GENERALIZED QUANTIZATION AND COLOUR ALGEBRAS

by

R. Kleeman, B.Sc. (Hons)

A thesis submitted in accordance with the requirements of the Degree of Doctor of Philosophy.

Department of Mathematical Physics
The University of Adelaide
South Australia

November 1985

AWARDED 22ND OCTOBER, 1986

Contents

ABSTRACT

An important solution of the paracommutation relations is the so-called Green ansatz. Recently it was observed that this may be constructed from an algebra which shall be termed in this thesis, a colour algebra. Colour algebras are natural generalizations of the better known superalgebras. Their generality suggests they may be the key to exploring further forms of quantization.

In Chapter 2 colour algebras are studied in their own right. It is observed that a colour algebra can be described by an abelian grading group and a complex valued commutation factor defined on this group. It is further observed that these two objects are, in general, not fixed for a particular colour algebra and in fact, a unique canonical pair may be found.

Another aspect of the classification problem for colour algebras is considered in section 3, where it is shown that there is an abstract algebraic map between colour algebras and "canonical" superalgebras. In section 4 it is shown how this abstract map may be implemented by a Klein transformation and how this allows one to show that a representation of a colour algebra can be obtained in a simple manner from a representation of its "canonical" superalgebra.

In Chapter 3 another method of quantization called modular quantization is examined. This is shown also to have a colour algebra ansatz solution— the relevant colour algebra being different to that for paraquantization. The uniqueness of this solution for Fock representations is examined and an algebraic vacuum condition (being a generalization of a similar paraquantization condition) is found which implies the solution. It is further shown that the only ansatz type solution is the one given. Relativistic complications are also examined.

In section 3 the question of suitable observables is discussed. A condition known as strong locality is imposed and a set of observables is demonstrated to satisfy the condition. Moreover these observables are shown to satisfy commutation relations that are a generalization of the paracommutation relations. Restrictions on the algebraic order of strongly local observables are then discussed.

Section 4 contains a comparison of a modular field theory and a normal field theory with a hidden $U(m)$ global gauge symmetry. This comparison is made possible by the Klein transfomation.

Finally in Chapter 4, a generalization of the modular quantization is examined.

STATEMENT

I hereby declare that this thesis contains no material which has been accepted for the award of any other degree or diploma in any University, and, to the best of my knowledge and belief, it contains no material previously published or written by any other person, except where due reference is made in the text of the thesis.

If this thesis is accepted for the award of the degree, then I give my consent that it be made available for photocopying and loan if applicable.

Richard Kleeman

ACKNOWLEDGEMENTS

The research in this thesis was carried out during the years 1981-1985 in the Department of Mathematical Physics, University of Adelaide, under the supervision of Professor H. S. Green.

I am indebted to Professor Green for his constant encouragement and for many useful discussions during the course of these studies.

I wish to thank Professor C. A. Hurst for a number of useful criticisms. In addition, I wish to thank R. Gatt for his help, particularly in the area of the general mathematical literature.

The financial support of a Commonwealth Postgraduate Scholarship is also gratefully acknowledged. Finally I wish to thank Ms Maureen Fletcher for her constant support during the past four years.

CHAPTER 1

INTRODUCTION

It is well known that observable particles may be divided according to their statistics into the two categories of bosons and fermions. Although there is no reason to suspect any other kind of particle statistics, it is not possible to rule out such possibilities mathematically. In fact, it was realized around 1950 by Wigner and others [1] that the Heisenberg equations of motion for quantum field theory do not neccessarily imply the usual bose and fermi equal-time commutation relations.

Somewhat later Green [2] showed that there are more general commutation relations which also satisfy the equations of motion. There is one set of relations for fermi-like spinor particles and one set for bose-like tensor particles. The two kinds of particles are referred to as parafermions and parabosons respectively. The relations have become known as the para commutation relations and the resulting field theory is usually referred to as parafield theory. While the para commutation relations are satisfied by the usual fermi and bose relations, Green also demonstrated the existence of further solutions. These are referred to as the *ansatz* solutions and are constructed by letting the parafields be sums of a certain number of ansatz fields which satisfy anomolous fermi or bose commutation relations. The number of such fields in each sum is then referred to as the *order* of parafield theory. In particular, the usual fermi and bose cases are of order one.

This scheme of generalized quantization remained of somewhat academic interest until 1964 (see however [3]) when Greenberg [4] made the suggestion that it be applied to the newly proposed quark model. The reason for this suggestion lay in the apparently anomolous statistics satisfied by the spin one half quarks within the baryon. They appeared to be symmetric with respect to interchange whereas their spin indicated that a fermionic antisymmetry should have been observed. By treating the quarks as parafermions of order three, Greenberg was able to construct, with the aid of Green's ansatz, a baryon state of three quarks which was symmetric with respect to interchange.

It was later shown [5], [6] that the Greenberg model was essentially equivalent to another suggestion made to overcome the above problem. This latter suggestion is the well-known colour model [7] which involves the introduction of the $SU(3)$ colour group as a further particle symmetry. The symmetry of the baryon state is then ensured by postulating that all observed particles are colour singlets. The baryonic colour singlet can then be shown by group theoretical means to be symmetric.

The possible physical relevance of parafield theory then stimulated a greater analysis of the subject. In 1965 Greenberg and Messiah [8] demonstrated that in the case of Fock representations, the only solutions to the para commutation relations are those given by Green's ansatz. Although non Fock representations have also been considered by several authors [9], even in these studies the ansatz fields still play a cental role. This indicates that the ansatz fields are of interest in their own right. We pursue this idea below.

In the late 1960s the properties of the Fock-space of parafield theory with respect to the symmetric group S_n were investigated by Landshoff and Stapp [10]. If this group is implemented by particle permutations (permutations of the momentum or spatial indices of the fields), it is well known that only the trivial representation of the group occurs for bosons and fermions. In the case of higher order parafermions (parabosons) however, Ohnuki and Kamefuchi [11] were able to demonstrate that for n-particle states exactly one representation of S_n occurs for each Young tableau with rows (columns) of length no more than p, where p is the order of the parafields.

This result was important because it allowed Drühl, Haag and Roberts [6] to demonstrate that parafield theory with a Fock representation is essentially equivalent to an ordinary field theory with a $U(p)$ symmetry. The proof given by these authors applied only to a non-relativistic theory in the sense that no consideration was given anti-particle states. The extension to anti-particle states was provided by Ohnuki and Kamefuchi [12].

The comparison with an ordinary theory was made possible by means of the Klein transformation [13], which enabled the authors to transform the ansatz fields, which satisfy anomolous commutation relations, into ordinary fermi or bose fields.

The notion that parafield theory is equivalent to a $U(p)$ global gauge theory was reinforced by Gray [14] who showed that the cluster decomposition principle is satisfied by parafield theory only if the observables of the theory are those which are left invariant under $U(p)$ in the corresponding normal theory. This conclusion was disputed however, by Ohnuki and Kamefuchi [15] who claimed that the cluster decomposition principle places no restriction on observables and that it is only conditions of locality which impose constraints on the theory. The differing conclusions appear to this author to be due to differences in how the theory should be physically interpreted.

The $U(p)$ symmetry of Drühl et al is, as was noted above, only a global gauge symmetry. Now since the colour group used in elementary particle theory has become a *local* gauge symmetry with the advent of Quantum Chromodynamics [16], one might hope that a similar extension of symmetry may be possible within the framework of parafield theory. In 1976 this problem was tackled by Freund [17] who concluded that it was impossible to construct the Yukawa term of an $SU(p)$ local gauge theory using parafields. Freund did acknowledge, however, that there is a possibility of constructing an $SO(3)$ theory using parafermions and parabosons of order three. This suggestion was made by Greenberg, and later Govorkov [18] explicitly constructed such a theory by means of the quarternions.

This lack of success in introducing the usual gauge fields led to a number of modified parafield theories. One of the first of these involved the introduction of octonions [19], which are a non-associative generalization of quarternions. In this theory the spinor fields are the the sum of three ansatz fields which are the products of ordinary fermi fields and certain complex octonion units. Gauge fields for an $SU(3)$ theory can then be introduced because the Lie algebra of derivations for the octonions contains $SU(3)$ as a subalgebra (the full algebra is the exceptional G_2). One of the difficulties involved in considering such a theory is the non-associativity of the octonions. This means that fields can no longer be considered as operators on an ordinary Hilbert space and the latter must be generalized to what is called an octonionic Hilbert space [20]. Another feature of such models which has not been sufficiently explored, is the problems involved in the "bracketing" of operators. This is obviously only a feature of non-associative theories, as in the usual formulation the composition of operators is unambiguous. The bracketing problem may have physical consequences as it appears to produce a profusion of new states, just as the *non-commutative* nature of parafields produces more states than the commuting and anti-commuting bose and fermi fields.

A further attempt at using a non-associative algebra in the context of ansatz constructions was made by Domokos et al [21]. In this approach the bracketing problem was resolved for states by assuming a fixed pattern. This was the "composition" bracketing, namely $(a_1(a_2(a_3(\ldots(a_n\phi)\ldots)))$.

It is interesting to examine the consequences of such bracketing in the case of the octonion theory mentioned above. If we consider the octonionic Hilbert space to be the direct product of an ordinary Hilbert space and the eight dimensional octonion algebra, then the above bracketing allows operators on states to be replaced by an associative algebra of operators acting on the direct product of the ordinary Hilbert space and an eight dimensional vector space. In this formulation, the octonion units in the ansatz are replaced by matrices corresponding to the "left multiplication" operators of the octonions [22]. In this reference, it is shown that these matrices form the complex Clifford algebra which has three pairs of starred and unstarred elements. It should be emphasized that the above conversion to an associative algebra is applicable only to the formation of states and the formation of observables still requires investigation.

In the light of the above discussion, it is interesting to note that Greenberg and Macrae [23] have considered ansatz fields which are products of ordinary fermi (or bose) fields and elements from a Clifford algebra. In the case of a real Clifford complex algebra however, a slightly modified theory results. The Clifford elements from the ansatz transform according to the fundamental (and conjugate in the complex case) representation of the groups $SO(p)$ in the real case and $SU(p)$ in the complex case. The transformation is implemented by quadratic Clifford elements which are then used to define gauge fields. A local gauge theory in the respective groups can then be constructed.
of bose fields and quadratic elements from the Clifford algebra.

All of the approaches to modifying parafield theory discussed above have as their central feature a modification of the ansatz solution to Green's original commutation relations. In view of this, it is of some interest to study the ansatz from a mathematical point of view. Now the ansatz "algebra" contains commutators and anti-commutators and one might at first sight conclude that it was an example of a superalgebra [24]. This is not the case however, and Rittenberg and Wyler [25] have demonstrated that it is an example of what they term a colour (super)algebra. This class of algebras is in fact, a generalization of the class of superalgebras.

Such algebras are graded by an arbitrary abelian group while superalgebras need only be graded by the cyclic group $\boldsymbol{Z}_2$. In addition the commutation relations in colour algebras are described by a complex valued commutation factor and thus need not be restricted to commutators and anti-commutators.

In 1979 Scheunert [26] analysed colour algebras in more detail and concluded that there is a "canonical" superalgebra for every colour algebra. In Chapter 2 of this thesis we continue the analysis initiated by Scheunert. It is shown that in certain cases, a grading group and commutation factor may be replaced or "covered" by a new such group and factor. This means that any colour algebra having the original grading group and commutation factor may also be considered as a colour algebra with the new group and factor. With the use of a particular kind of replacement called a covering homomorphism, it is shown that there exists a "canonical" set of grading groups and commutation factors. The canonical factor is almost determined by its canonical grading group. It is further shown that for a particular grading group and commutation factor there is a unique minimal replacement grading group and commutation factor from the canonical set. The word minimal means in this context, that no smaller grading group can cover the original group.

Klein transformations are also introduced within the general framework of colour algebra theory. It is shown that they allow the explicit transformation of colour algebras into their canonical superalgebras. This result is shown to have important implications for the representation theory of colour algebras.

As well as considering Klein transformations within the context of colour algebra theory, in Appendix B we consider them purely within the framework of ansatz algebras. Although the Klein transformations derived are no more general than their colour algebra counterparts, they are more convenient for the purposes of field theory.

It is interesting to compare the colour algebra ansatz solution of parafield theory with the modified ansatz constructions considered above. In the former case the fields are sums of products of ordinary fermi or bose fields and Klein operators which commute among themselves but not with the fields (they acquire numerical factors when "taken through" the fields). In the latter case the fields are again sums of products of ordinary fields and certain operators, however these latter operators commute with the fields and form a non-trivial algebra amongst themselves.

In 1975 Green [27] considered a further kind of generalized quantization which he termed "modular" quantization. The commutation relations satisfied by the modular fields are a generalization of a set of commutation relations discovered for parafield theory of order two (see Green [2] and Volkov [3]). By means of a unitary "permutation" operator* Green was able to derive a particularly simple billinear set of commutation relations for the modular fields. This operator was also used to demonstrate the existence of an energy-momentum operator satisfying the Heisenberg principle. In Chapter 3 a detailed analysis of modular quantization is undertaken. It is shown that the introduction of the permutation operator is equivalent to considering an ansatz solution of the original commutation relations. The modular ansatz fields form a colour algebra of a different kind to that of the parafield ansatz algebra. Given the interest in the literature in generalizing parafield theory and also in developing the mathematical theory of colour algebras, it is of some interest to study the implications of modular field theory. This is particularly so since, as far as this author is aware, modular field theory is the first example since parafield theory of a field theory with a colour algebra ansatz solution.

The permutation operator introduced by Green is shown to fit into the framework of colour algebra theory also, since it generates the Klein operators for the

---

* Such an operator had already been introduced by Carey [28] in the context of parafield theory of order two.

modular ansatz fields.

The similarity in the solutions for modular and parafield theory suggests that an analysis of the former along the lines described above for parafield theory may prove useful. This philosophy motivates the remainder of Chapter 3.

It is shown firstly that there is a condition on the Fock vacuum, involving modular creation and annihilation operators, which allows us to deduce the ansatz solution from the original modular commutation relations. It is not known whether such a condition can be derived from the commutation relations and the Fock-space properties, as it can be in parafield theory. It is shown, however, that the particular ansatz solution is the only such one. Relativistic complications are then considered and it is observed that anti-particle operators of modular field theories appear to be algebraically different to their particle counterparts. This is a situation not holding in parafield theory.

In the next section, the question of observables is considered and following Ohnuki and Kamefuchi [29], two notions of locality, strong and weak, are introduced. A neccessary and sufficient condition is derived for strongly local modular observables and a set of such observables are then derived. The algebraic form of such observables is far from trivial and may prove of interest in further developments of modular field theory (particularly in the area of interacting field theory). The fact that such observables are strongly local allows us to then introduce a new set of commutation relations for modular field theory. These relations are a generalization of the basic para commutation relations. It is possible that these relations may be more useful than the original relations although this question is not explored. Finally it is demonstrated that for modular field theories of order three or more, there are no quadratic strongly local observables while for order greater than four, it is shown that there are no such observables of order four or less. This latter observation is important because it may have consequences for constructing renormalizable interacting field theories.

In section 4, modular field theory is compared with an ordinary $U(m)$ gauge theory (m being the order of the modular field theory). It is firstly observed that it appears likely in modular field theory that not all the observables for the gauge theory can be constructed. Despite this, a number of observables which *are* invariant under $U(m)$ are constructed.

The states of the two theories are then compared and it is shown that all physically relevant states for the gauge theory are included in the modular theories' Fock-space. It is observed that in the modular case, the redundancy of these states is intermediate between the gauge theory and parafield theory (where there is no redundancy).

The conclusion of the above is that modular field theory is essentially equivalent to a $U(m)$ gauge theory which has some further restriction (apart from global gauge invariance) placed on its observables. This contrasts with parafield theory where there is no additional restriction. The mathematical and physical nature of this restriction awaits further investigation.

Finally in section 5, the question of the energy-momentum operator is considered. Various reasons are advanced as to why an interacting field theory may be a more natural setting for modular quantization. The most promising candidate in this regard is modular field theory of order three.

In the final chapter of the thesis a generalization of modular field theory is considered. This is introduced by considering a generalization of the modular ansatz algebra to a more general colour algebra. A possible application of the generalization to the rishon model of subconstituents [30] is then considered.

The *major* original results in this thesis are as follows:

Theorems 2.4 and 2.11 in Chapter 2, where a unique minimal commutation factor, grading group pair is demonstrated for all colour algebras graded by finite abelian groups.

Theorem 2.1 in Chapter 3 where a vacuum condition is shown to imply the ansatz solution for modular field theory.

Theorems 3.2 and 3.4 in Chapter 3 where the algebraic form of strongly local observables in modular field theory is explored.

Theorems 4.3 and 4.4 in Chapter 3 where it is shown that the usual modular field theory possesses all states relevant for a $U(m)$ gauge theory.

CHAPTER 2

COLOUR ALGEBRAS

The simplest non-trivial example of a colour algebra is the so-called Lie superalgebra. The study of superalgebras began, in mathematical physics at least, in the 1970s [31] when they were used to describe a postulated symmetry between bosons and fermions (supersymmetry). The idea behind such an algebra was that anti-commutation as well as the usual Lie commutation relations should be included in the one algebra.

In order to make such objects tractable one first supposes that the algebra is *graded* by Z_2 : If a_α and b_β are elements of the algebra and α and β are elements of Z_2 then

$$a_\alpha \circ b_\beta = c_{\alpha+\beta} ,$$

where $\circ$ is the product of the algebra. In addition a generalized symmetry of the product, together with a generalized Jacobi identity, are assumed to be satisfied. These algebras have been studied fairly intensively over the past decade and a classification analogous to the semi-simple classification of Cartan for Lie algebras has been obtained by Kac [24].

The object of this chapter is to study a further generalization of superalgebras to colour algebras*, which were introduced by Rittenberg and Wyler [25]. These will be defined more precisely below, but essentially all one does is extend the grading group Z_2 of superalgebras to a finitely generated (usually finite) abelian group. The symmetry property and the Jacobi identity are generalized in an obvious way and the anti-commutation and commutation relations are generalized by means of a complex-valued commutation factor.

For any particular colour algebra the grading group and commutation factor are not unique. This phenomenon is explored in section 2, where it is shown that

* These are also known in the literature as generalized Lie algebras. We adopt the name colour algebra in the the interests of brevity.

"canonical" forms are possible for these two objects. It is further shown that these are, in some sense, unique.

An important property of colour algebras was discovered by Scheunert [26] who showed that to every colour algebra there corresponds a "canonical" superalgebra. In section 3, proofs of this correspondence are given while in section 4, the correspondence is made more concrete by means of a generalization of a transformation due to Klein [13]. The usefulness of this transformation will become clear when we study the application of colour algebras to modular quantization in the next chapter.

1. Definitions and Examples

A vector space V is said to be graded by the abelian group Γ if it may be decomposed as a direct sum of subspaces each labelled by elements of Γ. Symbolically we write

$$V = \bigoplus_{\alpha \in \Gamma} V_\alpha \ . \tag{1.1}$$

We say further that an algebra $\mathbf{A}$ is graded by Γ if, as well as being graded as a vector space, its elements and product satisfy

$$\mathbf{A}_\alpha \mathbf{A}_\beta \subseteq \mathbf{A}_{\alpha+\beta} \ . \tag{1.2}$$

In order that this graded algebra becomes a *colour algebra* we need to impose further algebraic constraints.

Central to these extra conditions is the notion of the *commutation factor*. This is a mapping $\epsilon : \Gamma \times \Gamma \to C$ which satisfies the conditions

$$\begin{aligned} \epsilon(\alpha,\beta)\epsilon(\beta,\alpha) &= 1 \\ \epsilon(\alpha,\beta+\gamma) &= \epsilon(\alpha,\beta)\epsilon(\alpha,\gamma) \\ \epsilon(\alpha+\beta,\gamma) &= \epsilon(\alpha,\gamma)\epsilon(\beta,\gamma) \ . \end{aligned} \tag{1.3}$$

One now defines a colour algebra as satisfying (1.2) together with the two conditions

$$a_\alpha b_\beta = -\epsilon(\alpha,\beta) b_\beta a_\alpha \tag{1.4a}$$

$$\epsilon(\gamma,\alpha)a_\alpha(b_\beta c_\gamma) + \epsilon(\alpha,\beta)b_\beta(c_\gamma a_\alpha) + \epsilon(\beta,\gamma)c_\gamma(a_\alpha b_\beta) = 0 \ . \tag{1.4b}$$

The second of these conditions is a generalized form of the Jacobi identity.

One can thus think of a colour algebra as a graded algebra and a commutation factor. In the special case of a Lie algebra the grading group is trivial and the commutation factor is always 1 while in the case of a superalgebra the grading group is Z_2 and the commutation factor is $(-1)^{\alpha\beta} \quad \alpha, \beta \in Z_2 = \{0, 1\}$.

A few trivial consequences of (1.3) which prove useful below, are

$$\begin{aligned} \epsilon(\alpha, \alpha) &= \pm 1 \\ \epsilon(\alpha, 0) = \epsilon(0, \alpha) &= 1 \\ \epsilon(\alpha, n\beta) = \epsilon^n(\alpha, \beta) &= \epsilon(n\alpha, \beta) \ . \end{aligned} \tag{1.5}$$

0 is the identity of Γ and $n\beta \equiv \beta + \beta + \ldots + \beta$ (n times).

Associated with a Γ graded colour algebra is a natural Z_2 grading. This is obtained by the homomorphic mapping $\phi : \Gamma \rightarrow Z_2$ as follows

$$\begin{aligned} \phi(\alpha) = 0 \qquad &\text{if} \qquad \epsilon(\alpha, \alpha) = 1 \\ \phi(\alpha) = 1 \qquad &\text{if} \qquad \epsilon(\alpha, \alpha) = -1 \ . \end{aligned} \tag{1.6}$$

This map is well defined because of the first of (1.5), and is homomorphic due to (1.3). A "canonical" commutation factor ϵ_0 may now be defined on Γ:

$$\epsilon_0(\alpha, \beta) = (-1)^{\phi(\alpha)\phi(\beta)} \ . \tag{1.7}$$

The fact that ϵ_0 is a commutation factor follows from the homomorphic nature of ϕ. This grading and commutation factor will play a central role in sections 3 and 4 below.

As our first example of a colour algebra we consider the "ansatz" algebra of parastatistics [2]. This consists of N creation operators, N annihilation operators and the identity and satisfies the relations

$$\begin{aligned} &[a_i, a_j^*]_- = [a_i, a_j]_- = [a_i^*, a_j^*]_- = 0 \qquad i \neq j \ . \\ &[a_i, a_i]_+ = [a_i^*, a_i^*]_+ = 0 \\ &[a_i, a_i^*]_+ = 1 \ . \end{aligned} \tag{1.8}$$

The grading group for this algebra is taken to be $Z_2 \oplus Z_2 \oplus \ldots \oplus Z_2$ (N factors) and the elements of the algebra are assigned the gradings

$$\begin{aligned} a_i, a_i^* &\longrightarrow (0,0,\ldots,0,1,0,\ldots,0) \quad \text{(i'th place)} \\ 1 &\longrightarrow (0,0,\ldots,0) \ . \end{aligned}$$

With this assignment the commutation factor takes the form

$$\epsilon(\alpha,\beta) = (-1)^{\phi(\alpha,\beta)} \ ,$$

with $\phi(\alpha,\beta) = \sum_{i=1}^{N} \alpha_i \beta_i$ and where we are using the notation

$$\alpha \equiv (\alpha_1, \alpha_2, \ldots, \alpha_N) \ .$$

The product of the colour algebra is taken to be the brackets in (1.8). We shall meet another ansatz algebra with a similar interpretation in the next chapter.

Another simple example is the so-called generalized Clifford algebra of Ramakrishnan [32]. This consists of m elements a_i satisfying

$$a_i \circ a_j \equiv a_i a_j - \eta^{i-j} a_j a_i = 0$$

where η is the m'th primitive root of unity. The algebra is graded by $Z_m \oplus Z_m$ as follows

$$a_i \longrightarrow (i, 1) \ ,$$

and the commutation factor is simply

$$\epsilon(\alpha,\beta) = \eta^{\psi(\alpha,\beta)} \ ,$$

with $\psi(\alpha,\beta) = \alpha_1\beta_2 - \alpha_2\beta_1$.

Perhaps one of the most important examples of a colour algebra is the algebra denoted by $gl(V,\epsilon)$. This is the set of *graded* linear maps on the graded vector space V, with ϵ being the commutation factor with which they are turned into a colour algebra. More specifically they satisfy

$$g_\alpha(V_\beta) \subset V_{\alpha+\beta} \qquad \forall\ g_\alpha \in gl(V,\epsilon) \ , \tag{1.9}$$

and are turned into a colour algebra by the product

$$g_\alpha g_\beta = g_\alpha \circ g_\beta - \epsilon(\alpha, \beta) g_\beta \circ g_\alpha \ , \tag{1.10}$$

where $\circ$ is the composition of maps product. The associativity of composition maps, together with the axioms (1.3) for ϵ, ensure that the conditions (1.4) hold.

In perfect analogy with Lie algebra theory we are able to define a *representation* of a colour algebra $\mathbf{A}$ as a homomorphic mapping of $\mathbf{A}$ into $gl(V, \epsilon)$ which preserves the grading of the elements of $\mathbf{A}$ and ϵ obviously must be the same in both $\mathbf{A}$ and $gl(V, \epsilon)$.

2. Commutation factors

The major result of this section shall concern finite abelian groups however the preliminary results shall apply equally well to finitely generated abelian groups.

2.1. Introduction

The fundamental result [33] concerning finitely generated abelian groups is that they possess a *unique* (up to isomorphism) decomposition given by

$$\Gamma = \Gamma_{p_1} \oplus \Gamma_{p_2} \oplus \ldots \oplus \Gamma_{p_n} \oplus Z \oplus \ldots \oplus Z \ , \tag{2.1}$$

where Γ_{p_i} are abelian p_i-groups (with the p_i being distinct primes) and Z is the integers. The Γ_{p_i} have a further unique (up to isomorphism) decomposition into cyclic groups given by

$$\Gamma_{p_i} = Z\left[(p_i)^{r_1}\right] \oplus Z\left[(p_i)^{r_2}\right] \oplus \ldots \oplus Z\left[(p_i)^{r_m}\right] \ , \tag{2.2}$$

where $Z\left[(p_i)^{r_j}\right]$ is a cyclic group of order $(p_i)^{r_j}$.

If one confines oneself to finite groups then the copies of Z in (2.1) are omitted.

We begin by deriving a couple of basic results concerning commutation factors on finitely generated abelian groups. Firstly let us denote the generators* of the cyclic groups in the decompositions (2.1) and (2.2) by q_i. In otherwords q_i either generates $Z\left[(p_l)^{r_k}\right]$ or Z (the fact that Γ is finitely generated means the i ranges over a finite number of values). We now define

$$E_{ij} \equiv \epsilon(q_i, q_j) \neq 0 \ . \tag{2.3}$$

When use is made of (2.1) and (2.2) we can write an arbitrary $\alpha \in \Gamma$ as

$$\alpha = \sum_{i=1}^{M} n_i q_i \ , \tag{2.4}$$

where M is the number of cyclic summands in the unique decomposition. We can now deduce from (1.3) and (1.5) that an arbitrary commutation factor may be written as

$$\epsilon(\alpha, \beta) = \prod_{i=1}^{M} (E_{ii})^{n_i m_i} \prod_{j<k} (E_{jk})^{n_j m_k - n_k m_j} \ . \tag{2.5}$$

where

---

* These need not be unique.

$$\beta = \sum_{i=1}^{M} m_i q_i$$

and where (1.5) shows that

$$E_{ii} = \pm 1 \ . \tag{2.6}$$

A simple consequence of (2.5) which we will have cause to use in the next section is the following: For every ϵ satisfying $\epsilon(\alpha, \beta) = 1$ there exists a $\sigma : \Gamma \times \Gamma \to C$ which is non-zero and satisfies the relations

$$\begin{aligned} \epsilon(\alpha, \beta) &= \sigma(\alpha, \beta)\sigma^{-1}(\beta, \alpha) \\ \sigma(\alpha, \beta + \gamma) &= \sigma(\alpha, \beta)\sigma(\alpha, \gamma) \\ \sigma(\alpha + \beta, \gamma) &= \sigma(\alpha, \gamma)\sigma(\beta, \gamma) \ . \end{aligned} \tag{2.7}$$

It is simply given by

$$\sigma(\alpha, \beta) = \prod_{j<k} (E_{jk})^{n_j m_k} \ , \tag{2.8}$$

where the notation is the same as that of (2.5). The results expressed in equations (2.5), (2.7) and (2.8) are due to Scheunert [26].

The E_{ij} are not arbitrary and, by using (1.5), it is straightforward to see that

$$(E_{ij})^r = (E_{ij})^s = 1 \ , \tag{2.9}$$

where r and s are the order of the cyclic groups generated by q_i and q_j respectively (we take the order of Z to be zero for convenience). It immediately follows that if r and s are different prime powers then $E_{ij} = 1$ whereas if they are powers of the same prime then E_{ij} is a v'th root of unity, where

$$v = min(r, s) \ . \tag{2.10}$$

We now impose the restriction that Γ be finite in order to get a convenient decomposition of commutation factors. By use of the remarks following (2.9), and (2.5), we deduce that a commutation factor may be written as

$$\epsilon(\alpha, \beta) = \epsilon_1(\alpha_1, \beta_1)\epsilon_2(\alpha_2, \beta_2) \ldots \epsilon_n(\alpha_n, \beta_n) \ , \tag{2.11}$$

where the ϵ_i are commutation factors on the abelian p_i-groups of (2.1) and α_i, β_i are the projections of α and β onto these groups. Notice that if we had allowed copies of the integers in Γ then we could have cross terms between the integer groups and the p_i-groups (consider (2.9) with $r = 0$).

Now let the cyclic groups in (2.2) be generated by the elements $s_1, s_2, \ldots, s_m$. We can write then

$$\alpha_i = \sum_{j=1}^{m} k_i^j s_j \quad \beta_i = \sum_{j=1}^{m} l_i^j s_j \; , \tag{2.12}$$

and by using (2.5), together with the remarks preceding (2.10), we can write the commutation factors ϵ_i as

$$\epsilon_i(\alpha_i, \beta_i) = \eta_{i1}^{\psi(\alpha_i, \beta_i)}$$

with

$$\psi(\alpha_i, \beta_i) = \mathbf{k}_i^t \mathbf{M}_i \mathbf{l}_i \; . \tag{2.13}$$

η_{i1} is the *primitive* $(p_i)^{r_1}$ root of unity; the $\mathbf{k}_i$ and $\mathbf{l}_i$ are vectors of length m with elements $\left\{k_i^j\right\}$ and $\left\{l_i^j\right\}$ respectively and $\mathbf{M}_i$ is an $m \times m$ matrix of integers modulo $(p_i)^{r_1}$.

2.2. Covering

We now consider the central question of this section: The non-uniqueness of the commutation factor and grading group of a colour algebra.

Consider the class $C_{\Gamma\epsilon}$ of colour algebras with commutation factor ϵ and grading group Γ. We say that $< \Gamma', \epsilon' >$ *covers* $< \Gamma, \epsilon >$ if every member of $C_{\Gamma\epsilon}$ is *also* a member of $C_{\Gamma'\epsilon'}$ *and* any representation of an algebra in $C_{\Gamma\epsilon}$ is also a representation of the algebra when considered as a member of $C_{\Gamma'\epsilon'}$. Notice that this relation is not neccessarily symmetric and in fact is a partial ordering which is inherited from the class containment relation.

In order to use this relation we need a more technical definition of a colour algebra than that given in section 1: Suppose we have an algebra $\mathbf{A}$ whose elements we denote by a^i (i belonging to some set Ω) then we may define this algebra through its structure constants C_k^{ij}.* In otherwords the product on $\mathbf{A}$ is defined

* We assume for simplicity that these are complex.

through the equation

$$a^i \circ a^j = C_k^{ij} a^k \ , \tag{2.14}$$

where summation over Ω is implied by the repeated index. We now say that $\mathbf{A}$ is a *colour algebra* with abelian grading group Γ and commutation factor ϵ, or more briefly $\mathbf{A}$ is *coloured* by $< \Gamma, \epsilon >$, if

(i) There exists a map $\phi : \Omega \to \Gamma$ such that whenever $C_k^{ij} \neq 0$ then $\phi(i) + \phi(j) = \phi(k)$.

(ii) ϵ is a commutation factor in the sense of (1.2).

(iii) $C_k^{ij} = -\epsilon\,(\phi(i), \phi(j))\, C_k^{ji} \qquad \forall\ i, j, k \in \Gamma$.

(iv) $\sum\limits_{cycl(i,j,k)} \sum\limits_{l} \epsilon\,(\phi(k), \phi(i))\, C_m^{il} C_l^{jk} = 0 \qquad \forall\ i, j, k, m \in \Omega$.

It is clear from this definition that $\mathbf{A}$ will also be a colour algebra with grading group Γ' and commutation factor ϵ' if there exists a map $\phi' : \Omega \to \Gamma'$ satisfying condition (i), and if

$$\epsilon\,(\phi(i), \phi(j)) = \epsilon'\,(\phi'(i), \phi'(j)) \qquad \forall\ i, j \in \Omega\ . \tag{2.15}$$

Upon consideration of (1.10), (2.15) also implies that any representation of $\mathbf{A}$ with $< \Gamma, \epsilon >$ will be a representation with $< \Gamma', \epsilon' >$.

A very general situation where $< \Gamma', \epsilon' >$ covers $< \Gamma, \epsilon >$ is when there exists what we shall term a *covering-homomorphism* between Γ and Γ'.

We say $h : \Gamma \to \Gamma'$ is a covering-homomorphism if

(i) it is a homomorphism,

(ii) ϵ and ϵ' satisfy the relation

$$\epsilon(\alpha, \beta) = \epsilon'\,(h(\alpha), h(\beta)) \qquad \forall\ \alpha, \beta \in \Gamma\ . \tag{2.16}$$

To show that $< \Gamma', \epsilon' >$ covers $< \Gamma, \epsilon >$ we observe that given any algebra $\mathbf{A}$ with a colouring $< \Gamma, \epsilon >$ we can obtain a colouring by $< \Gamma', \epsilon' >$ with the new grading map ϕ' given by

$$\phi' = h \circ \phi\ .$$

This satisfies condition (i) of the colour algebra definition because h is a homomorphism. Finally, equation (2.15) follows directly from (2.16).

If we suppose that h is onto and satisfies the relation

$$\alpha \in ker(h) \quad \Rightarrow \quad \epsilon(\alpha,\beta) = 1 \quad \forall\, \beta \in \Gamma \ , \tag{2.17}$$

then ϵ' will be *induced* from ϵ via (2.16). This is the case since if h is onto then (2.16) will define ϵ'. This definition will make sense since if there is a $\gamma \neq \alpha$ such that $h(\alpha) = h(\gamma)$ then $\alpha - \gamma \in ker(h)$ and so $\epsilon(\alpha - \gamma, \beta) = 1$ or $\epsilon(\alpha,\beta) = \epsilon(\gamma,\beta)$. A similar argument holds for the second argument of ϵ. Finally it is easy to establish that ϵ' will be a commutation factor on Γ': The first equation of (1.3) follows from (2.16) and the fact that ϵ is a commutation factor. The other two also follow this way with the additional use of the homomorphic property of h.

Notice that if h were an isomorphism, that is $1:1$ *as well* as onto, condition (2.17) is fulfilled trivially because $ker(h) = \{0\}$ and so (2.17) follows from (1.5).

It should be observed at this point that Scheunert [26] has considered what he terms *equivalence* of commutation factors. Thus two commutation factors ϵ and ϵ', defined on the *same* Γ, are termed equivalent if there exists an *automorphism* $g : \Gamma \to \Gamma$ such that

$$\epsilon'(\alpha,\beta) = \epsilon\left(g(\alpha), g(\beta)\right) \ . \tag{2.18}$$

It is clear that in this case we can conclude that g is a covering-homomorphism, as is g^{-1}. Thus in our terminology $<\Gamma,\epsilon>$ covers $<\Gamma,\epsilon'>$ and vice-versa.

It is an interesting question as to whether a covering-homomorphism between Γ and Γ' is neccessarily implied when $<\Gamma',\epsilon'>$ covers $<\Gamma,\epsilon>$. We shall provide a partial answer to this question in Theorem 2.2 below. Before this result is proved we need a preliminary definition and lemma:

We say that $<\Gamma,\epsilon>$ is *reduced* if

$$\epsilon(\alpha,\beta) = 1 \quad \forall\, \beta \in \Gamma \quad \Rightarrow \quad \alpha = 0 \ .$$

Lemma 2.1. *There always exists an* <u>onto</u> *covering-homomorphism between* Γ *and a* Γ^r, *where* $<\Gamma^r,\epsilon^r>$ *is reduced.*

Proof: We define Γ_0, the ϵ*-trivial* subgroup of Γ, as follows:

$$\Gamma_0 \equiv \{\alpha \in \Gamma : \ \epsilon(\alpha,\gamma) = 1 \quad \forall\, \gamma \in \Gamma\} \ . \tag{2.19}$$

To see that it *is* a subgroup of Γ suppose $\alpha, \beta \in \Gamma_0$ then

$$\begin{aligned}\epsilon(\alpha-\beta,\gamma) &= \epsilon(\alpha,\gamma)\epsilon(-\beta,\gamma)\\ &= \epsilon(\alpha,\gamma)\epsilon^{-1}(\beta,\gamma)\\ &= 1 \quad \forall\, \gamma \in \Gamma \ .\end{aligned}$$

We identify the Γ^r with Γ/Γ_0 and choose the homomorphism $g: \Gamma \to \Gamma/\Gamma_0$ to be the *natural homomorphism* (Fuchs [34]) which is onto and has kernel Γ_0 and thus by (2.17) is a covering-homomorphism. As we have seen this means there is an induced ϵ^r. Finally $< \Gamma^r, \epsilon^r >$ *is* reduced since suppose $g(\alpha) \in \Gamma^r$ is an arbitrary element of Γ^r then

$$\begin{aligned} & \epsilon^r(g(\alpha),\gamma^r) = 1 \quad \forall\, \gamma^r \in \Gamma^r\\ \Rightarrow\quad & \epsilon(g(\alpha),g(\gamma)) = 1 \quad \forall\, \gamma \in \Gamma\\ \Rightarrow\quad & \epsilon(\alpha,\gamma) = 1 \quad \forall\, \gamma \in \Gamma\\ \Rightarrow\quad & \alpha \in \Gamma_0\\ \Rightarrow\quad & g(\alpha) = 0 \ . \end{aligned}$$

■

Theorem 2.2. *Suppose* $< \Gamma', \epsilon' >$ *covers* $< \Gamma, \epsilon >$ *then there exists a subgroup* $\Gamma^h \subseteq \Gamma'$ *such that there is a covering-homomorphism between* Γ *and the reduced* Γ^h/Γ^h_0 *(*Γ^h_0 *being the* ϵ'*-trivial subgroup of* Γ^h*).*

Proof: The proof depends mainly on the following proposition:

Proposition 2.3. *There exists a well defined mapping* $h: \Gamma \to \Gamma'$ *satisfying*

$$\epsilon(\alpha,\beta) = \epsilon'(h(\alpha),h(\beta)) \ .$$

The proof of this is somewhat technical and may be found in Appendix A.

We define Γ^h to be the subgroup of Γ generated by $h(\Gamma)$. The covering-homomorphism we require is just the composition of the map h and the onto covering-homomorphism $k: \Gamma^h \to \Gamma^h/\Gamma^h_0$ given by Lemma 2.1. To see this, firstly we observe that

$$\epsilon(\alpha,\beta) = \epsilon'(h(\alpha),h(\beta)) = \epsilon^r(kh(\alpha),kh(\beta)) \ ,$$

using Lemma 2.1 and proposition 2.3 and where ϵ^r is the induced commutation factor on Γ^h/Γ^h_0. It remains to be shown that kh is, in fact, a homomorphism. If we define

$$l(\alpha,\beta) \equiv kh(\alpha+\beta) - kh(\alpha) - kh(\beta)$$

then

$$\epsilon^r(l(\alpha,\beta), kh(\gamma)) = \epsilon(0,\gamma) = 1 \quad \forall\, \gamma \in \Gamma \; .$$

Now Γ^h/Γ^h_0 consists of elements of the form

$$k\left(\sum_i n^i h(\alpha_i)\right) = \sum_i n^i kh(\alpha_i) \quad \alpha_i \in \Gamma$$

but

$$\epsilon^r\left(l(\alpha,\beta), \sum_i n^i kh(\alpha_i)\right) = \prod_i \left[\epsilon^r(l(\alpha,\beta), kh(\alpha_i))\right]^{n_i} = 1 \; .$$

Now since Γ^h/Γ^h_0 is reduced this means that $l(\alpha,\beta) = 0$ which in turn means that kh must be homomorphic. ■

2.3. Canonical Forms

We consider now a canonical set of pairs $<\Gamma_c, \epsilon_c>$ which cover all possible pairs $<\Gamma,\epsilon>$.

Theorem 2.4. *Every colour algebra* **A** *which can be coloured by* $<\Gamma,\epsilon>$, *where* Γ *is a finite group, can also be coloured by a* $<\Gamma_c, \epsilon_c>$. *The* Γ_c *are of the following form*

$$\Gamma_c = \Gamma_{p_1} \oplus \Gamma_{p_1} \oplus \ldots \oplus \Gamma_{p_n} \; , \tag{2.19}$$

where the p_i *are distinct primes and each* p_i*-group* Γ_{p_i}, *with* $p_i \neq 2$, *is of the form*

$$\Gamma_{p_i} = Z\left[(p_i)^{r_1}\right]_1 \oplus Z\left[(p_i)^{r_1}\right]_2 \oplus \ldots \oplus Z\left[(p_i)^{r_1}\right]_{2j_1} \oplus Z\left[(p_i)^{r_2}\right]_1 \oplus \ldots$$
$$\oplus\, Z\left[(p_i)^{r_2}\right]_{2j_2} \oplus \ldots \oplus Z\left[(p_i)^{r_m}\right]_{2j_m} \; , \tag{2.20}$$

where $Z\left[(p_i)^{r_k}\right]_u$ *means the* u*'th copy of the cyclic group of order* $(p_i)^{r_k}$. *For* $p_i = 2$ *the group has the same form* <u>*except*</u> *that an odd number of copies of* Z_2 *are allowed. The* ϵ_c *defined on the* Γ_c *have the decomposition given by (2.11)*

$$\epsilon_c = \epsilon_1 \epsilon_2 \ldots \epsilon_n$$

with ϵ_i defined on the Γ_{p_i}. The ϵ_i are unique in the case $p_i \neq 2$, and are given by

$$\epsilon_i(\alpha,\beta) = \eta_{i1}^{\psi_1(\alpha_1,\beta_1)} \eta_{i2}^{\psi_2(\alpha_2,\beta_2)} \ldots \eta_{im}^{\psi_m(\alpha_m,\beta_m)} \tag{2.21}$$

with η_{iv} being the primitive $(p_i)^{r_v}$ root of unity; α_v and β_v the projections of α and β onto the copies of $Z\left[(p_i)^{r_v}\right]$ in Γ_{p_i}; and $\psi_v(\alpha_v,\beta_v)$ is the following antisymmetric bilinear form defined on copies of $Z\left[(p_i)^{r_v}\right]$

$$\psi_v(\alpha_v,\beta_v) = \sum_{i=1}^{j_v} \left[k_v^{2i-1} l_v^{2i} - k_v^{2i} l_v^{2i-1}\right] . \tag{2.22}$$

The k and l are as in (2.12). In the case of $p_i = 2$ the ϵ_i has the form

$$\epsilon_i(\alpha,\beta) = \eta_{i1}^{\psi_1(\alpha_1,\beta_1)} \ldots \eta_{im}^{\psi_m(\alpha_m,\beta_m)} (-1)^{\varphi(\alpha',\beta')} \tag{2.23}$$

where the η_{iv}, ψ_v, α_v and β_v are the same as before with the restriction that $r_v \neq 0$. φ is defined on the copies of Z_2 (α' and β' being the projections of α and β onto these copies) and has two possible forms. The first is

$$\varphi(\alpha',\beta') = \sum_{i=1}^{q} k'^i l'^i \tag{2.24a}$$

with q being the number of copies of Z_2 in Γ_2. The second is the antisymmetric form

$$\varphi(\alpha',\beta') = \sum_{i=1}^{q/2} \left[k'^{2i-1} l'^{2i} - k'^{2i} l'^{2i-1}\right] . \tag{2.24b}$$

Notice that in this case q must be even.

Proof: We shall show that a $< \Gamma_c, \epsilon_c >$ covers an arbitrary $< \Gamma, \epsilon >$ by considering a sequence of covering-homomorphisms between Γ and Γ_c. It shall be sufficient to restrict the covering-homomorphisms to a particular p_i-group with commutation factor obtained from the decomposition (2.11). It is obvious that these restricted covering-homomorphisms extend to the whole group– just set the action on the other p_i-groups to the identity.

Let us write the p_i-group of Γ as

$$Z\left[(p_i)^{r_1}\right]_1 \oplus Z\left[(p_i)^{r_1}\right]_2 \oplus \ldots \oplus Z\left[(p_i)^{r_1}\right]_{n_1} \oplus Z\left[(p_i)^{r_2}\right]_1 \oplus \ldots \oplus Z\left[(p_i)^{r_2}\right]_{n_2} \oplus \ldots \oplus Z\left[(p_i)^{r_m}\right]_{n_m} , \tag{2.25}$$

where $r_1 > r_2 > \ldots > r_m$. With respect to this basis of the p-group* the matrix $\mathbf{M}$ of (2.13), which determines the commutation factor on the p-group, can be written as

$$\mathbf{M} = \begin{pmatrix} M_{11} & M_{12} & \ldots & M_{1m} \\ M_{21} & M_{22} & \ldots & M_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m1} & M_{m2} & \ldots & M_{mm} \end{pmatrix} \tag{2.26}$$

where the M_{jk} are the submatrices of dimension $n_j \times n_k$. By use of (2.10) we deduce that these submatrices have the form

$$M_{jk} = p^{r_1 - min(r_j, r_k)} R_{jk} \ , \tag{2.27}$$

where the R_{jk} is an arbitrary $n_j \times n_k$ matrix of integers modulo p^{r_1}. In the same manner as $\mathbf{M}$, we can decompose the $\mathbf{k}$ and $\mathbf{l}$ of (2.13) into subvectors $\mathbf{k}_j$ and $\mathbf{l}_j$, where j runs from 1 to m.

Shoda [35] has shown that the automorphisms on the p-group have the following expression through the $\mathbf{k}$:

$$\mathbf{k}'_i = \sum_{j=1}^{m} P_{ij} \mathbf{k}_j \ , \tag{2.28}$$

where the P_{ij} have the following form: For $i > j$ the entries of the matrix P_{ij} are integers modulo p^{r_i}, while for $i \leq j$ $P_{ij} = p^{r_i - r_j} Q_{ij}$ with Q_{ij} having the form of P_{ij} when $i > j$. Finally $det(P_{ij})$ is required not to be a multiple of p. This is to ensure that the homomorphism is 1:1 and onto.

Now given a commutation factor ϵ on our p-group the automorphism of (2.28) induces a new commutation factor ϵ' via (2.18). This is given by

$$\epsilon'(\alpha, \beta) = \epsilon(\alpha', \beta') = \eta_{i1}^{\psi(\alpha', \beta')} \ ,$$

with

$$\psi(\alpha', \beta') = (\mathbf{Pk})^t \, \mathbf{M} \, (\mathbf{Pl}) = \mathbf{k}^t \left(\mathbf{P}^t \mathbf{M} \mathbf{P} \right) \mathbf{l} \, . \tag{2.29}$$

In otherwords the $\mathbf{M}$ is transformed to $\mathbf{M}'$ given by

$$\mathbf{M}' = \mathbf{P}^t \mathbf{M} \mathbf{P}. \tag{2.30}$$

---

* We drop the subscript i for notational ease.

As is usual in reduction problems of this kind we shall be interested in particular types of $\mathbf{P}$ which correspond to column and row operations on $\mathbf{M}$. From the form of (2.30) it is clear that a given row operation must always be followed by the *corresponding* column operation. The conditions on $\mathbf{P}$ outlined above mean that there are restrictions on the row (*and* corresponding column) operations allowed. These are easily seen to be the following:

(i) Let us denote the addition of a multiple of a row to another row by $sr_1 + r_2 = r_2'$ and suppose r_1 belongs to the i'th block row $(M_i)_j = M_{ij}$ and r_2 to the k'th block row $(M_k)_j = M_{kj}$. We then have the restriction that if $i < k$, then s must be a multiple of $p^{r_i - r_k}$. If $i \geq k$ then s is not restricted.

(ii) The multiplication of a row by a constant s has the restriction that s may not be divisible by p. This is a result of $det\,(P_{ii})$ not being a multiple of p.

(iii) The interchange of two rows is only possible when they belong to the same block row.

Apart from the above automorphisms we shall also be interested in the following non-automorphic covering-homomorphism: Suppose the m'th row in the first block row $(M_1)_j = M_{1j}$ is a multiple of p, then there is a covering-homomorphism ϕ which maps the $Z\,[(p_i)^{r_1}]_m$ summand of (2.25) into a $Z\,\left[(p)^{r_1-1}\right]$ summand and leaves all other summands unaffected. The map ϕ is defined as follows: We can write any element of $Z\,[(p_i)^{r_1}]_m$ uniquely as

$$kp^{r_1-1} + l\ , \tag{2.31}$$

with $k < p$ and l not divisible by p^{r_1-1}. Clearly l corresponds to an element of $Z\,\left[(p)^{r_1-1}\right]$ and then ϕ is simply given by

$$\phi(kp^{r_1-1} + l) = l\ . \tag{2.32}$$

To show that ϕ is a covering-homomorphism, we observe firstly that it is obviously an onto homomorphism by its definition. Secondly the kernel of ϕ just consists of elements of the form kp^{r_1-1} from $Z\,[(p_i)^{r_1}]_m$ and zeros from all the other summands in (2.25). Given now that the m'th row of the first block row is a multiple of p, it is clear from (2.13) that if $\alpha \in ker\phi$ then $\mathbf{k}^t\mathbf{M}$ is a vector which is a multiple of p^{r_1} and so $\epsilon(\alpha,\beta) = 1 \quad \forall\, \beta \in \Gamma$. This demonstrates that ϕ satisfies (2.17) and hence that it is a covering homomorphism.

We now use the above covering-homomorphisms to reduce M_{11}. We consider firstly the case $p \neq 2$ for which (2.5) gives

$$M_{11} = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n_1} \\ -a_{12} & 0 & \dots & a_{2n_1} \\ -a_{13} & -a_{23} & \dots & a_{3n_1} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n_1} & -a_{2n_1} & \dots & 0 \end{pmatrix}. \tag{2.33}$$

Consider now the first column: Two possibilities arise, either it is a multiple of p or else there exists an a_{1j} not a multiple of p. In the first case we apply the covering-homomorphism ϕ of (2.32) to the summand $Z\left[(p_i)^{r_1}\right]_1$ converting it to a $Z\left[(p)^{r_1-1}\right]$ summand and then relegate the column (and row) to the second block column (or row). Note that it may or may not be a *new* block column (or row) depending on whether $r_2 = r_1 - 1$. We then restart the analysis with a smaller $(n_1 - 1) \times (n_1 - 1)$ matrix M_{11}. In the second case we multiply the the column by the inverse of a_{1j} (which exists and is not a multiple of p because a_{1j} is not a multiple of p) and then interchange the second and j'th row. M_{11} has now been reduced to the following form

$$M'_{11} = \begin{pmatrix} 0 & 1 & a'_{13} & \dots & a'_{1n_1} \\ -1 & 0 & a'_{23} & \dots & a'_{2n_1} \\ -a'_{13} & -a'_{23} & 0 & \dots & a'_{3n_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a'_{1n_1} & -a'_{2n_1} & -a'_{3n_1} & \dots & 0 \end{pmatrix}. \tag{2.34}$$

We now eliminate all other elements in the first column (and row) by multiplying the second row by a'_{1j} and subtracting it from the j'th row. We can then multiply the first row by a'_{2j} and *add* it to the j'th row, thus eliminating all but the 1 from the second column. M_{11} now becomes

$$M''_{11} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & b_{34} & \dots & b_{3n_1} \\ 0 & 0 & -b_{34} & 0 & \dots & b_{4n_1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & -b_{3n_1} & -b_{4n_1} & \dots & 0 \end{pmatrix}.$$

It is obvious that the above analysis can be applied to the third column and so on and we therefore conclude that M_{11} may be reduced to the form

$$M_{11}^r = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

which is of dimension $n_1' \times n_1'$ with $n_1' \leq n_1$.

We can now use a row operation of type (i) to eliminate all elements in M_{j1}^r. This is because, by (2.27), elements in M_{j1} (and thus M_{j1}^r) have the form $kp^{r_1 - r_j}$, and the restriction (i) allows us to multiply a row from M_{11}^r by a number of this form and add it to a row in M_{j1}^r. It is to be noted that such an elimination would not have been possible in general, if we had not allowed the non-automorphic covering-homomorphism (consider the extreme example of when M_{11}^r consists entirely of zeros and hence all the rows in the first block row are multiples of p).

We have now reduced $\mathbf{M}$ to $\mathbf{M}^r$ with

$$\mathbf{M}^r = \begin{pmatrix} M_{11}^r & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \overline{\mathbf{M}} & \\ 0 & & & \end{pmatrix}. \tag{2.35}$$

Clearly now we can regard $\overline{\mathbf{M}}$ as determining a commutation factor on a group with cyclic summands of order strictly less than p^{r_1}. We may now repeat the analysis of above on this $\overline{\mathbf{M}}$ *without affecting* the decomposition in (2.35). The only complication with continuing the analysis iteratively is at the end where there may be rows (and columns) of zeros left. It is fairly obvious that the cyclic summands corresponding to these rows (and columns) may be mapped into the trivial group with a covering-homomorphism. Equations (2.21) and (2.22) are now clear.

In the case that $p = 2$, (2.5) shows that we may get diagonal elements in $\mathbf{M}$ and this will interfere with the reduction process outlined above. The approach we shall follow will be governed by the nature of these diagonal elements. Firstly if there are no such elements then clearly we may pursue the previous reduction. Secondly if there exists an $\alpha \in \Gamma$ such that*

$$\epsilon(\alpha, \alpha) = -1$$

and (2.36)

$$\epsilon(\alpha, \gamma) = \pm 1 \quad \forall\, \gamma \in \Gamma \;,$$

then we shall show that there exists an *onto* covering-homomorphism between Γ and a Γ_c. Thirdly if there exist elements satisfying the first equation of (2.36), but none of them satisfy the second, then we shall demonstrate a *non-onto* covering-homomorphism between Γ and a Γ_c. We proceed now to prove the second case.

Firstly we observe that if there is a $\beta \in \Gamma$ satisfying (2.36) then β cannot have the form 2γ, since in this case

$$-1 = \epsilon(\beta, \beta) = \epsilon(2\gamma, 2\gamma) = \epsilon^4(\gamma, \gamma)$$

which means that $\epsilon(\gamma, \gamma) \neq \pm 1$ and this contradicts (1.5). We conclude therefore, that β must have the form

$$\beta = e_1 + 2^{r_1} e_2 + \ldots + 2^{r_{n-1}} e_n \;,$$

where the e_i are the generators of the cyclic summands of Γ. Consider now the covering-homomorphism g, given by Lemma 2.1, onto a reduced Γ^r. It is clear that

$$\begin{aligned} &\epsilon^r(g(\beta), \gamma^r) \quad \text{for arbitrary } \gamma^r \in \Gamma^r \\ &= \epsilon^r(g(\beta), g(\gamma)) \quad \text{for some } \gamma \in \Gamma \\ &= \epsilon(\beta, \gamma) = \pm 1 \;. \end{aligned}$$

Now since Γ^r is reduced and

$$\epsilon^r(2g(\beta), \gamma^r) = 1 \quad \forall\, \gamma^r \in \Gamma^r \;,$$

* Γ shall be understood to be the 2-subgroup and ϵ the commutation factor restricted to this subgroup of the grading group.

it follows that $g(\beta)$ has order 2. Furthermore

$$\epsilon^r(g(\beta), g(\beta)) = \epsilon(\beta, \beta) = -1$$

and so

$$g(\beta) = e_1^r + 2^{s_1} e_2^r + \ldots + 2^{s_{n-1}} e_n^r$$

with

$$o(e_1^r) = o(2^{s_1} e_2^r) = \ldots = o(2^{s_{n-1}} e_n^r) = 2$$

and where the e_i^r are generators for the cyclic subgroups of Γ^r.

From the form of the isomorphisms given by (2.28) it follows that there exists an isomorphism f mapping $g(\beta)$ into e_1^r. Now consider any e_i^r with $o(e_i^r) > 2$ and $\epsilon^r(e_i^r, e_i^r) = -1$ then apply the following isomorphism k to the e_i^r:

$$k(e_i^r) = e_i^r + e_1^r$$

but

$$\epsilon^r(k(e_i^r), k(e_i^r)) = \epsilon^r(e_i^r + e_1^r, e_i^r + e_1^r) = -1.1. -1 = 1 \ ,$$

which means that there are no diagonal elements in the reduced $\mathbf{M}^r$ except those in the final Z_2 block. We can now apply the iterative process used in the case of $p \neq 2$ (which consists only of onto covering-homomorphisms) until we are left only with a sub-block corresponding to the Z_2 summands. The proof is now completed by use of a theorem of Scheunert [26].

In the third case mentioned above, consider an e_i (notation as in the second case) with least order such that $\epsilon(e_i, e_i) = -1$. Now let the other e_k satisfying $\epsilon(e_k, e_k) = -1$ have the following isomorphism applied to them:

$$e_k' = e_k + e_i \ .$$

It is fairly clear that after such transformations only e_i will contribute a diagonal element to $\mathbf{M}$. As we shall see later this diagonal element cannot be removed to the Z_2 sub-block by means of an *onto* covering-homomorphism and instead we put it there by the following non-onto covering-homomorphism: Let e_i generate a $Z[2^{r_i}]_j$ cyclic summand then we map $Z[2^{r_i}]_j$ into $Z_2 \oplus Z[2^{r_i}]$ as follows: It is clear that any element of $Z[2^{r_i}]_j$ can be written as $2k + l$ with $l = 0, 1$. The covering-homomorphism h is then given by

$$h(2k + l) = (l, 2k + l) \tag{2.37}$$

and the new commutation factor on the expanded Γ' has a new $\mathbf{M}'$ which is the same as $\mathbf{M}$ except that all diagonal entries not in the Z_2 sub-block are zero. In addition there is a new Z_2 summand in Γ' whose effect on $\mathbf{M}$ is to introduce a single diagonal element $2^{r_1 - 1}$; all new off diagonal elements are zeros.

It is obvious that (2.37) describes a homomorphic map and a little thought then shows that the new commutation factor we have defined satisfies the condition (2.16). We can now repeat the comments that applied for the final reduction in the second case and obtain the stated result. ∎

2.4. Uniqueness Results

Another important question to be considered is the uniqueness or otherwise of the canonical $< \Gamma_c, \epsilon_c >$ for a particular colour algebra. A little thought will show that if $< \Gamma_b, \epsilon_b >$ covers $< \Gamma, \epsilon >$ and if $\Gamma_b \subset \Gamma_a$ then it may be possible in general, to extend ϵ_b to a commutation factor on Γ_a and then obviously $< \Gamma_a, \epsilon_a >$ covers $< \Gamma, \epsilon >$. Clearly then, what we may hope for is that there is a unique *smallest* $< \Gamma_c, \epsilon_c >$ covering $< \Gamma, \epsilon >$. More precisely, what we shall prove is that there exists a $< \Gamma_c, \epsilon_c >$ covering $< \Gamma, \epsilon >$ such that any other $< \Gamma_c', \epsilon_c' >$ covering $< \Gamma, \epsilon >$ satisfies $\Gamma_c \subseteq \Gamma_c'$. Further we shall give criterion to determine what this minimal $< \Gamma_c, \epsilon_c >$ is.

In order to prove the above result we need to introduce a little machinery from elementary abelian group theory. This shall differ somewhat from the standard treatment (see Fuchs [33]) and so we shall be forced to prove a number of basic results in this field.

Firstly we define two notions of *linear independence*. We say that the set $\{\alpha_i\}$ of elements of Γ is *p-linearly independent* if $o(\alpha_i)$ is a power of p and if

$$\sum_i n^i \alpha_i = 0 \tag{2.38}$$

implies that for all i, $n^i \equiv 0 \bmod p$. Secondly we say the set $\{\alpha_i\}$ is *p-linearly independent with respect to* ϵ if again $o(\alpha_i)$ is a power of p, and if

$$\epsilon\left(\sum_i n^i \alpha_i, \gamma\right) = 1 \quad \forall\, \gamma \in \Gamma \tag{2.39}$$

means that for all i, $n^i \equiv 0 \bmod p$.

From these two definitions we are further able to define two notions of rank. We say the p^k*-rank* of Γ is the maximal number of p-linearly independent elements in $p^k\Gamma$. A similar definition holds for p^k*-rank with respect to* ϵ. In the interests of brevity, we use rank when we wish to refer to p^0-rank and l.i. when we wish to talk of p-linear independence.

Lemma 2.5. *The rank of Z_{p^r} is one.*

Proof: Let $0 \neq a, b \in Z_{p^r}$ and let e generate the group. It follows easily that

$$a = qne \quad b = qme \tag{2.40}$$

with $(m,n) = 1$. This means that one of m or n is non-zero mod p. Also (2.40) implies that

$$ma - nb = 0$$

which shows that a and b are linearly dependent. ∎

Lemma 2.6. *If $\{\alpha_j\}$ is linearly dependent then there exists an i such that*

$$\alpha_i = \sum_{j \neq i} n^j \alpha_j .$$

Proof: $\{\alpha_j\}$ linearly dependent means that $\exists\; m^i \not\equiv 0 \bmod p$ such that

$$\sum_j m^j \alpha_j = 0 \tag{2.41}$$

$\Rightarrow \; (m^i, o(\alpha_i)) = 1 \quad \Rightarrow \; \exists\; r, s \in Z$ such that

$$rm^i + so(\alpha_i) = 1$$

$$\Rightarrow \quad rm^i\alpha_i + so(\alpha_i)\alpha_i = \alpha_i$$

$$\Rightarrow \quad \alpha_i = rm^i\alpha_i = -\sum_{j \neq i} rm^j \alpha_j . \qquad ∎$$

Proposition 2.7. *rank($A \oplus B$)=rank(A)+rank(B)*

Proof: Let $\{a_i\}$ and $\{b_i\}$ be maximal sets of l.i. elements from A and B respectively. We show firstly that $\{a_i, b_i\}$ is an l.i. set. If

$$\sum_i n^i a_i + \sum_j m^j b_j = 0$$

$$\Rightarrow \quad \sum_i n^i a_i = \sum_j m^j b_j = 0 \ ,$$

where we have used the definition of the direct sum, namely $A \cap B = \{0\}$;

$$\Rightarrow \quad n^i \equiv 0 \quad \text{and} \quad m^j \equiv 0 \quad \text{mod } p \quad \forall \ i, j$$

which is what we require and shows

$$rank(A \oplus B) \geq rank(A) + rank(B) \ .$$

The following lemma is useful in demonstrating equality:

Lemma 2.8. *If $\{a_i\}$ is l.i. then $\{a_1 + \sum_{i>1} k^i a_i, a_2, \ldots\}$ is l.i.*

Proof: Let us suppose that

$$m\left(a_1 + \textstyle\sum_{i>1} k^i a_i\right) + \displaystyle\sum_{i>1} m^i a_i = 0$$

$$\Rightarrow \quad m a_1 + \sum_{i>1} \left(m k^i + m^i\right) a_i = 0$$

$$\Rightarrow \quad m \equiv 0 \quad \text{and} \quad m k^i + m^i \equiv 0 \quad \text{mod } p \quad \forall i$$

$$\Rightarrow \quad m^i \equiv 0 \quad \text{mod } p \quad \forall i \ .$$

■

Now let $\{c_j\}$ be an l.i. set in $A \oplus B$, then we can write c_j uniquely as $a_j + b_j$. Let us further assume that there are $k > m+n$ elements in $\{c_j\}$, where $m = rank(A)$ and $n = rank(B)$. We show, to begin with, that $m \geq 1$:

We deduce firstly that either there is a $b_l = 0$, in which case we are done, or else the b_j must be linearly dependent. In the latter case, select an i such that the statement of Lemma 2.6 holds. In otherwords we have

$$b_i = \sum_{i \neq j} k^j b_j \ . \tag{2.42}$$

Now transform the set $\{c_j\}$ to the set $\{c'_j\}$ with the same number of elements:

$$
\begin{aligned}
c'_j &= c_j \quad j \neq i \\
c'_i &= c_i - \sum_{j \neq i} k^j c_j \\
&= a_i - \sum_{j \neq i} k^j a_j \equiv a'_i \in A \ ,
\end{aligned}
$$

where (2.42) is being used for the last step. This new set is l.i. by Lemma 2.8 which means that a'_i is l.i. (non zero) since any subset of an l.i. set is obviously l.i. We conclude that $m \geq 1$ and the proposition is demonstrated if, in fact, $m = 0$. We now show that $m \geq 2$:

The set $\{b'_j\}$ has at most $k-1$ non-zero elements since $b'_i = 0$. If there are less than $k-1$ non-zero elements then clearly we are done. Now

$$
k - 1 > m + n - 1 \geq n
$$

since we have already shown that $m \geq 1$. It therefore follows that if $\{b'_j\}$ has $k-1$ non-zero elements they must be linearly dependent. Therefore from Lemma 2.6, there exists a $k \neq i$ such that

$$
b'_k = \sum_{\substack{j \neq i \\ j \neq k}} l^j b'_j \ . \tag{2.43}
$$

We redefine our $\{c'_j\}$ as follows

$$
\begin{aligned}
c''_j &= c'_j = c_j \quad j \neq i \text{ and } j \neq k \\
c''_i &= c'_i = a'_i \\
c''_k &= c'_k - \sum_{\substack{j \neq i \\ j \neq k}} l^j c'_j \\
&= a'_k - \sum_{\substack{j \neq i \\ j \neq k}} l^j a'_j \equiv a''_k \in A \ .
\end{aligned}
$$

By Lemma 2.8 this new set of k elements is l.i. and so $a''_i = a'_i$ and a''_k are l.i., which means that $m \geq 2$. The proposition has now been shown for $m = 0$ or 1. Obviously the above argument can be repeated until we finally conclude that $m = k$ which contradicts our assumption that $k > m + n$. ∎

The following corollary is used in a major way in Theorem 2.11 below:

Corollary. ***A finite abelian group is uniquely determined by its p^k-ranks.***

Proof: Lemma 2.5 and Proposition 2.7 imply that the rank of a p-group Γ_p is equal to the number of cyclic summands in its unique decomposition (2.2). Now the rank of $p\Gamma_p$ is the same as that for Γ_p *less* the number of Z_p summands in Γ_p. This argument extends in an obvious way to the rank of $p^k\Gamma_p$ which is equal to the p^{k-1}-rank *less* the number of Z_{p^k} cyclic summands in Γ_p. We therefore conclude that Γ_p is uniquely specified by its p^k-ranks. To extend this result to an arbitrary finite Γ it is sufficient to observe that, by the definition of p-linear independence, elements of non-prime power order do not contribute to the p^k-ranks of Γ. Hence the p^k-ranks of Γ determine uniquely, the unique p-groups making up the total group. ■

We now examine the connection between rank and rank w.r.t. ϵ:

Proposition 2.9. *The p^k-rank of Γ is at least as large as its p^k-rank w.r.t. ϵ. Equality holds when Γ is reduced.*

Proof: Let $a_i \in p^k\Gamma$ and suppose $\sum_i m^i a_i = 0$. Further suppose that $\{a_i\}$ is l.i. w.r.t. ϵ

$$\Rightarrow \quad \epsilon\left(\sum_i m^i a_i, \gamma\right) = 1 \quad \forall\, \gamma \in \Gamma$$
$$\Rightarrow \quad m^i \equiv 0 \mod p \quad \forall\, i$$

which means that $\{a_i\}$ is l.i.. For the second part of the proposition, suppose that Γ is reduced and that $\{a_i\}$ is l.i. If

$$\epsilon\left(\sum_i m^i a_i, \gamma\right) = 1 \quad \forall\, \gamma \in \Gamma$$

then because Γ is reduced

$$\Rightarrow \quad \sum_i m^i a_i = 0$$
$$\Rightarrow \quad m^i \equiv 0 \mod p \quad \forall\, i\,,$$

which means that $\{a_i\}$ is l.i. w.r.t. ϵ. ■

The reason for the usefulness, from our point of view, of rank w.r.t. ϵ is contained in the following:

Proposition 2.10. *The p^k-rank w.r.t. ϵ is preserved by an* <u>*onto*</u> *covering-homomorphism.*

Proof: Suppose $h : \Gamma \to \Gamma'$ is the onto covering-homomorphism and suppose that $h(a_i) \in \Gamma'$ are l.i. w.r.t. ϵ'. If

$$\begin{aligned} & \epsilon\left(\textstyle\sum_i m^i a_i, \gamma\right) = 1 \quad \forall\, \gamma \in \Gamma \\ \Rightarrow \quad & \epsilon'\left(h\left(\textstyle\sum_i m^i a_i\right), h(\gamma)\right) = 1 \quad \forall\, \gamma \in \Gamma\, , \end{aligned}$$

which implies, since h is onto, that

$$\begin{aligned} & \epsilon'\left(\textstyle\sum_i m^i h(a_i), \gamma'\right) = 1 \quad \forall\, \gamma' \in \Gamma' \\ \Rightarrow \quad & m^i \equiv 0 \mod p \quad \forall\, i \end{aligned}$$

and this therefore means that $\{a_i\}$ is l.i. w.r.t. ϵ.

Conversely suppose that $\{a_i\}$ is l.i. w.r.t. ϵ. If

$$\begin{aligned} & \epsilon\left(\textstyle\sum_i m^i h(a_i), \gamma'\right) = 1 \quad \forall\, \gamma' \in \Gamma' \\ \Rightarrow \quad & \epsilon'\left(h\left(\textstyle\sum_i m^i a_i\right), h(\gamma)\right) = 1 \quad \forall\, \gamma \in \Gamma \\ \Rightarrow \quad & \epsilon\left(\textstyle\sum_i m^i a_i, \gamma\right) = 1 \quad \forall\, \gamma \in \Gamma \\ \Rightarrow \quad & m^i \equiv 0 \mod p \quad \forall\, i \end{aligned}$$

and so we conclude that $\{h(a_i)\}$ is l.i. w.r.t. ϵ'. ∎

Having dispensed with the algebraic preliminaries, we are now able to prove the second major result of this section:

Theorem 2.11. *There exists a unique canonical $< \Gamma_c, \epsilon_c >$ covering $< \Gamma, \epsilon >$ such that if another canonical $< \Gamma'_c, \epsilon'_c >$ also covers $< \Gamma, \epsilon >$ then $\Gamma_c \subset \Gamma'_c$. Furthermore the p^k-rank of Γ_c is equal to the p^k-rank* <u>*w.r.t. ϵ*</u> *of Γ* <u>*unless*</u> *there exist $\beta \in \Gamma$ such that $\epsilon(\beta, \beta) = -1$ and none of these β satisfy*

$$\epsilon(\beta, \gamma) = \pm 1 \quad \forall\, \gamma \in \Gamma\, . \tag{2.44}$$

In this latter case the 2^0-rank of Γ_c is one greater than the 2^0-rank w.r.t. ϵ of Γ but all other p^k-ranks are identical. Finally the unique $< \Gamma_c, \epsilon_c >$ for every $< \Gamma, \epsilon >$ is the one achieved in the proof of Theorem 2.4.

Proof: We begin with the following essential Lemma:

Lemma 2.12. *If $\Gamma_a \subseteq \Gamma_b$ then the p^k-rank w.r.t. ϵ_a of Γ_a is no greater than the corresponding rank of Γ_b (providing, of course, that ϵ_a and ϵ_b agree on Γ_a). The same result holds for the p^k-ranks of Γ_a and Γ_b.*

Proof: Suppose $a_i \in p^k\Gamma_a$. Since $p^k\Gamma_a \subseteq p^k\Gamma_b$ this means that the a_i are also in $p^k\Gamma_b$. Further suppose that $\{a_i\}$ is l.i. w.r.t. ϵ_a *in* Γ_a. If

$$\begin{aligned} & \epsilon_b\left(\textstyle\sum_i m^i a_i, \gamma\right) = 1 \quad \forall\, \gamma \in \Gamma_b \\ \Rightarrow \quad & \epsilon_a\left(\textstyle\sum_i m^i a_i, \gamma'\right) = 1 \quad \forall\, \gamma' \in \Gamma_a \\ \Rightarrow \quad & m^i \equiv 0 \mod p \quad \forall\, i\ , \end{aligned}$$

which shows that $\{a_i\} \subset p^k\Gamma_b$ is l.i. w.r.t. ϵ_b. The second part of the lemma follows from the obvious observation that an l.i. set in $p^k\Gamma_a \subseteq p^k\Gamma_b$ is still one in $p^k\Gamma_b$. ■

Now if $< \Gamma', \epsilon' >$ covers $< \Gamma, \epsilon >$, then Theorem 2.2 tells us that the reduced Γ^h/Γ^h_0 contains the image of a covering-homomorphism from Γ. Lemma 2.12 and Proposition 2.10 then show that this quotient group must have p^k-ranks w.r.t. ϵ^r at least as large as the corresponding ranks of Γ w.r.t. ϵ. By Lemma 2.1, Γ^h/Γ^h_0 is the image of a covering-homomorphism from Γ^h and so, by Proposition 2.12 the p^k-ranks w.r.t. ϵ^r of the former group equal the p^k-ranks w.r.t. ϵ' of the latter group. Finally since $\Gamma^h \subseteq \Gamma'$, we deduce from Lemma 2.12 that the p^k-ranks w.r.t. ϵ' of Γ' are at least as great as the corresponding ranks w.r.t. ϵ of Γ. We also have the following:

Lemma 2.13. *Any canonical $< \Gamma_c, \epsilon_c >$ is reduced.*

Proof: Let $\alpha \in \Gamma_c$ satisfy

$$\epsilon_c(\alpha, \gamma) = 1 \quad \forall\, \gamma \in \Gamma_c$$

and for a $p_i \neq 2$, let s^t_j be the generator of $Z\left[(p_i)^{r_j}\right]_t$. We now deduce, using (2.21) and (2.22), that

$$1 = \epsilon(\alpha, s^t_j) = \epsilon_i(\alpha, s^t_j) = \eta_{ij}^{\pm k^t_j \mp 1}$$

where $k^t_j s^t_j$ is the projection of α onto $Z\left[(p_i)^{r_j}\right]_t$ and the $\pm$ depends on whether t is odd or even (t even gives the $+$). It follows immediately that $k^t_j \equiv 0 \mod p_i^{r_j}$

and so $k_j^t s_j^t = 0$. Since this holds for all j and t we conclude that α can have no components in p_i-subgroups of Γ_c. For $p_i = 2$ the argument is identical except in the case that copies of Z_2 have the form (2.24a) defined on them, in which case we get the simpler equation $(-1)^{k^t} = 1 \quad \forall\, t$, where $k^t s^t$ is the projection onto the t'th copy of Z_2 in Γ_c. Again this implies that $k^t s^t = 0$ and so we conclude that $\alpha = 0$. ∎

Proposition 2.9 and Lemma 2.13 now allow us to conclude that the p^k-rank of any covering Γ_c must be at least as large as the p^k-ranks w.r.t. ϵ of Γ. Now by the Corollary to Proposition 2.7, the p^k-ranks of a group determine it uniquely and so there must be a unique minimal canonical Γ_c^m with p^k-ranks *equal* to the p^k-ranks w.r.t. ϵ of Γ. Clearly then, any $< \Gamma_c, \epsilon_c >$ covering $< \Gamma, \epsilon >$, must satisfy $\Gamma_c^m \subseteq \Gamma_c$.

Except in the pathological case outlined in the statement of the theorem, the proof of Theorem 2.4 has shown that there exists an *onto* covering-homomorphism between Γ and a Γ_c. Proposition 2.10 therefore shows that this must be in fact Γ_c^m.

In the pathological case we have seen in the proof of Theorem 2.4 that there exists onto covering-homomorphisms at all but one place in the reduction— where we are forced to append an extra Z_2 summand. It follows, again from Proposition 2.10, that a reduction to $\Gamma_c^m \oplus Z_2$ is possible. Finally we complete the proof by showing that, in the pathological case, if Γ_c has 2^k-ranks equal to those of Γ_c^m then it cannot cover Γ. A little thought will show that this implies that any Γ_c covering Γ must satisfy $\Gamma_c^m \oplus Z_2 \subseteq \Gamma_c$.

Let us assume that a Γ_c with 2^k-ranks equal to those of Γ_c^m does cover Γ. The covering of Γ implies the existence of a covering-homomorphism $f : \Gamma \to \Gamma^h/\Gamma_0^h$ with $\Gamma^h \subseteq \Gamma_c$; also there is an onto covering-homomorphism $k : \Gamma^h \to \Gamma^h/\Gamma_0^h$. Consider now the 2-subgroups of Γ, Γ^h/Γ_0^h, Γ^h and Γ_c; denote them by Γ_2, Q_2, Γ_{2c}^h and Γ_{2c} respectively. Now by [33] the covering-homomorphisms f and k restrict to covering-homomorphisms $\Gamma_2 \to Q_2$ and $\Gamma_{2c}^h \to Q_2$ respectively; moreover it is easily seen that the latter must be onto. Denote by $r(\Gamma)$ and $r(\Gamma, \epsilon)$ the 2^k-rank and 2^k-rank w.r.t. ϵ of Γ respectively. We have, by the use of the technical lemmas and propositions above, the following inequalities

$$
\begin{aligned}
r(\Gamma_2,\epsilon) =& r(f(\Gamma_2),\epsilon_r) \le r(Q_2,\epsilon_r) = r(\Gamma^h_{2c},\epsilon_c) \le \\
& r(\Gamma_{2c},\epsilon_c) = r(\Gamma_{2c}) = r(\Gamma_2,\epsilon)
\end{aligned} \tag{2.47}
$$

which shows that equality must hold amongst all of them. Now we have seen that Γ^h/Γ^h_0 is reduced and we now show that this implies that $Q_2 \subseteq \Gamma^h/\Gamma^h_0$ is reduced. Suppose $\alpha \in Q_2$ satisfies

$$
\epsilon_r(\alpha,\gamma) = 1 \quad \forall\ \gamma \in Q_2\ ,
$$

then it is quite clear from (2.11) that

$$
\epsilon_r(\alpha,\gamma') = 1 \quad \forall\ \gamma' \in \Gamma^h/\Gamma^h_0
$$

and so $\alpha = 0$, which shows that Q_2 is reduced. We have thus $r(\Gamma_{2c}) = r(Q_2,\epsilon_r) = r(Q_2)$ and hence

$$
\Gamma_{2c} \cong Q_2\ . \tag{2.48}
$$

A further set of inequalities are the following:

$$
r(f(\Gamma_2),\epsilon_r) \le r(f(\Gamma_2)) \le r(Q_2) \tag{2.49}
$$

which, when (2.47) and (2.48) are considered, become equalities and lead to the conclusion that

$$
f(\Gamma_2) = Q_2
$$

and combining f with the isomorphism (2.48) leads one to conclude that there is an *onto* covering-homomorphism g between Γ_2 and Γ_{2c}.

We define a *diagonal* element β to be one satisfying $\epsilon(\beta,\beta) = -1$; such an element must exist in the pathological case. Now $\Gamma = \Gamma_2 \oplus \Gamma^t$ with Γ^t being a direct sum of $p \neq 2$ groups. The diagonal $\beta \in \Gamma$ must be able, therefore, to be written as $\beta = \alpha + \gamma$ with $\alpha \in \Gamma_2$ and $\gamma \in \Gamma^t$. Now

$$
-1 = \epsilon(\beta,\beta) = \epsilon(\alpha+\gamma,\alpha+\gamma) = \epsilon(\alpha,\alpha)\epsilon(\gamma,\gamma) = \epsilon(\alpha,\alpha)\ , \tag{2.50}
$$

where the last step follows from the results of section 2. Clearly then, $\alpha \in \Gamma_2$ is diagonal which implies, from the definition of the covering-homomorphism, that

$g(\Gamma_2) = \Gamma_{2c}$ possesses a diagonal element also. Examination of (2.24) then shows that this diagonal element must satisfy (2.44), but since Γ_{2c} is the image of a covering-homorphisms from Γ_2 it follows that this latter group must also contain a diagonal element satisfying (2.44). This however, contradicts the assumption that Γ is pathological and so we are done. ■

It is an obvious step now to extend the above results to the case where Γ is finitely-generated. This should not prove too difficult; however the result (2.11) will not now hold, and as a number of the proofs depend on this result, some reworking may be required. The concept of the covering-homomorphism should still, however, play a central role.

3. The canonical superalgebra

In this section we basically follow the work of Scheunert [26], however we provide proofs for a number of results stated by him and provide greater detail in the derivation of the main results.

The basic concept we shall require is the **σ map** which converts an ϵ colour algebra with grading group Γ into an ϵ' colour algebra with the same grading group. This new algebra will be the same set-theoretically as far as the grading is concerned, but will have a different bracket defined on it. We define this new bracket as

$$< a_\alpha, a_\beta >_\sigma = \sigma(\alpha, \beta) < a_\alpha, a_\beta > \tag{3.1}$$

where $\sigma : \Gamma \times \Gamma \to C$ is a non-zero valued map. Our first main result concerns the conditions that σ needs to satisfy in order that this new bracket still defines a colour algebra.

Proposition 3.1. *If σ is a multiplier [36] on Γ, that is, it satisfies*

$$\sigma(\alpha+\beta, \gamma)\sigma(\alpha, \beta) = \sigma(\alpha, \beta+\gamma)\sigma(\beta, \gamma) \qquad \forall\ \alpha, \beta, \gamma \in \Gamma\ , \tag{3.2}$$

then the map given by (3.1) defines a map from a ϵ colour algebra to a ϵ' colour algebra where

$$\epsilon'(\alpha, \beta) = \sigma(\alpha, \beta)\sigma^{-1}(\beta, \alpha)\epsilon(\alpha, \beta)\ . \tag{3.3}$$

Proof: In order to show this result we need to show firstly that ϵ' is a commutation factor and secondly that the bracket defined by (3.1) satisfies the conditions (1.4) required by a ϵ' colour algebra.

For convenience we define the maps

$$\begin{aligned} L(\alpha, \beta, \gamma) &\equiv \sigma(\gamma, \alpha+\beta)\sigma^{-1}(\gamma, \alpha)\sigma^{-1}(\gamma, \beta) \\ R(\alpha, \beta, \gamma) &\equiv \sigma(\alpha+\beta, \gamma)\sigma^{-1}(\alpha, \gamma)\sigma^{-1}(\beta, \gamma)\ , \end{aligned} \tag{3.4}$$

which satisfy the identities

$$\begin{aligned} L(\alpha, \beta, \gamma) &= L(\beta, \alpha, \gamma) \\ R(\alpha, \beta, \gamma) &= R(\beta, \alpha, \gamma)\ . \end{aligned} \tag{3.5}$$

It is now trivial to observe that the multiplier condition (3.2) is equivalent to

$$L(\beta,\gamma,\alpha) = R(\alpha,\beta,\gamma) \ . \tag{3.6}$$

By the use of (3.5) and (3.6) we conclude that

$$\begin{aligned} R(\alpha,\beta,\gamma) &= L(\beta,\gamma,\alpha) = L(\gamma,\beta,\alpha) \\ &= R(\alpha,\gamma,\beta) = R(\gamma,\alpha,\beta) = L(\alpha,\beta,\gamma) \ . \end{aligned} \tag{3.7}$$

In particular (3.4) and (3.7) show that

$$\sigma(\alpha+\beta,\gamma)\sigma^{-1}(\alpha,\gamma)\sigma^{-1}(\beta,\gamma) = \sigma(\gamma,\alpha+\beta)\sigma^{-1}(\gamma,\alpha)\sigma^{-1}(\gamma,\beta) \ . \tag{3.8}$$

Consider now ϵ' defined by (3.3), clearly it satisfies

$$\epsilon'(\alpha,\beta)\epsilon'(\beta,\alpha) = 1 \ ,$$

if ϵ does. In addition we have

$$\begin{aligned} \epsilon'(\alpha+\beta,\gamma) &= \epsilon(\alpha,\gamma)\epsilon(\beta,\gamma)\sigma(\alpha+\beta,\gamma)\sigma^{-1}(\gamma,\alpha+\beta) \\ &= \epsilon(\alpha,\gamma)\epsilon(\beta,\gamma)\sigma(\alpha,\gamma)\sigma^{-1}(\gamma,\alpha)\sigma(\beta,\gamma)\sigma^{-1}(\gamma,\beta) \ , \end{aligned}$$

from (3.8) and (1.3). However this is just equal to $\epsilon'(\alpha,\gamma)\epsilon'(\beta,\gamma)$, using (3.3). A similar argument shows that $\epsilon'(\alpha,\beta+\gamma) = \epsilon'(\alpha,\beta)\epsilon'(\alpha,\gamma)$ and so we conclude that ϵ' is a commutation factor.

We now use (1.4) for the ϵ bracket to conclude that

$$\begin{aligned} < a_\alpha, b_\beta >_\sigma &\equiv \sigma(\alpha,\beta) < a_\alpha, b_\beta > \\ &= -\sigma(\alpha,\beta)\epsilon(\alpha,\beta) < b_\beta, a_\alpha > \\ &= -\sigma(\alpha,\beta)\sigma^{-1}(\beta,\alpha)\epsilon(\alpha,\beta) < b_\beta, a_\alpha >_\sigma \\ &= -\epsilon'(\alpha,\beta) < b_\beta, a_\alpha >_\sigma \ . \end{aligned}$$

It remains therefore, to show that $<,>_\sigma$ satisfies the generalised Jacobi identity (1.4b). In otherwords we need to show that

$$\sum_{cycl(\alpha,\beta,\gamma)} \epsilon'(\gamma,\alpha) < a_\alpha, < b_\beta, c_\gamma >_\sigma >_\sigma = 0 \tag{3.9}$$

or

$$\sum_{cycl} \epsilon'(\gamma,\alpha)\sigma(\beta,\gamma)\sigma(\alpha,\beta+\gamma) < a_\alpha, < b_\beta, c_\gamma >> = 0 \,. \tag{3.10}$$

By the use of (3.3) we see that this will hold if

$$\begin{aligned} &\sigma(\gamma,\alpha)\sigma^{-1}(\alpha,\gamma)\sigma(\beta,\gamma)\sigma(\alpha,\beta+\gamma) \\ &= [\sigma(\gamma,\alpha)\sigma(\alpha,\beta)\sigma(\beta,\gamma)]\,\sigma^{-1}(\alpha,\gamma)\sigma^{-1}(\alpha,\beta)\sigma(\alpha,\beta+\gamma) \end{aligned}$$

is invariant under a cyclic permutation of α, β and γ. It is now easy to see that this is equivalent to $L(\beta,\gamma,\alpha)$ being invariant under cyclic permutations (this is because $\sigma(\gamma,\alpha)\sigma(\alpha,\beta)\sigma(\beta,\gamma)$ is). The invariance of L under cyclic permutations is an easy consequence of (3.7) and (3.6) and so the proof is complete. ■

The converse of the above proposition is not true as we can see from the following counter-example:

Let $\mathbf{A}$ be an algebra with grading group Z_3 and three elements a_1, a_2 and a_0 satisfying

$$< a_1, a_2 >= - < a_2, a_1 >= a_0 \,,$$

and with all other brackets zero. $\mathbf{A}$ is a colour algebra since (1.4) will hold if $\epsilon(1,2) = 1$; also the Jacobi identity is satisfied trivially.

If we allow σ to be symmetric in its arguments then (3.8) will hold and as a consequence the new bracket $<,>_\sigma$ will have the required symmetry property of (1.4); the Jacobi identity will obviously hold, again trivially. Thus the new bracket also defines a colour algebra. Now if we put $\alpha = \beta = 1$ and $\gamma = 2$ in (3.2) we obtain

$$\sigma(2,2)\sigma(1,1) = \sigma(1,0)\sigma(1,2) \,.$$

Evidently by a suitable choice of σ this may be violated and hence we have our counter-example to the converse of proposition 3.1.

The following result is one of the more important results in colour algebra theory:

Proposition 3.2. *Let Γ be finitely-generated then there exists a unique (up to isomorphism) "canonical" superalgebra associated with each colour algebra by the σ mapping of (3.1). The commutation factor is the one given by equation (1.7)* *in section 1. We denote it by ϵ_0 and the commutation factor from which it was derived by ϵ.*

Proof: Firstly it is straightforward to observe that $\epsilon_0\epsilon^{-1}$ is a commutation factor η which satisfies

$$\eta(\alpha,\alpha)=1 \qquad \forall\, \alpha \in \Gamma\ .$$

Section 2 now shows that there exists a σ satisfying (2.7), (providing Γ is finitely-generated) such that

$$\sigma(\alpha,\beta)\sigma^{-1}(\beta,\alpha)=\eta(\alpha,\beta)\ .$$

It is easy to see that conditions (2.7) imply that σ is a multiplier in the sense of (3.2) and so $<,>_\sigma$ is a colour algebra with commutation factor ϵ_0, in otherwords it is a superalgebra. It remains now to be shown that this superalgebra does not depend on a particular choice of σ.

Now any superalgebra produced by the σ mapping of (3.1) must have ϵ_0 as its commutation factor. This is because commutation factors ϵ_l on superalgebras are determined by the values $\epsilon_l(\alpha,\alpha)=\pm 1$ since these values determine the grading on the algebra. The result now follows from the equalities

$$\begin{aligned}\epsilon_l(\alpha,\alpha)&=\sigma(\alpha,\alpha)\sigma^{-1}(\alpha,\alpha)\epsilon(\alpha,\alpha)\\ &=\epsilon(\alpha,\alpha)=\epsilon_0(\alpha,\alpha)\ .\end{aligned}$$

From this we conclude that any two σ and σ_l producing a superalgebra, must satisfy

$$\sigma(\alpha,\beta)\sigma^{-1}(\beta,\alpha)=\sigma_l(\alpha,\beta)\sigma_l^{-1}(\beta,\alpha)$$

or

$$\sigma(\alpha,\beta)\sigma_l^{-1}(\alpha,\beta)=\sigma(\beta,\alpha)\sigma_l^{-1}(\beta,\alpha)\ .$$

In otherwords, the multiplier τ given by

$$\tau(\alpha,\beta)=\sigma(\alpha,\beta)\sigma_l^{-1}(\alpha,\beta),$$

must be symmetric. We now prove the following lemma which holds on finitely-generated grading groups Γ:

Lemma 3.3. *On finitely generated abelian groups all symmetric multipliers are* trivial. *That is, they satisfy*

$$\tau(\alpha,\beta) = s^{-1}(\alpha)s^{-1}(\beta)s(\alpha+\beta) \tag{3.11}$$

where s is some map $\Gamma \to C$.

Proof: Substituting $\alpha = \beta = 0$ into (3.2) we see that

$$\tau(0,\gamma) = \tau(\gamma,0) = constant \qquad \forall\, \gamma \in \Gamma\,.$$

Thus a simple rescaling of the multipliers allows us to set this constant to 1 and prove (3.11). We can then rescale the s to get our original unscaled multiplier. We now have

Lemma. *If Γ is generated by one element then τ is trivial.*

Proof: Denote the generator by 1 then we define

$$s(n) \equiv \prod_{k=1}^{n-1} \tau(1,k)s^{n}(1) \qquad n \geq 2\,, \tag{3.12}$$

where n has its obvious meaning as an element of Γ;

$$s(-n) \equiv \tau^{-1}(n,-n)s^{-1}(n)\ ; \tag{3.13}$$

and $s(1), s(0)$ are arbitrary. We now show that (3.11) holds:

Let $n \geq m \geq 2$ then

$$\begin{aligned} s^{-1}(n)s^{-1}(m)s(n+m) &= \left[\prod_{k=1}^{n-1}\tau(1,k)\prod_{l=1}^{m-1}\tau(1,l)\right]^{-1}\prod_{i=1}^{m+n-1}\tau(1,i) \\ &= \prod_{l=1}^{n-1}\tau^{-1}(1,l)\prod_{k=m}^{m+n-1}\tau(1,k) \\ &= \tau(1,n+m-1)\prod_{l=1}^{n-1}\tau(1,l+m-1)\tau^{-1}(1,l) \\ &= \tau(1,n+m-1)\prod_{l=1}^{n-1}\tau(m-1,l+1)\tau^{-1}(m-1,l) \\ &= \tau(m-1,n)\tau^{-1}(m-1,1)\tau(1,n+m-1) \\ &= \tau(m,n)\,, \end{aligned} \tag{3.14}$$

where we have used the fact that τ is a symmetric multiplier to derive lines 4 and 6. Consider next the following:

$$
\begin{aligned}
s^{-1}(n)s^{-1}(-m)s(n-m) &= \tau(m,-m)\prod_{k=1}^{n-1}\tau^{-1}(1,k)\prod_{l=1}^{m-1}\tau(1,l)\prod_{k=1}^{n-m-1}\tau(1,k) \\
&= \tau(m,-m)\prod_{k=n-m}^{n-1}\tau^{-1}(1,k)\prod_{l=1}^{m-1}\tau(1,l) \\
&= \tau(m,-m)\tau^{-1}(1,n-1)\prod_{k=1}^{m-1}\tau^{-1}(1,k+(n+m-1)) \\
&\qquad .\tau(1,k) \\
&= \tau(m,-m)\tau^{-1}(1,n-1)\prod_{k=1}^{m-1}\tau^{-1}(k+1,n-m-1) \\
&\qquad .\tau(k,n-m-1) \\
&= \tau(1,n-m-1)\tau^{-1}(m,n-m-1)\tau(m,-m)\tau^{-1}(1,n-1) \\
&= \tau(n,-m)\left[\tau^{-1}(n-1,-m)\tau^{-1}(m,n-m-1)\tau(m,-m)\right] \\
&= \tau(n,-m)\ , \qquad\qquad (3.15)
\end{aligned}
$$

where again we have used the fact that τ is a symmetric multiplier to derive lines 4, 6 and 7. In addition, $\tau(0,n-1)=1$ was used in the last line. Consider now arbitrary $p,q\in\Gamma$ and suppose we have

$$\tau(p,q)=s^{-1}(p)s^{-1}(q)s(p+q)\ ,$$

then using (3.13) we have

$$
\begin{aligned}
&s^{-1}(-p)s^{-1}(-q)s(-p-q) \\
&= s(p)s(q)s^{-1}(p+q)\tau(p,-p)\tau(q,-q)\tau^{-1}(p+q,-p-q) \\
&= \tau^{-1}(p,q)\tau(p,-p)\tau(q,-q)\tau^{-1}(p+q,-p-q) \\
&= \tau(p+q,-p-q)\tau^{-1}(q,-p-q)\tau(q,-q)\tau^{-1}(p+q,-p-q) \\
&= \tau(-q,-p) \\
&= \tau(-p,-q)\ .
\end{aligned}
$$

Where we have used the symmetry and multiplier nature of τ repeatedly. When this result is applied to (3.14) and (3.15) we see that only the cases $n=1$ or 0 remain to be shown. These are fairly easy consequences of the defining relations (3.12) and (3.13). Thus the proof is complete. ∎

Lemma. *Suppose τ is trivial on Γ_1 and Γ_2, then it is trivial on their direct sum $\Gamma \equiv \Gamma_1 \oplus \Gamma_2$.*

Proof: Suppose that

$$\tau(\alpha, \beta) = s_1^{-1}(\alpha) s_1^{-1}(\beta) s_1(\alpha + \beta) \qquad \forall\, \alpha, \beta \in \Gamma_1$$

and

$$\tau(\gamma, \delta) = s_2^{-1}(\gamma) s_2^{-1}(\delta) s_2(\gamma + \delta) \qquad \forall\, \gamma, \delta \in \Gamma_2 \,,$$

then we shall show that an appropriate trivial factor s for Γ is given by

$$s(\alpha + \gamma) = \tau(\alpha, \gamma) s_1(\alpha) s_2(\gamma) \qquad \forall\, \alpha \in \Gamma_1 \quad \forall\, \gamma \in \Gamma_2 \,.$$

This is consistent since Γ is a direct sum of Γ_1 and Γ_2. Consider now $\alpha, \beta \in \Gamma_1$ and $\gamma, \delta \in \Gamma_2$, then it follows that

$$\begin{aligned}
& s^{-1}(\alpha + \gamma) s^{-1}(\beta + \delta) s([\alpha + \beta] + [\gamma + \delta]) \\
&= \tau^{-1}(\alpha, \gamma) s_1^{-1}(\alpha) s_2^{-1}(\gamma) \tau^{-1}(\beta, \delta) s_1^{-1}(\beta) s_2^{-1}(\delta) \\
&\quad \times\ s_1(\alpha + \beta) s_2(\gamma + \delta) \tau(\alpha + \beta, \gamma + \delta) \\
&= \tau^{-1}(\alpha, \gamma) \tau(\alpha, \beta) \tau^{-1}(\beta, \delta) \tau(\gamma, \delta) \tau(\alpha + \beta, \gamma + \delta) \\
&= \tau(\alpha + \gamma, \beta) \tau^{-1}(\gamma, \alpha + \beta) \tau(\beta + \delta, \gamma) \tau^{-1}(\beta, \delta + \gamma) \tau(\alpha + \beta, \gamma + \delta) \\
&= \tau(\alpha + \gamma, \beta) \tau^{-1}(\gamma, \alpha + \beta) \tau(\alpha, \gamma) \tau(\alpha + \gamma, \beta + \delta) \\
&\quad \times\ \tau^{-1}(\alpha, \beta + \gamma + \delta) \tau(\alpha, \beta + \gamma + \delta) \tau^{-1}(\alpha, \beta) \\
&= \tau(\alpha + \gamma, \beta + \delta) \,,
\end{aligned}$$

where we are repeatedly using the fact that τ is a symmetric multiplier on Γ. The proof is now complete. ■

The main result (Lemma 3.3) now follows in a straightforward way from the above two lemmas. ■

We now complete the proof of Proposition 3.2 by showing that if $\sigma\sigma_l^{-1}$ is trivial then the brackets $<,>_\sigma$ and $<,>_{\sigma_l}$ define isomorphic superalgebras on the bracketless abstract vector-space $\mathbf{A}$ (which consists of elements from the colour algebra).

Denote the elements of $(\mathbf{A}, <>_\sigma)$ by l^σ_α and those of $(\mathbf{A}, <>_{\sigma_l})$ by $l^{\sigma_l}_\alpha$, where α refers to the grading on $\mathbf{A}$. We define the map $\phi : (\mathbf{A}, <>_\sigma) \to (\mathbf{A}, <>_{\sigma_l})$ by

$$\phi(l^\sigma_\alpha) = s^{-1}(\alpha) l^{\sigma_l}_\alpha$$

where we have from Lemma 3.3 that

$$\sigma(\alpha,\beta)\sigma_l^{-1}(\alpha,\beta) = s^{-1}(\alpha)s^{-1}(\beta)s(\alpha+\beta) \ . \tag{3.16}$$

We now show that ϕ is an isomorphism:

Firstly it is trivial to observe that ϕ is, by its definition, one to one and onto. Secondly from (3.1), (3.15) and (3.16) we have that

$$\begin{aligned}\phi\left(< l^\sigma_\alpha, l^\sigma_\beta >_\sigma\right) &= \phi\left(s^{-1}(\alpha)s^{-1}(\beta)s(\alpha+\beta)e\left(< l^{\sigma_l}_\alpha, l^{\sigma_l}_\beta >_{\sigma_l}\right)\right) \\ &= s^{-1}(\alpha)s^{-1}(\beta) < l^{\sigma_l}_\alpha, l^{\sigma_l}_\beta >_{\sigma_l} \\ &=< \phi(l^\sigma_\alpha), \phi(l^\sigma_\beta) > ,\end{aligned}$$

where $e : (\mathbf{A}, <>_{\sigma_l}) \to (\mathbf{A}, <>_\sigma)$ is given by $e(l^{\sigma_l}_\alpha) = l^\sigma_\alpha$. Proposition 3.2 now follows immediately. ■

4. Klein transformations

The above correspondence is rather abstract and not very well suited to the applications we shall later study. For this reason we study what we shall term Klein transformations. These generalize the original Klein transformations [13], introduced some years ago to change commutation relations to anti-commutation relations. As we shall see the transformations play a central role in the representation theory of colour algebras.

Consider firstly a graded algebra **A** with an associative product. As we saw in section 1, in a different context, this may be turned into a colour algebra by defining a bracket as

$$< a_\alpha, a_\beta > \equiv a_\alpha a_\beta - \epsilon(\alpha,\beta) a_\beta a_\alpha \ . \tag{4.1}$$

The closure of the associative product, the grading of the algebra and the fact that ϵ is a commutation factor ensure that with this bracket, **A** is a colour algebra.

The *Klein operators* K^σ_τ are a set of commuting operators with grading 0 which extend **A**. They satisfy

$$K^\sigma_\tau(\alpha) K^\sigma_\tau(\beta) = \tau(\alpha,\beta) K^\sigma_\tau(\alpha+\beta) \quad ; \quad K^\sigma_\tau(0) = 1 \tag{4.2}$$

$$K^\sigma_\tau(\alpha) a_\beta = \sigma(\beta,\alpha) a_\beta K^\sigma_\tau(\alpha) \ , \tag{4.3}$$

where the notation implies a unique Klein operator for each $\alpha \in \Gamma$. Also σ is a non-zero map $\Gamma \times \Gamma \to C$, satisfying $\sigma(\alpha, 0) = 1$ for consistency between (4.2) and (4.3). The confusion of notation with the σ of the previous section is deliberate as will become clearer below. τ, on the other hand, can easily be shown to be firstly symmetric and secondly, by the use of the law of associativity, a multiplier. As we have seen in section 3 this implies that if Γ is finitely-generated, then τ must be trivial. In otherwords (4.2) becomes

$$\begin{aligned} r(\alpha) K^\sigma_\tau(\alpha) r(\beta) K^\sigma_\tau(\beta) &= r(\alpha+\beta) K^\sigma_\tau(\alpha+\beta) \\ \tau(\alpha,\beta) &= r^{-1}(\alpha) r^{-1}(\beta) r(\alpha+\beta) \ , \end{aligned} \tag{4.2a}$$

which shows we can rescale our K^σ_τ to a new set K^σ which satisfy

$$K^\sigma(\alpha) K^\sigma(\beta) = K^\sigma(\alpha+\beta) \ . \tag{4.2b}$$

For the rest of this section we consider the K^σ only. We shall have cause to consider the original K^σ_τ again in Appendix B.

Consider now the factor σ, we can use (4.2), (4.3) together with the graded nature of $\mathbf{A}$ to show that in general* we must have the following relations

$$\begin{aligned} \sigma(\alpha+\beta,\gamma) &= \sigma(\alpha,\gamma)\sigma(\beta,\gamma) \\ \sigma(\alpha,\beta+\gamma) &= \sigma(\alpha,\beta)\sigma(\alpha,\gamma) \ , \end{aligned} \tag{4.4}$$

which imply, as we have seen before, that σ is a multiplier in the sense of (3.2).

The *Klein transformation* $\mathbf{A}^\sigma$ of the algebra $\mathbf{A}$ is a subalgebra of the above extended algebra and is defined elementwise from K^σ and $\mathbf{A}$:

$$a^\sigma_\alpha = K^\sigma(-\alpha)a_\alpha \qquad \forall\ a_\alpha \in \mathbf{A} \ . \tag{4.5}$$

Notice that if we had used K^σ_τ here, all we would obtain would be a rescaling of the a^σ_α.

The usefulness of the Klein transformation becomes apparent when one combines (4.1), (4.2), (4.3) and (4.5) obtaining

$$\begin{aligned} K^\sigma(\alpha+\beta)a^\sigma_{\alpha+\beta} &= < a_\alpha, a_\beta > \\ &= K^\sigma(\alpha)a^\sigma_\alpha K^\sigma(\beta)a^\sigma_\beta - \epsilon(\alpha,\beta)K^\sigma(\beta)a^\sigma_\beta K^\sigma(\alpha)a^\sigma_\alpha \\ &= K^\sigma(\alpha+\beta)\left[\sigma^{-1}(\alpha,\beta)a^\sigma_\alpha a^\sigma_\beta - \epsilon(\alpha,\beta)\sigma^{-1}(\beta,\alpha)a^\sigma_\beta a^\sigma_\alpha\right] \ . \end{aligned}$$

(We are assuming here that $< a_\alpha, a_\beta >= a_{\alpha+\beta}$). In otherwords we have the interesting relation

$$\sigma(\alpha,\beta)a^\sigma_{\alpha+\beta} = \sigma(\alpha,\beta) < a_\alpha, a_\beta >^\sigma = a^\sigma_\alpha a^\sigma_\beta - \epsilon(\alpha,\beta)\sigma(\alpha,\beta)\sigma^{-1}(\beta,\alpha)a^\sigma_\beta a^\sigma_\alpha \ . \tag{4.6}$$

We can see from this that $\mathbf{A}^\sigma$ is again a colour algebra with commutation factor given by equation (3.3). In fact, it is evident that the Klein transformation is just implementing the σ map of section 3. This follows since the colour algebra bracket $<,>_\sigma$ for $\mathbf{A}^\sigma$ is given by

$$< a^\sigma_\alpha, a^\sigma_\beta >_\sigma = \sigma(\alpha,\beta) < a_\alpha, a_\beta >^\sigma ,$$

* In certain special cases we may be able to avoid (4.4) — see the example following Proposition 3.1 for the kind of pathologies which may arise.

which is just (3.1), when one remembers that in section 3 the two colour algebras were identified set-theoretically whereas here they are different algebraic objects.

In a more general sense, if we restrict our transformations (3.1) to those for which σ is a multiplier, then the Klein transformation (4.5) will produce all such transformations (up to isomorphism), *providing* we assume that Γ is finitely-generated.

This statement follows because for σ a multiplier on a finitely-generated Γ, we have seen in section 2 that there will exist another σ_l *satisfying* (4.4), such that

$$\sigma_l(\alpha,\beta)\sigma_l^{-1}(\beta,\alpha) = \sigma(\alpha,\beta)\sigma^{-1}(\alpha,\beta) \ .$$

Then the final two results of section 3 show that two multipliers satisfying this relationship will produce isomorphic algebras under (3.1) (providing again that Γ is finitely-generated).

As a result of the above remarks we shall, for the rest of the section, restrict ourselves to the K^σ satisfying (4.4).

We now consider representations of colour algebras and it is here that the Klein transformation proves its usefulness.

Proposition 4.1. *Suppose we have a representation of a colour algebra* $\mathbf{A}$, *then it is possible to imbed this representation in a representation of the extended algebra* $<\mathbf{A}, K^\sigma>$. *In otherwords, the representation for* $\mathbf{A}$ *can provide a representation for* K^σ *as well.*

Proof: Let V be the graded vector-space upon which the representation $r(\mathbf{A})$ of $\mathbf{A}$ acts. Now we define $r(K^\sigma)$ as follows

$$r(K^\sigma(\beta))v_\alpha \equiv \sigma(\alpha,\beta)v_\alpha \qquad \forall\ v_\alpha \in V \ . \tag{4.7}$$

From this it follows that

$$r(a_\beta)r(K^\sigma(\gamma))v_\alpha = \sigma(\alpha,\gamma)r(a_\beta)v_\alpha$$

and

$$r(K^\sigma(\gamma))r(a_\beta)v_\alpha = \sigma(\beta+\alpha,\gamma)r(a_\beta)v_\alpha$$

because $r(a_\beta)v_\alpha$ has grading $\alpha+\beta$. We therefore conclude using (4.4) that

$$r(K^\sigma(\gamma))r(a_\beta) = \sigma(\beta,\gamma)r(a_\beta)r(K^\sigma(\gamma)) \ .$$

Also from above

$$r(K^\sigma(0))v_\alpha = \sigma(\alpha,0)v_\alpha = v_\alpha \ ,$$

and

$$\begin{aligned} r(K^\sigma(\alpha))r(K^\sigma(\beta))v_\gamma &= \sigma(\gamma,\alpha)\sigma(\gamma,\beta)v_\gamma \\ &= \sigma(\gamma,\alpha+\beta)v_\gamma \\ &= r(K^\sigma(\alpha+\beta))v_\gamma \ , \end{aligned}$$

which demonstrates the proposition. ∎

The concept of *irreducibility* of representations of colour algbras is the usual one for algebras: A representation is irreducible if the graded vector-space V upon which the representation r($\mathbf{A}$) is defined contains no non-trivial proper subspace U such that $r(\mathbf{A})U \subseteq U$. The following lemma now follows easily:

Lemma 4.2. *The Klein transformation of a representation of a colour algebra given by Proposition 4.1, preserves irreducibility.*

Proof: Suppose the Klein transformation of an irreducible representation was reducible. Then it follows that there exists a non-trivial $U \subset V$ such that

$$r(K^\sigma(-\alpha))r(a_\alpha)U \subseteq U \qquad \forall\ a_\alpha \in \mathbf{A} \ .$$

It then follows from the definition of $r(K^\sigma)$ and the non-zero nature of σ that

$$r(a_\alpha)U \subseteq U \qquad \forall\ a_\alpha \in \mathbf{A}$$

which is a contradiction. ∎

Schur's lemma holds for superalgebras (see Kac [24]) and we can now conclude that it holds for arbitrary colour algebras:

Lemma 4.3. *Let $r(\mathbf{A})$ be a finite-dimensional irreducible representation of the colour algebra $\mathbf{A}$ acting on the graded vector-space V and suppose s is a homomorphism $V \to V$ satisfying*

$$(i) \quad s(V_\alpha) \subset V_\alpha \qquad \forall\ a_\alpha \in \mathbf{A}$$

$$(ii) \quad sr(a_\alpha) = r(a_\alpha)s \qquad \forall\ a_\alpha \in \mathbf{A} \ ,$$

*then s is a multiple of the identity.**

Proof: By the results of section 3 there exists a σ such that $r(K^\sigma)r(\mathbf{A})$ forms a finite-dimensional representation of a superalgebra which, according to Lemma 4.2, is irreducible. Furthermore the definition of $r(K^\sigma)$ together with condition (i) ensures that $sr(K^\sigma) = r(K^\sigma)s$ which means, by condition (ii), that

$$sr(K^\sigma(-\alpha))r(a_\alpha) = r(K^\sigma(-\alpha))r(a_\alpha)s\ .$$

Hence using Schur's lemma for superalgebras we have the desired result. ∎

We can now deduce the following result:

Proposition 4.4. *Every representation of a colour algebra* $\mathbf{A}$ *is given by a Klein transformation of a representation of a superalgebra. Furthermore when the representation is irreducible and finite-dimensional this transformation is unique (up to a scalar multiple).*

Proof: Let $r(a_\alpha)$ be a representation of the colour algebra then, as in the proof of Lemma 4.3, there exists a σ such that $r(K^\sigma(-\alpha))r(a_\alpha)$ is a representation of a superalgebra. Now defining $r(K^{\sigma'}(\alpha)) \equiv r(K^\sigma(-\alpha))$ we have that $\sigma'(\alpha,\beta) = \sigma(\alpha,-\beta)$ and so by (4.4), σ' is a multiplier. It follows trivially that $r(K^{\sigma'}(\alpha))$ are Klein operators for the representation of the superalgebra and furthermore the Klein transformation of this representation is, using (4.2), just

$$r(K^{\sigma'}(-\alpha))r(K^\sigma(-\alpha))r(a_\alpha) = r(a_\alpha)\ ,$$

or, in otherwords, the original representation of the colour algebra.

For the second part let $r(K^\sigma)$ and $r(L^\sigma)$ be two different representations of Klein operators, with the same multiplier, on a finite-dimensional irreducible representation of a superalgebra. By the use of the definition of Klein operators (4.2) and (4.3) we have

$$r(K^\sigma(\alpha))r(a_\beta) = \sigma(\beta,\alpha)r(a_\beta)r(K^\sigma(\alpha))$$

and

* A more general result, with $s_\alpha(v_\beta) \subset V_{\alpha+\beta}$, may be possible — see Kac [24] for the superalgebra case.

$$r(L^{\sigma}(\alpha))r(a_{\beta}) = \sigma(\beta,\alpha)r(a_{\beta})r(L^{\sigma}(\alpha)) \ ,$$

or

$$\begin{aligned} r(L^{\sigma}(-\alpha))r(a_{\beta}) &= \sigma(\beta,-\alpha)r(a_{\beta})r(L^{\sigma}(-\alpha)) \\ &= \sigma^{-1}(\beta,\alpha)r(a_{\beta})r(L^{\sigma}(-\alpha)) \end{aligned}$$

when (4.4) is used;

$$\Rightarrow \quad r(L^{\sigma}(-\alpha))r(K^{\sigma}(\alpha))r(a_{\beta}) = r(a_{\beta})r(L^{\sigma}(-\alpha))r(K^{\sigma}(\alpha)) \ .$$

Since Klein operators have grading 0, we can apply Lemma 4.3 and conclude that

$$r(K^{\sigma}(\alpha)) = k \times r(L^{\sigma}(\alpha)) \qquad k \in C \ ,$$

which concludes the proof of the proposition. ■

It should be noted that the correspondence between representations given here by Klein tansformations has also been given by Scheunert [26] in a different way.

In the case where finite-dimensional representations are being considered, we have two possibilities.

(i) The representation is *completely* reducible, in which case Schur's lemma may be applied to each irreducible component of the representation and so Klein transformations may differ by scalar multiples on each component.

(ii) The representation is *incompletely* reducible. This situation applies only when the canonical superalgebra has a *non-trivial* Z_2 grading (see, for example, Scheunert [37]). The question of the uniqueness of the Klein transformation (up to *equivalence*) is open in this case because Schur's lemma is of no help.

There is another notion of uniqueness that is broader than the one we have considered here. Let us suppose that we have two sets of Klein operators $K^{\sigma}(\alpha)$ and $K^{\sigma'}(\alpha)$. As we have seen in section 3 and the beginning of this section, if $\sigma\sigma'^{-1}$ is trivial as a multiplier then the two Klein transformations will give rise to isomorphic colour algebras. On the assumption that this is the case here, we now consider representations of a colour algebra and the two sets of Klein operators. The two different Klein transformations will produce different representations of the same colour algebra. It is of interest then to know whether these representations are equivalent or not.

This question remains at present unsolved however the answer appears likely to be affirmative for the following reasons:

Consider a basis for our representations which has all its elements with a definite grading. This is certainly possible due to equation (1.1). It is clear that the only graded elements of the untransformed colour algebra with non-zero diagonal elements are those with grading 0 (this is a consequence of (1.9)).

Now due to the second equation of (4.2) and (4.5) these elements will be left invariant by both Klein transformations. In the case of finite-dimensional representations this implies that the characters of all elements in the transformed colour algebra will be the same in both representations.

On the assumption that the usual character theory for algebras [38] can be adapted to colour algebras, we can use the result that equal characters imply equivalent representations to derive the required result.

Obviously the above argument is incomplete and intuitive. It would be useful to prove it in the general situation without resort to character theory which applies usefully only in the case of finite-dimensional representations.

CHAPTER 3

MODULAR FIELD THEORY

In this chapter it is proposed to study a scheme of quantization introduced by Green in 1975 [27]. As was mentioned in the introduction, such a scheme is of interest because, like its parafield counterpart, it has an ansatz solution with the ansatz fields forming a colour algebra. We begin the chapter with a brief review of the basic features of the best known form of generalized quantization, namely parafield theory.

1. Basics of parafield theory

The parafield $\psi_\alpha(x)$ is assumed to satisfy the following equal-time* commutation relations:

$$\Big[\psi_\alpha(x_1), \big[\psi^*_\beta(x_2), \psi_\gamma(x_3)\big]_\mp\Big]_- = 2\delta(\mathbf{x}_1 - \mathbf{x}_2)\delta_{\alpha\beta}\psi_\gamma(x_3) \tag{1.1a}$$

$$\Big[\psi_\alpha(x_1), \big[\psi_\beta(x_2), \psi_\gamma(x_3)\big]_\mp\Big]_- = 0 \ . \tag{1.1b}$$

The subscripts refer to either spinor or vector indices and are omitted in the case of a scalar field. The $\mp$ indicates two different kinds of quantization known as parafermi and parabose quantization respectively.

The motivation behind (1.1a) lies in the Heisenberg principle which states that the energy-momentum operator P_μ must satisfy

$$[P_\mu, \psi_\alpha(x)]_- = -i\psi_{\alpha,\mu}(x) \ . \tag{1.2}$$

If one sets the energy-momentum operator equal to

$$P_\mu = i\int \textstyle\sum_\alpha \left[\psi^*_\alpha(x), \psi_{\alpha,\mu}(x)\right]_\mp d^3x \ , \tag{1.3}$$

then equation (1.1a) can be used to derive (1.2).

---

* In future bold face spatial indices within a delta function will imply that we are considering equal times

As one might expect the usual fermi and bose commutation relations are solutions to the parafermi and parabose relations respectively. A more general class of solutions can be given with the aid of the so-called *Green ansatz* [2]. This involves introducing an ancilliary set of ansatz fields $\phi_\alpha^{(r)}(x)$ $(r = 1, \ldots, p)$ which satisfy the following anomolous commutation relations:

$$\begin{gathered}\left[\phi_\alpha^{*(r)}(x_1), \phi_\beta^{(r)}(x_2)\right]_{\pm} = \delta_{\alpha\beta}\delta(\mathbf{x}_1 - \mathbf{x}_2) \\ \left[\phi_\alpha^{*(r)}(x_1), \phi_\beta^{(s)}(x_2)\right]_{\mp} = \left[\phi_\alpha^{(r)}(x_1), \phi_\beta^{(s)}(x_2)\right]_{\mp} = \left[\phi_\alpha^{(r)}(x_1), \phi_\beta^{(r)}(x_2)\right]_{\pm} = 0\end{gathered} \tag{1.4}$$

where $r \neq s$ and the upper signs refer to parafermions and the lower signs to parabosons. Compare these relations with the colour algebra given by (1.5) Chapter 1; these are the relations satisfied by the creation and annihilation operators in the discrete momentum representation of $\phi_\alpha^{(r)}(x)$. The ansatz solution then involves setting

$$\psi_\alpha(x) = \sum_{r=1}^{p} \phi_\alpha^{(r)}(x) \ . \tag{1.5}$$

It is a straightforward matter to confirm that (1.4) and (1.5) imply (1.1). The index p in (1.5) is referred to as the order of paraquantization and it is easy to see that for $p = 1$ (1.4) and (1.5) give the normal bose and fermi quantizations.

The significance of the ansatz solution becomes apparent when one considers Fock representations for (1.1). In this case Greenberg and Messiah [8] have shown that the ansatz provides *all* solutions to the relation (1.1). In fact, the order p of the paraquantization can be obtained independently of the ansatz via the relation

$$a_k a_l^* |\rangle = \delta_{kl} p |\rangle \ , \tag{1.6}$$

which holds in all Fock representations (the a_k and a_l^* are the annihilation and creation operators respectively).

For the case $p = 2$ a self-contained set of commutation relations are possible for the paraquantization:

$$\begin{aligned}&\psi_\alpha(x_1)\psi_\beta^*(x_2)\psi_\gamma(x_3) \pm \psi_\gamma(x_3)\psi_\beta^*(x_2)\psi_\alpha(x_1) \\ &\qquad\qquad = 2\delta(\mathbf{x}_1 - \mathbf{x}_2)\delta_{\alpha\beta}\psi_\gamma(x_3) \pm 2\delta(\mathbf{x}_2 - \mathbf{x}_3)\delta_{\beta\gamma}\psi_\alpha(x_1) \\ &\psi_\alpha^*(x_1)\psi_\beta(x_2)\psi_\gamma(x_3) \pm \psi_\gamma(x_3)\psi_\beta(x_2)\psi_\alpha^*(x_1) = 2\delta(\mathbf{x}_1 - \mathbf{x}_2)\delta_{\alpha\beta}\psi_\gamma(x_3) \\ &\psi_\alpha(x_1)\psi_\beta(x_2)\psi_\gamma(x_3) \pm \psi_\gamma(x_3)\psi_\beta(x_2)\psi_\gamma(x_1) = 0 \ .\end{aligned} \tag{1.7}$$

These relations can easily be shown to imply (1.1). In addition the hermitian conjugate of the first of the equations, when written in the momentum representation, yields

$$a_k^* a_l a_m^* \pm a_m^* a_l a_k^* = 2\delta_{lm} a_k^* \pm 2\delta_{kl} a_m^* \ . \tag{1.8}$$

When this is applied to the vacuum state, (1.6) implies that $p = 2$.

Consider now an arbitrary state

$$a_{k_1}^* a_{k_2}^* \dots a_{k_n}^* |\rangle \tag{1.9}$$

in order 2 paraquantization. The hermitian conjugate of the third equation of (1.7), when written in the momentum representation, is

$$a_k^* a_l^* a_m^* = \mp a_m^* a_l^* a_k^* \ . \tag{1.10}$$

When this is applied to (1.9) one concludes that there are two different species of particles in the state (1.9) which are fermions or bosons (depending on whether parafermi or parabose relations are under consideration) amongst themselves. The particles with momentum $k_1, k_3, \dots$ belong to one species and those with $k_2, k_4, \dots$ belong to the other. Unfortunately the above interpretation of paraquantization does not extend to higher order quantizations. In these cases the relations corresponding to (1.7) become quite complicated [39] thus precluding such a simple interpretation. This is part of the motivation for modular quantization, whose commutation relations are an obvious generalization of (1.7). We now examine this quantization.

2. Basics of modular field theory

2.1. Introduction

The basic commutation relations we shall adopt for modular quantization of order m are

$$\begin{aligned}
&\psi_\alpha(x_1)\psi^*_\beta(x_2)\psi_\gamma(x_3) \pm \psi_\gamma(x_3)\psi^*_\beta(x_2)\psi_\alpha(x_1) \\
&\qquad\qquad = \delta(\mathbf{x}_1-\mathbf{x}_2)\delta_{\alpha\beta}\psi_\gamma(x_3) \pm \delta(\mathbf{x}_2-\mathbf{x}_3)\delta_{\beta\gamma}\psi_\alpha(x_1) \\
&\psi^*_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\ldots\psi_{\alpha_{m+1}}(x_{m+1}) \pm \psi_{\alpha_3}(x_3)\ldots\psi_{\alpha_{m+1}}(x_{m+1})\psi_{\alpha_2}(x_2)\psi^*_{\alpha_1}(x_1) \\
&\qquad\qquad = \delta(\mathbf{x}_1-\mathbf{x}_2)\delta_{\alpha_1\alpha_2}\psi_{\alpha_3}(x_3)\ldots\psi_{\alpha_{m+1}}(x_{m+1}) \\
&\psi_{\alpha_1}(x_1)\ldots\psi_{\alpha_{m+1}}(x_{m+1}) \pm \psi_{\alpha_{m+1}}(x_{m+1})\psi_{\alpha_2}(x_2)\ldots\psi_{\alpha_m}(x_m)\psi_{\alpha_1}(x_1) = 0\ .
\end{aligned} \tag{2.1}$$

Apart from a factor of $\sqrt{2}$ it is clear that the commutation relations for modular quantization of order two are identical with those for paraquantization of the same order (equations (1.7)). For the case $m=1$, (2.1) reduces to the usual fermi and bose commutation relations with the first equation then being redundant. It is also clear that the third equation of (2.1) implies that

$$a^*_{k_1}a^*_{k_2}\ldots a^*_{k_m}a^*_{k_{m+1}} = \mp a^*_{k_{m+1}}a^*_{k_2}\ldots a^*_{k_m}a^*_{k_1} \tag{2.2}$$

which shows that modular quantization provides a generalization of the two species interpretation of paraquantization of order two and we may interpret modular quantization of order m as describing m different species of particles which are fermions or bosons amongst themselves.

Another way of introducing modular quantization (and the way initially chosen by Green [27]) is to introduce a unitary operator u satisfying

$$u^m = 1 \tag{2.3}$$

and then define a superscript on the $\psi_\alpha(x)$ via

$$\psi^{(t)}_\alpha(x) = u^{-t}\psi_\alpha(x)u^t\ . \tag{2.4}$$

The commutation relations are then assumed to take the form

$$\begin{aligned}
&\psi^{*(r)}_\alpha(x_1)\psi^{(s)}_\beta(x_2) \pm \psi^{(s+1)}_\beta(x_2)\psi^{*(r+1)}_\alpha(x_1) = \delta(\mathbf{x}_1-\mathbf{x}_2)\delta_{\alpha\beta}\delta^{rs} \\
&\psi^{(r)}_\alpha(x_1)\psi^{(s)}_\beta(x_2) \pm \psi^{(s-1)}_\beta(x_2)\psi^{(r+1)}_\alpha(x_1) = 0\ .
\end{aligned} \tag{2.5}$$

It is quite straightforward to show that (2.5) implies (2.1)— by simple substitution and use of the relations (2.5). As was pointed out by Green [27], one of the advantages of this formalism is the possibility of defining a time-ordering in a simple way. Thus, for example, the time-ordering of $\psi^*_\alpha(x_1)\psi_\beta(x_2)$ would be

$$\begin{aligned} T\left(\psi^*_\alpha(x_1)\psi_\beta(x_2)\right) &= \psi^*_\alpha(x_1)\psi_\beta(x_2) \quad t_1 > t_2 \\ &= \mp\psi^{(1)}_\beta(x_2)\psi^{*(1)}_\alpha(x_1) \quad t_2 > t_1 \\ &= \tfrac{1}{2}\left(\psi^*_\alpha(x_1)\psi_\beta(x_2) \mp \psi^{(1)}_\beta(x_2)\psi^{*(1)}_\alpha(x_1)\right) \quad t_1 = t_2 \ . \end{aligned} \tag{2.6}$$

The question of whether (2.5) is implied by (2.1) is quite difficult and not yet fully resolved. We consider now several aspects of it:

The first clue to an approach that might be followed is provided by an ansatz solution to (2.5). This is obtained through the following *non-singular* linear transformation of the $\psi^{(r)}_\alpha(x)$:

$$\phi^{(r)}_\alpha(x) \equiv \frac{1}{\sqrt{m}}\sum_{s=0}^{m-1}\eta^{-rs}\psi^{(s)}_\alpha(x) \ , \tag{2.7}$$

where η is the m'th primitive root of unity. When the inverse of this transformation is taken one is able to show that

$$\psi_\alpha(x) = \frac{1}{\sqrt{m}}\sum_{r=0}^{m-1}\phi^{(r)}_\alpha(x) \ . \tag{2.8}$$

In addition one can use (2.7) and (2.5) to derive the following relations for the $\phi^{(r)}_\alpha(x)$:

$$\phi^{(r)}_\alpha(x_1)\phi^{(s)}_\beta(x_2) \pm \eta^{r-s}\phi^{(s)}_\beta(x_2)\phi^{(r)}_\alpha(x_1) = 0 \tag{2.9a}$$

$$\phi^{*(r)}_\alpha(x_1)\phi^{(s)}_\beta(x_2) \pm \eta^{s-r}\phi^{(s)}_\beta(x_2)\phi^{*(r)}_\alpha(x_1) = \delta_{\alpha\beta}\delta(\mathbf{x}_1-\mathbf{x}_2)\delta^{rs} \tag{2.9b}$$

$$u^{-r}\phi^{(s)}_\alpha(x)u^r = \eta^{rs}\phi^{(s)}_\alpha(x) \ . \tag{2.9c}$$

The first two equations of (2.8) define a colour algebra: The grading group is $Z_m \oplus Z_m(\oplus Z_2)$, (the Z_2 summand being added when we consider the fermi modular quantization) with the gradings of the algebra being assigned as follows:

$$\begin{aligned} \phi^{(r)}_\alpha(x) &\longrightarrow (r,1,1) \\ \phi^{*(r)}_\alpha(x) &\longrightarrow (-r,-1,1) \\ \delta(\mathbf{x}_1-\mathbf{x}_2) &\longrightarrow (0,0,0) \ , \end{aligned} \tag{2.10}$$

while the commutation factor is given by

$$\epsilon(\alpha,\beta) = \eta^{\pi(\alpha,\beta)}(-1)^{\alpha_3\beta_3} \tag{2.11}$$

$$\pi(\alpha,\beta) = \alpha_1\beta_2 - \alpha_2\beta_1 \ , \tag{2.12}$$

where α_1, α_2 and α_3 are the projections of α onto the subgroups Z_m, Z_m and Z_2 of the grading group.

It is instructive to compare the relations (2.8) and (2.9) with those of (1.5) and (1.4). Obviously there is an analogy between the para and modular quantizations in that each has an ansatz solution with the component fields of the ansatz being, in both cases, elements of a colour algebra. Clearly because of the invertible nature of the transformation (2.7) the ansatz solution provides a complete solution to the relations (2.5). Whether the ansatz provides a complete solution to (2.1) in the case of Fock representations (as it does for (1.1) in the para case) is an open question. A partial answer is provided in the next subsection.

2.2. Fock representations and the modular ansatz

We shall confine our attention, for the present, to fermi modular quantization as the bose case appears somewhat more difficult. An outline of these difficulties and a comparison with the fermi case may be found at the end of this subsection.

A further complication arises when the possibility of anti-particles is considered. In this subsection and the next we shall assume a non-relativistic theory in the sense that the spatial wavefunction consists only of creation or annihilation operators. The relativistic complications are discussed in subsection 2.4 below.

In addition, for convenience we shall, for the rest of this section, work with the discrete momentum representation of the modular fields. In particular we shall consider the *modular ring* $\mathcal{A}$ to be finite linear combinations of monomials in the elements a_k and a_k^*; these latter elements will be assumed to satisfy the momentum analogs of (2.1):

$$\begin{gathered} a_j a_k^* a_l + a_l a_k^* a_j = \delta_{jk} a_l + \delta_{kl} a_j \\ a_{k_1}^* a_{k_2} a_{k_3} \ldots a_{k_{m+1}} + a_{k_3} \ldots a_{k_{m+1}} a_{k_2} a_{k_1}^* = \delta_{k_1 k_2} a_{k_3} \ldots a_{k_{m+1}} \\ a_{k_1} a_{k_2} \ldots a_{k_m} a_{k_{m+1}} + a_{k_{m+1}} a_{k_2} \ldots a_{k_m} a_{k_1} = 0 \ . \end{gathered} \tag{2.13}$$

Likewise we shall consider the *modular ansatz ring* $\mathcal{B}$ to be finite linear combinations of products of $b_k^{(r)}$ and $b_k^{*(r)}$; with these latter elements satisfying the discrete momentum analogs of (2.9):

$$\begin{aligned} b_j^{(r)} b_k^{(s)} + \eta^{r-s} b_k^{(s)} b_j^{(r)} &= 0 \\ b_j^{*(r)} b_k^{(s)} + \eta^{s-r} b_k^{(s)} b_j^{*(r)} &= \delta_{jk}\delta^{rs} \\ u^{-r} b_k^{(s)} u^r &= \eta^{rs} b_k^{(s)} \ . \end{aligned} \tag{2.14}$$

The *ansatz solution ring* $\mathcal{A}' \subseteq \mathcal{B}$ will be the subring generated by a'_j and a'^*_j which are given by

$$a'_j = \frac{1}{\sqrt{m}} \sum_{r=0}^{m-1} b_j^{(r)} \ . \tag{2.15}$$

Finally we shall consider, as usual, a Fock representation of, for instance, $\mathcal{A}$ to be a homomorphic mapping h of $\mathcal{A}$ into the ring of operators on a Hilbert space. This space shall possess a unique *vacuum* state which satisfies the usual relation $h(a_j)|\rangle = 0$. Moreover this state shall be *cyclic* with respect to the representation. In otherwords $h(\mathcal{A})|\rangle$ shall be dense in the Hilbert space. We call the Hilbert space the *Fock-space* and denote it symbolically by $\mathcal{F}(\mathcal{A})$.

As a partial solution to the problem posed at the end of the last subsection we have

Theorem 2.1. *If a Fock representation of the modular ring $\mathcal{A}$ satisfies**

$$a_{k_1} \ldots a_{k_n} a^*_{j_n} \ldots a^*_{j_1} |\rangle = \delta_{k_1 j_1} \ldots \delta_{k_n j_n} |\rangle \ , \tag{2.16}$$

for all $n \leq m$, then it is irreducible and unitarily equivalent to the Fock representation of the ansatz solution ring $\mathcal{A}'$.

Proof: We begin the proof with the following technical Lemma:

Lemma 2.2. *If $\phi \equiv a^*_{k_1} a^*_{k_2} \ldots a^*_{k_r}|\rangle$ for $r < m$ then $a_j a^*_k \phi = \delta_{jk}\phi$.*

Proof: For $r = m-1$ the result is immediate due to the second equation of (2.13). For $r < m-1$ consider firstly the case of $j = k$. The first equation of (2.13) shows that

$$a^*_j a_j a^*_j \phi = a^*_j \phi \ . \tag{2.17}$$

---

* for notational ease we write a_j for $h(a_j)$ and so on.

If we take the scalar product of both sides of this equation with the state $a_j^*\phi$, we obtain

$$\|a_j a_j^*\phi\|^2 = \|a_j^*\phi\|^2 = (\phi, a_j a_j^*\phi) \ . \tag{2.18}$$

Now if we assume that the vacuum state is normalized then we may use (2.16) to conclude that for $r < m-1$, which we are assuming,

$$\|a_j^*\phi\|^2 = (|\rangle, a_{k_r} \dots a_{k_1} a_j a_j^* a_{k_1}^* \dots a_{k_r}^* |\rangle) = 1 \ .$$

In an identical manner we deduce that $\|\phi\|^2 = 1$. Now from (2.18) we conclude that

$$1 = |(\phi, a_j a_j^*\phi)| = \|\phi\| \ \|a_j a_j^*\phi\| \ ,$$

and hence the Cauchy-Schwartz inequality demonstrates that

$$a_j a_j^*\phi = \alpha\phi \qquad \alpha \in C \ .$$

But (2.18) then shows that

$$1 = (\phi, a_j a_j^*\phi) = \alpha\|\phi\|^2 = \alpha$$

and so we have demonstrated the case $j = k$. For $j \neq k$ we have from the first of (2.13) that

$$a_j^* a_j a_k^* + a_k^* a_j a_j^* = a_k^* \ .$$

We now apply this equation to the state ϕ and take the inner product of the resulting state with $a_k^*\phi$ obtaining

$$\|a_j a_k^*\phi\|^2 + (a_j a_j^*\phi, a_k a_k^*\phi) = (\phi, a_k a_k^*\phi) \ ,$$

and using the results derived above, we immediately have

$$\|a_j a_k^*\phi\|^2 = 0 \quad \Rightarrow \quad a_j a_k^*\phi = 0 \ . \qquad \blacksquare$$

Consider now a state of the form

$$\phi = a_{k_1^i}^* \dots a_{k_1^m}^* a_{k_2^1}^* \dots a_{k_2^m}^* \dots a_{k_r^1}^* \dots a_{k_r^m}^* |\rangle \quad i \geq 1 \ . \tag{2.19}$$

This is an arbitrary product of creation operators applied to the vacuum. We shall say that particles with momentum k_s^j belong to *class* j. Notice that this is consistent with our interpretation of modular particles given previously. This interpretation is given further weight by the following:

Lemma 2.3. *The state ϕ will have norm 1 if and only if all the k^j_s for the class j are different; otherwise it vanishes. Furthermore a state ϕ' will be orthogonal to ϕ if and only if it possesses a particle in any class j with momentum different to any of the k^j_s for ϕ; otherwise $\phi' = \pm\phi$.*

Proof: If any of the k^j_s are the same, the vanishing of ϕ follows immediately from the first equation of (2.13).

Now consider the state

$$\begin{aligned} \pi = & a_{l^m_r} a_{l^{m-1}_r} \dots a_{l^1_r} \dots a_{l^m_1} \dots a_{l^i_1} \\ & . a^*_{k^i_1} \dots a^*_{k^m_1} a^*_{k^1_2} \dots a^*_{k^m_r} |\rangle \ ; \end{aligned} \tag{2.20}$$

we can move the $a_{l^i_1}$ to the right by use of the second of (2.13). After repeated use of this identity and finally with the use of Lemma 2.2 we obtain

$$\begin{aligned} \pi = \sum_{v=1}^{r} (-1)^{v+1} \delta_{l^i_1 k^i_v} a_{l^m_r} & \dots a_{l^{i+1}_1} a^*_{k^{i+1}_1} \dots a^*_{k^{i-1}_2} a^*_{k^i_1} a^*_{k^{i+1}_2} \\ & \dots a^*_{k^1_v} \dots a^*_{k^{i-1}_v} a^*_{k^i_{v-1}} a^*_{k^{i+1}_v} \dots a^*_{k^1_{v+1}} \dots \\ & a^*_{k^{i-1}_{v+1}} a^*_{k^i_{v+1}} a^*_{k^{i+1}_{v+1}} \dots a^*_{k^m_r} |\rangle \ . \end{aligned} \tag{2.21}$$

Now if $l^j_t = k^j_t$, $\forall j, t$, then it is clear that

$$\|\phi\|^2 = (|\rangle, \pi) \ ,$$

what is more, if all the momenta in class i are different only the first term in (2.21) will survive. This argument can be continued iteratively until we conclude that if all the momenta are different in all the classes then $\pi = |\rangle$; the final step in the argument follows from the equation (2.16). The first part of the lemma now follows trivially.

For the second part, if the state ϕ' has more particles than ϕ then either it will vanish or at least one of the momenta in one of the classes will be different to all the momenta in the same class in ϕ. In any case repeated use of the second equation of (2.13) will, a la (2.21), eventually show that $(\phi', \phi) = 0$. We need thus only consider ϕ' with the same number of particles. In this case we have $(\phi', \phi) = (|\rangle, \pi)$. Now in the class i if $l^i_1 \neq k^i_s$, $\forall s$ then this will be zero by (2.21).

In (2.21) we may interchange, using the first of (2.13), the $a_{l_1^i}$ with $a_{l_t^i}$ with only a change of sign to π and so we conclude that if any of the $l_t^i \neq k_s^i$, $\forall s$ then the inner product vanishes. Thus our result holds for the class i. To extend to the other classes is simply a matter of continuing the reduction began in (2.21) and this is, in principle, straightforward. ■

The following result is quite important, not only to the proof but also to arguments used later in the chapter.

Lemma 2.4. *There exists a subset of states in $\Omega = \{a_{k_1}^* \dots a_{k_n}^* |\rangle\}$ which provides a complete basis for $\mathcal{F}(\mathcal{A})$.*

Proof: Consider $z \in \mathcal{A}$ to be an arbitrary product of creation and annihilation elements, then $h(z)|\rangle$ may be rewritten, with the aid of Lemma 2.2 and the second of (2.13), as $h(z')|\rangle$ where z' is a linear combination of *only* creation elements.* It is thus clear that $h(\mathcal{A})|\rangle = h(\mathcal{D})|\rangle$ where $\mathcal{D} \subset \mathcal{A}$ is the ring of finite linear combinations of products of creation elements. Further Lemma 2.3 shows in an obvious way how an orthonormal basis for $h(\mathcal{D})|\rangle$ may be constructed from among the elements of Ω. Now since $h(\mathcal{A})|\rangle$ is dense in $\mathcal{F}(\mathcal{A})$ so is $h(\mathcal{D})|\rangle$ and thus from a standard result [41], the above basis is complete for $\mathcal{F}(\mathcal{A})$. ■

Denoting this basis by $v_i^*|\rangle$, with $v_0^* = 1$, $v_i^* \in \mathcal{D}$, it follows, again from [41], that any $\phi \in \mathcal{F}(\mathcal{A})$ can be written as

$$\phi = \sum_i \alpha_i v_i^* |\rangle \ . \tag{2.22}$$

Consider now an operator V which commutes with $h(\mathcal{A})$. Clearly

$$V\phi = \sum_i \alpha_i v_i^* V |\rangle \ ; \tag{2.23}$$

but from the definition of the vacuum we have

$$0 = V a_k |\rangle = a_k V |\rangle \quad \forall k \ ; \tag{2.24}$$

now $V|\rangle$ must be expressible in terms of the basis $v_i^*|\rangle$:

$$V|\rangle = \sum_i \beta_i v_i^* |\rangle \ . \tag{2.25}$$

* The same thing is possible in paraquantization, see [40]

Now from the proof of Lemma 2.3 it is clear that $a_k v_i^* |\rangle = 0$ unless the last created particle in $v_i^* |\rangle$ *or* one belonging to the same class, has momentum k. In this case a little consideration of Lemma 2.3 will show that $a_k v_i^* |\rangle = \pm v_j^* |\rangle$ for some j *and* for every different v_i^* there is a different v_j^*. Hence, applying a suitable a_k on the left to (2.25) leads, when (2.24) is used, to the conclusion that $\beta_i = 0$ for all i except 0. Thus we have shown that $V|\rangle = \alpha|\rangle$ $\alpha \in C$ and hence by (2.23) that V is a multiple of the identity. Schur's lemma [42] then shows that the Fock representation of $\mathcal{A}$ is irreducible.

We now show that all operators in the Fock representation of $\mathcal{A}$ are bounded: We begin by proving this result for a_k. Consider an arbitrary

$$\phi = \sum \alpha_i v_i^* |\rangle ,$$

now as we have seen above $a_k v_i^* |\rangle = \pm v_j^* |\rangle$ or 0 (with a unique j for every i). Thus we have

$$||a_k \phi|| = \sqrt{\sum |\alpha_i|^2 . P(i)}$$

where $P(i) = 0, 1$. However

$$||\phi|| = \sqrt{\sum |\alpha_i|^2}$$

and so $||a_k \phi|| \leq ||\phi||$. This means that $||a_k|| \leq 1$, and in fact equality holds as can be seen from the following:

$$||a_k(a_k^* |\rangle)|| = |||\rangle|| = 1 = ||a_k^* |\rangle|| .$$

From [43] we now have $||a_k^*|| = 1$, and now consider an arbitrary finite linear combination of monomials in a_k and a_k^*:

$$w = \sum_{j,n} \gamma^j c_{j_1} c_{j_2} \dots c_{j_n} \quad ; \quad c_{j_k} = a_{j_k}, a_{j_k}^* .$$

Now by the use of results from [44] we have

$$\begin{aligned} ||w|| &= || \textstyle\sum_{j,n} \gamma^j c_{j_1} \dots c_{j_n} || \\ &\leq \sum |\gamma^j| || c_{j_1} \dots c_{j_n} || \\ &\leq \sum |\gamma^j| ||c_{j_1}|| \dots ||c_{j_n}|| \\ &= \sum |\gamma^j| \end{aligned}$$

which is finite because the summation is finite, and so we are done.

Consider an arbitrary vacuum expectation value $(|\rangle, \omega|\rangle)$ of an element $\omega \in \mathcal{A}$. This may be rewritten with the aid of Lemma 2.4 as

$$\left(|\rangle, \sum_j \gamma^j v_j^* |\rangle\right) = \gamma^0 . \tag{2.26}$$

Thus what we have shown is that once (2.16) is specified for the Fock representation of $\mathcal{A}$ then all all V.E.V.s are determined. It now follows from [45], using the fact that $|\rangle$ is cyclic, that $\mathcal{A}|\rangle$ is determined up to equivalence.

Consider now a Fock representation of the modular ansatz ring $\mathcal{B}$ with vacuum state $|\rangle'$. Within $\mathcal{B}$ is the ansatz solution ring $\mathcal{A}'$ and one can consider the subspace of $h(\mathcal{B})\,|\rangle'$ generated by applying $h(\mathcal{A}')$ to $|\rangle'$. The closure of this subspace, which lies within $\mathcal{F}(\mathcal{A})$ by [46], is complete again by [46]. Letting $h(\mathcal{A}')$ act on this space it is clear from the definitions given earlier that we have defined a Fock representation of the relations (2.13). To show that this representation is equivalent to the one discussed above, it remains simply to show that

$$a'_{k_1} \dots a'_{k_n} a'^*_{j_n} \dots a'^*_{j_1} |\rangle' = \delta_{k_1 j_1} \dots \delta_{k_n j_n} |\rangle' \quad \forall\, n \leq m . \tag{2.27}$$

Consider the left hand side of this equation: By use of the momentum analogs of equation (2.5) we can shift the operator a'_{k_n} to the right until it is applied to the vacuum. We obtain

$$\begin{aligned} \text{L.H.S.} =& \delta_{k_n j_n} a'_{k_1} \dots a'_{k_{n-1}} a'^*_{j_{n-1}} \dots a'^*_{j_1} |\rangle' \\ &+ (-1)^n a'_{k_1} \dots a'_{k_{n-1}} a'^{*(-1)}_{j_n} \dots a'^{*(-1)}_{j_1} a'^{(-n)}_{k_n} |\rangle' . \end{aligned} \tag{2.28}$$

The second term in (2.28) is zero because of the Fock vacuum condition for the representation of $\mathcal{B}$. We can then reapply the argument above to the first term obtaining

$$\text{L.H.S.} = \delta_{k_n j_n} \delta_{k_{n-1} j_{n-1}} a'_{k_1} \dots a'_{k_{n-2}} a'^*_{j_{n-2}} \dots a'^*_{j_1} |\rangle' \tag{2.29}$$

Obviously this argument may be extended until we obtain the desired result. The proof of Theorem 2.1 is now complete. ■

To complete this subsection we make a few comments concerning the bose-like modular quantization:

It appears difficult to generalize Theorem 2.2 to this case. The basic problem stems from the proof of Lemma 2.2 which does not generalize because it uses the relation (2.17) in an essential way. There is no completely analogous relation in the bose case, as can be seen by setting $k = l = m$ in the relation

$$a_k a_l^* a_m - a_m a_l^* a_k = \delta_{kl} a_m - \delta_{lm} a_k ,$$

which is satisfied by the bose ring. It is possible that in this case further relations apart from simply (2.16) are needed to tie down the Fock representations to the ones given by the ansatz solution.

A further difficult arises because the proof of the uniqueness of the representation specified by equation (2.16) depends, in the fermi case, on a technical result [45] which applies only in the case of bounded operators. This appears to pose problems in the bose case where the operators are unbounded.. However given the fact that there exist general theorems (see [47]) reconstructing boson field theories from their V.E.Vs, this difficulty is not likely to be insummountable.

2.3. Uniqueness of modular ansatz

We now consider the question as to whether Fock representations of (2.13) other than that specified by the condition (2.16) are possible. We shall show that if they are given by an ansatz, such as (2.15), then the answer is negative. This result is significant because, as we shall see in section 4 below, the ansatz enables one to compare the theory with a normal theory of fermions.

We shall make a number of assumptions concerning the possible ansatz solutions. Firstly the solution will be given by the equation

$$a_j = \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} b_j^{(r)} . \tag{2.30}$$

Secondly the $b_j^{(r)}$ will be assumed to satisfy the algebraic relations

$$\begin{aligned} b_j^{(r)} b_k^{(t)} + \epsilon(r,t) b_k^{(t)} b_j^{(r)} &= 0 \\ b_j^{*(r)} b_k^{(t)} + \epsilon(\bar{r},t) b_k^{(t)} b_j^{*(r)} &= \delta_{jk} \delta^{rt} , \end{aligned} \tag{2.31}$$

with ϵ being an *arbitrary* non-zero map into the complex numbers. Notice that this algebra is more general than a colour algebra. Finally we shall assume that the ring $\mathcal{B}'$ generated by $b_j^{(r)}$ and $b_j^{*(r)}$ has a Fock representation. These three assumptions appear to this author to be minimum requirements if a theory is to be compared with a normal bose or fermi theory.

Consider now equation (2.13) with the momentum j_l all distinct. Substitution of (2.30) into this equation gives

$$\frac{1}{\sqrt{N^{m+1}}}\sum_{r_1,\ldots,r_{m+1}=0}^{N-1}\left[b_{j_1}^{(r_1)}\ldots b_{j_{m+1}}^{(r_{m+1})}+b_{j_{m+1}}^{(r_{m+1})}b_{j_2}^{(r_2)}\ldots b_{j_m}^{(r_m)}b_{j_1}^{(r_1)}\right]=0\ .$$

With the repeated use of the first equation of (2.31) this becomes

$$\sum\left[\prod_{l=1}^{m}\epsilon(r_{m+1},r_l)\prod_{l=2}^{m}\epsilon(r_l,r_1)-1\right]b_{j_1}^{(r_1)}\ldots b_{j_{m+1}}^{(r_{m+1})}=0\ .$$

Now consider $\mathcal{F}(\mathcal{B}')$ and the following inner product:

$$\begin{aligned}&\left(b_{j_{m+1}}^{*(r_{m+1})}\ldots b_{j_1}^{*(r_1)}|\rangle,b_{j_{m+1}}^{*(t_{m+1})}\ldots b_{j_1}^{*(t_1)}|\rangle\right)\\&=\left(|\rangle,b_{j_1}^{(r_1)}\ldots b_{j_{m+1}}^{(r_{m+1})}b_{j_{m+1}}^{*(t_{m+1})}\ldots b_{j_1}^{*(t_1)}|\rangle\right)\\&=\prod_{l=1}^{m+1}\epsilon^{-1}(\overline{t_l},r_l)\delta_{r_lt_l}\ ,\end{aligned}$$

where we have used the distinctness of the j_l, and the second equation of (2.31). The non-zero nature of ϵ implies that the state $b_{j_{m+1}}^{*(r_{m+1})}\ldots b_{j_1}^{*(r_1)}|\rangle$ is non-zero and orthogonal to $b_{j_{m+1}}^{*(t_{m+1})}\ldots b_{j_1}^{*(t_1)}|\rangle$ if this state is different. As a consequence of these considerations we deduce that all of the $b_{j_1}^{(r_1)}\ldots b_{j_{m+1}}^{(r_{m+1})}$ are linearly independent and hence that

$$\prod_{l=1}^{m}\epsilon(r_{m+1},r_l)\prod_{l=2}^{m}\epsilon(r_l,r_1)=1\ . \tag{2.32}$$

If we set $r_l=r$ for $2\le l\le m$ in this equation we can deduce that

$$\epsilon(r_{m+1},r_1)\epsilon^{m-1}(r_{m+1},r)\epsilon^{m-1}(r,r_1)=1\ . \tag{2.33}$$

Further, if we set $r_1=r_{m+1}=t$ and note that the first of (2.31) implies that $\epsilon(r,t)=\epsilon^{-1}(t,r)$, we may deduce that

$$\epsilon(t,t)=1 \tag{2.34}$$

and as a consequence $(b_j^{*(r)})^2=0$.

Now by the use of the second of (2.13) with all the j_i distinct except $j_1 = j_p$ for $3 \leq p \leq m+1$, we obtain

$$\sum_{r_1,\ldots,r_{m+1}=0}^{N-1} \left(b_{j_1}^{*(r_1)} b_{j_2}^{(r_2)} \ldots b_{j_{m+1}}^{(r_{m+1})} + b_{j_3}^{(r_3)} \ldots b_{j_{m+1}}^{(r_{m+1})} b_{j_2}^{(r_2)} b_{j_1}^{*(r_1)} \right) = 0 \, .$$

Repeated use of the second of (2.31) now leads to

$$\sum \Big[\Big(1 - \textstyle\prod_{l=2}^{m+1} \epsilon(\overline{r_1}, r_l) \prod_{l=3}^{m+1} \epsilon(r_2, r_l) \Big) b_{j_3}^{(r_3)} \ldots b_{j_{m+1}}^{(r_{m+1})} b_{j_2}^{(r_2)} b_{j_1}^{*(r_1)}$$
$$+ (-1)^p \prod_{q=2}^{p-1} \epsilon(\overline{r_1}, r_q) b_{j_2}^{(r_2)} \ldots b_{j_{p-1}}^{(r_{p-1})} b_{j_{p+1}}^{(r_{p+1})} \ldots b_{j_{m+1}}^{(r_{m+1})} \Big] = 0 \, .$$

If we take the hermitean conjugate of this equation and introduce an obvious abbreviation we obtain

$$\sum \Big[t(r_1, \ldots, r_{m+1}) b_{j_1}^{(r_1)} b_{j_2}^{*(r_2)} b_{j_{m+1}}^{*(r_{m+1})} \ldots b_{j_3}^{*(r_3)}$$
$$+ u(r_1, \ldots, r_{m+1}) b_{j_{m+1}}^{*(r_{m+1})} \ldots b_{j_{p+1}}^{*(r_{p+1})} b_{j_{p-1}}^{*(r_{p-1})} \ldots b_{j_2}^{*(r_2)} \Big] = 0 \, .$$

In $\mathcal{F}(\mathcal{B}')$ we apply this operator to the state $b_{j_1}^{*(w)} |\rangle$ obtaining

$$\sum_{r_p \neq w} t(w, r_2, \ldots, r_{m+1}) k(r_2, \ldots, r_{m+1}) b_{j_2}^{*(r_2)} b_{j_3}^{*(r_3)} \ldots b_{j_{m+1}}^{*(r_{m+1})} |\rangle$$
$$+ \sum u(r_1, \ldots, r_{m+1}) l(r_2, \ldots, r_{m+1}) b_{j_2}^{*(r_2)} \ldots b_{j_{p-1}}^{*(r_{p-1})} b_{j_p}^{*(w)} b_{j_{p+1}}^{*(r_{p+1})} \tag{2.35}$$
$$\times \ldots b_{j_{m+1}}^{*(r_{m+1})} |\rangle \, ,$$

where $k, l \neq 0$ and we have used $(b_{j_1}^{*(w)})^2 = 0$, together with the second of (2.31) applied to the vacuum. From the comments made above, the states $b_{j_2}^{*(r_2)} \ldots b_{j_{m+1}}^{*(r_{m+1})} |\rangle$ in the first sum of (2.35), together with the $b_{j_2}^{*(r_2)} \ldots b_{j_p}^{*(w)} \ldots b_{j_{m+1}}^{*(r_{m+1})} |\rangle$ from the second sum, are all non-zero and orthogonal. This leads to the conclusion that $t = u = 0$ or more explicitly to the equations

$$\prod_{l=2}^{m+1} \epsilon(\overline{r_1}, r_l) \prod_{l=3}^{m+1} \epsilon(r_2, r_l) = 1 \tag{2.36}$$

$$\sum_{r_1=0}^{N-1} \prod_{q=2}^{p-1} \epsilon(\overline{r_1}, r_q) = 0 \, . \tag{2.37}$$

If we set $r_l = r_2$ for $l \geq 2$ in (2.36) and use (2.34) we obtain

$$\epsilon^m(\bar{r}, t) = 1 \ . \tag{2.38}$$

A further equation may now be obtained from (2.36) by setting $r_l = r$ for $l \geq 3$ and using (2.38):

$$\epsilon(\bar{s}, t) = \epsilon(\bar{s}, r)\epsilon(t, r) \ . \tag{2.39}$$

Upon consideration of the second of (2.31) with $r = t$ we are led, after taking the hermitean conjugate, to the conclusion that $(\epsilon(\bar{r}, r))^* = \epsilon(\bar{r}, r)$. When (2.38) is considered this implies that

$$\epsilon(\bar{r}, r) = \pm 1 \ . \tag{2.40}$$

Suppose, for arguments sake, that $\epsilon(\bar{t}, t) = -1$ for some t. Now in $\mathcal{F}(\mathcal{B}')$ we have, from the second of (2.31), that

$$b_j^{(t)} b_k^{*(t)} |\rangle = \epsilon(\bar{t}, t)\delta_{jk}|\rangle = -\delta_{jk}|\rangle \ . \tag{2.41}$$

Next consider the V.E.V. $(|\rangle, b_k^{(t)} b_k^{*(t)} b_j^{(t)} b_j^{*(t)} |\rangle)$ for $j \neq k$. By (2.41) this has value $+1$, however it also equals

$$\begin{aligned}
&(|\rangle, +\epsilon(\bar{t}, t)\epsilon(t, t) b_j^{(t)} b_k^{(t)} b_k^{*(t)} b_j^{*(t)} |\rangle) \\
&= -(|\rangle, b_j^{(t)} b_k^{(t)} b_k^{*(t)} b_j^{*(t)} |\rangle) \qquad \text{(using (2.34))} \\
&= - \| \, b_k^{*(t)} b_j^{*(t)} |\rangle \, \|^2 \ ,
\end{aligned}$$

which certainly cannot equal $+1$ and so we conclude that

$$\epsilon(\bar{r}, r) = +1 \quad \forall r \ . \tag{2.42}$$

From (2.39) we now conclude that

$$\epsilon(\bar{s}, t) = \epsilon(t, s) \ , \tag{2.43}$$

and we may thus restrict our attention to the $\epsilon(\bar{s}, t)$. If we set $r_q = r$ for $q \geq 2$ in (2.37) then we obtain the useful equation

$$\sum_{r_1=0}^{N-1} \epsilon^n(\overline{r_1}, r) = 0 \qquad 1 \leq n \leq m-1 \ . \tag{2.44}$$

Now if we let $n = 1$ and note that (2.38) implies that all the $\epsilon(\overline{r_1}, r)$ must be powers of η, the m'th primitive root of unity, we can conclude that

$$\sum_{l=0}^{m-1} q_l \eta^l = 0 \tag{2.45}$$

where q_l is the number of occurrences of η^l amongst $\epsilon(\overline{r_1}, r)$ for r fixed. In general we have

$$\sum_{l=0}^{m-1} q_l (\eta^n)^l = 0 \qquad 1 \leq n \leq m-1 \ . \tag{2.46}$$

Now, up to a factor of proportionality, there is only one $(m-1)$'th degree polynomial which has *all* the roots of unity (apart from 1) as its roots, namely $\sum_{l=0}^{m-1} x^l$. We therefore conclude that the q_l in (2.45) and (2.36) are all equal to some integer which we call q.

From the above considerations we deduce that amongst the N values that the arguments of ϵ can take on, there must be exactly q sets of m values. These we denote by r^t_j with $t = 0, \ldots, q-1$, and $j = 1, \ldots, m$. By an elementary reordering, they satisfy the equation

$$\epsilon(\overline{r^t_0}, r^w_j) = \eta^{j-1} \ . \tag{2.47}$$

If (2.43) is now substituted into (2.39) we may generalize this to

$$\begin{aligned} \epsilon(\overline{r^t_i}, r^w_j) &= \epsilon(\overline{r^t_i}, r^t_0)\epsilon(\overline{r^t_0}, r^w_j) \\ &= \epsilon^*(\overline{r^t_0}, r^t_i)\epsilon(\overline{r^t_0}, r^w_j) \\ &= \eta^{j-i} \ , \end{aligned} \tag{2.48}$$

where we have taken the hermitean conjugate of the second equation of (2.31) to deduce the second step. Finally we form the new algebraic elements

$$b'^{(i-1)}_j \equiv \frac{1}{\sqrt{q}} \sum_{t=0}^{q-1} b^{(r^t_i)}_j \ . \tag{2.49}$$

By the use of (2.48) and (2.18) it is now straightforward to show that these new elements satisfy the same equations as the first two ansatz equations (2.14); moreover we have

$$a_j = \frac{1}{\sqrt{qm}} \sum_{t=0}^{q-1} \sum_{l=1}^{m} b^{(r^t_l)}_j = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} b'^{(k)}_j \tag{2.50}$$

which is identical to (2.15). As far as the last equation of (2.14) is concerned, it is shown in Appendix B that such a u always exists on a colour algebra satisfying the first two equations of (2.14).

The demonstration of the non-existence of other ansatz-like solutions of the fermi modular relations generalizes in an obvious way to the bose case (only the argument following equation (2.40) needs any significant modification).

We have not, as yet, considered the existence or otherwise of the Fock representation of (2.13) satisfying (2.16). This question is addressed in Appendix B in the context of the Klein transformation.

2.4. The relativistic case

We come now to the important consideration of a relativistic theory. In this case one would expect, as with the usual relativistic theory, that modular fields would be made up of two parts corresponding to positive and negative frequecies. Thus for example, one would write the free spinor modular field as [48]

$$\begin{aligned}\psi(x) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\Big\{&e^{i(\mathbf{k}.\mathbf{x}-Ex_0)}\textstyle\sum_{t=1}^{2} u^t(\mathbf{k})a^t(\mathbf{k})\\ &+e^{i(\mathbf{k}.\mathbf{x}+Ex_0)}\textstyle\sum_{t=3}^{4} u^t(\mathbf{k})c^{t*}(-\mathbf{k})\Big\}\end{aligned} \tag{2.51}$$

where the $u^t(\mathbf{k})$ are the usual Dirac spin components, V is the volume appropriate to the spatial fields, and the operators $a^t(\mathbf{k})$ and $c^{t*}(\mathbf{k})$ are to be interpreted as particle annihilation and anti-particle creation operators respectively.

In order that the relations (2.1) be satisfied by our relativistic spatial field, the relations (2.13) which apply in the non-relativistic case, need to be extended to deal with anti-particle operators. This may be achieved by following the prescription that a creation operator a_k^* in (2.13) may be replaced by an annihilation operator c_k *providing* that Kronecker deltas involving the momentum labels of particles and anti-particles are removed. In otherwords the c_k acts algebraically like a_k^* except with an extra degee of freedom corresponding to its status as an anti-particle operator. Similar comments apply for exchanging a_k with c_k^*. With this prescription the first equation of (2.13) expands to include the following extra equations:

$$
\begin{aligned}
c_j^* c_k c_l^* + c_l^* c_k c_j^* &= \delta_{kl} c_j^* + \delta_{jk} c_l^* \\
c_j^* a_k^* c_l^* + c_l^* a_k^* c_j^* &= 0 \\
c_j^* a_k^* a_l + a_l a_k^* c_j^* &= \delta_{kl} c_j^* \\
c_j^* c_k a_l + a_l c_k c_j^* &= \delta_{jk} a_l \\
a_j c_k a_l + a_l c_k a_j &= 0 \ .
\end{aligned}
\tag{2.52}
$$

Similar extensions occur for the other two equations. It is interesting to note that when $m \geq 3$ these extended equations are non-trivial in the sense that one cannot just consider the anti-particle to be an ordinary particle with an "anti-particle" label. To see this, we observe that the second equation of (2.52) has no counterpart involving just particle operators (except, of course, when $m = 2$). This non-trivial property distinguishes modular field theory from parafield theory where it *is* possible to consider anti-particles as simply ordinary particles with an anti-particle label (see [49]).

The extension of Theorem 2.1 to the relativistic case will present problems when there is a non-trivial extension of the relations (2.13) (the trivial extension in parafield theory can be easily dealt with by introducing an anti-particle label into the condition (2.16)). We simply remark that in the fermi modular case the relations satisfied by the anti-particle operators are identical in form to the relations (2.13). As a result, if we impose the condition (2.16) on anti-particle operators then the proof that the usual ansatz solution is implied by this condition will go through in an identical manner. This shows that the ansatz solution for anti-particles is the same as that for particles. The remaining problem concerns the commutation relations between particle and anti-paticle ansatz operators. It is not clear whether relations such as (2.52) are sufficient to determine these or whether further conditions such as (2.16) need to be imposed for mixtures of particle and anti-particle operators. We leave this question unresolved and merely demonstrate how a solution to the basic relativistic commutation relations of (2.1) may be constructed.

We begin by extending the modular ansatz ring $\mathcal{B}$ to include the elements

$e_k^{(r)}$ and $e_k^{*(r)}$ which we assume to satisfy the relations

$$\begin{aligned} e_j^{(r)} e_k^{(s)} \pm \eta^{r-s} e_k^{(s)} e_j^{(r)} &= 0 \\ e_j^{*(r)} e_k^{(s)} \pm \eta^{s-r} e_k^{(s)} e_j^{*(r)} &= \delta_{jk}\delta^{rs} \\ u^{-r} e_k^{(s)} u^r &= \eta^{-rs} e_k^{(s)} \end{aligned} \tag{2.53}$$

amongst themselves and also the relations

$$\begin{aligned} b_j^{(r)} e_k^{*(s)} \pm \eta^{r-s} e_k^{*(s)} b_j^{(r)} &= 0 \\ b_j^{*(r)} e_k^{*(s)} \pm \eta^{s-r} e_k^{*(s)} b_j^{*(r)} &= 0 \end{aligned} \tag{2.54}$$

with the original elements of $\mathcal{B}$. We now define an ansatz for the anti-particle operator c'_j:

$$c'_j = \frac{1}{\sqrt{m}} \sum_{r=0}^{m-1} e_j^{(r)} \ . \tag{2.55}$$

As in equation (2.4) we may also define

$$\begin{aligned} c_j'^{(r)} &\equiv u^{-r} c' u^r \\ a_j'^{(r)} &\equiv u^{-r} a'_j u^r \ . \end{aligned} \tag{2.56}$$

With these definitions and the equations (2.14), (2.15) and (2.53)—(2.55) we can derive the equations

$$\begin{aligned} c_j'^{(r)} c_k'^{(s)} \pm c_k'^{(s+1)} c_j'^{(r-1)} = a_j'^{*(r)} a_k'^{*(s)} \pm a_k'^{*(s+1)} a_j'^{*(r-1)} &= 0 \\ c_j'^{*(r)} c_k'^{(s)} \pm c_k'^{(s-1)} c_j'^{*(r-1)} = a_j'^{(r)} a_k'^{*(s)} \pm a_k'^{*(s-1)} a_j'^{(r-1)} &= \delta_{kj}\delta^{rs} \\ c_j'^{(r)} a_k'^{*(s)} \pm a_k'^{*(s+1)} c_j'^{(r-1)} = c_j'^{*(r)} a_k'^{*(s)} \pm a_k'^{*(s-1)} c_j'^{*(r-1)} &= 0 \ . \end{aligned} \tag{2.57}$$

Substitution of the expressions (2.15) and the hermitian conjugate of (2.55) into (2.51)* gives us our relativistic modular field. When the relations (2.57) are taken into account it is easily shown by the usual methods [48], that the fields so constructed satisfy (2.5) and hence (2.1). Moreover it is relatively easy to also see that equations (2.57) imply the extended relations such as (2.52).

Finally we come to the question of the *existence* of Fock representations of the relations (2.1). Clearly if we can show the existence of Fock representations of (2.14), (2.53) and (2.54) then this question will be resolved. This latter problem is solved in Appendix B where the Klein transformation again plays a central role.

* With an appropriate addition of a spin index.

3. Observables

3.1. Locality constraints

In ordinary field theory the consideration of what constitutes an observable is far from resolved. As a consequence of this, we shall follow the approach used by Ohnuki and Kamefuchi [29] to consider the analogous problem in parafield theory. This involves using locality conditions to restrict the possible algebraic form of observables.

The essential feature of this approach is that observables are defined in local regions of space. This is achieved as follows: Let g be a function of the fields $\psi(x_1), \psi(x_2), \ldots, \psi^*(y_1), \psi^*(y_2), \ldots$ and h_V a function of $x_1, x_2, \ldots, y_1, y_2, \ldots$ which vanishes if any of it arguments lie spatially outside the region V. An *observable* $F(V)$ for the region V is now defined to be of the form

$$F(V) = \int_{space} h_V g(\psi(x_1), \psi(x_2), \ldots, \psi^*(y_1), \psi^*(y_2), \ldots) dx_1 dx_2 \ldots dy_1 dy_2 \ldots \quad , \tag{3.1}$$

with $x_1, x_2, \ldots, y_1, y_2, \ldots$ having the same time component. In practice we shall possibly require derivatives of the fields in our observables. The complications introduced by this generalization are discussed at the end of this section.

A first requirement of our theory is that measurement of two observables defined at equal times in non-connected regions should be independent. This is simply an expression of the principle of causality and can be achieved through the following equal time equation

$$[F(V), F'(V')]_- = 0 \qquad V \sim V' \ , \tag{3.2}$$

where $V \sim V'$ means that V and V' are disjoint. We shall refer to (3.2) as a condition of *weak locality*.

A stronger condition than (3.2) is the equal time relation

$$[F(V), \hat{\psi}(x)]_- = 0 \qquad x \notin V \ , \tag{3.3}$$

where $\hat{\psi}(x) = \psi(x)$ or $\psi^*(x)$. This relation ensures that measurement of $F(V)$ is unaffected by the existence of particles in regions which cannot have any causal influence* on V. Condition (3.3) shall be referred to as *strong locality*. It is fairly

* See [29] p88 for a more detailed dicussion on this point.

clear that (3.3) implies (3.2), however, as we shall see below, the converse is certainly not true.

We turn now to the particular case of modular quantization. We make the assumption here that the modular fields satisfy the conditions (2.3)–(2.5). In otherwords, we are considering the ansatz solution of the relations (2.1). We also restrict our attention here to the fermi modular quantization. These two assumptions will remain for the rest of this chapter.

It is fairly easy to construct observables from modular fields which obey weak locality. An example is

$$F(V) = \int_{space} h_V(x,y)\psi^*(y)\psi(x)dxdy \ . \tag{3.4}$$

Relations (2.5) easily confirm that $[F(V), F(V')]_- = 0$ for $V \sim V'$. In general however, these observables do not satisfy the condition of strong locality*. In order to consider the form of observables which are strongly local it proves convenient to allow them to be constructed from the ansatz fields $\phi^{*(r)}(x)$ and $\phi^{(r)}(x)$, or equivalently, by (2.7), from the fields $\psi^{*(r)}(x)$ and $\psi^{(r)}(x)$. The following result now holds:

Theorem 3.1. *Observables $F(V)$ constructed from the ansatz fields obey strong locality if and only if*

(i) They are functions of $\phi^{*(r)}(x)\phi^{(t)}(y)$, $\phi^{(r_1)}(x_1)\ldots\phi^{(r_q)}(x_q)$ and its hermitean conjugate; where $q = m$ for m even and $q = 2m$ for m odd.

(ii) $u^{-1}F(V)u = F(V)$.

Proof: We firstly demonstrate the sufficiency of the two conditions: Using a Taylor series expansion of g in (3.1), we may rewrite it, with the aid of (2.5) and a change of variables, as:

$$g(x_1,\ldots,y_1,\ldots) = \sum_{l,n,r_i,t_j} c_{ln}(r_i,x_i,t_j,y_j)\psi^{(r_1)}(x_1)\ldots\psi^{(r_l)}(x_l)\psi^{*(t_1)}(y_1) \ldots\psi^{*(t_n)}(y_n) \ . \tag{3.5}$$

* For $m = 2$ candidates may be constructed— see [29]. For $m > 2$ see Theorem 3.4 below.

Now if we take $\psi(z)$ with $z \notin V$, we obtain after repeated use of (2.5)

$$\psi(z)F(V) = \int h_V \sum_{l,n,r_i,t_j} c_{ln}\psi^{(r_1+1)}(x_1)\ldots\psi^{(r_l+1)}(x_l)\psi^{*(t_1+1)}(y_1)\ldots$$
$$\psi^{*(t_n+1)}(y_n)\psi^{(l-n)}(z)(-1)^{n+l}dx_1\ldots dy_1\ldots \quad (3.6)$$

By the use of condition (i) we have that $n-l \equiv 0$ mod m and $n+l \equiv 0$ mod 2. It follows now from (2.4) and (3.6) that

$$\psi(z)F(V) = u^{-1}F(V)u\psi(z)$$
$$= F(V)\psi(z) ,$$

when (ii) is used.

To demonstrate neccessity we firstly rewrite (3.5) with the aid of (2.7):

$$g(x_1,\ldots,y_1,\ldots) = \sum_{l,n,r_i,t_j} d_{ln}(r_i,x_i,t_j,y_j)\phi^{(r_1)}(x_1)\ldots\phi^{(r_l)}(x_l)\phi^{*(t_1)}(y_1)$$
$$\ldots\phi^{*(t_n)}(y_n) . \quad (3.7)$$

Secondly we regroup terms in this sum as follows:

$$F_{qt}^{(c)} \equiv \sum_{\substack{x-y\equiv c \\ mod\ m}} \sum_{\substack{l-n\equiv q \\ mod\ m}} \sum_{\substack{l+n\equiv t \\ mod\ 2}} d_{ln}(r_i,x_i,t_j,y_j)\phi^{(r_1)}(x_1)\ldots\phi^{(r_l)}(x_l)$$
$$.\phi^{*(t_1)}(y_1)\ldots\phi^{*(t_n)}(y_n) \quad (3.8)$$

with $x \equiv \sum_{i=1}^{l} t_i$ and $y \equiv \sum_{i=1}^{n} r_i$. We now have the following commutation relations:

$$\phi^{(b)}(z)h_V F_{qt}^{(c)} = \eta^{qb+c}(-1)^t h_V F_{qt}^{(c)}\phi^{(b)}(z) . \quad (3.9)$$

Since

$$F(V) = \int h_V \sum_{c,q,t} F_{qt}^{(c)} dx_1\ldots dy_1\ldots , \quad (3.10)$$

strong locality therefore demands that

$$\int \mathbf{dx}\mathbf{dy} h_V \sum_{b,c,q,t} [1-\eta^{qb+c}(-1)^t]F_{qt}^{(c)}\phi^{(b)}(z) = 0 , \quad (3.11)$$

where $d\mathbf{x} = dx_1 dx_2\ldots$ and $d\mathbf{y} = dy_1 dy_2\ldots$. Let W be a region of space containing z but not intersecting V. Define the following operator:

$$u(W) \equiv exp(A)$$

with

$$A = \frac{i2\pi}{m} \int \chi_W(x) \left[\sum_{r=0}^{m-1} r\phi^{*(r)}(x)\phi^{(r)}(x) \right] dx \tag{3.12}$$

and

$$\begin{aligned} \chi_W(x) &= 1 \qquad x \in W \\ &= 0 \qquad x \notin W \ . \end{aligned}$$

This operator will be unitary since A is evidently anti-hermitean. One of the Baker-Campbell-Hausdorff identities [50] is:

$$\begin{aligned} &exp(-A)\phi \ exp(A) = exp(-ad_A)\phi \\ \text{where} \quad &exp(-ad_A)\phi = 1 - [A,\phi]_- + \tfrac{1}{2!}[A,[A,\phi]_-]_- - \dots \ . \end{aligned} \tag{3.13}$$

Now

$$\begin{aligned} [A, \phi^{(b)}(z)]_- &= \frac{i2\pi}{m} \left[\int \chi_W(x) (\textstyle\sum_{m=0}^{m-1} r\phi^{*(r)}(x)\phi^{(r)}(x)) dx, \ \phi^{(b)}(z) \right]_- \\ &= \frac{i2\pi}{m} \int \chi_W(x). - \delta(x-z) b\phi^{(b)}(x) dx \\ &= \frac{-i2\pi b}{m} \chi_W(z)\phi^{(b)}(z) \ . \end{aligned} \tag{3.14}$$

Therefore (3.13) allows us to conclude that

$$\begin{aligned} u^{-1}(W)\phi^{(b)}(z)u(W) &= exp(\tfrac{i2\pi b}{m}\chi_W(z))\phi^{(b)}(z) \\ &= \eta^{b\chi_W(z)}\phi^{(b)}(z) \ . \end{aligned} \tag{3.15}$$

The operator $u(W)$ might be considered to be a "local" Klein operator. If (3.11) is premultiplied by $u^{-r}(W)$ and post-multiplied by $u^r(W)$ we obtain

$$\int \mathbf{dxdy} h_V \sum_{b,c,q,t} \eta^{rb} [1 - \eta^{qb+c}(-1)^t] F_{qt}^{(c)} \phi^{(b)}(z) = 0 \ . \tag{3.16}$$

If we multiply this by η^{-dr}, sum over r and use the following

$$\sum_{r=0}^{m-1} \eta^{-dr}\eta^{rb} = \sum_{r=0}^{m-1} \eta^{r(b-d)} = m\delta_{bd} \ , \tag{3.17}$$

then we can conclude that

$$\int \mathbf{dxdy} h_V \sum_{c,q,t} [1 - \eta^{qd+c}(-1)^t] F_{qt}^{(c)} \phi^{(d)}(z) = 0 \ . \tag{3.18}$$

Consider now the operator $u(V)$. By the use of (3.15) and its hermitean conjugate, together with (3.8) we conclude that

$$u^{-1}(V)F_{qt}^{(c)}u(V) = \eta^c F_{qt}^{(c)} \ . \tag{3.19}$$

We can now premultiply (3.18) by $u^{-r}(V)$ and post-multiply by $u^r(V)$ and, using an analogous argument to the one developed above, conclude that

$$\int \mathbf{dxdy}h_V \sum_{q,t}[1-\eta^{qd+i}(-1)^t]F_{qt}^{(i)}\phi^{(d)}(z) = 0 \ . \tag{3.20}$$

We introduce now a further unitary operator given by

$$v = exp\left(\frac{i2\pi}{m}\int \sum_{r=0}^{m-1} \phi^{*(r)}(x)\phi^{(r)}(x)dx\right) \ . \tag{3.21}$$

When (3.13) is used in conjunction with the commutation relations for $\phi^{(t)}(y)$ we obtain

$$v^{-1}\phi^{(t)}(y)v = \eta\phi^{(t)}(y)$$

and hence, by (3.8),

$$v^{-1}F_{qt}^{(i)}v = \eta^q F_{qt}^{(i)} \ . \tag{3.22}$$

The argument used twice above, leads then to

$$\int \mathbf{dxdy}h_V \sum_{t}[1-\eta^{jd+i}(-1)^t]F_{jt}^{(i)}\phi^{(d)}(z) = 0 \ . \tag{3.23}$$

Finally we eliminate the sum over t by introducing the unitary operator

$$w = exp\left(i\pi\int \sum_{r=0}^{m-1} \phi^{*(r)}(x)\phi^{(r)}(x)dx\right) \tag{3.24}$$

which satisfies

$$w^{-1}\phi^{(t)}(y)w = -\phi^{(t)}(y) \tag{3.25}$$

and hence

$$w^{-1}F_{jt}^{(i)}w = (-1)^t F_{jt}^{(i)} \ . \tag{3.26}$$

Premultiply (3.23) by w^{-1} and post-multiply by w; adding the result to (3.23) then shows that

$$\int \mathbf{dxdy}h_V[1-\eta^{jd+i}(-1)^s]F_{js}^{(i)}\phi^{(d)}(z) = 0 \ . \tag{3.27}$$

The above arguments are easily modifiable to the case $\psi^*(z)$ instead of $\psi(z)$. We are led then to

$$\int \mathbf{dxdy} h_V [1 - \eta^{-jd-i}(-1)^s] F_{js}^{(i)} \phi^{*(d)}(z) = 0 \, . \tag{3.28}$$

The bracketed quantity in (3.27) is the complex conjugate of the corresponding quantity in (3.28) and so therefore they vanish simultaneously. Suppose they vanish for all values of d. A little thought will show that this can only occur when $j = i = s = 0$ (consider $d = 1$ and $m - 1$ and solve the relevant equations). Thus unless this situation occurs we may conclude that for some d

$$\int \mathbf{dxdy} h_V F_{js}^{(i)} \phi^{(d)}(z) = \int \mathbf{dxdy} h_V F_{js}^{(i)} \phi^{*(d)}(z) = 0 \, . \tag{3.29}$$

Consider now the integral

$$\int \mathbf{dxdy} \int h_V F_{js}^{(i)} \left(\phi^{*(d)}(z) \phi^{(d)}(z') + \phi^{(d)}(z') \phi^{*(d)}(z) \right) dz$$

with $z' \notin V$. By (3.29) this is zero, however by (2.9b) it is also

$$\int \mathbf{dxdy} \int h_V F_{js}^{(i)} \delta(z - z') dz = \int \mathbf{dxdy} h_V F_{js}^{(i)}$$

and so therefore we are led to

$$\int \mathbf{dxdy} h_V F_{js}^{(i)} = 0 \, , \tag{3.30}$$

whenever $i = j = s = 0$ does not apply. Conditions (i) and (ii) now follow from (3.1), (3.7), (3.8) and (2.9c). ∎

The question now arises as to the form of strongly local observables which are solely functions of the modular fields $\psi(x)$ and $\psi^*(x)$. This problem is partially solved in the following theorem:

Theorem 3.2. *Let us define the following polynomials of modular fields:*

$$M(x_1, \ldots, x_{m-1}, y_1, \ldots, y_{m-1}) = \sum_{perm(1,\ldots,m-1)} \sum_{l=0}^{m-1} \psi(x_{m-l}) \ldots \psi(x_{m-1}) \psi^*(y_{m-1})$$
$$\ldots \psi^*(y_1) \psi(x_1) \ldots \psi(x_{m-l-1}) (-1)^l \tag{3.31}$$

$$B_N(x_1,\dots,x_N) = \sum_{cycl(1,\dots,N)} (-1)^{cycl(N)} \psi(x_1)\dots,\psi(x_N) \tag{3.32}$$

$$C_N(x_1,\dots,x_m) = \sum_{cycl(1,\dots,N)} \psi(x_1)\dots\psi(x_N)\ . \tag{3.33}$$

An observable $F(V)$ is strongly local if the function g in (3.1) is a function of the following modular field polynomials (and their hermitean conjugates): M, $C_{lm}C'_{lm}$, $C_{lm}C'^{*}_{lm}$ and when km is even, B_{km}.

Proof: Contemplation of Theorem 3.1 shows that it is sufficient to show that the relevant field polynomials are invariant under u. The case of B is considered firstly:

It is easily seen from (2.3), (2.4) and (2.5) that when N is even and equal to km then

$$\begin{aligned}\psi(x_N)\psi(x_1),\dots\psi(x_{N-1}) &= -\psi^{(-1)}(x_1)\dots\psi^{(-1)}(x_{N-1})\psi^{(N-1)}(x_N)\\ &= -\psi^{(-1)}(x_1)\dots\psi^{(-1)}(x_{N-1})\psi^{(-1)}(x_N)\end{aligned}$$

and therefore

$$B_N(x_1,\dots,x_N) = k\sum_{r=0}^{m-1} \psi^{(r)}(x_1)\dots\psi^{(r)}(x_N)\ .$$

The form of the right-hand side of this equation now gives the desired result. The case of C is proved in a similar way; we content ourselves with a proof that $u^{-1}C_{lm}C'^{*}_{lm}u = C_{lm}C'^{*}_{lm}$ as the proof for the other C polynomial is almost identical. Now

$$\begin{aligned}C_{lm}C'^{*}_{lm} &= \sum_{\substack{cycl(x_1,\dots,x_{lm})\\ cycl(y_1,\dots,y_{lm})}} \psi(x_1)\dots\psi(x_{lm})\psi^*(y_{lm}),\dots\psi(y_1)\\ &= \sum_{cycl(1,\dots,lm)}\ \sum_{cycl(x_1,\dots,x_{lm})} \psi(x_1)\dots\psi(x_{lm})\psi^*(y_{lm})\dots\psi^*(y_1)\ .\end{aligned}$$

However by (2.3)–(2.5)

$$\begin{aligned}\psi(x_1)\dots\psi(x_{lm})\psi^*(y_{lm})\dots\psi^*(y_1) =&\psi^{(1)}(x_{lm})\psi^{(1)}(x_1)\dots\psi^{(1)}(x_{lm-1})\\ &.\psi^{*(1)}(y_{lm-1})\dots\psi^{*(1)}(y_1)\psi^{*(1)}(y_{lm})\end{aligned}$$

so therefore we have

$$C_{lm}C'^{*}_{lm} = \sum_{cycl(x_1,\ldots,x_{lm})} \sum_{r=0}^{m-1} \psi^{(r)}(x_1)\ldots\psi^{(r)}(x_{lm})\psi^{*(r)}(y_{lm})\ldots\psi^{*(r)}(y_1)$$

which demonstrates the required result.

For the case of M we move the fields $\psi(x_1),\ldots,\psi(x_{m-l-1})$ in equation (3.31) to the left using equation (2.5) repeatedly. After a straightforward but tedious calculation we obtain the result:

$$\begin{aligned}\sum_{perm}\sum_{q=0}^{m-1}\sum_{l=0}^{m-q-1}\sum_{n_1=0}^{k_1}\sum_{n_2=0}^{k_2}\ldots\sum_{n_q=0}^{k_q} & X^{(0)}(m-l,m-1)X^{(-1)}(1,n_1) \\ & X^{(-2)}(n_1+2,n_1+n_2+1)\ldots X^{(-q)}(t-n_q-1,t-2) \\ & X^{(-q-1)}(t,m-l-1)Y^{(-l-q-1)}(m-1,t)Y^{(-l-q)}(t-2,t-n_q-1)\ldots \\ & Y^{(-l-1)}(n_1,1)\delta^{n_1+1}_{n_1+1}\,\delta^{n_1+n_2+2}_{n_1+n_2+2}\ldots\delta^{t-1}_{t-1}(-1)^x \end{aligned} \tag{3.34}$$

where

$$X^{(r)}(a,b) \equiv \psi^{(r)}(x_a)\ldots\psi^{(r)}(x_b)\ ; \quad Y^{(r)}(a,b) \equiv \psi^{*(r)}(y_a)\ldots\psi^{*(r)}(y_b)$$

$$t \equiv q+1+\sum_{i=1}^{q} n_i\ ; \quad x \equiv (l+1)(q-m+1)+l\ ; \quad \delta^j_i \equiv \delta(x_j-y_i)$$

$$k_i \equiv m-1-l-\sum_{j=1}^{i-1} n_j\ .$$

This may now be rearranged by use of (2.5):

$$\begin{aligned}&\sum X^{(-l-1)}(1,n_1)\ldots X^{(-l-q)}(t-n_q-1,t-2)X^{(-l-q-1)}(t,m-1) \\ &\quad Y^{(-l-q-1)}(m-1,t)\ldots Y^{(-l-1)}(n_1,1)\delta^{n_1+1}_{n_1+1}\ldots\delta^{t-1}_{t-1}(-1)^{q-m-1}\ .\end{aligned} \tag{3.35}$$

By the rearrangement of the summations and by setting $n_{q+1} \equiv m-t$, we obtain

$$\begin{aligned}\sum_{perm}\sum_{q=0}^{m-1}(-1)^s \sum_{\sum_{i=1}^{q+1} n_i = s}\ \sum_{l=0}^{n_{q+1}} & X^{(-l-1)}(1,n_1)\ldots X^{(-l-q-1)}(m-n_{q+1},m-1) \\ & Y^{(-l-q-1)}(m-1,m-n_{q+1})\ldots Y^{(-l-1)}(n_1,1) \\ & \delta^{n_1+1}_{n_1+1}\ldots\delta^{m-n_{q+1}-1}_{m-n_{q+1}-1}\ ,\end{aligned} \tag{3.36}$$

where s stands for $m-1-q$. Consider terms in this sum with fixed n_i with $i=1,\ldots,q+1$. Corresponding to these terms are other terms with their n_i being a *cyclic* permutation of these fixed values. After an appropriate permutation of the spatial indices these latter terms become

$$\begin{aligned}
&\sum_{l=0}^{n_j} X^{(-l-1)}(f(j)+1,f(j+1)-1)\ldots X^{(-l-1+j-q)}(m-n_{q+1},m-1)\\
&\quad X^{(-l-2+j-q)}(1,n_1)\ldots X^{(-l-q-1)}(f(j-1)+1,f(j)-1)\\
&\quad Y^{(-l-q-1)}(f(j)-1,f(j-1)+1)\ldots Y^{(-l-1)}(f(j+1)-1,f(j)+1)\\
&\quad \delta^{n_1+1}_{n_1+1}\ldots\delta^{m-n_{q+1}-1}_{m-n_{q+1}-1}
\end{aligned}\tag{3.37}$$

where $f(j)\equiv j+\sum_{i=1}^{j} n_i$ with $j=1,\ldots,q$. This may be rearranged using (2.5) repeatedly:

$$\begin{aligned}
&\sum_{l=0}^{n_j} X^{(-l-1+f(j))}(1,n_1)X^{(-l-2+f(j))}(n_1+2,n_1+n_2+1)\ldots\\
&\quad X^{(-l-1-q+f(j))}(m-n_{q+1},m-1)Y^{(-l-1-q+f(j))}(m-1,m-n_{q+1})\ldots\\
&\quad Y^{(-l-1+f(j))}(n_1,1)\delta^{n_1+1}_{n_1+1}\ldots\delta^{m-n_{q+1}-1}_{m-n_{q+1}-1}
\end{aligned}\tag{3.38}$$

Now as l goes from 0 to n_j the index $-l-1+f(j)$ goes from $-1+f(j)$ to $f(j-1)$ or when $j=1$, to 0. In the original unpermuted term the corresponding index goes from -1 to $-1-n_{q+1}=t-1=q+\sum_{i=1}^{q} n_i=f(q)$. It is clear now that this index will cover all values mod m when (3.38) is summed over all values of j and added to the original term. Hence this sum will be invariant under u. A little thought about how the terms in (3.38) are produced will show that all terms in (3.36) may be grouped into such sums in a *non-overlapping* way. Hence $u^{-1}Mu=M$. ■

3.2. New modular commutation relations

In the special cases of $m=2$ and $m=3$, the polynomial M has the form

$$M(x_1,y_1)=[\psi(x_1),\psi^*(y_1)]_- \tag{3.39}$$

$$\begin{aligned}
M(x_1,x_2,y_1,y_2)=\sum_{perm(1,2)} &\psi^*(y_2)\psi^*(y_1)\psi(x_1)\psi(x_2)\\
&-\psi(x_2)\psi^*(y_2)\psi^*(y_1)\psi(x_1)\\
&+\psi(x_1)\psi(x_2)\psi^*(y_2)\psi^*(y_1)\,.
\end{aligned}\tag{3.40}$$

In the case of $m = 2$ we are dealing with parastatistics of order two and in that case the strong locality of M follows directly from the fundamental commutation relation of paraquantization, namely (1.1a). This observation tends to suggest that a generalization of this fundamental equation may be possible. This turns out to be the case as the following theorem shows:

Theorem 3.3. *The field polynomial M given in equation (3.31) satisfies the following commutation relations:*

$$[M, \psi(z)]_- = \sum_{perm(1,\ldots,m-1)} \sum_{l=0}^{m-2} (-1)^{l+1} \delta(z - y_{m-l-1}) \psi^*(y_{m-l-2}) \ldots \psi^*(y_1) \\ \psi(x_1) \ldots \psi(x_{m-1}) \psi^*(y_{m-1}) \ldots \psi^*(y_{m-l}) \quad (3.41)$$

$$[M, \psi^*(z)]_- = \sum_{perm(1,\ldots,m-1)} \sum_{l=0}^{m-2} (-1)^{l} \delta(z - x_{m-l-1}) \psi(x_{m-l}) \ldots \psi(x_{m-1}) \\ \psi^*(y_{m-1}) \ldots \psi^*(y_1) \psi(x_1) \ldots \psi(x_{m-l-2}) \quad (3.42)$$

Proof: We have firstly, the following interesting lemma:

Lemma.

$$\Big[\psi(x_1) \ldots \psi(x_l) \psi^*(y_l) \ldots \psi^*(y_1), \\ \psi^*(y_{m-1}) \ldots \psi^*(y_{l+1}) \psi(x_{l+1}) \ldots \psi(x_{m-1})\Big]_- = 0 \quad (3.43)$$

Proof: We introduce the abbreviations $\psi(x_i) = x_i$ and $\psi^*(y_i) = y_i$. Now by (2.5) we have

$$\begin{aligned} & x_1 \ldots x_l y_l \ldots y_1 y_{m-1} \ldots y_{l+1} x_{l+1} \ldots x_{m-1} \\ &= x_1 \ldots x_l y^{(l)}_{m-1} \ldots y^{(l)}_{l+1} y^{(l+1)}_l \ldots y^{(l+1)}_1 x_{l+1} \ldots x_{m-1} (-1)^{lm} \\ &= y_{m-1} x^{(-1)}_1 \ldots x^{(-1)}_l y^{(l)}_{m-2} \ldots y^{(l)}_{l+1} y^{(l+1)}_l \ldots y^{(l+1)}_1 x_{l+1} \ldots x_{m-1} (-1)^{l(m+l)} . \end{aligned} \quad (3.44)$$

Now for there to be any fields in the range $y^{(l)}_{m-2} \ldots y^{(l)}_{l+1}$, the index l must be less than $m-2$. It follows from (2.5) that we may move $y^{(l)}_{m-2}$ to the left without picking up a delta function, thus:

$$y_{m-1} y_{m-2} x^{(-2)}_1 \ldots x^{(-2)}_l y^{(l)}_{m-3} \ldots y^{(l)}_{l+1} y^{(l+1)}_l \ldots y^{(l+1)}_1 x_{l+1} \ldots x_{m-1} (-1)^{lm}$$

and now for there to be any fields in the range $y^{(l)}_{m-3}\dots y^{(l)}_{l+1}$ we must have $l < m-3$ and hence we can also move $y^{(l)}_{m-3}$ to the left without picking up a delta function. Obviously this argument can be extended until we obtain

$$y_{m-1}\dots y_{l+1}x_1^{(l+1)}\dots x_l^{(l+1)}y_l^{(l+1)}\dots y_1^{(l+1)}x_{l+1}\dots x_{m-1}\ .$$

By the use of an argument similar to the one just described we can move the fields $x_{l+1}\dots x_{m-1}$ to the left obtaining

$$y_{m-1}\dots y_{l+1}x_1^{(l+1)}\dots x_l^{(l+1)}x_{l+1}^{(l)}\dots x_{m-1}^{(l)}y_l\dots y_1(-1)^{lm}$$

and finally this may be rewritten as

$$y_{m-1}\dots y_{l+1}x_{l+1}\dots x_{m-1}x_1\dots x_l y_l\dots y_1$$

which demonstrates the lemma. ∎

As a corollary to the above lemma we have the following alternative form for M:

$$M(x,y) = \sum_{perm(1,\dots,m-1)}\sum_{l=0}^{m-1}\psi^*(y_{m-l-1})\dots\psi^*(y_1)\psi(x_1)$$
$$\dots\psi(x_{m-1})\psi^*(y_{m-1})\dots\psi^*(y_{m-l})(-1)^l\ . \tag{3.45}$$

We begin the proof of Theorem 3.3 by demonstrating (3.42). By the use of the abbreviations introduced above we have, using (3.31),

$$\begin{aligned}M(x,y)\psi^*(z) &= \sum_{perm}\sum_{l=0}^{m-1}(-1)^l x_{m-l}\dots x_{m-1}y_{m-1}\dots y_1x_1\dots x_{m-l-1}z\\ &= \sum(-1)^{m-1}x_{m-l}\dots x_{m-1}y_{m-1}\dots y_1z^{(l+1)}x_1^{(-1)}\dots x_{m-l-1}^{(-1)}\\ &+\sum_{perm}\sum_{l=0}^{m-2}(-1)^l x_{m-l}\dots x_{m-1}y_{m-1}\dots y_1\\ &\qquad x_1\dots x_{m-l-2}\,\delta(z-x_{m-l-1})\end{aligned}\tag{3.46}$$

The first term in (3.46) is equal to

$$\begin{aligned}&\sum x_{m-l}\dots x_{m-1}z^{(l)}y_{m-1}^{(-1)}\dots y_1^{(-1)}x_1^{(-1)}\dots x_{m-l-1}^{(-1)}\\ &= z\sum(-1)^l x_{m-l}^{(-1)}\dots x_{m-1}^{(-1)}y_{m-1}^{(-1)}\dots y_1^{(-1)}x_1^{(-1)}\dots x_{m-l-l}^{(-1)}\\ &=\psi^*(z)M^{(-1)}(x,y)\\ &=\psi^*(z)M(x,y)\ .\end{aligned}$$

Consideration of the second term in (3.46) then gives equation (3.42).

Now the hermitean conjugate of (3.31) is

$$
\begin{aligned}
M^*(x_1,\ldots,x_{m-1},y_1,\ldots y_{m-1}) &= \sum(-1)^l \psi^*(x_{m-l-1})\ldots\psi^*(x_1)\psi(y_1)\ldots\psi(y_{m-1}) \\
&\qquad \psi^*(x_{m-1})\ldots\psi^*(x_{m-l}) \\
&= M(y_1,\ldots y_{m-1},x_1,\ldots,x_{m-1}) \ , \qquad (3.47)
\end{aligned}
$$

where (3.45) has been used. We have now

$$
\begin{aligned}
[M(x,y),\psi(z)]_- &= -\left([M^*(x,y),\psi^*(z)]_-\right)^* \\
&= -\left([M(y,x),\psi^*(z)]_-\right)^* \qquad (3.48)
\end{aligned}
$$

and then (3.41) follows from (3.42) and (3.48). ■

In the case $m = 2$, the commutation relations given in (3.41) and (3.42) evidently have other solutions apart from simply $m = 2$ modular field theory. These are, of course, the higher order parafield theories. One might expect, therefore, that the relations (3.41) and (3.42) will have further solutions when $m > 2$. Whether this is so is, at present, unclear. In the special case of $m = 3$ this author has attempted without success to find other *ansatz* solutions. This suggests that the above expectation may not be realized.

A further question deserving investigation is whether the new commutation relations can serve as the fundamental defining relations for modular field theory. This is of some interest since we have been unable to show that the original modular relations (2.1) imply the Fock-condition (2.16), which selects out the ansatz solution to these relations. The relations (3.41) and (3.42) may be stronger in this regard.

3.3. Order restrictions

A classification of all strongly local observables remains an open question. In the case of $m = 2$ the parafield classification applies (see [29] for details). In the more general setting the following theorem is of some interest:

Theorem 3.4. *For $m > 2$ there are no strongly local observables which are of second order or less in modular fields. For $m > 4$ there are no such observables which are fourth order or less.*

Proof: Consideration of Theorem 3.1 shows that first and third order polynomials are impossible for strongly local observables. We show now that second order polynomials are impossible for $m > 2$:

By Theorem 3.1 such polynomials must involve both a ψ and a ψ^* and must therefore have the form*

$$F_2(V) = \int [a\psi(x_1)\psi^*(x_2) + b\psi^*(x_2)\psi(x_1)]dx_1dx_2 \tag{3.49}$$

where a and b are functions of x_1 and x_2 which vanish when these variables are not in V. By the use of (2.8) and (2.9) this may be rewritten as

$$F_2(V) = \int \left[\sum_{r,t}(a - \eta^{t-r}b)\phi^{(t)}(x_1)\phi^{(r)}(x_2)\right] dx_1dx_2 + K \tag{3.50}$$

where K is a c-number. When the notation of equation (3.8) in the proof of Theorem 3.1 is used, we may rewrite this as

$$F_2(V) = \int \left[\sum_{v=0}^{m-1}(a - \eta^{-v}b)F_{00}^{(v)}\right] dx_1dx_2 + K \tag{3.51}$$

where we have

$$F_{00}^{(v)} = \sum_{r-t=v} \phi^{(t)}(x_1)\phi^{*(r)}(x_2) \ . \tag{3.52}$$

It follows from the proof of Theorem 4.1 that

$$F_v \equiv \int (a - \eta^{-v}b)F_{00}^{(v)} = 0 \qquad v \neq 0 \ . \tag{3.53}$$

Evaluate now

$$0 = F_v\phi^{(w)}(z_2) - \eta^v\phi^{(w)}(z_2)F_v \equiv G_{v-w}$$

and then

$$G_{v-w}\phi^{*(w-v)}(z_1) + \eta^{v-w}\phi^{*(w-v)}(z_1)G_{v-w} \ .$$

After these calculations are carried out, we obtain the equation

$$a(z_1, z_2) - \eta^{-v}b(z_1, z_2) = 0 \qquad v \neq 0 \ .$$

If v can take on more than one value, as it can when $m > 2$, then this implies that $a = b = 0$ which shows that $F_2(V) = 0$.

---

* Apart from the addition of a c-number, of course.

If $m > 4$ it is clear from Theorem 4.1 that the only possible strongly local observables of fourth order involve two ψ and two ψ^* fields. Some thought as to the possible permuations of these fields leads us to conclude that such observables must have the form*

$$
\begin{aligned}
F_4(V) = \int dx_1 dx_2 dy_1 dy_2 [& a_1 \psi(x_1)\psi(x_2)\psi^*(y_1)\psi^*(y_2) + a_2 \psi(x_1)\psi^*(y_1)\psi(x_2)\psi^*(y_2) \\
& + a_3 \psi^*(y_1)\psi(x_1)\psi(x_2)\psi^*(y_2) + a_4 \psi^*(y_1)\psi(x_1)\psi^*(y_2)\psi(x_2) \\
& + a_5 \psi^*(y_1)\psi^*(y_2)\psi(x_1)\psi(x_2)] \\
+ \int & dxdy[b_1\psi(x)\psi^*(y) + b_2\psi^*(y)\psi(x)] \ ,
\end{aligned} \tag{3.54}
$$

where a_i and b_i are functions vanishing when their arguments lie outside V.

When (2.4), (2.5) and (2.8) are used, we can rewrite the terms in the first bracket (apart from a c-number) as

$$
\begin{aligned}
& \sum_{r_1,r_2,t_1,t_2} \Lambda(r_1,r_2,t_1,t_2)\phi^{(t_1)}(x_1)\phi^{(t_2)}(x_2)\phi^{*(r_1)}(y_1)\phi^{*(r_2)}(y_2) \\
& \qquad + \sum_{r,t}\sum_{i,j=1}^{2} T^{ij}(r,t)\phi^{(t)}(x_i)\phi^{*(r)}(y_j)
\end{aligned} \tag{3.55}
$$

where

$$
\begin{aligned}
\Lambda(r_1,r_2,t_1,t_2) &= a_1 - \eta^{t_2-r_1}a_2 + \eta^{v+r_1-r_2}a_3 - \eta^{v+t_2-r_1}a_4 + \eta^{2v}a_5 \\
v &= t_1 + t_2 - r_1 - r_2
\end{aligned} \tag{3.56}
$$

$$
\begin{aligned}
T^{12}(r,t) &= \delta(x_2 - y_1)(a_2 + \eta^{t-r}a_4 - \eta^{2(t-r)}a_5) \\
T^{21}(r,t) &= -\delta(x_1 - y_2)\eta^{t-r}a_5 \\
T^{11}(r,t) &= -\delta(x_2 - y_2)\eta^{t-r}a_4 \\
T^{22}(r,t) &= \delta(x_1 - y_1)(a_3 - \eta^{t-r}a_4) \ .
\end{aligned} \tag{3.57}
$$

By the use of a change of variables, the second integral in (3.54) can be combined with the second term from (3.55) and we may write (3.54) as

$$
\begin{aligned}
F_4(V) = \int dx_1 dx_2 dy_1 dy_2 \Big[& \textstyle\sum_{r_i,t_i} \Lambda(r_1,r_2,t_1,t_2)\phi^{(t_1)}(x_1)\phi^{(t_2)}(x_2)\phi^{*(r_1)}(y_1)\phi^{*(r_2)}(y_2) \\
& + \textstyle\sum_{r,t}\sum_{i,j} U^{ij}(r,t)\phi^{(t)}(x_i)\phi^{*(r)}(y_j) \Big] + K
\end{aligned} \tag{3.58}
$$

---

* We are using $\psi^*(y_1)\psi(x_1)\psi(x_2)\psi^*(y_2) = \psi(x_2)\psi^*(y_2)\psi^*(y_1)\psi(x_1)$, which follows from the lemma above.

$$U^{ij}(r,t) = T^{ij}(r,t) \quad \text{except when } i=j=1\ , \text{ when}$$

where

$$U^{11}(r,t) = T^{11}(r,t) + b_1(x_1,y_1) - \eta^{t-r} b_2(x_1,y_1)\ ;$$

and where K is a c-number.

By the use of a similar proof to the one used for second order observables above, we may conclude, using the proof of Theorem 4.1, that

$$\int dx_1 dx_2 dy_1 dy_2 F_{00}^{(v)} = 0 \qquad v \neq 0 \tag{3.59}$$

with

$$\begin{aligned} F_{00}^{(v)} = & \sum_{t_1+t_2-r_1-r_2=v} \Lambda \phi^{(t_1)}(x_1)\phi^{(t_2)}(x_2)\phi^{*(r_1)}(y_1)\phi^{*(r_2)}(y_2) \\ & + \sum_{t-r=v} \sum_{ij} U^{ij} \phi^{(t)}(x_i)\phi^{*(r)}(y_j)\ . \end{aligned} \tag{3.60}$$

We introduce the following notation:

$$F_v \equiv \int F_{00}^{(v)} dx_1 dx_2 dy_1 dy_2$$

$$\Lambda_{t_1 t_2 r_1 r_2}^{x_1 x_2 y_1 y_2} \equiv \Lambda(r_1, r_2, t_1, t_2)$$

where, in the second line, Λ is evaluated at x_1, x_2, y_1, y_2.

Now when (3.59), (3.60) and (2.9) are used we obtain

$$\begin{aligned} 0 &= \phi^{(u)}(z_1)F_v - \eta^{-v} F_v \phi^{(u)}(z_1) \\ &= -\eta^{-v} \Bigg[\int \sum_{t_1+t_2-r=v+a} [\Lambda_{t_1 t_2 r a}^{x_1 x_2 y z_1} - \eta^{a-r} \Lambda_{t_1 t_2 a r}^{x_1 x_2 z_1 y}] \\ &\qquad .\phi^{(t_1)}(x_1)\phi^{(t_2)}(x_2)\phi^{*(r)}(y) dx_1 dx_2 dy \\ &\qquad + \int U^{ij} \phi^{(v+a)}(x_i)\delta(z_1 - y_j) dx_1 dx_2 dy_1 dy_2 \Bigg] \\ &\equiv -\eta^{-v} G_{v+a} \qquad v \neq 0\ . \end{aligned} \tag{3.61}$$

Similarly we have

$$\begin{aligned} 0 &= \phi^{(b)}(z_2)G_{v+a} + \eta^{b-v-a} G_{v+a}\phi^{(b)}(z_2) \\ &= -\eta^{-v-a} \int \sum_{t_1+t_2=v+a+b} [\Lambda_{t_1 t_2 b a}^{x_1 x_2 z_2 z_1} - \eta^{a-b} \Lambda_{t_1 t_2 a b}^{x_1 x_2 z_1 z_2}] \\ &\qquad .\phi^{(t_1)}(x_1)\phi^{(t_2)}(x_2) dx_1 dx_2 \\ &\equiv -\eta^{-v-a} H_{v+a+b} \qquad v \neq 0\ . \end{aligned} \tag{3.62}$$

If we continue in this manner we obtain

$$
\begin{aligned}
0 &= \phi^{*(c)}(z_3)H_{v+a+b} - \eta^{v+a+b-2c}H_{v+a+b}\phi^{*(c)}(z_3) \\
&= -\eta^{v+a+b-2c}\int dx \Big[\Lambda^{xz_3z_2z_1}_{dcba} - \eta^{a-b}\Lambda^{xz_3z_1z_2}_{dcab} - \eta^{c-d}\Lambda^{z_3xz_2z_1}_{cdba} \\
&\qquad\qquad + \eta^{c+a-b-d}\Lambda^{z_3xz_1z_2}_{cdab}\Big]\phi^{(d)}(x) \\
&\equiv -\eta^d J_d \qquad d+c-a-b \neq 0 \,,
\end{aligned}
\tag{3.63}
$$

where we have defined $d \equiv v+a+b-c$. Finally taking the anticommutator of J_d with $\phi^{*(d)}(z_4)$ we obtain

$$
\Lambda^{z_4z_3z_2z_1}_{dcba} - \eta^{a-b}\Lambda^{z_4z_3z_1z_2}_{dcab} - \eta^{c-d}\Lambda^{z_3z_4z_2z_1}_{cdba} + \eta^{c+a-b-d}\Lambda^{z_3z_4z_1z_2}_{cdab} = 0 \tag{3.64}
$$

for $d+c-a-b \neq 0$. By the use of (3.56) and the notation

$$
a_i^{4321} \equiv a_i(z_4, z_3, z_2, z_1) \,, \tag{3.65}
$$

we obtain the equation

$$
\begin{aligned}
& a_1^{4321} - \eta^{a-b}a_1^{4312} - \eta^{c-d}a_1^{3421} + \eta^{c+a-b-d}a_1^{3412} \\
& - \eta^{c-b}(a_2^{4321} - a_2^{4312} - a_2^{3421} + a_2^{3412}) \\
& + \eta^{d+c-a-b}(\eta^{a-b}a_3^{4321} - a_3^{4312} - \eta^{c+a-b-d}a_3^{3421} + \eta^{c-d}a_3^{3412}) \\
& - \eta^{d+c-a-b}\eta^{c-b}(a_4^{4321} - a_4^{4312} - a_4^{3421} + a_4^{3412}) \\
& + \eta^{2(d+c-a-b)}(a_5^{4321} - \eta^{a-b}a_5^{4312} - \eta^{c-d}a_5^{3421} + \eta^{c+a-b-d}a_5^{3412}) = 0
\end{aligned}
\tag{3.66}
$$

when $d+c-a-b \neq 0$.

If we take $d = a+b-c+1$ and $d = a+b-c+2$ and remember that $m > 3$ then we may subtract the resulting equations obtaining:

$$
\begin{aligned}
& (\eta^{-1} - \eta^{-2})[\eta^{2(c-b)}a_1^{3412} - \eta^{2c-a-b}a_1^{3421}] + (\eta - \eta^2)[\eta^{a-b}a_3^{4321} - a_3^{4312}] \\
& - (\eta - \eta^2)\eta^{c-b}[a_4^{4321} - a_4^{4312} - a_4^{3421} + a_4^{3412}] \\
& + (\eta^2 - \eta^4)[a_5^{4321} - \eta^{a-b}a_5^{4312}] \\
& - (\eta - \eta^2)[\eta^{2c-a-b}a_5^{3421} - \eta^{2(c-b)}a_5^{3412}] = 0 \,.
\end{aligned}
\tag{3.67}
$$

If we let $a = 0, 1$ and subtract the resulting equations we obtain

$$
\begin{aligned}
&(\eta^{-1} - \eta^{-2})(\eta^{-1} - 1)\eta^{2c-b} a_1^{3421} + (\eta - \eta^2)(1 - \eta)\eta^{-b} a_3^{4321} \\
&\quad + (\eta^2 - \eta^4)(\eta - 1)\eta^{-b} a_5^{4312} + (\eta - \eta^2)(\eta^{-1} - 1)\eta^{2c-b} a_5^{3421} = 0 \ . \quad (3.68)
\end{aligned}
$$

If $c = 0, 1$ is then substituted, subtraction of these equations gives

$$
a_1^{3421}(\eta^{-1} - \eta^{-2}) + a_5^{3421}(\eta - \eta^2) = 0 \ . \tag{3.69}
$$

The above procedure may be repeated, in the case of $m > 3$, with $d = a+b-c-1$ and $d = a+b-c+1$ in (3.66). The resulting expression is

$$
a_1^{3421}(\eta^{-1} - \eta) + a_5^{3421}(\eta - \eta^{-1}) = 0
$$

or in otherwords $a_1^{3421} = a_5^{3421}$. Substitution of this into (3.69) gives

$$
a_1^{3421}(\eta^{-1} - \eta^{-2} + \eta - \eta^2) = 0
$$

or

$$
a_1^{3421}(\cos\tfrac{2\pi}{m} - \cos\tfrac{4\pi}{m}) = 0 \ .
$$

If $\cos\frac{2\pi}{m} = \cos\frac{4\pi}{m}$ we have, using $\cos 2\theta = 2\cos^2\theta - 1$,

$$
2\cos^2\tfrac{2\pi}{m} - \cos\tfrac{2\pi}{m} + 1 = 0
$$

which has solution $\cos\frac{2\pi}{m} = 1, -\frac{1}{2}$. This in turn has the solution $\frac{2\pi}{m} = n2\pi$ and $\frac{2\pi}{m} = \frac{2\pi}{3} + n2\pi$ which means that $m = 1, 3$, which we have excluded. We therefore conclude that $a_1^{3421} = a_5^{3421} = 0$. Substitution into (3.68) gives also $a_3^{4321} = 0$. Equations (3.67) and (3.66) finally give

$$
\begin{aligned}
&a_4^{4321} - a^{4312} - a_4^{3421} + a_4^{3412} = 0 \\
&a_2^{4321} - a_2^{4312} - a_2^{3421} + a_2^{3412} = 0 \ .
\end{aligned} \tag{3.70}
$$

By the use of the first equation of (2.1) and its hermitean conjugate the following identity may be derived:

$$
\begin{aligned}
&\psi(x_1)\psi^*(y_1)\psi(x_2)\psi^*(y_2) - \psi(x_2)\psi^*(y_1)\psi(x_1)\psi^*(y_2) - \psi(x_1)\psi^*(y_2)\psi(x_2)\psi^*(y_1) \\
&+ \psi(x_2)\psi^*(y_2)\psi(x_1)\psi^*(y_1) \equiv 4\psi(x_1)\psi^*(y_1)\psi(x_2)\psi^*(y_2) \\
&\quad - 3\delta(x_2 - y_1)\psi(x_1)\psi^*(y_2) + \delta(x_1 - y_2)\psi(x_2)\psi^*(y_1) \\
&\quad - \delta(x_1 - y_1)\psi(x_2)\psi^*(y_2) - \delta(x_2 - y_2)\psi(x_1)\psi^*(y_1) \ .
\end{aligned} \tag{3.71}
$$

If we multiply the left hand side by $a_2(x_1, x_2, y_1, y_2)$ and integrate over the four variables we may, after a change of variables in the final three integrals and use of (3.70), conclude that the result is zero. Examination of the right hand side of the identity will then show that the second term in (3.54) becomes a quadratic term. A similar argument holds for the fourth term and so we conclude that $F_4(V)$ must be a second order polynomial and hence, from the first part of the proof, zero. ■

3.4. Derivative fields

As a final topic in this section we consider the possibility of observables constructed from the derivatives of field operators. Clearly such observables will be used in constructing useful physical observables such as the energy–momentum operator P_μ.

Firstly we have, as usual, the equal–time commutation relations

$$\left[\Phi^{*(r)}(x_1),\ \Phi^{(t)}(x_2)_{,\mu_2}\right]_+ = \delta_{,\mu_2}(\mathbf{x}_2 - \mathbf{x}_1)\,\delta^{rt}$$

where $\Phi^{(r)}(x)$ are the Klein transformed fermi fields and the subscript $_{,\mu_2}$ is understood to mean the covariant derivative with respect to $(x_2)_\mu$. The Klein transformation of this equation gives

$$\phi^{*(r)}(x_1)\phi^{(t)}(x_2)_{,\mu_2} + \eta^{t-r}\phi^{(t)}(x_2)_{,\mu_2}\phi^{*(r)}(x_1) = \delta_{,\mu_2}(\mathbf{x}_2 - \mathbf{x}_1)\,\delta^{rt}\,. \tag{3.72}$$

It is now demonstrated that Theorems 3.1–3.4 are true with replacements (in any position) of $\psi(x_i)$, $\psi^*(x_i)$ and $\delta(x_i - x_j)$ by $\psi(x_i)_{,\mu_i}$, $\psi^*(x_i)_{,\mu_i}$ and $\delta_{,\mu_i}(x_i - x_j)$ or $\delta_{,\mu_i\mu_j}(x_i - x_j)$ respectively. It is clear from (3.72) that the proofs of these modified theorems will change only in that there may be a mixture of delta functions and their derivatives rather than simply delta functions. This will only affect Theorem 3.1 in that the rearrangement given in (3.5) may need the $c_{ln}(r_i, x_i, t_j, y_j)$ modified. Obviously though, this does not affect the proof in any essential way.

In the case of Theorem 3.2, the collection of terms (3.36) and (3.37) whose sum is shown to be invariant under u, will have the same product of delta functions. Hence if some are changed to derivatives of delta functions, the invariance under u is unaffected, and so the proof goes through in the same way. In Theorem 3.3 delta functions make one appearance in both of the commutation relations and examination of the proof shows that no other use of delta functions is made. Thus this proof also goes through in a similar way if these delta functions are replaced by the derivatives of delta functions.

Finally the proof of Theorem 3.4 needs more extensive modification. The details are rather technical and may be found in Appendix C.

4. Relationship to a normal field theory

In this section we use the Klein transformation developed in Appendix B to compare modular quantization with a field theory which is quantized normally but which has a "gauge" invariance.

4.1. Introduction

In order to introduce the techniques that are required for such a comparison we briefly review the situation which applies in the parafield theory case. For the purposes of clarity we shall restrict ourselves to the parafermi alternative.

In this case it has been shown [29] that observables satisfying the condition of strong locality must be functions of the field polynomial

$$P(x_1, x_2) = [\hat{\psi}(x_1), \hat{\psi}(x_2)]_- \tag{4.1}$$

where

$$\hat{\psi}(x) = \psi(x) \quad \text{or} \quad \psi^*(x) \ .$$

When equations (1.5), (B.2), (B.31), (B.32) and (B.37) are used $P(x_1, x_2)$ may be rewritten in terms of the fermi fields $\hat{\Phi}^r(x)$ as

$$P(x_1, x_2) = \sum_{r=1}^{p} [\hat{\Phi}^r(x_1), \hat{\Phi}^r(x_2)]_- \ . \tag{4.2}$$

The parafermi states may also be transformed into fermi states once the following condition is imposed:

$$K_r|\rangle = |\rangle \ . \tag{4.3}$$

The transformation is possible for the following reasons: Theorem I.1 from [51] shows that the Fock-space for parafield theory is spanned by states of the form $M(b^*_{k_1} b^*_{k_2} \dots b^*_{k_n})|\rangle$, where M is a monomial of para creation operators. By use of the ansatz equation and the Klein transformation (B.3), this may be rewritten as

$$\sum_{r_1,\dots,r_n=1}^{p} M(K_{\bar{r}_1} a^{*(r_1)}_{k_1} \dots K_{\bar{r}_n} a^{*(r_n)}_{k_n})|\rangle \ . \tag{4.4}$$

When (B.2) is used, the Klein operators may be shifted to the right within M until they all act upon the vacuum state. Usage of (B.6) and (4.3) allows us to then conclude the desired result.

We can see then that once a Fock representation is assumed, parafield theory is equivalent to a normal theory in which both the form of the observables and the states, are restricted in some way.

An explanation for this restriction can be found through the notion of "gauge" invariance. To explore this, we introduce the following transformation on the fermi fields and vacuum:

$$\begin{aligned} \Phi^r(x) &\longrightarrow \sum_{t=1}^{p} g^{rt}\Phi^t(x) \\ a^{(r)} &\longrightarrow \sum_{t=1}^{p} g^{rt} a^{(t)} \\ |\rangle &\longrightarrow |\rangle\,, \end{aligned} \tag{4.5}$$

where the matrices g belong to a p-dimensional unitary representation of some compact Lie group G. Now, as is observed in [52], $P(x_1, x_2)$ is invariant under the group $O(p)$* and moreover this is the minimal group showing this invariance. In addition to this invariance of observables, $O(p)$ also leaves invariant the basic fermi commutation relations and the Fock vacuum condition $a^{(r)}|\rangle = 0$. Because of this overall invariance we say that a theory with observables restricted in the manner implied by (4.2) is $O(p)$ *gauge invariant.*

In [6] a stronger result is derived. It is shown that in an ordinary field theory, if we a priori require that the theory be gauge invariant under $O(p)$ then our observables are restricted precisely to those generated by $P(x_1, x_2)$. Thus in parafield theory the requirement of strong locality is the same as the requirement of $O(p)$ gauge invariance in the corresponding fermi field theory.

We cannot, however, conclude from the above that the two theories are equivalent since we have not considered the effect of the restriction of states implied by parafield theory.

As we have seen, the transformation (4.5) is an automorphism. If we use this property and the invariance of the Fock vacuum condition, it is straightforward to show that all V.E.V.s are also left invariant. As is well known [53] this means that the transformation (4.5) induces a continuous unitary representation of G on the Fock-space of the fermi quantization. As a consequence (see [54]), this implies

* g obviously belonging to the fundamental representation of $O(p)$.

that this representation is decomposable into a direct sum of finite-dimensional irreducible representations of G and hence the Fock-space decomposes into a direct sum of orthogonal finite-dimensional subspaces, each of which carries an irreducible representation of G.

Now within each subspace, a basis may be labelled using the generators of the unitary operator implementing G. It is clear then that the Fock-space has a basis of states of the form $|R, x, d\rangle$, where R denotes the irreducible representation which acts on the vector; x denotes the labelling by group generators which specifies the "direction" within its subspace; and d is another label which indicates that there may be more than one subspace for each irreducible representation.

Now the quantities that are actually physically accessible, are the expectation values of the observables. On the basis states mentioned above these have the form

$$\langle R, x, d|F(V)|R, x, d\rangle \ . \tag{4.6}$$

Let $U(g)$ be the unitary operators implementing the gauge group G, then the invariance of the observables implies that

$$U^{-1}(g)F(V)U(g) = F(V) \qquad \forall g \in G \tag{4.7}$$

and hence that

$$\begin{aligned}\langle R, x, d|F(V)|R, x, d\rangle &= \langle R, x, d|U^{-1}(g)F(V)U(g)|R, x, d\rangle \\ &= \langle R, y, d|F(V)|R, y, d\rangle \end{aligned} \tag{4.8}$$

with

$$U(g)|R, x, d\rangle = |R, y, d\rangle \ .$$

This shows that all states within a particular irreducible subspace are physically equivalent (since $U(g)$ acts transitively on irreducible subspaces). Now it is possible [55] to label *inequivalent* representation subspaces of G using Casimir operators constructed from the field algebra. Since these operators are invariant under the gauge group it follows that they are potential observables and hence we conclude that states coming from inequivalent subspaces are physically inequivalent.

In parafield theory it may be shown [6] that if $G = O(p)$ then the restriction of states implied by the theory forbids the appearance of states belonging to most of the irreducible representations of the gauge group. Given the discussion above, this implies that the parafield theory is not physically equivalent to a normal field theory with an $O(p)$ gauge symmetry. In view of this negative result, one seeks to alter the appropriate gauge group by restricting the possible form of observables—this restriction being clearly above and beyond the requirement of strong locality.

Thus we require that observables be generated only by the following kind of field polynomials:

$$Q(x_1, x_2) = [\psi^*(x_1), \psi(x_2)]_- \ . \tag{4.9}$$

Upon Klein transformation this polynomial becomes

$$Q(x_1, x_2) = \sum_{r=1}^{r} [\Phi^{*r}(x_1), \Phi^r(x_2)]_- \tag{4.10}$$

and such polynomials are easily shown to be invariant under the larger gauge group $U(p)$.

With this larger group, the commutation relations and Fock condition are still invariant. Moreover one can show [52] once again that *all* observables invariant under $U(p)$ can be generated from $Q(x_1, x_2)$. Thus with our restricted set of observables, we can again regard $U(p)$ as a gauge group.

The situation with regard to the restricted set of para states is more positive in the case that $U(p)$ is the gauge group. In fact, one can show [6], [12] that from every irreducible representation subspace of $U(p)$ in the Fock-space of the normal theory, there is exactly *one* state in the Fock-space of the para theory. Since all states within an irreducible subspace are physically equivalent, it follows that the para theory contains all states* physically relevant for a $U(p)$ gauge invariant normal field theory. Thus, with the restriction on observables mentioned above, parafield theory becomes physically equivalent to a $U(p)$ gauge theory since they share the same set of observables and physically relevant states. The description of such a gauge theory with a parafield theory might be regarded as convenient since there are no physically redundant states in the latter theory.

---

* See the note at the end of the proof of theorem 4.3.

4.2. Observables

We turn now to the case of modular field theory and consider firstly the effect of the Klein transformation on strongly local observables. We make the assumption that this Klein transformation is given by (B.27) and (B.37). There are still unresolved questions concerning the equivalence of the representations of normal fields obtained by different Klein transformations (see section 4 Chapter 1), however we shall not go into these here.

If one takes an arbitrary product of modular fields then there is no guarantee that after the tranformations (2.8) and (B.37) are applied, the resulting products of fermi fields will not involve the non-local Klein operator. Certainly if modular field theory is to be compared with a normal field theory then observables in the two theories should coincide. The following result is therefore reassuring:

Theorem 4.1. *After Klein transformation, strongly local observables involving modular fields consist of only normal fermi fields and may be considered as strongly local observables in the fermi field theory.*

Proof: As was seen in the proof of Theorem 3.1 strongly local observables may be written as

$$F(V) = \int h_V F_{00}^{(0)} dx_1 \ldots dy_1 \ldots \tag{4.11}$$

where $F_{00}^{(0)}$ involves field polynomials of the form

$$\phi^{(r_1)}(x_1) \ldots \phi^{(r_l)}(x_l) \phi^{*(t_1)}(y_1) \ldots \phi^{*(t_n)}(y_n) \ , \tag{4.12}$$

with $\sum r_i \equiv \sum t_i \mod m$; $l - n \equiv 0 \mod m$; $l + n$ even.

Upon use of the transformation (B.37), such polynomials become

$$u^{1-r_1}\Phi^{r_1}(x_1) \ldots u^{1-r_l}\Phi^{r_l}(x_l)\Phi^{*t_1}(y_1)u^{t_1-1} \ldots \Phi^{*t_n}(y_n)u^{t_n-1} \ . \tag{4.13}$$

The spatial analogs of (B.2) then allow us to write this as

$$ku^{(l-n-\sum r_i+\sum t_i)}\, \Phi^{r_1}(x_1) \ldots \Phi^{r_l}(x_l)\Phi^{*t_1}(y_1) \ldots \Phi^{*t_n}(y_n) \ , \tag{4.14}$$

where k is some phase factor involving the m'th primitive root of unity. Now because of the restrictions following (4.12) and the fact that $u^m = 1$, we have demonstrated the first part of the theorem. The second part is trivial since any observable consisting of an even product of fermi fields is easily shown to be strongly local. The third restriction following (4.12) requires this for strongly local observables.■

We turn now to the question of identifying a suitable normal gauge theory with which modular field theory may be compared. Since the classification problem for strongly local observables is as yet incomplete, we obviously cannot, as has been done in parafield theory, identify a gauge group which will select out the strongly local observables.

There may, moreover, be fundamental problems in this regard since Theorem 3.4 appears* to rule out the possibility of observables of second degree when $m > 2$. Since the fundamental invariants of the simple Lie-groups are quadratic the existence of a "selecting" gauge group appears problematical. One possible solution to this difficulty lies in the area of non-linear representations [56]. Thus one implements the gauge group not through (4.5) but through a non-linear generalization of it. One might hope that the linear part of the representation (namely the stablity group of the related coset space) would be a group which selected out certain invariant polynomials in the usual way and that the non-linear part of the representation would leave only higher order linear invariants, invariant.

Naturally the above discussion is purely speculative and awaits further investigation for confirmation.

In view of the above difficulties we confine ourselves here to comparing modular field theory with a normal field theory having a $U(m)$ gauge symmetry. We choose such a gauge group for two reasons: Firstly it plays a central role in parafield theory and secondly the physical applications of modular field theory are hoped [27] to lie primarily in the area of colour where the appropriate gauge theory is unitary. As we shall see some progress is possible in the proposed comparison.

We begin by constructing strongly local observables which, when expressed in terms of normal fermi fields, are invariant under $U(m)$. In this regard we have the following results:

Theorem 4.2. *Observables constructed from the following modular field poly-*

* There remains the peculiar possibility that the observables might be of higher than second order in the modular fields but of second order only in the fermi fields.

nomials are strongly local and when Klein transformed, are invariant under the gauge group $U(m)$:

(i) $\overline{C}_m \overline{C}^{*\prime}_m$, *where*

$$\overline{C}_m = \sum_{perm(x_0,\ldots,x_{m-1})} \psi(x_0)\ldots\psi(x_{m-1}) \tag{4.15}$$

$$= \sum_{perm(0,\ldots,m-1)} C_m(x_0,\ldots,x_{m-1}) \,. \tag{4.16}$$

(ii) For $m=3$ *the special case*

$$\overline{M}(x_1,x_2,y_1,y_2) = M(x_1,x_2,y_1,y_2) + M(x_2,x_1,y_1,y_2) \,. \tag{4.17}$$

Proof: Consider firstly the case (i): When the ansatz (2.8) is used, $\overline{C}_m$ becomes (apart from a numerical factor)

$$\overline{C}_m = \sum_{perm} \sum_{r_i=0}^{m-1} \phi^{(r_0)}(x_0)\ldots\phi^{(r_{m-1})}(x_{m-1}) \,. \tag{4.18}$$

Consider now the terms $\phi^{(r_0)}(x_0)\ldots\phi^{(r_j)}(x_j)\ldots\phi^{(r_k)}(x_k)\ldots\phi^{(r_{m-1})}(x_{m-1})$ from this sum. By use of the commutation relations (2.9a) the $\phi^{(r_j)}(x_j)$ and $\phi^{(r_k)}(x_k)$ fields may be interchanged with the result

$$-\eta^{r_j-r_k}\phi^{(r_0)}(x_0)\ldots\phi^{(r_k)}(x_k)\ldots\phi^{(r_j)}(x_j)\ldots\phi^{(r_{m-1})}(x_{m-1}) \,.$$

Thus if $r_j = r_k$ then this term will not be present in the sum (4.18) since this sum extends over all permuations of the spatial indices. Hence it becomes

$$\overline{C}_m = \sum_{perm} \sum_{r_i \neq r_j} \phi^{(r_0)}(x_0)\ldots\phi^{(r_{m-1})}(x_{m-1}) \,. \tag{4.19}$$

If we now apply the Klein transformation (B.37), we obtain

$$\overline{C}_m = \sum_{perm} \sum_{r_i \neq r_j} u^{1-r_0}\Phi^{r_0}(x_0)\ldots u^{1-r_{m-1}}\Phi^{r_{m-1}}(x_{m-1})$$

$$= \sum_{perm} \sum_{r_i \neq r_j} f(r_0,\ldots,r_{m-1})\Phi^{r_0}(x_0)\ldots\Phi^{r_{m-1}}(x_{m-1})\, u^{\frac{-m(m-1)}{2}} \tag{4.20}$$

using (2.9c), the fact that $r_0, \ldots, r_{m-1}$ must be a permutation of $0, \ldots, m-1$ and where

$$f(r_0, \ldots, r_{m-1}) \equiv \eta^{\sum_{i \geq j} r_i r_j - \sum_{i=0}^{m-1} (i+1) r_i} . \tag{4.21}$$

Now (4.20) may be re-expressed as

$$\sum_{perm(x_0,\ldots,x_{m-1})} \sum_{\gamma} f(\gamma(0), \gamma(1), \ldots, \gamma(m-1)) \Phi^{\gamma(0)}(x_{\gamma(0)}) \ldots \Phi^{\gamma(m-1)}(x_{\gamma(m-1)}) \times u^{\frac{-m(m-1)}{2}} . \tag{4.22}$$

where γ is an arbitrary permutation of $0, \ldots, m-1$. When the anticommuting nature of the fermi fields is used, this becomes

$$\left\{ \sum_{perm(x_0,\ldots,x_{m-1})} \Phi^0(x_0) \ldots \Phi^{m-1}(x_{m-1}) \right\} \sum_{\gamma} sign(\gamma) f(\gamma(0), \ldots, \gamma(m-1)) \times u^{\frac{-m(m-1)}{2}} . \tag{4.23}$$

We show now that the sum to the right of the curly brackets is non-zero: Now it is obvious that $\sum_{i=0}^{m-1} \gamma(i) = \sum_{i=0}^{m-1} i$; and also that $\sum_{i=0}^{m-1} \gamma^2(i) = \sum_{i=0}^{m-1} i^2$; so therefore we have

$$\sum_{i=0}^{m-1} i \sum_{j=0}^{m-1} j = \sum_{i=0}^{m-1} \gamma(i) \sum_{j=0}^{m-1} \gamma(j) = \sum_{i=0}^{m-1} \gamma^2(i) + 2 \sum_{i>j} \gamma(i)\gamma(j)$$

which implies that

$$\sum_{i \geq j} \gamma(i)\gamma(j) = \sum_{i \geq j} ij \tag{4.24}$$

and hence

$$f(\gamma(0), \ldots, \gamma(m-1)) = \eta^{\sum_{i \geq j} ij - \sum_{i=0}^{m-1} i} \times \eta^{-\sum_{i=0}^{m-1} i\gamma(i)} . \tag{4.25}$$

If we call the first term on the right hand side η^c, and remember that the determinant of an $m \times m$ matrix $A = \{a_{ij}\}$ is

$$det(A) = \sum_{\gamma} sign(\gamma) a_{0\gamma(0)} a_{1\gamma(1)} \cdots a_{m-1\gamma(m-1)} , \tag{4.26}$$

then we can derive the result

$$\sum_{\gamma} sign(\gamma) f(\gamma(0), \ldots, \gamma(m-1)) = \eta^c det(S) \tag{4.27}$$

where $$S_{ij} = \eta^{-ij} .$$

Consider the matrix $\overline{S}_{ij} = \eta^{ij}$; we have

$$\begin{aligned}(\overline{S}S)_{ij} &= \sum_{k=0}^{m-1} \eta^{ik}\eta^{-kj} \\ &= \sum_{k=0}^{m-1} \eta^{(i-j)k} \\ &= m\delta_{ij}\end{aligned}$$

and so therefore $det(S) \neq 0$. The matrix S was used in (2.7) and the matrix $\frac{1}{\sqrt{m}}\overline{S}$ is the so-called Sylvester matrix [57] and is unitary.

We may now rewrite (4.23) as

$$\left\{ \sum_{r_0,\ldots,r_{m-1}} \varepsilon_{r_0\ldots r_{m-1}} \Phi^{r_0}(x_0)\ldots\Phi^{r_{m-1}}(x_{m-1}) \right\} \eta^c det(S) u^{\frac{-m(m-1)}{2}} \tag{4.28}$$

where $det(S) \neq 0$ and ε is the completely antisymmetric tensor of m'th order.

Let the quantity in brackets in (4.28) be called D_m. This has been shown [6] to transform as follows under $U(m)$:

$$D_m \longrightarrow \frac{1}{det(G)} D_m , \tag{4.29}$$

where G is the matrix implementing $U(m)$ through (4.5). If we take the hermitean conjugate of (4.28), we may deduce that

$$\overline{C}_m \overline{C}'^*_m = k D_m D'^*_m \tag{4.30}$$

where k is a real constant. Now since G is unitary it follows that $det(G)$ is a phase factor and hence $\overline{C}_m \overline{C}'^*_m$ is left invariant by G. Finally Theorem 3.2 and (4.16) show that $\overline{C}_m \overline{C}'^*_m$ can be written as a sum of field polynomials which give rise to strongly local observables and hence any observable constructed from it will obviously also be strongly local. This completes the demonstration of the first case.

In the second case we have

$$\overline{M} = \sum_{\substack{perm(x_1,x_2)\\perm(y_1,y_2)}} \psi(x_1)\psi(x_2)\psi^*(y_2)\psi^*(y_1) - \psi(x_2)\psi^*(y_2)\psi^*(y_1)\psi(x_1)$$
$$+ \psi^*(y_2)\psi^*(y_1)\psi(x_1)\psi(x_2) \ . \tag{4.31}$$

(2.5) allows us to rewrite this as

$$\begin{aligned}\overline{M} = \sum_{perm}\Big[\sum_{r=0}^{2} &\psi^{(r)}(x_1)\psi^{(r)}(x_2)\psi^{*(r)}(y_2)\psi^{*(r)}(y_1)\\ &- \psi(x_2)\psi^*(y_2)\,\delta(x_1-y_1) + \psi^*(y_2)\psi(x_2)\,\delta(x_1-y_1)\\ &- \psi^{(2)}(x_1)\psi^{*(2)}(y_1)\,\delta(x_2-y_2)\Big]\\ = \sum_{perm}\Big[\sum_{r=0}^{2}[&\psi^{(r)}(x_1)\psi^{(r)}(x_2)\psi^{*(r)}(y_2)\psi^{*(r)}(y_1) - \psi^{(r)}(x_2)\psi^{*(r)}(y_2)\,\delta(x_1-y_1)\\ &+ \delta(x_2-y_2)\,\delta(x_1-y_1)\Big] \ . \end{aligned} \tag{4.32}$$

We show that the various order terms in this sum are invariant under $U(3)$. The scalar terms (δ functions) are obviously so. For the second order terms we have, using (2.8) and (2.9c):

$$\begin{aligned}\sum_{r=0}^{2}\psi^{(r)}(x)\psi^{*(r)}(y) &= \tfrac{1}{3}\sum_{r=0}^{2}\sum_{s,t=0}^{2}\eta^{r(s-t)}\phi^{(s)}(x)\phi^{*(t)}(y)\\ &= \sum_{s=0}^{2}\phi^{(s)}(x)\phi^{*(s)}(y) \ . \end{aligned} \tag{4.33}$$

Upon application of the Klein transformation (B.37) this becomes

$$\sum_{s=0}^{2}\Phi^{s}(x)\Phi^{*s}(y) \tag{4.44}$$

and, as is well known, this transforms as the fundamental invariant of $U(3)$. Consider now the fourth order terms:

$$\begin{aligned}&\sum_{\substack{perm(x_1,x_2)\\perm(y_1,y_2)}}\sum_{r=0}^{2}\psi^{(r)}(x_1)\psi^{(r)}(x_2)\psi^{*(r)}(y_2)\psi^{*(r)}(y_1)\\ &= \tfrac{1}{9}\sum_{perm}\sum_{r=0}^{2}\sum_{s,t,v,w}\eta^{r(s+t-v-w)}\phi^{(s)}(x_1)\phi^{(t)}(x_2)\phi^{*(v)}(y_2)\phi^{*(w)}(y_1)\\ &= \tfrac{1}{3}\sum_{perm}\sum_{s+t\equiv v+w}\phi^{(s)}(x_1)\phi^{(t)}(x_2)\phi^{*(v)}(y_2)\phi^{*(w)}(y_1)\end{aligned} \tag{4.45}$$

where we have used (2.8) and (2.9c) and the equivalence sign is modulo 3. Now since the sum is over permutations of the x and y spatial indices, it follows that we must have $s \neq t$ and $v \neq w$ within the summation. When the Klein transformation is applied we obtain

$$\frac{1}{3}\sum_{perm}\sum_{\substack{s+t\equiv v+w\\ s\neq t\ v\neq w}} u^{1-s}\Phi^{s}(x_1)u^{1-t}\Phi^{t}(x_2)\Phi^{*v}(y_2)u^{v-1}\Phi^{*w}(y_1)u^{w-1} \tag{4.46}$$

$$= \frac{1}{3}\sum_{perm}\sum_{\substack{s+t\equiv v+w\\ s\neq t\ v\neq w}} \eta^{s(1-t)}u^{2-s-t}\Phi^{t}(x_1)\Phi^{t}(x_2)\Phi^{*v}(y_2)\Phi^{*w}(y_1)$$

$$\times\, u^{v+w-2}\eta^{w(v-1)}\ . \tag{4.47}$$

The fact that the sum only extends over $s+t\equiv v+w$ and also (2.9c), means this may be rewritten as

$$\frac{1}{3}\sum_{perm}\sum_{\substack{s+t\equiv v+w\\ s\neq t\ v\neq w}} \eta^{s-w+wv-st}\,\Phi^{s}(x_1)\Phi^{t}(x_2)\Phi^{*v}(y_2)\Phi^{*w}(y_1)\ . \tag{4.48}$$

Consider the possible values for s and t in this sum: Clearly, apart from order, these are 0,1 0,2 and 1,2. Given the restriction $s+t\equiv v+w$ mod 3, these values must be matched by the same values (apart from order) for v and w. When the sum over spatial indices is taken into account, (4.48) becomes

$$\frac{1}{3}\sum_{perm}\sum_{a>b}F(a,b)\Phi^{a}(x_1)\Phi^{b}(x_2)\Phi^{*b}(y_2)\Phi^{*a}(y_1)$$

$$F(a,b) = 1-\eta^{b-a}-\eta^{a-b}+\eta^{a-b+b-a} = 3$$

and so therefore the sum is

$$\sum_{perm}\sum_{a>b}\Phi^{a}(x_1)\Phi^{b}(x_2)\Phi^{*b}(y_2)\Phi^{*a}(y_1)\ .$$

The sum over permutations means that this becomes

$$\sum_{perm}\sum_{i,j}\Phi^{i}(x_1)\Phi^{j}(x_2)\Phi^{*j}(y_2)\Phi^{*i}(y_1)$$

$$= \sum_{perm}\sum_{i,j}[\Phi^{i}(x_1)\Phi^{*i}(y_1)\Phi^{j}(x_2)\Phi^{*j}(y_2)$$

$$-\,\delta(x_2-y_1)\,\Phi^{i}(x_1)\Phi^{*i}(y_2)] \tag{4.49}$$

which is manifestly invariant under $U(3)$. Given (4.17), $\overline{M}$ is, by Theorem 3.2, strongly local. ∎

For future reference we express $\overline{M}$ in terms of the fermi fields. By use of (4.32), (4.44) and (4.49), we deduce that

$$\overline{M} = \sum_{\substack{perm(x_1,x_2)\\ perm(y_1,y_2)}} \sum_{i,j=0}^{2} [\Phi^i(x_1)\Phi^{*i}(y_1)\Phi^j(x_2)\Phi^{*j}(y_2) - 2\delta(x_1-y_1)\,\Phi^i(x_2)\Phi^{*i}(y_2)$$
$$+\,\delta(x_2-y_2)\,\delta(x_1-y_2)]\;. \tag{4.50}$$

A few comments are appropriate regarding Theorem 4.2: Firstly, in the case that $m = 2$ we are dealing with parafield theory and so the field polynomial $[\psi^*(y), \psi(x)]_-$ is strongly local and invariant under $U(2)$. Secondly one might expect that strongly local $U(m)$ invariant field polynomials could be constructed from the polynomial M for arbitrary m. Whether this is so is not clear. We have, however, the following conjecture:

Conjecture: *The polynomial*

$$\overline{M} \equiv \sum_{perm(x_1,\ldots,x_{m-1})} M(x_1,\ldots,x_{m-1},y_1,\ldots,y_{m-1}) \tag{4.51}$$

is strongly local and invariant under $U(m)$.

The proof of this conjecture appears to involve complicated algebraic computations and is not attempted here.

4.3. The non-relativistic states

Having shown that there exist appropriate observables invariant under $U(m)$, we now consider the properties of states with respect to this gauge group. In particular we shall be interested in the question of whether the Fock-space of modular quantization contains all the physically relevant states for a $U(m)$ gauge theory. We begin by considering this question for what we shall term the "non-relativistic sector" of the Fock-space. By this we shall mean the subspace of the relativistic Fock-space generated by linear combinations of states obtained by applying a number of *particle* as opposed to *anti-particle* creation operators to the vacuum.

To consider the decomposition of the non-relativistic sector into direct sums of subspaces invariant under $U(m)$, we consider the subalgebra of $\mathcal{C}$ obtained by restricting the momentum indices to a finite set of N values. The Fock-space obtained by applying creation operators from the restricted set is finite-dimensional and it is straightforward to show that the representation of the restricted algebra on this space is irreducible.

In the same way as we introduced a unitary group on the upper indices of elements from $\mathcal{C}$, we may introduce another such group $U(N)$ which acts on the N momentum indices of the restricted algebra $\mathcal{C}_N$:

$$d_{k_i}^{(r)} \to \sum_{j=1}^{N} g^{ij} d_{k_j}^{(r)} . \tag{4.52}$$

Such a transformation leaves the commutation relations of $\mathcal{C}_N$ invariant and if we assume that the group leaves the vacuum invariant, then we may use the argument given above for $U(m)$ to conclude that $U(N)$ is implemented on $\mathcal{F}(\mathcal{C}_N)$ as a continuous unitary representation and hence $\mathcal{F}(\mathcal{C}_N)$ decomposes into finite-dimensional irreducible subspaces of this group as well. Consider now the subspace $V_n \subset \mathcal{F}(\mathcal{C}_N)$ which consists of states of the form

$$d_{k_1}^{*(r_1)} \dots d_{k_n}^{*(r_n)} |\rangle ,$$

where $n \leq N$. Given that non-zero distinct states of this form are orthogonal, it follows from the usual theory of the representations of the unitary group (see Boerner [58]) that states within the possible irreducible subspaces of the two groups $U(N)$ and $U(m)$ may be projected out by means of the so-called Young symmetrizers. These antisymmetrize and symmetrize with respect to certain of the r_i and k_i indices (depending on whether $U(m)$ or $U(N)$ respectively is being considered) according to a Young tableau. An example of such a tableau may be seen in Figure 1.

Into such a diagram one puts the integers $1, \dots, n$ in order to specify how the Young symmetrizer is to operate. Thus the numbers in the diagram refer to the indices to be symmetrized or antisymmetrized (1 stands for k_1 or r_1 and so on). One then *antisymmetrizes* with respect to all indices in the columns of the tableau and then *symmetrizes* with respect to those in the rows (or vice versa).

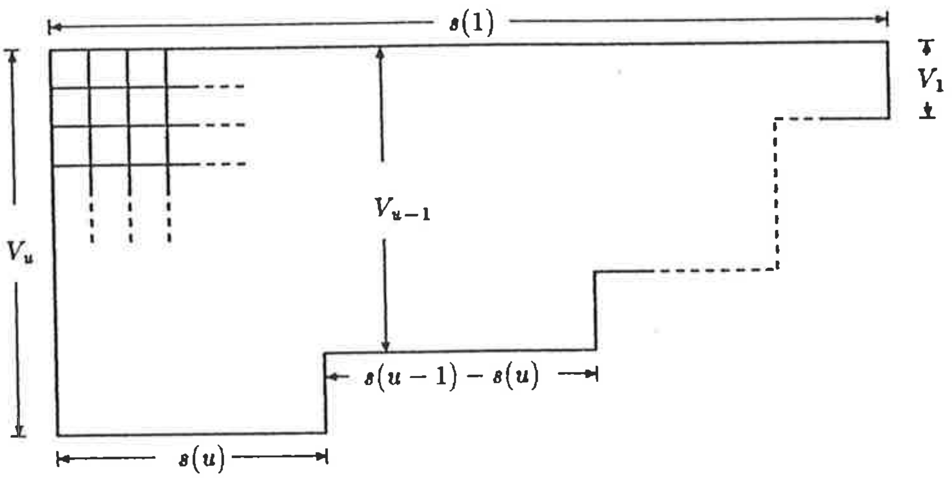

Figure 1. **The Young tableau referred to in the text.**

It is worth noting at this point that since the r_i indices cannot take on more than m values, any attempt to antisymmetrize with respect to more than m indices will result in zero. As a consequence there are no irreducible subspaces of $U(m)$ in V_n corresponding to tableaux with columns of length more than m.

Consider now the operation of antisymmetrization with respect to a certain set of momentum indices $k_{i_1}, \ldots, k_{i_q}$. By rearrangement of such an antisymmetrization it is easy to see, using the anticommuting nature of the fermi operators, that it corresponds to a *symmetrization* of the indices $r_{i_1}, \ldots, r_{i_q}$. Similarly symmetrization with respect to the momentum indices corresponds to antisymmetrization with respect to the "gauge" indices.

It follows from the above that if we project into a particular irreducible subspace of $U(N)$ specified by a particular Young tableau then we will also be in an irreducible subspace of $U(m)$ specified by the so-called *conjugate* tableau (the transposed tableau). Now it is straightforward to check that the actions of the groups $U(m)$ and $U(N)$ commute on $\mathcal{F}(C_N)$. It follows that this space decomposes into irreducible subspaces of the product group $U(m) \otimes U(N)$. Given the nature of such subspaces (see, for example, van der Waerden [59]), it follows that the projection mentioned above is into a particular irreducible subspace of $U(m) \otimes U(N)$.

We shall now demonstrate that there cannot be two equivalent irreducible representations of $U(m) \otimes U(N)$ on V_n:

Suppose to the contrary that there were two such representations. Denote the subspaces corresponding to these two representations by U^1 and U^2. Since the representations are equivalent it is possible to construct bases for the two subspaces which transform in the same way under $U(m) \otimes U(N)$. Thus

$$\begin{aligned} U_m(g_1)U_N(g_2)v^1_{ij} &= \sum_{k,l} h_1^{ik} h_2^{jl} v^1_{kl} \\ U_m(g_1)U_N(g_2)v^2_{ij} &= \sum_{k,l} h_1^{ik} h_2^{jl} v^2_{kl} , \end{aligned} \tag{4.53}$$

where v^1_{ij} and v^2_{ij} are the desired bases vectors for U^1 and U^2 respectively; h_1 and h_2 are irreducible representations of $U(m)$ and $U(N)$ respectively and $U_m(g_1) \in U(m)$, $U_N(g_2) \in U(N)$. Define now the following linear operator on $\mathcal{F}(\mathcal{C}_N)$:

$$\begin{aligned} W v^1_{ij} &\equiv v^2_{ij} \\ W w_{ij} &\equiv w_{ij} \end{aligned} \tag{4.54}$$

where w_{ij} are the bases vectors of all other irreducible subspaces of $U(m) \otimes U(N)$ in $\mathcal{F}(\mathcal{C}_N)$. It follows easily from (4.53) and (4.54) that W commutes with all elements from $U(m) \otimes U(N)$. In otherwords

$$U_m(g_1)U_N(g_2)W U_N^{-1}(g_2)U_m^{-1}(g_1) = W . \tag{4.55}$$

Now since the representation of $\mathcal{C}_N$ on $\mathcal{F}(\mathcal{C}_N)$ is irreducible it follows that the commutant of $\mathcal{C}_N$ consists of multiples of the identity (Schur's Lemma). As a consequence the bicommutant $\mathcal{C}''_N$ is equal to the set of linear operators on $\mathcal{F}(\mathcal{C}_N)$. Now a very well known result (orginally due to Von Neumann, for a modern reference see [60]) tells us that $\mathcal{C}''_N$ is equal to the closure of $\mathcal{C}_N$ under, amongst others, the strong topology. Now the operators in $\mathcal{C}_N$ belong to a finite-dimensional space since they are finite matrices. As is well known [61], all the relevant topologies are eqiuvalent on such a topological vector space and as a result we deduce that W is in the norm closure of $\mathcal{C}_N$ (the operator norms generate the *uniform* topology which is not covered in the bicommutant theorem). Now it has been shown in [62] that if an element in the norm closure of a Clifford algebra is invariant under $U(m)$

then it may be approximated in norm by invariant polynomials from the algebra. It follows that W may be approximated in norm by polynomials of creation and annihilation operators from $\mathcal{C}_N$ which are invariant under $U(m)$. Now it is further shown in [63] that the invariant polynomials of $\mathcal{C}_N$ must be polynomials in the fundamental invariants of $U(m)$. In otherwords they must be polynomials in

$$P_{kl} = \sum_{r=0}^{m-1} d_k^{*(r)} d_l^{(r)} . \tag{4.56}$$

Once the relation

$$[P_{kl},\ d_m^{(s)}]_- = -\delta_{km} d_l^{(s)}$$

is noted it becomes clear that the P_{kl} are merely the generators for the group $U(N)$. Let $f_q(P_{kl})$ be the sequence of invariant polynomials which approximates W in norm. It follows that

$$\begin{aligned}
& \|f_q(P_{kl}) - W\| \to 0 \quad \text{as} \quad q \to \infty \\
\Rightarrow \quad & \|(f_q(P_{kl}) - W)v_{ij}^1\| \to 0 \quad \text{as} \quad q \to \infty \\
\Rightarrow \quad & \| \textstyle\sum_{p,t} a_{ij}^{pt}(q) v_{pt}^1 - v_{ij}^2\| \to 0 \quad \text{as} \quad q \to \infty .
\end{aligned} \tag{4.57}$$

The last step follows from the fact that U^1 is transformed into itself by $U(N)$ and hence also by the generators of this group. Equation (4.56) cannot hold because v_{ij}^2 is linearly independent of U^1 and so we have a contradiction.

We complete the description of the decomposition of V_n into irreducible subspaces of $U(m) \otimes U(N)$ by showing that every irreducible subspace characterized by a Young tableau for $U(N)$ with rows of length less than or equal to m is, in fact present: Into the tableau given in Figure 1, we place integers sequentially across the rows starting at the top left-hand corner. With reference to such a tableau we consider the state

$$\begin{aligned}
& d_{k_1}^{*(0)} \ldots d_{k_{s(1)}}^{*(s(1)-1)} d_{k_{s(1)+1}}^{*(0)} \ldots d_{k_{2s(1)}}^{*(s(1)-1)} d_{k_{2s(1)+1}}^{*(0)} \\
& \ldots d_{k_{V_1 s(1)+1}}^{*(0)} \ldots d_{k_{V_1 s(1)+s(2)}}^{*(s(2)-1)} \ldots d_{k_n}^{*(s(u)-1)} |\rangle ,
\end{aligned} \tag{4.58}$$

with the k_i *all* distinct. When the antisymmetrization process is carried out, the anticommuting nature of the fermi operators ensures that the state is simply multiplied by an integer. After the symmetrization process is carried out we may use the linear independence of distinct non-zero fermi states to conclude that the result is non-zero.

The above decomposition of V_n into irreducible subspaces of the group $U(m) \otimes U(N)$ has been considered in a slightly different context by Bracken and Green [64].

We now consider the non-relativistic modular Fock space $\mathcal{F}(\mathcal{A})$ for which we have the following result:

Theorem 4.3. *The modular Fock-space $\mathcal{F}(\mathcal{A})$ possesses all physically relevant states from the non-relativistic sector of a $U(m)$ gauge theory.*

Proof: Since the subspaces V_n discussed above span $\mathcal{F}(\mathcal{C})$ it suffices* to show that there is a state in $\mathcal{F}(\mathcal{A})$ belonging to every irreducible subspace of $U(m)$ in V_n. Consider the subspace $V_n^m \subset \mathcal{F}(\mathcal{A})$ generated by states of the form

$$a^*_{k_1} \dots a^*_{k_n} |\rangle \tag{4.59}$$

where the k_i are allowed to take on the N values of momenta present in the states of V_n. Now by the Klein transformation (B.3), equation (2.15) and the action of Klein operators on $\mathcal{F}(\mathcal{A})$ given by equations (B.20), (B.29) and (4.7) Chapter 2, we deduce that $V_n^m \subset V_n$. States within the irreducible subspaces of $U(m) \otimes U(N)$ may obviously now be obtained by applying Young symmetrizers to the momentum indices of the states in V_n^m. Suppose there is a state $\phi_i \in V_n^m$ which belongs to a particular irreducible subspace of $U(m) \otimes U(N)$. Now since this group acts transitively on its irreducible subspaces, any state ϕ_i' within the subspace may be written as

$$\phi_i' = U_m(g_1) U_N(g_2) \phi_i \ . \tag{4.60}$$

It follows that there exists a state

$$\phi_i'' = U_m(g_1^{-1}) \phi_i' = U_N(g_2) \phi_i \tag{4.61}$$

---

* See the note at the end of this proof.

which belongs to the same irreducible subspace of $U(m)$ as ϕ_i' does. Now since the group $U(N)$ acts on the momentum indices, it takes states within V_n^m into other states within the same subspace and as a result $\phi_i'' \in V_n^m$. This demonstrates that if there exists just one state in V_n^m within a particular $U(m) \otimes U(N)$ irreducible subspace of V_n then there exist states from V_n^m which are in every irreducible subspace of $U(m)$ within the $U(m) \otimes U(N)$ irreducible subspace.

In the light of this and also the discussion about the possible irreducible subspaces of $U(m) \otimes U(N)$ in V_n, it suffices to demonstrate that there is a state $\phi \in V_n^m$ such that

$$\delta\phi \neq 0 \tag{4.62}$$

with δ being a Young symmetrizer acting on the momentum indices and corresponding to an arbitrary tableau with row lengths less than or equal to m. We proceed now to a proof of this:

We write δ as

$$\delta = \eta\theta \ , \tag{4.63}$$

where η symmetrizes with respect to all arguments within rows of the corresponding tableau and θ antisymmetrizes with respect to arguments within columns. Now consider an arbitrary $\phi \in V_n^m$ which has *all* its momenta distinct.

By use of the Klein transformation we obtain

$$\phi = \sum_{r_i} d_{k_1}^{*(r_1)} u^{r_1 - 1} d_{k_2}^{*(r_2)} u^{r_2 - 1} \dots d_{k_n}^{*(r_n)} u^{r_n - 1} |\rangle \ . \tag{4.64}$$

Now since the vacuum has grading 0 (see Appendix B) it follows that $u|\rangle = |\rangle$. Usage of (B.2) and (B.27) now shows that

$$\phi = \sum_{r_i} f(r_1, \dots, r_n) d_{k_1}^{*(r_1)} \dots d_{k_n}^{*(r_n)} |\rangle$$

with (4.65)

$$f(r_1, \dots, r_n) \equiv \eta^{\sum_{i=1}^{n}(i-1)r_i - \sum_{i<j} r_j r_i} \ .$$

Consider now a permutation γ of the momentum indices of ϕ. It is straightforward to show that

$$\gamma\phi = \sum_{r_i} sign(\gamma) f(r_{\gamma(1)}, \dots, r_{\gamma(n)}) d_{k_1}^{*(r_1)} \dots d_{k_n}^{*(r_n)} |\rangle \ . \tag{4.66}$$

It follows from the usual properties of fermi states that the states

$$d_{k_1}^{*(r_1)} \dots d_{k_n}^{*(r_n)} |\rangle \tag{4.67}$$

are non-zero and linearly independent for different choices of $r_1, \dots, r_n$. It therefore suffices to show that

$$\begin{aligned} &\delta\big(f(0,\dots,s(1)-1,0,\dots,s(1)-1,0,\dots,0,\dots,s(2)-1,\dots,s(u)-1) \\ &\qquad d_{k_1}^{*(0)} \dots d_{k_{s(1)}}^{*(s(1)-1)} d_{k_{s(1)+1}}^{*(0)} \dots d_{k_{2s(1)}}^{*(s(1)-1)} d_{k_{2s(1)+1}}^{*(0)} \\ &\qquad \dots d_{k_{V_1 s(1)+1}}^{*(0)} \dots d_{k_{V_1 s(1)+s(2)}}^{*(s(2)-1)} \dots d_{k_n}^{*(s(u)-1)} |\rangle\big) \neq 0 \,. \end{aligned} \tag{4.68}$$

The $s(j)$ and V_j in equation (4.68) refer to the notation used in Figure 1. Apply now the antisymmetrizer θ to the state in equation (4.68). Clearly this antisymmetrization will apply only to arguments with equal values of r_i. As a result, equation (4.66) implies that the result will simply be a non-zero numerical multiple of the original state. It is thus clear that the only non-trivial part of the proof is the symmetrization. If we define

$$\gamma(f(r_1,\dots,r_n)) \equiv sign(\gamma) f(r_{\gamma(1)},\dots,r_{\gamma(n)}) \;, \tag{4.69}$$

then it is clear from equation (4.66) that we need only show that $\eta(f) \neq 0$, where f has the form given in (4.68). Now with reference to the Young symmetrizer δ and Figure 1, we may rewrite $f(r_1,\dots,r_n)$ as η^g, with

$$g = \sum_{l=1}^{u} \sum_{k=1}^{V_l} \left[\textstyle\sum_{i=1}^{s(l)} (i + t_{lk} - 1) r_{i+t_{lk}} - \sum_{1 \le i < j}^{s(l)} r_{i+t_{lk}} r_{j+t_{lk}} \right] \tag{4.70a}$$

$$t_{lk} = \sum_{p=1}^{l-1} (V_p - V_{p-1}) s(p) + k s(l) \quad ; \; V_0 \equiv 0 \,. \tag{4.70b}$$

When the symmetrizer η is applied to $f(0,\dots,s(1)-1,0,\dots,s(u)-1)$ we obtain

$$\eta(f) = \prod_{l=1}^{u} \prod_{k=1}^{V_l} \sum_{\gamma_l} sign(\gamma_l) \eta^{g_{lk}} \tag{4.71}$$

$$g_{lk} = \sum_{i=0}^{s(l)-1} (i + t_{lk}) \gamma_l(i) - \sum_{0 \le i \le j}^{s(l)-1} \gamma_l(j) \gamma_l(i) \;, \tag{4.72}$$

where γ_l permutes $0, \ldots, s(l)-1$. By use of arguments similar to those developed in the proof of the first part of Theorem 4.2, it follows that the only part of (4.72) not left invariant under γ_l is the term

$$\sum_{i=0}^{s(l)-1} i\gamma_l(i) \ .$$

From the form of (4.71) it clearly now suffices to show that

$$\sum_{\gamma_l} sign(\gamma_l) \prod_{i=0}^{s(l)-1} \eta^{i\gamma_l(i)} \neq 0 \ . \tag{4.73}$$

We deduce immediately that the left-hand side of (4.73) is the determinant of the $s(l) \times s(l)$ Sylvester matrix S^l which has components

$$(S^l)_{ij} = \eta^{ij} \ . \tag{4.74}$$

The determinant of this matrix is known [57], and is

$$det(S^l) = \prod_{k>j\geq 0}^{s(l)-1} (\eta^k - \eta^j) \ . \tag{4.75}$$

This will obviously be non-zero when $s(l) \leq m$. If $s(l) > m$ then we are dealing with a Young tableau with a row of length greater than m. As we saw previously such tableaux are not relevant to our considerations. The proof of Theorem 4.3 is now complete. ■

Additional note: The above proof is not quite complete for the following reason: Although we have shown there is a state in $\mathcal{F}(\mathcal{A})$ for a spanning set of irreducible subspaces of $U(m)$ in $\mathcal{F}(C)$, we have not shown that there are states in $\mathcal{F}(\mathcal{A})$ corresponding to *sums* of states from the irreducible subspaces. Let ψ_1, $\psi_2 \in \mathcal{F}(C)$ belong to two different irreducible subspaces. We have shown that there are states ϕ_1, $\phi_2 \in \mathcal{F}(\mathcal{A})$ such that $U_m(g_1)\phi_1 = \psi_1$ and $U_m(g_2)\phi_2 = \psi_2$. It is clear, however, that $\psi_1 + \psi_2 \neq U_m(g_3)(\phi_1 + \phi_2)$ in general, which shows that the state $\phi_1 + \phi_2$ is not neccessarily physically equivalent to the state $\psi_1 + \psi_2$. We indicate a possible resolution of this difficulty:

It has been observed in [6] that superselection rules operate between states in $\mathcal{F}(\mathcal{C})$ which belong to inequivalent irreducible subspaces of $U(m)$. As a consequence sums of states from inequivalent subspaces are not physical in the gauge theory and hence need not be demonstrated to exist in $\mathcal{F}(\mathcal{A})$. Now the irreducible representations of $U(m)$ are such (see [58]) that any distinct standard Young tableau specifies a distinct (up to equivalence) irreducible representation. As a consequence all irreducible subspaces of a particular character must lie in the union of the subspaces V_n discussed above (notice that this is not the case for the gauge group $SU(m)$). It is clear from the decomposition of these subspaces that by choosing the finite set of momenta appropriately (and hence N large enough), any *finite* sum χ of states from the spanning $U(m)$ subspaces can be placed in an irreducible subspace of $U(m) \otimes U(N)$ and hence by the use of the argument at the beginning of the proof of the last theorem, a state $\chi_m \in \mathcal{F}(\mathcal{A})$ can be found such that $\chi = U_m(g)\chi_m$ which demonstrates physical equivalence. Obviously infinite sums of states need also to be considered and in order to do this the group $U(N)$ needs to be extended to the infinite unitary group which acts on a countable set of indices (the full set of momentum values). It is conjectured that the results derived at the beginning of the present subsection also hold in this case. The proofs given there appear to require modification to deal with topological complications.

We have shown above that there is *at least* one state in the modular Fock-space for a spanning set of irreducible subspaces of $U(m)$ in the non-relativistic sector of the gauge theory. In fact, there are in general more. Consider equation (2.16), this may be rewritten as

$$(a^*_{k_n} \dots a^*_{k_1}|\rangle, a^*_{j_n} \dots a^*_{j_1}|\rangle) = \delta_{k_1 j_1} \dots \delta_{k_n j_n} \,. \tag{4.76}$$

Clearly this equation implies that any non-trivial permutation of the indices k_i for the state $a^*_{k_1} \dots a^*_{k_n}|\rangle$ results in another non-zero, *orthogonal*, state (providing that the k_i are distinct). If we now consider the subspace of $U^n \subset V_n^m$ formed by allowing permutations only of *distinct* k_i then it follows that for $n \leq m$, $dim(U^n) = n!$. The orthogonality of the permuted states allows us then to define a representation of the symmetric group S_n on U^n. The decomposition of U^n into irreducible subspaces of S_n in this instance is well known [65]: There are d_l

of the d_l-dimensional subspaces which carry the l'th irreducible representation. Each irreducible representation has the symmetry property of a Young tableau with n boxes. Consequently states within each of the d_l-dimensional irreducible subspaces of S_n also belong to the one irreducible subspace of $U(m) \otimes U(N)$ in V_n.

Consider a state ω within a particular d_l-dimensional irreducible subspace. In the light of previous discussion it may obviously be written as

$$\omega = \sum_{r_i} g(r_1, \ldots, r_n) d_{k_1}^{*(r_1)} \ldots d_{k_n}^{*(r_n)} |\rangle \ . \tag{4.77}$$

Now it is clear from the action of the group $U(N)$ that a spanning set of states for the $U(N)$ irreducible subspace of ω may be obtained by allowing the momentum indices in (4.77) to take on all possible N values. If the new values are simply a permutation of the original set then the new state will belong to the same irreducible subspace of S_n as did ω (in fact since the permutation of distinct momentum indices is an operator belonging to $U(N)$, these states will belong to the same $U(N)$ irreducible subspace as ω). On the other hand, if the new values are not such a permutation then clearly the state will not belong to U^n. The consequence of this is that states belonging to different irreducible subspaces of equivalent representations of S_n, belong also to different irreducible subspaces of $U(N)$ within the one irreducible subspace of $U(m) \otimes U(N)$. From this we may deduce that within a different $U(N)$ irreducible subspace lies a state which belongs both to V_n^m and to the same $U(m)$ irreducible subspace as ω. Given the distinctness of the $U(N)$ irreducible subspace for this other state, it is clear from the transitivity of $U(N)$ that this new state must be distinct from ω.

We now have the following conjecture:

Conjecture: *For $n \leq m$ particle states, every $U(m)$ irreducible subspace of of $\mathcal{F}(\mathcal{C}_N)$ contains d_l linearly independent states from $\mathcal{F}(\mathcal{A})$. The number d_l is the dimension of the irreducible representation of S_n corresponding to the Young tableau which characterizes the particular $U(m)$ representation.*

It is interesting to contrast this "degeneracy" of description with the situation in parafield theory where there is only one state for every irreducible subspace of $U(p)$. The difference is caused by the fact that in parafield theory states belonging to U^n are in general not linearly independent under the full group of permutations (this is due to the relation (1.1b)). As a consequence the space U^n has dimension less than $n!$ and in fact contains every representation of S_n just once.

4.4. The relativistic states

We now extend the result of the previous subsection to states involving anti-particles as well as particles. We shall prove that all physically relevant $U(m)$ states occur here as well. The relativistic extension is non-trivial because the anti-particle operators transform according to the conjugate rather than fundamental representation of $U(m)$. The proof we shall give is close conceptually to the corresponding proof for parafield theory [12].

Theorem 4.4. *The modular Fock-space $\mathcal{F}(\mathcal{A})$ possesses all* the physically relevant states of a $U(m)$ gauge theory.*

Proof: Consider a particular irreducible subspace of $U(m)$ from a spanning set of such subspaces in the Fock space $\mathcal{F}(\mathcal{C})$. The space will be spanned by states of the form

$$\phi = \sum_n \sum_{r_i} \Gamma(r_1, \ldots, r_n) d_{i_1}^{*(r_1)}(k_1) \ldots d_{i_n}^{*(r_n)}(k_n) \| \rangle \ , \tag{4.78}$$

where the index i_q takes on the value 1 or -1 to indicate that $d_{i_q}^{*(r_q)}(k_q)$ is a particle or anti-particle creation operator respectively. We shall show that the subspace contains at least one state which belongs to $\mathcal{F}(\mathcal{A})$, the modular Fock-space.

Now the state ϕ may contain factors that are invariant under $U(m)$. These will be products of the polynomials

$$\mathcal{M} \equiv \sum_{r=0}^{m-1} d_1^{*(r)}(k) d_{-1}^{*(r)}(l) \ . \tag{4.79}$$

Such polynomials, *when included in* ϕ, may be rewritten in terms of modular particle and anti-particle operators. This may be deduced from the following lemma:

---

* The incompleteness of the proof of the non-relativistic case applies here as well. The proof easily extends to finite sums of states such as ϕ. It is conjectured that the result also holds for infinite sums.

Lemma 4.5. *Let ϕ' be a state in $\mathcal{F}(C)$ which consists of particles (or anti-particles) of momentum $l_1, \ldots, l_q$ then*

$$\mathcal{M}\phi' = \Big\{[a_1^*(k), a_{-1}^*(l)]_- + \sum_{j=1}^{m-2} a_{-1}(n_1)\ldots a_{-1}(n_j)a_1^*(k)a_{-1}^*(l)$$

$$.a_{-1}^*(n_j)\ldots a_{-1}^*(n_1)\Big\}\phi' \qquad (4.80)$$

providing the n_i are all distinct from all the l_i and l. The $a_{i_q}(k)$ and $a_{i_q}^*(k)$ are particle or anti-particle modular creation and annihilation operators.

Proof: For notational purposes let the quantity in braces on the left-hand side of (4.80) be called $\mathcal{M}_{mod}$.

Now the relations (2.57) from section 2 allow us to shift the operators $a_{-1}(n_i)$ to the right in $\mathcal{M}_{mod}$, obtaining the expression

$$\begin{aligned}\mathcal{M}_{mod} =& a_1^*(k)a_{-1}^*(l) + a_1^{*(-1)}(k)a_{-1}^{*(-1)}(l) \\ &+ \sum_{j=1}^{m-2} a_1^{*(j)}(k)a_{-1}^{*(j)}(l)a_{-1}(n_1)\ldots a_{-1}(n_j)a_{-1}^*(n_j)\ldots a_{-1}^*(n_1).\end{aligned} \qquad (4.81)$$

Consider now

$$\begin{aligned}&a_{-1}(n_1)\ldots a_{-1}(n_j)a_{-1}^*(n_j)\ldots a_{-1}^*(n_1)\phi' \\ &= \Big\{a_{-1}(n_1)\ldots a_{-1}(n_{j-1})a_{-1}^*(n_{j-1})\ldots a_{-1}^*(n_1) \\ &\qquad + (-1)^j a_{-1}(n_1)\ldots a_{-1}(n_{j-1})a_{-1}^{*(1)}(n_j)\ldots a_{-1}^{*(1)}(n_1)a_{-1}^{(j)}(n_j)\Big\}\phi' .\end{aligned}$$

The second term of the last line vanishes due to the condition that $n_j \neq l_i \ \forall i$. Evidently the above argument can be extended until we conclude that

$$a_{-1}(n_1)\ldots a_{-1}(n_j)a_{-1}^*(n_j)\ldots a_{-1}^*(n_1)\phi' = \phi' . \qquad (4.82)$$

When (4.82) is combined with (4.81), we obtain

$$\mathcal{M}_{mod}\phi' = \sum_{r=0}^{m-1} a_1^{*(r)}(k)a_{-1}^{*(r)}(l)\phi' . \qquad (4.83)$$

With the use of equations (2.55) and (2.53), this becomes

$$\begin{aligned}&\frac{1}{m}\sum_{r=0}^{m-1}\sum_{s,t=0}^{m-1} \eta^{r(s-t)}b^{*(t)}(k)e^{*(s)}(l)\phi' \\ &= \sum_{s=0}^{m-1} b^{*(s)}(k)e^{*(s)}(l)\phi'\end{aligned} \qquad (4.84)$$

When use is made of the Klein transformations (B.3) and (B.35) this becomes

$$\sum_{s=0}^{m-1} d_{-1}^{*(s)}(k) d_{-1}^{*(s)}(l)\phi' \tag{4.85}$$

which is what was required. ■

As an obvious extension to the above lemma, if we have $\mathcal{M}_1 \ldots \mathcal{M}_l \phi'$ then this may be replaced by $\mathcal{M}^1_{mod} \cdots \mathcal{M}^l_{mod}\phi'$ providing the n_i from different $\mathcal{M}^k_{mod}$ are distinct.

The consequence of the above discussion is that we can write invariant factors in ϕ in terms of modular fields. It suffices therefore to consider the "lowest configuration" corresponding to the particular irreducible representation (in otherwords one not involving invariant factors). As was observed in [12] in such a situation the state ϕ is homogenous in $d_1^{*(r)}(k)$ and $d_{-1}^{*(r)}(k)$ seperately. This means that we may write

$$\phi = \sum_{r_i} \Gamma(r_1, \ldots, r_n) d_1^{*(r_1)}(k_1) \ldots d_1^{*(r_p)}(k_p) d_{-1}^{*(r_{p+1})}(k_{p+1}) \ldots d_{-1}^{*(r_n)}(k_n)|\,|\rangle \tag{4.86}$$

with p and n fixed.

Consider now the following state:

$$\begin{aligned}
&\frac{1}{(m-1)!} \sum_{t_i, s_i} \varepsilon_{t_1 t_2 \ldots t_m}\, \varepsilon_{s_1 \ldots s_{m-1} r_n} d_{-1}^{*(t_1)}(k_n) d_1^{(t_2)}(j_1) \ldots d_1^{(t_m)}(j_{m-1}) \\
&\qquad\qquad\qquad . d_1^{*(s_{m-1})}(j_{m-1}) \ldots d^{*(s_1)}(j_1)|\,|\rangle \\
&= \frac{1}{(m-1)!} \sum_{t_i, s_i} \varepsilon_{t_1 t_2 \ldots t_m}\, \varepsilon_{s_1 \ldots s_{m-1} r_n}\, \delta^{t_m s_{m-1}} \delta^{t_{m-1} s_{m-2}} \ldots \delta^{t_2 s_1} d_{-1}^{*(t_1)}(k_n)|\,|\rangle \\
&= \frac{1}{(m-1)!} \sum_{t_i} \varepsilon_{t_1 t_2 \ldots t_m}\, \varepsilon_{t_2 \ldots t_m r_n} d_{-1}^{*(t_1)}(k_n)|\,|\rangle \\
&= \sum_{t_1} \delta_{t_1 r_n} d_{-1}^{*(t_1)}(k_n)|\,|\rangle \\
&= d_{-1}^{*(r_n)}(k_n)|\,|\rangle \,.
\end{aligned} \tag{4.87}$$

We may therefore replace the latter state by the former. Now as we saw at the end of section 2 the operator $a^*_{-1}(k)$ acts algebraically like $a_1(k)$ apart from a degree of freedom associated with being an anti-particle. Consider now the operator $\overline{C}_m$

from Theorem 4.2. When the momentum analog of this operator is considered we may use the proof of this theorem, in particular equation (4.28), to show that

$$\sum_{perm(k_1,\ldots k_m)} a_1(k_1)\ldots a_1(k_m) = c. \sum_{r_1,\ldots,r_m} \varepsilon_{r_1\ldots r_m} d_1^{(r_1)}(k_1)\ldots d_1^{(r_m)}(k_m) u^{\frac{-m(m-1)}{2}}$$

and hence that

$$\begin{aligned}\overline{G}_m \equiv & \sum_{perm(k_1,\ldots,k_m)} a^*_{-1}(k_1)a_1(k_2)\ldots a_1(k_m) \\ = c. & \sum_{r_1,\ldots,r_m} \varepsilon_{r_1\ldots r_m} d_{-1}^{*(r_1)}(k_1) d_1^{(r_2)}(k_2)\ldots d_1^{(r_m)}(k_m) u^{\frac{-m(m-1)}{2}} \quad . \end{aligned} \tag{4.88}$$

When (4.87) and (4.88) are combined, we conclude that

$$d_{-1}^{*(r_n)}(k_n)\|\rangle = c'\overline{G}_m^n \sum_{s_i} \varepsilon_{s_1\ldots s_{m-1}r_n} d_1^{*(s_{m-1})}(j_{m-1})\ldots d_1^{*(s_1)}(j_1)\|\rangle \tag{4.89}$$

where $\overline{G}_m^n$ involves the momenta $k_n, j_1, \ldots, j_{m-1}$ rather than those in (4.88). If the momenta $j_1, \ldots, j_{m-1}$ are chosen to be distinct from $k_1, \ldots, k_{m-1}$ then in (4.86) we may shift the operator $\overline{G}_m^n$ to the left.

If the above process is repeated for all the anti-particle operators in ϕ, we may eventually rewrite it as a product of $n-p$ $\overline{G}_m$ factors and $p+(m-1)(n-p)$ particle creation operators. Now given that $d_{-1}^{*(r)}(k)$ transforms under $U(m)$ in the same way as $d_1^{(r)}(k)$, it follows from (4.29) that the $\overline{G}_m$ transform as singlets under $U(m)$. In otherwords, they acquire phase factors upon transformation. In addition they can be written in terms of modular fields. The remaining particle creation operators applied to the vacuum belong to an irreducible representation subspace of $U(m)$ in the non-relativistic sector of the Fock-space. As Theorem 4.3 has shown, it is always possible to produce a state from such a subspace by applying only modular particle creation operators to the vacuum. We have therefore shown that the irreducible subspace containing ϕ also contains a state which belongs to the modular Fock-space. ∎

In summary, we can conclude that modular field theory is essentially equivalent to a normal field theory with a $U(m)$ gauge symmetry in which the observables have been further restricted by some, as yet unknown, requirement. Furthermore modular field theory lies between a normal field theory and parafield theory with respect to degeneracy of physically relevant states.

5. The energy-momentum operator

One of the very basic equations of second quantization is the expresssion of Heisenberg's principle [66]:

$$[P_\mu, \psi(x)]_- = -i\psi,_\mu(x) \; . \tag{5.1}$$

This simply expresses the fact that the energy-momentum operator generates space-time translations of the fields in the theory.

In modular field theory, as Green [27] has pointed out, it is possible construct a P_μ satisfying this condition. It has the form

$$P_\mu = \int d^3x \left(\sum_{r=0}^{m-1} i\psi^{*(r)}(x)\psi^{(r)},_\mu(x) \right) \; . \tag{5.2}$$

A short calculation using equation (2.5) confirms that this indeed satisfies (5.1). Notice that when $m = 2$ this operator is precisely the same as the P_μ introduced for parafield theory in equation (1.3). The parallel is, in fact, stronger: If the fields in (5.2) are Klein transformed then we obtain

$$P_\mu = \int d^3x \left(\sum_{r=0}^{m-1} \Phi^{*r}(x)\Phi^{r},_\mu(x) \right) \; , \tag{5.3}$$

which is precisely the expression obtained when (1.3) is also Klein transformed. It is also easy to see that (5.3) is invariant under $U(m)$ and so, as an observable, P_μ is consistent with the discussion of the previous section. It would appear that if a free field theory is desired then (5.2) is the correct choice for P_μ. This expression is, however, undesirable in one respect: apparently it cannot be written purely in terms of the modular fields and the use of the Klein operator u would appear to be mandatory.

To see why this is likely to be so, observe firstly that since (5.2) is invariant under u and consists of operators of the form $\phi^{*(r)}\phi^{(t)}$ then by Theorem 3.1 its local form*

$$P_{\mu V} = \int h_V d^3x \left(\sum_{r=0}^{m-1} i\psi^{*(r)}(x)\psi^{(r)}(x) \right) \tag{5.4}$$

* This will correspond to an energy-momentum operator for the region V

is strongly local (a desirable feature naturally). The proof of Theorem 3.4 then tells us that for $m > 2$ the density

$$T_{4\mu}(x) = \sum_{r=0}^{m-1} \psi^{*(r)}(x)\psi^{(r)}(x)$$

cannot be written as a quadratic expression in $\psi(x)$ and so the same applies to P_μ. There is still the rather unlikely possibility mentioned in the footnote on page in which it may be possible to write $T_{4\mu}(x)$ as a higher order polynomial in $\psi(x)$ and $\psi^*(x)$.

In view of the above remarks it would appear that interacting field theories may be a more appropriate setting for modular field theory. Such theories are non-renormalizable when their Lagrangians, and hence energy-momentum operators, are greater than fourth order in the fields. It is desirable therefore to avoid such operators. If one further requires that the local energy-momentum operator be strongly local then Theorem 3.4 would appear to rule out the cases $m > 4$. Furthermore if one also requires that P_μ be invariant under $U(m)$ then the only suitable candidate discovered by this author is given in equations (3.40) and (4.17) and these apply only in the case that $m = 3$. One might, therefore, consider a P_μ containing terms such as*

$$\begin{aligned} P'_\mu = \int d^3x \Big\{ & \psi^*(x)\psi^*(x)[\psi_{,\mu}, \psi(x)]_+ \\ & - \psi_{,\mu}(x)\psi^*(x)\psi^*(x)\psi(x) \\ & - \psi(x)\psi^*(x)\psi^*(x)\psi_{,\mu}(x) \\ & + [\psi_{,\mu}(x), \psi(x)]_+ \psi^*(x)\psi^*(x) \Big\} , \end{aligned} \tag{5.5}$$

which is invariant under $U(3)$. It is rather fortuitous that possibilities such as (5.5) occur when the gauge group is, apart from a $U(1)$ summand, precisely the group usually used to describe colour. One might expect in such a theory that the equations of motion would be obtained by the requirement that P_μ satisfy Heisenbergs principle.

The above discussion is naturally only tentative and is presented only to indicate possible future avenues of enquiry.

---

* There may also be terms involving gauge fields.

CHAPTER 4

A GENERALIZATION OF MODULAR QUANTIZATION

1. Introduction

There is evidently much scope for generalization of the quantizations which have been considered to date. The form of the ansatz solutions for both para and modular quantizations suggest that it should be possible to consider arbitrary ansatz solutions. More explicitly, one could consider fields constructed from the ansatz

$$\psi(x) = \sum_{r=1}^{N} \phi^{(r)}(x) \tag{1.1}$$

where the fields $\phi^{(r)}(x)$ are elements from some arbitrary colour algebra. Given this large mathematical diversity it seems likely that *suitable* quantizations could only be identified on the basis of physical criteria. One might hope that such criteria could rule out certain possiblities and render others equivalent. A "classification" such as this is, however, beyond the scope of this thesis.

A further generalization can be considered by allowing several fields rather than the single $\psi(x)$:

$$\psi_s(x) = \sum_{r=1}^{N_s} \phi_s^{(r)}(x) \ . \tag{1.2}$$

Once again the fields may be assumed to be elements of a colour algebra. Such a generalization has been considered by Ohnuki and Kamefuchi [67] in the case that the individual $\psi_s(x)$ are parafields. In this chapter we consider a particular generalization in which the individual fields are modular. We shall content ourselves with a fairly introductory discussion and shall not go into the details of the comparison with an ordinary gauge theory (this has been done for the parafield case in [67]). In addition we shall consider a possible application of this generalization to the rishon model.

2. Generalized modular fields

We introduce this generalization by considering a generalization of the colour algebra of ansatz fields for modular field theory (the fields given by equation (2.7) Chapter 3).

Consider the fields $\phi_l^{r_1,\ldots,r_n}$ with $r_i = 0,\ldots,m_i - 1$ and $l = 1,\ldots,n$ (we are omitting the spatial index for clarity). Let them satisfy the relations

$$\phi_l^{r_1,\ldots,r_n}\phi_q^{t_1,\ldots,t_n} + \eta_l^{-t_l}\eta_q^{r_q}\phi_q^{t_1,\ldots,t_n}\phi_l^{r_1,\ldots,r_n} = 0 \tag{2.1}$$

$$\begin{aligned}\phi_l^{*r_1,\ldots,r_n}\phi_q^{t_1,\ldots,t_n} + \eta_l^{t_l}\eta_q^{-r_q}\phi_q^{t_1,\ldots,t_n}\phi_l^{*r_1,\ldots,r_n} &= \delta^{r_1,t_1}\ldots\delta^{r_n t_n} \\ &\quad .\delta_{ql}\delta(\mathbf{x}_l - \mathbf{x}_q)\ ,\end{aligned} \tag{2.2}$$

where η_l is the m_l'th primitive root of unity.

Such an algebra is a colour algebra: The appropriate grading group is

$$\Gamma = (Z_{m_1} \oplus Z_{m_1}) \oplus \ldots \oplus (Z_{m_n} \oplus Z_{m_n}) \oplus Z_2 \tag{2.3}$$

and one assigns gradings as follows:

$$\begin{aligned}\phi_l^{r_1,\ldots,r_n} &\longrightarrow (r_1, 0, r_2, 0, \ldots, r_l, 1, \ldots, r_n, 0, 1) \\ \phi_l^{*r_1,\ldots,r_n} &\longrightarrow (-r_1, 0, -r_2, 0, \ldots, -r_l, -1, \ldots, -r_n, 0, 1) \\ 1 &\longrightarrow (0,\ldots,0)\ .\end{aligned} \tag{2.4}$$

With such gradings the commutation factor implicit in equations (2.1) and (2.2) becomes

$$\begin{aligned}\epsilon(\alpha,\beta) &= \eta_1^{\alpha_1\beta_2-\alpha_2\beta_1}\ldots\eta_n^{\alpha_{2n-1}\beta_{2n}-\alpha_{2n}\beta_{2n-1}}(-1)^{\alpha_{2n+1}\beta_{2n+1}} \\ \alpha &\equiv (\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}\alpha_{2n}, \alpha_{2n+1})\ .\end{aligned} \tag{2.5}$$

The above equations apply to the case of fermi-like fields. In order to obtain the bose case one changes the + signs in (2.1) and (2.2) and also drops the Z_2 summand at the end of Γ. We make no further comment on the bose case.

To obtain the Klein transformations of the ϕ fields into fermi Φ fields, we need to define a σ factor (see Appendix B and Chapter 2 section 4). We choose this to be a straightforward generalization of the one chosen in the modular case (see equation (B.29)):

$$\sigma(\alpha,\beta) = \eta_1^{(\beta_2-\beta_1)\alpha_1}\eta_2^{(\beta_4-\beta_3)\alpha_3}\ldots\eta_n^{(\beta_{2n}-\beta_{2n-1})\alpha_{2n-1}}\ . \tag{2.6}$$

It is easily confirmed that $\epsilon(\alpha, \beta) = (-1)^{\alpha_{2n+1}\beta_{2n+1}}\sigma(\alpha, \beta)\sigma^{-1}(\beta, \alpha)$, which is what is required in the fermi case.

In terms of the indices r_i and l we may use (2.4) to show that

$$\sigma(\mathbf{r}, l, \mathbf{t}, q) = \eta_1^{-t_1 r_1}\eta_2^{t_2 r_2} \ldots \eta_l^{(1-t_l)r_l} \ldots \eta_n^{-t_n r_n} \ . \tag{2.7}$$

In view of (B.2), the Klein operators for the fields $\phi_l^{r_1,\ldots,r_n}$ satisfy

$$K_{\mathbf{r}l}\phi_q^{t_1,\ldots,t_n} = \sigma^{-1}(\mathbf{t}, q, \mathbf{r}, l)\phi_q^{t_1,\ldots,t_n} K_{\mathbf{r}l} \ . \tag{2.8}$$

If we now define the operators S_l as satisfying

$$\begin{aligned} S_l^{-1}\phi_q^{r_1,\ldots,r_n} S_l &= \eta_l^{r_l}\phi_q^{r_1,\ldots,r_n} \\ (S_l)^{m_l} &= 1 \\ [S_l, S_q]_- &= 0 \end{aligned} \tag{2.9}$$

then a short calculation using (2.7)–(2.9) will show that the operators $K_{\mathbf{r}l}$ and $S_1^{-r_1}S_2^{-r_2}\ldots S_l^{1-r_l}\ldots S_n^{-r_n}$ have the same commutation relations with respect to the fields $\phi_l^{r_1,\ldots,r_n}$. If it is assumed that S_l are unitary then we may conclude that the latter operators are suitable Klein operators.

We can now define generalized modular fields through the following ansatz:

$$\psi_l = \sum_{r_1,\ldots,r_n} \phi_l^{r_1,\ldots,r_n} \ . \tag{2.10}$$

The summation over the r_i extends over all possible values.

If we now make the definition

$$\psi_l^{(r_1,\ldots,r_n)} \equiv (S_1)^{-r_1}(S_2)^{-r_2}\ldots(S_n)^{-r_n}\psi_l(S_n)^{r_n}\ldots(S_2)^{r_2}(S_1)^{r_1} \ , \tag{2.11}$$

then it is easily shown, using (2.1), (2.2), (2.9) and (2.10), that these ancilliary fields satisfy

$$\begin{aligned} &\psi_l^{(r_1,\ldots,r_q,\ldots,r_n)}\psi_q^{(t_1,\ldots,t_l,\ldots,t_n)} + \psi_q^{(t_1,\ldots,t_l-1,\ldots,t_n)}\psi_l^{(r_1,\ldots,r_q+1,\ldots,r_n)} = 0 \\ &\psi_l^{*(r_1,\ldots,r_q,\ldots,r_n)}\psi_q^{(t_1,\ldots,t_l,\ldots,t_n)} \\ &\qquad + \psi_q^{(t_1,\ldots,t_l+1,\ldots,t_n)}\psi_l^{*(r_1,\ldots,r_q+1,\ldots,r_n)} = \delta^{r_1 t_1}\ldots\delta^{r_n t_n}\,\delta_{lq}\,\delta(\mathbf{x}_l - \mathbf{x}_q) \end{aligned} \tag{2.12}$$

These relations generalize the relations (2.5) in Chapter 2.

As was indicated in the introduction we shall not attempt to carry out a comparison of this generalized theory with an ordinary gauge theory. We simply point out that deciding on the appropriate gauge theory for comparison may be a little less straightforward than in the modular case: After Klein transformation the fields $\phi_l^{r_1,\ldots,r_n}$ become

$$\Phi_l^{r_1,\ldots,r_n} \equiv S_1^{r_1} \ldots S_l^{r_l-1} \ldots S_n^{r_n} \phi_l^{r_1,\ldots,r_n} \tag{2.13}$$

which are the fermi fields for the ordinary gauge theory. One must now decide which way such fields are to transform under the gauge group. An obvious candidate for such a group is

$$G = U(m_1) \otimes U(m_2) \otimes \ldots \otimes U(m_n) \ . \tag{2.14}$$

The indices r_i would then transform according to m_i-dimensional representations of the group $U(m_i)$. There are however, two m_i-dimensional representations of such a group, namely the fundamental and its conjugate. Obviously one could envisage various mixtures of such representations. Deciding on the "correct" transformation properties of the fermi fields would probably be governed by the kind of application one was looking at.

As a final comment we observe that it is possible to construct an energy-momentum operator for free fields which satisfies the Heisenberg principle. This has the form

$$P_\mu = i \int d^3x \sum_{r_1,\ldots,r_n} \sum_l \psi_l^{*(r_1,\ldots,r_n)} \psi_l^{(r_1,\ldots,r_n)}{}_{,\mu} \tag{2.15}$$

and like its modular counterpart appears to require Klein operators for its expression. By use of the relations (2.12) one can easily verify that this satisfies the equation

$$[P_\mu, \psi_l]_- = -i\psi_{l,\mu} \ . \tag{2.16}$$

3. The Rishon model

We study here a possible application of the generalized modular quantization to the Rishon hypothesis [30]. In this it is proposed to unify leptons and quarks by assuming that they are both composed of two different kinds of fundamental particles T and V which are called rishons. The T is assumed to have charge one third and the V is assumed to be neutral. Both particles are assumed to carry spin one half. Quarks and leptons are then built up as certain combinations of the T and V. Thus the positron becomes TTT and the ν_e-neutrino becomes VVV. In addition combinations such as TTV become u-quarks while combinations such as TVV become $\bar{d}$-antiquarks. One of the interesting hypotheses made by Harari is that the particular order of the T and V within quarks indicates the colour of that quark. Thus TTV, TVT and VTT give three different colours to the u-quark. This ordering effect strongly suggests the application of a generalized quantization as it is obviously not possible when T and V are fermions.

A number of criticisms have been leveled at the original Harari-Shupe model (see Lyons [68]). Apart from the above mentioned problem with ordinary statistics, there are a number of others. Two of these, which are of interest here, are

(i) Why are combinations of rishons such as TT, $\mathrm{VV\overline{V}}$ and so on, never observed?

(ii) Why are quarks confined but leptons not, when both have similar substructures consisting of three quarks?

A possible solution to the above problems was proposed by Jarvis and Green [69]. In their model the rishons T and V were modular fields of order three when considered seperately. The algebraic relations satisfied *between* the two different rishons required the introduction of the Z-metacyclic group $C_3 \times D_3$ (see [70]) In this section we introduce these relations from a different perspective— namely as an example of the generalized modular fields of the previous section.

If the T and V particles are both required to be modular fields of order three, then the generalization of section 2 forces the fields (which we call $\psi_1(x)$ and $\psi_2(x)$) to satisfy the relations

$$\psi_1^{(r,s)}(x)\psi_2^{(t,v)}(y) + \psi_2^{(t-1,v)}(y)\psi_1^{(r,s+1)}(x) = 0 \tag{3.1}$$

$$\psi_1^{*(r,s)}(x)\psi_2^{(t,v)}(y) + \psi_2^{(t+1,v)}(y)\psi_2^{*(r,s+1)}(y) = 0\ . \tag{3.2}$$

Now up until the present, only bosons and fermions have been observed as free particles. This suggests that only "modules" of the fields ψ_1 and ψ_2 which commute or anticommute should be allowed as free particles. Naturally this requirement does not explain the mechanism by which non-modules are never observed as free particles. This is the well-known dynamical problem of confinement which has not, as yet, even been resolved within ordinary field theory. The requirement is, however, suggestive and it is possible that it may point in the direction of a correct dynamical theory.

We shall now demonstrate a collection of modules which commute or anticommute amongst themselves. We shall also demonstrate that this collection is maximal in the sense that any other product of fields fails to commute or anticommute with at least one kind of module.

This collection consists of "conglomerates" of fields in which the number of particle fields of type i minus the number of antiparticle fields of the same type is equal to zero modulo 3.

Particle and anti-particle fields arise from the splitting of the relativistic fields into positive and negative frequency parts:

$$\psi_i(x) = \psi_i^p(x) + \psi_i^{*ap}(x) \ . \tag{3.3}$$

As was observed at the end of section 2 in Chapter 3, if the anti-particle field $\psi_i^{ap}(x)$ satisfies the same algebraic relations as the particle field $\psi_i^{*p}(x)$ (apart from the extra degree of freedom associated with being an anti-particle) then the relativistic field $\psi_i(x)$ will satisfy the same algebraic relations (namely (3.1) and (3.2)) as its non-relativistic (or "particle") counterpart.

Consider products of operators which create particle and anti-particle fields:

$$\mathcal{P} = \psi_{i_1}^{*f_1}(x_1)\psi_{i_2}^{*f_2}(x_2)\dots\psi_{i_n}^{*f_n}(x_n) \ , \tag{3.4}$$

where $f_j = p$ or ap and $i_j = 1,2$. Now by use of relation (2.12) and the comments concerning anti-particles above, we deduce that if $\mathcal{P}$ is a module in the sense described above, then

$$\begin{aligned} \psi_1^{*p}\mathcal{P} &= (-1)^n\mathcal{P}^{(1,0)}\psi_1^{*p} \qquad & \psi_1^{*ap}\mathcal{P} &= (-1)^n\mathcal{P}^{(-1,0)}\psi_1^{*ap} \\ \psi_2^{*p}\mathcal{P} &= (-1)^n\mathcal{P}^{(0,1)}\psi_2^{*p} \qquad & \psi_2^{*ap}\mathcal{P} &= (-1)^n\mathcal{P}^{(0,-1)}\psi_2^{*ap} \ . \end{aligned} \tag{3.5}$$

We have defined

$$\mathcal{P}^{(r,t)} \equiv S_1^{-r} S_2^{-t} \mathcal{P} S_2^{+t} S_1^{+r} . \tag{3.6}$$

It is reasonably clear now from the relations (3.5) that different modules will either commute or anticommute with each other. In addition if we consider products Q of the fields, which *do not* satisfy the conditions imposed on modules above we shall have the relations

$$Q\mathcal{P} = (-1)^{ln} \mathcal{P}^{(r,t)} Q . \tag{3.7}$$

In (3.7) l is the number of fields in Q; r is the difference (modulo 3) in the number of particles and anti-particles for type 1 fields and t is the same thing for type 2 fields. In general the requirement for a module does not imply that

$$\mathcal{P}^{(r,t)} = \mathcal{P} \qquad \text{for } r, t \neq 0 . \tag{3.8}$$

This may be demonstrated by considering the simple module

$$\mathcal{P}_0 = \psi_1^{*p}(x)\psi_1^{*p}(y)\psi_1^{*p}(z) . \tag{3.9}$$

By the use of arguments similar to those used in the proof of Theorem 3.1 Chapter 3 one can show, with (2.9) and (2.10), that

$$S_1^{-1} \mathcal{P}_0 S_1 \neq \mathcal{P}_0 \neq S_2^{-1} \mathcal{P}_0 S_2 \tag{3.10}$$

which shows that (3.8) must be false in general.

In view of the above construction of modules we may conclude that objects such as TTT, VVV, TVVVTVVVT and TVTVTV are allowable as free particles whereas objects such as TT, VV$\overline{\text{V}}$ and TTV are not. In otherwords objects identified by Harari as positrons, neutrinos, baryons and W-particles are observable as free particles whereas exotic rishon combinations and more importantly, quarks, are not.

One possible problem with this proposal concerns colour. As was mentioned above, Harari originally proposed that colour be dealt with through the ordering of rishons within quarks (and other coloured particles). Consider now a baryon state constructed from three quarks:

$$(\text{TTV})(\text{TVT})(\text{VTT})|\rangle .$$

One might at first conclude that the state consisted of quarks of three colours. This is not quite clear however, if one considers that the *order* in which the quarks are applied to the vacuum is also important in our proposal. Thus the state

$$(\mathrm{TVT})(\mathrm{TTV})(\mathrm{VTT})|\rangle$$

is not neccessarily a numerical multiple of the original baryon state. Another problem with interpreting ordering as a colour effect lies with the W-particle. It is clearly possible to obtain different orderings of rishons in its construction: TVTVTV and VTVTVT are two. Despite this, W-particles are assumed usually to be colourless. This latter difficulty has already been pointed out by other authors [68] in the context of the original Harari-Shupe model.

A resolution of the above problems will probably require a careful comparison with a normal gauge theory: The appropriate gauge group would appear to be $U(3) \times U(3)$ (consider (2.10) and (2.13)). It is interesting to note in this regard that a dynamical rishon model based upon almost the same gauge group ($SU(3) \times SU(3) \times U(1)$) has been proposed by Harari and Seiberg [71]. In this model the groups denote colour, hypercolour and electromagnetism, and particles such as quarks, leptons and W-particles are hypercolour singlets (the latter two are also colour singlets).

Whether the proposal made here can be shown to be equivalent, at the global gauge symmetry level, to the Harari-Seiberg model is not straightforward.

In order to see this, consider, for instance, the neutrino which in the Harari-Seiberg model is a colour and hypercolour singlet. Within the context of the present proposal it may be written as combinations of states of the form

$$\psi_2^{*p}(x_1)\psi_2^{*p}(x_2)\psi_2^{*p}(x_3)|\rangle \ . \tag{3.11}$$

When use is made of equation (2.10) this becomes

$$\sum_{r_i,t_i,s_i} \phi^{*r_1r_2}(x_1)\phi^{*t_1t_2}(x_2)\phi^{*s_1s_2}(x_3)|\rangle \tag{3.12}$$

(we are dropping the particle labels for notational ease). Now if the superscripts of the fields in (3.12) are interpreted in the obvious way then they would transform as the colour and hypercolour indices. It is unclear then how to obtain colour and hypercolour singlets from linear combinations of states such as (3.11). The only possibility would appear to be permutations of the fields in (3.11). As was noted above the ψ_2 fields act like modular fields of order three amongst themselves. In particular relations (2.1) show that only the second index of the ϕ_2^* fields play a role in the commutation relations. One can therefore construct a singlet with respect to the group acting on the second index (see (4.15) in Chapter 3) by permutations of fields. The first index remains however, problematical. Similar considerations with respect to the electron will show that it is possible there to construct singlets with respect to the *first* index only. The above comments indicate that the present proposal may not be compatible with the Harari-Seiberg model and its notion of hypercolour.

APPENDIX A

Proof of Proposition 2.3, Chapter 2.

Consider the usual fermi-bose creation and annihilation algebra:

$$\begin{aligned} a_i^* a_j \pm a_j a_i^* &= \delta_{ij} \\ a_i a_j \pm a_j a_i = a_i^* a_j^* \pm a_j^* a_i^* &= 0 \end{aligned} \tag{A.1}$$

where the indices belong to an arbitrary finite set. We create a colour algebra with grading group Γ as follows: Take an arbitrary non-zero $\alpha \in \Gamma$; if $\epsilon(\alpha,\alpha) = 1$, select a bose a_j^* and a_j or if $\epsilon(\alpha,\alpha) = -1$, select a fermi a_j^* and a_j. Now if

(i) $2\alpha \neq 0$ then assign a_j^* the grading α and a_j the grading $-\alpha$.

(ii) $2\alpha = 0$ then assign $p_j = a_j^* + a_j$ the grading α.

Now rewrite the a_j^*, a_j or p_j as $a_\alpha^*, a_{-\alpha}$ or p_α. Repeat the above procedure for the other elements of Γ *unless* $-\alpha$ has been considered previously, in which case make no assignment. Finally assign the identity the grading 0 and delete any surplus creation or annihilation operators.

It is clear that the constructed colour algebra is a canonical superalgebra and has an associated colour algebra coloured by $< \Gamma, \epsilon >$. Furthermore for every $\alpha \in \Gamma$ there is a *unique* element in the algebra.

Consider now the Fock representation of (A.1) and consider the states obtained by applying the $a_\alpha^*, a_{-\alpha}$ and p_α to the vacuum: That is, states of the form

$$(a_\alpha^*)^{i_1}(a_{-\alpha})^{i_2}\dots(a_\beta^*)^{j_1}(a_{-\beta})^{j_2}(p_\gamma)^k\dots(p_\delta)^l|\rangle \; .$$

If we assign such states the grading

$$i_1\alpha - i_2\alpha + \dots + j_1\beta - j_2\beta + k\gamma + \dots + l\delta$$

and let the operators $a_\alpha^*, a_{-\alpha}$ or p_α act only on linear combinations of such states, then it follows from (A.1) and the assignments of gradings, that these operators, when applied to the above states, will satisfy the fundamental equation (1.9) from Chapter 2. We have therefore defined a colour algebra representation for our canonical superalgebra. Moreover it is clear from the usual Fock construction [63], that any product of two elements of our algebra will be represented by a non-zero operator *except* $a^*_\alpha a^*_\alpha$ or $a_{-\alpha}a_{-\alpha}$, where the creation and annihilation operators concerned are fermi.

By the results of section 4, Chapter 2, there exists a Klein transformation on our graded vector space which converts the representation of the canonical superalgebra into a representation of the associated colour algebra coloured by $<\Gamma,\epsilon>$. The Klein transforms of the $a^*_\alpha, a_{-\alpha}$ and p_α we denote by $b^*_\alpha, b_{-\alpha}$ and q_α respectively. They satisfy the relations

$$b^*_\alpha b_{-\beta} - \epsilon(\alpha,-\beta)b_{-\beta}b^*_\alpha = \delta_{\alpha\beta} \tag{A.2a}$$

$$q_\alpha q_\beta - \epsilon(\alpha,\beta)q_\beta q_\alpha = 2\delta_{\alpha,\beta}(1-\epsilon(\alpha,\beta)) \tag{A.2b}$$

$$b_\alpha b_\beta - \epsilon(\alpha,\beta)b_\beta b_\alpha = b^*_\alpha b^*_\beta - \epsilon(\alpha,\beta)b^*_\beta b^*_\alpha = 0 \tag{A.2c}$$

$$q_\alpha b_\beta - \epsilon(\alpha,\beta)b_\beta q_\alpha = q_\alpha b^*_\beta - \epsilon(\alpha,\beta)b^*_\beta q_\alpha = 0\ . \tag{A.2d}$$

Furthermore the non-zero nature of the Klein transformation will ensure that the product property mentioned above will remain true. Now if the algebra given by (A.2) is coloured by $<\Gamma',\epsilon'>$ as well as $<\Gamma,\epsilon>$, then since there is only one element of (A.2) for each element of Γ, it follows that there must be a well-defined map h between Γ and Γ'.

Now since we require the operator algebra (A.2) to also be a representation when coloured by $<\Gamma',\epsilon'>$, then (A.2) together with the non-zero product property, imply that

$$\epsilon(\alpha,\beta) = \epsilon'(h(\alpha),h(\beta))$$

unless $\alpha=\beta$ and $2\alpha\neq 0$ when the product property fails in (A.2c). In this event, we consider (A.2a) with $\alpha=\beta$ and deduce that

$$\epsilon(\alpha,-\alpha) = \epsilon'(h(\alpha),h(-\alpha))\ .$$

Now since the identity commutes with all the other elements of the algebra, we have

$$\epsilon'(h(0),h(\gamma)) = 1 \qquad \forall\gamma\in\Gamma \tag{A.3}$$

and so

$$\begin{aligned}
\epsilon(\alpha,\alpha) &= \epsilon(\alpha,-\alpha) \\
&= \epsilon'(h(\alpha),h(-\alpha)) \\
&= \epsilon'(h(\alpha),h(0)-h(\alpha)) \\
&= \epsilon'(h(\alpha),-h(\alpha))\epsilon'(h(\alpha),h(0)) \\
&= \epsilon'(h(\alpha),h(\alpha)) \ .
\end{aligned}$$

In the above we have used $h(\alpha)+h(-\alpha) = h(0)$ which we can deduce from (A.2a) with $\alpha = \beta$ and from (i) in the definition of colouring. We have also used (A.3) to get the last step. ■

APPENDIX B

Klein transformations in field theory

In this appendix we consider Klein transformations independently of the colour algebra formalism developed in Chapter 2. Despite this, we show below that nothing extra is gained. The formalism developed here, however, proves more convenient for considering a number of applications in generalized quantizations.

We shall be interested in creation-annihilation rings satisfying the relations

$$\begin{aligned} d_j^{(r)} d_k^{(t)} \pm d_k^{(t)} d_j^{(r)} &= 0 \\ d_j^{*(r)} d_k^{(t)} \pm d_k^{(t)} d_j^{*(r)} &= \delta_{jk}\delta_{rt} \ . \end{aligned} \tag{B.1}$$

The j and k refer to a denumerably infinite momentum set, while the r and t take on a finite* number N of integral values. For our purpose, only the latter set of indices are relevant and so for convenience we rewrite $d_j^{(r)}$ as d_r and so on.

The Klein operators K_r and $K_{\bar{r}}$ will be assumed to commute and also to satisfy the following quite general relations:

$$\begin{aligned} \sigma(t,r)K_r d_t &= d_t K_r & \qquad \sigma(t,\bar{r})K_{\bar{r}} d_t &= d_t K_{\bar{r}} \\ \sigma(\bar{t},r)K_r d_t^* &= d_t^* K_r & \qquad \sigma(\bar{t},\bar{r})K_{\bar{r}} d_t^* &= d_t^* K_{\bar{r}} \ , \end{aligned} \tag{B.2}$$

where σ is an *arbitrary non-zero* mapping of our index set into the complex numbers.

The Klein transformation is then given through the equations

$$b_r = K_r d_r \qquad b_r^* = K_{\bar{r}} d_r^* \ . \tag{B.3}$$

The b_r and b_r^* shall be required to satisfy the relations

$$\begin{aligned} b_r b_t \pm \epsilon(r,t) b_t b_r &= 0 \\ b_r^* b_t \pm \epsilon(\bar{r},t) b_t b_r^* &= \delta_{rt} \ , \end{aligned} \tag{B.4}$$

---

* Generalizations to an infinite number of values are no doubt possible but are not central to this thesis.

with ϵ being a non-zero map into C which can be determined fron σ. To make this determination, we substitute (B.3) into (B.4) and use (B.2) and (B.1) to conclude that

$$\epsilon(v,u) = \sigma(v,u)\sigma^{-1}(u,v) \ , \tag{B.5}$$

with $u = t, \bar{t}$ and $v = r, \bar{r}$. In addition to this, the second equation of (B.4) leads to the requirement that

$$K_{\bar{r}}K_r = \sigma^{-1}(\bar{r}, r) \ . \tag{B.6}$$

This ensures that the right-hand side of the equation is δ_{rt} after the application of the Klein transformation.

Thus given the equations (B.1), (B.2), (B.3), (B.5) and (B.6) then (B.4) follows. We therefore adopt these first five equations as our definition of the Klein transformation. A number of consequences for σ can be deduced from these equations. Firstly, if we multiply the first equation of (B.2) on the left by $K_{\bar{r}}$ and then use the second equation in conjunction with (B.6), we conclude* that

$$\sigma(t,r)\sigma(t,\bar{r}) = 1 \ , \tag{B.7}$$

and in a similar way that

$$\sigma(\bar{t},r)\sigma(\bar{t},\bar{r}) = 1 \ . \tag{B.8}$$

Consider now the second of (B.1) with $r = t$ and $j = k$:

$$d_t^* d_t \pm d_s d_s^* = 1 \ .$$

If we multiply on the left by K_r and use (B.2) we can deduce that

$$\sigma(\bar{t},r)\sigma(t,r)K_r = K_r \ ;$$

and when (B.6) is used this becomes

$$\sigma(\bar{t},r)\sigma(t,r) = 1 \ . \tag{B.9}$$

Similarly we deduce that

$$\sigma(\bar{t},\bar{r})\sigma(t,\bar{r}) = 1 \ . \tag{B.10}$$

---

* We are assuming that $d_r \neq 0$; this follows from the second of (B.1) with $j = k$, $r = t$.

It is now possible to show that (B.1) and (B.4) define colour algebras; that ϵ defines a commutation factor and finally that the Klein operators introduced here are just special cases of the ones introduced in section 4 of Chapter 2.

To carry out this program, our first step is to grade the elements d_r, d_r^* and b_r, b_r^* with the grading group* $\Gamma_Z \equiv Z \oplus Z \oplus \ldots \oplus Z$ (N copies):

$$\begin{aligned} d_r,\ b_r &\longrightarrow (0,0,\ldots,0,1,0,\ldots,0) \quad r\text{'th place} \\ d_r^*,\ b_r^* &\longrightarrow (0,0,\ldots,0,-1,0,\ldots,0) \quad\quad '' \\ 1 &\longrightarrow (0,0,\ldots,0)\ . \end{aligned} \tag{B.11}$$

Now let $\alpha, \beta \in \Gamma_Z$ have the form $(r_1,\ldots,r_N)$ and $(t_1,\ldots,t_N)$ respectively. We induce a $\sigma : \Gamma_Z \times \Gamma_Z \to C$ as follows:

$$\sigma(\alpha,\beta) = \prod_{i,j}^{N} [\sigma(i,j)]^{r_i t_j} \tag{B.12}$$

where $\sigma(i,j)$ are the σ defined previously. If the index set is mapped into Γ_Z in the obvious way, namely

$$\begin{aligned} \alpha(r) &= (0,\ldots,1,\ldots,0) \quad r\text{'th place} \\ \alpha(\bar{r}) &= (0,\ldots,-1,\ldots,0) \quad\quad '' , \end{aligned}$$

it is clear from (B.12) and (B.7)-(B.10) that

$$\sigma(\alpha(u),\alpha(v)) = \sigma(u,v)\ . \tag{B.13}$$

Moreover we can induce an ϵ from (B.12):

$$\begin{aligned} \epsilon(\alpha,\beta) = \sigma(\alpha,\beta)\sigma^{-1}(\beta,\alpha) &= \prod_{i,j}^{N} [\sigma(i,j)]^{r_i t_j} \left[\sigma^{-1}(j,i)\right]^{r_i t_j} \\ &= \prod_{i,j}^{N} \epsilon(i,j)^{r_i t_j}\ , \end{aligned} \tag{B.14}$$

which consequently satisfies

$$\epsilon(\alpha(u),\alpha(v)) = \epsilon(u,v)\ , \tag{B.15}$$

---

* It is to be observed that such a choice of grading group is not, in general, unique.

and, as a result of (B.5) and (B.14), also satisfies the commutation factor rules (1.3) of Chapter 2. It is clear now that (B.4) is a colour algebra*. Finally we can identify the K_r, $K_{\bar{r}}$ with the unscaled K^σ_τ of section 4, Chapter 2. This identification is

$$K_r = K^\sigma_\tau(-\alpha(r)) \qquad K_{\bar{r}} = K^\sigma_\tau(+\alpha(r)) \; . \tag{B.16}$$

To see this, firstly observe that the form of (B.12) ensures that $\sigma(\alpha,\beta)$ satisfies (4.4), Chapter 2 and $\sigma(\alpha,0) = \sigma(0.\alpha) = 1$. Secondly with the help of these relations for σ, together with equation (B.13), equations (B.2) become special cases of equation (4.3), Chapter 2. Thirdly we deduce from (4.2), Chapter 2 and (B.6), that the τ of K^σ_τ need only satisfy

$$\tau(\alpha(r), -\alpha(r)) = \sigma^{-1}(\bar{r}, r) \; . \tag{B.17}$$

When τ is decomposed into its trivial form $r^{-1}(\alpha)r^{-1}(\beta)r(\alpha+\beta)$ we see that (B.17) amounts to the restriction

$$r(-\alpha(r)) = r(0)r^{-1}(\alpha(r))\sigma^{-1}(\bar{r}, r) \; ; \tag{B.18}$$

so, providing our scaling set $r(\alpha)$ satisfy this relation, the Klein operators defined through (B.16) will satisfy equation (4.2), Chapter 2. Finally we can induce the full set of Klein operators $K^\sigma_\tau(\beta)$ through the equation

$$\begin{aligned} K^\sigma_\tau(\beta) &= r^{-1}(\beta)\prod_{i=1}^{N}\left[r(sign(t_i)\alpha(i))K^\sigma_\tau(sign(t_i)\alpha(i))\right]^{|t_i|} \\ &= r^{-1}(\beta)\prod_{i=1}^{N}\left[r(sign(t_i)\alpha(i))K_i\right]^{|t_i|} \end{aligned} \tag{B.19}$$

which can be shown, in a straightforward manner, to agree with (B.16) and more importantly to satisfy (4.2), Chapter 2.

We now examine the question of representations. Fock representations of the relations (B.1) have been explicitly constructed by, amongst others, Berezin [63] and we use these here.

* In the fermi case the commutation factor for such a colour algebra is actually $-\epsilon$.

An interesting feature of such representations is that they may be graded by an arbitrary grading group Γ. More specifically we assign gradings from Γ for d_r and d_r^* as follows:

$$\begin{aligned} d_r^* &\longrightarrow -\alpha(r) \\ d_r &\longrightarrow +\alpha(r) \qquad\qquad (B.20) \\ 1 &\longrightarrow 0 \,, \end{aligned}$$

where $\alpha(r)$ is an *arbitrary* map of the index set into Γ. Products of elements from (B.20) will have gradings that are sums of the gradings of the elements in the product. Consider now the Fock-space and observe that any state $d|\rangle$, with d an arbitrary element of the ring $\mathcal{C}$ associated with d_r and d_r^*, can be easily rewritten, with the aid of (B.1), as $d'|\rangle$ where d' consists only of sums of products of creation operators. Now as we have seen in the discussion preceding (2.32), Chapter 3, it is possible to choose an orthogonal set of states $d_{j_1}^{*(r_1)} \ldots d_{j_n}^{*(r_n)}|\rangle$, each being non-zero, which will span the set of states $d'|\rangle$. Now since such a set is dense in the Fock-space we conclude, as we have done before, that an arbitrary state in this space may be written as a linear combination of the above orthogonal states. Now if we assign $d_{j_1}^{*(r_1)} \ldots d_{j_n}^{*(r_n)}|\rangle$ the grading $-\sum_{l=1}^{n} \alpha(r_l)$ then it is clear that $\mathcal{F}(\mathcal{C})$ has been graded according to (1.1), Chapter 2. It remains to be shown that the grading assignments made for $\mathcal{C}$ respect those of $\mathcal{F}(\mathcal{C})$ (see equation (1.9), Chapter 2).

A little thought shows that it suffices to consider the action of $d_j^{*(r)}$ and $d_j^{(r)}$ on the state $d_{j_1}^{*(r_1)} \ldots d_{j_n}^{*(r_n)}|\rangle$. The first has the result $d_j^{*(r)} d_{j_1}^{*(r_1)} \ldots d_{j_n}^{*(r_n)}|\rangle$ which has grading $-\alpha(r) - \sum_{l=1}^{n} \alpha(r_l)$ which is what we require. The second has the result $d_j^{(r)} d_{j_1}^{*(r_1)} \ldots d_{j_n}^{*(r_n)}|\rangle$, which when the second of (B.1) is used repeatedly, becomes

$$\sum_{l=1}^{n} (-1)^{l+1} \delta_{r r_l} \delta_{j j_l} d_{j_1}^{*(r_1)} \ldots d_{j_{l-1}}^{*(r_{l-1})} d_{j_{l+1}}^{*(r_{l+1})} \ldots d_{j_n}^{*(r_n)}|\rangle \,. \qquad (B.21)$$

Now a term in (B.21) will only be non-zero if $\alpha(r) = \alpha(r_l)$ and so it follows that the state (B.21) has grading $\alpha(r) - \sum_{l=1}^{n} \alpha(r_l)$. This completes the demonstration that $\mathcal{C}$ has a graded representation for arbitrary grading group Γ.

Given this fact and the fact that the Klein operators considered above are just examples of those considered in section 4, Chapter 2, it follows from that section that these Klein operators have a representation on $\mathcal{F}(C)$. An interesting feature of this representation is that the Klein operators turn out to be unitary and the σ factor a phase factor. To see this we firstly rewrite the second of (B.3) using (B.2):

$$b_r^* = \sigma^{-1}(\bar{r},\bar{r})d_r^* K_{\bar{r}} \tag{B.22}$$

and now taking the hermitean conjugate of both sides and using the first of (B.3) together with (B.8), we have

$$K_r d_r = \sigma^*(\bar{r},r)K_{\bar{r}}^* d_r \tag{B.23}$$

or, multiplying through by $K_{\bar{r}}$ and using (B.6)

$$d_r = |\sigma(\bar{r},r)|^2 K_{\bar{r}} K_{\bar{r}}^* d_r \ . \tag{B.24}$$

Now consider an arbitrary basis state ϕ of $\mathcal{F}(C)$ given by $\phi = d_{j_1}^{*(r_1)} \dots d_{j_n}^{*(r_n)}|\rangle$. By choosing $r \neq r_i$ and $j \neq j_i$ for all i, we deduce, by use of (B.1), that $d_j^{(r)} d_j^{*(r)}\phi = \pm\phi$ (notice that this argument holds for $\phi = |\rangle$). It follows now from (B.24) and (B.7) that

$$K_{\bar{r}} K_{\bar{r}}^* \phi = |\sigma(r,r)|^2 \phi$$

or

$$K_{\bar{r}} K_{\bar{r}}^* = |\sigma(r,r)|^2 \ . \tag{B.25}$$

If we take the hermitean conjugate of the second of (B.3) and use (B.2) on the first of (B.3), we obtain

$$d_r K_{\bar{r}}^* = d_r K_r \sigma^{-1}(r,r) \ .$$

If this is multiplied through on the right by $K_{\bar{r}}$ and we use the fact that it commutes with both $K_{\bar{r}}^*$ and K_r (due to (B.25)), then we obtain, with the use of (B.25), (B.6) and (B.9),

$$d_r|\sigma(r,r)|^2 = d_r \sigma^{-1}(\bar{r},r)\sigma^{-1}(r,r) = d_r \ ,$$

which implies that σ is a phase factor and that $K_{\bar{r}}$ is unitary by (B.25). If we multiply this latter equation on the left by K_r and use (B.6) then we may deduce that

$$\sigma^{-1}(\bar{r},r)K_{\bar{r}}^* = K_r \ , \tag{B.26}$$

and since σ is a phase factor and $K^*_{\bar{r}}$ unitary, it follows that K_r is also unitary.

We now consider the Klein operators which allow a transformation to the para and modular ansatz algebras. This allows us to demonstrate the existence of the usual Fock representations of para and modular quantization.

In the modular case we choose σ to have the form

$$\sigma(r,t) = \eta^{(1-t)r} \tag{B.27}$$

and because of (B.5) this results in ϵ having the forms

$$\epsilon(r,t) = \eta^{r-t} \qquad \epsilon(\bar{r},t) = \eta^{t-r} \tag{B.28}$$

which means (B.4) agrees with (2.14) of Chapter 3.

Consider now the u operator of section 2, Chapter 3. A little calculation using the commutation relations (2.14) and the expression (B.27) for σ, will show that the operators u^{1-r} and K_r obey the same commutation rules with respect to d_r and d^*_r. Since the Fock representation of C is irreducible [63], they are therefore numerical multiples of each other. It is clear then how the operator u may be defined on our representation, moreover it will be unitary providing the numerical factor is a phase factor.

The index set can be graded in this case by the finite group $Z_N \oplus Z_N(\oplus Z_2)$ (contrast this with the Γ_Z grading above) as was done in equation (2.10) of Chapter 3. With this grading the σ map becomes

$$\sigma(\alpha,\beta) = \eta^{(\beta_2-\beta_1)\alpha_1} \tag{B.29}$$

and this leads to

$$\epsilon(\alpha,\beta) = \eta^{\alpha_1\beta_2-\alpha_2\beta_1}. \tag{B.30}$$

In the para case we choose σ to have the following form

$$\begin{aligned} \sigma(s,r) &= h(r-s) \qquad && s \text{ even} \\ &= h(r-s-1) && s \text{ odd} \\ h(t) &= +1 && t \geq 0 \\ &= -1 && t < 0\,. \end{aligned} \tag{B.31}$$

It is easily confirmed then that $\epsilon(r,s)$ has the form

$$\epsilon(r,s) = 2\delta_{rs} - 1 \ ,$$

which is the correct commutation factor for the para algebra (c.f. (1.8) of Chapter 2). An interesting feature of the choice made in (B.31) is that for r odd, K_r and K_{r+1} satisfy the same commutation relations with respect to d_t and d_t^*; hence, as before, they are numerical factors of each other. If we make the choice $K_r = -iK_{r+1}$, then (B.26) and (B.31) combine to show that $K_r^* = K_{\bar{r}}$. It is also apparent from (B.31), (B.7) and (B.8) that $K_{\bar{r}}$ and K_r satisfy the same commutation relations and are therefore multiples of each other. If we choose them to be equal, we may conclude that

$$K_r = K_r^* = K_{\bar{r}} = K_r^{-1} \ . \tag{B.32}$$

If one assumes that the grading is as it was in the equations following (1.8), Chapter 2 then it easily checked that the σ map may be written as

$$\begin{aligned}
&\sigma(\alpha,\beta) = (-1)^{\psi}(\alpha,\beta) \\
&\psi(\alpha,\beta) = \sum_{i<j} \alpha_i\beta_j + \sum_{k=1}^{x} \alpha_{2k-1}\beta_{2k-1} \\
&x \equiv p/2 \qquad \text{for } p \text{ even} \\
&\quad \equiv (p+1)/2 \quad \text{for } p \text{ odd} \ .
\end{aligned} \tag{B.33}$$

This then leads to the following expression for ϵ:

$$\epsilon(\alpha,\beta) = (-1)^{\sum_{i\neq j}\alpha_i\beta_j} \ . \tag{B.34}$$

Finally we turn our attention to the extension of the above discussion to the relativistic case.

To do this, we introduce another copy of the fermi operators d_r and d_r^*. We shall call these further fermi operators g_r and g_r^*. Instead of grading these as we did in (B.20) for d_r and d_r^*, we grade g_r with $-\alpha(r)$ and g_r^* with $\alpha(r)$.

Since g_r and g_r^* both anticommute with d_r and d_r^*, it follows that the arguments following (B.20) will generalize. In otherwords, the extended ring which includes g_r and g_r^* will have a graded Fock representation and this grading may be with an arbitrary group.

The Klein transformations for g_r and g_r^* are determined by their gradings, equation (4.5) of Chapter 2 and equation (B.3). Thus we have

$$e_r = K_{\bar{r}} g_r \qquad e_r^* = K_r g_r^* \ . \tag{B.35}$$

In otherwords, the g_r transforms as d_r^* and the g_r^* as d_r.

Consider now a spinor spatial fermi field $\Phi_r(x)$ [48]. We may write it in terms of the d_r and the g_r^*:

$$\Phi_r(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \Big\{ e^{i(\mathbf{k}.\mathbf{x} - Ex_0)} \textstyle\sum_{t=1}^{2} v^t(\mathbf{k}) d_r^t(\mathbf{k})$$

$$+ e^{i(\mathbf{k}.\mathbf{x} + Ex_0)} \textstyle\sum_{t=3}^{4} v^t(\mathbf{k}) g_r^{*t}(\mathbf{k}) \Big\} \tag{B.36}$$

where, as in section 2 of Chapter 3, v^t are the Dirac spin components and V is the volume in which the field theory is being considered. It is clear now that as a result of (B.3) and (B.35), a consistent Klein transformation is possible for the relativistic spinor fields. This is simply given by

$$\phi_r(x) = K_r \Phi_r(x) \ . \tag{B.37}$$

The possibility for such consistency flows directly from our choice of gradings for g_r and g_r^*.

We consider now the special cases of the modular and para ansatz algebras. It is clear in the former case that the relations (B.2), (B.27) and (B.35) will lead to the relations (2.53) and (2.54) of Chapter 3. These were the relations which were needed to construct a solution to the modular quantization relations (2.1) in the relativistic case. In the para case the extension to the relativistic case is trivial since by (B.32) we may choose $K_r = K_{\bar{r}}$ which means that d_r, d_r^*, g_r and g_r^* all Klein transform in the same way. As a result the addition of the latter two operators is no different from simply adding an extra label for anti-particles to the d_r and d_r^* operators. This situation appears to derive from the fact that the gradings for the para-ansatz algebra satisfy $\alpha(r) = -\alpha(r)$ or $2\alpha(r) = 0$.

APPENDIX C

Proof of Theorem 3.4, Chapter 3 for derivative fields.

We consider here the case in which possibly one field $\psi(x_1)$ is replaced by $\psi(x_1)_{,\mu_1}$. The extension to the more general case involves no essential difficulties.

In the case of $F_2(V)$ we have the more general possible form:

$$F_2(V) = \int \sum_{i=0}^{1} \left[a_i \psi^i(x_1)\psi^*(x_2) + b_i \psi^i(x_1)\right] dx_1 dx_2 \tag{C.1}$$

with

$$\begin{aligned} \psi^i(x_1) &\equiv \psi(x_1) \qquad i = 0 \\ &\equiv \psi(x_1)_{,\mu_1} \qquad i = 1 \, . \end{aligned}$$

By the use of the same arguments as those presented in the restricted proof, we may deduce that for $v \neq 0$,

$$F_v \equiv \int \sum_i \sum_{r-t=v} \phi^{i(t)}(x_1)\phi^{*(r)}(x_2)(a_i - \eta^{-v} b_i) = 0 \, . \tag{C.2}$$

Introduce now the following operator:

$$U(f) = exp\left[i \int f(x) \sum_{r=0}^{m-1} \phi^{*(r)}(x)\phi^{(r)}(x) dx\right] \, . \tag{C.3}$$

By the use of (3.13), (3.72)* and the properties of both the delta and derivative delta function, we may conclude that

$$\begin{aligned} U^{-1}(f)\phi^{(t)}(x_1)U(f) &= e^{if(x_1)}\phi^{(t)}(x_1) \\ U^{-1}(f)\phi^{1(t)}(x_1)U(f) &= e^{-if'(x_1)}\phi^{1(t)}(x_1) \, . \end{aligned} \tag{C.4}$$

Upon calculation of $U^{-1}(f)F_v U(f)$ with an appropriate choice of f (that is, one with $e^{if(x_1)} \neq e^{-if'(x_1)}$), we obtain two equations one of which we have considered in the restricted proof and which leads to the conclusion (for $m > 2$) that $a_0 = b_0 = 0$. The other equation is

$$F'_v \equiv \int \sum_{r-t=v} \phi^{1(t)}(x_1)\phi^{*(r)}(x_2)(a_1 - \eta^{-v} b_1) = 0 \, . \tag{C.5}$$

* Equation numbers in this appendix refer to those in Chapter 3.

If we carry out the calculations $F_v'\phi^{(w)}(z_2) - \eta^v\phi^{(w)}(z_2)F_v' \equiv G_{v-w}$ and then $G_{v-w}\phi^{*(w-v)}(z_1) + \eta^{v-w}\phi^{*(w-v)}(z_1)G_{v-w}$, we may conclude, in the same way as the restricted proof (using the properties of the derivative delta function), that for $m > 2$

$$a_{1,\mu_1} = b_{1,\mu_1} = 0 \tag{C.6}$$

where $_{,\mu_1}$ now means $\dfrac{\partial}{\partial z_{1\mu}}$. This equation implies that $a_1(z_1, z_2)$ and $b_1(z_1, z_2)$ are not functions of z_1. We now evaluate

$$0 = G_{v-w}\int tan^{-1}(z_{1\mu})\phi^{*(w-v)}(z_1)dz_{1\mu} + \eta^{v-w}\int tan^{-1}(z_{1\mu})\phi^{*(w-v)}(z_1)dz_{1\mu}G_{v-w} \tag{C.7}$$

and obtain the equations

$$\int \frac{a_1(x_1, z_2)}{1 + (x_{1\mu})^2}dx_1 = \int \frac{b_1(x_1, z_2)}{1 + (x_{1\mu})^2}dx_1 = 0\ . \tag{C.8}$$

Since the $a_1[x_1, z_2)$ and $b_1(x_1, z_2)$ do not depend on their first variable and since the rest of the integrands above are positive it follows that $a_1 = b_1 = 0$.

In the case of $F_4(V)$ the form of the possible observables is modified in a way exactly analogous to (C.2). Application of the argument above involving the operator $U(f)$ allows us to reduce the problem to one in which each term in the observable has a $\psi^1(x_1)$. Following the argument given in the restricted proof we obtain equations analogous to (3.66) except with, for example, a_1^{klmn} replaced by $a_{11}^{klmn}{}_{,\mu_k}$ (the second subscript has the same meaning as the subscripts in (C.1)). By use of the argument following this, we conclude that $a_{11}^{klmn}, a_{31}^{klmn}$ and a_{51}^{klmn} do not depend on their first variable. The proof given above for $F_2(V)$ is then easily modified to show that these functions are in fact zero. For the coefficients a_{21}^{klmn} and a_{41}^{klmn} we again have equations (3.70), with the two coefficients replaced by $a_{21}^{klmn}{}_{,\mu_k}$ and $a_{41}^{klmn}{}_{,\mu_k}$. When the identity (3.71) is modified by replacing $\psi(x_1)$ by $\psi(x_1)_{,\mu_1}$ and similarly the appropriate delta functions replaced by derivative delta functions, the argument below (3.71) will hold with the modified (3.70) showing that the right hand side of the integrated expression is zero. This allows us again to reduce the $F_4(V)$ to a $F_2(V)$ as we did in the restricted proof.

BIBLIOGRAPHY

[1] E. P. Wigner, *Phys. Rev.*, **77**, 711, (1950)
S. S. Schweber, *Phys. Rev.*, **78**, 613, (1950)
C. R. Putnam, *Phys. Rev.*, **83**, 1047, (1951)
L. M. Yang, *Phys. Rev.*, **84**, 788, (1951)

[2] H. S. Green, *Phys. Rev.*, **90**, 270, (1953)

[3] I. E. Macarthy, *Proc. Camb. Phil. Soc.*, **51**, 131, (1955)
D. V. Volkov, *Sov. Phys. JETP*, **9**, 1107, (1959); **11**, 375, (1960)

[4] O. W. Greenberg, *Phys. Rev. Lett.*, **13**, 598, (1964)

[5] O. W. Greenberg and D. Zwanziger, *Phys. Rev.*, **150**, 1177, (1966)

[6] K. Drühl, R. Haag and J. E. Roberts, *Comm. Math. Phys.*, **18**, 204, (1970)

[7] O. W. Greenberg and C. A. Nelson, *Phys. Reports*, **32C**, 69, (1977)

[8] O. W. Greenberg and A. M. L. Messiah, *Phys. Rev.*, **138**, B1155, (1965)

[9] A. B. Govorkov, *Sov. Phys. JETP*, **27**, 960, (1968)
A. B. Govorkov, *Ann. Phys.(N.Y.)*, **53**, 349, (1969)

[10] P. V. Landshoff and H. P. Stapp, *Ann. Phys.(N.Y.)*, **45**, 72, (1967)

[11] Y. Ohnuki and S. Kamefuchi, *Ann. Phys.(N.Y.)*, **51**, 337, (1969)

[12] Y. Ohnuki and S. Kamefuchi, *Prog. Theor. Phys.*, **50**, 258, (1973)

[13] O. Klein, *J. Phys. Radium*, **9**, 1, (1938)

[14] D. A. Gray, *Prog. Theor. Phys.*, **49**, 1027, (1973)

[15] Y. Ohnuki and S. Kamefuchi, *Quantum Field Theory and Parastatistics*, (Univ. of Tokyo Press, Springer 1982) pp120-127

[16] S. Weinberg, *Phys. Rev. Lett.*, **31**, 494, (1973)
D. J. Gross and F. Wiczek, *Phys. Rev.*, **D8**, 3633, (1973)
H. Fritzsch, M. Gell-Mann and H. Leutwyler, *Phys. Lett.*, **47B**, 365, (1973)

[17] P. G. O. Freund, *Phys. Rev.*, **D13**, 2322, (1976)

[18] A. B. Govorkov, Dubna preprints P2-81-749, P2-82-296, E-2-82-470

[19] M. Günaydin and F. Gürsey, *Phys. Rev.*, **D9**, 3387, (1974)

[20] M. Günaydin, *Journ. Math. Phys.*, **17**, 1875, (1976)

[21] G. Domokos and S. Kövesi-Domokos, *Phys. Rev.*, **D19**, 2984, (1979)

[22] M. Günaydin and F. Gürsey, *Journ. Math. Phys.*, **14**, 1651, (1973)

[23] O. W. Greenberg and K. I. Macrae, *Nucl. Phys.*, **B219**, 358, (1983)
see also K. I. Macrae, *Journ. Math. Phys.*, **26**, 1695, (1985)

[24] V. G. Kac, *Adv. Math.*, **26**, 8, (1977)
P. G. O. Freund and I. Kaplansky, *Journ. Math. Phys.*, **17**, 228, (1976)

[25] V. Rittenberg and D. Wyler, *Nucl. Phys.*, **B139**, 189, (1978)

[26] M. Scheunert, *Journ. Math. Phys.*, **20**, 712, (1979)

[27] H. S. Green, *Aust. J. Phys.*, **28**, 115, (1975)

[28] A. L. Carey, *Prog. Theor. Phys.*, **49**, 658, (1972)

[29] Reference [15], Chapter 9

[30] H. Harari, *Phys. Lett.*, **86B**, 83, (1979)
M. A. Shupe, *Phys. Lett.*, **86B**, 87, (1979)

[31] J. Wess and B. Zumino, *Nucl. Phys.*, **B70**, 39, (1974);
Phys. Lett., **49B**, 52, (1974); *Nucl. Phys.*, **B78**, 1, (1974)

[32] A. Ramakrishnan, *L-Matrix Theory*, (Tata-Macgraw Hill, 1972) Part 2

[33] L. Fuchs, *Abelian Groups*, (Pergamon, 1960), Chapter 2

[34] ibid. Chapter 1

[35] K. Shoda, *Math. Ann.*, **100**, 674, (1928)

[36] K.R. Parthasarathy, *Multipliers on locally compact groups, Lect. Notes in Math.*, **93**, (Springer, Berlin 1969)

[37] M. Scheunert, *The Theory of Superalgebras, Lect. Notes in Math.*, **716**, (Springer, 1979) p239

[38] B. L. van der Waerden, *Algebra*, (Unger, N.Y. 1970) Vol2, p82

[39] Y. Ohnuki and S. Kamefuchi, *Prog. Theor. Phys.*, **54**, 1860, (1975)

[40] Reference [15], p26

[41] M. A. Naimark, *Normed Rings*, (Noordhoff, Groningen 1964) p92

[42] ibid. p283

[43] ibid. p105

[44] ibid. pp75-76

[45] ibid. pp241-242

[46] J. D. Pryce, *Basic methods of linear functional analysis*, (Hutchinson, London 1973) pp12-13

[47] R. F. Streater and A. S. Wightman, *P.C.T., Spin and Statistics and all that*, (Benjamin, New York 1964) pp117-126

[48] G. Källén, *Quantum Electrodynamics*, (Unwin, Springer 1972) Chapter 3

[49] Reference [15], p207

[50] R. Gilmore, *Lie Groups, Lie Algebras and some of their applications*, (Wiley, 1974) and references therein

[51] Reference [15], p26

[52] Reference [15], p98

[53] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics, Vol 1*, (Springer, 1979) pp56-57

[54] J. Dixmier, *C^*-Algebras*, (North Holland, 1982) pp319-320

[55] Reference [15], pp111-116

[56] S. Coleman, J. Wess and B. Zumino, *Phys. Rev.*, **177**, 2239, (1969)
C. Callen, S. Coleman, J. Wess and B. Zumino, *Phys. Rev.*, **177**, 2247, (1969)

[57] P. H. Hanus, *Theory of Determinants*, (Ginn, Boston, 1903) p187ff

[58] H. Boerner, *Representations of Groups*, (North Holland, Amsterdam, 1969) Chapter 5

[59] B. L. van der Waerden, *Group Theory and Quantum Mechanics*, (Springer, 1974) pp178-179

[60] Reference [53] p72

[61] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of Operator Algebras, Vol 1*, (Academic, 1983) p23

[62] Reference [6], see the proof of Theorem 2.2

[63] ibid, see the Appendix

[64] A. J. Bracken and H. S. Green, *Journ. Math. Phys.*, **14**, 1764, (1973)

[65] H. Weyl, *The Theory of Groups and Quantum Mechanics*, (Dover, 1950) Chapter 5

[66] Reference [48] p14

[67] Reference [15] Part 3

[68] L. Lyons, *Progr. in Part. and Nucl. Phys.*, **10**, 227, (1983)

[69] P. D. Jarvis and H. S. Green, *Aust. J. Phys.*, **36**, 123, (1983)

[70] H. S. M. Coxeter and W. O. J. Moser, *Ergeb. Math.*, **41**, 1, (1957)

[71] H. Harari and N. Seiberg, *Phys. Lett.*, **98B**, 269, (1981); **100B**, 41, (1981); **102B**, 263, (1981) and *Nucl. Phys.*, **B204**, 141, (1982).

PUBLICATIONS

Aspects of modular quantization
R. Kleeman
Journ. Math. Phys., **24**, 166, (1983)

Commutation factors on generalized Lie algebras
R. Kleeman
Journ. Math. Phys., **26**, 2405, (1985)