)$, and with the directions of E_+ and E_- fixed, the boundary of $K(\psi)$ meets that of $\overline{K(\psi)}$ in just two points, for all values of ψ making J suitably thin. Hence $s(K(\psi)) = 1$ for all values of ψ sufficiently close to ψ_*. Therefore condition (f) occurs before O is a boundary point of the transformed set. The modification therefore stops before $\psi = \psi_*$, and so O is an interior point of $K(\psi)$. Hence $K(\psi)$ is t-admissible.

By the above argument, it is clear that one of conditions (d) (e) and (f) must occur, at say $\psi = \psi_*$, and that there is a neighbourhood of O common to all the transformed sets $K(\psi)$, for $\psi \leqslant \psi_*$. By Cohn's result, Theorem 1.3.2, the sets $K(\psi)$ are uniformly bounded. As in Chapter 3.6, we can deduce then that if (f) occurs either with or before either (d) or (e), that $A(K(\psi_*)) \leqslant 2(t+2)$.

Otherwise, if (d) or (e) occur, the set $K(\psi_*)$ is that required by lemma 5.4.4.

Lemma 5.4.5 If K is a triangle, we may assume that at least two edges of K are multiple lattice point edges. If K is a quadrilateral, we may assume that at least one edge of K is a multiple lattice point edge.

Proof. We first adapt the modification to K detailed in lemma 5.4.4. We suppose that E is a lattice midpoint edge of K, centred at L. We denote by E_+ and E_- the edges of K adjacent to E, and suppose that E_+ and E_- meet in the halfplane bounded by E and containing K. We claim that we may modify K even though L is the midpoint of E. For, if we replace E by an edge $E(\psi)$ at angle ψ to E (ψ small, either orientation), the set $K(\psi)$ so formed has greater area than K. The incidence property of E_+ and E_- guarantees this, for a congruent copy of the triangle $K \sim K(\psi)$ is contained in the triangle $K(\psi) \sim K$. In fact, since ψ may be chosen with either orientation, an edge E, with edges E_+ and E_- adjacent to it as above, cannot be modified by the modification of lemma 5.4.4 to an edge satisfying condition (d). The edge $E(\psi)$ can be further modified exactly as in lemma 5.4.4, since L is not the midpoint of $E(\psi)$, until either condition (e) or (f) occurs, as condition (d) cannot.

In order to claim that the sets $K(\psi)$ so obtained are uniformly bounded, it is sufficient to note that by the same argument used in the proof of lemma 5.4.4, O is an interior point of $K(\psi^*)$ when condition (e) or (f) occurs. Similarly, as in lemma 5.4.4, $K(\psi)$ contains no lattice points in its interior which are not interior points of K. Hence $K(\psi)$ is t-admissible, for all ψ between 0 and ψ^*. Also, by Cohn's result, Theorem 1.3.2 the sets $K(\psi)$, for ψ between 0 and ψ^*, are uniformly bounded. As in Chapter 3.6, we deduce that if condition (f) occurs either with or before (e), then $A(K(\psi^*)) \leq 2(t+2)$. We therefore assume that $E(\psi^*)$ has property (e).

The above modification to lemma 5.4.5, allows us to modify an edge E of K for which condition (d) applies, provided E_+ and E_- meet to the side of E containing K. We can clearly always use this result for such an edge E, if K is a triangle.

We first suppose that K is a triangle with no multiple lattice point edges. We suppose that E is an edge of K containing a single lattice point L in its interior. By the modification of lemma 5.4.4, started, if necessary, as above, we may assume that E may be replaced by a multiple lattice point edge $E(\psi)$ through L. The set $K(\psi)$ so formed has the same number of chords of symmetry as K, and has area no less than $A(K)$.

We may therefore suppose that a triangular set K has at least one multiple lattice point edge. We now suppose that K is a triangle with a single multiple lattice point edge E, and that one of the other edges E_1 of K has its single interior lattice point L closer to the vertex V of K not on E than it is to the vertex $F \cap E$ of K. We apply the modification of lemma 5.4.4 to E_1, replacing E_1 by a multiple lattice point edge. The edge E adjacent to E_1 is so extended, and remains a multiple lattice point edge. The triangle K so formed satisfies the requirement of this lemma.

We therefore suppose that K is a triangle with a single multiple lattice point edge E, and that the other edges of K, E_1 and E_2, have lattice points L_1, L_2 in their relative interiors, respectively. We suppose that L_i is no further from the vertex $E \cap E_i$ than it is from the vertex $V = E_1 \cap E_2$, for $i=1,2$. We apply the modification of lemma 5.4.4 to E_1, started if necessary as above, and replace E_1 by a multiple lattice point edge E_1', and in so doing shorten edge E of K. The lattice point L_2 on E_2 lies no further from $E \cap E_2$ than from the vertex $V' = E_1' \cap E_2$ of K', and so edge E_2 of K' can be modified, if necessary, by lemma 5.4.4 into a multiple lattice point edge E_2'', taking K' to K". We note that E need no longer be a multiple lattice point edge of K", for in the modifications its lattice points can become its endpoints. However, by reapplying the modification of lemma 5.4.3 to E if necessary, we can assume that E contains a lattice point in its relative interior. The resulting set K'" satisfies the statement of the lemma.

We next suppose that K is a quadrilateral, each of whose edges contains only one lattice point in its relative interior. If K is a parallelogram, then it is a centrally symmetric convex set with $s(K) \geqslant 3$, and so by lemma 5.3.3, it has an extremal chord of symmetry. By Corollary 5.2.1, we deduce that $A(K) \leqslant 2(t+2)$ in this case.

We may therefore suppose that the quadrilateral K has an edge E such that the edges E_1 and E_2 adjacent to E meet in the halfplane bounded by E containing K. By the modification of lemma 5.4.4, applied to E and started as above if the edge E has a lattice midpoint L, we may replace E with a multiple lattice point edge E'. The set K' so formed satisfies the conditions required for lemma 5.4.5, unless E' passes through the sole lattice point in the relative interior of either E_1 or E_2. Without loss of generality we suppose that E' passes through L_1 the sole lattice point in the relative interior of E_1. The edge E_1' of K contains no lattice point in its relative interior, and so by lemma 5.4.3 we can replace E_1' by a parallel edge E_1'' either of length zero or which contains a lattice point in its relative interior. Hence K'' so formed is either a triangle or satisfies the requirement of lemma 5.4.5. As we have dealt with the case of modifying such a triangle above, we may assume that K has the properties stated in the lemma.

Lemma 5.4.6 We may assume that every edge of K is a lattice midpoint edge or a multiple lattice point edge.

Proof. The convex hull of the set of lattice points on the relative interior of the edges of K is a convex lattice n-gon Y. We claim that $n \geqslant 5$. By lemma 5.4.3, we have been able to assume that each edge of K has a lattice point in its relative interior. Hence if K has five or more edges, $n \geqslant 5$. Otherwise, K is either a triangle or quadrilateral and $n \geqslant 5$ by lemma 5.4.5.

By lemma 5.3.2(ii), the lattice polygon Y contains in its interior a fixed disc about an interior lattice point L of K, not necessarily O. Further, all the sets K' which can be obtained by repeatedly modifying K by the modifications of lemma 5.4.3 and 5.4.4 contain the lattice polygon Y, and so contain this same neighbourhood of the common interior point L. By translating any such K' by the integral translation taking L to O, we can apply Cohn's Theorem, Theorem 1.3.2 to this translate K". We deduce that K" is bounded by a bound dependent only on the disc about O, and so that the sets K' are uniformly bounded.

We may thus apply the Blaschke Selection Theorem, Theorem 1.5.1 to any sequence of sets (K_i) obtained by successively modifying K as above. We obtain a limiting figure K* which can no longer be modified. However $s(K_i) = s(K)$ is odd for each of these t-admissible sets K_i, and the sequence (K_i) converges to K* in the Hausdorff metric. Hence by lemma 5.3.1, we deduce that if $s(K^*) \neq s(K)$ then $A(K) \leqslant 2(t+2)$.

We may therefore assume that $s(K^*) = s(K)$, and as K* is immutable by the modifications of lemma 5.4.2, lemma 5.4.3 and lemma 5.4.4, it is a convex polygon, with at least s(K) edges, and with only lattice midpoint edges or multiple lattice point edges. As $A(K^*) \geqslant A(K)$, it suffices to prove Theorems 5.1.2 and 5.1.3 for the set K*.

<u>Lemma 5.4.7</u> If K is a triangle, we may assume that all edges of K are multiple lattice point edges. If K is a quadrilateral, we may assume that at least two edges of K are multiple lattice point edges.

<u>Proof</u>. By lemma 5.4.5, we may assume that if K is a triangle, it has at least two multiple lattice point edges, and if K is a quadrilateral, it has a multiple lattice point edge. By lemma

5.4.6 we may further assume that K has only multiple lattice point edges and lattice midpoint edges.

We first suppose that K is a triangle, with a lattice midpoint edge E_1 with midpoint L_1, and with E_2 and E_3 multiple lattice point edges. As in lemma 5.4.5, we replace E_1 by an edge $E_1(\psi)$ through L_1, at angle ψ to E, where ψ is oriented so that edge E_2 is shortened and E_3 lengthened when E_1 is replaced by $E_1(\psi)$. As argued in the proof of lemma 5.4.5, we may increase ψ until $E_1(\psi)$ is a multiple lattice point edge of the modified set $K(\psi)$. The set $K(\psi)$ then has three multiple lattice point edges as required, unless the endpoint of $E_1(\psi)$ on E_2 is one of only two lattice points that were originally in the relative interior of E_2.

In this case, since lattice points are regularly placed along lattice lines, the remaining single lattice point L_2 on the relative interior of the new shortened edge E_2 lies no further from $E_1 \cap E_2$ than from $E_2 \cap E_3$. Hence the edge E_2 can be modified to $E_2(\psi)$, in such a way that E_1 is lengthened.

Again ψ can be increased until $E_2(\psi)$ is a multiple lattice point edge, as in the proof of lemma 5.4.5. The edge E_1 now contains at least 3 lattice points in its relative interior, since the lattice point endpoint of $E_1(\psi)$ on E_2 becomes a point in the relative interior of E_1 under the modification of $E_2(\psi)$. The modified set $K(\psi)$ then has 3 multiple lattice point edges as required, unless the endpoint of $E_2(\psi)$ on E_3 is one of only two lattice points on E_3 before E_2 was modified.

In this case, we modify E_3 as above, lengthening E_2 and shortening E_1. As E_1 has at least 3 lattice points in its relative interior, in this case the resulting set $K(\psi)$ has indeed

3 multiple lattice point edges. We have therefore proved the result if K is a triangle.

We secondly suppose that K is a quadrilateral. By lemma 5.4.5 we may suppose that K is not a parallelogram, and that any edge E of K, whose adjacent edges E_+ and E_- meet in the halfplane bounded by E containing K, is a multiple lattice point edge. By lemma 5.4.5, K has at least one such edge. By lemma 5.4.6, the remaining edges of K may be assumed to be lattice midpoint edges. We claim that we may assume that K has a second edge with the above property.

To the contrary, we suppose that K has parallel edges E_1 and E_3, where E_1 is a multiple lattice point edge, and where E_2 and E_4 meet in the halfplane bounded by E_1 containing K. We suppose that E_2, E_3 and E_4 are lattice midpoint edges with midpoints L_2 L_3 and L_4 respectively. We claim that there is a lattice point in the interior of K on the line segment L_2L_4. For, otherwise we let L_0 and L_1 be two adjacent lattice points in the relative interior of E_1, and let S denote the strip with sides L_0L_4 and L_1L_2. Since S contains two lattice points on its boundary and none in its interior on each of the equally spaced lattice lines E_1, and L_2L_4, the same can be said of the equally spaced line through E_3. However, as L_0 and L_1 lie in the relative interior of E_1 and as L_2 and L_4 are boundary points of K, the edge E_3 of K lies wholly within S. As E_3 has midpoint L_3, a lattice point, we have a contradiction. There is therefore a lattice point X in the interior of K on the segment L_2L_4, as claimed.

We next claim that there is a chord of K through X, bisected by X, such that K has parallel supporting lines at its endpoints. We let Y be the point $E_2 \cap E_4$. The line XY lies between E_2 and E_4 in the pencil of lines through Y, and so meets E_1 and E_3 at points in their relative interiors. The chord $XY \cap K$ has parallel supporting lines at its endpoints, along the edges E_1 and E_3, and has midpoint X since the chord L_2L_4 lies midway between E_1 and E_3. Since area and the incidences of K with the integer lattice are unchanged under an integral translation of X to O, we may assume that $X = O$, and so that $XY \cap K$ is an extremal chord of symmetry of K. By Corollary 5.3.1, $A(K) \leqslant 2(t+2)$.

We may therefore make the assumption that K has no pair of parallel edges, and so that, by the modification of lemma 5.4.5, K has at least two multiple lattice point edges. We have therefore proven the lemma 5.4.7.

Proof of Theorem 5.4.1 By lemma 5.4.2, K may be assumed to be a polygon. By lemma 5.4.6 we may assume that each edge of K is a lattice midpoint edge or a multiple lattice point edge. By lemma 5.4.7 we may assume that if K is a triangle, all of its edges are multiple lattice point edges, while if it is a quadrilateral, at least two of its edges are multiple lattice point edges. Hence we have shown that we may assume the properties for K listed in Theorem 5.4.1.

5.5 An Analogue to the Minkowski-Vander Corput Theorem for a Class of Polygons.

Definition 5.5.1 A convex lattice 6-gon all of whose interior lattice points are collinear we call a *flat lattice hexagon.*

In order to prove Theorems 5.1.2 and 5.1.3, I would like to use the following result, which I have been unable to prove if $c \geqslant 5$.

Conjecture 5.5.2 Let K be a convex polygon in the plane which has c lattice points in its interior. Let Y be the convex lattice n-gon, with $n \geqslant 6$, which is the convex hull of all the lattice points on the boundary of K which are not vertices of K. We suppose that the edges of K either contain two vertices of Y in their relative interior, or have a vertex of Y as midpoint.

Then we have either

(a) $A(K) \leqslant 2(c+1)$ or

(b) Y is a flat lattice hexagon, and $A(K) \leqslant 2(c+1) + (2c)^{-1}$.

The remainder of this section 5.5 is occupied in proving

Theorem 5.5.3 Conjecture 5.5.2 is true if $c \leqslant 4$.

We shall prove Theorem 5.5.3 by enumerating cases. Our plan is as follows. After a preliminary lemma (lemma 5.5.4) we show that

(a) there are just four types of convex lattice 7-gon having at most 4 interior lattice points (lemma 5.5.5)

(b) there are 24 types of non-flat convex lattice 6-gon having at most 4 interior lattice points (lemma 5.5.6).

We then examine the convex polygons K of Theorem 5.5.3 investigating in turn the case when Y is a convex 8-gon (lemma 5.5.7), a convex

7-gon (lemma 5.5.8), a non-flat convex 6-gon (lemma 5.5.9) and a flat convex 6-gon (lemma 5.5.10). In the process we use results (a) and (b) above. The proof of Theorem 5.5.3 now follows easily.

Lemma 5.5.4 Let O be a convex lattice polygon, and let I be the convex lattice polygon given by the convex hull of the set of interior points of O which belong to the integer lattice Λ_0. Let E be an edge of I, and let H_E be the open halfplane not containing I bounded by the line E. We let ℓ be the lattice line parallel and adjacent to E in H_E. Then all vertices of O in H_E lie on ℓ.

Proof. Let P_1 and P_2 be two lattice points on E such that no lattice point lies on the segment of E between them. Let T_1 be any triangle with vertices P_1, P_2 and P_3, where P_3 is a lattice point on ℓ. By Pick's Theorem 1.4.1, as T_1 contains no lattice points besides its vertices, the area of T_1 is 0.5. Any triangle T_2 with vertices P_1, P_2 and P_4, a lattice point belonging to H_E but not ℓ, has an area at least 1, since its height above E is at least double that of T_1 as parallel lattice lines are spaced equally apart. By Picks Theorem 1.4.1, T_2 therefore contains at least one lattice point other than its vertices.

We now suppose that O has a vertex V in H_E but not on ℓ. By the above, the triangle with vertices P_1, P_2 and V contains at least one lattice point X besides these vertices. Since P_1 and P_2 are adjacent on E, X is not contained in I. However, since P_1 and P_2 are interior points of O, so is X, in

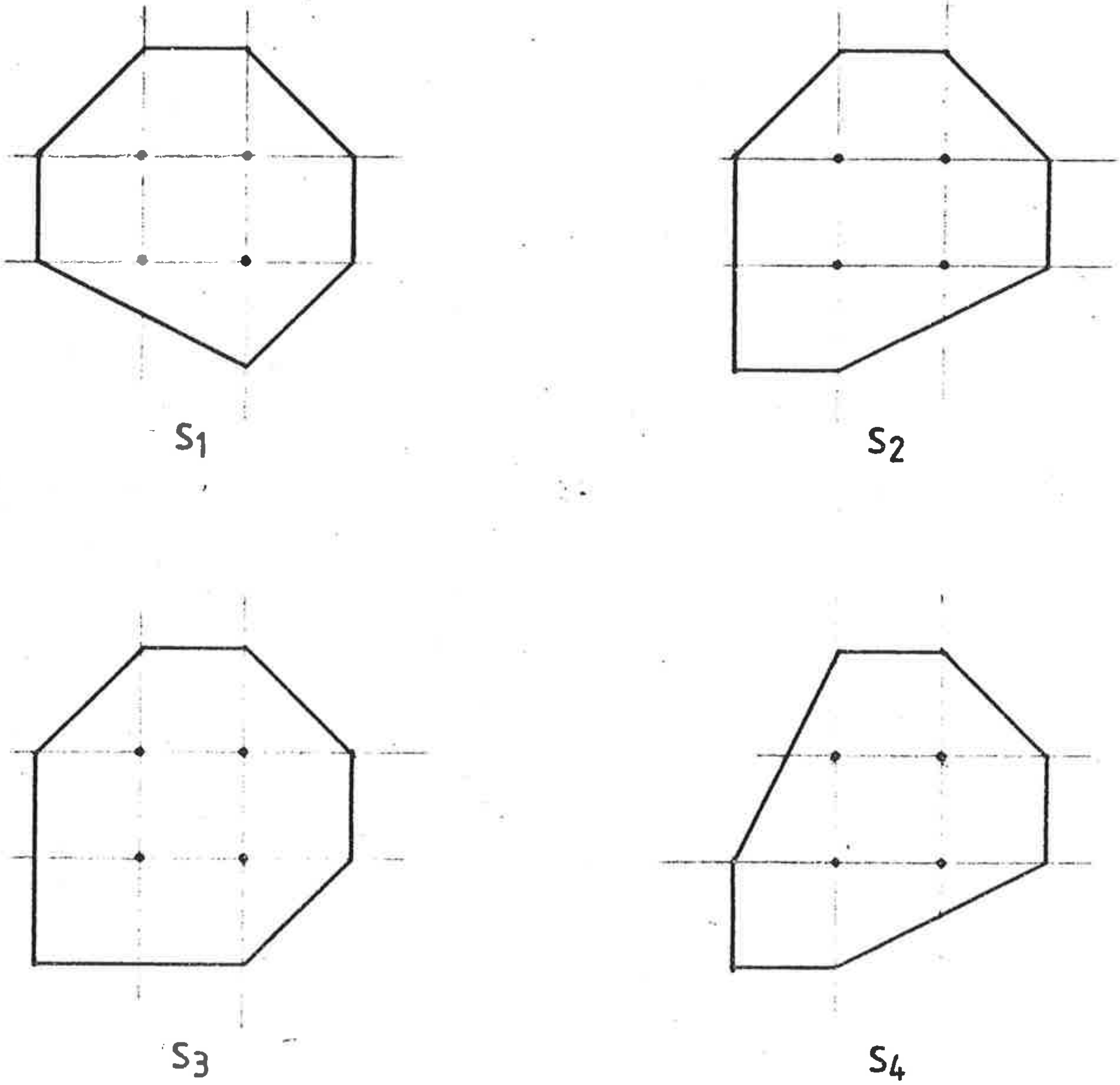


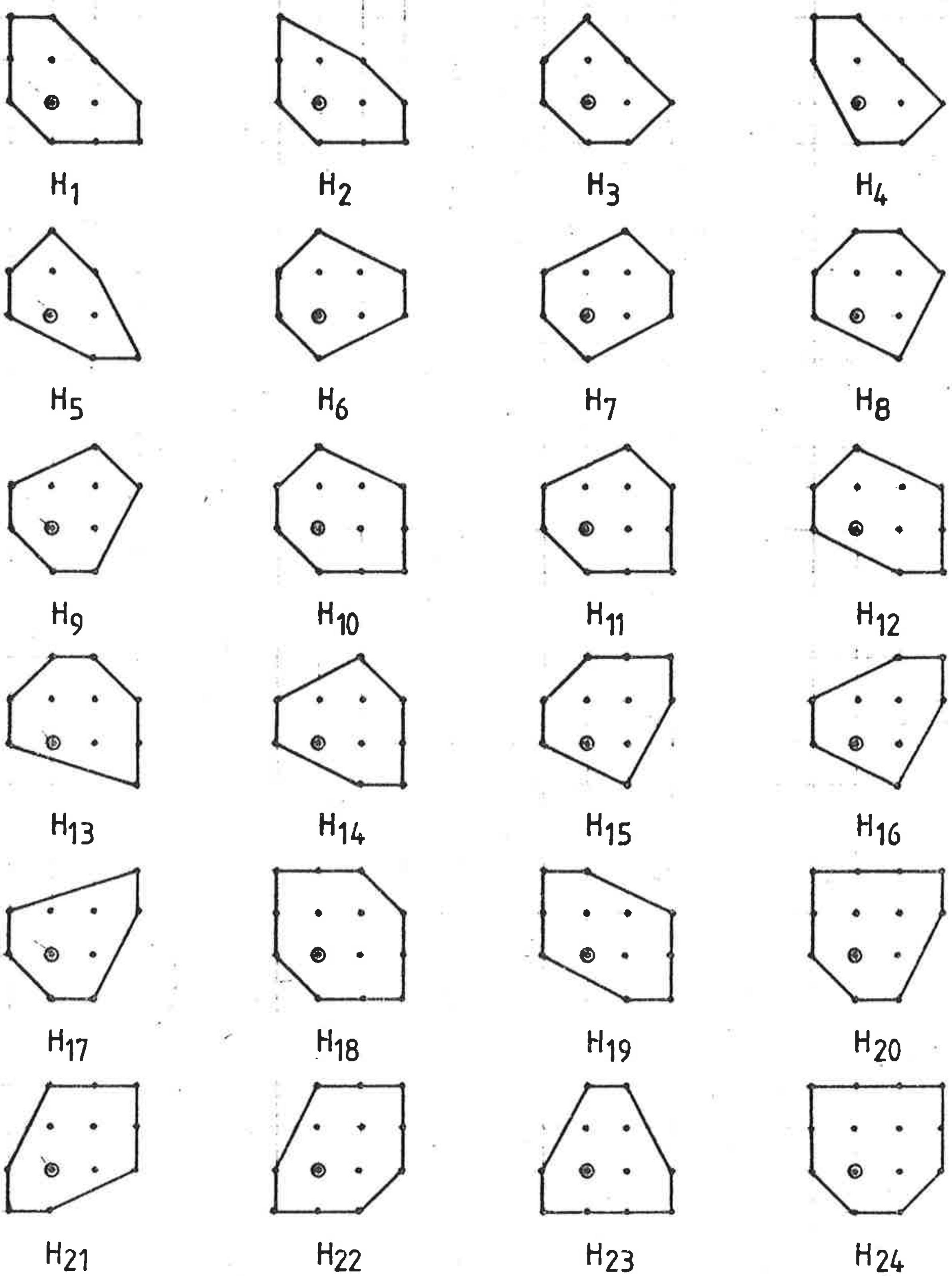
FIGURE 5.2

contradiction to the definition of I. Hence the lemma is proved.

Lemma 5.5.5 Let S be a convex lattice 7-gon which contains 4 lattice points in its interior. Then S is an integral unimodular transform of one of 4 lattice 7-gons S_1, S_2, S_3, S_4 tabulated below and shown in Figure 5.2, opposite.

Proof. By lemma 5.3.2 (Vii) a convex lattice 7-gon contains just four lattice points in its interior only if these four lattice points form a parallelogram. Since area and the configurations of polygons and lattice points are invariant under an integral unimodular transformation, with no loss of generality we assume that this parallelogram is the square Q with vertices (0,0) (1,0) (0,1) and (1,1). By lemma 5.5.4, the vertices of S lie on the boundary of the square U with vertex set $A = \{(-1,-1), (-1,2), (2,2), (2,-1)\}$. Since S has no more than two vertices on each of the edges $y = -1$ and $y = 2$ of U, the number a_1 of vertices of S in the set $A_1 = \{(-1,0), (-1,1), (2,0), (2,1)\}$ satisfies $a_1 \geqslant 3$. Similarly, the number a_2 of vertices of S in $A_2 = \{(0,2), (1,2), (0,-1), (1,-1)\}$, that is the number of vertices of S not on the lines $x = -1$ or $x = 2$, satisfies $a_2 \geqslant 3$.

Hence, by reflecting S in the lines $x = \frac{1}{2}$ or $y = \frac{1}{2}$ if necessary, we may assume that four of the vertices of S are (0,2) (1,2) (2,1) and (2,0). As three of the vertices of U are collinear with two of these vertices of S, the only vertex of U which can be a vertex of S is (-1,-1). Hence a_3, the number of vertices of S in A is at most one. We now list the four 7-gons $S_1 S_2 S_3 S_4$, and their parameters.



H1
H2
H3
H4
H5
H6
H7
H8
H9
H10
H11
H12
H13
H14
H15
H16
H17
H18
H19
H20
H21
H22
H23
H24

FIGURE 5.3

Parameters a_1	a_2	a_3	Representative 7-gon, and vertex set
4	3	0	S_1 $A_1 \cup A_2 \sim \{(0,-1)\}$
3	3	1	S_2 $A_1 \sim \{(-1,0)\} \cup A_2 \sim \{(1,-1)\} \cup \{(-1,-1)\}$
3	3	1	S_3 $A_1 \sim \{(-1,0)\} \cup A_2 \sim \{(0,-1)\} \cup \{(-1,-1)\}$
3	3	1	S_4 $A_1 \sim \{(-1,1)\} \cup A_2 \sim \{(1,-1)\} \cup \{(-1,-1)\}$

We claim that, up to an integral unimodular transformation, these four 7-gons represent all convex lattice 7-gons which contain the vertices of Q as their sole interior points. Clearly, if $a_3 = 0$, the vertices of S are all but one element of $A_1 \cup A_2$, and so S can be transformed to the set S_1 by a suitable rotation about $(\frac{1}{2},\frac{1}{2})$ and/or reflection in $x = \frac{1}{2}$. If $a_3 = 1$, S may have only one other vertex on each of the lines $x = -1$ and $y = -1$. Since the two 7-gons obtained by taking the choices $(-1,1)$ and $(0,-1)$, and $(-1,0)$ and $(1,-1)$ respectively are images of each other under a reflection in $x = y$, we have just three distinguishable choices of one vertex on each of $x = -1$ and $y = -1$. As these choices correspond to 7-gons $S_2 S_3$ and S_4, we have verified our claim.

<u>Lemma 5.5.6</u> Let H be a convex lattice 6-gon which is not a flat convex lattice hexagon (see definition 5.5.1). If H contains 3 lattice points in its interior, then H is an integral unimodular transform of one of five hexagons $H_1,\ldots,H_5$ tabulated below. If H contains 4 lattice points in its interior, then H is an integral unimodular transform of one of 19 hexagons $H_6,\ldots,H_{24}$ tabulated below. Hexagons $H_1,\ldots,H_{24}$ are shown in the figure 5.3, opposite.

<u>Proof</u>. If H contains three lattice points in its interior, they form a triangle, as this is the only configuration of three non-collinear points. Without loss of generality, if necessary employing an integral unimodular transformation, we may assume that these points are $(0,0)$, $(1,0)$ and $(0,1)$, and we let R be the triangle with these vertices. We first show just which lattice hexagons H contain these points as their only interior lattice points. By lemma 5.5.4, the vertices of H must lie on the triangle T with edges $x = -1$, $y = -1$ and $x+y = 2$. Further, since H has 6 vertices (no three collinear), no vertex of T is a vertex of H. Since integral unimodular transformations exist which interchange any pair of vertices of R while leaving the third vertex fixed, the three sides of T are equivalent. We can thus characterise the possible hexagons H by the numbers of lattice points of H belonging to each of the three sides of T, respectively.

Choice from edges of T			Representative Hexagon: Vertices
3	3	3	H_1: (-1,2) (-1,0) (0,-1) (2,-1) (2,0) (0,2)
3	3	2	H_2: (-1,2) (-1,0) (0,-1) (2,-1) (2,0) (1,1)
2	2	3	H_3: (-1,1) (-1,0) (0,-1) (1,-1) (2,0) (0,2)
2	2	3	H_4: (-1,2) (-1,1) (0,-1) (1,-1) (2,0) (0,2)
2	2	2	H_5: (-1,1) (-1,0) (1,-1) (2,-1) (1,1) (0,2)

It is easily confirmed that the five hexagons listed above exhaust all lattice hexagons containing T, up to integer unimodular transformation.

If H contains four lattice points in its interior, we first show that they form a quadrilateral Q. The only other possibilities are that they form a triangle T_1 containing one lattice point on the relative interior of one side, or a triangle T_2 containing one lattice point in its interior. By mapping with an integer unimodular transformation if necessary, we may assume with no loss of generality that T_1 has vertices $(0,0)$ $(2,0)$ and $(0,1)$ and that T_2 has vertices $(1,0)$ $(0,1)$ and $(-1,-1)$ respectively. By lemma 5.5.4, the vertices of H lie on the sides of the triangles S_1 with vertices $(-1,-1)$ $(5,-1)$ and $(-1,2)$ or S_2 with vertices $(2,0)$ $(0,2)$ and $(-2,-2)$ respectively. Now $(-1,2)$ is the only lattice point on the boundary of S_1 in the halfplane $y > 1$, and since $(0,1)$ is an interior point of H, $(-1,2)$ must be a vertex of H. However $(-1,2)$ is a vertex of the triangle S_1, and so as before, at most 4 further vertices of H can be chosen on the boundary of S_1, a clear contradiction. In the case of T_2, S_2 has only 3 lattice points on its boundary which are not vertices of S_2, and so no lattice hexagon H exists with all its vertices on the boundary of S_2. By these contradictions, we deduce that the four interior lattice points of H lie at the vertices of the quadrilateral Q.

From lemma 5.3.2(i), we deduce that Q has a pair of parallel sides, and since lattice points are regularly placed on parallel lattice lines, Q is a parallelogram. By mapping with an integral unimodular transformation we may assume with no loss of generality that Q has vertices $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$.

By lemma 5.5.4, the vertices of H lie on the sides of the square U, with sides $x = -1$, $y = -1$, $x = 2$ and $y = 2$. As no 3 vertices of H are collinear, no more than 2 vertices of H lie on any given side of U, and so at least one vertex of H lies on every side of U. Hence there are a_1, $2 \leqslant a_1 \leqslant 4$, vertices of H belonging to the set $A_1 = \{(-1,0),(-1,1),(2,0),(2,1)\}$ of lattice points of U not on $y = -1$ or $y = 2$. Similarly there are a_2, $2 \leqslant a_2 \leqslant 4$, vertices of H belonging to the set $A_2 = \{(0,-1),(1,-1),(0,2),(1,2)\}$ of lattice points of U not on $x = -1$ or $x = 2$. Since no three vertices of H are collinear, the number of vertices of H which are vertices of U, $a_3 \leqslant 2$. We now list 19 hexagons, by their parameters, and their vertices.

Parameters a_1	a_2	a_3	Representative Hexagon: Vertex Set
4	2	0	H_6: $A_1 \cup \{(0,-1),(0,2)\}$
4	2	0	H_7: $A_1 \cup \{(0,-1),(1,2)\}$
3	3	0	H_8: $A_1 \sim \{(2,0)\} \cup A_2 \sim \{(0,-1)\}$
3	3	0	H_9: $A_1 \sim \{(2,0)\} \cup A_2 \sim \{(0,2)\}$
3	2	1	H_{10}:$A_1 \sim \{(2,0)\} \cup \{(0,-1),(0,2)\} \cup \{(2,-1)\}$
3	2	1	H_{11}:$A_1 \sim \{(2,0)\} \cup \{(0,-1),(1,2)\} \cup \{(2,-1)\}$
3	2	1	H_{12}:$A_1 \sim \{(2,0)\} \cup \{(0,2),(1,-1)\} \cup \{(2,-1)\}$
3	2	1	H_{13}:$A_1 \sim \{(2,0)\} \cup \{(0,2),(1,2)\} \cup \{(2,-1)\}$
3	2	1	H_{14}:$A_1 \sim \{(2,0)\} \cup \{(1,2),(1,-1)\} \cup \{(2,-1)\}$
3	2	1	H_{15}:$A_1 \sim \{(2,0)\} \cup \{(1,-1),(0,2)\} \cup \{(2,2)\}$
3	2	1	H_{16}:$A_1 \sim \{(2,0)\} \cup \{(1,-1),(1,2)\} \cup \{(2,2)\}$
3	2	1	H_{17}:$A_1 \sim \{(2,0)\} \cup \{(1,-1),(0,-1)\} \cup \{(2,2)\}$
2	2	2	H_{18}:$\{(-1,0),(2,1)\} \cup \{(0,-1),(1,2)\} \cup \{(-1,2),(2,-1)\}$
2	2	2	H_{19}:$\{(-1,0),(2,1)\} \cup \{(0,2),(1,-1)\} \cup \{(-1,2),(2,-1)\}$
2	2	2	H_{20}:$\{(-1,0),(2,1)\} \cup \{(0,-1),(1,-1)\} \cup \{(-1,2),(2,2)\}$
2	2	2	H_{21}:$\{(-1,0),(2,0)\} \cup \{(0,-1),(0,2)\} \cup \{(2,2),(-1,-1)\}$
2	2	2	H_{22}:$\{(-1,0),(2,0)\} \cup \{(1,-1),(0,2)\} \cup \{(2,2),(-1,-1)\}$
2	2	2	H_{23}:$\{(-1,0),(2,0)\} \cup \{(0,2),(1,2)\} \cup \{(2,-1),(-1,-1)\}$
2	2	2	H_{24}:$\{(-1,0),(2,0)\} \cup \{(0,-1),(1,-1)\} \cup \{(-1,2),(2,2)\}$

We claim that, up to an integral unimodular transformation, these 19 hexagons represent all convex lattice hexagons which contain Q. Both Q and U are invariant figures under the symmetry group of Q, a subgroup of the integral unimodular transformations of the plane. We thus have lost no generality by assuming that $a_1 \geqslant a_2$, that $(2,0)$ is not a vertex of H when $a_1 = 3$ or that $(-1,0)$ is a vertex of H when $a_1 = 2$.

It is clear that hexagons $H_6, \ldots, H_9$ represent all the hexagons for which $a_3 = 0$. With $a_3 = 1$, and so with the vertices of A_1 chosen as indicated above, there are only two vertices of U, $(2,-1)$ and $(2,2)$, which can be the vertex of H. The hexagons $H_{10}, \ldots, H_{14}$ represent all the possible choices for the remaining vertices from A_2, if $(2,-1)$ is a vertex of H. Similarly if $(2,2)$ is a vertex of H, $(1,-1)$ must be then a vertex of H in order that the vertices of Q are interior to H, and so there are exactly three hexagons, H_{15}, H_{16} and H_{17} with this vertex. Finally, if $a_3 = 2$, it is impossible for the four vertices of H in A_1 and A_2 to lie on just two sides of U, for there is only one vertex of U not on these two sides. Hence we may assume that $(-1,0)$ is the only vertex of H on the edge of U given by $x = -1$, which belongs to A_1. If H has two vertices collinear with an edge of Q, we may assume without loss of generality, applying a symmetry of Q if necessary, that those vertices are $(-1,0)$ and $(2,0)$. There are four choices for the vertices of H on A_2, since with the vertices so far specified, H is symmetric in $x = \frac{1}{2}$. These four choices correspond to $H_{21}, \ldots, H_{24}$. If H has no two vertices collinear with an edge of Q, $(2,1)$ is a vertex of H and the vertices of H which belong to A_2 may be chosen in just three ways, since with the vertices so far specified, H is invariant under a reflection in the point $(\frac{1}{2},\frac{1}{2})$. These choices correspond to hexagons $H_{18},\ldots,H_{20}$. The two vertices of U which are vertices of H are fixed by our choice of the vertices taken from A_1 and A_2, with the exception of H_{19}. In this case the vertices of H which are vertices of U can be chosen in two ways, but the two hexagons so attained are symmetric, under a rotation by $\frac{\pi}{2}$ about $(\frac{1}{2},\frac{1}{2})$.

Thus the hexagons $H_6,\ldots,H_{24}$ represent all hexagons which contain just the four vertices of Q as interior lattice points.

Lemma 5.5.7 Let K be a convex polygon in the plane which has at most four lattice points in its interior and which contains on its boundary, not as vertices, the vertices of a convex lattice 8-gon O. If each edge of K contains either two vertices of O in its relative interior, or has a vertex of O as midpoint, then the area of K is at most 9.

Proof. Since K contains at most four lattice points in its interior, so does its subset O. However, by lemma 5.3.2(viii), a convex lattice 8-gon contains at least four lattice points in its interior, and contains four only if it is an integral unimodular transform of the symmetric 8-gon O_0. Therefore O is such a transform of O_0, and since area and the configurations of lattice points in the statement of the lemma are invariant under an integral unimodular transformation we assume without loss of generality that O is O_0, with interior points (0,0),(1,0),(0,1) and (1,1).

We now show that not all four vertices of O_0 in any of the halfplanes $x \geq \frac{1}{2}$, $x \leq \frac{1}{2}$, $y \geq \frac{1}{2}$ and $y \leq \frac{1}{2}$ are midpoints of sides of K. By the symmetry of K it suffices to show this for the four vertices of O_0 in the halfplane $x \geq \frac{1}{2}$. We therefore suppose to the contrary that the edges V_1V_2, V_2V_3, V_3V_4 and V_4V_5 of K have midpoints $(1,2),(2,1),(2,0)$ and $(1,-1)$ respectively. By the convexity of K, since K has the vertices of O_0 on its boundary, the vertex V_1 of K lies in the triangle T bounded by the lines $y = 2$, $y = x+2$ and $y+x = 3$. As V_2 is the image of V_1 under a halfturn about $(1,2)$, and V_3 is the image of V_2 under a halfturn about $(2,1)$, the point V_3 therefore lies in the triangle T_3 obtained by translating T_1 by the vector $(2,-2)$. Similarly, since V_5 lies in the triangle T_5 bounded by $y = -1$, $y = x-2$ and $y+x+1 = 0$ the point V_3 lies in the triangle T'_3 obtained by translating T_5 by the vector $(2,2)$. However, the only common point of triangles T_3 and T'_3 is $(2\frac{1}{2},\frac{1}{2})$. Hence V_3 must be this point. But now V_1V_2 and V_3 are collinear, contradicting our assumption that $(1,2),(2,1)$ are midpoints of distinct edges of K. Thus the vertices of O_0 lying in any of the given halfplanes can not all be midpoints of edges of K.

We deduce that at least two edges of K contain two vertices of O_0. Since O_0 has eight vertices, it follows that K is neither an 8-gon nor a 7-gon.

We now suppose that K has an edge E which has two vertices of O_0 in its relative interior. By the symmetry of O_0, it suffices to consider two cases, according to whether E is parallel, or at 45°, to one of the coordinate axes.

We suppose first that E with endpoints V_1 and V_2 lies parallel to a coordinate axis.

Without any loss of generality, we suppose that E contains the vertices $(0,2)$ and $(1,2)$ of O_0 in its relative interior. We claim that V_1 and V_2 must be $(-1,2)$ and $(2,2)$. The point $(-1,2)$ cannot be in the relative interior of E, for then by convexity $(-1,1)$ is interior to K, contrary to the fact that vertices of O_0 lie on the boundary of K. If $(-1,2)$ is exterior to K, and so to E, the point V_1 has coordinates $(-1+\varepsilon,2)$ where $0 < \varepsilon < 1$. The edge of K with $(-1,1)$ in its relative interior is therefore a lattice midpoint edge of K, and so K has the vertex $(-1-\varepsilon,0)$. However, this contradicts the fact that $(-1,0)$ and $(0,-1)$ lie on the boundary of K. Hence V_1 is $(-1,2)$. A precisely similar argument shows that $V_2 = (2,2)$; hence the edges of K adjacent to E are multiple lattice point edges parallel to the other coordinate axis. By reapplying this argument to one of these edges, we deduce that K is a square of side length 3, and so $A(K) = 9$.

We may therefore assume that K has an edge E at 45° to the coordinate axis, which contains two vertices of O_0 in its relative interior. By the symmetry of O_0, we choose E to pass through $(1,2)$ and $(2,1)$, with no loss of generality. We now show that $A(K) = 8$.

We first show that K has two parallel edges which each contain two vertices of O_0 in their relative interiors. This is so if both the edges of K adjacent to E contain two vertices of O_0 in their relative interiors, for they are parallel to $y = x$ by the only possible choice of four such vertices of O_0. Otherwise, we may suppose that K has an edge E_1 adjacent to E which has a vertex of O_0 as its midpoint. Without loss of generality, since O_0 is symmetric about $y = x$, we take this midpoint to be $(2,0)$. Thus E_1 has an endpoint on the line $x+y = 1$, strictly between the point $(2,-1)$ and the line $y+2 = x$. The next edge E_2 of K adjacent to E_1 is therefore also a lattice midpoint edge of K, with midpoint $(1,-1)$, and so has endpoint on the line $x+y+1 = 0$. Since the vertices $(-1,0)$ and $(0,-1)$ of O_0 also lie on this line, the next edge E_3 of K is a multiple lattice point edge parallel to E. Thus in this case too, K has two parallel edges at 45^o to the axes, which contain two vertices of O_0 in their relative interiors.

If all the vertices of K lie on edges of the above type, K is then a square of side length $2\sqrt{2}$, and so has area 8. We denote this square U. Without loss of generality, we suppose that K has parallel edges E and E' containing $(1,2)$ and $(2,1)$, and $(-1,0)$ and $(0,-1)$ respectively in their relative interiors. Since the remaining four vertices of O_0 lie midway between the line $x+y = 1$ and E and E', the triangular components of $K \sim U$ and of $U \sim K$ to either side of $x+y = 1$ are congruent. Hence $A(K) = A(U) = 8$, and we have proved the lemma.

Lemma 5.5.8 Let K be a convex polygon in the plane which has at most four lattice points in its interior and which contains on the relative interior of its edges the vertices of a convex lattice 7-gon S. If all the edges of K contain either two vertices of S in their relative interior or have as midpoint such a vertex, then $A(K) = 7\frac{1}{2}$.

Proof. Since K contains at most four lattice points, so does its subset S. However, by lemma 5.3.2(vii) a convex lattice 7-gon contains at least four lattice points in its interior, and so S contains exactly four interior lattice points. By lemma 5.5.5, S is therefore an integral unimodular transform of one of the sets S_1, S_2, S_3 or S_4 specified there. Since area and the incidences of K with S and the lattice are preserved under an integral unimodular transform, we assume that S is S_i for some $i=1,\ldots,4$, with no loss of generality. We shall show that S must be S_4.

It is easy to show that S is not S_2 or S_3. In each case, the point $(-1,0)$ is not interior to K; the collinearity of $(-1,1)$ $(-1,0)$ and $(-1,-1)$ now implies the existence of an edge E of K containing these three points. For S_3 the same argument shows that K has an edge through $(-1,-1)$ $(0,-1)$ and $(1,-1)$. But now $(-1,-1)$ is a vertex of K, and so does not belong to the set S, contrary to the supposition $S = S_3$. In the case of S_2, let E' be the edge of K adjacent to E and passing through $(0,2)$. If $(0,2)$ is the midpoint of E', the endpoint of E' not on E has coordinates $(1,2+\varepsilon)$, $\varepsilon \in (0,1)$, contradicting the fact that $(1,2)$ and $(2,1)$, vertices of S_2, belong to the boundary of K. Hence E' contains the two lattice points $(0,2)$ and $(1,2)$. By repeating this argument, we can show that the edge E'' of K adjacent to E', which passes through $(2,1)$ is also a multiple lattice point edge of K. Now edges E and E'' are parallel, and the remaining vertex $(0,-1)$ of S_2 is not midway between E and E''. We deduce that no fourth edge of K, either a lattice midpoint edge or multiple lattice point edge, can exist, i.e. there is no such set K.

We now suppose that S is S_1, with vertices $(1,-1)$ $(2,0)$ $(2,1)$ $(1,2)$ $(0,2)$ $(-1,1)$ and $(-1,0)$, so that the vertices of S_1 are those of O_0 other than $(0,-1)$. We claim that the vertices $(-1,0)$ and $(1,-1)$ are both in the relative interior of a common edge of K. To the contrary, we suppose that there is a vertex V_1 of K, on the boundary of K between $(-1,0)$ and $(1,-1)$. Unless V_1 lies on the line $x = -1$, it is the endpoint of a lattice midpoint edge of K centred at $(-1,0)$. By convexity, the other endpoint of this edge is contained in the closed triangle bounded by the lines $x = -1$, $y = x+2$ and $x+2y+1 = 0$. Hence V_1 either lies on the line $x = -1$ or in the closed triangle bounded by the lines $x = -1$, $y = x$ and $x+2y+1 = 0$. Similarly V_1 either lies on the line $y+2 = x$, or V_1 is the endpoint of a lattice midpoint edge centred at $(1,-1)$, and so belongs to the closed triangle bounded by the lines $x = 0$, $x+2y+1 = 0$ and $y+2 = x$. Since the only intersection of these two loci for V_1 is the point $(-1,-3)$, we deduce that $V_1 = (-1,-3)$. However, since $(1,1)$ is an interior point of K, the midpoint of V_1 and $(1,1)$, namely $(0,-1)$ is interior to K; this contradicts our assumption that the only lattice points in the interior of K are the four interior to S_1. Hence the vertices $(-1,0)$ and $(1,-1)$ lie in the relative interior of a common edge of K. We label this edge of K V_2V_3, with V_2 closer to $(-1,0)$ than $(1,-1)$.

Either V_3 lies on the line $x = 2$, or is the endpoint of a lattice midpoint edge V_3V_3' of K centred at $(2,0)$. Now the position of V_3 implies that the y-coordinate of V_3' is greater than 1; this precludes the point (2,1) from being a boundary point of K, as assumed. Hence V_3 lies on $x = 2$, and so is the point $(2,-1\frac{1}{2})$. As (2,0) and (2,1) are vertices of S_1, they are boundary points of K, and so lie on a multiple lattice point edge of K along $x = 2$ which is of course parallel to the y-axis. By the argument given in the 5th argument of the proof of lemma 5.5.7, which is applicable since the vertices of S_1 are all vertices of O_0, K has also multiple lattice point edges along the lines $y = 2$ and $x = -1$. But now vertex V_2 of K is the point (-1,0), contrary to our assumption that (-1,0) is a vertex of S_1, and so not a vertex of K. Hence S is not S_1.

Finally, we suppose that S is S_4, with vertices (2,0) (2,1) (1,2) (0,2) (-1,0) (-1,-1) and (0,-1). Note that S_4 is symmetric about the line $y = x$. We first claim that the vertices (0,-1) and (2,0) are both in the relative interior of a common edge of K. To the contrary, we suppose that there is a vertex V_1 of K, on the boundary of K between (0,-1) and (2,0). Either V_1 lies on the line $y = -1$, or is the endpoint of a lattice midpoint edge V_1V_2 of K centred at (0,-1). By convexity, the endpoint of such a lattice midpoint edge, V_2 lies in the closed triangle bounded by lines $x = -1$, $y = -1$ and $2y+2 = x$. Therefore V_1 either lies on $y = -1$, or in the closed triangle bounded by lines $x = -1$, $y = -1$ and $2y+2 = x$. Similarly, either V_1 lies on the line $x = 2$, or is the endpoint of a lattice midpoint edge V_1V_3 of K centred at (2,0). The other endpoint of this edge of K, V_3, lies in the closed triangle bounded by the lines $x = 2$, $x+y = 3$ and $2y+2 = x$, and so has x-coordinate no greater than $2\frac{2}{3}$. Hence V_1 either lies on $x = 2$, or has x-coordinate no less than $1\frac{1}{3}$. We deduce that V_1 is the only point of intersection of these two loci for V_1, namely (2,-1). However, then (1,-1) must also belong to the relative interior of the multiple lattice point edge of K through V_1 and (-1,-1). Therefore the 7-gon S composed of all the lattice points of the boundary of K other than vertices, is not S_4 but is S_2, and we have a contradiction. It follows that the vertices (0,-1) and (2,0) are both in the relative interior of a common edge E_1 of K, along the line $2y+2 = x$. By the symmetry of S_4, we deduce that the vertices (-1,0) and (0,2) are both in the relative interior of a common edge E_2 of K, along the line $y = 2x+2$. Since the vertex (-1,-1) of S_4 is the only vertex of S_4 between the lines $2y+2 = x$ and $y = 2x+2$ with $x+y < -1$, we deduce that (-1,-1) is the midpoint of an edge of K along the line $x+y = -2$.

We now claim that the vertices (1,2) and (2,1) of S_4 are both in the relative interior of a common edge of K. To the contrary, we suppose there is a vertex V of K, on the boundary of K between (1,2) and (2,1). V cannot lie on the line $x = 2$, else the point (2,0) is a vertex of K not in the relative interior of edge E_1. Hence V is the endpoint of a lattice midpoint edge of K centred at (2,1) and having its other endpoint on E_1. It follows that V lies on the reflection of E_1 in the point (2,1). Similarly, V lies on the reflection of E_2 in the point (1,2). This implies that V is the point (2,2), contradicting the known fact that V cannot lie on $x = 2$. Hence (1,2) and (2,1) lie in the relative interior of a common edge.

Finally, it is trivially confirmed that K as specified above, has area $7\frac{1}{2}$. Thus the lemma is proved.

Lemma 5.5.9 Let K be a convex polygon which has c lattice points in its interior. We suppose that H, the convex hull of all lattice points on the boundary of K which are not vertices of K, is a convex lattice hexagon. We further suppose that H is not a flat lattice hexagon, and that all the edges of K contain either two vertices of H in their relative interior, or have as midpoint a vertex of H.

Then, if $c \leqslant 4$, $A(K) \leqslant 2(c+1)$.

Proof. If $c = 1$ or 2, then H contains at most two lattice points in its interior. By lemma 5.3.2(iii), H contains at least one lattice point in its interior, and so H is a flat convex lattice hexagon. We may therefore assume that the number of lattice points in the interior of H is either $c = 3$ or $c = 4$, and that not all these points are collinear.

By lemma 5.5.6, we may assume that H is one of the 24 hexagons there listed. We will show that there are only two sets K with $c = 3$, and no sets K with $c = 4$ which have as set H any one of $H_1,\ldots,H_5$. Subsequently, we will show that only two sets K have $c = 4$ and any of $H_6,\ldots,H_{24}$ as their set H. These sets are triangles of area 8 and $7\frac{1}{120}$ about H_{23} and H_{17} respectively.

K contains H_1

The hexagon H_1 has three edges whose midpoints are lattice points. Consequently if $c = 3$ and $H = H_1$, K has three edges with three lattice points in their relative interior, and so K is the triangle T with vertices $(-2,2)$ $(2,2)$ and $(2,-2)$. Since $A(T) = 8$, we are within the bounds required for $c = 3$.

We now suppose that $c = 4$ and $H = H_1$. For each edge E of H_1, the only points of K in the exterior halfplane bounded by E belong to the triangle with edges E and the extensions of the edges of H adjacent to E. Since $(1,-1)$, $(1,1)$ and $(-1,1)$ are the only three lattice points, besides the three interior lattice points of H_1, in the union of H_1 with these six triangles, the fourth interior lattice point of K is one of these points. The set H_1 is invariant under the six integral unimodular transformations which map the triangle of interior points of H_1 onto itself. Hence, without loss of generality, we may assume that $(1,1)$ is an interior point of K, and that the other two points belong to the boundary of K. Hence K has edges along the lines $x = -1$ and $y = -1$. As $(1,1)$ is an interior point of K, the remaining two vertices $(0,2)$ and $(2,0)$ of H_1 are midpoints of two lattice midpoint edges of K. The common endpoint of these edges, a vertex of K, therefore lies on the reflections of $x = -1$ and $y = -1$ in the points $(0,2)$ and $(2,0)$ respectively. Hence this common endpoint is $(1,1)$, but as we assumed $(1,1)$ to be interior to K, this is a contradiction. Hence there is no such set K.

<u>K contains H_2</u>

The hexagon H_2 has two edges whose midpoint is a lattice point. Consequently, for a set K with $c = 3$ and $H = H_2$, neither midpoint is interior to K, and so K has two edges along the lines $x = -1$ and $y = -1$, each of which contains 3 lattice points in its relative interior. We claim that the edge of K through $(2,0)$ is also a multiple lattice point edge of

K. For, if this edge were a lattice midpoint edge, it would have an endpoint on the line $y = 1$, between $(1,1)$ and $(2,1)$, contrary to the fact that $(-1,2)$ and $(1,1)$ are boundary points of the convex set K. Hence K is the triangle T bounded by the lines $x = -1$, $y = -1$ and $x+y = 2$. However, since for this triangle $H = H_1$, we obtain a contradiction.

We show that no K exists with $c = 4$ and $H = H_2$. We suppose that K is such a set. Since H_2 has two edges whose midpoint is a lattice point, the set K has a multiple lattice point edge along at least one of the lines $x = -1$ and $y = -1$. If K has a multiple lattice point edge along the line $y = -1$, we can apply the argument given for $c = 3$, to deduce that K has a multiple lattice point edge along the line $x+y = 2$. If K has also a multiple lattice point edge along $x = -1$, we deduce that $K = T$ and obtain the contradiction $H = H_1$ as above. We may therefore assume that $(-1,2)$ is the midpoint of a lattice midpoint edge of K, one endpoint of which lies on $x+y = 2$. The other endpoint P of this edge therefore lies on the line $x+y = 0$, between $(-1,1)$ and $(-3,3)$. The point P is also the endpoint of a lattice midpoint edge of K, centred at $(-1,0)$, whose other endpoint lies on the line $y = -1$. Hence P lies on the line $y = 1$, and so P is $(-1,1)$. This contradicts our assumption that the edge of K through $(-1,2)$ is a lattice midpoint edge.

We may therefore assume that K has no multiple lattice point edge along the line $y = -1$, and so that K has a multiple lattice point edge along the line $x = -1$. We claim that the edge of K through $(1,1)$ is a multiple lattice point edge of

K. For otherwise, it is a lattice midpoint edge and, as one endpoint lies on the line $x = -1$ between $(-1,2)$ and $(-1,3)$, it has an endpoint on the line $x = 3$ between $(3,0)$ and $(3,-1)$. However, as $(2,0)$ is a vertex of H_2 and so a boundary point of K, this endpoint must be $(3,-1)$, which contradicts our assumption that the edge through $(1,1)$ is not a multiple lattice point edge, and proves our claim. Since K has multiple lattice point edges on the lines $x = -1$ and $x+y = 2$, the point $(0,2)$ belongs to the relative interior of an edge of K. Since $(0,2)$ is not a point of H_2, we have contradicted the assumption $H = H_2$. Hence no set K exists with $c = 4$ and $H = H_2$.

<u>K contains H_3 or H_4</u>

The hexagons H_3 and H_4 each have an edge between $(0,2)$ and $(2,0)$ which contains $(1,1)$ as midpoint. Consequently, a set K with $c = 3$ and $H = H_3$ or $H = H_4$ has a multiple lattice point edge E_1 along the line $x+y = 2$. We shall next show that no such K exists.

We claim that the edge E_2 of K through $(1,-1)$ is a multiple lattice point edge. For, if E_2 is a lattice midpoint edge, its endpoint V not on E_1 lies on the line $x+y = -2$. By the convexity of K and the fact that vertices of H are boundary points of K, the vertex of K given by $E_1 \cap E_2$ lies strictly between the lattice points $(2,0)$ and $(3,-1)$. Therefore V lies strictly between the lattice points $(0,-2)$ and $(-1,-1)$. The vertices $(1,-1)$ and $(-1,1)$ of H are boundary points of K, and so the triangle T with vertices V, $(-1,1)$ and $(1,-1)$ is contained in K. From our knowledge of the position of V, we can deduce that $(0,-1)$ is an interior point of T, and so an interior point of K. However, $(0,-1)$ is a vertex of H_3 or H_4, and so belongs to the boundary of K. By this contradiction, we deduce that our claim is true.

Since E_1 and E_2 are both multiple lattice point edges of K, the point $(2,-1)$ is a boundary point of K which is not a vertex of K. Since $(2,-1)$ belongs to neither H_3 nor H_4, we deduce that no such set K exists.

We now claim that a set K with $c = 4$ and $H = H_3$ or $H = H_4$ also has a multiple lattice point edge E_1 along the line $x+y = 2$. Since the above argument applies equally well in this instance, this claim is all we need to show that no such K exists.

We now prove the claim. To the contrary, we suppose that V is the vertex of K which is the common endpoint of two distinct edges of K through $(0,2)$ and $(2,0)$ respectively. As both H_3 and H_4 have an edge along the line $y = -1$, V lies in the halfplane $y \leqslant 1$, or on the line $y = x-2$ according to whether the edge of K through $(2,0)$ is a lattice midpoint edge or a multiple lattice point edge, necessarily through $(1,-1)$. As both H_3 and H_4 have an edge along the lines $x = -1$ and $x+y = 2$, V lies in the halfplanes $x \leqslant 1$ and $y \geqslant 1$ if K has a lattice midpoint edge through $(0,2)$. If the edge of K through $(0,2)$ is a multiple lattice point edge, it must lie on either the lines $y = 2$ or $y = x+2$, as $H = H_3$ or $H = H_4$ respectively. The only intersection points of these loci for V are the points $(1,1)$ and $(4,2)$. As V is a vertex of K, V is not $(1,1)$. If $(4,2)$ is a vertex of K, K contains the points $(1,1)$ and $(2,1)$ as interior points, as well as the three lattice points in the interior of H. Hence, if $c = 4$, we have obtained a contradiction. Hence our claim is verified, and so no such set K exists.

<u>K contains H_5</u>

The hexagon H_5 has the following easily verified property. For any edge E of H_5, the triangle formed in the exterior halfplane bounded by E, with edges E and the extensions of the two edges of H_5 adjacent to E, contains no lattice point in its interior. Since a set K with $H = H_5$ is contained in the union of H_5 with these six triangles, and as no edge of H_5 contains a lattice point in its relative interior, we deduce that if $H = H_5$ then $c = 3$.

Let K have $c = 3$ and $H = H_5$. We show that K is a triangle having area 6. We begin by showing that K has a multiple lattice point edge. To the contrary, we suppose that all the edges of K are lattice midpoint edges. We label the vertices of K anticlockwise, $V_1,V_2,\ldots,V_6$ with V_1 between the boundary points $(-1,1)$ and $(-1,0)$ of K. By the convexity of K, V_1 lies in the triangle with vertices $(-1,1)$ $(-1,0)$ and $(\frac{-5}{3},\frac{1}{3})$. The vertex V_2 of K therefore lies in the reflection of this triangle in $(-1,0)$, and so the vertex V_3 of K lies in the further reflection of this triangle in $(1,-1)$. This locus for V_3 has vertices $(3,-1)$ $(3,-2)$ and $(2\frac{1}{3},-1\frac{2}{3})$. Since the line through the vertices of H $(1,1)$ and $(2,-1)$, with equation $y = -2x+3$, meets the locus for V_3 in just the point $(2\frac{1}{3},-1\frac{2}{3})$, K must have a multiple lattice point edge along this line. As this is contrary to our choice of K, we deduce that K has a multiple lattice point edge.

We next show that K has no multiple lattice point edge along the line $x = -1$. To the contrary, we suppose that K has such an edge E, containing (-1,0) and (-1,1) in its relative interior. The edge of K through (0,2) is not a lattice midpoint edge, for such an edge would have as endpoint a point on the line $x = 1$ with $y > 1$. This would contradict the assumption that (1,1) is a boundary point of K. Hence K has a second multiple lattice point edge passing through (0,2) and (1,1). We deduce that (-1,3) is a vertex of K, and so that (-1,2) is in the relative interior of E. This contradicts our choice of H as H_5. Hence K has no multiple lattice point edge along the line $x = -1$.

There are six integral unimodular transformations of the plane which map the triangle T with vertices (0,0) (1,0) and (0,1) onto itself. These correspond to the permutations of the vertices of T, and are generated as a group by a reflection in $y = x$, and τ, given in matrix form by $\tau(\underset{\sim}{x}) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\underset{\sim}{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. It is easily seen that τ cycles the vertices of T anticlockwise. Only the three powers of τ map H_5 onto itself, since a reflection in $y = x$ maps H_5 to an equivalent but different hexagon. The edge E of H_5 with endpoints (-1,1) and (-1,0) is mapped successively to the edge $\tau(E)$ with endpoints (1,-1) and (2,-1) and the edge $\tau^2(E)$ with endpoints (1,1) and (0,2). Similarly, the remaining three edges of H_5 are also transformed to each other by powers of τ.

Since K can have no multiple lattice point edge which contains the edge E of H_5 above, by applying the symmetry τ of H_5, we deduce that neither edge $\tau(E)$ nor $\tau^2(E)$ of H_5 is contained in a multiple lattice point edge of K.

We can thus assume that K has a multiple lattice point edge on any of the remaining edges of H_5, since they too are permuted by τ. Without loss of generality, we suppose that K has a multiple lattice point edge E_1 passing through $(-1,1)$ and $(0,2)$. We let E_2 denote the edge of K through $(1,1)$, and let $V_1 = E_1 \cap E_2$. If E_2 is a multiple lattice point edge through $(1,1)$ and $(2,-1)$, the vertex V_2 of K between $(2,-1)$ and $(1,-1)$ lies on this edge, and so on the line $2x+y = 3$. Otherwise E_2 is a lattice midpoint edge of K, and since one endpoint V_1 of E_2 lies on E_1, the other endpoint V_2' lies on the line $y = x-2$, between $(2,0)$ and the line $2x+y = 3$. Since then V_2' lies on a lattice midpoint edge $V_2'V_3'$ centred at $(2,-1)$, the vertex V_3' of K between $(2,-1)$ and $(1,-1)$ lies on the line $y = x-4$, between $(2,-2)$ and the line $2x+y = 3$. Hence the vertex of K between $(2,-1)$ and $(1,-1)$ lies on either the line $2x+y = 3$, or the line $y = x-4$, between $(2,-2)$ and $2x+y = 3$.

Similarly, if the edge of K through $(-1,0)$ and $(1,-1)$ is a multiple lattice point edge, the vertex of K between $(2,-1)$ and $(1,-1)$ lies on the line $x+2y+1 = 0$. Otherwise, as above, we deduce that this vertex lies on the line segment of $y = x-4$, between $(3,-1)$ and the line $x+2y+1 = 0$. However, the lines $2x+y = 3$, $x+2y+1 = 0$ and $y+4 = x$ meet at the point $(\frac{7}{3},-\frac{5}{3})$, which is an interior point of the segment from $(2,-2)$ to $(3,-1)$. Therefore this point is a vertex of K, on a multiple lattice point edge of K through each of the pairs of points $(1,1)$ and $(2,-1)$, and $(-1,0)$ and $(1,-1)$.

Hence K is a triangle with these three multiple lattice point edges. It is readily calculated that this triangle has area 6, which is well within the bounds required by this lemma.

Each of the remaining hexagons $H_6,\ldots,H_{24}$ contains 4 interior lattice points. Therefore any set K, with H any of these hexagons, necessarily has $c = 4$. In order to complete the proof of lemma 5.5.9 we show that there are only two sets K, with $c = 4$, having any of these hexagons as its set H. In fact we shall show that these sets are a triangle of area 8 with $H = H_{23}$, and a triangle of area $7\frac{1}{120}$, with $H = H_{17}$.

K contains $H_{10},H_{11},H_{18},H_{19},H_{20},H_{21},H_{22}$ or H_{24}

Fortunately, 8 of the hexagons $H_6,\ldots,H_{24}$ can easily be eliminated from consideration. Each of the 7 hexagons H_{10},H_{11}, $H_{18},H_{20},H_{21},H_{22}$ and H_{24} has a pair of adjacent edges, each of which contains in its relative interior at least one lattice point. A set K with $c = 4$ and H one of these sets must, by the convexity of K, have each of the edges in this pair as a multiple lattice point edge, for otherwise K would have more than 4 lattice points in its interior. However, the lattice point vertex of H common to this pair of adjacent edges of H is therefore not in the relative interior of an edge of K. This contradicts the defining property of H, and so no such set K exists.

The hexagon H_{19} has a pair of parallel edges, each containing at least one lattice point in its relative interior. Hence a set K with $c = 4$ and $H = H_{19}$ has two parallel multiple lattice point edges E_1 and E_2. The remaining two vertices of H_{19} lie between E_1 and E_2, and are not consecutive vertices of H_{19}. Thus K has a lattice midpoint edge through each. This is impossible, however, since these two vertices are not midway between E_1 and E_2. Hence no such set K exists.

K has all lattice midpoint edges

In all but one case it is easy to show that no K exists having all lattice midpoint edges. Hexagons $H_{12}, H_{13}, H_{14}, H_{15}$ and H_{23} each have an edge which contains a lattice point in its relative interior, which must, since $c = 4$, correspond to a multiple lattice point edge of K. If all the vertices of H are midpoints of edges of K, by lemma 3.4.3(ii) the triangles whose vertices are the alternate vertices of H share a common centroid with K. As the centroid of a triangle is one third the vector sum of its vertices, the sums of alternate vertices of H must be equal. However, these sums are not equal in the cases when H is H_6, H_7, H_8, H_{16} or H_{17}, as shown in the following table.

Hexagon:	The sums of alternate vertices are not equal
H_6 :	$(0,2)+(-1,0)+(2,0) = (1,2) \neq (1,1) = (-1,1) + (0,-1) + (2,1)$
H_7 :	$(-1,1)+(0,-1)+(2,1) = (1,1) \neq (2,2) = (-1,0) + (2,0) + (1,2)$
H_8 :	$(-1,1)+(1,-1)+(1,2) = (1,2) \neq (1,3) = (-1,0) + (2,1) + (0,2)$
H_{16}:	$(-1,1)+(1,-1)+(2,2) = (2,2) \neq (2,3) = (-1,0) + (2,1) + (1,2)$
H_{17}:	$(-1,1)+(0,-1)+(2,1) = (1,1) \neq (2,1) = (-1,0) + (1,-1) + (2,2)$

The set H_9 is the only set H we have not precluded from being the set H for a set K all of whose edges are lattice midpoint edges. We next do so.

We suppose that K with $H = H_9$ and $c = 4$ is a hexagon with vertices $V_1,\ldots,V_6$ labelled anticlockwise. We suppose that all edges of K are lattice midpoint edges and that V_1 lies between the boundary points $(2,1)$ and $(1,2)$ of K.

We show that no such K exists. From the convexity of K, the vertex of K, V_3 lies in the triangle T_3 with vertices $(-1,1)$, $(-1,0)$ and $(-\frac{5}{3},\frac{2}{3})$. The vertex V_2 of K lies in the triangle T_2 given by reflecting T_3 in the point $(-1,1)$, since $(-1,1)$ is the midpoint of the edge V_2V_3 of K. Similarly, V_1 lies in the triangle T_1 given by reflecting T_2 in the point $(1,2)$. As T_1 has vertices $(3,3)$ $(3,2)$ and $(\frac{7}{3},\frac{8}{3})$, the point $(2,2)$ is contained in the triangle with vertices $(1,2)$ $(2,1)$ and V_1. We therefore have obtained a contradiction, since either $(2,2)$ is a fifth interior lattice point of K, and $c = 4$, or $(2,2)$ is a boundary point and not a vertex of K but is not on the boundary of H_9. Hence no such set K exists.

We may therefore assume throughout the remainder of the proof of this lemma that K has at least one edge which is a multiple lattice point edge.

K contains H_6, H_{14}, H_{16} or H_{23}

By a suitable rotation of H_6, H_{14}, H_{16} and H_{23} about the point $(\frac{1}{2},\frac{1}{2})$, (an integral unimodular transformation), we may assume that H has two vertices on the line $x = -1$, and the vertices $(0,2)$, $(2,1)$, $(2,0)$ and $(0,-1)$.

Let K have $c = 4$, and H be one of the above sets, in this standard position. We claim that K has a vertex on its boundary between $(2,0)$ and $(2,1)$. To the contrary, we suppose that K has a multiple lattice point edge E_1 along the line $x = 2$. Since lattice points are regularly spaced on lattice lines, an edge of K passing through $(0,2)$ and a lattice point on the line $x = -1$ also has in its relative interior a lattice point on the line $x = 1$. As H has no such point on its boundary, the edge of K through $(0,2)$ is not a multiple lattice point edge. Neither, however, can K have a lattice midpoint edge E_2 through $(0,2)$, since as one endpoint of E_2 lies on E_1, the other must lie on the line $x = -2$. This is contrary to the fact that the two vertices of H on the line $x = -1$ are boundary points of K. Since no edge of either type can pass through $(0,2)$, we conclude that our claim is true.

Thus K has a vertex V_1 between $(2,0)$ and $(2,1)$. By convexity V_1 lies in the triangle T_1 with vertices $(2,0)$, $(2,1)$ and $(3,\frac{1}{2})$. We next show that $V_1 = (3,\frac{1}{2})$.

To the contrary, we suppose that V_1 is not $(3,\frac{1}{2})$. The point $(3,\frac{1}{2})$ is the intersection of the edges of H along the lines $2y+2 = x$ and $2y+x = 4$. Since $V_1 \neq (3,\frac{1}{2})$, V_1 is not on both these lines, and so is the endpoint of a lattice midpoint edge of K, centred either at $(2,0)$ or $(2,1)$. Our standard position for H is a specification symmetric about the line $y = \frac{1}{2}$. We may therefore suppose with no loss of generality, reflecting K about $y = \frac{1}{2}$ if necessary, that K has a lattice midpoint edge E_1 centred at $(2,1)$, with endpoints V_1 and V_2. The point V_2 therefore lies in the triangle T_2 with vertices $(2,1)$, $(2,2)$ and $(1,1\frac{1}{2})$, but is not this last vertex nor is V_2 on the line $x = 2$. Thus V_2 is the endpoint of a further lattice midpoint edge E_2 of K centred at $(0,2)$, for a multiple lattice point edge of K through $(0,2)$ and a vertex of H on the line $x = -1$ can meet T_2 only at the point $(2,2)$. However, the other endpoint of E_2, V_3, lies in the reflection of T_2 about $(0,2)$, and so V_3 is in the halfplane $x < -1$. This contradicts the fact that two vertices of H on $x = -1$ are boundary points of K. Hence V_1 is $(3,\frac{1}{2})$, and so K has two multiple lattice point edges along the lines $2y+2 = x$ and $2y+x = 4$.

Finally, we show that K has a multiple lattice point edge along the line $x = -1$. In the case $H = H_{23}$ this is trivially so by the convexity of K, as the edge of H_{23} along $x = -1$ (in standard position) contains two lattice points in its interior. In the other three cases, we note that H has just two lattice points on the line $x = -1$. To the contrary then, we suppose that K has a vertex V_4 in the halfplane $x < -1$. Both of the edges of K with endpoint V_4 are lattice midpoint edges of K centred at the two vertices of H on $x = -1$, since these are the only vertices of H unaccounted for. As these vertices of H are unit distance apart, the other two endpoints V_2 and V_3 of the lattice midpoint edges with endpoint V_4 are distance 2 apart. But this contradicts the fact that these vertices lie in the halfplane $x < 0$ on the lines $2y+2 = x$ and $2y+x = 4$. Hence K has a multiple lattice point edge on $x = -1$ also, and so K is a triangle with $H = H_{23}$, whose area is easily calculated to be 8. This area is within the bounds required by this lemma.

<u>K contains H_{12}, H_{13} or H_{15}</u>

Each of these sets H are distinguished by the fact that they have an edge which has a lattice point as midpoint. As $c = 4$, any set K with one of these sets as H must have a multiple lattice point edge along this edge of H. For each of $H_{12} H_{13}$ and H_{15} we shall show that no such set K exists.

We first suppose that K has $H = H_{12}$. As $(2,1)$ and $(2,-1)$ are adjacent vertices of H_{12}, whose midpoint is a lattice point, they are in the relative interior of a multiple lattice point edge E_1 of K. The edge E_2 of K passing through

(0,2) cannot be a multiple lattice point edge, since then E_2 lies along the line $y = x+2$, and the point (1,2) would then be a fifth interior lattice point of K. Hence E_2 must be a lattice midpoint edge, whose vertices V_1 and V_2 are on E_1 and the line $x = -2$ respectively. However, as two vertices of H_{12} lie on the line $x = -1$, this contradicts the convexity of K. Hence there is no such set K.

We next suppose that K has $H = H_{13}$. Again (2,1) and (2,-1) are adjacent vertices of H_{13}, and so lie in the relative interior of a multiple lattice point edge E_1 of K. The endpoint V_1 of E_1 in the halfplane $y < 0$ lies between the points (2,-1) and (2,-2), since (2,-1) is a vertex of H_{13}. The edge E_2 of K passing through (-1,0) is not a multiple lattice point edge, since such an edge would pass through the vertex (-1,1) of H_{13}, be parallel to E, and so not contain V_1. Therefore E_2 is a lattice midpoint edge, and since V_1 lies on the line $x = 2$, its other endpoint V_2 lies on the segment of the line $x = -4$, between (-4,1) and (-4,2). As this segment lies completely in the outer halfplane bounded by the line $y = x+2$ through vertices (-1,1) and (0,2) of H_{13}, this again contradicts the convexity of K. Hence there is no such set K.

We now suppose that K has $H = H_{15}$. Again H_{15} has adjacent vertices (0,2) and (2,2), whose midpoint is a lattice point. K therefore has a multiple lattice point edge E_1 through these two points, along the line $y = 2$. If the edge E_2 of K through (-1,1) is a lattice midpoint edge, it has one endpoint on E_1, and so has its other endpoint V on the line $y = 0$. As (-1,0) is a vertex of H_{15} it is a boundary point of K, and so V is not (-1,0). Since the point (1,-1) is a vertex of H_{15}, by the convexity of K we have a contradiction. Therefore E_2 is a multiple lattice point edge along the line $x = -1$.

The edge E_3 of K through (1,-1) is not a multiple lattice point edge of K, for it would then pass through (2,1), and K would then contain the additional two points (0,-1) and (0,-2) in its interior. Hence E_3 is a lattice midpoint edge, whose endpoints lie on E_2 $(x = -1)$ and the line $x = 3$, respectively. As K has two boundary points (2,1) and (2,2) on the line $x = 2$, this is contrary to the convexity of K. Hence again there is no such set K.

<u>K contains $H = H_7$</u>

The hexagon H_7 is centrally symmetric about the point $(\frac{1}{2},\frac{1}{2})$. We let K have $H = H_7$.

We first show that K has no multiple lattice point edge along the line $x = -1$. To the contrary, we let E_1 be such an edge. The edge E_2 of K passing through (1,2) cannot be a multiple lattice point edge, for such a multiple edge would pass through the point (2,1), and then the point (0,2) would be a fifth interior lattice point of K. Hence E_2 is a lattice midpoint edge, with endpoints V_1 on E_1 and V_2 on the line $x = 3$. However, since (2,1) and (2,0) are two boundary points of K on the line $x = 2$, this is contrary to the convexity of K. Hence K has no multiple lattice point edge on the line $x = -1$. By the central symmetry of H_7, we can symmetrically argue that K has no lattice point edge on the line $x = 2$.

We next show that K has no multiple lattice point edge along the line $2y = x+3$, passing through (-1,1) and (1,2).

To the contrary we let E_1 be such an edge. The edge E_2 of K through $(2,1)$ is not a multiple lattice point edge of K, since by the above argument, there is no such edge along the line $x = 2$. Hence E_2 is a lattice midpoint edge, with endpoints V_1 on E_1 and V_2 on the line $2y = x-3$. As $(2,0)$ and $(0,-1)$ are two boundary points of K on the line $2y = x-2$, this contradicts the convexity of K. Hence no such edge E_1 exists on the line $2y = x+3$, and by the symmetric argument, K has no multiple lattice point edge along the line $2y = x-2$ either.

Finally, we show that K has no multiple lattice point edge along the line $x+y = 3$. To the contrary, we again let E_1 be such an edge, and let E_2 be the edge of K through $(-1,1)$. Since K has no multiple lattice point edge along the line $x = -1$, E_2 is a lattice midpoint edge. The endpoints of E_2 are V_1 on E_1 and V_2 on the line $x+y = -3$. As the points $(-1,0)$ and $(0,-1)$ lie on the boundary of K, and on the line $x+y = -1$, this again contradicts the convexity of K. Symmetrically again, we conclude that K has also no multiple lattice point edge along the line $x+y = -1$.

By the above three arguments, K has no multiple lattice point edge. Since we have previously argued that K, with $H = H_7$, has such an edge, we have obtained a contradiction. We conclude that no set K exists.

<u>K contains $H = H_8, H_9$ or H_{17}</u>

We note that H_8, H_9 and H_{17} may be put in a standard form, in which they have five vertices in common. By reflecting H_8 in the line $y = \frac{1}{2}$ and H_{17} in the line $x = y$, we may assume that H_8, H_9 and H_{17} share the following vertices; (1,2) (-1,1) (-1,0) (0,-1) and (1,-1). The sixth vertex X of H lies on the line $x = 2$, and X is (2,0),(2,1) and (2,2) in the case of H_8, H_9 and H_{17} respectively. We assume that H is in this standard form.

We first show that K has a multiple lattice point edge through (-1,1) and (1,2), along the line $2y = x+3$. To the contrary, we suppose that there is a vertex V_1 of K between (-1,1) and (1,2). By the convexity of K any vertex V_2 of K lying on the boundary of K between (-1,1) and (-1,0), should there be such, a vertex, lies in the triangle T_2 with vertices (-1,1) (-1,0) and $(-\frac{5}{3},\frac{2}{3})$. If (-1,1) is the midpoint of a lattice midpoint edge of K, we conclude that V_1 is in the triangle T_1 with vertices (-1,1) (-1,2) and $(-\frac{1}{3},\frac{4}{3})$. Otherwise K has a multiple lattice point edge along the line $x = -1$, and so V_1 is on this line. However the point (0,2) is external to K, and (1,2) is a boundary point of K. Therefore, by the convexity of K, V_1 lies strictly below the point (-1,2), and so is a point of $T_1 \sim \{(-1,2)\}$.

The edge of K through (1,2) is a lattice midpoint edge, since the lines through (1,2) and each of the three possible points X do not intersect $T_1 \sim \{(-1,2)\}$. Therefore K has a vertex V_3 in the triangle T_3 with vertices (3,3) (3,2) and $(\frac{7}{3},\frac{8}{3})$. Unless V_3 is the point (3,2), the point (2,2) is a strict convex combination of the points V_3, (1,2) and (1,-1), and so is a fifth interior point of K. If V_3 is the point (3,2) then the point (2,1) is a strict convex combination of V_3, (1,2) and (1,-1). Therefore, K contains a fifth interior lattice point, which contradicts $c = 4$. Hence we conclude that K has a multiple lattice point edge E_1 along the line $2y = x+3$.

The endpoint V_3 of E_1, which is a vertex of K between $(1,2)$ and X, must lie in the closed segment of the line $2y = x+3$ between $(1,2)$ and $(\frac{11}{5},\frac{3}{5})$. For, as the points $(1,-1)$ $(2,2)$ and $(\frac{11}{5},\frac{3}{5})$ are collinear on $y+4 = 3x$, the point $(2,2)$ would otherwise be a fifth interior lattice point of K.

The other endpoint V_2 of E_1, which is a vertex of K between $(-1,1)$ and $(-1,0)$, lies on the closed segment of the line $2y = x+3$ between $(-1,1)$ and $(-\frac{5}{3},\frac{2}{3})$, since it must lie in T_2.

We next show that V_2 is $(-\frac{5}{3},\frac{2}{3})$. To the contrary, we suppose that V_2 belongs to the open segment between $(-1,1)$ and $(-\frac{5}{3},\frac{2}{3})$. As V_2 is not collinear with the vertices $(-1,0)$ and $(0,-1)$ of H, the edge of K through $(-1,0)$ is a lattice midpoint edge, whose other endpoint is therefore a vertex of K on the open segment between $(-1,-1)$ and $(-\frac{1}{3},-\frac{2}{3})$. No point of this segment lies on the line $y = -1$ through the vertices $(0,-1)$ and $(1,-1)$ of H, and so K has a further lattice midpoint edge through $(0,-1)$. The other endpoint of this edge is therefore a vertex of K on the open segment of the line $2y = x-3$ between $(\frac{1}{3},-\frac{4}{3})$ and $(1,-1)$. As the line $2y = x-3$ does not pass through X, K has a further lattice midpoint edge centred at $(1,-1)$. Therefore K has a vertex V on the open segment of the line $2y = x-3$ between $(1,-1)$ and $(\frac{5}{3},-\frac{2}{3})$.

The only vertex of H unaccounted for by these edges of K is X. Hence X is the midpoint of a lattice midpoint edge of K with endpoints V and V_3, which lie on the parallel lines $2y = x-3$ and $2y = x+3$ respectively. Therefore X lies on the line $2y = x$, and so $X = (2,1)$. However, as V belongs to the segment between $(1,-1)$ and $(\frac{5}{3},-\frac{2}{3})$, V_3 therefore belongs to the segment between $(3,3)$ and $(\frac{7}{3},\frac{8}{3})$. This is contrary to the fact, deduced above, that V_3 belongs to the segment between $(1,2)$ and $(\frac{11}{5},\frac{13}{5})$, as $\frac{11}{5} < \frac{7}{3}$.

We deduce from this contradiction that $V_2 = (-\frac{5}{3},\frac{2}{3})$. Hence K has an edge E_2 along the line $x+y = -1$, with endpoint V_2. We label the other endpoint of E_2, V_1. As $(1,2)$, X and $(1,-1)$ are boundary points of K, V_1 belongs to the open segment of the line $x+y = -1$ between $(0,-1)$ and $(1,-2)$.

We now show that K has no lattice midpoint edge through $(1,-1)$. To the contrary we let E be such an edge. Since V_1 is one endpoint of E, the other endpoint V of E lies on the open segment of the line $x+y = 1$ between $(1,0)$ and $(2,-1)$. As all the vertices of H besides X are accounted for on edges of K, X is the midpoint of a lattice midpoint edge of K between V and V_3. However, if X is either $(2,0)$ or $(2,1)$, the lattice point $(2,2)$ is then a fifth interior point of K. Also, since V_3 is at most distance 1 from $(2,2)$ (if $V_3=(1,2)$) and V is at least distance $\sqrt{5}$ from $(2,2)$ (if $V = (1,0)$), we conclude that X is not the midpoint of VV_3. In either case then, X is not the midpoint of VV_3, and we have so obtained a contradiction. Hence K has an edge, which we label E_3, which is a multiple lattice point edge through $(1,-1)$ and X.

If X is $(2,0)$ or $(2,1)$, the point $(2,2)$ is therefore a fifth interior lattice point of K. Hence if $H = H_8$ or $H = H_9$, we conclude that no set K exists, with $c = 4$.

If $X = (2,2)$, the set K bounded by the three edges $E_1 E_2$ and E_3 is a triangle, all of whose edges are multiple lattice point edges, with $H = H_{17}$. The area of this triangle is readily calculated to be $7\frac{1}{120}$. As this is within the bounds required by the lemma, the lemma is verified in this case.

To complete the proof of lemma 5.5.9, we only need comment that all 24 hexagons $H_1, \ldots, H_{24}$ have been dealt with, and that the four sets K found all have area within the bounds required by the lemma.

Lemma 5.5.10. Let K be a convex polygon which has c lattice points in its interior. We let H be the convex lattice polygon which is the convex hull of all the lattice points on the boundary of K which are not vertices of K. We suppose that H is a flat lattice hexagon, and that each edge of K contains either two vertices of H in its relative interior, or has a vertex of H as midpoint.

Then $A(K) \leq 2(c+1)$

unless K is a triangle, all of whose interior lattice points are collinear, in which case

$$A(K) = 2(c+1) + \frac{1}{2c}.$$

Proof. Since H is a flat lattice hexagon, the interior lattice points of H are collinear on a line ℓ. Therefore, by lemma 5.3.2(vi), the hexagon H has two vertices on ℓ, and two vertices on each of the lattice lines adjacent, and parallel, to ℓ. We denote these lines ℓ_+ and ℓ_- for convenience. Since $A(K)$, and the incidences of K, H and the lattice are unaltered by the application of an integral unimodular transformation, we can with no loss of generality assume that ℓ is the x-axis and that ℓ_+ and ℓ_- are the lines $y = \pm 1$ respectively. Further, since a reflection of K about ℓ is an integral unimodular transformation, we may assume that K has no more lattice points, either interior or boundary points, on ℓ_+ than ℓ_-. We label the vertices of H anticlockwise, with L_1 and L_2 on ℓ_+, L_3 and L_6 on ℓ and L_4, L_5 on ℓ_- respectively.

Edges along both ℓ_+ and ℓ_-.

We first suppose that K has an edge E_1 along either ℓ_+ or ℓ_-. As H has two vertices on each of these lines E_1 is therefore a multiple lattice point edge of K. We now also suppose that K has a lattice midpoint edge E_2 incident with E_1. Since the midpoint of E_2 lies on ℓ, the endpoints of E_2 lie on ℓ_+ and ℓ_-. The next edge E_3 of K therefore is a multiple lattice point edge of K through the two vertices of H not on E_1 or ℓ. Therefore E_1 and E_3 are edges of K along ℓ_+ and ℓ_-. The lattice quadrilateral $L_1L_2L_4L_5$ meets ℓ in a segment of length at least one, and so there is a lattice point P on this segment. As H is a hexagon, P is an interior point of H, and so K. Further, since E_1 and E_3 contain $L_1L_2L_4$ and L_5 in their relative interiors, P belongs to the convex hull of E_1 and E_3. The lines ℓ_+ and ℓ_- are therefore parallel lines of support of K at the endpoints of a chord C of K through P which has endpoints on E_1 and E_3. By translating L to O, if necessary, by an integral translation along the x-axis, we change neither the area of K nor the number of lattice points of K. Hence with no loss of generality L is the point O, and C is therefore an extremal chord of symmetry of K. By Corollary 5.2.1, we then deduce that $A(K) \leqslant 2(c+1)$.

Edge along ℓ_-

We may therefore assume also that K has two multiple lattice point edges E_2 and E_3 adjacent to E_1, which each contain in its relative interior two vertices of H. Since all six vertices of H belong to $E_1 E_2$ and E_3, K is a triangle bounded by these edges. As we originally assumed that K has no more lattice points on ℓ_+ than ℓ_-, E_1 must lie on ℓ_-. The triangle K lies entirely in the halfplane $y \leqslant 2$, for otherwise edges E_2 and E_3, which have lattice points in their relative interiors on both ℓ and ℓ_+, must contain lattice points on $y = 2$ in their relative interiors. Thus the interior lattice points of K lie only on ℓ and ℓ_+, where we suppose there are n and m respectively; i.e. $c = m+n$. The area of the trapezium section of K in the halfplane $y \leqslant 1$ is $2(n+1)$, and the area of the triangle of height h sectioned from K in the halfplane $y \geqslant 1$ is $\frac{1}{2}h(m+1)$. Hence the area of K is given by

(1) $A(K) = 2(n+1) + \frac{1}{2}h(m+1) = 2(c+1) + (\frac{1}{2}h-1)m + (\frac{1}{2}h-m)$.

By our above remark, $h \leqslant 1$. If $m \geqslant 1$, then the final two terms in (1) are both negative, and so $A(K) < 2(c+1)$ as required. If $m = 0$, then $n = c$ and all the interior points of K are collinear on ℓ; by the similarity of K and T we deduce that $h = \frac{1}{n}$. In this case then, $A(K) = 2(c+1) + \frac{1}{2c}$, which is the area required in the statement of the lemma for such a triangle.

No edge along either ℓ_+ or ℓ_-

By the above, we may suppose that K has no edge along either ℓ_+ or ℓ_-. In the remainder of this proof we shall, on three occasions, compare the area of K to the area of a triangle T. We let T be the triangle, with three multiple lattice point edges $D_1 D_2$ and D_3 through L_6 and L_1, L_2 and L_3, and L_4 and L_5 respectively. We denote the vertices of T by R_1, R_2 and R_3, where $R_i = D_j \cap D_k$, $\{i,j,k\} = \{1,2,3\}$.

K has two incident multiple lattice point edges

We first suppose that K has two multiple lattice point edges E_1 and E_2 which meet at the vertex V_1. Since K has no edge along either ℓ_+ or ℓ_-, the point V_1 lies outside the strip $|y| \leqslant 1$. We claim that V_1 lies in the halfplane $y > 1$.

To the contrary, we suppose that V_1 lies in the halfplane $y < -1$, and that E_1 and E_2 pass through L_3, L_4 and L_5, L_6 respectively. Hence the edges E_3 and E_4 of K through L_1 and L_6 respectively are both lattice midpoint edges, which meet at vertex V_3 of K. The distance between the other two vertices of K, $|V_2V_4|$ is twice $|L_1L_2|$, since triangles $L_1V_3L_2$ and $V_2V_3V_4$ are similar. As neither E_1 nor E_2 contain any lattice points in their relative interiors besides L_3, L_4 and L_5, L_6 respectively the length $|L_3L_6|$ is at least twice $|L_4L_5|$. Since the segment V_2V_4 is parallel to L_3L_6, $|V_2V_4| > |L_3L_6|$. So

$$2|L_4L_5| \leqslant |L_3L_6| < |V_2V_4| = 2|L_1L_2|.$$

Therefore $|L_4L_5| < |L_2L_1|$. This is contrary to our choice of orientation of K at the beginning of this proof. Hence our claim is verified.

Thus V_1 lies in the halfplane $y > 1$. We relabel K, so that E_1 passes through L_6, L_1 and E_2 through L_2, L_3. Since the remaining two edges of K, E_3 through L_4 and E_4 through L_5, are lattice midpoint edges, the vertices $V_2 = E_2 \cap E_3$ and $V_4 = E_4 \cap E_1$ of K have the same y-coordinate. As edges E_1 and E_2 of K lie on the same vertices of H as edges D_1 and D_2 of T, we deduce that the vertex R_3 of T is V_1, and so that T contains no lattice point in the halfplane $y > 1$ in the relative interior of edges D_1 or D_2.

We claim that $A(T) > A(K)$. Let m_1 be the line V_1V_3, and let m_2 and m_4 be lines parallel to m_1 through V_2 and V_4 respectively. Let $X_i = m_i \cap \ell_-$, for $i = 1,2$ and 4. The set $K \sim T$ is made up of the disjoint triangles $L_4V_3X_1$ and $L_5V_3X_1$.

These triangles are congruent to the triangles $L_4V_2X_2$ and $L_5V_4X_4$ respectively, as E_3 and E_4 are lattice midpoint edges. Since these triangles belong to disjoint components of $T \sim K$, our claim is verified.

Notice though, that the lattice points interior to T form a subset of the lattice points interior to K; in fact if K has any interior points on the line $y = -1$, (i.e. $|L_4L_5| \neq 1$), this is a proper subset. The convex hull of the lattice points on the relative interiors of the edges of T also forms a flat lattice hexagon. Hence by the previous case, when K has an edge along ℓ_-, $A(T) < 2(c+1)$ unless K has no lattice points in its interior on ℓ_-. We show this cannot occur.

For suppose K contains no interior lattice point on ℓ_-. By our normalisation of K, K has no interior lattice point on

ℓ_+, either. Hence $|L_1L_2| = |L_4L_5| = 1$. As E_3 and E_4 are lattice midpoint edges, the segment V_2V_4 has length 2 and is parallel to ℓ. Since edges E_1 and E_2 meet at V_1, $|V_2V_4| > |L_3L_6| > |L_1L_2| = 1$. Therefore $|L_3L_6|$ is an integer length strictly between 1 and 2, and so we have a contradiction. Thus no such set K exists.

Hence K has area within the given bounds.

K has two nonincident multiple lattice point edges

We now suppose that K has two multiple lattice point edges E_1 and E_3, which are not incident nor parallel to ℓ. As a reflection of K (and H) about the y-axis is an integral unimodular transformation, we may assume with no loss of generality that E_1 passes through L_1 and L_6 and E_3 passes through L_3 and L_4. The other two edges of K are lattice midpoint edges E_2 and E_4 through L_2 and L_5 respectively. By our normalisation of K, the chord L_1L_2 has length no greater than the length of L_4L_5.

Since $V_1 = E_1 \cap E_2$ is the endpoint of lattice midpoint edge E_2, the chord c_1 of K, parallel to ℓ through $V_2 = E_2 \cap E_3$, has length $|c_1| = 2|L_1L_2|$. Similarly, the chord c_2 of K parallel to ℓ through $V_4 = E_1 \cap E_4$, has length $|c_2| = 2|L_4L_5|$. The chord L_3L_6 of K lies between c_1 and c_2, and so has length between $|c_1|$ and $|c_2|$. As $|L_1L_2| \leqslant |L_4L_5|$, we deduce that $|L_3L_6| \geqslant 2|L_1L_2|$.

Since $|L_3L_6| \geqslant 2|L_1L_2|$, the triangle T has no lattice point in the relative interior of its edges between R_3 and ℓ_+. Hence as before, the convex hull of the lattice points on the relative interiors of sides of T is a flat lattice hexagon. Again, the lattice points interior to T form a subset of the lattice points interior to K, which, if K has any interior points on the line $y = -1$, is a proper subset.

We claim that $A(T) > A(K)$. Let $\ell_1 \ell_2$ and ℓ_4 be lines parallel to ℓ through vertices V_1, V_2 and V_4 of K respectively. Let $X_i = D_2 \cap \ell_i$, for $i = \{1,2,4\}$. As $|c_1| \leqslant |c_2|$, the edges E_1 and E_3 of K are either parallel or meet in the halfplane $y > 0$. Therefore the triangle $L_4 L_5 V_3$, a component of $K \sim T$, has a congruent copy contained in the triangle $R_2 L_5 V_4$, a component of $T \sim K$. The remaining components of $K \sim T$, triangles $L_2 X_2 V_2$ and $L_3 X_2 V_2$, are congruent to components $L_2 X_1 V_1$ and $L_3 X_4 X$, of $T \sim K$, where $X = c_2 \cap E_3$. As these three components of $T \sim K$ are disjoint, our claim is verified.

As above, $A(T) < 2(c+1)$ unless K has no lattice points in its interior on ℓ_-. We show that this only occurs when $A(K) \leqslant 2(c+1)$.

For suppose K contains no interior lattice point on ℓ_-. By our normalisation of K, $|L_1 L_2| = |L_4 L_5| = 1$. Therefore $|c_1| = |c_2| = 2$, and so $|L_3 L_6| = 2$. Hence K has a single interior lattice point L, which is the midpoint of L_3 and L_6. Further, as $|c_1| = |c_2|$, the edges E_1 and E_3 are parallel. By translating L to O, if necessary, the chord $L_3 L_6$ of K becomes an extremal chord of symmetry of K. By Corollary 5.2.1 we can therefore deduce that $A(K) \leqslant 2(c+1)$.

Hence K has area within the given bounds.

K has at least four lattice midpoint edges.

We have above covered every possibility in which K has two or more multiple lattice point edges. We first deal with the case in which K has all lattice midpoint edges. Then we shall show that the case of K having one multiple lattice point edge is easily explained as a degenerate case of the first.

We suppose that K has six lattice midpoint edges $E_1,\ldots,E_6$ with midpoints $L_1,\ldots,L_6$ respectively. We let $V_1 = E_1 \cap E_2$ and label the vertices $V_2,\ldots,V_6$ counterclockwise. The chord V_2V_6 is parallel to and twice the length of L_1L_2. Similarly, V_3V_5 is parallel to and twice the length of L_4L_5. Thus the chord L_3L_6 of K, midway between these chords has length $|L_3L_6| = |L_1L_2| + |L_4L_5|$, and so $|L_3L_6| \geqslant 2|L_1L_2|$. Hence T contains no lattice points in the relative interior of its edges adjacent to R_3 besides $L_1L_2L_3$ and L_6, and T contains no lattice points in its interior which are not interior to K.

We again claim that $A(K) \leqslant A(T)$. We denote the lines parallel to ℓ through V_1, V_2 and V_6, and V_3 and V_5 by ℓ_1, ℓ_2 and ℓ_3 respectively. We let the lines parallel to V_1V_4 through V_3, V_4, and V_5 be m_1, m_2 and m_3 respectively. We denote by X_{ij}, $i,j = 1,2,3$ the 9 points on the boundary of T given by $X_{ij} = D_i \cap \ell_j$ if i=1 or 2 and $X_{3j} = D_2 \cap m_j$. The set $K \sim T$ is the disjoint union of the triangles $L_2V_2X_{22}$, $L_3V_2X_{22}$, $L_4V_4X_{32}$, $L_5V_4X_{32}$, $L_6V_6X_{12}$, $L_1V_6X_{12}$. These triangles are in turn respectively congruent to $L_1V_1X_{21}$, $L_3V_3X_{23}$, $L_4V_3X_{31}$, $L_5V_5X_{33}$, $L_6V_5X_{13}$, $L_1V_1X_{11}$, as is easily confirmed as all corresponding edges are parallel and each pair of triangles has a pair of edges L V equal, due to the six lattice midpoint edges of K. The second set of six triangles has a disjoint union contained in $T \sim K$, and so we have proved our claim.

Unless K has no interior point on the line $y = -1$, we can deduce that $A(T) < 2(c+1)$, where c is the number of interior points of K. Since $A(K) \leqslant A(T) < 2(c+1)$, the result easily follows in this case.

We therefore assume that K has no interior point on the line $y = -1$. The chords L_1L_2 and L_4L_5 have length 1, by the normalisation of K. By the above, $|V_2V_6| = |V_3V_5| = 2$, and so $|L_3L_6| = 2$. The single lattice point L on L_3L_6 is the only interior lattice point of K. Also, again, edges E_3 and E_6 of K are parallel, and so the chord L_3L_6 is a chord of symmetry of K about L. By Corollary 5.2.1, we therefore deduce that $A(K) \leqslant 2(c+1)$ in this case also.

Finally we must consider the case in which K has one multiple lattice point edge. Relabelling K, we may assume that V_1 is the vertex of K, outside the strip $|y| \leqslant 1$, which is the endpoint of the multiple lattice point edge E_1. We label the vertices of K, $V_1,\ldots,V_5$ cyclically, so that E_1 has endpoints V_1 and V_5. We let $\mathcal{V}_6$ be the point on E_1 such that the chord $\mathcal{V}_6V_2$ is parallel to ℓ. We claim that segments $V_1\mathcal{V}_6$ and $\mathcal{V}_6V_5$ have lattice point midpoints. For as $\mathcal{V}_6$ has the same y-coordinate as V_2, and V_3 and V_5 have the same y-coordinate, this follows from the facts that E_1 has two lattice points in its relative interior and that both V_1V_2 and V_2V_3 are lattice midpoint edges. The 5-gon K is then a degenerate 6-gon; this degeneracy in no way affects the proof that $A(K) \leqslant 2(c+1)$ above.

We therefore have shown that $A(K) \leqslant 2(c+1)$ for all sets K other than a particular triangle. Hence lemma 5.5.10 is proved.

<u>Corollary 5.5.11</u> Let K and H be sets as in the hypothesis of lemma 5.5.10, where K is a triangle, the interior lattice points of which are collinear, and $A(K) = 2(c+1) + (2c)^{-1}$. If the origin O is an interior point of K, then K has three chords of symmetry.

<u>Proof.</u> By lemma 5.5.10, K is a triangle, all of whose interior lattice points are collinear on a line ℓ. Using the notation of the proof of this lemma, the boundary of K has two adjacent lattice points L_1, L_2 on the line ℓ_+, and two boundary lattice points L_3 and L_6 on ℓ, between which lie all the interior lattice points of K, including O. K also has an edge E along ℓ_-, whose endpoints are lattice points L_4 and L_5. We choose $L_1, \ldots, L_6$ to circuit the boundary of K anticlockwise, and as in lemma 5.5.10, assume with no loss of generality that ℓ, ℓ_+ and ℓ_- are the lines $y = 0$, and $y = \pm 1$ respectively.

Since lattice points are regularly placed along lattice lines, the lines L_1O and L_2O meet ℓ_- at the lattice points M_1 and M_2 respectively. We claim that M_1 and M_2 belong to the relative interior of E. Since O is an interior point of K on ℓ, the distance $|L_3O|$ is at least one and so by the similarity of triangles L_2L_3O and $L_2L_4M_2$, the distance $|L_4M_2|$ is at least two, with M_2 to the right of L_4 on ℓ_-. Similarly, the distance $|M_1L_5|$ is at least two, with M_1 to the left of L_5 along ℓ_-. However, since $|M_1M_2| = |L_1L_2| = 1$, and since M_2 is to the right of M_1 along ℓ_-, we deduce that both M_1 and M_2 belong to the relative interior of E.

Hence K has two chords L_1OM_1 and L_2OM_2 which are chords of symmetry, but not extremal chords of symmetry, for no other side of K can be parallel to ℓ_-, and none of $L_1L_2M_1M_2$ are vertices of K. These two chords are not the only chords of symmetry of K, for if they were, by Theorem 5.1.1, $A(K) \leqslant 2(c+1)$. Since K is a triangle, K has at most three chords of symmetry, with endpoints given by the intersections of K with $\bar{K}$, the reflection of K in O.

Hence K has exactly three chords of symmetry, and the lemma is proved.

We now prove Theorem 5.5.3.

Proof of Theorem 5.5.3 Since K contains at most $c \leqslant 4$ lattice points in its interior, Y contains at most c interior lattice points. Thus if $c \leqslant 3$, Y has exactly six vertices, for at least six are specified in the statement of the theorem, and by lemma 5.3.2(vii), if Y has seven or more vertices, it must contain at least four lattice points in its interior. If Y is not a flat lattice hexagon, lemma 5.5.9 then gives the required bound on A(K). If Y is a flat lattice hexagon, the bounds required are given by lemma 5.5.10.

If $c = 4$, the result is given by the previous paragraph, unless each of K and Y contain exactly four lattice points in its interior. If Y contains 4 lattice points in its interior, by lemma 5.3.2(viii), Y has at most eight edges. By lemmas 5.5.6, 5.5.5 and 5.3.2(viii) we can classify Y as a transform of one of 24 hexagons, 4 7-gons or O_0 respectively, unless Y is a flat lattice hexagon. However lemmas 5.5.9, 5.5.8, 5.5.7 and 5.5.10 specifically give the bounds on A(K) required by the theorem in these cases respectively.

By exhaustion of cases then we have proved Theorem 5.5.3.

5.6 Completion of the Proof of Theorems 5.1.2 and 5.1.3

By Theorem 5.4.1, we know it is sufficient to prove Theorems 5.1.2 and 5.1.3 for a set K, which is a polygon with at least $s(K)$ edges each of which is a lattice midpoint edge or a multiple lattice point edge. If K is a 3-gon, we may assume that no edge of K is a lattice midpoint edge, while if K is a 4-gon, K has at most two lattice midpoint edges.

Hence the convex hull Y of the lattice points on the boundary of K, but not vertices of K, is a convex lattice polygon with at least five vertices. Indeed, by the above, Y is a 5-gon only if K is a 5-gon with all edges lattice midpoint edges. In this case, by lemma 5.3.4, we can deduce that $A(K) \leqslant 2(t+2)$.

We may therefore assume that Y is a convex lattice n-gon with $n \geqslant 6$. By Theorem 5.5.3, $A(K) \leqslant 2(t+2)$ unless K is a triangle with area $A(K) = 2(t+2) + (2(t+1))^{-1}$. By Corollary 5.5.11, $s(K) = 3$ in this case.

Hence if $s(K) \geqslant 5$, $A(K) \leqslant 2(t+2)$

while if $s(K) = 3$, $A(K) \leqslant 2(t+2) + (2(t+1))^{-1}$

We have thus completed the proof of Theorems 5.1.2 and 5.1.3.

We note that the fact that $t \leqslant 3$ is used only to ensure that $c \leqslant 4$, in Theorem 5.5.3. Should Conjecture 5.5.2 ever be proved, we can replace the above application of 5.5.3 with 5.5.2, and deduce the bounds of Theorems 5.1.2 and 5.1.3 without restricting t to be at most 3.

The following simple corollary to Theorem 5.1.3 is a generalisation of Ehrhart's Theorem 1.2.1.

Theorem 5.6.1 Let K be a convex set in the plane with centre of gravity O, and with area $A(K) > 8\frac{1}{6}\det(\Lambda)$. Then K contains at least three points of the lattice Λ besides O.

Proof. By lemma 1.2.3, $s(K) \geqslant 3$. If K was 2-admissible, by Theorem 5.1.3, its area $A(K) \leqslant [2(2+2) + (2(2+1))^{-1}]\det(\Lambda)$

$$= 8\tfrac{1}{6}.\det(\Lambda).$$

Hence K is not 2-admissible, and so contains at least three points of Λ besides O. We note that the bound $8\frac{1}{6}\det(\Lambda)$ is not the least possible bound for which this result is true, since the centre of gravity of the 2-admissible triangle of area $8\frac{1}{6}\det(\Lambda)$ is not at O. I conjecture that the bound 8 is sufficient. Certainly no lower bound will do, since the 2-admissible, O-symmetric rectangle with vertices $(\pm 2, \pm 1)$ has area 8.

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