# CLASSICAL AND QUANTUM QUADRATIC HAMILTONIANS

- a thesis submitted by

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August, 1982.

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**SUMMARY**


This thesis is concerned with the properties of dynamical systems with Hamiltonians which are general quadratic combinations of classical or quantum canonical variables or of Boson or Fermion annihilation and creation operators.

Chapter I, consisting of six sections, deals mainly with sets of canonical forms for quadratic Hamiltonians under the action of the group of linear canonical transformations. By viewing the group of C.C.R. Bogoliubov transformations as Sp(2N,R), a full set of canonical Boson Hamiltonians, labelled by the invariants for symplectic conjugacy classes, is obtained. Since the conjugacy class of a real orthogonal matrix is determined by its eigenvalues, there is only one type of canonical form for quadratic Fermion Hamiltonians. Unlike the Fermion case, indefinite Boson Hamiltonians can not in general be reduced to a sum of quasi-particle number operators. However, it is shown how spectral theory in Krein space rigorizes the full reduction of a strictly positive quadratic Boson Hamiltonian for which the single particle space is infinite dimensional. In the case of infinite degrees of freedom, there are various existence criteria for a unitary operator which implements a given Bogoliubov transformation and the equivalence of these is discussed in Section 6.

Chapter II, consisting of seven sections, applies the techniques of Chapter I to algebraic quantization. The Boson commutation relations, among formal mode operators obtained in heuristic quantization, are shown to be incompatible with the canonical commutation relations among conjugate variables unless the motion is stable. Segal quantization of any real orthogonal dynamics, according to Fermi-Dirac statistics, is shown to be possible. The orthogonal component, in the polar decomposition of the dynamical generator $-\hat{A}$, is a complex structure which enables the classical

dynamics to be viewed as being unitary. When a symplectic dynamical group has a strictly positive generator $\hat{A}$, the complex structure J is uniquely determined as the pseudo-orthogonal component in the polar decomposition of $\hat{A}$ viewed as an operator on Krein space with indefinite inner product $i\,B(\cdot,\cdot)$, B being the symplectic form. Equivalently, $J = -i(1-2E(o))$, E(o) being the projection onto the maximal dynamically invariant subspace on which $i\hat{A}$ is negative definite. Unitarization of unstable symplectic dynamics is impossible but for a large class of Hamiltonians, determined in section 13, the unstable motion can be viewed as being pseudo-unitary.

Chapter III is devoted to applications of the general theory developed earlier. In Section 14, fields with external potential, non-local fields, the Schwinger model and the Thirring-Narnhofer model are considered. Section 15 treats the Ising model and assimilates the theory of quasi-free states of the C.A.R. C* algebra. A similar analysis of statistical systems of Bosons is discussed in Section 16.

## STATEMENT

This thesis contains no material which has been accepted for the award of any other degree, and to the best of my knowledge and belief, contains no material previously published or written by any other person except where due reference is made in the text.

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Philip Broadbridge

## ACKNOWLEDGEMENTS

This thesis results from work carried out in the Department of Mathematical Physics, University of Adelaide, during the years 1979 to 1982. During that time, all members of the Department's staff have encouraged a positive attitude to research and have attracted many distinguished visitors, including Professors E. Stormer, J. Roberts, D. Robinson, J. Blatt, R. Dalitz, F. Dyson, I. Segal, A. Jaffe, E. Lieb, A. Broyles, R. Leipnik and A. Wightman. This has been not only a source of motivation but also an influence on the direction of enquiry.

I am indebted to Professor C. Angas Hurst, whose supervision has been characterized by encouragement and determination. Our many hours of discussion have been both stimulating and rewarding, resulting in three joint publications and influencing several others of sole authorship.

Drs. D. O'Brien and A. Carey have shown continual interest and have always been willing to offer advice when called upon. I have benefitted from stimulating discussions with fellow students, including J. Wright, K. Hinton, R. Kleeman, H. Grundling and M. Gould.

I am grateful for the kind assistance of the staff of the Commonwealth Department of Education, in their administration of the Postgraduate Research Awards. Finally, at the end of my formal education, it would be remiss of me not to express my heartfelt gratitude for the support of my father, my late mother and my wife.

# CHAPTER I - CANONICAL TRANSFORMATIONS



## SECTION 1 - CLASSICAL HAMILTONIAN SYSTEMS

It is generally accepted that the variety of solutions to a system of physical dynamical equations should form a differentiable manifold $V$. The phase manifold $M$ can then be defined as the set of points $\underline{z} = (\underline{q},\underline{p})$ in the cotangent bundle $T^*V$. This leads to a natural canonical symplectic structure on $M$, $\omega^2 = \sum dp_j \wedge dq_j$. Given any classical observable H (a differentiable function on $M$), a Hamiltonian flow is defined by the Hamiltonian vector field

$$\dot{\underline{z}} = -GdH, \tag{1.1}$$

where $-G$ is the natural mapping from the cotangent bundle $T^*M$ to the tangent bundle $TM$ defined in such a way that a tangent vector $\underline{\xi} \in T_{\underline{z}}M$ is the image of the differential 1-form

$$\omega^1_{\underline{\xi}} : T_{\underline{z}} \to R \tag{1.2}$$

$$\eta \to \omega^2(\underline{\eta},\underline{\xi})$$ (see, for example, chap.8 of [1]).

In this thesis, we shall be concerned almost entirely with linear systems, so that $V$ is flat and we may identify $M$ with $V \oplus V'$ ($V'$ is the dual of $V$) or more simply, with $R^{2N}$ when $\text{Dim}\, V = N < \infty$. When linear dynamical systems are described in the language of symplectic geometry, the notation which emerges is clearly defined (e.g. [2]). In this section, we shall be dealing with a finite number N of degrees of freedom. In this case we may always choose a basis $\{\underline{e}_j\}_{j=1,..,2N}$ so that a given symplectic form $B(\cdot,\cdot)$ on $R^{2N}$ may be expressed $B(\underline{\xi},\underline{\zeta}) = (\underline{\xi},G\underline{\zeta})$,

$G = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ ; I is the identity on $R^N$ and

$(\cdot,\cdot)$ is the Euclidean inner product.

Such a basis, satisfying $B(\underline{e}_j,\underline{e}_k) = 0$

$B(\underline{e}_j,\underline{e}_{k+N}) = -\delta_{jk}$ for all j, k $\leq$ N

is called a symplectic basis. For any such basis, $\{e_j\}_{j\leq N}$ and $\{\underline{e}_{j+N}\}_{j\leq N}$

each span a Lagrange subspace, a maximal subspace on which the symplectic form vanishes.

A general quadratic Hamiltonian H can always be made homogeneous by choosing the dynamical critical point as the origin [3].

$$H = \frac{1}{2}\,(\underline{z}, \hat{H}\,\underline{z}),$$

where $\underline{z} = \begin{pmatrix} \underline{q} \\ \underline{p} \end{pmatrix}$ and $H = \begin{pmatrix} A & B \\ B^T & F \end{pmatrix}$, $A = A^T$, $F = F^T$ (real symmetric) (1.3)

Then the Hamilton equations are

$$\dot{\underline{z}} = -G\hat{H}\,\underline{z}\ ,\ G = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = -G^{-1} \tag{1.4}$$

or

$$\underline{z}(t) = e^{-G\hat{H}t}\,\underline{z}(0)\ . \tag{1.5}$$

(1.5) describes a one parameter group of symplectic transformations generated by $-G\hat{H}$, an element of the lie algebra sp(2N,R). The eigenvalues of $iG\hat{H}$ are known as the frequencies of system (1.5). If s is a frequency, then so are the complex conjugate $\bar{s}$ and the additive inverse $-s$. A real symplectic transformation

$$\underline{z}' = C^{-1}\,\underline{z} : C^T\,GC = G\ \ (C \in Sp(2N,R)) \tag{1.6}$$

preserves

(a) The form of Hamilton's equations

$$\dot{\underline{z}}' = -G\,\hat{K}\,\underline{z}'\ ;\ \hat{K} = C^T\,\hat{H}\,C\ .$$

(b) The Poisson brackets

$$\{z'_\mu(z),\ z'_\nu(\underline{z})\} = -G_{\mu\nu} \tag{1.7}$$

and

(c) The value of the Hamiltonian

$$K = \frac{1}{2}\,\underline{z}'^T\,\hat{K}\,\underline{z}' = \frac{1}{2}\,\underline{z}^T\,\hat{H}\,\underline{z} = H\ .$$

A general linear canonical transformation need only preserve the form of Hamilton's equations. If C is relatively symplectic [4],

$$C^T\,G\,C = g\,G\ ,\ g \in R \sim \{0\}\ , \tag{1.8}$$

then (1.4) implies $\dot{\underline{z}}' = -G\,\hat{K}\,\underline{z}'$, $\hat{K} = g^{-1}\,C^T\,\hat{H}\,C$ . (1.9)

However, the Poisson brackets are no longer preserved

$$\{z'_\mu, z'_\nu\} = - g^{-1} G_{\mu\nu} \tag{1.10}$$

and the Hamiltonian is no longer a scalar quantity

$$K = \frac{1}{2} \underline{z}'^T \hat{K} \underline{z}' = g^{-1} \underline{z}^T \hat{H} \underline{z} = g^{-1} H . \tag{1.11}$$

When considering the action of the group of relatively symplectic canonical transformations on symmetric matrices H, it will suffice to assume that $g = \pm 1$. If $\hat{H}_1 = g^{-1} C^T \hat{H} C$ with $C^T G C = gG$, we may define $C_1 = |g|^{-1/2} C$, so that $C_1{}^T G C_1 = g/|g| \; G$ and $\hat{H} \to \hat{H}_1 = |g|/g \, C_1{}^T \hat{H} C_1$. (1.11a)

We shall first consider the action of the symplectic group and then the inclusion of anti-symplectic transformations will produce some minor modifications. Each real 2N × 2N symmetric matrix $\hat{H}$ is contained in a canonical orbit

$$\{C^T \hat{H} C \; ; \; C \in Sp(2N,R)\} . \tag{1.12}$$

In any treatment of the symplectic transformations $C \in Sp(2N,C)$, the following facts are used repeatedly :

$$C^{-1} = - G C^T G \tag{1.13}$$

$$C^{-1} G = G C^T \tag{1.14}$$

$C^T$ is symplectic. (1.15)

From (1.14) $$\hat{H}_2 = C^T \hat{H}_1 C \Leftrightarrow -G\hat{H}_2 = C^{-1}(-G\hat{H}_1)C \tag{1.16}$$

Under the symplectic similarity transformation (1.16), and indeed under any similarity transformation, the elementary divisors $(s-s_i)^{N_i}$ of the characteristic pencil $iG\hat{H} - sI$ will be invariant [5,6]. The invariants determine the eigenvalues $s_i$ of $iG\hat{H}$ but because we are dealing only with a (symplectic) subgroup of the similarity transformations, they will not always determine the orbit of $\hat{H}$ uniquely. In fact, it turns out that all eigenvalues of $iG\hat{H}$ need to have non-vanishing imaginary parts before the elementary divisors are certain to have this ability. By Sylvester's law of inertia, all elements $\hat{H}$ within a canonical orbit (1.12) must have the same signature. For example, if $\hat{H} = I$, then $iG\hat{H} - sI$ and $-iG\hat{H} - sI$ have the same elementary divisors $s \pm 1$ and hence $+iG\hat{H}$ and $-iG\hat{H}$ have the same Jordan canonical form. However, $+\hat{H}$ is positive definite while $-\hat{H}$ has the opposite signature and so can not belong to the same canonical orbit.

By (1.16), representatives $-G\hat{K}$ of the conjugacy classes for the Lie algebra sp(2N,R) may be pre-multiplied by G to give representatives $\hat{K} = \hat{K}^T$ of the canonical orbits. The conjugacy classes of sp(2N,R) may be obtained from those of the group Sp(2N,R). The latter were first obtained by Williamson [7] and this work was expanded by Wall [8] to accommodate symplectic spaces over arbitrary division rings or fields. One should note, in this context, that some symplectic transformations do not have an exponential representation [9]. In general, representatives of the algebra are placed in 1-1 correspondence with representatives of the group by means of a generalized Cayley transform [10]: $f(z) = u(z) + (1+u^2(z))^{1/2}$, with $u$ an odd function analytic at $z = 0$.

Examples include $u(z) = \sinh z$, $f(z) = \exp z$,

the Cayley transform $u(z) = -2z/(1-z^2)$, $f(z) = (1-z)(1+z)^{-1}$ (applied to Sp(2N,R) by Laub and Meyer [2])

and $u(z) = z = \frac{1}{2}(f-f^{-1})$ (applied to Sp(2N,R) by Cushman [11]).

The canonical orbits were first obtained in a separate work by Williamson [12], who showed how to construct canonical forms for $\hat{H}$. The list of canonical forms, to which we shall refer, has been devised by Galin [13].

Theorem (1.17) A real symplectic vector space with a given quadratic form $H = \frac{1}{2}(\underline{x}, \hat{H}\underline{x})$ can be decomposed into a direct sum of pairwise skew orthogonal real symplectic subspaces so that the form H is a sum of forms of the types $\frac{1}{2}(\underline{y}, \hat{K}_j^{(2k)}\underline{y})$ on these subspaces, with $\hat{K}_j^{(2k)}$ the fundamental normal form of type j and order 2k listed below.

$$\hat{K}_j^{(2k)} = \begin{pmatrix} A & B \\ B^T & F \end{pmatrix},$$

with order k sub-matrices as listed:

$$\hat{K}_1^{(2k)}: \quad A = F = 0 \ \& \ B = \begin{bmatrix} -a & & & & \\ 1 & -a & & & \\ & 1 & -a & & \\ & & 1 & -a & \end{bmatrix};$$

$$\hat{K}_2^{(4k)}: \quad A = F = 0 \ \& \ B = \begin{bmatrix} W & & & \\ I_2 & W & & \\ & I_2 & \cdot & \\ & & \cdot & \cdot & \\ & & 0 & I_2 & W \end{bmatrix}; \quad W = \begin{pmatrix} -a & -b \\ b & -a \end{pmatrix};$$

$$\hat{K}_3^{(2k)}: \quad A = F = 0 \ \& \ B = \begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 1 & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 1 & 0 \\ & & & & & 1 & 0 \end{bmatrix};$$

$$\hat{K}_4^{(2k)}(\rho): \quad A = \begin{bmatrix} & & & & -\rho \\ & & & -\rho & \\ & & \cdot & & \\ & \cdot & & & \\ -\rho & & & & \\ -\rho & & & & \end{bmatrix},$$

$$F = \begin{bmatrix} & & & \rho & 0 \\ & & \rho & 0 & \\ & \cdot & \cdot & & \\ \cdot & \cdot & & & \\ \rho & 0 & & & \\ 0 & & & & \end{bmatrix}$$

$$B = -\begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 1 & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 1 & 0 \\ & & & & & 1 & 0 \end{bmatrix}, \quad \text{with } \rho = \pm 1;$$

$$\hat{K}_5^{[2\times(2k+1)]}(\rho): \quad A = \begin{bmatrix} & & & & & -\rho \\ & & & & +\rho & \\ & & & -\rho & & \\ & & \cdot & & & \\ & +\rho & & & & \\ -\rho & & & & & \end{bmatrix}$$

$F = b^2$ A,B as in $\hat{K}_4$ (above), with $\rho = \pm 1$;

$$\hat{K}_6^{(4k)}(\rho): \quad A = \begin{bmatrix} & & & & Y \\ & & & Y & \\ & & \cdot & & \\ & \cdot & & & \\ Y & & & & \end{bmatrix}, \text{ with } Y = \rho\begin{pmatrix} b^{-2} & 0 \\ 0 & 1 \end{pmatrix} \; \rho = \pm 1,$$

$$F = \begin{bmatrix} & & & & & 0_2 \\ & & & & 0_2 & Q \\ & & & 0_2 & Q & \\ & & \cdot & \cdot & & \\ & 0_2 & Q & & & \\ 0_2 & Q & & & & \end{bmatrix}, \text{ with } Q = -\rho\begin{pmatrix} b^2 & 0 \\ 0 & 1 \end{pmatrix}, \; 0_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} M & & & & \\ & M & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & M \end{bmatrix}, \text{ with } M = \begin{pmatrix} 0 & 1 \\ -b^2 & 0 \end{pmatrix}$$

Some invariants associated with orbits containing $\hat{K}_j^{(2k)}$ are listed in the following:

TABLE (1.18)

| Canonical form | Elementary divisors | Frequencies | Geometric Multiplicity | Signature |
|---|---|---|---|---|
| $\hat{K}_1^{(2k)}$ | $(s+ia)^k, (s-ia)^k$ | $\pm ia$ | 1 | $(k,k)$ |
| $\hat{K}_2^{(4k)}$ | $(s\pm ia\pm b)^k$ | $\pm ia\pm b$ | 1 | $(2k,2k)$ |
| $\hat{K}_3^{(2k)}$ | $s^k, s^k$ | 0 | 2 | $(k-1,k-1)$ |
| $\hat{K}_4^{(2k)}(\rho)$ | $s^{2k}$ | 0 | 1 | $(\frac{1}{2}(1-\rho),\frac{1}{2}(1+\rho))$ (mod.2) for $k = 1,2$ |
| $\hat{K}_5^{(2(2k+1))}(\rho)$ | $(s\pm b)^{2k+1}$ | $\pm b$ | 1 | $(1-\rho,1+\rho)$ for $k = 0$ |
| $\hat{K}_6^{(4k)}(\rho)$ | $(s\pm b)^{2k}$ | $\pm b$ | 1 | $(2,2)$ for $k = 1$ |

In the above, the elementary divisors are those of $i\hat{G}H-sI$. The signature is that of $\hat{H}$, expressed as an ordered pair $(n_+,n_-)$, $n_\pm$ being the number of positive and negative eigenvalues.

In the general case, a canonical form $\hat{K}$ can be decomposed as a skew orthogonal (ie with respect to the symplectic form) direct sum of indecomposable canonical forms

$$\hat{K} = \hat{K}_{(1)} \oplus \hat{K}_{(2)} \oplus \ldots \oplus \hat{K}_{(p)} \tag{1.19}$$

with each component $\hat{K}_{(j)}$ belonging to the above list. In a coordinate representation of a symplectic direct sum

$$V = V_1^{(2k)} \oplus V_2^{(2\ell)} ,$$

the sum $(q_1,\ldots,q_k, p_1,\ldots,p_k) \oplus (q_{k+1},\ldots,q_{k+\ell}, p_{k+1},\ldots,p_{k+\ell})$ should be written as $(q_i,q_{k+i},p_j,p_{k+j})$. This reordering ensures that the basis $\{\underline{e}_\mu\}$ for $V$ is a symplectic basis $\underline{e}_\mu^T G \underline{e}_\nu = G_{\mu\nu}$, where $G = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. (1.20)

For some applications, other authors [14,15] have found it convenient to work with the usual direct sum and with a symplectic form given by the matrix

$$G = \begin{bmatrix} 0 & -I^{(2k)} & & \\ I^{(2k)} & 0 & & \\ & & 0 & -I^{(2\ell)} \\ & & I^{(2\ell)} & 0 \end{bmatrix}$$

However, we prefer to use a sympletic direct sum in (1.19) and to retain (1.20). The normal forms for $\hat{H}$ given in (1.19) are

$$\hat{K} = \begin{bmatrix} A_1 & & & & B_1 & & & \\ & A_2 & & & & B_2 & & \\ & & \ddots & & & & \ddots & \\ & & & A_p & & & & B_p \\ B_1^T & & & & F_1 & & & \\ & B_2^T & & & & F_2 & & \\ & & \ddots & & & & \ddots & \\ & & & B_p^T & & & & F_p \end{bmatrix}$$

and the indecomposable components in (1.19) are $\hat{K}_{(j)} = \begin{pmatrix} A_j & B_j \\ B_j^T & F_j \end{pmatrix}$.

In table 1.18, $\hat{K}_6^{(4k)}(\rho)$, with $\rho = \pm 1$, is the only listed example of a pair of distinct indecomposable canonical forms with the same elementary divisors and same signature. To find an extra distinguishing invariant, we note that for every element $\hat{A} = -G\hat{H}$ of the Lie algebra $sp(2N,R)$, there is a unique Jordan decomposition $\hat{A} = \hat{S} + \hat{N}$, with $\hat{S}$ semi-simple and $\hat{N}$ nilpotent, such that $[\hat{N},\hat{S}] = 0$. It is well known that $\hat{S}$ and $\hat{N}$ are polynomials in $\hat{A}$. Any zero order terms in the polynomials will be absorbed in the diagonalizable component $\hat{S}$. Therefore, $\hat{N}$ is a linear combination of terms $\hat{A}^j = -G[(-1)^{j-1} \hat{H} \underbrace{G \hat{H} \ldots G\hat{H}}_{j-1 \text{ factors}}]$ with $j \geq 1$. Thus, $\hat{N} = -G\hat{H}_N$, with $\hat{H}_N$ symmetric and therefore $\hat{S} = -G\hat{H}_s$, with $\hat{H}_s$ $(= \hat{H} - \hat{H}_N)$ symmetric. Under a symplectic canonical transformation,

$$\hat{H} \longrightarrow C^T \hat{H} C, \quad (C^T G C = G)$$

$$-G\hat{H} \longrightarrow -G C^T \hat{H} C = C^{-1}(-G\hat{H})C,$$

so that $\hat{N} = -G\hat{H}_N \longrightarrow C^{-1}(-G\hat{H}_N)C = -G C^T \hat{H}_N C.$

That is, $$\hat{H}_N \longrightarrow C^T \hat{H}_N C \, . \tag{1.21}$$

Therefore, by Sylvester's law of inertia, the signature of $\hat{H}_N$ is invariant. When $\hat{H} = \hat{K}_6^{(4k)}(\rho)$, $\hat{H}_N$ includes precisely those matrix elements of $\hat{H}$ in which $\rho$ appears. The only eigenvalues of $\hat{H}_N$ are 1, -1 and 0. The dimensions of the eigenspaces $\ker(\hat{H}_N - \lambda I)$ can then be obtained by standard Gaussian elimination, from which

$$\text{sig } \hat{H}_N = (2k-1 + \rho(-1)^{k+1}, \; 2k-1 + \rho(-1)^k) \tag{1.22}$$

for all members $\hat{H}$ of the same canonical orbit as $\hat{K}_6^{(4k)}(\rho)$.

A full set of invariants for conjugacy classes of the *group* $Sp(2N,R)$ has previously been published by Cushman [11], who made extensive use of the unique factorization $C = C_S C_N$, with symplectic factors $C_S$ and $C_N$ being respectively semi-simple and $I + N$, $N$ being nilpotent.

The invariants which we have used above, namely elementary divisors of $iG\hat{H} - sI$, sig.$\hat{H}$ and sig.$\hat{H}_N$ lead to a very practical determination of the canonical orbit of $\hat{H}$.

Having listed the canonical forms for $\hat{H}$ under changes of symplectic basis, we shall now ask how the canonical orbits are modified by allowing, in addition, anti-symplectic transformations, as in (1.11a). With $g = -1$, the transformation (1.11a) changes the signature $(n_+, n_-)$ of $\hat{H}$ to $(n_-, n_+)$. Although the indecomposable canonical forms $\hat{K}_j^{(2m)}(\rho)$ with $\rho = +1, -1$ belong to different canonical orbits, in each case $j = 4, 5$ or $6$, they are equivalent via the transformation law (1.11a) with $C = \hat{I} = \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix}$, an anti-symplectic transformation. In fact, no other anti-symplectic transformations need be considered, since every anti-symplectic transformation $C$ may be expressed $C = \hat{I}\, C_1$, with $C_1 (= \hat{I}\, C)$ symplectic. The effect of the symplectic transformations has already been accounted for. In other words, the group $G$ of symplectic and anti-symplectic transformations is $\{I, \hat{I}\} \otimes Sp(2N,R)$. $Sp(2N,R)$ is a normal subgroup of $G$ and the factor group $G/Sp(2N,R)$ contains only two left cosets.

The anti-symplectic matrix $\hat{I}$ merely changes the sign of all conjugate momenta and may be associated with time-reversal. As discussed by Kaellen [16], under time inversion

$$\xi = -t \,, \tag{1.23a}$$

define $$Q(\xi) = q(t) \tag{1.23b}$$

and $$P(\xi) = \frac{\partial L}{\partial \dot{Q}^1} \,, \quad \text{with} \quad \dot{Q}^1 = \frac{dQ}{d\xi} = -\frac{dq}{dt}$$

Thus $$P(\xi) = -p(t) \tag{1.23c}$$

Assuming (1.23) and the Hamilton equations among $q(t)$, $p(t)$, we obtain

$$\frac{dQ}{d\xi} = \frac{\partial}{\partial P} K \quad \text{with} \quad K = \frac{1}{2}(Q,P)\hat{I}^T \hat{H} \hat{I} \begin{pmatrix} Q \\ P \end{pmatrix} = H \,.$$

In Wintner's formulation [4], the anti-symplectic transformation $\hat{I}$, without the associated time-reversal, results in $H$ being pseudo-scalar, $\frac{dQ}{dt} = \frac{\partial}{\partial P}(-H)$, as in (1.11).

At first appearances, it seems that our classification of the canonical orbits contradicts Mal'cev's result [17] that conjugacy classes of the symplectic Lie algebra are completely determined by the Jordan canonical forms, just as if general similarity transformations were allowed in (1,16). However, Mal'cev's construction clearly admits complex symplectic transformations. Application of this larger group lowers the number of canonical orbits. For example, the transformation of $\hat{H}$ under the anti-symplectic transformation

$$\hat{I} : \hat{H} = \begin{pmatrix} A & B \\ B^T & F \end{pmatrix} \rightarrow -\hat{I}\,\hat{H}\,\hat{I} = \begin{pmatrix} -A & B \\ B^T & -F \end{pmatrix} = \hat{H}_1$$

could have been achieved by the complex symplectic transformation $C = i\,\hat{I}$. $C^T\,\hat{H}\,C = \hat{H}_1$ and $C^T\,G\,C = G$. Hence, under the action of $Sp(2N,C)$, the orbits containing $\hat{K}_j(\rho)$ ($\rho = +1$ and $-1$) merge, just as when anti-symplectic transformations are introduced. The action of the complex symplectic group on a real symmetric matrix is the same as that of the group of real symplectic and anti-symplectic transformations.

Besides $Sp(2N,C)$, there is another commonly used complex extension of $Sp(2N,R)$ namely the group

$$\{C \in GL(C^{2N}) \; ; \; C^{\dagger}\,G\,C = G\}$$

This is the group $U(N,N)$ which preserves the indefinite complex inner product $(\underline{\xi}, iG\underline{\zeta})$, with $(\underline{\xi},\underline{\zeta})$ the complex Euclidean inner product on $C^{2N}$. This group will appear later in our treatment of Bogoliubov transformations. The extensions $Sp(2N,C)$ and $U(N,N)$ of $Sp(2N,R)$ are analogous to the familiar extensions $O(2N,C)$ and $U(2N)$ of $O(2N,R)$.

For practical applications of symplectic transformations, the following three theorems are often useful:

<u>Theorem</u> (1.24)(Weierstrass [18]). If, in the dynamical matrix $\hat{H}$ of (1.3), the kinetic energy term $F$ is positive definite, then $\hat{H}$ can be completely diagonalized by a symplectic transformation.

Theorem (1.25)(Moshinsky and Winternitz [15]). If $\hat{H}$ is invariant under the rotation, displacement and permutation subgroups of the canonical point transformations, which do not mix canonical coordinates with their conjugate momenta, then H can be diagonalized by a canonical transformation.

It can easily be checked that $G\hat{H}$, with $\hat{H}$ diagonal, can be decomposed into a symplectic direct sum of $2 \times 2$ matrices $G\hat{H}_{(j)}$, corresponding to independent subsystems with Hamiltonian $H_j = \frac{1}{2}(\alpha P_j^2 + \beta q_j^2)$; $\alpha, \beta \in R$.

An elementary calculation of signature and of elementary divisors, followed by a comparison with table (1.18), shows that if $H_j$ is a harmonic oscillator Hamiltonian ($\alpha, \beta > 0$), then $\hat{H}_j$ belongs to the same class as $\hat{K}_5^{(2)}(-1)$. If $\alpha, \beta < 0$, $H_j$ still generates simple harmonic motion but now $\hat{H}_j$ is in the same class as $\hat{K}_5^{(2)}(+1)$. If $\alpha\beta < 0$, then $\hat{H}$ has imaginary frequencies, so that it belongs to the same class as $\hat{K}_1^{(2)}$, sometimes called the "inverted oscillator" or "repulsive oscillator" class. However, such a system, being exponentially unstable, certainly does not oscillate. One realization of such a system is a massive object in frictionless contact with a parabolic hill, in the presence of a uniform vertical gravitational field, so that potential energy is proportional to $y(\alpha - x^2)$. If exactly one of $\alpha$ and $\beta$ vanish, then $\hat{H}_j$ belongs to the same class as $\hat{K}_4^{(2)}(\rho)$. The free particle case, $\alpha > 0$ and $\beta = 0$, is associated with $\hat{K}_4^{(2)}(-1)$.

Theorem (1.26)(Whittaker [3]). Suppose that $\hat{H}$ is positive definite. Then $\hat{H}$ may be diagonalized to $\hat{H}_0 = \text{diag.}[s_1,\ldots,s_N,s_1,\ldots,s_N]$ by a change of symplectic basis.

Whittaker's symplectic basis consists of vectors $2^{-\frac{1}{2}}[\underline{W}_k + \overline{\underline{W}}_k]$ and $2^{-\frac{1}{2}}i[\underline{W}_k - \overline{\underline{W}}_k]$, where $\underline{W}_k = \begin{pmatrix} \underline{U}_k \\ \underline{V}_k \end{pmatrix}$ are independent eigenvectors corresponding to eigenvalues $-s_k$ of $i\,G\,\hat{H}$. Placed in an appropriate order, these symplectic basis vectors are the columns of the diagonalizing symplectic matrix. By Whittaker's theorem, any positive definite quadratic Hamiltonian can be transformed, by a change of symplectic basis, to that of a collection of independent harmonic oscillators.

In Newtonian mechanics, the kinetic energy term is $\sum_j \frac{p_j^2}{2M_j}$, which must be positive definite, as specified in Weierstrass's theorem (1.24). However, one should beware of Newtonian notions in the general Hamiltonian mechanics. For example, the classical sleeping top can be described by a quadratic Hamiltonian [19] which contains gyroscopic terms $B_{ij}\,q_i\,p_j$. We know that such a system has a stable equilibrium point at which the energy is not minimal. Let us take such an equilibrium point as the origin of our canonical coordinate system. Since the system is stable, $iG\hat{H} - sI$ must have elementary divisors $(s - s_j)$, $s_j \in R$, (e.g. [20]). In canonical form

$$\hat{K} = \hat{K} \oplus \ldots \oplus \hat{K}_p \tag{1.27}$$

for $\hat{H}$, the only possible indecomposable blocks $\hat{K}_j$ are

$$\hat{K}_5^{(2)}(\pm 1, s_j) \equiv \text{diag.}[s_j^2, 1] . \tag{1.28}$$

Since the stable equilibrium point $(\underline{0},\underline{0})$ does not minimize the energy, some block $\hat{K}_j$ must be $-\text{diag.}[s_j^2, 1]$, corresponding to an independent subsystem with Hamiltonian $-\frac{1}{2}\,p_j^2 - \frac{1}{2}\,s_j^2\,q_j^2$, including a negative kinetic energy term, which allows no Newtonian interpretation.

Consider now a quadratic Hamiltonian (1.3) in which the symmetric matrix $\hat{H}$ is time-dependent. Then we have

$$\dot{\underline{z}} = -G\hat{H}(t)\,\underline{z}(t) . \tag{1.29}$$

If $\hat{H}(t)$ is to be reduced by a symplectic transformation $C$ to a canonical form which allows simple solution of the Hamiltonian equations at all times, then we must allow $C$ to be time dependent. In the case that $\hat{H}(t)$ is positive definite, the construction of $C(t)$ by a perturbation series was first investigated by Lanczos [21].

Suppose that C(t) is a differentiable symplectic transformation. Then

$$\underline{z}(t) = C(t)\,\underline{z}'(t) \tag{1.30}$$

$$\Rightarrow \dot{\underline{z}}(t) = -G\,\hat{K}(t)\,\underline{z}'(t) \tag{1.31}$$

for some $\hat{K} = \hat{K}^T$. By (1.29-30)

$$\begin{aligned} -G\hat{H}\,C(t)\,\underline{z}'(t) &= -G\hat{H}\,\underline{z}(t) \\ &= \dot{\underline{z}}(t) \\ &= \frac{d}{dt}[C(t)\,\underline{z}'(t)] \\ &= \dot{C}(t)\,\underline{z}'(t) - C(t)\,G\hat{K}(t)\,\underline{z}'(t) \end{aligned}$$

$$\Leftrightarrow -G\hat{H}\,C(t) = \dot{C}(t) - C(t)\,G\hat{K}(t) \tag{1.32}$$

(since $\underline{z}'(t)$ is an arbitrary element of phase space). It is known [22] that whenever $\hat{K}(t)$ and $\hat{H}(t)$ are continuous over a closed interval [a,b], there exists a solution C(t) to (1.32) which is differentiable over (a,b). Furthermore, if C(0) is symplectic, then C(t) is symplectic for all t [23]. This means that any two time-dependent quadratic Hamiltonians are related by some time-dependent symplectic transformation. However, we may re-express (1.32) as

$$\hat{K} = C^T\,\hat{H}C - C^T\,G\dot{C}\ . \tag{1.33}$$

From (1.33), we see that, as in the case of relatively symplectic transformations (1.11), the Hamiltonian is no longer a scalar invariant [21].

$$\begin{aligned} K = \tfrac{1}{2}\,\underline{z}'^T\,\hat{K}\,\underline{z}' &= \underline{z}^T\,C^{T^{-1}}\,C^T\,\hat{H}C\,C^{-1}\,\underline{z} - \underline{z}^T\,G\dot{C}\,C^{-1}\,\underline{z} \\ &= H - \tfrac{1}{2}\,\underline{z}^T\,G\dot{C}\,C^{-1}\,\underline{z}\ . \end{aligned} \tag{1.34}$$

Using the time-dependent symplectic transformations, the invariants of, for example, a harmonic oscillator with time-dependent frequency, can be derived [22] from the invariants of a harmonic oscillator with constant frequency $\omega = 1$. With one degree of freedom, equation (1.32) for C(t) can be solved explicitly [24]. However, with extra degrees of freedom, the direct construction of C(t) which reduces an arbitary indefinite quadratic Hamiltonian H to that of a collection of independent harmonic oscillators may be no easier than a direct assault on the Hamiltonian equations generated by H.

For some purposes, it might be necessary to list a full set of canonical quadratic Hamiltonians which are not singular (i.e. for which the kinetic energy matrix F of (1.3) is invertible). In the next section, we shall derive such a set. Later, we shall indirectly derive another set of canonical forms in which the gyroscopic matrix B is skew-symmetric. In the common examples with F = I, a symmetric term in B only serves to add a total time derivative $\frac{d}{dt} f(\underline{q},t)$ to the Lagrangian and this is known to have no effect on the canonical formulation of dynamics.

## SECTION 2 - RECONSTRUCTIVE SURGERY ON SINGULAR HAMILTONIANS

We recall that the general quadratic Hamiltonian

$$H = \tfrac{1}{2}(\underline{q}, A\,\underline{q}) + (\underline{q}, B\,\underline{p}) + \tfrac{1}{2}(\underline{p}, F\,\underline{p})$$

leads to the Hamilton equations, which include

$$\dot{\underline{q}} = B^T \underline{q} + F\,\underline{p} \tag{2.1}$$

This allows us to express $\underline{p}$ in terms of $\underline{q}$ and $\dot{\underline{q}}$

$$\underline{p} = F^{-1}(\dot{\underline{q}} - B^T \underline{q}) \tag{2.2}$$

The Legendre transform then produces a Lagrangian

$$L = (\dot{\underline{q}},\underline{p}) - H$$

$$= -\tfrac{1}{2}(\underline{q}, W\,\underline{q}) + (\underline{q}, X\,\dot{\underline{q}}) + \tfrac{1}{2}(\underline{q}, Y\,\dot{\underline{q}}) \tag{2.3}$$

where

$$W = A - B\,F^{-1}\,B^T \tag{2.4a}$$

$$X = -B\,F^{-1} \tag{2.4b}$$

$$Y = F^{-1} \tag{2.4c}$$

In the case of a singular Hamiltonian, $F^{-1}$ does not exist. Nevertheless, as discussed by Duffin [10], $Y$ need only be one of the generalized inverses satisfying $F\,Y\,F = Y$. Since $F$ is symmetric, many such generalized inverses exist, as can be seen in the basis in which $F$ is diagonal. In particular, the symmetric matrix $F$ must be normal, $F^T F = F\,F^T$, and therefore, there exists a unique Moore-Penrose inverse [25] satisfying

$$F\,Y\,F = Y, \quad Y\,F\,Y = F \quad \text{and} \quad 0 = (F\,Y)^T - F\,Y = (Y\,F)^T - Y\,F. \tag{2.5}$$

Alternatively, it is known from general considerations by Sudarshan and Mukunda [26] that a singular Hamiltonian may always be transformed to a non-singular one, although the transformations have not been given explicitly. We shall now extract a non-singular Hamiltonian from each canonical orbit and shall state the symplectic transformation C which relates it to the previously listed canonical form.

Given any decomposable linear Hamiltonian system, we could transform each indecomposable subsystem separately to one of the canonical forms already listed. The canonical transformation which achieves this could be constructed explicitly using the methods of Laub and Meyer [2]. Then, to $\hat{K}_1^{(2k)}$, we would apply $C = \begin{pmatrix} I & I \\ -\frac{1}{2} I & \frac{1}{2} I \end{pmatrix}$, so that $C^T \hat{K}_1 C$ has a lower diagonal (kinetic energy) block

$$F = \begin{bmatrix} -a & \frac{1}{2} & & & & \\ \frac{1}{2} & -a & \frac{1}{2} & & & \\ & \frac{1}{2} & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \frac{1}{2} \\ & & & \frac{1}{2} & -a \end{bmatrix} .$$

The same transformation $C$ applied to $\hat{K}_2^{(4k)}$ changes $F$ to

$$\begin{bmatrix} -a & 0 & -\frac{1}{2} & & & & \\ 0 & -a & \cdot & \cdot & & & \\ \frac{1}{2} & \cdot & \cdot & \cdot & & & \\ & \cdot & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & -\frac{1}{2} \\ & & & \cdot & \cdot & 0 \\ & & & -\frac{1}{2} & 0 & -a \end{bmatrix}$$

and applied to $\hat{K}_3^{(2k)}$, it yields

$$F = \begin{bmatrix} 0 & 1 & & & \\ 1 & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & 1 \\ & & & 1 & 0 \end{bmatrix}$$

To $\hat{K}_4^{(2k)}(\rho)$, we may apply $C = G = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, to obtain

$$F = \begin{bmatrix} & & & & -\rho \\ & & & -\rho & \\ & & \cdot & & \\ & \cdot & & & \\ -\rho & & & & \end{bmatrix}$$

The above transformation $C = G$, applied to $\hat{K}_6^{(4k)}(\rho)$, yields

$$F = \begin{bmatrix} & & & & & & \alpha & 0 \\ & & & & & 0 & 0 & 1 \\ & & & & \alpha & 0 & 0 & \\ & & & \cdot & \cdot & 1 & & \\ & & \cdot & \cdot & \cdot & & & \\ & \cdot & \cdot & \cdot & & & & \\ \alpha & \cdot & \cdot & & & & & \\ 0 & 1 & & & & & & \end{bmatrix} \quad \text{with } \alpha = b^{-2} .$$

Finally, the matrix $\hat{K}_5^{(2k)}(\rho)$ already has a non-singular kinetic energy block.

## SECTION 3 - QUADRATIC HAMILTONIANS IN THE SCHROEDINGER REPRESENTATION OF QUANTUM MECHANICS

A most transparent account of the construction of stationary states for the general quadratic combination of canonical variables $(\underline{Q},\underline{P}) = (\underline{x}, -i\underline{\nabla})$, using time-independent canonical transformations, can be found in a recent article by Moshinsky and Winternitz [15]. Given a general quadratic Hamiltonian $H = \frac{1}{2}\underline{Z}^T \hat{H}\, \underline{Z}$ , their approach involves the following steps:

(A) Find the canonical form $\hat{K}$ for $\hat{H}$ under change of symplectic basis. We have discussed this step in section 1. $\hat{K}$ is a direct sum of indecomposable canonical forms $\hat{K}_j^{(2k)}(\rho,a)$, each of which defines an independent Hamiltonian subsystem. Here, a denotes the frequency. (3.1)

(B) Find the symplectic transformation $C$ such that $C^T \hat{H} C = \hat{K}$. $C$ may be found by the methods of Laub and Meyer [2].

(C) Find a maximal set of commuting self-adjoint quadratic invariants $K_{jk} = \frac{1}{2}\underline{Z}^T \hat{K}_{jk}\, \underline{Z}$ , which commute with $K_j$. Then $K_j$ may be expressed as a real linear combination $K_j = \sum_{k=1}^{p} \gamma_k K_{jk}$ . The Lie algebra of homogeneous quadratic expressions $K(Z_\mu)$ is isomorphic to sp(2N,R). Therefore, this step may be approached through classifying the maximal Abelian subalgebras (M.A.S.A.) of sp(2N,R) into canonical equivalence classes. This step has been developed by the Montreal group [27].

(D) Each possible decomposition (3.1C) leads to a block separation of variables (see e.g. Miller and Kalnins [28]) in the Schroedinger equation

$$H'\,\psi'_E = E\,\psi'_E\,, \quad H' = \frac{1}{2}\underline{Z}^T \hat{K}\, \underline{Z}\,.$$

$$\underline{Z} = \begin{pmatrix}\underline{Q}\\ \underline{P}\end{pmatrix}, \quad \underline{Q} \equiv \underline{x}\,, \quad \underline{P} \equiv -i\underline{\nabla}\,.$$

If possible, the decomposition (3.1C) must be chosen in such a way that this separation of variables allows an exact solution.

(E) Find the unitary transformation $U$ on $L^2(R^N)$ such that $U^{-1} H U = H'$. This allows a direct solution of the original Schroedinger equation

$$H' \psi'_E = E \psi'_E \Leftrightarrow H \psi_E = E \psi_E \text{ , with } \psi_E = U \psi'_E \text{ .}$$

In (3.1E), the existence of $U$ follows from the fact that $C$ is symplectic, so that $\underline{Z}' = C^{-1} \underline{Z}$ satisfy the canonical commutation relations (C.C.R.)

$$[Z'_\mu \text{ , } Z'_\nu] = -i \, G_{\mu\nu} \text{ .} \tag{3.2}$$

Since, by the Stone-von Neumann theorem [29], all irreducible representations of the C.C.R. are unitarily equivalent, there exists a unitary operator $U$ on $L^2(R^N)$ such that

$$Z'_\mu = U \, Z_\mu \, U^\dagger$$

$$\Rightarrow H' = \frac{1}{2} \sum_{\mu,\nu=1}^{2N} Z_\mu \, \hat{K}_{\mu\nu} \, Z_\nu$$

$$= \frac{1}{2} \sum_{\mu,\nu} U^\dagger \, Z'_\mu \, \hat{K}_{\mu\nu} \, Z'_\nu \, U$$

$$= \frac{1}{2} \sum_{\mu,\nu} U^\dagger \, Z_\mu \, \hat{H}_{\mu\nu} \, Z_\nu \, U \quad \text{(as in (1.7c))}$$

$$= U^\dagger \, H \, U \text{ .}$$

For an arbitrary symplectic transformation $C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$, the transformation brackets $\langle x|U|x'\rangle$ in the $L^2(R)$ position representation have been presented by Moshinsky and Quesne [30]:

$$\langle x|U|x'\rangle = [(2\pi)^N |\det C_2|]^{-\frac{1}{2}}$$
$$\times \exp [-\tfrac{1}{2} i (\underline{x}^T C_2^{-1} C_1 \underline{x} - 2\underline{x}^T C_2^{-1} \underline{x}' + \underline{x}'^T C_4 C_2^{-1} \underline{x}')]. \tag{3.3}$$

When $C_2$ is singular, it is possible to express $C$ as a product of two symplectic factors in which the off-diagonal block is non-singular, so that

(3.3) can be re-applied [30]. Even in the case of infinite degrees of freedom, the implementating unitary operator, in some representations of the C.C.R., may be obtained from the theory of Bogoliubov transformations, which is discussed in a later section.

For general quadratic Hamiltonians with irreducible blocks $\hat{K}_j$ of order 3 or higher, the Moshinsky-Winternitz programme (3.1) has not yet been completed. There is much freedom in the choice of M.A.S.A. in step (3.1D). This procedure is not to be confused with the construction of the Cartan subalgebra of a simple complex Lie algebra.

When the number N of degrees of freedom exceeds three, there is an infinite number of inequivalent M.A.S.A. of sp(2N,R) and it is not yet known whether it is always possible to separate variables in such a way that the Schroedinger equation can be solved exactly. However, this is possible when the indecomposable blocks $\hat{K}_j$ of $\hat{H}$ are of order 2 or 1. In table 3.4, we list one representative Hamiltonian for each canonical orbit with indecomposable canonical form $\hat{K}_j^{(2k)}(\rho,s)$ (k = 1 or 2), give a complete orthonormal set (c.o.n.s.) of eigendistributions and list the reference sources in historical order.

By the process (3.1), a c.o.n.s. of eigendistributions for any quadratic Hamiltonian in one or two degrees of freedom can be derived from the eigendistributions listed in table 3.4. For example, for the repulsive "oscillator" Hamiltonian $H_r = \frac{1}{2}(P^2 - Q^2)$, $\hat{H} = \text{diag.}\ [-1,1]$, which is in the same canonical orbit as $K_1^{(2)}(i)$. In the Schroedinger representation, the eigenstates for $H_r$ are [28]:

$$\psi'_{E,\pm}(x) = 2^{-3/4}\ \pi^{-1} \exp\left[-(i/4)\ \pi(iE' + 1/2)\right] \times \Gamma(-iE' + 1/2) D_{iE'-1/2}(\pm 2^{1/2} \exp(3i\pi/4)x)\ ;\ E' \in R\ , \tag{3.5}$$

$D_\nu(r)$ is the parabolic cylinder function [35].

TABLE (3.4) Complete Orthonormal Set of Eigendistributions for Representative Quadratic Hamiltonians on $L^2(R^N)$

| Example | Normal form for H | c.o.n.s. of eigendistributions | Spectrum | Reference |
|---|---|---|---|---|
| $H = 0$ | $K_3^{(2)}$ | any c.o.n.s. | $\{0\}$ | |
| $H = \frac{1}{2} p^2$ (free particle) | $K_4^{(2)}(-1,0)$ | $e^{i\underline{k}\cdot\underline{x}}$; $\underline{k} \in R^N$ | $R^+ \cup \{0\}$ | |
| $H = \frac{1}{2}(p^2 + s^2Q^2)$ | $K_5^{(2)}(-1,s)$ | $\psi_n(x) = \left[\frac{s^{\frac{1}{2}}\pi^{-\frac{1}{2}}}{2^n\, n!}\right]^{\frac{1}{2}} e^{-\frac{1}{2} sx^2} H_n(s^{\frac{1}{2}}x)$<br>$H_n$ = nth Hermite polynomial | $\{(n+\frac{1}{2})s;\ n=0,1,2,\ldots\}$ | Schroedinger [31] (1926) |
| $H = \frac{1}{2}(Q_1^2+Q_2^2+s(P_1Q_2-P_2Q_1))$<br>2 coupled oscillators with same real frequency s | $K_6^{(4)}(1,s)$ | $\psi_{E';n}(r,\theta) = (2\pi r')^{-\frac{1}{2}}\, \delta(r-r')e^{in\theta}$<br>$r' = (2E'-2ns)^{\frac{1}{2}}$; $n \in Z$; $E'-2ns > 0$<br>$(x_1,x_2) = (r\cos\theta,\ r\sin\theta)$ | $E' \in R$<br>countably infinite degeneracy | Pais and Uhlenbeck [32] (1950) |
| $H = -\frac{b}{2}(Q_1P_1+Q_2P_2)$<br>$-\frac{a}{2}(Q_1P_2-Q_2P_1)$<br>+ adjoint | $K_2^{(4)}(a+bi)$ | $\psi_{E',n}(r,\theta) = (2\pi)^{-1} r^{-i\lambda-1}\ e^{in\theta}$<br>$\lambda = b^{-1}(E' - an)$; $n \in Z$ | $E' \in R$<br>countably infinite degeneracy | Pais and Uhlenbeck [32] (1950) |
| $H = -\frac{b}{2}[Q_1P_1+Q_2P_2]$<br>+ adjoint | $K_1^{(2)} \oplus K_1^{(2)}$ | as for $K_2^{(4)}(a+bi)$, but with $a = 0$ | $E' \in R$<br>countably infinite degeneracy | Schroer and Swieca [33] (1970) |
| $H = -\frac{b}{2}(QP+PQ)$ | $K_1^{(2)}(bi)$ | $\psi_{E';\pm} = (2\pi)^{-\frac{1}{2}}\ x_{\pm}^{-i\lambda-\frac{1}{2}}$<br>$x_{\pm} = \frac{1}{2}(x \pm \lvert x\rvert)$ | $E' \in R$<br>2-fold degeneracy | Kalnins and Miller [28] (1974) |
| $H = -\frac{b}{2}(Q_1P_1+Q_2P_2)+\frac{1}{2}Q_2P_1$<br>+ adjoint | $K_1^{(4)}(bi)$ | $\psi_{E';\alpha\pm}(r,\theta) = [u(\pi-\theta)\pm(-1)^{i\alpha+1}u(\theta-\pi)]$<br>$\times\ 2^{3/2}\ \pi(r\sin\theta)^{-i\alpha-1}\ e^{i\beta\cot\theta}$<br>$\beta = E' - b\alpha$; $\alpha \in R$; $u$ = unit step function | R ; degeneracy labelled by $R \oplus R$, for even and odd parity | unpublished but derivable from Moshinsky and Winternitz [15] (1980) |
| $H = Q_2P_1$ | $K_3^{(4)}(0)$ | $\psi_{E';y'}(x_1,x_2) = (4\pi y')^{-\frac{1}{2}}\delta(x_2-y')e^{iE'x_1/2y'}$<br>$E',\ y' \in R$ | R ; uncountably infinite degeneracy due to ignorable coordinate $x_2$ | |
| $H = \frac{1}{2}\rho(-Q_1Q_2+P_1^2) - Q_2P_1$<br>+ adjoint | $K_4^{(4)}(\rho,0)$ | $\psi_{E';y'}(x_1,x_2) = 2^{1/3}\lvert y'\rvert^{-1/6}\delta(x_2-y')$<br>$e^{iy'\rho x_1} \times Ai(-(2y')^{-2/3}$<br>$[(y')^2 + 2\rho E' + 2y'\ x_1])$<br>Ai is the Airy function [34], $y' \in R$ | R ; uncountably infinite degeneracy due to ignorable coordinate $x_2$ | |

With N degrees of freedom, it is possible to solve the Schroedinger equation whenever all the Jordan blocks of $i\hat{GH}$ are of order 1 or 2. This is the case when the conditions of theorems 1.24, 1.25 or 1.26 are satisfied.

To complete table (3.4), it was necessary to find stationary states for the last three examples.

$$K_1^{(4)} \text{ may be expressed } bM_1 + M_2 \,, \tag{3.6}$$

with $M_1$ and $M_2$ commuting Hermitean operators

$$M_1 = -Q_1P_1 - Q_2P_2 + i$$

$$M_2 = Q_2P_1$$

In the Schroedinger $L_2(R^2)$ representation and using polar coordinates $(x_1,x_2) = r(\cos\theta, \sin\theta)$,

$$M_1 = i + i\, r \frac{\partial}{\partial r} \tag{3.7a}$$

and
$$M_2 = -i\, r \sin\theta \cos\theta \frac{\partial}{\partial r} + i \sin^2\theta \frac{\partial}{\partial\theta} \tag{3.7b}$$

It is easy to construct a complete set of simultaneous eigenstates of $M_1$ and $M_2$ because of the elementary form (3.7a) of $M_1$ in polar coordinates.

With $H = K_3^{(4)} = Q_2P_1$, the Schroedinger equation is very easy to solve because $Q_2$ is a constant of the motion, $[Q_2,H] = 0$. Finally, with $H = K_4^{(4)}(\rho) = \rho(\frac{1}{2} P_1^2 - Q_1Q_2) - Q_2P_1$, $Q_2$ is a constant $y$ of the motion but the Schroedinger equation, reduced to one independent variable $x = x_1$, is non-trivial:

$$-\frac{1}{2}\rho\, \theta'' + i\, y\, \theta' + (-E-\rho\, x\, y)\theta = 0 \tag{3.8}$$

Suppose that $\theta(x) = u(x)v(x)$ (3.9)

then, substituting (3.9) in (3.8), the coefficient of $u'$ is $-\rho\, v' + i\, y\, v$, which vanishes if we choose

$$v(x) = \exp(i\, y\, \rho\, x) \tag{3.10}$$

Then (3.8) becomes

$$u'' + z\,u = 0, \quad \text{where } z = y^2 + 2\rho E + 2\,y\,x \tag{3.11}$$

Now, in terms of the variable z, (3.11) becomes

$$4y^2 \frac{d^2u}{dz^2} + u\,z = 0 \tag{3.12}$$

After trying yet another substitution $u(z) = \nu w(z)$, $z = \mu\, s$ with $\mu$ and $\nu$ constant, (3.12) reduces to Airy's equation

$$\frac{d^2w}{ds^2} + w\,s = 0 \tag{3.13}$$

when $\nu = (2y)^{-2/3}$ and $\mu = \nu^{-1}$.

This explains the appearance of Airy functions in the listed stationary states.

The correct normalization of the listed stationary states $\psi_{E,y}$

$$\int \psi_{E_2,y_2}(\underline{x})^* \,\psi_{E_1,y_1}(\underline{x})\, d\underline{x} = \delta(y_1-y_2)\;\;\delta(E_1-E_2)$$

and the completeness relations

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dE\;\psi_{y,E}(\underline{x})^*\,\psi_{y,E}(\underline{y}) = \delta(\underline{x}-\underline{y})$$

may both be verified using the integral representation

$$A_i(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{t^3}{3} + xt\right)dt \tag{3.14}$$

(page 131 of [34])

From table (3.4). we see that the simplest cases of imaginary frequency ($\hat{H} = \hat{K}_1^{(2)}$ or $\hat{K}_1^{(4)}$) and of complex frequency ($\hat{H} = \hat{K}_2^{(4)}$) give rise to a quadratic quantum mechanical Hamiltonian with continuous spectrum. We wish to extend this result to the general case $K_j^{(2k)}$ j=1 or 2. First, we can gain much information about the spectrum of H by proving that it is essentially self adjoint.

Proposition 3.15  Let $H = \frac{1}{2}\sum_{\mu,\nu=1}^{2M} \hat{H}_{\mu\nu} Z_\mu Z_\nu$, with $\hat{H}$ real symmetric and $\{Z_i\}_{i=1,\dots,2M}$ a collection of self adjoint operators obeying the C.C.R. in an irreducible representation on Hilbert space $\mathcal{F}$. Then H is essentially self-adjoint.

Proof: We may work in any particular irreducible representation of the C.C.R., since by the Stone-von Neumann theorem [29], these are all unitarily equivalent.

Define $\underline{Z} = \binom{Q}{\underline{P}}$ by

$$Q_j = 2^{-\frac{1}{2}} (a_j + a_j^\dagger) \tag{3.16}$$

$$P_j = -2^{-\frac{1}{2}} i\, (a_j - a_j^\dagger)$$

where $a_j$ are annihilation operators acting on Fock space. (See e.g. [36-38])

We shall demonstrate that for all finite n, any n-particle state $\psi \in \mathcal{H}^{(n)}$ is an analytic vector for H. Hence, there exists a total set of analytic vectors for H, so that by Nelson's analytic vector theorem (chap. X of [36]), the Hermitean operator H is essentially self-adjoint.

Let $\psi \in \underline{\mathcal{H}}^{(n)}$

then $H\psi \in \mathcal{F}_0 = \bigcup_n \mathcal{H}^{(n)}$

$$\|H^k \psi\| = 2^{-k} \Big\| \sum_{\mu_1,\nu_1,\dots,\mu_k,\nu_k=1}^{2M} \hat{H}_{\mu_k,\nu_k} \cdots \hat{H}_{\mu_k,\nu_k} Z_{\mu_1} Z_{\nu_1} \cdots Z_{\mu_k} Z_{\nu_k} \psi \Big\|$$

$$\le \left(\frac{\alpha}{2}\right)^k \sum \| Z_{\mu_1} \cdots Z_{\nu_1} \psi \|, \text{ (triangle inequality)}$$

(where $\alpha = \max_{\mu,\nu} |\hat{H}_{\mu\nu}|$)

$$\le (4\,M^2\alpha)^k \, ((n+2k)!)^{\frac{1}{2}} \|\psi\|$$

(since $\|a_j \psi\| = \sqrt{n+1}\, \|\psi\|$ - see §X.7 of [35])

Now $$\sum_{k=0}^{\infty} \| H^k \psi \| \frac{t^k}{k!} \leq \sum_{k=0}^{\infty} (4M^2 \alpha t)^k \frac{((n+2k)!)^{\frac{1}{2}}}{k!} \| \psi \| \tag{3.17}$$

From Stirling's approximation $k^{-k} e^k k! \to 1$,

$$\exists\, k_0,\ \forall\, k > k_0,\ ((n+2k)!)^{\frac{1}{2}}/k! < (\tfrac{n}{2}+k+1)2^k\, k^{n/2}\, e^{-n/2} \tag{3.18}$$

Using the comparison (3,18), (3,17) converges for some $t > 0$, provided $\sum_{k=1}^{\infty} k^{n/2+1} s^k$ converges for some $s > 0$. This is indeed the case, since $\int_0^{\infty} k^{1+n/2} s^k \, dk$ converges for all $s \in (0,1)$. Therefore $\psi$ is an analytic vector for H, as required.

<u>Theorem</u> 3.16 For all k, $K_1^{(2k)}$ and $K_2^{(4k)}$ have no point spectrum.

<u>Proof</u>: $$K_1^{(2k)} = \frac{1}{2} \underline{Z}^T \hat{K}_1^{(2k)} \underline{Z} = -\frac{b}{2} \sum_{r=1}^{k} Q_r P_r + P_r Q_r + \sum_{s=2}^{k} Q_s P_{s-1}$$

Note that for $\psi$ to belong to the domain of $K_1^{(2k)}$, $\psi$ must also belong to dom $Q_r$. To show that $K_1^{(2k)}$ has continuous spectrum only, let us assume the contrary. i.e. $K_1^{(2k)}$ has a point spectrum.

$$K_1^{(2k)} \psi_j = E_j \psi_j \quad \text{for some } \psi_j \in \mathcal{F},\ E_j \in \mathbb{R}. \tag{3.19}$$

However, from the C.C.R.,

$$\left[K_1^{(2k)}, Q_k\right] = ib\, Q_k \tag{3.20}$$

$Q_k$ is a raising operator which increments the eigenvalue $E_j$ by ib.

$$K_1^{(2k)} Q_k \psi_j = (E_j + ib) Q_k \psi_j \tag{3.21}$$

However, the complex number $E_j + ib$ can not be a spectral value of the essentially self adjoint operator $K_1^{(2k)}$. Therefore,

$$Q_k \psi_j = 0 \quad \text{for all } j. \tag{3.22}$$

This too, is contradictory, since $Q_k$ is known to have no point spectrum. Therefore, the original assumption of the existence of a point spectrum for $K_1^{(2k)}$ was false. Similarly, the operator $Q_{2k-1} - i\,Q_{2k}$ adds $a + ib$ to eigenvalues of $K_2^{(4k)} = -\frac{b}{2}\sum_{r=1}^{2k} Q_r P_r + P_r Q_r + a\sum_{s=1}^{k} - Q_{2s-1}P_{2s} + Q_{2s}P_{2s-1}$

$$+ \sum_{t=3}^{2k} Q_t P_{t-2}\ .$$

By the same argument, $K_2^{(4k)}$, as well as $K_1^{(2k)}$, lacks a point spectrum. Both of these, being essentially self adjoint, must have a real continuous spectrum.

So far, in this section, we have considered the stationary states of a general quadratic Hamiltonian in quantum mechanics. Another point of view is that the quantum mechanical system is completely soluble if we can determine the propagator $K(\underline{x}^1,\underline{x};t)$ such that

$$\psi(\underline{x}^1,t) \;=\; \int K(\underline{x}^1,\underline{x};t)\;\psi(\underline{x})\,d\underline{x} \tag{3.23}$$

The propagator may be thought of as the integral kernel of the unitary operator $U(t) = \exp[-iHt]$, which implements time evolution. This is the same operator, which in the Heisenberg picture transforms an operator $A$ to $A(t) = U(t)^{\dagger} A\,U(t)$. However, for quadratic Hamiltonians, the Heisenberg time evolution has exactly the same form as the classical Hamilton equations

$$\dot{Z}_\mu \;=\; i[H,Z_\mu] \;=\; \sum_\nu -(G\hat{H})_{\mu\nu}\,Z_\nu \tag{3.24}$$

$$Z(t) = \exp[-G\hat{H}t]\underline{Z}(0) \tag{3.25}$$

Given any symplectic transformation C, the kernel for the implementing unitary operator U in the position representation can be constructed from C using the techniques of Moshinsky and Quesne [30]. In principle, the kernel $K(\underline{x}^1,\underline{x};t)$ can be constructed in this way for the one parameter group of symplectic transformations $C(t) = \exp[-GHt]$ given in (3.25). Alternatively, in the representation (3.16), U(t) can be constructed in terms of creation and annihilation operators as in Berezin's book [38]. $K(\underline{x}';\underline{x},t)$ has been expressed in terms of $\hat{H}$ by Malkin, Dodonov and Man'ko [39], who have also studied the integral kernel of U(t) in mixed representations, in which the parameters $\underline{x}$ of (3.23) are spectral values of one complete set of commuting observables, while $\underline{x}'$ are spectral values of another such set (e.g. the Weyl-Moyal mixed position-momentum representation). Alternative path integral expressions for the propagator expressed in terms of $\hat{H}$ are available [40-42]. However, Marshall and Pell [40] have pointed out that numerical evaluation of these often involves negotiating awkward infinite series and products.

To conclude this section, we comment on the quantum mechanical analogues of the time dependent classical canonical transformations which were referred to at the end of section 1.

In the case that $\hat{H}$ is time-dependent, $\hat{H}$ can be reduced to any chosen time-independent canonical form $\hat{K}$ by a time-dependent symplectic transformation (1.33). If, as in (3.1D), we define $H' = \frac{1}{2} \underline{Z}^T \hat{K} \underline{Z}$, then from (1.34),

$$H' = U^{-1}(t) \, [H - \tfrac{1}{2} \underline{Z}'^T \, G \, \dot{C} \, C^{-1} \underline{Z}'] \, U(t) \qquad (3.26)$$

When $\dot{C}$ does not vanish, we can no longer assume that H and H' are related by a unitary transformation as in (3.1D). For systems of one degree of freedom and time-independent $\hat{H}$, (3.26) can be written in terms of an integral transform [43] instead of a unitary transformation. In the case that H has canonical form $K_j^{(2)}(\rho,s)$, then we may choose $\hat{K} = \hat{K}_j^{(2)}(\rho,s')$ with $|s'| = 0$ or 1. The transformation $C(t) = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$ may then be chosen so that $C_2 = 0$. The integral transform then reduces to a geometric transform [20b] $T_c$ on $L^2(R)$.

$$T_c : \psi(x) \to \mu(x) \, \psi \, (g_c(x)). \qquad (3.27)$$

When a 2N × 2N symmetric time-dependent matrix $\hat{H}(t)$ is positive definite, theorem 1.26 tells us that at each time t, the canonical orbit containing $\hat{H}(t)$ also contains a direct sum of blocks $K_5^{(2)}(-1,s_j(t))$.

Under these circumstances, $\hat{H}$ is equivalent to a direct sum of N copies of $K_5^{(2)}(-1,1)$ by a time-dependent symplectic transformation (1.33) implemented by a time-dependent geometric transform [22b]. If, for example, $K_j^{(2)}(t)$ changes from j = 5 (harmonic oscillator class) at $t = t_1$ to j = 1 (repulsive "oscillator" class) at $t = t_2$, then the spectrum of H(t) would change from being discrete to continuous and there certainly could be no 1-1 correspondence between the stationary states of $H(t_1)$ and $H(t_2)$. The time dependent canonical transformations have not proved to be useful in deriving the eigenstates of a quadratic Hamiltonian from those of another whose structure is vastly different. Once again, we are forced to consider the canonical orbits (1.12) separately.

## SECTION 4 - QUADRATIC BOSON AND FERMION HAMILTONIANS

Models of systems of Bosons or Fermions, in which the Hamiltonian (4.1) is a complex quadratic combination of annihilation and creation operators on a Hilbert space, are ubiquitous in quantum statistical mechanics and quantum field theory.

$$H = \frac{1}{2} \sum_{i,j=1}^{N} A_{ij}\, a_i^{\dagger} a_j + B_{ij}\, a_i^{\dagger} a_j^{\dagger} + B_{ij}^{\dagger}\, a_i a_j + C_{ij}\, a_i a_j^{\dagger} \qquad (4.1)$$

with

$$A = A^{\dagger}, \quad C = C^{\dagger}.$$

($\dagger$ will signify the Hermitean conjugate of a matrix or the adjoint of an operator and $*$ will signify the complex conjugate). The Hamiltonian (4.1) may be specified by the dynamical matrix $D = \begin{pmatrix} A & B \\ B^{\dagger} & C \end{pmatrix}$ which is a Hermitean operator on the Hilbert space $L \oplus L$, where $L$ is single particle space.

$$H = \frac{1}{2} \underline{\alpha}^{\dagger} D\, \underline{\alpha}\ ; \quad \underline{\alpha}^{\dagger} = (a_1^{\dagger},\ldots,a_N^{\dagger},a_1,\ldots,a_N) \qquad (4.1a)$$

For systems of Bosons, the construction operators obey the canonical commutation relations (C.C.R.)

$$\begin{aligned} [a_i,\ a_j] &\subset 0 \\ [a_i^{\dagger},a_j^{\dagger}] &\subset 0 \\ [a_i,a_j^{\dagger}] &\subset \delta_{ij}\ I\ . \end{aligned} \qquad (4.2)$$

In (4.2), we use an inclusion symbol because the Boson operators must be unbounded in any representation. For systems of Fermions, the C.C.R. are replaced by the canonical anti-commutation relations (C.A.R.)

$$0 = [a_i,a_j]_+ = [a_i^{\dagger},a_j^{\dagger}]_+ = \delta_{ij}\ I - [a_i,a_j^{\dagger}]_+ \qquad (4.3)$$

A linear canonical transformation

$$\begin{pmatrix} \underline{a} \\ \underline{a}^{\dagger} \end{pmatrix} = T \begin{pmatrix} \underline{c} \\ \underline{c}^{\dagger} \end{pmatrix} \quad \text{or} \quad \underline{\alpha} = T\, \underline{\gamma} \qquad (4.4a)$$

$$D_1 = T^{\dagger} D\, T$$

preserves the relations (4.2) or (4.3) and maintains that $c_j^\dagger$ is the adjoint of $c_j$. This latter imposition is equivalent to a restricted structure for $T$.

$$T = \begin{pmatrix} T_1 & T_2 \\ T_2^* & T_1^* \end{pmatrix} . \tag{4.5}$$

In the case that single particle space $L$ is a general Hilbert space, such a canonical transformation may be expressed [38]:

$$T : A(\psi) = C(T_1^*\psi) + C^\dagger(T_2\psi) \tag{4.6}$$

$$A^\dagger(\psi) = C(T_2\psi) + C^\dagger(T_1^*\psi) \quad \psi \in L$$

In the finite dimensional case, this reduces to (4.4), since on $L$, C is anti-linear and $C^\dagger$ is linear.

The Boson algebra, generated by the Boson construction operators $\{a_j, a_j^\dagger\}$, is just one example of a para-Boson algebra, which in abstract terms is the free algebra generated by symbols $\{a_j, a_j^\dagger\}_{j \leq N}$, factored by the ideal defined by the equations

$$\left[a_i, [a_s, a_t]_+\right]_- = 0 \tag{4.7a}$$

$$\left[a_i^\dagger, [a_s^\dagger, a_t^\dagger]_+\right]_- = 0 \tag{4.7b}$$

$$\left[a_i, [a_s^\dagger, a_t]_+\right]_- = 2\, \delta_{is}\, a_t \tag{4.7c}$$

$$\left[a_i^\dagger, [a_s^\dagger, a_t]_+\right]_- = -2\, \delta_{it}\, a_s^\dagger \quad , \tag{4.7d}$$

in which the minus brackets and plus brackets represent the commutator and anti-commutator respectively. The equations (4.7) were found by Green [44] to be sufficient to ensure the Heisenberg relations for a quantum mechanical coordinate operator expanded in a Fourier series with coefficients $a_j$ and $a_j^\dagger$ (the adjoint of $a_j$).

Suppose that the relations (4.7) are preserved by a canonical transformation of the form (4.4-5). Assuming that $C_i, C_j^\dagger$ obey (4.7), it follows that

$$\left[C_i,[C_j^\dagger,C_k^\dagger]_+\right]_- = 2\,\delta_{ij}\,C_k^\dagger + 2\,\delta_{ik}\,C_j^\dagger \tag{4.8}$$

and

$$\left[C_\ell^\dagger,[C_m,C_n]_+\right]_- = -2\,\delta_{\ell m}\,C_n - 2\,\delta_{\ell n}\,C_m \tag{4.9}$$

Now assuming (4.4a) and using (4.8-9)

$$\left[a_i,[a_s^\dagger,a_t]_+\right]_- = 2\left[T_1T_2^T\right]_{it} a_s^\dagger - \left[T_2T_1^T\right]_{it} a_s^\dagger + 2\left[T_1T_1^\dagger - T_2T_2^\dagger\right]_{is} a_t \tag{4.10}$$

Therefore, if the para-Bose relations (4.7) are to hold among the operators $a_i$ and $a_j^\dagger$, we must have

$$T_1T_2^T - T_2T_1^T = 0 \tag{4.11a}$$

and

$$T_1T_1^\dagger - T_2T_2^\dagger = I \tag{4.11b}$$

(4.11) may be conveniently expressed

$$\mathcal{T}\,\hat{I}\,\mathcal{T}^\dagger = \hat{I} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\Longleftrightarrow \quad \mathcal{T}^\dagger\,\hat{I}\,\mathcal{T} = \hat{I} \qquad \text{(that is, } \mathcal{T} \in U(N,N)\text{)} \tag{4.12}$$

The Fermion algebra, generated by the Fermion construction operators (4.3), is just one example of a para-Fermion algebra, given by the defining relations [44]:

$$[a_r,[a_s^\dagger,a_t]] = 2\,\delta_{rs}\,a_t \tag{4.13a}$$

and

$$[a_r,[a_s,a_t]] = 0 \tag{4.13b}$$

(plus similar relations obtained by assuming that $a_t^\dagger$ is the adjoint of $a_t$). (4.13) defines an ordinary Lie algebra, in fact SO(2N+1) according to [45], and the brackets appearing in (4.13) are not graded but always represent the commutator.

The defining relations (4.13) among $C_i$ and $C_j^\dagger$ imply

$$[C_r^\dagger,[C_s,C_t]] = 2\,\delta_{rs}\,C_t - 2\,\delta_{rt}\,C_s \tag{4.14a}$$

$$[C_r,[C_s^\dagger,C_t^\dagger]] = 2\,\delta_{rs}\,C_t^\dagger - 2\,\delta_{rt}\,C_s^\dagger \tag{4.14b}$$

$$[C_r^\dagger,[C_s,C_t^\dagger]] = 2\,\delta_{rs}\,C_t^\dagger \tag{4.14c}$$

Assuming that $a_r,a_r^\dagger$ are constructed from $C_s,C_s^\dagger$ as in (4.4a) and that $C_s,C_s^\dagger$ obey (4.14), we obtain

$$[a_r,[a_s^\dagger,a_t]] = -2(ST^T-TS^T)_{rt}\,a_s^\dagger + 2(SS^\dagger+TT^\dagger)_{rs}\,a_t$$

Therefore, if the para-Fermi relations (4.13) are to hold among $a_r$ and $a_s^\dagger$, we must have

$$T_1T_1^\dagger + T_2T_2^\dagger = I \tag{4.15a}$$

and

$$T_1T_2^T + T_2T_1^T = 0\,, \tag{4.15b}$$

which may be re-expressed

$$\mathcal{T}\,\mathcal{T}^\dagger = I \qquad (\mathcal{T} \in U(2N)) \tag{4.16}$$

In the case of ordinary Bose statistics or Fermi-Dirac statistics, the relations (4.11) and (4.15) which are equivalent to the canonical character of $\mathcal{T}$, are well-known and they appear in the two definitive books on the subject by Friedrichs [46] and Berezin [38], after whom these transformations have sometimes been called "the Friedrichs-Berezin transformations" [47]. The 1-1 relationship between the automorphisms of the para-Fermion algebra of order k and those of the Fermion algebra can be seen most clearly in the reducible representation given by the Green ansatz [44]:

$$a_r = \sum_{j=1}^{k} a_r^{(j)}$$

with $\left[a_r^{(j)},a_s^{(j)}\right]_+ = \left[a_r^{(j)\dagger},a_s^{(j)}\right]_+ - \delta_{rs}\,I = 0$

and $\left[a_r^{(j)},a_s^{(\ell)}\right]_- = \left[a_r^{(j)},a_s^{(\ell)\dagger}\right]_- = 0, \;\; \text{if } j \neq \ell$

A canonical transformation $\alpha_\mu = \sum_{\nu=1}^{2N} T_{\mu\nu}\, \alpha_\nu$ on the concrete para-Fermion algebra, in this case results from an identical canonical transformation on each of the k copies of the Fermion algebra : $\alpha_\mu^{(j)} = \sum_{\nu=1}^{2N} T_{\mu\nu}\, \alpha_\nu^{(j)}$; $j=1,\ldots,k$. The situation is similar in the Green ansatz for constructing a representation of the para-Bose algebra of order k.

Another approach to the automorphism group of the para-Fermion algebra would have been to begin with the observation of Ryan and Sudarshan [45] that this algebra is isomorphic to SO(2N+1). In the para-Bose case, the procedure of constructing Lie algebras from particle construction operators is well-developed in the Boson calculus [48].

The group of canonical transformations used by Friedrichs was defined as the subgroup of Sp(2N,C) given by our condition (4.5). In fact, assuming the condition (4.5), the defining relations among $T_1$, $T_2$ so that $T \in Sp(2N,C)$ are exactly the same as the defining relations for $T$ to belong to U(N,N).

Suppose that $T$ has the form (4.5). $T = \begin{pmatrix} U & V \\ V^* & U^* \end{pmatrix}$ . Then

$$
\begin{aligned}
T \in Sp(2N) &\Leftrightarrow T^T G\, T = T\, G\, T^T = G \\
&\Leftrightarrow V\, U^T - U\, V^T = 0\ , \\
&\qquad V^\dagger U - U^T V^* = 0\ , \\
&\qquad U\, U^\dagger - V\, V^\dagger = I\ , \\
\text{and} &\qquad V^\dagger V - U^T U^* = -I\ .
\end{aligned}
\tag{4.17}
$$

Also,

$$
\begin{aligned}
T \in U(N,N) &\Leftrightarrow T^\dagger \hat{I}\, T = \hat{I}\ , \\
&\Leftrightarrow U^\dagger U - V^T V^* = I\ , \\
&\qquad U^\dagger V - V^T U^* = 0\ , \\
&\qquad U\, U^\dagger - V^T U^\dagger = 0\ , \\
\text{and} &\qquad U\, V^T - V\, U^T = 0\ . \\
&\Leftrightarrow (4.17)\ .
\end{aligned}
\tag{4.18}
$$

Hence, $$T = \begin{pmatrix} U & V \\ V^* & U^* \end{pmatrix} \Rightarrow [T \in U(N,N) \Leftrightarrow T \in Sp(2N,C)] \quad (4.19)$$

Therefore,

$$U(N,N)/(4.5) = Sp(2N,C)/(4.5) \quad (4.20)$$

and

$$Sp(2N,C)/(1.5) \subseteq Sp(2N,C) \cap U(N,N). \quad (4.21)$$

To prove that (4.21) is an equation, let $T$ be an arbitrary element of the matrix group $Sp(2N,C) \cap U(N,N)$. To show that $T$ has the form (4.5) is to verify that

$$\Gamma T \Gamma = T , \quad (4.22)$$

where $\Gamma$ is the conjugation operator defined by

$$\Gamma \begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix} = \begin{pmatrix} \underline{v}^* \\ \underline{u}^* \end{pmatrix}$$

$$= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix}^*$$

$$= G \hat{I} \begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix}^* ,$$

$$\Gamma T \Gamma = [G \hat{I} T G \hat{I}]^*$$

$$= [G \hat{I} G \tilde{T}^{T^{-1}} \hat{I}]^* \quad (\text{since } T \in Sp(2N,C))$$

$$= [G \hat{I} G \hat{I} T^* \hat{I} \hat{I}]^* \quad (\text{since } T \in U(N,N))$$

$$= [T^*]^* \quad (\text{since } (G\hat{I})^2 = \hat{I}^2 = I)$$

$$= T, \text{ which verifies (4.22)}$$

By (4.21-22)

$$Sp(2N,C) \cap U(N,N) = Sp(2N,C)/(1.5) \quad (4.23)$$

The usual aim in applying a canonical transformation is to achieve a diagonal dynamical matrix $D'$ so that H is represented as a linear combination of number operators, the spectral properties of which are trivial. For finite N, the Stone-von Neumann theorem [29] on equivalence of representations of the C.C.R. ensures that for any C.C.R. canonical transformation, there exists a unitary operator U on the Fock space for $\underline{a}$, $\underline{a}^\dagger$ such that

$$c_j = U^\dagger a_j U ; \quad j = 1,\ldots,N . \tag{4.24}$$

Similarly, for finite N, any C.A.R. canonical transformation can be implemented by a unitary operator whose existence is guaranteed by the Jordan-Winger theorem [49]. The set of quasi-particle number operators $c_j^\dagger c_j$ constitutes a complete set of commuting dynamically invariant observables of the system and so may be used to determine the spectrum of H. The unitary operator U in (4.24) may then be interpreted as achieving a basis of stationary states of non-interacting quasi-particles. Chevalier and Rideau [50] used this technique to simplify the Marshak-Wentzel pair theory [51] of nuclear forces, in which muon pair creation terms and annihilation terms were used to model a nuclear exchange force, the pion not yet having been observed. In the case of Boson systems, such transformations had been used by Holstein and Primakoff [52]. At this time, the most common name for the algebra automorphism (4.4) is the "Bogoliubov transformation", after Bogoliubov who applied such techniques in his theory of superfluids [53].

In the early 1960's, the target for the armoury of Bogoliubov transformations began to broaden from specific examples of (4.1) to the abstract case. Thouless [54] who studied the vibrational states of nuclei, was able to prove that with N finite, for any positive definite Boson Hamiltonian, the dynamical matrix $D$ can be fully diagonalized. Since then, detailed proofs of the same result have appeared in works by Tyablikov [55], Colpa [56] and Tikochinsky [57]. The case $N < \infty$ corresponds to a finite number of states being available to a single classical particle and such a situation occurs, for example in lattice models of matter with N large and finite. The studies of Lieb, Schultz and Mattis [58] of anti-ferromagnetic chains of Fermions, also undertaken in the early 1960's, produced diagonalization by a C.A.R. Bogoliubov transformation in the general case of real

$D = D^*$ and N finite. However, their method of constructing a diagonalizing C.A.R. Bogoliubov transformation $T$ no longer produces a unitary $T$ when complex coefficients A, B and C are allowed in (4.1). The situation is similar for Boson models, for which a real dynamical matrix can be characterised by an involutive matrix F, which can be simplified by the method of Shoon K. Kim [59]. However, in the case that $D$ is complex, there is no characteristic matrix F that satisfies a polynomial equation and Kim's method is no longer applicable. This deficiency for Fermion models was rectified by Araki [60], who demonstrated that even in the general case of $N \le \infty$ and $D$ complex, the off-diagonal blocks B and $B^\dagger$ can be removed from $D$ so that (4.1) contains only number-conserving terms. In the case that N is finite, it is then clear that after a simple unitary transformation (4.5) with $T_2 = 0$, H can be fully reduced to a finite scalar multiple of the identity plus a linear combination of number operators. The concrete approach to diagonalization of the general quadratic Fermion Hamiltonian $(N < \infty)$ with a linear term added to (4.1) has been developed further by Colpa [61].

At this point, the reader should be aware that general diagonalization theorems for quadratic Boson Hamiltonians are known only in the case that $D$ is positive definite. Tsallis [62] (1978) has made a tentative suggestion on how $D$ might be diagonalized in the general case. In footnote 21 of [62], he comments on his construction of the diagonalizing transformation,

> "This is not a *sufficient* condition for the existence of the solution; therefore strictly speaking it guarantees nothing beyond a strong suspicion."

In Colpa's paper [56] of the same year, the author admitted (p.345) that the most unsatisfactory aspect of that work was its failure to treat even *semi*-definite matrices $D$.

All of the above-mentioned general approaches involved direct application of Bogolibov transformations in their primary form (4.4-5). The restriction (4.5) is harsh because it does not appear in the definition of any classical matrix group. Therefore, it would appear that the well-oiled machinery which constructs conjugacy classes for classical groups can not be used to find canonical forms for quadratic Hamiltonians. However, we shall show that the subgroup of U(N,N) given by restriction (4.5) is isomorphic to Sp(2N,R). This enables us to deduce a complete set of canonical forms for quadratic Boson Hamiltonians, using the results of section 1. For many quadratic Boson Hamiltonians, the diagonalizing transformation suspected by Tsallis does not exist and it is not possible to transform to a system of free quasi-particles.

Let $$P = 2^{-\frac{1}{2}} \begin{bmatrix} I & I \\ -iI & iI \end{bmatrix} \quad \text{and} \quad G = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{4.25}$$

P is unitary and P intertwines $\hat{I}$ and $-iG$, $P\,\hat{I}\,P^{\dagger} = -iG$ (4.25a)

In Bose field theory, P is used in the construction of essentially self adjoint Segal field operators [35,63] from annihilation and creation operators, as in (3.16). For any 2N × 2N complex matrix F, define

$$\rho(F) = P\,F\,P^{\dagger}. \tag{4.26}$$

<u>Proposition</u> (4.27). ρ defined in (4.25-26) is an isomorphism that

(a) maps the C.C.R. Bogoliubov subgroup of U(N,N) onto Sp(2N,R)

(b) maps the C.A.R. Bogoliubov subgroup of U(2N) onto O(2N,R).

<u>Proof</u>: Let $T$ be of the form (4.5)

$$\rho(T) = P\,T\,P^{\dagger}$$

$$= \frac{1}{2} \begin{pmatrix} S_1+S_1^* & i(S_2-S_2^*) \\ -i(S_1-S_1^*) & S_2+S_2^* \end{pmatrix} \; ; \quad S_{1,2} = T_1 \pm T_2 \tag{4.28}$$

which is a real matrix.

(a) Now suppose $\mathcal{T} \in U(N,N)$

$$\mathcal{T}^{\dagger}\,\hat{I}\,\mathcal{T} = \hat{I}$$

$$\Rightarrow [\rho(\mathcal{T})]^{\dagger}\, G\, \rho(\mathcal{T}) = G, \quad \text{by (4.25a)}$$

$$\Rightarrow \rho(\mathcal{T}) \in Sp(2N,R) \quad \text{(since } \rho(\mathcal{T}) \text{ is real, by (4.28))}$$

(b) Suppose $\mathcal{T} \in U(2N)$

Since P is unitary, $P(\mathcal{T})$ is unitary.

Since $\rho(\mathcal{T})$ is real, $\rho(\mathcal{T}) \in O(2N,R)$.

Conversely,

Let $C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$ be a real matrix.

$$P^{-1}(C) = \begin{pmatrix} T_1 & T_2 \\ T_2^{*} & T_1^{*} \end{pmatrix}$$

where $$T_{1,2} = \frac{1}{2}[C_1 \pm C_4 \mp iC_2 + iC_3] \qquad (4.29)$$

Thus, $\rho^{-1}(C)$ has the form (4.5).

(a) Suppose $C \in Sp(2N,R)$

$$C^T\, G\, C = G$$

$$\Rightarrow P^{-1}(C)^{\dagger}\,\hat{I}\,P^{-1}(C) = \hat{I} \qquad \text{(by (4.25a))}$$

(b) Suppose $C \in O(2N,R)$ (a real unitary matrix)

Since P is unitary, $P^{-1}(C)$ must be unitary.

Since $\rho$ is a similarity transformation, $\rho$ is a homomorphism and by the above results it is bijective. Therefore, $\rho$ is an isomorphism.

Note that proposition (4.27) resulted from the existence of P satisfying unitarity ($P^{\dagger} P = I$), the intertwining property (4.25a) and the condition that $P\,\mathcal{T}\,P^{\dagger}$ is real, provided $\mathcal{T}$ has the form (4.5). P could have been replaced by any other matrix satisfying these properties. For example, another possibility is

$$P^{\dagger} = 2^{-\frac{1}{2}} \begin{pmatrix} I & iI \\ iI & I \end{pmatrix} .$$

This transformation was used by Lanczos [21] to represent a classical linear canonical transformation in U(N,N)/(4.5). Since then, it has become known as the Bargmann transformation [64].

Recently, van Hemmen [65] has considered some similarities and differences in the processes of diagonalization of quadratic Boson and Fermion Hamiltonians. The differences must be traced to the exchanged role between symmetrization and anti-symmetrization in the concepts of Bose statistics and Fermi-Dirac statistics. This difference in symmetry is not obvious in the groups U(N,N) and U(2N) which respectively contain as subgroups the C.C.R. Bogoliubov transformations and the C.A.R. Bogoliubov transformations. However, by proposition (4.27), these groups are respectively Sp(2N,R), which preserves a skew-symmetric form and O(2N,R), which preserves a symmetric form.

Let $\mathcal{D}^s$ be the real vector space of complex Hermitean 2N × 2N matrices. $\mathcal{D}^s$ is left invariant by the Bogoliubov transformations (4.4-5). Therefore, by proposition (4.27), we have a representation of either O(2N,R) or Sp(2N,R) on $\mathcal{D}^s$. It is easily verified that

$$\mathcal{D}^s = \mathcal{D}^s_1 \oplus \mathcal{D}^s_2 , \tag{4.30}$$

where elements $\mathcal{D}_1$, $\mathcal{D}_2$ of $\mathcal{D}^s_1$, $\mathcal{D}^s_2$ have the form

$$\mathcal{D}_1 = \begin{pmatrix} A_1 & B_1 \\ B_1^* & A_1^* \end{pmatrix} \quad A_1 = A_1^\dagger, \quad B_1 = B_1^{*\dagger} = B^T \text{ (symmetric)} \tag{4.31}$$

$$\mathcal{D}_2 = \begin{pmatrix} A_2 & B_2 \\ -B_2^* & -A_2^* \end{pmatrix} \quad A_2 = A_2^\dagger, \quad B_2 = -B_2^T \text{ (skew-symmetric)} \tag{4.32}$$

$$\mathcal{D}_1 = \alpha(\mathcal{D}) = \frac{1}{2}\begin{pmatrix} C^*+A & B+B^T \\ (B+B^T)^* & (C^*+A)^* \end{pmatrix} \tag{4.33}$$

$$\mathcal{D}_2 = \beta(\mathcal{D}) = \frac{1}{2}\begin{pmatrix} A-C^* & B-B \\ -(B-B)^* & -(A-C^*)^* \end{pmatrix} \tag{4.34}$$

$$D_{1,2} = \frac{1}{2}\left[D \pm \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} D^* \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right] \tag{4.35}$$

To interpret (4.35), we recall the conventions adopted by Källen [16] for time reversal in ordinary quantum mechanics:

$$\psi(\underline{x},t) \to \psi'(\underline{x},t) = \psi^*(\underline{x},-t) \tag{4.36}$$

$$H^* \psi'(\underline{x},t) = i \frac{\partial \psi'(\underline{x},t)}{\partial t} \tag{4.37}$$

In addition, we can define time-reversed construction operators by

$$\begin{pmatrix} \underline{b} \\ \underline{b}^\dagger \end{pmatrix} = \begin{pmatrix} \underline{a}^\dagger \\ \underline{a} \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \underline{a} \\ \underline{a}^\dagger \end{pmatrix}, \tag{4.38}$$

since time reversal interchanges annihilation and creation. Therefore, $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} D^* \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ is the dynamical matrix which gives $H^*$, constructed from time-reversed operators according to (4.1). $H^*$ is the Hamiltonian which generates temporal evolution in a time-reversed picture. Accordingly, $D_{1,2}$ given by (4.35) are respectively even and odd with respect to time reversal.

It is easily verified that $D_1^s$ and $D_2^s$ are invariant under transformations of the type (4.4-5). This means that the representation of the group of symplectic or orthogonal transformations $\rho(T)$ on $D^s$ decomposes according to (4.30).

<u>Proposition</u> (4.39) (a) $\rho(D_1^s)$ is the real vector space of real symmetric $2N \times 2N$ matrices.

(b) $\rho(D^s)$ is the real vector space of purely imaginary skew-symmetric $2N \times 2N$ matrices.

<u>Proof</u>: Consider $D_1$ and $D_2$ of (4.32-32)

Then
$$\rho(D_1) = \frac{1}{2}\begin{pmatrix} K_1 & L_1 \\ L_1^T & M_1 \end{pmatrix} \tag{4.40}$$

$$\text{with}\quad K_1,M_1 = [A_1 \pm B_1] + [A_1 \pm B_1]^*$$

$$L_1 = i[A_1 - B_1] - i[A_1 - B_1]^*$$

$$L_1^T = -i[A_1 + B_1] + i[A_1 + B_1]^*$$

$K_1$, $L_1$ and $M_1$ are obviously real. Since $A_1$ is Hermitean and $B_1$ is symmetric, $K_1$ and $M_1$ are symmetric.

$$\rho(D_2) = P\, D_2\, P^\dagger$$

$$= \frac{1}{2}\begin{bmatrix} A_2 + B_2 - (A_2^* + B_2^*) & i(A_2 + A_2^* - [B_2 + B_2^*]) \\ -i(A_2 + A_2^* + [B_2 + B_2^*]) & A_2 - B_2 - (A_2^* - B_2^*) \end{bmatrix} \qquad (4.41)$$

which is purely imaginary and skew-symmetric, since $A_2 = A_2^\dagger$ and $B_2 = -B_2^T$.

Conversely, let $\hat{H}$ be real and symmetric,

$$\hat{H} = \begin{pmatrix} K_1 & L_1 \\ L_1^T & M_1 \end{pmatrix} \quad \text{for some real symmetric } K_1, M_1 .$$

$$\text{Then}\quad \rho^{-1}(\hat{H}) = P^\dagger\, \hat{H}\, P = \begin{pmatrix} A_1 & B_1 \\ B_1^* & A_1^* \end{pmatrix} \qquad (4.42)$$

$$\text{where}\quad A_1 = \frac{1}{2}(K_1 + M_1 - iL_1 - iL_1^T) = A_1^\dagger$$

$$\text{and}\quad B_1 = \frac{1}{2}(K_1 - M_1 + iL_1 + iL_1^T) = B_1^T$$

Therefore, $\rho^{-1}(\hat{H}) \in D_1^S$

Now let $i\hat{A}$ be purely imaginary and skew-symmetric.

$$A = \begin{pmatrix} K_2 & L_2 \\ -L_2^T & M_2 \end{pmatrix} \quad \text{for some real skew-symmetric } K_2, M_2 \text{ and real } L_2.$$

$$\rho^{-1}(i\hat{A}) = P^\dagger\, i\, \hat{A}\, P = \begin{pmatrix} A_2 & B_2 \\ -B_2^* & -A_2^* \end{pmatrix} \in D_2^2 \qquad (4.43)$$

where $A_2, B_2 = \tfrac{1}{2}(L_2^T \pm L_2) + \tfrac{1}{2}\, i(K_2 \pm M_2) = A_2^\dagger, -B_2^T$

The linear non-singular transformation $\rho$ is a vector space isomorphism.

$D_{1,2}^S$ correspond respectively to the symmetric and antisymmetric tensor representations of O(2N) for the C.A.R. case and of Sp(2N) for the C.C.R.

case. The reduction (4.30) can not be further decomposed with respect to the group of Bogoliubov transformations. In the C.C.R. case, $D_2$ contributes a real number

$$H_2 = \frac{1}{2}\begin{pmatrix} \underline{a} \\ \underline{a}^{\dagger} \end{pmatrix}^{\dagger} D_2 \begin{pmatrix} \underline{a} \\ \underline{a}^{\dagger} \end{pmatrix} = (-\frac{1}{2}\,\mathrm{Tr}\,A_2)I, \tag{4.44}$$

which is a finite scalar multiple of the identity for all $D_2 \in D_2^s$ with $N < \infty$. Similarly, in the C.A.R. case, $D_1$ contributes $\frac{1}{2}(\mathrm{Tr}\,A_1)I$ to the quantum mechanical Hamiltonian. Therefore, the problem of finding canonical forms for all Boson and Fermion quadratic Hamiltonians is reduced to the problem of finding canonical forms for $D_1$ and $D_2$ respectively under the action of Bogoliubov transformations.

In recent years, several authors [56,57,62] have investigated the special case $\alpha(D) > 0$, for which diagonalization is always possible. With the aid of propositions (4.27) and (4.39), it is now easy to demonstrate that this result is equivalent to the diagonalization of a general positive definite real symmetric matrix by a change of symplectic basis, which appeared in Whittaker's treatise [3] of 1904.

<u>Theorem</u> (4.41) (Thouless [54] and others). If $D_1$ is a positive definite $2N \times 2N$ matrix belonging to class $D_1^s$, then by a transformation of type (4.4-4.5), $D_1$ can be reduced to diagonal form

$$D_2 = \mathrm{diag.}[|w_1|,\ldots,|w_N|,|w_1|,\ldots,|w_N|]$$

The frequencies $w_j$ are eigenvalues of $\hat{I}\,D_1$.

To show that theorem (4.41) is a consequence of theorem (1.26), suppose that $D_1 \in D_1^s$ and that $D_1 > 0$. By proposition (4.39a), this is equivalent to $\rho(D_1)$ being real symmetric and since $\rho$ is a similarity transformation, $\rho(D_1) > 0$. Therefore, by theorem (1.26),

$$(\exists\, C_1 \in S_p(2N,R))\quad C_1^T\,\rho(D_1)C_1 = \mathrm{diag.}[s_1^2,\ldots,s_N^2,1,\ldots,1]$$

$$\Rightarrow\quad (\exists\, C \in Sp(2N,R))\ C^T\,\rho(D_1)\,C = \mathrm{diag.}[|s_1|,\ldots,|s_N|,|s_1|,\ldots,|s_N|]$$

(e.g. let $C = C_1 C_2$ where $C_2 = \text{diag.}[|s_1|^{-\frac{1}{2}},\ldots,|s_N|^{-\frac{1}{2}},|s_1|^{\frac{1}{2}},\ldots,|s_N|^{\frac{1}{2}}]$)

$$\Rightarrow \quad (\exists\, C \in Sp(2N,R))$$

$$P^{\dagger} C^{T} P\, D_1\, P^{\dagger} C P = P^{\dagger}\, \text{diag.}[|s_1|,\ldots,|s_N|,|s_1|,\ldots,|s_N|]P$$

$$= \text{diag.}[|s_1|,\ldots,|s_N|,|s_1|,\ldots,|s_N|]$$

$$\Rightarrow \exists\, C \in Sp(2N,R), \rho^{-1}(C)^{\dagger} D_1\, \rho^{-1}(C) = \text{diag}[|s_1|,\ldots,|s_N|,|s_1|,\ldots,|s_N|]\ .$$

Finally, $\rho^{-1}(C)$ is the matrix for a C.C.R. Bogoliubov transformation by proposition (4.27a).

The converse result, that theorem (1.26) follows from theorem (4.41), may be proven in an analogous way.

To obtain a full list of canonical forms for quadratic Boson Hamiltonians, we note that by propositions (4.27) and (4.39), there will be exactly one canonical class of Boson Hamiltonians for every canonical class of classical Hamiltonians and we can derive normal forms for $D$ from the corresponding normal forms for $\hat{H}$. Corresponding to the normal form $\hat{K}$ of (1.19) for $\hat{H}$, there will be a normal form $A = \rho^{-1}(\hat{K})$ for $D$.

$$A = \rho^{-1}(\hat{K}) = \frac{1}{2}\begin{bmatrix} M_1 & & & & N_1 & & & \\ & M_2 & & & & N_2 & & \\ & & \ddots & & & & \ddots & \\ & & & M_P & & & & N_P \\ N_1^* & & & & M_1^* & & & \\ & N_2^* & & & & M_2^* & & \\ & & \ddots & & & & \ddots & \\ & & & N_P^* & & & & M_P^* \end{bmatrix} \qquad (4.42)$$

where $\qquad M_j = A_j + F_j - i\, B_j + i\, B_j^T$

and $\qquad N_j = A_j - F_j + i\, B_j + i\, B_j^T$ , with $A_j, B_j$ and $F_j$ as listed in theorem (1.17).

<u>Proposition</u> (4.43): Let $H = \frac{1}{2}\underline{\alpha}^{\dagger} D \underline{\alpha}$ be the quadratic Boson Hamiltonian of (4.1). There exists a Bogoliubov transformation which reduces H to a sum of terms $\frac{1}{2}\underline{\Upsilon}^{\dagger} A_j^{(2\ell)}\underline{\Upsilon}$, corresponding to independent subsystems, with the indecomposable canonical forms $A_j^{(2\ell)}$ belonging to the following list:

$$A_j^{(2\ell)} = \frac{1}{2}\begin{pmatrix} M & N \\ N^* & M^* \end{pmatrix},$$ with the order $\ell$ matrices M and N given by

$$A_1^{(2\ell)}: \quad M = \begin{bmatrix} 0 & i & & & & \\ -i & 0 & i & & & \\ & -i & \cdot & & & \\ & & & \cdot & & \\ & & & & 0 & i \\ & & & & -i & 0 \end{bmatrix}, \quad N = i\begin{bmatrix} -2a & 1 & & & & \\ 1 & -2a & 1 & & & \\ & 1 & \cdot & & & \\ & & & \cdot & & \\ & & & & -2a & 1 \\ & & & & 1 & -2a \end{bmatrix};$$

$$A_2^{(4\ell)}: \quad M = i\begin{bmatrix} Q & I & & & \\ -I & Q & I & & \\ & -I & \cdot & & \\ & & & \cdot & I \\ & & & -I & Q \end{bmatrix}, \quad \text{with } Q = \begin{pmatrix} 0 & 2b \\ -2b & 0 \end{pmatrix}$$

$$N = i\begin{bmatrix} 2a & 0 & 1 & & & & \\ 0 & -2a & 0 & & & & \\ 1 & 0 & \cdot & & & & \\ & 1 & & \cdot & & & \\ & & & & \cdot & 0 & 1 \\ & & & & 0 & -2a & 0 \\ & & & & 1 & 0 & -2a \end{bmatrix};$$

$$A_3^{(2\ell)}: \quad M = \begin{bmatrix} 0 & i & & & & \\ -i & 0 & i & & & \\ & -i & \cdot & & & \\ & & & \cdot & & \\ & & & & 0 & i \\ & & & & -i & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & i & & & & \\ i & 0 & i & & & \\ & i & \cdot & & & \\ & & & \cdot & & \\ & & & & 0 & i \\ & & & & i & 0 \end{bmatrix};$$

$$A_4^{(4\ell)}(\rho): \quad M = \begin{bmatrix} 0 & -i & & & \rho & -\rho \\ i & 0 & -i & & -\rho & \\ & i & \cdot & & & \\ & \rho & & \cdot & & \\ \rho & -\rho & & & 0 & -i \\ -\rho & & & & i & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & -i & & & -\rho & -\rho \\ -i & 0 & -i & & & \\ & -i & \cdot & & & \\ & -\rho & & \cdot & & \\ -\rho & -\rho & & & 0 & -i \\ -\rho & & & & -i & 0 \end{bmatrix} \quad \text{with } \rho = \pm 1;$$

$$A_5^{((2\ell+1))}(\rho):\quad M = \begin{bmatrix} 0 & -i & & -\rho(1+b^2) \\ i & 0 & -i & + \\ & i & \ddots & \\ + & & 0 & -i \\ -\rho(1+b^2) & & i & 0 \end{bmatrix},\quad N = \begin{bmatrix} 0 & -i & & -\rho(1+b^2) \\ -i & 0 & -i & + \\ & -i & \ddots & \\ & & 0 & -i \\ -\rho(1\overset{+}{-}b^2) & & -i & 0 \end{bmatrix}, \text{ with } \rho = \pm 1;$$

$A_6^{(4\ell)}(\rho)$: with $\rho = \pm 1$

$$M = \begin{bmatrix} 0 & -i(1+b^2) & & & & & \rho/b^2 & 0 \\ i(1+b^2) & 0 & 0 & & & 0 & 0 & \rho \\ & 0 & 0 & -i(1+b^2) & & 0 & -b^2\rho & 0 \\ & & i(1+b^2) & & & & 0 & -\rho \\ & & \rho/b^2 & & & & & \\ & 0 & 0 & \rho & -\rho & 0 & 0 & \\ \rho/b^2 & 0 & -b^2\rho & 0 & 0 & 0 & 0 & -i(1+b^2) \\ 0 & \rho & 0 & -\rho & & & i(1+b^2) & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & i(1-b^2) & & & & \rho/b^2 & 0 \\ i(1-b^2) & 0 & & & 0 & 0 & \rho \\ & 0 & 0 & i(1-b^2) & 0 & b^2\rho & 0 \\ & & i(1-b^2) & & \rho & 0 & \rho \\ & & & & & \rho & \\ & 0 & 0 & & & & \\ \rho/b^2 & 0 & b^2\rho & 0 & \rho & 0 & i(1-b^2) \\ 0 & \rho & 0 & \rho & & i(1-b^2) & 0 \end{bmatrix}$$

The single particle frequencies $\omega_j$ of a system of Bosons, with quadratic Hamiltonian $H = \frac{1}{2}\ \underline{\alpha}^{\dagger}\ D\ \underline{\alpha}$, are defined as the eigenvalues of $\hat{I}\ D_1$, with $D_1$ defined as in (4.35). Since P of (4.25) is unitary, the single particle frequencies $\omega_j$ may be identified as the classical frequencies $s_j$ of the classical dynamical system with dynamical matrix $\hat{H} = \rho(D_1)$. Since $P\ \hat{I}\ P\ = -iG$, the elementary divisors $(\omega-\omega_j)^{N_j}$ of $-\hat{I}\ D_1 - \omega I$ may be identified with the elementary divisors of the pencil $iG\hat{H} - \omega I$. The signature of $D_1$ will be the same as that of $\hat{H} = P\ D_1\ P^{\dagger}$. Hence, the invariants associated with a classical dynamical matrix $\hat{H}$ are invariants of a system of Bosons with quadratic Hamiltonian $H = \underline{\alpha}^\dagger \rho^{-1}(\hat{H})\underline{\alpha}$, under the action of the group of Bogoliubov transformations. Table 4.44 lists some invariants associated with orbits of the group of C.C.R. Bogoliubov transformations, containing indecomposable canonical forms $A_j^{(2\ell)}$.

TABLE 4.44

| Canonical form | Elementary divisors | Frequencies | Signature |
|---|---|---|---|
| $A_1^{(2\ell)}$ | $(\omega\pm ia)^\ell$ | $\pm ia$ | $(\ell,\ell)$ |
| $A_2^{(4\ell)}$ | $(\omega\pm ia\pm b)^\ell$ | $\pm ia\pm b$ | $(2\ell,2\ell)$ |
| $A_3^{(2\ell)}$ | $\omega^\ell,\ \omega^\ell$ | 0 | $(\ell-1,\ell-1)$ |
| $A_4^{(2\ell)}(\rho)$ | $\omega^{2\ell}$ | 0 | $(\frac{1}{2}(1-\rho),\frac{1}{2}(1+\rho))\ (\mathrm{mod}\ 2)$ for $\ell=1,2$ |
| $A_5^{(2(2\ell+1))}(\rho)$ | $(\omega\pm b)^{2\ell+1}$ | $\pm b$ | $(1-\rho,1+\rho)$ for $\ell=0$ |
| $A_6^{(4\ell)}(\rho)$ | $(\omega\pm b)^{2\ell}$ | $\pm b$ | $(2,2)$ for $\ell=0$ |

In addition, the signature of $D_N = \rho^{-1}(\hat{H}_N)$, with $\hat{H} = \rho(D_1)$ and $\hat{H}_N$ as in (1.21), is given by (1.21) when $D_1$ is equivalent to $A_6^{(4k)}(\rho)$.

Contrary to popular belief, a wide variety of quadratic Boson Hamiltonians can not be reduced to a linear combination of number operators by a Bogoliubov transformation. If the off-diagonal blocks, corresponding to pair creation terms $N_{ij}\ a_i^\dagger\ a_j^\dagger$ and pair annihilation terms $N_{ij}^*\ a_i\ a_j$, are absent, then $\hat{I}\ D_1 = \begin{pmatrix} A_1 & 0 \\ 0 & -A_1^* \end{pmatrix}$, with $A_1$ Hermitean, obviously has real eigenvalues and linear elementary divisors. This severely limits the types of orbits to which $D_1$ may belong, in comparison with the full range of table 4.44.

Just as it is impossible to block diagonalize $D_1$ in the general case, so too is it impossible to avoid the use of the complex field. If $D_1$ is real,

$$D_1 = \begin{pmatrix} M & N \\ N & M \end{pmatrix} \text{ with M and N both real symmetric (by (4.31)),}$$

then $$P\,D_1\,P^{-1} = 2^{-1}\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}\begin{pmatrix} M & N \\ N & M \end{pmatrix}\begin{pmatrix} I & iI \\ I & -iI \end{pmatrix}$$

$$= \begin{pmatrix} M+N & 0 \\ 0 & M-N \end{pmatrix} \qquad (4.41)$$

This means that $D_1$ must have real eigenvalues and linear elementary divisors. However, we can show that the symmetric off-diagonal block N of $D_1$ can always be chosen to be real.

If $$D_1 = \begin{pmatrix} M & N \\ N & M^* \end{pmatrix} \text{ with M Hermitean and N real symmetric,}$$

then $$\rho(D_1) = P\,D_1\,P^{\dagger} = \begin{bmatrix} N + \frac{1}{2}(M+M^*) & \frac{1}{2}i(M-M^*) \\ -\frac{1}{2}i(M-M^*) & -N+\frac{1}{2}(M+M^*) \end{bmatrix} \qquad (4.42)$$

Since M is Hermitean, the off-diagonal block of $\hat{H} = \rho(D_1)$ in (4.42) is real skew symmetric. Conversely, if the real symmetric $\hat{H}$ has an off-diagonal block B which is skew symmetric, then the off-diagonal block of $\rho^{-1}(\hat{H})$ is real. Jackiw and Rossi [19] have reminded us that when F = I in (1.3), the off-diagonal block of $\hat{H}$ can always be assumed to be anti symmetric since the remaining symmetric term then contributes a total time derivative to the Lagrangian. If $B = B^{(a)} + B^{(s)}$ with $B^{(a)}$ anti-symmetric and $B^{(s)}$ symmetric, then by (2.4),

$$L = -\frac{1}{2}(\underline{q},(A-BB^T)\underline{q}) - (\underline{q},B^{(a)}\dot{\underline{q}}) + \frac{1}{2}(\dot{\underline{q}},\dot{\underline{q}}) - \frac{d}{dt}\,\frac{1}{2}(\underline{q},B^{(s)}\underline{q}) \qquad (4.43)$$

It is well-known that neglect of the last term, a total time derivative has no effect on the Lagrangian dynamics. If we ignore this last term and then re-apply the involutory Legendre transform (2.3), we obtain the adjusted Hamiltonian

$$H = \frac{1}{2}(\underline{q},A^1\underline{q}) + (\underline{q},B^{(a)}\underline{p}) + \frac{1}{2}(\underline{p},\underline{p}), \qquad (4.44)$$

where $$A^1 = A - BB^T - B^{(a)}B^{(a)} = A - B^{(s)}B^{(s)} + [B^{(s)},B^{(a)}]$$

When F is allowed to be a general symmetric matrix, we can still derive from the canonical forms listed in section 1, another complete list in which the gyroscopic term B is always anti-symmetric.

In the cases $\hat{K}_j^{(2\ell)}$, with j = 1,2 or 3, the diagonal blocks are absent,

$$\hat{K}_j = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \quad ; \quad B = B^{(s)} + B^{(a)}$$

We may then apply the symplectic transformation $C = \begin{pmatrix} I & I \\ -\frac{1}{2} I & \frac{1}{2} I \end{pmatrix}$ to obtain

$$C^T \hat{K}_j C = \begin{pmatrix} -B^{(s)} & B^{(a)} \\ -B^{(a)} & B^{(s)} \end{pmatrix}$$

The canonical orbit containing $\hat{K}_4^{(2m)}$ is that for which zero is the only eigenvalue of $G\hat{H}$ and for which this eigenvalue has geometric multiplicity 1. One representative of this orbit is

$$\hat{H} = \begin{bmatrix} & & & & -\rho & & & & & \\ & & & \cdot & & & & & & \\ & & \cdot & & & & & & & \\ & -\rho & & & & & & & & \\ -\rho & & & & & & & & & \\ & & & & & & & & \rho & 0 \\ & & & & & & & \cdot & \cdot & \\ & & & & & & \rho & \cdot & & \\ & & & & & \rho & 0 & & & \\ & & & & & 0 & & & & \end{bmatrix}$$

and this example has trivial anti-symmetric off-diagonal blocks.

To $K_5^{(2(2m+1))}$, we may apply the symplectic transformation $C = \begin{pmatrix} -b^{-1} I & I \\ \frac{1}{2} I & -\frac{1}{2} b I \end{pmatrix}$ to obtain, in the off-diagonal block,

$$B = \frac{1}{2} \begin{bmatrix} 0 & 1 & & & & & \\ -1 & 0 & 1 & & & & \\ & 1 & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{bmatrix}$$

Finally, to $K_6^{(4m)}$, we may apply the symplectic transformation

$C = \text{diag.}[b^{-\frac{1}{2}}, b^{\frac{1}{2}}, \ldots, b^{-\frac{1}{2}}, b^{\frac{1}{2}}, b^{\frac{1}{2}}, b^{-\frac{1}{2}}, \ldots, b^{\frac{1}{2}}, b^{-\frac{1}{2}}]$ , to obtain a skew-symmetric gyroscopic block

$$B = b \begin{bmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & & \\ & -1 & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{bmatrix}$$

We have now shown how to explicitly construct a full set of canonical forms $\hat{K}$ for $\hat{H}$, in which the off-diagonal block is anti-symmetric. From this set, we may construct a full set of canonical forms $\rho^{-1}(\hat{K})$, representing Boson Hamiltonians (4.1) with $D = \rho^{-1}(\hat{H})$, in which the off diagonal (pair creation) block is real.

In the historical introduction to this section, it was recalled that Bogoliubov transformations were initially used as a device to transform the interaction terms away from a quadratic Boson Hamiltonian. In the more general case that diagonalization is not possible, the approach of Solomon [66] is to express H as a linear combination of operators which satisfy the commutation relations of fundamental basis elements of a spectrum-generating complex semi-simple Lie algebra. A Lie algebra automorphism then maps the Hamiltonian to an element of the Cartan sub-algebra and the spectrum of H is deduced after evaluating the Casimir invariants for the particular representation in use. Since, by the correspondence (3.16), H could have been expressed as a quadratic expression in quantum canonical observables, this approach is closely related to that of Moshinsky and Winternitz, discussed in section 3. In the latter approach, one attempts to represent H as a linear combination of basis elements of a (non-unique) maximal Abelian sub-algebra of the real algebra Sp(2N,R).

We have deduced the canonical forms for quadratic Boson Hamiltonians from the conjugacy classes of Sp(2N,R). For the quadratic Fermion Hamiltonians, the single diagonal type of canonical form for $D_2$ could have been deduced from the single type of canonical form for the skew symmetric generators of O(2N,R). Explicit construction of the simplifying orthogonal transformation will appear in a later section on the Fermion analogues of classical mechanics. This work will later be used in algebraic quantization.

The constructions which have been used in this section to produce canonical forms for quadratic Boson Hamiltonians, are applicable only to systems of a finite number of degrees of freedom. In the next section, the equivalent diagonalization theorems of Whittaker and Thouless are shown to follow from a general spectral theory of pseudo self-adjoint operators on a space with indefinite metric. Recent advances in this spectral theory enable us to extend these diagonalization results to infinite degrees of freedom.

## SECTION 5 - DIAGONALIZATION VIA SPECTRAL THEORY IN KREIN SPACE

In the case of finite N(= Lim $L$) and $D_1 > 0$, the diagonalizing U(N,N) matrix $T$ was constructed in such a way [54-57] that the columns of $T$ were a complete $\hat{I}$-orthogonal set of eigenvectors $\begin{pmatrix} \underline{u}_j \\ \underline{v}_j \end{pmatrix}$ and $\begin{pmatrix} \underline{v}_j^* \\ \underline{u}_j^* \end{pmatrix}$ satisfying

$$\hat{I}\, D_1 \begin{pmatrix} \underline{u}_j \\ \underline{v}_j \end{pmatrix} = \omega_j \begin{pmatrix} \underline{u}_j \\ \underline{v}_j \end{pmatrix} \quad , \quad \hat{I} = I \oplus -I \text{ on } L \oplus L \tag{5.1}$$

$$\left( \begin{pmatrix} \underline{u}_j \\ \underline{v}_j \end{pmatrix} , \hat{I} \begin{pmatrix} \underline{u}_k \\ \underline{v}_k \end{pmatrix} \right) = \delta_{jk} = \hat{I}_{jk} \tag{5.2}$$

$$\left( \begin{pmatrix} \underline{v}_j^* \\ \underline{u}_j^* \end{pmatrix} , \hat{I} \begin{pmatrix} \underline{v}_k^* \\ \underline{u}_k^* \end{pmatrix} \right) = -\delta_{jk} = \hat{I}_{j+N,k+N}$$

The frequencies $\omega_j$ are eigenvalues of the matrix $\hat{I}\, D_1$ which is not always Hermitean. Nevertheless, $\hat{I}\, D_1$ is pseudo-Hermitean on the indefinite inner product space ($L \oplus L$, $(\cdot,\hat{I}\cdot)$), as we shall discuss below. Therefore, we should enquire whether the diagonalization of quadratic Boson Hamiltonians could be a direct corollary of a more widely encompassing abstract spectral theory of pseudo-self-adjoint operators on an indefinite inner product space. This point of view was expressed by Araki at the end of his paper [60] of 1968. However, the required spectral theory was largely undeveloped at that time and it seems that Araki was not yet familiar with Thouless's successful diagonalization of positive definite $D_1$ in the case of finite N, albeit as an isolated mathematical task [54]. It is now time to re-examine the state of spectral theory with indefinite metric to see whether Araki's general results on quadratic Fermion Hamiltonians and infinite dimensional $L$ can be extended to Bose fields.

Suppose that ($H_c$,<·,·>) is a complex Hilbert space and that $\eta$ is a self-adjoint involution on $H_c$,

$$\eta^\dagger = \eta = \eta^{-1} \tag{5.3}$$

Then $\eta$ is the metric or Gram operator for a particular indefinite inner product $[\cdot,\cdot]$ on $H_c$ .

$$[\underline{\zeta},\underline{\xi}] = \langle\underline{\zeta},\eta\underline{\xi}\rangle \quad . \tag{5.4}$$

Definition (5.5)

(a) A densely defined operator M on $H_c$ has $\eta$-adjoint $M^+$ defined by $M^+\underline{\phi} = \underline{\psi}$, provided $[\underline{\phi},M\underline{\chi}] = [\underline{\psi},\underline{\chi}]$ for all $\underline{\chi} \in$ dom. M.

(b) M is self $\eta$-adjoint if $M^+ = M$;

(c) a self $\eta$-adjoint operator M is $\eta$-non-negative if $[\underline{\psi},M\underline{\psi}] \geq 0$ for all $\underline{\psi} \in$ dom. M.

If, in addition, $[\underline{\psi},M\underline{\psi}] = 0 \Rightarrow \underline{\psi} = 0$, then M is strictly $\eta$-positive.

Proposition (5.6)

(a) $M^+ = \eta M^\dagger \eta$ ,

(b) M is self $\eta$-adjoint if and only if $M = \eta D$, where D is self adjoint on $H_c$ ,

(c) M is $\eta$-non-negative if and only if D given in (b) is non-negative,

(d) M is strictly $\eta$-positive if and only if D is strictly positive.

Proof: (a) $M^+\underline{\phi} = \underline{\psi}$

$\Leftrightarrow \forall\ \underline{\chi} \in$ dom. M, $\langle\underline{\phi},\eta M\underline{\chi}\rangle = \langle\underline{\psi},\eta\underline{\chi}\rangle$

$\Leftrightarrow \forall\ \underline{\chi} \in$ dom. M, $\langle\eta\underline{\phi},M\underline{\chi}\rangle = \langle\eta\underline{\psi},\underline{\chi}\rangle$

$\Leftrightarrow M^\dagger\ \eta\ \underline{\phi} = \eta\ \underline{\psi}$

$\Leftrightarrow \eta\ M^\dagger\ \eta\ \underline{\phi} = \underline{\psi}$

(b) $M^+ = M$

$\Leftrightarrow [\forall\ \underline{\chi} \in$ dom. M, $\langle \underline{\phi}, \eta M \underline{\chi} \rangle = \langle \eta \underline{\psi}, \underline{\chi} \rangle \Leftrightarrow \underline{\phi} \in$ dom. M and $\underline{\psi} = M\underline{\phi}$

$\Leftrightarrow \underline{\phi} \in$ dom. $\eta M$ and $\eta\underline{\psi} = \eta M \underline{\phi}]$

$\Leftrightarrow$ $\eta M$ is self-adjoint

$M = \eta D$, for some self-adjoint operator D.

(c) $\forall\ \underline{\psi} \in$ dom. M, $\langle \underline{\psi}, \eta M \underline{\psi} \rangle \geq 0$

$\Leftrightarrow$ $\forall\ \underline{\psi} \in$ dom. $D(=\eta M)$, $\langle \underline{\psi}, D\underline{\psi} \rangle \geq 0$

(d) as for (c).

<u>Definition</u> (5.7): E is an $\eta$-orthogonal projector on $H_c$ if E is self $\eta$-adjoint with domain $H_c$ and E is idempotent, $E^2 = E$.

If E is an $\eta$-orthogonal projector on $H_c$, then it is easy to verify that

$$\forall\ \underline{\phi}, \underline{\psi} \in H_c \quad , \quad [E\underline{\phi}, (1 - E)\underline{\psi}] = 0.$$

<u>Definition</u> (5.8): An $\eta$-spectral function defined on the interval (a,b) with $a < 0 < b$, is an operator valued function E(t) on (a,b) satisfying

(a) E(t) is an $\eta$-orthogonal projector on $H_c$ .

(b) $E(s)\ E(t) = E(\min\{s,t\})$.

(c) In the strong operator topology,

$$\lim_{t \to s^+} E(t) = E(s) \quad \text{for all} \quad s \in (a,b)$$

and (d) In the strong operator topology,

$$\lim_{t \to b^-} E(t) = I \quad \text{and} \quad \lim_{t \to a^+} E(t) = 0 .$$

<u>Theorem</u> (5.9): Let M be a densely defined strictly $\eta$-positive operator whose resolvent set $\delta(M)$ is non-empty. Then $H_c$ is a direct sum of M-invariant $\eta$-orthogonal subspaces $H_1$ and $H_2$ such that

(a) $M_1 = M / H_1 = \int_{-\infty}^{\infty} t \, d \, E_1(t)$ (converging in the strong topology)

for some $\eta$-spectral function $E_1(t)$.

(b) For all $\underline{\phi} \in H_2 \cap$ dom. M ,

$$M\phi = \int_{-\infty}^{\infty} t^{-1} \, d \, E_2(t)\underline{\phi} \quad \text{(converging in the strong topology)}$$

for some $\eta$-spectral function $E_2(t)$.

(c) $[\underline{\psi}, \pm \underline{\psi}]$ is positive definite on $E_j(t) \, H_j$ for $t \gtrless 0$ .

For bounded operators M, $H_2$ may be neglected, and in this case, the above spectral theorem was founded by Krein and Smul'jan [67]. A clear discussion of this subject, including the uniqueness of $E_1(t)$, may be found in [68]. The extension to unbounded M has recently been achieved by Harvey [69].

If M is allowed to be a general $\eta$-non-negative operator, then theorem (5.9) requires some modifications [68-69]. Namely, (5.9a) becomes

$$M_1 = \int_{-\infty}^{\infty} t \, d \, E_1(t) + S \, , \tag{5.10}$$

where $S^2 = 0$ and $S(E(t) - E(r)) = 0$ if $r < t < 0$ or $0 < r < t$ .

If M is iG-non-negative ($\hat{H} = iGM \geq 0$) and has zero as an eigenvalue, then (5.10) restricts the index of nilpotency of the nilpotent part S of M to be 1 or 2. From section 1, if $\hat{H}$ has zero as an eigenvalue, then the classical system with Hamiltonian $\frac{1}{2} \underline{z}^T \hat{H} \underline{z}$ includes one or more independent subsystems for which the matrix $\hat{H}$ can be reduced to $\hat{K}_3^{(2\ell)}$ or $\hat{K}_4^{(2\ell)}$, ($\ell \leq N$). $\hat{K}_3^{(2\ell)}$ has eigenvalues 0, 1 and -1 with eigenspaces of dimension 2, $\ell$-1 and $\ell$-1 respectively. Therefore, if $\hat{H}$ is non-negative, indecomposable canonical blocks $\hat{K}_3^{(2\ell)}$ appear only in the trivial case $\ell=1$, so that $\hat{K}_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The index of nilpotency for $-G\hat{K}_4^{(2\ell)}$ is $2\ell$, since $-iG\hat{K}_4^{(2\ell)} - sI$ has elementary divisor $s^{2\ell}$. Therefore, indecomposable canonical blocks $K_4^{(2\ell)}$ result from positive semi-definite quadratic Hamiltonians only in the trivial case $\ell=1$, $\hat{K}_4^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The Hamiltonian for the corresponding classical subsystem is $\frac{1}{2} q_1^2$, which is in the same canonical orbit as $\frac{1}{2} p_1^2$, the Hamiltonian for the classical free particle.

Naturally, the spectral resolution of a η-non-positive self-η-adjoint operator M follows from theorem (5.9) applied to -M. A minor extension to (5.9) would then accommodate the direct sum of a η-non-positive self-adjoint operator and a η-non-negative self-η-adjoint operator. However, a η-indefinite self-η-adjoint operator can not necessarily be decomposed in this way and there is no possible extension of theorem (5.9) which guarantees that the spectrum of such an operator is real. On the contrary, it was realised long ago [70] that the frequencies of a classical linear Hamiltonian system (eigenvalues of $iG\hat{H}$) could take any complex value if the Hamiltonian were indefinite.

For physical applications, it has been convenient, following Dirac [71] , to interpret the spectral decomposition of an unbounded normal operator Q on Hilbert space as the existence of a complete orthonormal set of eigenfunctions. In some circumstances, distribution theory rigorously justifies this notion. In particular, by theorem 6 in chapter II of [72], if there exists a dense space $S$ of $C_\infty$ vectors for M($S = \bigcap_{n \in N}$ dom $Q^n$, with N the set of natural numbers), then there exists a nuclear space $N$ which is dense in $S$ and $H_c$ can be continuously embedded in the rigged Hilbert space consisting of Gelfand triplet $N \subset H_c \subset N'$, with the dual space $N'$ being the space of continuous linear functionals on $N$. It then follows (theorem 5 in chapter II of [72]) that there exists in $N'$ a complete orthonormal set of eigendistributions $\psi_\alpha$ for Q. In the case that Q is self-adjoint, there exists a dense set of analytic vectors for Q, belonging to the domain of $\sum_{n=0}^{\infty} (i\,t\,Q)^n$ (see e.g. section X.6 of [35]), which is a subset of $S$.

The formal expansion $H_c \ni \underline{\psi} = \int_S \underline{\psi}_\alpha\, a(\alpha)\, d\sigma(\alpha)$ with S a set of spectral indicies and $\sigma$ an appropriate measure, has the following meaning:

if $$\underline{\phi} \in N,\ (\underline{\psi},\underline{\phi}) = (\underline{\psi},\underline{\phi}) = \int_S \underline{\psi}_\alpha(\underline{\phi})\, a(\alpha)\, d\sigma(\alpha) \tag{5.11}$$

With intuitive physical applications in mind, we now reformulate theorem (5.9) in terms of eigendistributions.

Proposition (5.12): Let M be as specified in theorem (5.9). Then there exists a dense nuclear space $N \subset H_c$ such that $N'$ contains a complete $\eta$-orthonormal set of eigendistributions $\underline{\psi}_\alpha$ for M.

Proof: Define $$V_+ = (I - E_1(0))H_1 \cup (I - E_2(0))H_2$$

and $$V_- = E\,(0)H_1 \cup E\,(0)H_2\ .$$

By (5.9c), the inner product $\langle\underline{\phi}, \pm\,\eta\underline{\phi}\rangle$ is positive definite on $V_\pm$. Therefore, $(V_\pm, \langle\cdot, \pm\,\eta\cdot\rangle)$ may be completed to form Hilbert spaces $H_\pm$ .

Now consider a sequence $\{\underline{\phi}s\}$ in $V_\pm$ which is a Cauchy sequence in $H_\pm$. Given $\epsilon > 0$, $\exists$ N such that

$$r,s > N \Rightarrow |[\underline{\phi}r - \underline{\phi}s,\ \underline{\phi}r - \underline{\phi}s]| < \epsilon$$

$$\Rightarrow \langle\underline{\phi}r - \underline{\phi}s,\ \underline{\phi}r - \underline{\phi}s\rangle = |[\underline{\phi}r - \underline{\phi}s,\ \eta(\underline{\phi}r - \underline{\phi}s)]|$$

$$\leq |[\underline{\phi}r - \underline{\phi}s, \underline{\phi}r - \underline{\phi}s]|^{\frac{1}{2}} |[\eta(\underline{\phi}r - \underline{\phi}s), \eta(\underline{\phi}r - \underline{\phi}s)]|^{\frac{1}{2}}$$

(by the Schwartz inequality)

The second factor, in the right hand side above, equals $|\langle\underline{\phi}r-\underline{\phi}s, \eta(\underline{\phi}r-\underline{\phi}s\rangle|^{\frac{1}{2}}$, since $\eta = \eta^{\dagger} = \eta^{-1}$. Therefore, the right hand side is $|[\underline{\phi}r-\underline{\phi}s, \underline{\phi}r-\underline{\phi}s]|$, which is less than $\epsilon$. Hence, $\{\underline{\phi}s\}$ must also be a Cauchy sequence in the original $H_c$ topology. Thus, we see that the completion of $V_{\pm}$ in the $H_{\pm}$ topology is nothing more than the completion of $V_{\pm}$ in the $H_c$ topology. Since $M = M^{+}$, the restrictions $M_{\pm}$ of M to $H_{\pm}$ must be self-adjoint. This allows us to apply the spectral theory in ordinary Hilbert space. Therefore, there exists in $H_-$ a dense M-invariant nuclear space $N_-$. Similarly, there exists in $H_+$ a dense M-invariant nuclear space $N_+$. Hence, there exists in the dual spaces $N'_{\pm}$ a complete $\eta$-orthornormal system of eigendistributions of $M_{\pm}$. These eigendistributions $\psi^{(\pm)}_{\underline{k},\underline{\ell}}$ are considered to be labelled by the discrete spectral values $\underline{k}$ and continuous spectral values $\underline{\ell}$ of a complete set of commuting self-adjoint operators $\{K_1^{\pm}, K_2^{\pm}, \ldots;$ $L_1^{(\pm)}, L_2^{(\pm)}, \ldots\}$

$$M\,\psi^{(\pm)}_{\underline{k},\underline{\ell}} = m^{\pm}(\underline{k},\underline{\ell})\,\psi^{(\pm)}_{\underline{k},\underline{\ell}} \quad \text{with} \quad m^{\pm}(\underline{k},\underline{\ell}) \gtrless 0 \,, \tag{5.13}$$

$$0 = \langle \psi^{(+)}_{\underline{k},\underline{\ell}} \,,\; \eta\,\psi^{(-)}_{\underline{k}',\underline{\ell}'} \rangle \,, \tag{5.13a}$$

$$\delta_{\underline{k},\underline{k}'}\;\delta(\underline{\ell}-\underline{\ell}') = \langle\psi^{(+)}_{\underline{k},\underline{\ell}},\; \eta\,\psi^{(+)}_{\underline{k}',\underline{\ell}'}\rangle = -\langle\psi^{(-)}_{\underline{k},\underline{\ell}}\;\; \eta\,\psi^{(-)}_{\underline{k}',\underline{\ell}'}\rangle \tag{5.13b}$$

To obtain theorem (1.26) (Whittaker's reduction of a classical Hamiltonian system with positive definite quadratic Hamiltonian to a system of independent harmonic oscillators) from proposition (5.12), we take $J = iG = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}$, $A = iG\hat{H}$, $\hat{H}$ real symmetric and positive definite and $H_c$ = 2N dimensional Euclidean space. Then, by (5.12), there exists in $H_+$ and $H_-$, a complete iG-orthonormal set of eigenvectors $\underline{\psi}_j$ of $iG\hat{H}$ corresponding to positive and negative eigenvalues respectively.

$$iG\hat{H}\,\underline{\psi}_\mu = \alpha_\mu\,\underline{\psi}_\mu \qquad (\mu = 1,\ldots,2N)$$

$$\Leftrightarrow\; -iG\hat{H}\,\underline{\psi}^*_\mu = \alpha_\mu\,\underline{\psi}^*_\mu \tag{5.14}$$

$\Leftrightarrow$ $\underline{\psi}^*_\mu$ is an eigenvector of $iG\hat{H}$ corresponding to eigenvalue $-\alpha_\mu$.

Now by (5.14), we may choose $\underline{\psi}_j$ $(j = 1,\ldots,N)$ so that the corresponding eigenvalues $\alpha_j$ are negative. By the property (5.13b), the $\underline{\psi}_j = \left(= \begin{pmatrix} \underline{u}_j \\ \underline{v}_j \end{pmatrix}\right)$ may be normalised so that

$$(\underline{\psi}_j, \, i \, G \, \underline{\psi}_k) = -\delta_{jk} \quad . \tag{5.15}$$

$$\Leftrightarrow \quad (\underline{v}_j, \underline{u}_k) - (\underline{u}_j, \underline{v}_k) = i \, \delta_{jk} \, .$$

Also, by (5.14), $\underline{\psi}_j^*$ is an eigenvector of $i G \hat{H}$ corresponding to eigenvalue $-\alpha_j > 0$ and from (5.15),

$$(\underline{\psi}_j^*, \, i \, G \, \underline{\psi}_k^*) = \delta_{jk} \, . \tag{5.16}$$

In addition, there is a well-known property [73] of pseudo self-adjoint operators on an indefinite inner product space that eigenvectors $\underline{\phi}_j$, $\underline{\phi}_k$ corresponding to eigenvalues $\alpha_j$, $\alpha_k$ are orthogonal unless $\alpha_k = \alpha_j^*$ .

$$(\alpha_k - \alpha_j^*) \, \langle \underline{\phi}_k, \underline{\phi}_j \rangle = 0 \, . \tag{5.17}$$

From (5.13) and (5.16), we have

$$(\underline{\psi}_j, \, G \, \underline{\psi}_k^*) = 0 \qquad (j,k = 1,\ldots,N) \, . \tag{5.18}$$

Now we are able to construct a real basis $\{\underline{\phi}_j\}_{j=1,\ldots,N} \cup \{\underline{\phi}_{N+j}\}_{j=1,\ldots,N}$ defined by

$$\underline{\phi}_j = 2^{-\frac{1}{2}}(\underline{\psi}_j + \underline{\psi}_j^*) \tag{5.19}$$

$$\underline{\phi}_{N+j} = 2^{-\frac{1}{2}} \, i(\underline{\psi}_j - \underline{\psi}_j^*) \, .$$

From (5.14-18), we observe that (5.19) constitutes a real symplectic basis

$$(\underline{\phi}_\mu, \, G \, \underline{\phi}_\nu) = G_{\mu\nu} \, , \tag{5.20}$$

and that in this basis, $\hat{H}$ is diagonal, as required.

$$(\underline{\phi}_\mu, \, \hat{H} \, \underline{\phi}_\nu) = |\alpha_\mu| \delta_{\mu\nu}$$

To obtain theorem 4.41 (diagonalization of a positive definite quadratic Boson Hamiltonian) from proposition (5.12), we take $J = \hat{I} = \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix}$, $N < \infty$ , $A = \hat{I} \, D_1$ , $D_1$ a positive definite matrix in

$D_1^s$ and $H = E^{2N}$ . (5.12) then provides a complete $\hat{I}$-orthonormal set of eigenvectors $\underline{\psi}_j$ of $\hat{I}\, D_1$.

$$\hat{I}\, D_1\, \underline{\psi}_j = \alpha_j\, \underline{\psi}_j$$

$$\Leftrightarrow \quad \hat{I}\, D_1^*\, \underline{\psi}_j^* = \alpha_j\, \underline{\psi}_j^*$$

$$\Leftrightarrow \quad \hat{I} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} D_1 \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \underline{\psi}_j^* = \alpha_j\, \underline{\psi}_j^*$$

$$\Leftrightarrow \quad \hat{I}\, D_1 \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \underline{\psi}_j^* = -\alpha_j \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \underline{\psi}_j^* \qquad (5.22)$$

By (5.22), we can construct a basis $\left\{ \begin{pmatrix} \underline{u}_j \\ \underline{v}_j \end{pmatrix}, \begin{pmatrix} \underline{u}_{N+j} \\ \underline{v}_{N+j} \end{pmatrix} \right\}$ of eigenvectors of $\hat{I}\, D_1$ for which $\underline{u}_j = \underline{v}_{N+j}^*$ and $\underline{v}_j = \underline{u}_{N+j}^*$ and these basis vectors form the columns of a matrix $T \in U(N,N)$ of the form (4.5) such that $T^\dagger\, D_1\, T$ is diagonal.

For field models, the single particle Hilbert space $L$ is infinite dimensional and the matrices A,B and C of (4.1) become integral kernels of operators on $L$ (see [38,45]). In order that the operator H has a self-adjoint extension on the Fock space for $\{a(\phi), a^\dagger(\phi); \phi \in L\}$, or even that the domain of H includes the vacuum $\psi_0$ for $a(\phi)$, it is necessary that B is of the Hilbert-Schmidt class $C_2(L)$ and that C is of the trace class $C_1(L)$. Araki's method succeeds in producing a unitary transformation $T$ on $L \oplus L$ , of the form (4.5), which diagonalizes $\beta(D) \in D_2^s$. However, one can not assume that A belongs to the trace class and can therefore not assume that the component $H_1$ of H, due to $\alpha(D) \in D_1^s$, is merely a scalar multiple of the identity, as in (4.54). Nevertheless, the off-diagonal blocks $B_1$, $B_1^*$ of $\alpha(D)$ are symmetric ($B_1 = B_1^{*\dagger}$) and so, by the C.A.R., they do not contribute to the Fermion Hamiltonian H. After diagonalization of $\beta(D)$, H contains only number-conserving terms and in this sense, the off-diagonal blocks of $D$ have been removed.

So far, the most general conditions for block diagonalizability of a quadratic Boson Hamiltonian, with $L$ infinite dimensional, have been provided by Dadashev and Kuliev [74]. However, they, like Berezin, whose results they generalized, considered only the real case $D = D^*$. Although Harvey's work [69] was carried out in a purely mathematical context, our reformulation (1.12) in terms of eigendistributions can be directly applied to linear Bose fields in the case of complex $D$.

Proposition (5.23): For a Boson Hamiltonian of type (4.1) with A,B,C arbitrary operators on a complex Hilbert space $L$ such that H is essentially self-adjoint, a sufficient condition that H can be reduced to the form

$$H = \frac{1}{2}\begin{pmatrix} \underline{c} \\ \underline{c}^{\dagger} \end{pmatrix}^{\dagger} D'' \begin{pmatrix} \underline{c} \\ \underline{c}^{\dagger} \end{pmatrix} \qquad (D'' \text{ block diagonal}) ,$$

by a Bogoliubov transformation is that $\alpha(D)$ has a strictly positive self-adjoint extension $\bar{\alpha}(D)$.

$$\bar{\alpha}(D) > \epsilon I > 0 \quad \text{on} \quad L \oplus L \ .$$

Proof: Let $D_1 = \bar{\alpha}(D) > \epsilon I > 0$ . Let $M = \hat{I}\, D_1$ .

$\frac{\epsilon}{2}$ belongs to the resolvent set of M ,

since $M^{\dagger} M = D_1^2$

$\Rightarrow \quad \sigma(M^{\dagger}M) \subseteq (\epsilon^2, \infty)$

$\Rightarrow \quad \sigma(M) \subseteq C \sim N(0,\epsilon)$ (i.e. spectral elements of M have modulus no smaller than $\epsilon$)

Now M is self $\hat{I}$-adjoint and strictly $\hat{I}$-positive on $L \oplus L$ and the resolvent set $\delta(A)$ is non-empty. Therefore, we may apply proposition 5.12, with $H_c = L \oplus L$ and $J = \hat{I}$. Just as in the case Dim $L = N < \infty$, when we were able to construct an $\hat{I}$-orthonormal basis $\left\{ \begin{pmatrix} \underline{u}_j \\ \underline{v}_j \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_j \\ \underline{v}_j \end{pmatrix}^* \right\}$ of eigenvectors of $\hat{I}\, D_1$ , satisfying

$$\hat{I} \, D_1 \begin{pmatrix} \underline{u}_j \\ \underline{v}_j \end{pmatrix} = m_j \begin{pmatrix} \underline{u}_j \\ \underline{v}_j \end{pmatrix} \cdot \qquad (m_j > 0) ,$$

we can now construct an $\hat{I}$-orthonormal basis

$$\left\{ \begin{pmatrix} \underline{u}_\alpha \\ \underline{v}_\alpha \end{pmatrix} , \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_\alpha \\ \underline{v}_\alpha \end{pmatrix}^* \right\} \text{ of eigendistributions of } \hat{I} \, D_1 .$$

The transformation to this basis is a Bogoliubov transformation $T$ which diagonalizes $D_1$. The off-diagonal blocks of the transformed $D_2$ component of $D$ remain anti-symmetric and by the C.C.R., they make no contribution to the Hamiltonian H. Hence, by the decomposition (4.30), $D$ has been effectively block diagonalized.

Berezin's work (theorem 8.1 of [38]) involved dynamical matrices of the type

$$D = \begin{pmatrix} 2A & B \\ B^* & 0 \end{pmatrix} , \text{ with } A = A^\dagger = A^*, \ B = B^* = B^\dagger, \ ||A|| < \infty \qquad (5.24)$$

with the extra condition

$$A \pm B > \epsilon E > 0 . \qquad (5.25)$$

In this case, we have

$$\alpha(D) = \begin{pmatrix} A & B \\ B & A \end{pmatrix} .$$

If $\begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix}$ is an eigendistribution of $\alpha(D)$ corresponding to real spectral value $\lambda$, then

$$\begin{aligned} A\underline{u} + B\underline{v} &= \lambda \underline{u} \\ B\underline{u} + B\underline{v} &= \lambda \underline{v} , \end{aligned} \qquad (5.26)$$

which implies

$$\begin{aligned} (A + B)(\underline{u} + \underline{v}) &= \lambda(\underline{u} + \underline{v}) \\ (A - B)(\underline{u} - \underline{v}) &= \lambda(\underline{u} - \underline{v}) . \end{aligned} \qquad (5.27)$$

Since $\underline{u} \pm \underline{v}$ do not both vanish, (5.25) then requires $\lambda > \epsilon > 0$. Therefore, $\alpha(D) > \epsilon I > 0$ and the conditions of our proposition (5.23) apply, so that Berezin's theorem and its extension [7] to unbounded $D$ are corollaries.

Since publishing this application of spectral theory in Krein space [II], we have found a report by Kuliev [75], originally published in the Russian language, of a very similar programme aimed at extending the results of Dadashev and Kuliev to allow complex $D$. However, it appears that they were not aware of any extension of the Krein-Šmul'jan spectral theory to unbounded $\eta$-non-negative operators.

## SECTION 6 - UNITARY IMPLEMENTABILITY OF BOGOLIUBOV TRANSFORMATIONS

When the dimension N of the single particle space $L$ is finite, the group of one-parameter continuous unitary transformations generated by a quadratic Hamiltonian on Fock space, is equivalent to the group of Bogoliubov transformations. The dynamics of a linear system of Bosons or Fermions is specified by a quadratic Hamiltonian of the form (4.1). As discussed in section 4, it follows that $D_1$ and $D_2$ contribute nothing more than a real multiple of the identity to H in the cases of Fermi-Dirac statistics and Bose statistics respectively. Then, using Friedrichs' first similarity rule [46],

$$e^{itH}\,\alpha_\mu\,e^{-itH} = \sum_{\nu=1}^{2N}\left[e^{-iI_jD_jt}\right]_{\mu\nu}\alpha_\nu \tag{6.1}$$

where $$I_j = \begin{cases} \hat{I} & \text{for } j = 1 \text{ (Bose statistics)} \\ I & \text{for } j = 2 \text{ (Fermi-Dirac statistics)} \end{cases}$$

For a general Bogoliubov transformation (4.5), the implementing unitary transformation U, such that

$$\gamma_\mu = U^{-1}\,\alpha_\mu\,U = \sum_{\nu=1}^{2N} T_{\mu\nu}\,\alpha_\nu \tag{6.2}$$

has been given explicitly in [38] and in [63]. This has been achieved even for those C.C.R. Bogoliubov transformations for which, corresponding to the non-exponential symplectic transformations mentioned in section 1, there is no logarithm $i\,\hat{I}\,D_1$.

When $L$ is infinite dimensional, there is a continuum of inequivalent irreducible representations of the C.C.R. or C.A.R. [76]. The representation of the C.C.R. or C.A.R. obtained by a Bogoliubov transformation (4.6) may no longer be unitarily equivalent to the original Fock representation $\{a(\phi), a^\dagger(\phi), F(L)\}$ over $L$. From (6.2), it is clear that if U exists, then a vacuum $\psi_0'$ satisfying $C(\phi)\psi_0' = 0$ for all $\phi \in L$, may be obtained from the original vacuum $\psi_0$ (annihilated by $a(\phi)$) according to $\psi_0' = U^{-1}\,\psi_0$.

Conversely, as shown by Berezin [38], if $\psi_0'$ exists, then the transformation $T$ must be unitarily implementable.

In the general case, it can not be assumed that $\psi_0'$ exists in the same irreducible representative space as $\psi_0$. Both $\psi_0$ and $\psi_0'$ exist in a representative space which is reducible into a direct sum of many copies of the Fock space, the vectors $\psi_0$ and $\psi_0'$ contained in different orthogonal irreducible subspaces. This phenomenon was discovered by van Hove [77]. In field theoretic language, the clothed vacuum $\psi_0'$ may be related to the bare vacuum $\psi_0$ by a procedure of infinite renormalization.

The well-known result that $T$ is unitarily implementable if and only if the off-diagonal block $T_2$ of $T$ belongs to the Hilbert-Schmidt class $C_2(L)$, appeared in the books by Friedrichs [46] and Berezin [38] and has since been rediscovered many times.

If the one parameter group of Bogoliubov transformations $T(t)$ represents time-evolution of a Fermi-Dirac field, then $T(t)$ is usually assumed to be charge-conserving.

$$T(t) \quad \text{has matrix} \quad \begin{pmatrix} T_1(t) & T_2(t) \\ T_2(t)^* & T_1(t)^* \end{pmatrix} \quad \text{on} \quad L \oplus L$$

However, $L = L_+ \oplus L_-$ , where $L_\pm$ are the spaces of single particles with positive and negative charges respectively. For a charge-conserving transformation $T(t)$,

$$T_1(t) = \begin{pmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{pmatrix}$$

and

$$T_2(t) = \begin{pmatrix} 0 & S_3(t) \\ S_4(t) & 0 \end{pmatrix}$$

Then the matrix for $T(t)$ may be written conveniently as

$$S = \begin{pmatrix} S_1 & S_3 \\ S_4 & S_2 \end{pmatrix} \qquad \text{(see e.g. [78,79])}$$

The condition for unitary implementability may still be expressed succinctly as

$$U \text{ exists} \Leftrightarrow T_2 \in C_2(L) \Leftrightarrow S_3 \in C_2(L_+,L_-) \text{ and } S_4 \in C_2(L_-,L_+) \qquad (6.3)$$

In the process of second quantization, the dynamics of the quantized system follows from the dynamics of the corresponding classical system, whose Hamiltonian is determined, according to section 4, by the generator $D_j$ of the one parameter group $T(t)$ of Bogoliubov transformations. Therefore, one often bases criteria for the unitary implementability of $T(t)$ on the properties of $D_j$. One important result in this direction is the Lundberg criterion [80] for unitary implementability of C.A.R. Bogoliubov evolutions:

Theorem (6.4): Suppose that $T(t) = \exp[it\, D_2]$, with $D_2 (\in D_2^s)$ bounded and self-adjoint on $L \oplus L$. If $D_2$ has an off-diagonal block which is in the Hilbert-Schmidt class $C_2(L)$, then so does $T(t)$, which is unitarily implementable.

With $T(t)$ charge-conserving, the Lundberg criterion could be expressed as the Hilbert-Schmidt condition on $P_+ D_2 P_-$ and $P_- D_2 P_+$. To prove theorem (6.4), Lundberg first demonstrated that $T_0(t) = \exp(-it\, D_0)\exp(it\, D_1)$, with $D_0 = \begin{pmatrix} A_2 & 0 \\ 0 & -A_2^* \end{pmatrix}$ and $D_2 = \begin{pmatrix} A_2 & B_2 \\ -B_2^* & -A_2^* \end{pmatrix}$, could be expressed as a Dyson series $T_0(t) = \sum_{n=0}^{\infty} R_n(t)$, where $R_0(t) = I$, $R_n(t) = i\int_0^t e^{-is D_0}[D_2 - D_0]e^{is D_0} R_{n-1}(s)\, ds$ for $n \geq 1$, with absolute convergence in the Hilbert-Schmidt norm. This device had been used earlier by Bongaarts [79] for the electron-positron field. We note here that Lundberg's proof carries over, without any significant modification, to the C.C.R. Bogoliubov transformations.

Proposition (6.5): Suppose that $\mathcal{T}(t) = \exp[it\,\hat{I}\,\mathcal{D}_1]$, with $\mathcal{D}_1 (\in \mathcal{D}_1^s)$ and $\mathcal{D}_1$ bounded, $\mathcal{D}_1 = \begin{pmatrix} A_1 & B_1 \\ B_1^* & A_1^* \end{pmatrix}$ with $A_1 = A_1^\dagger$ and $B_1 \in \mathcal{C}_2(L)$. Then the off-diagonal block of $\mathcal{T}(t)$ also belongs to $\mathcal{C}_2(L)$, so that $\mathcal{T}(t)$ is unitarily implementable.

Proof: Let $\mathcal{T}_0(t) = \exp(-it\,\hat{I}\,\mathcal{D}_0)\exp(it\,\hat{I}\,\mathcal{D}_1)$, with $\mathcal{D}_0 = \begin{pmatrix} A_1 & 0 \\ 0 & A_1^* \end{pmatrix}$

Differentiation of $\mathcal{T}_0(t)$, followed by integration, yields

$$\mathcal{T}_0(t) = I + i\int_0^t F(s)\,\mathcal{T}_0(s)ds\,, \tag{6.5}$$

where $\quad F(s) = \exp(-is\,\hat{I}\,\mathcal{D}_0)[\mathcal{D}_1-\mathcal{D}_0]\exp(is\,\hat{I}\,\mathcal{D}_0)$

$F(s) \in \mathcal{C}_2(L \oplus L)$, since $(\mathcal{D}_1-\mathcal{D}_0) = \begin{pmatrix} 0 & B_1 \\ B_1^* & 0 \end{pmatrix}$, $B_1 \in \mathcal{C}_2(L)$ and $\exp(-is\,I\,\mathcal{D}_0)$ is unitary. Iteration of (6.5) gives the Dyson series

$$\mathcal{T}_0(t) = \sum_{n=0}^{\infty} R_n(t)\,, \tag{6.6}$$

where $R_0(t) = I$, $R_n(t) = i\int_0^t F(s)\,R_{n-1}(s)ds$ for $n \geq 1$ $\qquad$ (6.7)

To show that (6.6) converges in the Hilbert-Schmidt norm $\|\ \|_2$, (6.7) implies

$$\|R_1(t)\|_2 = \|i\int_0^t F(s)ds\|_2 \leq |t|\ \|F(t)\|_2 \tag{6.8a}$$

$$\|R_2(t)\|_2 \leq |\int_0^t F(s)\|R_1(s)\|_2\,ds| \leq \frac{|t|^2}{2}\|F(t)\|_2^2 \tag{6.8b}$$

$$\|R_n(t)\|_2 \leq \frac{|t|^n}{n!}\|F(t)\|_2^n \tag{6.8c}$$

Now if $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ on $L \oplus L$, one off-diagonal block of $T(t)$ is

$$P\, e^{it\hat{I}D_1}(1-P) = P\, e^{it\hat{I}D_0}\, T_0(t)\ (1-P)$$
$$= e^{it\hat{I}D_0}\, P\, T_0(t)\ (1-P)$$

which belongs to the Hilbert-Schmidt class, as required.

One interesting aspect of the unitarily implementable C.A.R. Bogoliubov transformations is the fact that the new vacuum $\psi_0' = U^{-1}\psi_0$ may have an even or odd charge, as may be seen for example from equation (6.2) or [63]. Carey, Hurst and O'Brien have recently shown that two C.A.R. Bogoliubov transformations $T_1$ and $T_2$ are connected by a continuous curve of implementable transformations if and only if the two corresponding vacua $\psi_0'(1)$ and $\psi_0'(2)$ have the same charge [81]. This work enabled a full resolution of the converse to the Lundberg criterion.

<u>Theorem</u> (6.9): Suppose that $T(t) = e^{itD_2}$ is a *uniformly* continuous one parameter group. Then $T(t)$ is unitarily implementable for all t if and only if $D_2$ is bounded and $P\, D_2(1-P) \in C_2$.

An explicit example in [81] shows that a strongly (but not *uniformly*) continuous one-parameter group of implementable C.A.R. Bogoliubov transformations may have a generator $D_2$ which has an off-diagonal block outside of $C_2$.

Since a Bogoliubov transformation has a matrix $T$ on $L \oplus L$ of the structure $T = \begin{pmatrix} T_1 & T_2 \\ T_2^* & T_1^* \end{pmatrix}$ (as in (4.6)), the group of Bogoliubov transformations leaves invariant the real subspace $L \oplus L^*$ of $L \oplus L$, consisting of elements of the type $\psi \oplus \psi^*$ with $\psi \in L$. Using the transformation $P : \begin{pmatrix} \psi \\ \psi^* \end{pmatrix} \rightarrow 2^{-\frac{1}{2}} \begin{pmatrix} \psi + \psi^* \\ -i(\psi-\psi^*) \end{pmatrix}$ of (4.25), we may identify the group of Bogoliubov transformations with the group of symplectic (preserving $\mathrm{Im}(\psi_1,\psi_2)$) or orthogonal (preserving $\mathrm{Re}(\psi_1,\psi_2)$) transformations on $L$, considered as a real space. Following the results of Segal [82] and Shale [83], it soon became well-known that such a symplectic transformation C is unitarily implementable, if and only if $C^T C - I \in C_2$. For the C.A.R. case, the analogous result, due to Shale and Stinespring [84], is that such an orthogonal transformation C is unitarily implementable if and only if $GC - CG \in \mathcal{C}_2$, where $G = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. These conditions are quoted everywhere in the literature on this subject but, with the exception of Seiler's discussion of the Klein-Gordon field with an external classical potential [85], it is very hard to find a proof of the equivalence of these and the previously mentioned criterion $P\,\mathcal{T}(1-P) \in \mathcal{C}_2$. With the notation that we have set up here and in section 4, this task will be straight-forward but not trivial.

Proposition (6.10): Let $\mathcal{T}$ be the matrix (on $\mathcal{L} \oplus \mathcal{L}$) for a C.C.R. Bogoliubov transformation. Then $P\,\mathcal{T}(1-P) \in \mathcal{C}_2$ if and only if $\rho(\mathcal{T})^T \rho(\mathcal{T}) - I \in \mathcal{C}_2$, with $\rho$ defined in (4.26).

Proof: Let $\mathcal{T} = \begin{pmatrix} T_1 & T_2 \\ T_2^* & T_1^* \end{pmatrix}$

The pseudo-unitarity $\mathcal{T}^\dagger \hat{I}\, \mathcal{T} = \hat{I} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

implies $$T_1^\dagger T_1 = T_2^T T_2^{T\dagger} + I \tag{6.11a}$$

and $$T_1^\dagger T_2 = T_2^T T_1^* \tag{6.11b}$$

From (6.11a), it is clear that $T_1$ has a bounded inverse [38]. (This contrasts with the C.A.R. case, in which $T_1$ may have a kernel, the dimension of which is related to the charge of the vacuum [63,81]). Suppose

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = \rho(\mathcal{T}) = P\,\mathcal{T}\,P^\dagger,$$

satisfies $$C^T C - I \in \mathcal{C}_2 \,. \tag{6.12}$$

$$C^T C - I = \begin{pmatrix} C_1^T C_1 + C_3^T C_3 - I & C_1^T C_2 + C_3^T C_4 \\ C_2^T C_1 + C_4^T C_3 & C_2^T C_2 + C_4^T C_4 - I \end{pmatrix} \tag{6.13}$$

By (6.12-13)

$$[C_1^T C_1 + C_3^T C_3 - I] - i[C_1^T C_2 + C_3^T C_4] - i[C_2^T C_1 + C_4^T C_3] - [C_2^T C_2 + C_4^T C_4 - I] \in \mathcal{C}_2 \tag{6.14}$$

The symplectic condition $C^T G C = G$ implies

$$C_3^T C_1 = C_1^T C_3 \tag{6.15a}$$

$$C_1^T C_4 - C_3^T C_2 = I \tag{6.15b}$$

$$C_4^T C_1 - C_2^T C_3 = I \tag{6.15c}$$

$$C_4^T C_2 = C_2^T C_4 \tag{6.15d}$$

From (6.15) and the relationship (4.28) between C and $T$, the expression in (6.14) equals $4\, T_1^T T_2^*$.

Thus, $\quad T_1^T T_2^* \in \mathcal{C}_2$

$\Rightarrow \quad T_2^* \in \mathcal{C}_2$ (since $T_1$ has a bounded inverse)

$\Rightarrow \quad T_2 \in \mathcal{C}_2$ , as required.

Conversely, suppose $T_2 \in \mathcal{C}_2$ .

Then $T_1$ is bounded by (6.11a).

$\Rightarrow \quad T_1^T T_2 \in \mathcal{C}_2$

$$\Rightarrow [C_1^T C_1 + C_3^T C_3 - I] - [C_2^T C_2 + C_4^T C_4 - I] - i[C_1^T C_2 + C_3^T C_4] - i[C_2^T C_1 + C_4^T C_3] \in \mathcal{C}_2 \tag{6.16}$$

Similarly, we have, using (6.15),

$$T_2^\dagger T_2 = [C_1^T C_1 + C_3^T C_3 - I] + [C_2^T C_2 + C_4^T C_4 - I] + i[C_1^T C_2 + C_3^T C_4] - i[C_2^T C_1 + C_4^T C_3] \in \mathcal{C}_2 \tag{6.17}$$

The addition or subtraction of the Hilbert-Schmidt operators given in (6.16,17) must result in another Hilbert-Schmidt operator, whence it follows that the four bracketed terms in (6.17) are separately Hilbert-Schmidt. These four terms are just the four operator-valued entries in the matrix for $C^T C - I$.

Proposition (6.18): Let $T = \begin{pmatrix} T_1 & T_2 \\ T_2^* & T_1^* \end{pmatrix}$ be the matrix (on $L \oplus L$) for a C.A.R. Bogoliubov transformation. Then $P\,T(1-P) \in C_2$ if and only if $G\,\rho(T) - \rho(T)G \in C_2$.

Proof:

$$G\,\rho(T) - \rho(T)G \in C_2$$

$$\Leftrightarrow \quad G\,P\,T\,P^{\dagger} - P\,T\,P^{\dagger}\,G \in C_2$$

$$\Leftrightarrow \quad P^{\dagger}G\,P\,T - T\,P^{\dagger}G\,P \in C_2 \quad \text{(since P is unitary)}$$

$$\Leftrightarrow \quad \hat{I}\,T - T\,\hat{I} \in C_2 \quad (\text{since } P^{\dagger}G\,P = i\,I)$$

$$\Leftrightarrow \quad \begin{pmatrix} 0 & T_2 \\ -T_2 & 0 \end{pmatrix} \in C_2$$

$$\Leftrightarrow \quad T_2 \in C_2$$

The work of Carey, Hurst and O'Brien [81] on the topological structure of the group of unitarily implementable C.A.R. Bogoliubov transformations, makes use of the fact that $T$ is unitary on $L \oplus L$. However, in the case of the C.C.R. Bogoliubov transformations, $T$ is pseudo-unitary. Nevertheless, it can be shown that if $T$ is unitarily implementable, then $T$, as well as being pseudo-unitary, belongs to the group $U_2(L \oplus L)$ of operators on $L \oplus L$ which are unitary, modulo the Hilbert-Schmidt class.

Proposition (6.19): Let $T = \begin{pmatrix} T_1 & T_2 \\ T_2^* & T_1^* \end{pmatrix}$ be the matrix for a unitarily implementable C.C.R. Bogoliubov transformation. Then $T_1 = U + K$, with U a unitary operator and K a Hilbert-Schmidt operator on $L$.

Proof: $T_2$ must be a Hilbert-Schmidt operator. $T_2 \in C_2(L)$

From (6.11a), $T_1^{\dagger}T_1 - I = T_2^T T_2^{T\dagger} \in C_2(L)$

That is, $(|T_1| - I)(|T_1| + I) = T_2^T T_2^{T\dagger}$

$$\Rightarrow \quad |T_1| - I = T_2^T T_2^{T\dagger}\,(|T_1| + I)^{-1} \in C_2(L) \tag{6.20}$$

(since $|T_1| + I$ must have a bounded inverse)

Now applying the theorem of polar decomposition, $T_1 = |T_1| U$ , (6.21)

where U is a uniquely determined isometric operator.

In fact, U must be unitary, since by (6.11a), $T_1$ is invertible.

Substituting (6.20) in (6.21),

$$T_1 = U + K$$

where $K = T_2^T T_2^{T\dagger} (|T_1| + I)^{-1} U \in C_2(L)$

Corollary (6.22): Under the conditions specified in proposition (6.19), $\mathcal{T} = \mathcal{U} + \mathcal{K}$, where $\mathcal{U}$ is unitary and $\mathcal{K}$ is a Hilbert-Schmidt operator on $L \oplus L$.

Proof: Take $\mathcal{U} = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix}$ ,

and $\mathcal{K} = \begin{pmatrix} K & T_2 \\ T_2^* & K^* \end{pmatrix}$ with U and K the unitary and Hilbert-Schmidt operators established in proposition (6.19).

If the Bogoliubov transformations are restricted to be unitarily implementable, then this naturally restricts the class of Hamiltonians to which a given Hamiltonian may be euqivalent. For example, for a free Bose field, $D = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, with A positive definite.

Proposition (6.23): Suppose that a Bose field has Hamiltonian $H = \frac{1}{2} \underline{\alpha}^\dagger D \underline{\alpha}$ , with $D = \begin{pmatrix} K & L \\ L^* & M \end{pmatrix}$ . (That is,

$$H = \int [K(\mu,\nu)a^\dagger(\mu)a(\nu) + L(\mu,\nu)a^\dagger(\mu)a^\dagger(\nu) + L(\mu,\nu)^* a(\mu)a(\nu)$$
$$+ M(\mu,\nu)a(\mu)a^\dagger(\nu)] \, d\sigma(\mu) \, d\sigma(\nu) \quad ,$$

where $K(\mu,\nu)$ and $M(\mu,\nu)$ are the integral kernels of self-adjoint operators K and M, $L(\mu,\nu)$ is the integral kernel of a symmetric operator L and $d\sigma(\mu)$ is the spectral measure derived from some complete set of commuting self-adjoint operators on single particle space $L$.) If H is unitarily equivalent to the Hamiltonian of a free linear Bose field, then

(1) K is invertible

(2) $L^* K^{-1} \in C_2(L)$

and (3) $M K^{-1} \in C_2(L)$

Proof: There must exist a Bogoliubov transformation

$$T = \begin{pmatrix} T_1 & T_2 \\ T_2^* & T_1^* \end{pmatrix}; \quad T_2 \in C_2(L) \text{ , such that}$$

$$D = T^\dagger D_0 T \text{ , with } D_0 \text{ having the form } D_0 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}; \ A > 0$$

$$\text{Now } T^\dagger D_0 T = \begin{pmatrix} T_1^\dagger A T_1 & T_1^\dagger A T_2 \\ T_2^\dagger A T_2 & T_2^\dagger A T_2 \end{pmatrix} = D = \begin{pmatrix} K & L \\ L^* & M \end{pmatrix}$$

(1) Since $T_1$ is invertible by (6.11a) and $A > 0$, K is invertible.

(2) $L^* K^{-1} = T_2^\dagger A T_1 T_1^{-1} A^{-1} T_1^{-1\dagger}$

$= T_2^\dagger T_1^{-1\dagger} \in C_2(L)$, since $T_2 \in C_2(L)$

and $T_1^{-1}$ is bounded, by (6.11a).

(3) $M K^{-1} = T_2^\dagger A T_2 T_1^{-1} A^{-1} T_1^{-1\dagger}$

$\in C_2(L)$, since $T_2 T_1^{-1} \in C_2(L)$

$A T_2 T_1^{-1} A^{-1} \in C_2(L)$ (by similarity),

$T_2 \in C_2(L)$ and $T_1^{-1}$ is bounded.

# CHAPTER II - ALGEBRAIC QUANTIZATION

## SECTION 7 - HEURISTIC MODE SPACE QUANTIZATION

It is well-known from the theory of homogeneous linear systems of ordinary differential equations with constant coefficients [86] that each solution of the linear Hamiltonian equations can be expressed as an exponential polynomial

$$z_\mu(t) = \sum_k \sum_{\ell=0}^{N_k-1} b_{\mu,k,\ell}\, t^\ell\, e^{is_k t} \tag{7.1a}$$

In (7.1a), the k-summation accounts for all elementary divisors $(s-s_k)^{N_k}$ of $-i\hat{GH} - sI$. Since $\underline{z}(t)$ is determined by its initial value, there are 2N independent parameters among the coefficients $b_{\mu,k,\ell}$. To quantize (7.1a) in the Heisenberg scheme, $z_j(t)$ is replaced by a time dependent operator $Z_j(t)$, while the coefficients b become constant operators B. This straight-forward substitution follows for quadratic Hamiltonians since in this case, the Heisenberg equations have the same form as the classical Hamilton equations, even though the use of Heisenberg's equations is problematical for Hamiltonians of order higher than three [87].

$$z_\mu(t) = \sum_k \sum_\ell B_{\mu,k,\ell}\, t^\ell\, e^{is_k t} . \tag{7.1b}$$

If there is a symplectic transformation which reduces the system to a collection of independent harmonic oscillators, as in the case of a free Bose field, then for each k , $s_k$ is real non-zero and $N_k = 1$. For example, if $\hat{H} = I^{(2)}$ and $H = \frac{1}{2}(q^2 + p^2)$, then the elementary divisors of $iG\hat{H} - sI$ are $s \pm 1$. Then (7.1a) reduces to

$$q(t) = b_{1,1}\, e^{it} + b_{1,2}\, e^{-it} \tag{7.2a}$$

$$p(t) = b_{2,1}\, e^{it} + b_{2,2}\, e^{-it} \tag{7.2b}$$

Since q(t) and p(t) are real variables,

$$b_{1,2} = b^*_{1,1} \quad \text{and} \quad b_{2,2} = b^*_{2,1} \tag{7.3}$$

Now the Hamiltonian equations are

$$\dot{q} = p \quad \text{and} \quad \dot{p} = -q \tag{7.4}$$

Substituting (7.2) in (7.4), we obtain

$$ib_{1,1} = b_{2,1} \quad \text{and} \quad -ib_{1,2} = b_{2,2} . \tag{7.5}$$

By (7.3,7.5), (7.2) is simplified to

$$q(t) = b\, e^{-it} + b^{*}\, e^{it} \tag{7.6}$$

$$p(t) = -ib\, e^{-it} + ib^{*}\, e^{it} \text{ , where } b = b_{1,2} \tag{7.7}$$

or in the quantized version,

$$Q(t) = B^{\dagger}\, e^{it} + B\, e^{-it} \tag{7.8}$$

$$P(t) = iB^{\dagger}\, e^{it} - iB\, e^{-it}$$

Assuming the C.C.R.,

$$[Z_{\mu}(t),\ Z_{\nu}(t)] = -i\, G_{\mu\nu} \ , \tag{7.9}$$

it is well known that (7.8) implies $[B,B^{\dagger}] = 1$, the canonical commutation relation between the lowering and raising operator of a quantum mechanical harmonic oscillator. In exactly the same way, we may deduce that the mode operators for a system of N independent harmonic oscillators obey the Boson commutation relations

$$[B_j,\ B_k^{\dagger}] = \delta_{jk}\, I \ . \tag{7.10}$$

The problem which motivates this section is that the C.C.R. (7.9) may be incompatible with the Boson commutation relations (B.C.R.) among an independent set of mode operators $B_{\mu,k,\ell}$. When this problem arises, it is not clear whether the B.C.R., C.C.R. or neither should be retained. For example, for a charged repulsive "oscillator",

$$H = P^{\dagger}\, P - Q^{\dagger}\, Q \tag{7.11a}$$

$$= \tfrac{1}{2}[(P_1^{\,2} + P_2^{\,2}) - (Q_1^{\,2} + Q_2^{\,2})] \tag{7.11b}$$

with $$Q = 2^{-\frac{1}{2}} (Q_1 - iQ_2) \tag{7.12a}$$

$$P = 2^{-\frac{1}{2}} (P_1 + iP_2) \tag{7.12b}$$

According to (7.1b),

$$Q(t) = A e^{-t} + B e^{t} \quad . \tag{7.13}$$

From Hamilton's equations, $\dot{Q}^{\dagger}(t) = \frac{\partial H}{\partial P^{\dagger}} = P$ .

Therefore, from (7.13).

$$P = -A^{\dagger} e^{-t} + B^{\dagger} e^{t} \quad . \tag{7.14}$$

If we choose to assume the B.C.R.,

$$[A,A^{\dagger}] = I \ , \quad [B,B^{\dagger}] = I$$

and all other commutators among $A, A^{\dagger}, B$ and $B^{\dagger}$ vanish and this results in the following commutator relation:

$$\begin{aligned}[Q_1,Q_2] &= 2^{-1}i[Q + Q^{\dagger}, Q - Q^{\dagger}] \\ &= i[Q^{\dagger},Q] \\ &= i[A^{\dagger} e^{-t} + B^{\dagger} e^{t}, A e^{-t} + B e^{t}] \\ &= -2i \cosh 2t \end{aligned} \tag{7.15}$$

Commutators of the type (7.15) are the finite dimensional prototypes of a-causal non-vanishing commutators among field operators $\phi(x),\phi(x')$ with $(x-x')$ space-like. Sudarshan, Arons and Dhar [88] have used this a-causality in the quantized imaginary mass Klein-Gordon system, to model a tachyon field. In fact, the classical imaginary mass Klein-Gordon system, based on the equation

$$0 = (\partial_t^{\,2} - \nabla^2 - m^2)\ (x) \qquad (\nabla^2 = \text{the Laplacian operator}) \tag{7.16}$$

may be analysed as a set of independent harmonic oscillators together with a set of independent repulsive "oscillators" [89], in the same way as a free massive Klein-Gordon field may be treated as a collection of independent harmonic oscillators (e.g. [46,71]). Schroer [89] quantized this system using the other approach, namely to assume the C.C.R. among $Q_j$ and $P_j$

and to deduce the commutation relations among the mode operators. Assuming

$$0 \;=\; [Q_j,\, Q_k] \;=\; [P_j,\, P_k] \;=\; i\delta_{jk}\, I - [Q_j,\, P_k] \quad , \tag{7.16}$$

(7.13-14) imply

$$iI = [Q(t),\, P(t)] = -e^{-2t}[A,A^\dagger] + [A,B^\dagger] - [B,A^\dagger] + e^{2t}[B,B^\dagger] \tag{7.16a}$$

$$0 \;=\; [Q(t),\, P^\dagger(t)] \;=\; 2[A,B] \tag{7.16b}$$

$$0 \;=\; [Q(t),\, Q^\dagger(t)] = e^{-2t}[A,A^\dagger] + e^{2t}[B,B^\dagger] + [A,B^\dagger] + [B,A^\dagger] \tag{7.16c}$$

From (4.16), we deduce the following commutation relations among the mode operators:

$$0 \;=\; [A,A^\dagger] = [B,B^\dagger] = [A,B] \tag{7.17a}$$

$$\frac{1}{2}\, iI \;=\; [A,B^\dagger] \tag{7.17b}$$

Let us now assume the existence of a normalised cyclic vector $\psi_0$ satisfying

$$A\,\psi_0 \;=\; B\,\psi_0 \;=\; 0 \tag{7.18}$$

We shall interpret linear combinations of states $A^\dagger\psi_0$ and $B^\dagger\,\psi_0$ as belonging to a space $\mathcal{H}^{(1)}$ of single mode states. The modes can no longer be interpreted as single particles because of the explosive behaviour exhibited in (7.13). On single mode space, we then deduce the inner product

$$<A^\dagger\,\psi_0,\; A^\dagger\,\psi_0> \;=\; <\psi_0,\; [A,A^\dagger]\,\psi_0> \;=\; 0 \tag{7.19a}$$

$$<B^\dagger\,\psi_0,\; B^\dagger\,\psi_0> \;=\; 0 \tag{7.19b}$$

$$<A^\dagger\,\psi_0,\; B^\dagger\,\psi_0> \;=\; <\psi_0,\; [A,B^\dagger]\,\psi_0> \;=\; \frac{1}{2}\, i \tag{7.19c}$$

In the basis $(A^\dagger\,\psi_0,\; B^\dagger\,\psi_0)$, the metric for $\mathcal{H}^{(1)}$ is $g = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, which has eigenvalues $\pm\,\frac{1}{2}$. The necessity for indefinite metric in this case arises from the fact that a single mode evolves with complex frequency $\pm\, i$ and therefore the Hamiltonian on single mode space has complex eigenvalues.

$$\begin{aligned} H \;&=\; P^\dagger P - Q^\dagger Q \\ &=\; -2\,[A^\dagger B + B^\dagger A] \;, \quad \text{by (7.13-14)} \end{aligned} \tag{7.20}$$

$$H\,A^\dagger\,\psi_0 = i\,A^\dagger\,\psi_0 \qquad \text{by (7.17).} \tag{7.21}$$

In fact, an operator H which is self-adjoint with respect to an indefinite complex inner product may have an eigenvalue $\alpha \in ¢ \sim R$ only if the corresponding eigenvector $\psi_\alpha$ has zero norm [68,73], as in (7.19a).

$$\begin{aligned} 0 &= <\psi_\alpha, H\,\psi_\alpha> - <H\,\psi_\alpha, \psi_\alpha> \\ &= (\alpha - \alpha^*)\,<\psi_\alpha, \psi_\alpha> . \end{aligned} \tag{7.21}$$

The next question is whether the indefinite metric is necessitated only by complex frequencies. We shall test the simple example $\hat{H} = \hat{K}_6^{(4)}(-1)$, for which the elementary divisors of $-iG\hat{H} - sI$ are $(s \pm a)^2$. Such a system can not be canonically reduced to two independent harmonic oscillators, since the invariant elementary divisors would then be linear.

$$\hat{H} = \hat{K}_6^{(4)}(-1) = \begin{pmatrix} -a^{-2} & 0 & 0 & 1 \\ 0 & -1 & -a^2 & 0 \\ 0 & -a^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

$$H = \frac{1}{2}\,\underline{Z}^T \hat{H} \underline{Z} = -\frac{1}{2}\,(a^{-2} Q_1^2 + Q_2^2) + Q_1 P_2 - a^2 Q_2 P_1 . \tag{7.23}$$

The Heisenberg equations are

$$\begin{aligned} \dot{Q}_1 &= -a^2 Q_1 \\ \dot{P}_2 &= Q_2 + a^2 P_1 \end{aligned} \tag{7.24}$$

and these yield the self-adjoint solutions

$$Q_1(t) = A\,e^{iat} + A^\dagger\,e^{-iat} \tag{7.25a}$$

$$Q_2(t) = -ia^{-1} A\,e^{iat} + ia^{-1} A^\dagger\,e^{-iat} \tag{7.25b}$$

$$P_1(t) = C\,e^{iat} + C^\dagger\,e^{-iat} + a^{-2} At\,e^{iat} + a^{-2} A^\dagger t\,e^{-iat} \tag{7.25c}$$

$$P_2(t) = -iaC\,e^{iat} + ia\,C^\dagger e^{-iat} - ia^{-1} At\,e^{iat} + ia^{-1} A^\dagger t\,e^{-iat} \tag{7.25d}$$

From (7.25),

$$[Q_1(t),\ Q_2(t)] = 0 \Rightarrow [A,A^{\dagger}] \ = \ 0 \tag{7.26}$$

$$[P_1(t),\ P_2(t)] = 0 \Rightarrow 2ia[C,C^{\dagger}] + 2ia^{-1}t([A,C^{\dagger}] + [C,A^{\dagger}]) = 0$$

$$0 = [C,C^{\dagger}] = [A,C^{\dagger}] + [C,A^{\dagger}] \tag{7.27}$$

$$[Q_1(t),\ P_2(t)] = 0 \Rightarrow [A,C] = 0 \quad \text{(using 7.25-2 )} \tag{7.28}$$

$$[Q_1(t),\ P_1(t)] = iI \Rightarrow [A,C^{\dagger}] = \frac{i}{2}\, I \quad \text{(using 7.25-28)}\ . \tag{7.29}$$

Assuming $A\ \psi_0 = B\ \psi_0 = 0$ and using the relations (7.27-29), the Hermitean metric on single mode space, in basis $(A^{\dagger}\ \psi_0,\ C^{\dagger}\ \psi_0)$ again turns out to be $g = \begin{pmatrix} 0 & i/2 \\ -i/2 & 0 \end{pmatrix}$ . In this case, the metric on modal Fock space is indefinite, even though the system is devoid of non-real frequencies. The indefinite metric in this case is necessitated by the fact that the (pseudo) self-adjoint Hamiltonian does not have a complete set of eigenvectors. From (7.23-25),

$$H \ = \ -2a^{-2}\ A^{\dagger}\ A + 2ia(AC^{\dagger} - A^{\dagger}C)$$

$$H\ A^{\dagger}\ \psi_0 \ = \ -2a\ A^{\dagger}\ \psi_0$$

$$H\ C^{\dagger}\ \psi_0 \ = \ -ia^{-2}\ A^{\dagger}\ \psi_0 - 2a\ C^{\dagger}\ \psi_0\ . \tag{7.30}$$

Therefore, on $\mathcal{H}^{(1)}$, H may be represented by the matrix $\begin{pmatrix} -2a & -ia^{-2} \\ 0 & -2a \end{pmatrix}$ which has only one independent eigenvector.

We recall from section 4 that a general quadratic quantum mechanical Hamiltonian $H = \frac{1}{2}\ \underline{Z}^T\ \hat{H}\,\underline{Z}$ may be reviewed as a quadratic Boson Hamiltonian $H = \frac{1}{2}\ \underline{\alpha}^{\dagger}\ \mathcal{D}_1\ \underline{\alpha}$ , with $\mathcal{D}_1 = P^{\dagger}\ \hat{H}\, P$. However, the total number operator $N \ = \ \sum_{j=1}^{N} a_j^{\dagger} a_j$ might not commute with the Hamiltonian. This can be rectified by a Bogoliubov transformation $\underline{\alpha} = \mathcal{T}\ \underline{\gamma}$ $\left(\underline{\gamma} = \begin{pmatrix} \underline{c} \\ \underline{c}^{\dagger} \end{pmatrix}\right)$, only if the Hamiltonian belongs to the same canonical orbit as that of a collection of independent harmonic oscillators. Otherwise, H must contain $C_j\ C_k$ terms which can not commute with $N' = \sum_{j=1}^{N} C_j^{\dagger} C_j$ .

In spite of the possible lack of a time-invariant quasi-particle number operator, it may be possible to define a time-invariant mode number. Assume that each operator $B_{\partial,k,\ell}$ in (7.1b) can be expressed in terms of N independent operators Br and their adjoints $Br^{\dagger}$. Assume the existence of a normalized cyclic vector $\psi_0$ satisfying

$$Br\ \psi_0 = 0 \quad \text{for all } r. \tag{7.31}$$

<u>Definition</u> (7.32): A mode number operator N must satisfy

(a) formal Hermiticity $N^{\dagger} = N$

(b) $N\ \psi_0 = 0$

(c) $[N,Br^{\dagger}] = Br^{\dagger}$ for all $r = 1,\ldots,N$

(i.e. the $Br^{\dagger}$ are creation operators)

(d) $[N,H] = 0$

In the example of (7.20), we may take $N = 2i[A^{\dagger}B - B^{\dagger}A]$ and in the example (7.30), we may take $N = 2i[A^{\dagger}C - C^{\dagger}A]$. Although in each of these examples, a mode number operator exists and $\psi_0$ is an eigenvector corresponding to a real eigenvalue of the Hamiltonian, the single mode space has indefinite metric. We can show that this problem remains whenever the Hamiltonian does not belong to the same canonical orbit as that of a system of independent harmonic oscillators.

<u>Proposition</u> (7.33): Let H be a quadratic Hamiltonian $H = \frac{1}{2}\underline{Z}^T \hat{H} \underline{Z}$, with $Z_{\mu}$ obeying (7.9). Suppose that the formal single mode space for the mode operators of (7.31) has positive definite metric, with respect to which time evolution is unitary. Suppose also that $H\ \psi_0 = \beta\ \psi_0$ for some $\beta \in R$. Then $-iG\hat{H}$ has real diagonal Jordan canonical form, so that the system can be reduced by a symplectic transformation to a collection of independent harmonic oscillators.

<u>Proof</u>: Assume that for some real symmetric $\hat{H}$ of order 2N, there exists a single mode space $\mathcal{H}^{(1)}$ with positive definite metric $g_{ij} = \langle A_i^\dagger \psi_0, A_j^\dagger \psi_0 \rangle_1$. $Q_j(0)$, $P_j(0)$ may be expressed as linear combinations of the mode construction operators $A_j$ and $A_j^\dagger$. This defines a representation of the C.C.R. on the Fock space $\mathcal{F}(\mathcal{H}^{(1)})$ over $\mathcal{H}^{(1)}$.

Define a complex-valued sesquilinear form $\langle.,.\rangle_M$ on real symplectic space $(M,B)$ $(\equiv(R^{2N},G))$ by

$$\langle \underline{\zeta},\underline{\xi}\rangle_M = \langle \sum_{\mu=1}^{2N} \zeta_\mu \, Z_\mu(0)\, \psi_0 \, , \, \sum_{\nu=1}^{2N} \xi_\nu \, Z_\nu(0)\, \psi_0 \rangle_1 \tag{7.34}$$

Assume that $\langle \psi_0, \psi_0 \rangle = 1$. Hence,

$$\begin{aligned}\text{Im}\, \langle \underline{\zeta},\underline{\xi}\rangle_M &= -\frac{i}{2}\,(\langle \underline{\zeta},\underline{\xi}\rangle_M - \langle \underline{\xi},\underline{\zeta}\rangle_M) \\ &= -\frac{i}{2} \sum_{\mu,\nu} \zeta_\mu \, \xi_\nu \, \langle \psi_0, [Z_\mu(0), Z_\nu(0)]\, \psi_0 \rangle \\ &= -\frac{1}{2}\,(\underline{\zeta}, G\, \underline{\xi}) \quad \text{(by the C.C.R.)}\, .\end{aligned} \tag{7.35}$$

where ( , ) is the Euclidean inner product. By (7.34) and positivity of the metric in $\mathcal{H}^{(1)}$, Re. $\langle \underline{\zeta},\underline{\xi}\rangle_M$ must be a positive definite symmetric form.

$$\text{Re.}\, \langle \underline{\zeta},\underline{\xi}\rangle_M = (\underline{\zeta}, \hat{K}\, \underline{\xi}) \quad ; \quad \hat{K} = \hat{K}^T > 0\, . \tag{7.36}$$

Time evolution in the Heisenberg picture is given by

$$Z_\mu(t) = e^{iHt}\, Z_\mu(0)\, e^{-iHt} \quad ; \quad H = \frac{1}{2}\, \underline{Z}(0)^T\, \hat{H}\, \underline{Z}(0) \tag{7.37}$$

$$= \sum_{\nu=1}^{2N} \left[ e^{-G\hat{H}t} \right]_{\mu\nu} Z_\nu(0) \tag{7.38}$$

(7.38) reflects the simple correspondence between the Heisenberg equations and the Hamilton equations for quadratic Hamiltonians. It may be proved directly [42], using the commutation relations or may be shown to be equivalent to (6.1), the first similarity rule of Friedrichs for Bogoliubov transformations.

Since $e^{-iHt}\,\psi_0 = e^{-i\beta t}\,\psi_0$ and because time evolution in $\mathcal{H}^{(1)}$ is unitary,

$$\frac{d}{dt} < \sum_{\mu=1}^{2N} \zeta_\mu \, Z_\mu(t)\, \psi_0,\ \sum_{\nu=1}^{2N} \xi_\nu \, Z_\nu(t)\, \psi_0 >_1 \;=\; 0 \tag{7.39}$$

However, from (7.38) ,

$$\sum_{\alpha=1}^{2N} \zeta_\alpha(0)\, Z_\alpha(t) = \sum_{\alpha=1}^{2N} \zeta_\alpha(0) \left[e^{-G\hat{H}t}\right]_{\alpha\beta} Z_\beta(0)$$

$$= \sum_{\alpha,\beta=1}^{2N} \left[e^{(-G\hat{H})^T t}\right]_{\beta\alpha} \zeta_\alpha(0)\, Z_\beta(0)$$

$$= e^{\hat{H}Gt}\, \underline{\zeta}(0)\cdot \underline{Z}(0) \tag{7.40}$$

Therefore, by (7.39) ,

$$\frac{d}{dt} <e^{\hat{H}Gt}\, \underline{z}(0).\underline{Z}(0)\, \psi_0,\ e^{\hat{H}Gt}\, \underline{z}(0)\cdot\underline{Z}(0)\, \psi_0>_1 = 0$$

$$\Rightarrow \quad \frac{d}{dt} <e^{\hat{H}Gt}\, \underline{z}(0),\ e^{\hat{H}Gt}\, \underline{z}(0)>_M = 0$$

$$\Rightarrow \quad \frac{d}{dt} (e^{\hat{H}Gt}\, \underline{z}(0),\ \hat{K}\, e^{\hat{H}Gt}\, \underline{z}(0)) = 0$$

$\Rightarrow$ $\hat{H}\,G$ is skew-symmetric with respect to $(\cdot,\ \hat{K}\ \cdot)$

$$\Rightarrow \quad G\,\hat{H}\,\hat{K} = \hat{K}\,\hat{H}\,G\,. \tag{7.41}$$

Condition (7.41) greatly restricts the form which $\hat{H}$ may take. Since $\hat{K}$ is positive definite, it may be diagonalised by a change of symplectic basis, according to theorem (1.26). There exists an ordered symplectic basis $(\underline{C}_1.,,,.\underline{C}_{2N})$ for the symplectic space $M = (R^{2N}, G)$, such that for $j = 1,\ldots,N$ ,

$$\underline{C}_j = 2^{-\frac{1}{2}}(\underline{w}_j + \underline{w}_j^*)$$

$$\underline{C}_{j+N} = -2^{-\frac{1}{2}}\, i(\underline{w}_j - \underline{w}_j^*) \tag{7.42}$$

where (1) $\underline{w}_j = \begin{pmatrix} \underline{u}_j \\ \underline{v}_j \end{pmatrix}$ is a complex eigenvector of $iG\hat{K}$ corresponding to eigenvalue $\alpha_j > 0$ .

(2) $(\underline{w}_j, G\,\underline{w}_j^*) = -(\underline{u}_j, \underline{v}_j^*) + (\underline{u}_j^*, \underline{v}_j) = -i$ .

In the basis depicted in Whittaker's theorem,

$$\hat{K} = \text{diag. } [\alpha_1, \ldots, \alpha_N, \alpha_1, \ldots, \alpha_N] .$$

Now let $W_\alpha$ be the complex eigenspace for eigenvalue $\alpha$ of $iG\hat{K}$. $W_\alpha$ has basis $\{\underline{w}_{\alpha,1}; \ldots; \underline{w}_{\alpha,M}\}$ , where the $\underline{w}_{\alpha,k}$ are the $\underline{w}_j$ of (7.42), provided $\alpha_j = \alpha$ . If $W_{-\alpha}$ denotes the eigenspace for eigenvalue $-\alpha$, then $W_{-\alpha} = W_\alpha^*$ .

Lemma (7.43): Let $\Delta$ be the spectrum of $iG\hat{K}$. The for all $\alpha \in \Delta$ , $W_\alpha$ is $\hat{H}G$-invariant.

Proof: Let

$$\underline{w}_\alpha \in W_\alpha$$

$$i\,G\,\hat{K}\,\underline{w}_\alpha = \alpha\,\underline{w}_\alpha$$

Now

$$\begin{aligned} iG\hat{K}\,\hat{H}G\,\underline{w}_\alpha &= iG(G\,\hat{H}\,\hat{K})\,\underline{w}_\alpha \qquad \text{by (7.41)} \\ &= -\,i\,\hat{H}\,\hat{K}\,\underline{w}_\alpha \\ &= \hat{H}\,G(iG\hat{K})\,\underline{w}_\alpha \\ &= \alpha\,\hat{H}\,G\,\underline{w}_\alpha \end{aligned}$$

$$\Rightarrow \quad \hat{H}\,G\,\underline{w}_\alpha \in W_\alpha .$$

Now define

$$V_\alpha = W_\alpha \oplus W_{-\alpha} .$$

From lemma (7.43), $V_\alpha$ is $\hat{H}G$-invariant. From (7.42), $V_\alpha$ has an ordered real symplectic basis

$$(\underline{C}_{\alpha,1}; \ldots; \underline{C}_{\alpha,2M}) ,$$

obeying

$$(\underline{C}_{\alpha,j}\, ,\, G^{(2M)}\,\underline{C}_{\alpha,k}) = G_{jk} .$$

$V_\alpha$ is an $\hat{H}G$-invariant symplectic subspace of $(M, B)$ on which $\hat{K} = \alpha I$ .

Then by (7.41)

$$\hat{H}\,G - G\,\hat{H} = 0 \quad \text{on} \quad V_\alpha \tag{7.44}$$

$$\Rightarrow \quad G^T\,\hat{H}\,G = \hat{H}\,. \tag{7.45}$$

The symplectic transformation $G$ exchanges the role of $q_j$ and $p_j$. Therefore, (7.45) is equivalent to the Born-Green reciprocity [90] of the classical Hamiltonian system defined by the restriction of $\hat{H}$ to $V_\alpha$, in basis (7.42). In this basis, we conclude immediately that $iG\hat{H}$ is Hermitean.

$$(i\,G\,\hat{H})^\dagger = i\,\hat{H}\,G$$

$$= i\,G\,\hat{H} \quad \text{by (7.44)}\,.$$

Therefore, $iG\hat{H} - sI$, restricted to $V_\alpha$, has elementary divisors $(S \pm \alpha)$, $\alpha \in R$. However, by (7.42), the whole symplectic space $M$ decomposes into a symplectic direct sum

$$M = \underset{\alpha \in \Delta^+}{\oplus} V_\alpha\,,$$

where

$$\Delta^+ = \Delta \cap R^+\,.$$

Therefore, $iG\hat{H}$ has real diagonal Jordan canonical form. The only non-trivial indecomposable blocks for $\hat{H}$ allowing this possibility are $\pm K_5^{(2)}(-1)$, which must lie in the same canonical orbit as $\hat{H} = \pm\text{diag.}[s_j^{\,2}, 1]$, corresponding to $H = \pm\frac{1}{2}(p_j^{\,2} + s_j^{\,2}\,q_j^{\,2})$, which generates simple harmonic motion. We have now established proposition (7.33) by exhibition a symplectic basis in which $G\hat{H}$ is skew-symmetric.

$$\exists\, C \in Sp(2N,R),\ G\,C^T\,\hat{H}\,C \quad \text{is skew-symmetric}\,.$$

$$\Rightarrow C^{-1}\,G\,\hat{H}\,C \quad \text{is skew-symmetric}$$

$$(\text{since} \quad C \in Sp(2N,R) \Rightarrow G\,C^T = C^{-1}\,G).$$

To relate this to the theory of classical stability, we quote the following:

<u>Theorem</u> (7.46): (Sz. Nagy, Daleckii and Krein [20])

Let $A$ be a bounded operator on some Hilbert space $H$. In order that the commutative group $K = \{e^{At} ; t \varepsilon R\}$ is bounded, it is necessary and is sufficient that A is similar to a skew Hermitean operator B.

In order for the linear Hamiltonian system (1.4), to be stable about the origin, it is necessary and sufficient that the commutative group $\{e^{-G\hat{H}t}\}$ is bounded on Euclidean space. By theorem 7.46, this is equivalent to $-G\hat{H}$ being similar to a skew-symmetric matrix.

## SECTION 8 - SEGAL QUANTIZATION OF LINEAR SYSTEMS

In section 7, we heuristically constructed the model Fock space, simply by postulating the existence of a cyclic vector $\psi_0$ which is annulled by each mode annihilation operator. When each classical mode is stable, the modal Fock space has positive definite metric. A Hilbert space $F(H)$ with all the properties of the modal Fock space can then be constructed by the well-known method of Cook [37].

$$F(H) = \bigoplus_{n=0}^{\infty} H^{(n)} , \tag{8.1}$$

where $H^{(0)} = C$ and the space $H^{(n)}$ consists of the closure of all symmetrized tensor products $S_n \underline{\phi}_1 \otimes \ldots \otimes \underline{\phi}_n$ of n vectors of $H$. Here, $S_n$ is the symmetrisation operator

$$S_n = \frac{1}{n!} \sum_{\sigma \epsilon P_n} \sigma ,$$

where $P_n$ is the group of permutations of the n symbols 1,...,n. The construction operators are then defined by

$$a(\phi)\, S_n\, \underline{\phi}_1 \otimes \ldots \otimes \underline{\phi}_n = n^{\frac{1}{2}}\, S_n \,\langle \underline{\phi}, \underline{\phi}_1 \rangle_1\, \underline{\phi}_2 \otimes \ldots \otimes \underline{\phi}_n \quad \text{for } n \geq 1$$

$$a(\phi)\, (H^{(0)}) = \{0\}$$

$$a^{\dagger}(\phi_{n+1})\, S_n\, \underline{\phi}_1 \otimes \ldots \otimes \underline{\phi}_n = (n+1)^{\frac{1}{2}}\, S_{n+1}\, \underline{\phi}_1 \otimes \ldots \otimes \underline{\phi}_n \otimes \underline{\phi}_{n+1} \tag{8.2}$$

In Cook's construction of Fock space for a Fermi-Dirac field, the n-particle states $\psi^{(n)} \epsilon H^{(n)}$ are anti-symmetric, rather than symmetric. The annihilation operators $a(\underline{\zeta})$ and their adjoints, the creation operators $a^{\dagger}(\underline{\zeta})$ satisfy

$$[a(\underline{\zeta}), a^{\dagger}(\underline{\xi})]_{\pm} = \langle \underline{\zeta}, \underline{\xi} \rangle_1 \, I \tag{8.3}$$

$$[a(\underline{\zeta}), a(\underline{\xi})]_{\pm} = [a^{\dagger}(\underline{\zeta}), a^{\dagger}(\underline{\xi})] = 0 ,$$

where + and - denote commutation or anti-commutation appropriate to Bose statistics or to Fermi-Dirac statistics respectively.

For systems of infinite degrees of freedom, there exist uncountably many inequivalent irreducible representations of the C.C.R. However, it would be fair to say that the only representations which have found universal favour among particle physicists are those which are unitarily equivalent to the Fock representation.

In a workable scheme of quantization, a physically appropriate representation of the C.C.R. should be constructed in some well-defined way from the classical representative space. The bridge between classical mechanics and appropriate representations of the C.C.R. is usually associated with the work of Segal [91]. Given the classical phase space $(M,B)$, quantization in accordance with Bose-Einstein statistics involves the construction of a Weyl system:

$$M \ni \underline{\zeta} \rightarrow W(\underline{\zeta}) \ , \tag{8.4}$$

with $W(\underline{\zeta})$ unitary operators on a complex Hilbert space $(F,\langle\cdot,\cdot\rangle)$ satisfying the canonical commutation relations in Weyl form

$$W(\underline{\zeta})\, W(\underline{\xi}) \ = \ \exp[\tfrac{1}{2}i\, B(\underline{\zeta},\underline{\xi})]\, W(\underline{\zeta} + \underline{\xi}) \ , \tag{8.5}$$

and weak ray continuity of $\{W(t\underline{\zeta}) \ ; \ t \in R\}$ for fixed $\underline{\zeta}$. The Fermion analogue of the above involves the construction of a Hilbert space representation of the canonical anti-commutation relations in Shale-Stinespring form [84]:

$$[R(\underline{\zeta}),\, R(\underline{\xi})]_{+} \ = \ S(\underline{\zeta},\underline{\xi})I \ , \tag{8.6}$$

with $\underline{\zeta}$ and $\underline{\xi}$ arbitary elements of a real Hilbert space $(V,S)$.

In the second quantization procedure, $B(\underline{\zeta},\underline{\xi})$ (or $S(\underline{\zeta},\underline{\xi})$) appears as a scalar multiple of $\mathrm{Im}\langle\underline{\zeta},\underline{\xi}\rangle_1$ (or $\mathrm{Re}\langle\underline{\zeta},\underline{\xi}\rangle_1$), with $\underline{\zeta}$ and $\underline{\xi}$ belonging to single particle space, a complex Hilbert space $(H, \langle\cdot,\cdot\rangle_1)$ and $C(t)$ is unitary. Although for infinite classical degrees of freedom, there is a continuum of inequivalent representations of (8.5) or (8.6), there is a natural way of defining a unique abstract C* algebra $A$ of second quantized operators. In the Boson case, *A* is the Weyl C* algebra of conceptual observables which was defined by Segal [91,92] while in the Fermion case it is the Clifford C* algebra over *H* which was defined by Shale and Stinespring [84]. The minimal C* algebra containing unitary operators $W(\underline{\zeta})$, satisfying (8.5), was determined by Slawny [93].

Given that a complex Hilbert space structure on the classical space *V* allows algebraic second quantization to run so naturally, the next question is how to select a physically appropriate Hilbert space representation of *A* in a direct abstract manner. It is now well-known [94] that uniqueness of a linear functional E on *A* can be guaranteed by the following conditions:

(8.7a) that the generating function $E(W(\underline{\zeta}))$ is continuous;

(8.7b) that E is time-invariant; $E(W(\underline{\zeta}(t))) = E(W(\underline{\zeta}(o)))$;

(8.7c) that in the G.N.S. representation $\pi_E$ of *A* induced by E, there exists a continuous one-parameter unitary group U(t) with positive generator and implementing time evolution in *A*,

$$\pi_E \circ W(\underline{\zeta}(t)) = U(t)\,[\pi_E \circ W(\underline{\zeta}(0))]\,U^{-1}(t)\ .$$

Also, it was shown by Chaiken [95] that condition (8.7c) is equivalent to the existence of a self-adjoint number operator N satisfying

$$e^{itN}\, W_E(\underline{\xi})\, e^{-itN} = W_E(e^{it}\underline{\xi}) \qquad (8.8)$$

($W_E(\underline{\xi})$ are the bounded Weyl operators $\pi_E \circ W(\underline{\xi})$), and having the set of non-negative integers as its spectrum. This settles the unique specification of the vacuum state functional.

In the Gelfand-Naimark-Segal representation $\pi_E$ given by the unique functional E satisfying (8.7), there exists a cyclic vector $\psi_o \in F$ such that $E(A) = \langle\psi_o, \pi_E(A)\,\psi_o\rangle$. Then $\psi_o$ may be identified with the vacuum in Fock space $F(H)$ over *H*, which has a consistent physical interpretation.

For Bose fields, the generators of the continuous ray $W_E(t\underline{\xi})$ are identified with the unbounded essentially self-adjoint Segal field operators (section X.7 of [38])

$$\Phi(\underline{\zeta}) \quad = \quad 2^{-\frac{1}{2}}\,[a(\underline{\zeta}) + a^{\dagger}(\underline{\zeta})] \tag{8.9}$$

which satisfy

$$[\Phi(\underline{\zeta}),\Phi(\underline{\xi})]_- = i\ \mathrm{Im}\langle\underline{\zeta},\underline{\xi}\rangle_1 \ \text{(on a dense domain in } F) \tag{8.10}$$

For $\alpha \in R$,

$$\exp[i\alpha\Phi(\underline{\zeta})]\ \exp[i\alpha\Phi(\underline{\xi})] = \exp[-\tfrac{1}{2}\, i\,\alpha^2\ \mathrm{Im}\langle\underline{\zeta},\underline{\xi}\rangle_1]\ \exp[i\alpha\Phi(\underline{\zeta}) + i\alpha\Phi(\underline{\xi})] \tag{8.11}$$

Thus, $W(\underline{\zeta}) \supset \exp[i\alpha\Phi(\underline{\zeta})]$ provides a Weyl system, conforming with (8.5), provided

$$\mathrm{Im}\ \langle\underline{\zeta},\underline{\xi}\rangle_1 \quad = \quad -\ \alpha^{-2}B(\underline{\zeta},\underline{\xi}). \tag{8.12}$$

For Fermi-Dirac fields, the operators $\Phi(\underline{\zeta})$, constructed as in (8.9), are bounded and self-adjoint, and satisfy

$$[\Phi(\alpha\underline{\zeta}),\ \Phi(\alpha\underline{\xi})]_+ \quad = \quad \alpha^2\ \mathrm{Re}\langle\underline{\zeta},\underline{\xi}\rangle_1 \tag{8.13}$$

$R(\underline{\zeta}) = \Phi(\alpha\underline{\zeta})$ conforms with (8.6), provided

$$\mathrm{Re}\ \langle\underline{\zeta},\underline{\xi}\rangle_1 \quad = \quad \alpha^{-2}S(\underline{\zeta},\underline{\xi})\ . \tag{8.14}$$

Since the above procedure of rigorous second quantization runs so smoothly, we are motivated to ask which classical systems ($M$, B (or S), C(t)) can be viewed as complex Hilbert spaces ($H$, $\langle\,,\,\rangle_1$) with unitary dynamics C(t) and with the inner product satisfying (8.12) or (8.14). On classical space $M$, there must exist a complex structure J, a real linear operator satisfying $J^2 = -I$. Since $\langle\underline{\zeta},\underline{\xi}\rangle_1$ must be a J-sesquilinear form,

$$\langle J\underline{\zeta},J\underline{\xi}\rangle_1 \quad = \quad \langle\underline{\zeta},\underline{\xi}\rangle_1\ . \tag{8.15}$$

From (8.12, 8.14), this implies that J must be symplectic in the Boson case and orthogonal in the Fermion case. The J-sesquilinearity condition $\langle\underline{\xi},J\underline{\zeta}\rangle_1 \ = \ i\langle\underline{\xi},\underline{\zeta}\rangle_1$ then determines $\langle\underline{\xi},\underline{\zeta}\rangle_1$ uniquely as

$$\langle\underline{\xi},\underline{\zeta}\rangle_1 = -\alpha^{-2}B(\underline{\xi},J\underline{\zeta}) - \alpha^{-2}iB(\underline{\xi},\underline{\zeta}) \quad \text{(Boson case)}, \tag{8.16a}$$

$$\langle\underline{\xi},\underline{\zeta}\rangle_1 \ = \ \alpha^{-2}S(\underline{\xi},\underline{\zeta}) - \alpha^{-2}iS(\underline{\xi},J\underline{\zeta}) \quad \text{(Fermion case)}\ . \tag{8.16b}$$

In order that the above sesquilinear form extends to a positive definite inner product, it is necessary that $-\alpha B(\underline{\xi},J\underline{\xi})$ is strictly positive (Boson case) or that $\alpha > 0$ (Fermion case). We require that C(t) is a one parameter group of unitary transformations on $\mathcal{H} = (\mathcal{M}, J, \langle\cdot,\cdot\rangle_1)$. This condition reduces to $[C(t),J] = 0$.

Given that in $\mathcal{M}$, there are no non-trivial invariant vectors v such that C(t)v = v and that C(t) has a positive generator on $\mathcal{H}$, the uniqueness of J, when it exists, is known from the work of Weinless [96]. For the Boson case, this result was made more explicit by Kay [97]. More recently, Sparzani and Gallone [98] have proven that if there exists any group $\{C(g);\ g \in G\}$ of symplectic transformations which is an irreducible G-representation on the real space such that C(g) are unitary on complex space $\mathcal{H} = (\mathcal{M}, J, \langle\cdot,\cdot\rangle_1)$, then J is unique. For example, the real solution space of the Klein-Gordon equation $(\partial_t^2 - \nabla^2 + m^2)\ \phi(\underline{x},t) = 0$ carries an irreducible unitarizable representation of the inhomogeneous restricted Lorentz group. Therefore, the unitarization can be effected by only one complex structure J on $\mathcal{M}$. The existence of J for the Klein-Gordon system is related to the unambiguous separation of $\mathcal{M}$ into particle solutions and anti-particle solutions [91]. However, in some cases, the complex structure J does not exist. For example, it has long been known [96] that unitarization of the imaginary mass Klein-Gordon equation (7.16) is impossible. Although the existence or non-existence of J has been settled for isolated cases, the existence problem still requires a general resolution, as pointed out quite recently by Gallone and Sparzani [98]. We aim, in this thesis, to peg back the current lead of uniqueness results over existence results. To this end, we first discuss, in section 9, the problem for finite degrees of freedom. This restriction enables a simple and complete characterization of the classical dynamical systems which are and are not unitarizable. When J is unique, we derive an expression which enables its explicit construction. To see how this can be extended to infinite degrees of freedom, it proves to be illuminating to first find an expression for J in the Fermion case. This is done in Section 11. However, before that, it is necessary to discuss the Fermion analogues of classical mechanics in Section 10.

## SECTION 9 - COMPLEX STRUCTURES FOR SYMPLECTIC DYNAMICS

Given a 2N-dimensional symplectic space $(M,B)$, a suitable choice of basis $\{\underline{e}_\mu\}$ reduces the symplectic form $B(\underline{\zeta},\underline{\xi})$ to $(\underline{\zeta},G\underline{\xi})$, with $G = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. We wish to find a complex structure $J$ on $M$ which allows $M$ to be viewed as a complex Hilbert space $H^{(1)}$, to be identified with the single particle space in the Cook construction. On $M$, there will be specified a symplectic dynamical flow

$$\underline{\xi}(t) = e^{-G\hat{H}t}\,\underline{\xi}(o) \tag{9.1}$$

The unitarizing complex structure J, a linear operator on real space $M$, must satisfy

$$J^2 = -I \tag{9.2a}$$

$$J^T G J = G \qquad \text{(from (8.12,15))} \tag{9.2b}$$

$$-GJ > 0 \qquad \text{(positivity of the metric (8.16a))} \tag{9.2c}$$

$$[J,G\hat{H}] = 0 \qquad \text{(unitarity of } e^{-G\hat{H}t} \text{ on } H^{(1)}) \tag{9.2d}$$

In the Cook construction,

$$\underline{\xi}(t) = a^\dagger(\underline{\xi}(t))\,\psi_o$$

$$= 2^{\frac{1}{2}}\,\Phi(\underline{\xi}(t))\,\psi_o \qquad \text{by (8.9)}$$

$$= 2^{\frac{1}{2}} \sum_{\mu,\nu=1}^{2N} [e^{-G\hat{H}t}]_{\mu\nu}\, \xi^\nu(0)\, \Phi(\underline{e}_\mu)\,\psi_o$$

$$= 2^{\frac{1}{2}} \sum_{\mu,\nu=1}^{2N} \xi^\nu(0)\, [e^{\hat{H}Gt}]_{\nu\mu}\, \Phi(\underline{e}_\mu)\,\psi_o$$

$$= 2^{\frac{1}{2}} \sum_{=1}^{2N} \xi^\nu(0)\, e^{-iHt}\, \Phi(\underline{e}_\nu)\, e^{iHt}\,\psi_o$$

(where, in accordance with (7.38), $H = \frac{1}{2} \sum_{\mu,\nu=1}^{2N} \Phi(\underline{e}_\mu)\,[-G\hat{H}G]_{\mu\nu}\,\Phi(\underline{e}_\nu)$)

$$= 2^{\frac{1}{2}}\, e^{-iHt}\,\Phi(\underline{\xi}(0))\, e^{iHt}\,\psi_o \tag{9.3}$$

In the case that $H\,\psi_o = \alpha\,\psi_o$ with $\alpha \in R$, (9.3) becomes

$$a^\dagger(\underline{\xi}(t))\,\psi_o = e^{-i(H+\alpha)t}\, a^\dagger(\underline{\xi}(0))\,\psi_o$$

and by the same reasoning, this extends to n-particle space

$$a^{\dagger}(\underline{\xi}_1(t)) \ldots a\,(\underline{\xi}_n(t))\,\psi_o = e^{-i(H+\alpha)t}\, a^{\dagger}(\underline{\xi}_1(o)) \ldots a^{\dagger}(\underline{\xi}_n(o))\,\psi_o$$

Before addition of the vacuum energy $\alpha$, H may be seen to be the exact analogue of the classical Hamiltonian

$$H = \tfrac{1}{2}\,\underline{Z}^T \hat{H}\,\underline{Z}\ , \text{ where } \ Z_\mu = \sum_{\nu=1}^{2N} G_{\mu\nu}\,\Phi(\underline{e}_\nu)$$

($Z_\mu$ obey the C.C.R.)

In a Hamiltonian system of one classical degree of freedom, $J \in Sp(2,R) = SL(2,R)$ and $J^2 = -I$.

Therefore,

$$J = \begin{pmatrix} \pm(ab-1)^{\frac{1}{2}} & b \\ -a & \mp(ab-1)^{\frac{1}{2}} \end{pmatrix} \quad \text{for some } a,b \in R. \qquad (9.4)$$

Since J in this case is specified by only two real parameters, it is easy to determine the commutant of $\hat{H}G$, to which J must belong, by (9.2d). It turns out that $\hat{H}$ must either be trivial ($\hat{H} = 0$) or definite ($\hat{H} > 0$ or $\hat{H} < 0$). In these latter cases, the Hamiltonian H can be transformed by a symplectic transformation to that of a harmonic oscillator, $H = \pm\tfrac{1}{2}(p^2 + \omega^2 q^2)$. Evidently, the other one dimensional systems, which must lie in the same canonical class as either the free particle with semi-definite Hamiltonian $H = \pm\tfrac{1}{2}p^2$, or the repulsive "oscillator" with indefinite Hamiltonian $H = \tfrac{1}{2}(p^2 - \omega^2 q^2)$, cannot be unitarized. This was first exhibited by Sparzani and Gallone [99]. When considering arbitrary finite degrees of freedom, extra complexity arises because the frequencies (eigenvalues of $i\hat{GH}$) may be mixed complex, not just pure imaginary as for the repulsive "oscillator", and even if the frequencies are real, $iG\hat{H}$ may not necessarily have a complete set of eigenvectors, due to the non-trivial Jordon canonical blocks.

By the following result, the existence of a unitarizing complex structure J for the symplectic one-parameter group $C(t) = \exp(-G\hat{H}t)$, ensures the existence of some unitarizing complex structure $J_1$ for $\exp(-G\hat{H}_1 t)$ where $\hat{H}_1$ is any member of the same canonical orbit as $\hat{H}$.

<u>Proposition</u> (9.5): Suppose that $J$ is a complex structure which unitarizes $C(t) = \exp(-G\hat{H}t)$ and that $\hat{H}_1 = C^T \hat{H} C$ with $C \in Sp(2N,R)$. Then $J_1 = C^{-1} J C$ is a complex structure which unitarizes $C_1(t) = \exp(-G\hat{H}_1 t)$.

<u>Proof</u>: $$J_1^2 = C^{-1} J^2 C = -I \tag{9.2a}$$

$J_1$ is a product of symplectic matrices and is therefore symplectic (9.2b)

$$\begin{aligned} G J_1 &= G C^{-1} J C \\ &= C^T G J C, \quad \text{by (1.14)} \\ &> 0, \quad \text{since } G J > 0 \quad \text{(by Sylvester's law of inertia)} \end{aligned} \tag{9.2c}$$

$$\begin{aligned} [G H_1, J_1] &= [G C^T \hat{H} C, C^{-1} J C] \\ &= [C^{-1} G \hat{H} C, C^{-1} J C] \\ &= C^{-1} [G \hat{H}, J] C \\ &= 0 \end{aligned} \tag{9.2d}$$

The following is a direct consequence of (9.5):

<u>Proposition</u> (9.6): Suppose that the dynamical flow $\underline{\xi}(t) = \exp(-G\hat{H}t)\, \underline{\xi}(0)$ is stable. Then there exists a unitarizing complex structure given by $J = C G C^{-1}$, where $C$ is the transformation, given in theorem (1.26), for which $C^T \hat{H} C$ is a symplectic direct sum of irreducible $2 \times 2$ components $\pm s_k I$, with $s_k > 0$.

<u>Proof</u>: If the given symplectic flow is stable, then by theorem (7.46), $iG\hat{H}$ is similar to a Hermitean matrix and therefore it must have real eigenvalues $\pm s_k$, with corresponding linear elementary divisors $s \pm s_k$ of $iG\hat{H} - s_k I$. From the canonical forms listed in theorem (1.17), it is evident that this

property uniquely determines the canonical orbit containing $\hat{H}$. In particular, there exists $C \in Sp(2N,R)$ such that $C^T \hat{H} C$ is a symplectic direct sum of $2 \times 2$ matrices $\pm s_k I$ with $s_k > 0$. As in theorem (1.26), let C consist of columns

$$(\underline{x}_1, \ldots, \underline{x}_n, \underline{y}_1, \ldots, \underline{y}_n) \quad ,$$

where
$$\underline{x}_k = 2^{-\frac{1}{2}}(\underline{\omega}_k + \underline{\omega}_k^*)$$
$$\underline{y}_k = 2^{-\frac{1}{2}} i(\underline{\omega}_k - \underline{\omega}_k^*) \quad ,$$

with $\underline{\omega}_k = \begin{pmatrix} \underline{u}_k \\ \underline{v}_k \end{pmatrix}$ independent eigenvectors of $-iG\hat{H}$ corresponding to positive eigenvalues $s_k$ and normalized so that

$$\underline{v}_k \cdot \underline{u}_k^* - \underline{v}_k^* \cdot \underline{u}_k = -i \quad \text{(which is always possible [3])}$$

If $J^{(2)}$ is a $2 \times 2$ matrix and $\hat{H}^{(2)} = s_k I^{(2)}$, the conditions (9.2a-d) impose relations among the matrix elements of $J^{(2)}$, giving a solution $J^{(2)} = G^{(2)}$, which is unique when no frequency $s_k$ is zero. It follows that the complex structure which unitarizes $\exp[-G\hat{H}_1 t]$, with $\hat{H}_1 = C^T \hat{H} C$ (= symplectic direct sum $s_1 I^{(2)} \oplus_s \ldots \oplus_s s_N I^{(2)}$), is $J_1 = G^{(2)} \oplus_s \ldots \oplus_s G^{(2)} = G^{(2N)}$.

Hence, by proposition (9.6), $J = C G C^{-1}$ is a unitarizing complex structure for $\exp(-G\hat{H}t)$.

As an example, for the harmonic oscillator, $\hat{H} = \begin{pmatrix} b^2 & 0 \\ 0 & 1 \end{pmatrix}$, the elementary divisors of $-iG\hat{H} - sI$ are $s \pm b$, $b > 0$. The eigenvectors corresponding to b are $\underline{\omega}_b = \alpha \begin{pmatrix} 1 \\ -ib \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$. The normalization condition is $vu^* - v^*u = -i$, so that we may choose $\alpha = (2b)^{-\frac{1}{2}}$. Hence, $J = C G C^{-1}$, where $C = \begin{pmatrix} b^{-\frac{1}{2}} & 0 \\ 0 & b^{\frac{1}{2}} \end{pmatrix}$. That is,

$$J = \begin{pmatrix} 0 & -b^{-1} \\ b & 0 \end{pmatrix} \qquad (9.7)$$

In order to find the full class of dynamical systems for which J exists, as in the case of one degree of freedom, we shall first deduce the structure of J and then find which elements $G\hat{H}$ of the Lie algebra Sp(2N,R) commute with J.

Proposition (9.8): Every symplectic complex structure satisfying (9.2a-c) must belong to the same symplectic conjugacy class as G.

Proof: From (9.2a,b), $GJ = -J^T G = (GJ)^T$

Therefore, the matrix $\hat{K} = -GJ$ is symmetric and by (9.2c) $\hat{K}$ is positive definite. Therefore, by theorem (1.26), there exists a symplectic matrix $C \in Sp(2N,R)$ such that $C^T \hat{K} C = D = \text{diag}[s_1,\ldots,s_N, s_1,\ldots,s_N]$ for some $s_j > 0$ ; $j = 1,\ldots,N$ .

$$\text{Therefore, } J\ (=G\hat{K}) = G\, C^{T^{-1}}\, D\, C^{-1}$$

$$= C\, G\, D\, C^{-1} \quad \text{by (1.13-15)} \qquad (9.9)$$

Then $J^2 = -I$ implies $(GD)^2 = -I$ from which we deduce that $s_j = 1$ and $D = I$. Thus, (9.9) becomes $J = C\,G\,C^{-1}$, as required.

An alternative well-known proof (e.g. [97]) is that given any matrix $J_1$ satisfying (9.2), the system $(M, J_1, \langle\cdot,\cdot\rangle_1)$ with $\langle\cdot,\cdot\rangle_1$ defined as in (8.16a), constitutes an N-dimensional complex Hilbert space, which is unique up to unitary equivalence. Given any two such complex structures $J_1$ and $J_2$, there must exist a transformation C on $M$ such that for all $\underline{\xi} \in M$, $C J_1 \underline{\xi} = J_2 C \underline{\xi}$ and C must be unitary from $(M, \langle\cdot,\cdot\rangle_1)$ to $(M, \langle\cdot,\cdot\rangle_2)$, with $\langle\cdot,\cdot\rangle_2$ constructed from $J_2$, as in (8.16a). Since $Im\langle\underline{\xi},\underline{\zeta}\rangle_1 = Im\langle\underline{\xi},\underline{\zeta}\rangle_2 = -\alpha^{-2} B(\underline{\xi},\underline{\zeta})$, C must be symplectic.

Proposition (9.5) effectively reduces the question of existence of J to an inspection of the canonical forms for $\hat{H}$. This was completed in our paper [III]. Here, we shall demonstrate, by a more concise method, that J exists only if $\hat{H}$ belongs to the same canonical orbit as that of a collection of independent subsystems with Hamiltonian $H_j = 0$ or $H_j = \pm\frac{1}{2}(p_j^2 + s_j^2 q_j^2)$, with $s_j > 0$. In an appropriate symplectic basis, each variable $q_j$ exhibits simple harmonic motion.

Proposition (9.10): Suppose that there exists a complex structure J which unitarizes $C(t) = \exp(-G\hat{H}t)$. Then $\hat{H}$ can be reduced, by a change of symplectic basis, to a symplectic direct sum of $2 \times 2$ components

$$\hat{H}_j = \pm\begin{pmatrix} 1 & 0 \\ 0 & s_j^2 \end{pmatrix} \text{ or } \hat{H}_j = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} .$$

Proof: Suppose that J satisfies (9.2a-d).

By proposition (9.8), there exists $C \in Sp(2N,R)$ such that $C^{-1} J C = G$. By (9.2d), $[G\hat{H}, C G C^{-1}] = 0$

$$\Rightarrow C^{-1} [G\hat{H}, C G C^{-1}]C = 0$$

$$\Rightarrow [C^{-1} G \hat{H} C, G] = 0$$

$$\Rightarrow [G C^T \hat{H} C, G] = 0 \quad \text{by (1.14)}$$

$$\Rightarrow [C^T \hat{H} C, G] = 0 \tag{9.11}$$

As discussed at the end of Section 7, the condition (9.11) ensures that $i G C^T \hat{H} C$ is Hermitean. Therefore, by (1.14), $C^{-1}(iG\hat{H})C$ is Hermitean. As discussed in the proof of proposition (9.6), this fact completely determines the canonical orbits to which $\hat{H}$ may belong and hence, proposition (9.10) is established.

From propositions (9.10) and (7.33), we observe that Segal quantization is restricted to the same class of finite dimensional systems as those which have a positive definite heuristic single mode space. In this sense, Segal quantization is a rigorized version of heuristic mode space quantization.

We now proceed to solve a problem which is inverse to that of proposition (9.6). Namely, given a complex Hilbert space $(M,J,\langle\cdot,\cdot\rangle_1)$, with $(M,B)$ a 2N-dimensional real symplectic space, the problem is to derive the quadratic Hamiltonians $\frac{1}{2}\,\underline{z}^T \hat{H}\,\underline{z}$ for which $\exp(-G\hat{H}t)$ has a positive definite self-adjoint generator on $H^{(1)}$.

Proposition (9.12): Let $J = C\,G\,C^{-1}$ with $C \in Sp(2N,R)$. Then $\hat{H}$ satisfies (9.2d) if and only if $\hat{H} = C^{T^{-1}} \hat{H}_o\, C^{-1}$, where

$$\hat{H}_o = \begin{pmatrix} L & K \\ -K & L \end{pmatrix} \text{ for some } K = -K^T,\ L = L^T .$$

Proof: Suppose that J satisfies (9.2a-d).
As in (9.11), $[C^T \hat{H} C, G] = 0$. The matrix $C^T \hat{H} C$, being symmetric, may be expressed as $\begin{pmatrix} L & K \\ K^T & M \end{pmatrix}$, with L and M symmetric. Then (9.11) implies $M = L$ and $K^T = -K$, as required. The converse follows from (9.5).

Given an arbitrary symplectic complex structure J, proposition (9.12) characterizes the Lie algebra $\{G\hat{H}\}_J$ of a subgroup $G_J$ of $Sp(2N,R)$ which is isomorphic to $U(N)$. In addition, a one parameter group of orthogonal transformations $\exp(\hat{A}t)$ on $E^{2N}$ is a subgroup of $Sp(2N,R) \cap O(2N,R)$ if and only if $\hat{A} = -G\hat{H} = -\hat{A}^T$ for some real symmetric $\hat{H}$. This reduces to $\hat{A} = -G\hat{H}_o$, where $\hat{H}_o$ has the form specified in proposition (9.12). Since $-G\,C\,\hat{H}_o\, C^T = C^{T^{-1}}(-G\hat{H}_o)C^T$, by (1.14), if $C \in Sp(2N,R)$, the Lie algebra $\{G\hat{H}\}_J$ is isomorphic to that of $Sp(2N,R) \cap O(2N,R)$. Combining this information with (4.20-21), the relevant chain of group embeddings which we have encountered so far is

$$G_J \simeq U(N) \simeq Sp(2N,R) \cap O(2N,R) < Sp(2N,R) \simeq Sp(2N,R) \cap U(N,N)$$
$$\simeq U(N,N)/(4.5) < U(N,N) \qquad (9.13)$$

If time evolution $\exp(-G\hat{H}t) = \exp(-J[-JG\hat{H}]t)$ is to have a positive definite self-adjoint generator $-JG\hat{H}$ on $H^{(1)}$, then by (8.16a), $-GJ(-JG\hat{H}) > 0$

$$\Rightarrow \quad \hat{H} > 0 \qquad (9.14)$$

Combining this with proposition (9.12),

$$\hat{H} = C^{T^{-1}} \hat{H}_o \, C^{-1}$$

$$\text{with } \hat{H}_o = \begin{pmatrix} L & K \\ -K & L \end{pmatrix} > 0 \; ; \; L = L^T \; , \; K = -K^T$$

$$\Rightarrow \quad P^\dagger \hat{H}_o \, P > 0 \; , \quad \text{with } P = 2^{-\frac{1}{2}} \begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} L-iK & 0 \\ 0 & L+iK \end{pmatrix} > 0$$

$C(t) = \exp(-G\hat{H}t)$ has a positive definite self-adjoint generator on $H^{(1)} = (M, J, \langle\cdot,\cdot\rangle_1)$, with $J = C\,G\,C^{-1}$, if and only if $\hat{H} = C \begin{pmatrix} L & K \\ -K & L \end{pmatrix} C^T$, with L and K the real and imaginary parts of a positive definite Hermitean matrix.

In proposition (9.6) we have an explicit construction of the complex structure J for any unitarizable one-parameter symplectic group $\exp(-G\hat{H}t)$. The explicit determination of J which allows quantization according to Fermi-Dirac statistics will be more easily generalized to infinite degrees of freedom and this will shed some light on Bose fields. Therefore, the next section is devoted to the Fermion analogues of classical mechanics.

# SECTION 10 - THE FERMIONIC ANALOGUE OF CLASSICAL MECHANICS

Before we can discuss quantization according to Fermi-Dirac statistics, we need to define the classical analogue of a system of Fermions. In fact, there is more agreement on the structure of the quantized Fermion system than on that of its classical counterpart, which must be deduced. As discussed in Section 8, there is a well-defined abstract C* algebra $A$ of classical observables which follows straight from the C.A.R. (8.3), which reflect the Fermi-Dirac statistics. Usually, in practice, there is a distinguished conjugation on single particle space $H^{(1)}$, for example, charge conjugation on the solution space of some wave equation. Then the N-dimensional space $H^{(1)}$ may be identified with a 2N-dimensional real space $V \oplus V$, with $V$ the real space of self-conjugate elements of $H^{(1)}$.

Now we define the self-adjoint operators

$$Z(\underline{u} \oplus \underline{v}) = \left(\frac{\hbar}{2}\right)^{\frac{1}{2}} [a(\underline{u}) + a^{\dagger}(\underline{u}) - i[a(\underline{v}) - a^{\dagger}(\underline{v})]] \tag{10.1}$$

for all $\underline{u}$, $\underline{v} \in V$.

In this section, we do not assume $\hbar = 1$, since we need to consider a classical limit $h \to 0$. In Shale-Stinespring form [84], the C.A.R. become

$$[Z(\underline{u} \oplus \underline{v}), Z(\underline{u}' \oplus \underline{v}')]_+ = \hbar I[\text{Re} <\underline{u}, \underline{u}'>_1 + \text{Re} <\underline{v}, \underline{v}'>_1] . \tag{10.2}$$

The operators $Z(\underline{u} \oplus \underline{v})$ and I, obeying $Z(\underline{u} \oplus \underline{v})^{\dagger} = Z(\underline{u} \oplus \underline{v})$ and (10.2), generate the Clifford C* algebra $A$ over real Hilbert space $(V,s) \oplus (V,s)$, with $s(\underline{u}, \underline{u}') = \text{Re} <\underline{u}, \underline{u}'>_1$. When $\dim H^{(1)} = N < \infty$, by a suitable choice of basis $\{\underline{e}_1,\ldots,\underline{e}_{2N}\}$, $(V,S) \oplus (V,S)$ may be taken to be Euclidean space $E^{2N}$. The C.A.R. reduce to (4.3) and

$$[Z_\mu, Z_\nu]_+ = \hbar\, \delta_{\mu\nu}\, I , \tag{10.3}$$

where $Z_\mu = Z(\underline{e}_\mu)$

(10.1) may then be expressed $\underline{Z} = P\alpha\_$, as in Section 4.

The dynamics of a linear system of Fermions is specified by a quadratic Fermion Hamiltonian of the form (4.1). Then, the Heisenberg time evolution may be seen to be a continuous one-parameter group of C.A.R. Bogoliubov transformations as in (6.1).

$$e^{iHt}\,\alpha_\mu\,e^{-iHt} = \sum_{\nu=1}^{2N}\left[e^{-itD_2}\right]_{\mu\nu}\alpha_\nu \qquad (10.4)$$

where $H = \tfrac{1}{2}\,\underline{\alpha}^\dagger D_2\,\underline{\alpha}$ + real constant (from Section 4)

$$= \tfrac{1}{2}i\,\underline{Z}^T \hat{A}\,\underline{Z} + \text{real constant} \qquad (10.5a)$$

where $\hat{A} = -iP\,D_2\,P^\dagger$ (real anti-symmetric, by (4.49b)) (10.5b)

Therefore, substituting $\underline{\alpha} = P^\dagger\,\underline{Z}$ in (10.4),

$$e^{iHt}\,Z_\mu\,e^{-iHt} = \sum_{\nu=1}^{2N}\left[e^{t\hat{A}}\right]_{\mu\nu} Z_\nu$$

$$\text{or} \quad \underline{Z}(t) = e^{t\hat{A}}\,\underline{Z}(0) \qquad (10.6)$$

By analogy with Boson systems, we shall assume that the self-adjoint generators $Z_\mu$ correspond to classical Fermionic phase variables $z_\mu$, which are self-conjugate with respect to some fundamental complex conjugation and that anti-commutators correspond to symmetric brackets of the generalized Poisson type

$$\{f,\,W,\,g\} = f\,\frac{\overleftarrow{\partial}}{\partial z_\mu}\,W_{\mu\nu}\,\frac{\overrightarrow{\partial}}{\partial z_\nu}\,g\,. \qquad (10.7)$$

Symmetric Poisson brackets were first introduced by Martin [100], in this very context of Fermi-Dirac quantization. Later, Duffin [10] introduced the plus-Poisson bracket ($W = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ in (10.7)) in order to extend Hamiltonian mechanics to dissipative systems. The general Poisson bracket was then moulded into the differential geometric framework by Droz-Vincent [101].

If we adhere to a strict Fermionic analogue of the Dirac correspondence rule for linear observables, we have

$$[\zeta^\mu Z_\mu,\,\xi^\nu Z_\nu]_+ = \hbar\,\{\zeta^\mu z_\mu,\,W,\,\xi^\nu z_\nu\} \qquad (10.8)$$

$$\text{for all } \underline{\zeta},\,\underline{\xi} \in R^{2N}\,.$$

Then by (10.3), $W = I$. The brackets appearing in the Dirac correspondence rule and in (10.8) are respectively imaginary skew-symmetric ($W = -iG = \begin{pmatrix} o & i \\ -i & o \end{pmatrix}$) and real symmetric ($W = I$) and these take a special significance in the work of Droz-Vincent [101]. In the generalized classical mechanics involving the Poisson bracket $\{f, W, g\}$, a complex $C^1$ function $f(\underline{z})$ of the real phase variables defines a Hamiltonian vector field

$$\Phi(W, f) = W_{\mu\nu} \frac{\partial f}{\partial z_\nu} \frac{\partial}{\partial z_\mu} , \tag{10.9}$$

which generates a phase flow. Droz-Vincent's fundamental transform $\theta$ is defined by

$$\theta\Phi(W, f) = \Phi(W, f^*) . \tag{10.10}$$

It turns out $\theta$ is an adjunction over the operations of commutation or anti-commutation if and only if $W$ is Hermitean, a circumstance which is satisfied by $W = -iG$ or $W = I$.

The symmetric bracket (10.7), with $W = I$, is preserved by the group of real orthogonal transformations, which will be the group of linear canonical transformations for the Fermionic phase variables $\underline{z}$.
From (10.5),

$$\dot{Z}_\mu(t) = \hat{A}_{\mu\nu} Z_\nu(t) \tag{10.11}$$

$$= i/\hbar \, [H, Z] . \tag{10.12}$$

However, if we assume that the classical phase variables $z_\mu$ are ordinary commuting variables, then the the classical Hamiltonian $\frac{1}{2}i \, \underline{z}^T \hat{A} \, \underline{z}$ corresponding to (10.5a) must vanish, because of the anti-symmetry of $\hat{A}$. To avoid this difficulty, we may take $z_\mu$ to be anti-commuting first order elements of a Grassmann algebra

$$[z_\mu, z_\nu]_+ = 0 \tag{10.13}$$

The Grassmann technique was first employed in this way by Martin [100]. Also, Casalbuoni [102] pointed out that this Grassmann algebra is the classical $(h \to 0)$ limit of the quantum mechanical Clifford algebra (10.2) and then proceeded to develop the technique further. Other applications to field theory were made by Berezin and Marinov [103].

The Grassmann variables themselves have no numerical significance and therefore do not contribute to physical predictions until quantization is effected. However, the Grassmannian algebra of classical observables supports a natural definition of spin [103] and facilitates the construction of the quantum mechanical propagator via a simple one to one relationship with a Grassmannian path integral which is analogous to the Feynman path integral in many ways [103, 100b].

The classical analogue of (10.11),

$$\dot{z}_\mu(t) = \hat{A}_{\mu\nu}\, z_\nu(t) , \tag{10.14}$$

can now be formulated as a generalized Hamiltonian equation

$$\dot{z}_\mu(t) = -i\ \{z_\mu, I, H\} = -i\ z_\mu \frac{\overleftarrow{\partial}}{\partial z_\mu} \frac{\overrightarrow{\partial}}{\partial z_\nu} H , \tag{10.15a}$$

where

$$H = i\ z_\mu\, \hat{A}_{\mu\nu}\, z_\nu . \tag{10.15b}$$

In (10.15), the left and right derivatives are defined on monomials by [103, 97a] :

$$\frac{\overrightarrow{\partial}}{\partial z_\nu} z_\alpha z_\beta z_\gamma \dots = [\delta_{\nu\alpha} z_\beta z_\gamma \dots] - [z_\alpha \delta_{\nu\beta} z_\gamma \dots] + \dots \tag{10.16a}$$

$$\dots z_\beta\, z_\alpha \frac{\overleftarrow{\partial}}{\partial z_\nu} = [\dots z_\alpha z_\beta \delta_{\alpha\nu}] - [\dots z_\gamma \delta_{\beta\nu} z_\alpha] + \dots \tag{10.16b}$$

Definition (10.16) is extended linearly to polynomials. From (10.16) it follows that

$$\frac{\overrightarrow{\partial}}{\partial z_\nu} \frac{\overrightarrow{\partial}}{\partial z_\mu} = - \frac{\overrightarrow{\partial}}{\partial z_\mu} \frac{\overrightarrow{\partial}}{\partial z_\nu} . \tag{10.17}$$

If the variables $z_\mu$ were ordinary numbers, it would follow from the argument of Duffin [10] that every Poisson bracket-preserving canonical transformation must be linear, while ordinary skew-symmetric Poisson bracket mechanics is not so restricted. However, Duffin thrice used the identity

$$\frac{\partial^2}{\partial z_\nu \, \partial z_\mu} = \frac{\partial^2}{\partial z_\mu \, \partial z_\nu} , \tag{10.18}$$

which must be replaced by (10.17) when Grassmann variables are employed. Duffin's argument would then merely establish $J_\nu = J_\nu$, rather than $J_\nu = -J_\nu = 0$ ($J_\nu$ being the derivatives $\frac{\vec{\partial}}{\partial z_\nu}$ J of the Jacobian matrix J of a canonical transformation). This further supports the view that an exact Fermion analogue of the classical observable algebra is achieved only within the Grassmann framework.

Now the symmetric bracket appearing in (10.15) can be related to the plus-Poisson bracket. To see this, we first define complex classical variables

$$\psi_j = 2^{-\frac{1}{2}} (z_j + i \, z_{j+N}) \tag{10.19a}$$

$$\pi_j = i \, \bar{\psi}_j = 2^{-\frac{1}{2}} i (z_j - i \, z_{j+N}) . \tag{10.19b}$$

In ordinary Hamiltonian mechanics, the Poisson bracket was extended linearly to complex functions by Strocchi [104], who found it convenient to have classical analogues for the non-self-adjoint Boson annihilation and creation operators. The variables $\psi_j$, $\pi_k$ obey symmetric bracket relations

$$\{\psi_\ell, I, \pi_k\} = i \, \delta_{\ell k} , \tag{10.20}$$

$$0 = \{\psi_\ell, I, \psi_k\} = \{\pi_\ell, I, \pi_k\} ,$$

which correspond to the C.A.R. among conjugate non-Hermitean Fermion fields. Now

$$\begin{aligned}\{f, I, g\} &= f \frac{\overleftarrow{\partial}}{\partial z_\mu} \frac{\vec{\partial}}{\partial z_\mu} g \\ &= \left[ f \frac{\overleftarrow{\partial}}{\partial \psi_k} \frac{\partial \psi_k}{\partial z_\mu} + f \frac{\overleftarrow{\partial}}{\partial \pi_k} \frac{\partial \pi_k}{\partial z_\mu} \right] \left[ \frac{\partial \psi_\ell}{\partial z_\mu} \frac{\vec{\partial}}{\partial \psi_\ell} g + \frac{\partial \pi_\ell}{\partial z_\mu} \frac{\vec{\partial}}{\partial \pi_\ell} g \right] \\ &= i \left[ f \frac{\overleftarrow{\partial}}{\partial \psi_j} \frac{\vec{\partial}}{\partial \pi_j} g + f \frac{\overleftarrow{\partial}}{\partial \pi_j} \frac{\vec{\partial}}{\partial \psi_j} g \right] , \text{ by (10.19)} \\ &= i \, \{f, g\}_+ , \text{ the plus-Poisson bracket with respect to } \underline{\psi} \text{ and } \underline{\pi} .\end{aligned} \tag{10.21}$$

Therefore, the Fermionic correspondence rule (10.8) may be expressed

$$
\begin{aligned}
[\zeta^{\mu} Z_{\mu},\ \xi^{\nu} Z_{\nu}]_{+} &= \hbar\ \{\zeta^{\mu} z_{\mu},\ I,\ \xi^{\nu} z_{\nu}\} \\
&= i\hbar\ \{\zeta^{\mu} z_{\mu},\ \xi^{\nu} z_{\nu}\}_{+}\ , \qquad (10.22)
\end{aligned}
$$

which now appears to be exactly analogous to the Dirac correspondence rule for ordinary Poisson brackets [105].

The canonical forms for the Grassmannian Hamiltonian (10.15b) are determined by representatives $\hat{A}$ of the conjugacy classes of the Lie algebra consisting of real skew-symmetric matrices, which generate the orthogonal group O(2N,R). It is well-known that these conjugacy classes are completely determined by the positive eigenvalues of $i\hat{A}$. Therefore, we may choose the canonical forms to be of the type

$$
\hat{A}_o = \left[\begin{array}{c|c}
 & \begin{matrix}\omega_1 & & & & & \\ & \ddots & & & & \\ & & \omega_M & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0\end{matrix} \\ \hline
\begin{matrix}-\omega_1 & & & & & \\ & \ddots & & & & \\ & & -\omega_M & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0\end{matrix} &
\end{array}\right] \qquad (10.23)
$$

where $\omega_j : j = 1,\ldots, M$ are the positive eigenvalues of $i\hat{A}$ .

To construct the unitarizing complex structure J described in Section 8, we will need to explicitly construct an orthogonal transformation C which reduces $\hat{A}$ to $\hat{A}_o$.

Since iA is Hermitean and purely imaginary, it follows that

$$
i\,A\,\underline{V} \;=\; \omega\,\underline{V} \Leftrightarrow i\,A\,\underline{V}^{*} \;=\; -\,\omega\,\underline{V}^{*}\ . \qquad (10.24)
$$

Let $\Delta_{+,-,o}$ be subspaces of $C^{2N}$ spanned by eigenvectors of iA corresponding respectively to positive, negative and zero eigenvalues. Since iA is Hermitean, $\Delta_+ \cup \Delta_- \cup \Delta_o = C^{2N}$ .

Lemma (10.25): There exists a basis $\{\underline{V}_1, \ldots, \underline{V}_M\}$ for $\Delta_+$, such that $\{\underline{V}_1^*, \ldots, \underline{V}_M^*\}$ is a basis for $\Delta_-$.

Proof: Obtained directly from (10.24).

From Lemma (10.25),

$$\dim. \Delta_o = 2N - 2M \quad (M = \dim. \Delta_+) \text{ , which is even.}$$

Lemma (10.26): There exists an orthonormal basis of real vectors for $\Delta_o$.

Proof: Let $\underline{W}_1 \in \Delta_o$ and $\underline{W}_1 \neq \underline{0}$. By (10.24),

$$\underline{W}_1^* \in \Delta_o .$$

Define

$$\underline{X}_1 = 2^{-\frac{1}{2}} |\underline{W}_1|^{-1} (\underline{W}_1 + \underline{W}_1^*)$$

$$\underline{Y}_1 = -2^{-\frac{1}{2}} i |\underline{W}_1|^{-1} (\underline{W}_1 - \underline{W}_1^*) .$$

$\underline{X}_1$ and $\underline{Y}_1$ are real vectors in $\Delta_o$ and at least one of them is non-null. From $\underline{X}_1$, $\underline{Y}_1$, we now choose $\underline{Z}_1$ to be a non-null real unit vector. This procedure may be reiterated. Given mutually orthogonal real unit vectors $\underline{Z}_1, \ldots, \underline{Z}_r \in \Delta_o$, choose

$$\underline{W}_{r+1} \in \Delta_o \cap \{\underline{Z}_1, \ldots, \underline{Z}_r\}^{\perp} .$$

By the above procedure, construct $\underline{X}_{r+1}$ and $\underline{Y}_{r+1}$ and choose $\underline{Z}_{r+1}$. Eventually, this Gram-Schmidt process produces a basis of 2(N-M) real unit vectors for $\Delta_o$.

Lemma (10.27): In $\Delta_o$, there exists an orthonormal basis

$$\{\underline{K}_1, \ldots, \underline{K}_L \; ; \; \underline{K}_L^*, \ldots, \underline{K}_1^*\} \qquad (L = N - M)$$

Proof: Consider $\underline{Z}_1, \ldots, \underline{Z}_{2L}$ given by lemma (10.26). Define

$$\underline{K}_j = 2^{-\frac{1}{2}} (\underline{Z}_{2j-1} + i\, \underline{Z}_{2j}) \quad \text{for } j = 1, \ldots, L .$$

By lemmas (10.25, 26), there exists in $C^{2N}$, an orthonormal basis

$$\{\underline{V}_1,\ldots, \underline{V}_M ; \underline{K}_1,\ldots, \underline{K}_L ; \underline{K}_L^*,\ldots, \underline{K}_1^* ; \underline{V}_M^*,\ldots, \underline{V}_1^*\} \qquad (10.28)$$

such that

$$i\hat{A}\underline{V}_j = \omega_j \underline{V}_j , \quad \omega_j > 0$$

$$i\hat{A}\underline{V}_j = -\omega_j \underline{V}_j^*$$

$$i\hat{A}\underline{K}_j = i\hat{A}\underline{K}_j^* = \underline{0} .$$

Define U to be the unitary matrix with columns given by the above vectors in the order specified in (10.28)

$$U^\dagger i\hat{A} U = D_1 \qquad (10.29)$$

where

$$D_1 = \text{diag.}[\omega_1,\ldots, \omega_M, 0,\ldots,0, -\omega_M,\ldots, -\omega_1]$$

Furthermore, $U_1^{-1} D_1 U_1 = i\hat{A}_o$ , where (10.30a)

$$U_1 = 2^{-\frac{1}{2}} \begin{bmatrix} -1 & & & -i & & \\ & \ddots & & & \ddots & \\ & & -1 & & & -i \\ & & -1 & & & i \\ & \cdot^{\cdot^{\cdot}} & & & \cdot^{\cdot^{\cdot}} & \\ -1 & & & i & & \end{bmatrix} \qquad (10.30b)$$

Combining (10.29) and (10.30),

$$C^{-1}\hat{A} C = \hat{A}_o \qquad (10.31a)$$

where $C = UU_1$ (with U and $U_1$ unitary, defined above) (10.31b)

$C^{-1}$ is obviously unitary and the columns of $C^{-1}$ are $-2^{-\frac{1}{2}}(\underline{V}_j^* + \underline{V}_j)$, $-2^{-\frac{1}{2}}(\underline{K}_j^* + \underline{K}_j)$, $-2^{-\frac{1}{2}}i(\underline{V}_j - \underline{V}_j^*)$ and $-2^{-\frac{1}{2}}i(\underline{K}_j - \underline{K}_j^*)$, which are real. Therefore, $C^{-1}$ and C are real orthogonal.

By propositions (4.27b) and (4.49b), the reduction of an arbitrary real skew symmetric matrix $\hat{A}$ to the canonical form $\hat{A}_o$, is equivalent to the reduction of a Fermion dynamical matrix $\mathcal{D}_2 \in \mathcal{D}_2^s$ to diagonal form $iP^\dagger \hat{A}_o P = \text{diag.}[\omega_1,\ldots,\omega_M,0,\ldots,0,-\omega_M,\ldots,-\omega_1]$ by a C.A.R. Bogoliubov transformation $\mathcal{T} = P^\dagger C P$. Diagonalization of quadratic Fermion Hamiltonians is possible in the general case, even though this is not true for the quadratic Boson Hamiltonians.

## SECTION 11 - COMPLEX STRUCTURES FOR FERMI-DIRAC QUANTIZATION

Let $\underline{\zeta}$, $\underline{\xi}$ be arbitrary elements of $R^{2N}$ and let $z_\mu(t)$ be classical phase variables. A generalized Poisson bracket $\{\cdot, W, \cdot\}$ defines a bilinear form

$$\begin{aligned} w(\underline{\zeta}, \underline{\xi}) &= \{\zeta^\mu z_\mu(0), W, \xi^\nu z_\nu(0)\} \\ &= \zeta^\mu W_{\mu\nu} \xi^\nu \end{aligned} \tag{11.1}$$

In the case that W is skew-symmetric and non-singular, w is a symplectic form and $(R^{2N}, w)$ is a real symplectic space. For systems of finite degrees of freedom, we may assume that $w(\underline{\zeta}, \underline{\xi}) = (\underline{\zeta}, G\, \underline{\xi})$, as in Section 9. In the case that W is symmetric and non-singular, w is an inner product and $(R^{2N}, w)$ is a real Hilbert space. When $z_\mu(t)$ are anti-commuting Fermionic phase variables, $W_{\mu\nu}$ may be taken to be $\delta_{\mu\nu}$, as in Section 10, and w reduces to the inner product on real Euclidean space $E^{2N}$.

From (10.5) and (10.15), the Grassmannian Hamiltonian

$$H = \tfrac{1}{2}\, i\, \underline{z}^T \hat{A}\, \underline{z} \tag{11.2}$$

is the classical analogue of the quadratic Fermion Hamiltonian

$$H = \tfrac{1}{2}\, \underline{\alpha}^\dagger \mathcal{D}_2\, \underline{\alpha}\ , \text{ with } \mathcal{D}_2 = i\, P^\dagger \hat{A}\, P\ . \tag{11.3}$$

Therefore, quantization should achieve a Fermion Hamiltonian of the form (11.3) from the classical dynamical matrix $\hat{A}$. All number-changing terms $a^\dagger_\mu a^\dagger_\nu$ and $a_\mu a_\nu$ can be removed from (11.3) by a C.A.R. Bogoliubov transformation. With the Hamiltonian in this form, single particle space, in the Fock representation, is time-invariant. In the Segal approach, one begins with a real Hilbert Space $V$ which is complexified in such a way that the orthogonal dynamics, $\underline{\xi}(t) = e^{-\hat{A}t}\, \underline{\xi}(0)$ on $V$, becomes the unitary dynamics on complex single particle space $H^{(1)}$. Since quantum statistics is a many body concept, there should be no a-priori method for determining, from the single particle dynamics alone, whether quantization should proceed according to Bose statistics or to Fermi-Dirac statistics. In the case that $V$ is finite dimensional, we can not even invoke the spin-statistics theorem for non-trivial (necessarily infinite dimensional) representations of the Lorentz group. Therefore, if a real orthogonal transformation is the correct Fermionic analogue of a real symplectic canonical transformation, then the single particle dynamics resulting from a number conserving Hamiltonian (11.3), should be both orthogonal and symplectic, so that quantization can proceed according to either statistics. We verify below that this is indeed the case.

A matrix $\hat{A}$ generates a one parameter group of transformations in $Sp(2N,R) \cap O(2N,R)$ if $\hat{A}$ is skew symmetric and $G\hat{A}$ is symmetric. That is,

$$\hat{A} = \begin{pmatrix} K & L \\ -L & K \end{pmatrix} \text{, with } K^T = -K \text{ and } L^T = L \text{ .} \qquad (11.4)$$

<u>Proposition</u> (11.5): If a quadratic Fermion Hamiltonian or a quadratic Boson Hamiltonian is number-conserving, then the corresponding classical dynamical matrix $\hat{A} = i\,\rho(D_2)$ or $G\hat{H} = G\,\rho(D_1)$, as defined in (4.26), generates a one parameter subgroup of $Sp(2N,R) \cap O(2N,R)$ .

<u>Proof</u>: (a) Let H be a number-conserving quadratic Fermion Hamiltonian.

$$H = \tfrac{1}{2}\,\underline{\alpha}^{\dagger}\, D_2\, \underline{\alpha} \text{ ,}$$

with

$$D_2 = \begin{bmatrix} A_2 & 0 \\ 0 & -A_2^{*} \end{bmatrix} \text{ ; } A_2 = A_2^{\dagger}$$

Then the corresponding dynamical matrix is

$$\hat{A} = -i\, P\, D_2\, P^{\dagger}$$

$$= -\tfrac{1}{2} i \begin{bmatrix} A_2 - A_2^{*} & i(A_2+A_2^{*}) \\ -i(A_2+A_2^{*}) & A_2 - A_2^{*} \end{bmatrix}$$

which conforms to (11.4), since $A_2 = A_2^{\dagger}$ .

(b) Let H be a number-conserving quadratic Boson Hamiltonian.

$$H = \tfrac{1}{2} \underline{\alpha}^{\dagger} \mathcal{D}_1 \underline{\alpha} ,$$

with

$$\mathcal{D}_1 = \begin{bmatrix} A_1 & 0 \\ 0 & A_1^* \end{bmatrix}$$

Then the corresponding generator of classical time evolution is

$$-G\hat{H} = -G\,P\,\mathcal{D}_1\,P^{\dagger}$$

$$= -\tfrac{1}{2}i \begin{bmatrix} A_1 - A_1^* & i(A_1+A_1^*) \\ -i(A_1+A_1^*) & (A_1 - A_1^*) \end{bmatrix}$$

which conforms to (11.4), since $A_1 = A_1^{\dagger}$ .

Given a one-parameter continuous group of orthogonal transformations $C(t) = e^{\hat{A}t}$ , the unitarizing complex structure J must satisfy

$$J^2 = -I \quad \text{(J is anti-involutory)} \qquad (11.6a)$$

$$J^T J = I \quad \text{(J is orthogonal, from (8.14, 15))} \qquad (11.6b)$$

and $$[J,\hat{A}] = 0 \quad \text{(unitary of } C(t) \text{ on } \mathcal{H}^{(1)}) \qquad (11.6c)$$

Unlike the Boson case, from (8.16b), the inner product $\langle\cdot,\cdot\rangle_1$ on $\mathcal{H}^{(1)}$ is already positive definite. It turns out that in order to specify J uniquely, we can add a fourth condition

$$J\hat{A} > 0 \quad \text{(positivity of the single particle Hamiltonian)} \qquad (11.6d)$$

To find the necessary and sufficient conditions on $\hat{A}$ for the existence of J, we shall first reduce the problem to an examination of conjugacy classes for O(2N,R).

<u>Proposition</u> (11.7): Suppose that J is a unitarizing complex structure for $\exp(\hat{A}t)$ and that $A_1 = C^{-1}\hat{A}\,C$, with $C \in O(2N,R)$. Then $J_1 = C^{-1} J\,C$ is a unitarizing complex structure for $\exp(\hat{A}_1 t)$.

<u>Proof</u>: It is straightforward to show that $J_1$, as defined, satisfies (11.6).

Proposition (11.8): Given any real $2N \times 2N$ skew-symmetric matrix $A$, there exists $J$ satisfying (11.6). We may take $J = C\,G\,C^{-1}$, where $G = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and $C$ is the orthogonal transformation, constructed explicitly in Section 10, which reduces $\hat{A}$ to the canonical form $\hat{A}_o = C^{-1}\hat{A}\,C$ of (10.23).

Proof: With $\hat{A}$ reduced to the form $\hat{A}_o$, it is easily verified that $J = G$ is a solution to (10.23a-d). Then we apply proposition (11.7) to achieve a solution $J = C\,G\,C^{-1}$ in the general case.

In order to discuss the uniqueness of $J$, it is convenient to reduce $\hat{A}$ to an alternative canonical form

$$\hat{A}_1 = C_1^{-1}\,\hat{A}_0\,C_1 = \begin{bmatrix} 0 & \omega_1 & & & & & & & \\ -\omega_1 & 0 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 0 & \omega_2 & & & & \\ & & & -\omega_2 & 0 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 0 & \omega_M & \\ & & & & & & -\omega_M & 0 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 0 \\ & & & & & & & & & & \ddots \\ & & & & & & & & & & & 0 \end{bmatrix} \tag{11.9}$$

where the real orthogonal matrix $C_1$ is given by

$$C_1 = \begin{bmatrix} 0 & -1 & 0 & 0 & & & & & \\ & & & -1 & 0 & 0 & & & \\ & & & & & -1 & & & \\ & & & & & & \ddots & & \\ & & & & & & -1 & 0 & 0 \\ & & & & & & & 0 & -1 \\ 1 & 0 & 0 & & & & & & \\ & & 1 & 0 & 0 & & & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & 1 & 0 & 0 & \\ & & & & & & 0 & 1 & 0 \end{bmatrix} \tag{11.10}$$

Hence, $\hat{A}_1 = C_3^{-1}\,\hat{A}\,C_3$, where $C_3 = C\,C_1$. (11.11)

In the orthonormal basis in which $\hat{A} = \hat{A}_1$ , the dynamical group $e^{\hat{A}t}$ decomposes according to

$$e^{\hat{A}_1 t} = e^{i\,\omega_1 t\,\sigma_2} \oplus \ldots \oplus e^{i\,\omega_M t\,\sigma_2} \oplus I^{(2P)} \qquad (11.12)$$

where $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $P = N - M$, so that 2P is the nullity of $\hat{A}$. Hence, the original Grassmann dynamical system has been reduced to a collection of independent Fermi oscillators [100a] plus a subsystem on which the dynamics is trivial. In this basis, the most general complex structure satisfying (11.6a-c) is

$$J_1 = \begin{bmatrix} J_1^{(2)} & & & & \\ & J_2^{(2)} & & & \\ & & \ddots & & \\ & & & J_M^{(2)} & \\ & & & & J^{(2P)} \end{bmatrix} \qquad (11.13)$$

where

$$J_k^{(2)} = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ ; \quad k = 1, \ldots, M ,$$

and $J^{(2P)}$ is any orthogonal complex structure of order 2P.

The uniqueness of $J_1$ follows after we impose two extra conditions, firstly that no non-trivial time-invariant vectors exist and secondly the requirement (11.6d) that the single particle Hamiltonian is positive definite. In (11.13), the first condition ensures that P = 0, so that $J^{(2P)}$ is not considered. The second condition forces $J_k^{(2)}$ to be $+\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, as opposed to $-\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

In the orthonormal basis $\{\underline{e}_1, \ldots, \underline{e}_{2N}\}$ in which $\hat{A} = \hat{A}_o$ and J = G, the constructed complex Hilbert space $\mathcal{H}^{(1)}$ is $(R^{2N}, G)$, with inner product defined as in (8.16b). In the Fock-Cook representation over $\mathcal{H}^{(1)}$, we have

$$a(\underline{\zeta})\, a^*(\underline{\zeta}_1) \ldots a^*(\underline{\zeta}_n)\, \psi_0$$

$$= \sum_{j=1}^{n} (-1)^{j+1} \langle \underline{\zeta}, \underline{\zeta}_j \rangle_1 \, \Big( \prod_{k \neq j} a^*(\underline{\zeta}_k) \Big) \psi_0 \tag{11.14}$$

Now for all $r = 1,\ldots,N$ , $\quad G\,\underline{e}_r = \underline{e}_{N+r}$ .

Therefore, $a(\underline{e}_{N+r}) = a\,(G\,\underline{e}_r)$

$= -i\; a\,(\underline{e}_r)$ , using (11.14) and $\langle \cdot, G\cdot \rangle_1 = i \langle \cdot , \cdot \rangle_1$ .

In this way, the Segal procedure produces a vacuum representation of the C.A.R. in which there are defined N complex linearly independent annihilation operators, thereby constructing a concrete representation of the Fermion systems discussed in Section 4, from their classical counterparts.

We shall now solve a certain problem which is inverse to that of proposition (11.8). Namely, given a finite dimensional complex Hilbert space $(\mathcal{H}^{(1)}, \langle \cdot , \cdot \rangle_1)$ which is obtained from a real Hilbert space $\mathcal{V}$ via a complex structure J, which classical quadratic Grassmann Hamiltonians give rise, in the Segal procedure, to a positive definite self-adjoint generator of time evolution on $\mathcal{H}^{(1)}$?

<u>Lemma</u> (11.15): Suppose that J is an orthogonal complex structure on $E^{2N}$. Then there exists an orthogonal transformation C such that

$$C^{-1}\, J\, C = G = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

<u>Proof</u>: Since $J^2 = -I$ and $J^T J = I$, J is skew-symmetric, $J^T = -J$. Therefore, as in Section 10, we may explicitly construct an orthogonal matrix C such that

$$C^{-1}(-J)\, C = \begin{bmatrix} 0 & \begin{matrix} \omega_1 & & \\ & \ddots & \\ & & \omega_N \end{matrix} \\ \begin{matrix} -\omega_1 & & \\ & \ddots & \\ & & -\omega_N \end{matrix} & 0 \end{bmatrix} ; \quad \omega_j > 0 \tag{11.16}$$

However, since $J^2 = -I$ , $\omega_j = 1$ for all $j$ .

Hence, (11.16) becomes $C^{-1} J C = G$.

Proposition (11.17): Suppose that $J$ is an orthogonal complex structure on $E^{2N}$. Then $\hat{A}$ satisfies conditions (11.6c) and (11.6d) if and only if $\hat{A} = C \begin{pmatrix} K & L \\ -L & K \end{pmatrix} C^{-1}$ , with $C$ the orthogonal matrix of lemma (11.15) and $L,K$ the real and imaginary parts of an arbitrary positive definite Hermitean matrix $L + iK$.

Proof: From lemma (11.15),

$$J = C G C^{-1} \text{ , with } C \in O(2N,R) .$$

Then condition (11.6c) becomes

$$[C G C^{-1}, \hat{A}] = 0$$

$$\Leftrightarrow [G, C^{-1} \hat{A} C] = 0$$

$$\Leftrightarrow C^{-1} \hat{A} C = \begin{bmatrix} K & L \\ -L & K \end{bmatrix} \text{ for some } K = -K^T , \quad L = L^T \tag{11.18}$$

The additional positivity condition (11.6d) is

$$J \hat{A} > 0$$

$$\Leftrightarrow C G \begin{pmatrix} K & L \\ -L & K \end{pmatrix} C^{-1} > 0 \quad \text{(assuming (11.18))}$$

$$\Leftrightarrow G \begin{pmatrix} K & L \\ -L & K \end{pmatrix} = \begin{pmatrix} -L & K \\ -K & -L \end{pmatrix} > 0$$

$$\Leftrightarrow P^{-1} \begin{pmatrix} -L & K \\ -K & -L \end{pmatrix} P > 0 \quad \left(\text{with } P = 2^{-\frac{1}{2}} \begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}\right)$$

$$\Leftrightarrow \begin{pmatrix} L+iK & 0 \\ 0 & L-iK \end{pmatrix} > 0$$

$$\Leftrightarrow L+iK > 0 \tag{11.19}$$

With $J$ known, the Grassmann Hamiltonian, discussed in Section 10, must have the form $\frac{1}{2} i \, \underline{z}^T \hat{A} \, \underline{Z}$ , with $\hat{A}$ having the above structure.

The reader may recall that our motivation for solving the existence problem for J (Fermion) at this point was that the solution extends readily to infinite degrees of freedom and this sheds light on the extension of existence results for J (Boson) to infinite degrees of freedom. Accordingly, we now recast proposition (11.8) in a more suitable form.

Proposition (11.20): Suppose that $\hat{A}$ is a real $2N \times 2N$ skew symmetric matrix. Then if J is the orthogonal component in the polar decomposition of $-\hat{A}$, J satisfies (11.6a-d).

Proof: Since $-\hat{A}$ is skew symmetric, the orthogonal factor in the polar decomposition of $-\hat{A}$ is a complex structure (e.g. Manuceau in [106]). From well-known results on the polar decomposition, J satisfies all conditions (11.6). It can be verified that this agrees with out previous solution (11.8) for J.

$$J = -[\hat{A}^T \hat{A}]^{-\frac{1}{2}} \hat{A} \qquad (11.21)$$

$$= -[-\hat{A}^2]^{-\frac{1}{2}} \hat{A} \qquad (11.22)$$

$$= -[-C \hat{A}_o^2 C^{-1}]^{-\frac{1}{2}} C \hat{A}_o C^{-1} \text{ , with } \hat{A}_o \text{ as in (10.23)}$$

$$= -C \operatorname{diag.}[|\omega_1|^{-1},\ldots,|\omega_m|^{-1}, |\omega_1|^{-1},\ldots,|\omega_m|^{-1}] \hat{A}_o C^{-1}$$

$$= C G C^{-1} \text{ , in agreement with (11.8).}$$

In the case that $\hat{A}$ is a skew symmetric operator on a real Hilbert space $(V,s)$, the polar decomposition is still well-defined through spectral theory. The spectral theory in real Hilbert space follows from the more familiar spectral theory in complex Hilbert space.

Theorem (11.23): Let M be a self adjoint operator on a separable real Hilbert space $(V,s)$. Then there exists a unique spectral function $E(\lambda)$ such that (1) for all $\lambda$, $E(\lambda)$ is a self-adjoint projection on $V$.

$$E(\lambda)^T = E(\lambda) = E(\lambda)^2 .$$

(2) $E(\lambda)\, E(\mu) = E(\min\{\lambda,\mu\})$, $E(\infty) = I - E(-\infty) = I$

(3) $M = \int_{-\infty}^{\infty} \lambda \, d\,E(\lambda)$ , converging in the strong topology

(4) $\lim_{t\to s^+} E(t) = E(s)$

Proof: Define a complex structure $K = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ on $V \oplus V$ .

The usual inner product $(\psi_1,\phi_1) + (\psi_2,\phi_2)$ between elements $\Psi = \psi_1 \oplus \psi_2$ and $\Phi = \phi_1 \oplus \phi_2$ will be donoted $(\Psi,\Phi)$. Define a K-sesquilinear form

$$\langle\Psi,\Phi\rangle = (\Psi,\Phi) - i(\Psi, K\Phi) . \tag{11.24}$$

$C = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ is a K-conjugation, since

$$C(a+bK)\Psi = aC\Psi - bKC\Psi \quad \text{for all } a,b \in R . \tag{11.25}$$

Since $\langle\Psi,\Psi\rangle = (\Psi,\Psi)$, the elements of $V \oplus V$, furnished with complex structure K and inner product $\langle\cdot,\cdot\rangle$, constitute a complex Hilbert space $V^c$. $V^c$ is equivalent to the extension of $V$ to the complex field by the formal root i of -1, with S extended sesquilinearly.

In the following, if A is an operator on $V$, $A^{[2]}$ will denote $A \oplus A$. Since $M^{[2]}$ is self-adjoint on $V^c$, the spectral theory for complex Hilbert space tells us that there exists a unique complex spectral function $F(\lambda)$ such that $M^{[2]} = \int_{-\infty}^{\infty} \lambda \, d\,F(\lambda)$.

Since $C\, M^{[2]}\, C = M^{[2]}$, $\int_{-\infty}^{\infty} \lambda \; C\, d\,F(\lambda) C = \int_{-\infty}^{\infty} \lambda \, d\,F(\lambda)$

Therefore, $C\, F(\lambda) C = F(\lambda)$ , by uniqueness of the spectral function. That is, $F(\lambda) = \begin{pmatrix} E_1(\lambda) & 0 \\ 0 & E_2(\lambda) \end{pmatrix}$, with $E_{1,2}(\lambda)$ spectral functions on $V$.

However, we must have $E_1(\lambda) = E_2(\lambda)$, otherwise $F(\lambda)$ could be replaced by $\begin{pmatrix} E_2(\lambda) & 0 \\ 0 & E_1(\lambda) \end{pmatrix}$ , contradicting uniqueness of F.

Therefore, $M = \int_{\infty}^{\infty} \lambda \, dE(\lambda)$ , where $E(\lambda) = E_1(\lambda) = E_2(\lambda)$ .

$E(\lambda)$ must be unique, otherwise it would be easy to construct two different spectral functions $F_{1,2}(\lambda)$ for the self-adjoint operator $M^{[2]}$ on $V^c$ .

<u>Proposition</u> (11.24): Let $C(t) = \exp[-\hat{A}t]$ be a one parameter group of orthogonal transformations on a real separable Hilbert space $(V,s)$. If zero does not belong to the point spectrum of $\hat{A}$, then $C(t)$ may be unitarized, with the complex structure $J$ given by the orthogonal factor in the polar decomposition of $-\hat{A}$.

<u>Proof</u>: $[\hat{A}^T \hat{A}]^{-\frac{1}{2}}$ is uniquely defined as a real self-adjoint operator, via the functional calculus based on theorem (11.23). Therefore, $[\hat{A}^T \hat{A}]^{-\frac{1}{2}} \hat{A}^T$ is a real operator which, by the well-known theory of the polar decomposition, satisfies all conditions (11.6).

Since $i\hat{A}$ is a self-adjoint operator on $V^c$, $i\hat{A} = \int_{\infty}^{\infty} \lambda \, dE(\lambda)$, where $E(\lambda)$ is the unique spectral function for $i\hat{A}$ on $V^c$.

Then
$$J = -[-\hat{A}^2]^{-\frac{1}{2}} \hat{A}$$
$$= i[(i\hat{A})^2]^{-\frac{1}{2}} i\hat{A}$$
$$= i \int_{-\infty}^{\infty} \frac{\lambda}{|\lambda|} \, dE(\lambda)$$
$$J = i(I - 2E(0)) \tag{11.25}$$

## SECTION 12 - COMPLEX STRUCTURES FOR BOSE QUANTIZATION OF CLASSICAL FIELDS

In this section, the classical real symplectic space $(M,B)$ will be assumed to be of a special type. Namely, $M$ will be a real Hilbert space with inner product $(\underline{\zeta},\underline{\xi})$ and the symplectic form will be

$$B(\underline{\zeta},\underline{\xi}) = (\underline{\zeta},G\underline{\xi}) , \tag{12.1}$$

where G is a pre-determined skew adjoint complex structure on $M$. As discussed in Section 9, this assumption can always be made for finite degrees of freedom. In addition, familiar infinite symplectic systems such as the Klein-Gordon system can be integrated into this framework [91].

The dynamics of a classical system may be viewed as a continuous one-parameter group of symplectic transformations C(t). In the terminology of Section 5, C(t) is a group of iG-unitary transformations. Therefore, if C(t) is strongly continuous, $C(t) = \exp(-tG\hat{H})$, by an extension of Stone's theorem to indefinite inner product space [108].

In finite degrees of freedom and with $\hat{H} > 0$ the dynamics is stable, by theorem (1.26). Proposition (9.6) then gives the unique solution for J. If we write the generator $-G\hat{H}$ of C(t) as $\hat{A}$, we can verify that (11.22) remains valid, as in the Fermion case.

$$\begin{aligned} J &= -[-\hat{A}^2]^{-\frac{1}{2}}\,\hat{A} \qquad\qquad (12.2) \\ &= [-G\hat{H}\,G\hat{H}]^{-\frac{1}{2}}\,G\hat{H} \\ &= [-GC^{T^{-1}}\,\hat{H}_o\,C^{-1}\,G\,C^{T^{-1}}\,\hat{H}_o\,C^{-1}]^{-\frac{1}{2}}\,GC^{T^{-1}}\,\hat{H}_o\,C^{-1} \\ &\qquad\qquad \text{(with C and } \hat{H}_o \text{ as in (1.26))} \\ &= [-CG\,\hat{H}_o\,C^{-1}\,CG\,\hat{H}_o\,C^{-1}]^{-\frac{1}{2}}\,CG\,\hat{H}_o\,C^{-1} \quad (\text{since } C, C^T \in Sp(2N,R)) \\ &= C\,\text{diag.}[|s_1|^{-1},\ldots,|s_N|^{-1},|s_1|^{-1},\ldots,|s_N|^{-1}]\,\hat{A}_o\,C^{-1} \\ &= CGC^{-1}\ , \quad \text{in agreement with (9.6).} \end{aligned}$$

The expression (12.2) for J is not simply the orthogonal component in the polar decomposition of $\hat{A}$, as in the Fermion case, since

$$\hat{A}^T \hat{A} = (G\hat{H})^T G\hat{H} = \hat{H}^T G^T G\hat{H} = \hat{H}^2 ,$$

which is not necessarily equal to $(G\hat{H})^2$, since $\hat{H}$ does not necessarily commute with G. However, $iG\hat{H}$ is self-adjoint with respect to the indefinite metric $(\cdot, iG\cdot)$. Therefore,

$$-(G\hat{H})^2 = (i\,G\hat{H})^+ \, i\,G\hat{H} ,$$

where $\hat{A}^+$ denotes the adjoint $-G\,\hat{A}^T G$ of $\hat{A}$, with respect to the indefinite metric. Hence, we recover an expression similar to (11.21).

$$J = -[\hat{A}^+ \hat{A}]^{-\frac{1}{2}} \hat{A} \tag{12.3}$$

J is the (pseudo) orthogonal component in the polar decomposition of an operator on Krein space. To extend this result to infinite degrees of freedom, we shall use the spectral theory in Krein space, which was presented in Section 5. In the applications to classical mechanics, we are concerned with real symplectic transformations on a real classical space. We need to modify the spectral theory for complex Krein space to accommodate a real symplectic space. The following result achieves the simultaneous canonical reduction of a strictly positive real self-adjoint operator $\hat{H}$ and a real skew-symmetric complex structure G.

<u>Proposition</u> (12.4): Let $(\mathcal{H},(\cdot,\cdot))$ be a real separable Hilbert space, G be a skew adjoint complex structure on $\mathcal{H}$ and $\hat{H}$ be a self adjoint operator on $\mathcal{H}$ such that for some $\varepsilon > 0$, $H - \varepsilon I$ is strictly positive. Then there exists a dense nuclear space $\mathcal{N} \subset \mathcal{H}$, such that the dual $\mathcal{N}'$ of $\mathcal{N}$ contains a basis $\{\psi_\alpha;\ \alpha \in S\} \cup \{\psi'_\alpha;\ \alpha \in S\}$ (S may be considered to be a set of spectral indices) in which simultaneously $G = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and $\hat{H} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}$, with D positive definite diagonal.

**Proof:** In the following, the complex structure K on $H \oplus H$, the K-sesquilinear form $<\cdot,\cdot>$ on $H \oplus H$, the complex Hilbert space $H^c$, the conjugation $C$ and the operator $A^{[2]} = A \oplus A$, will be defined as in the proof of proposition (11.23).

The involution $\eta = KG^{[2]} = \begin{pmatrix} 0 & -G \\ G & 0 \end{pmatrix}$ is self-adjoint on $H^c$. Define a strictly $\eta$-positive operator $M = \eta \hat{H}^{[2]}$. M satisfies all requirements given in proposition (5.12).

Therefore, there exists $\eta$-orthogonal subspaces $H^c_\pm$ such that

(1) $<\cdot, \pm\eta\cdot>$ is positive definite on $H^c_\pm$ .

(2) There exist dense nuclear spaces $N_\pm \subset H^c_\pm$ such that the dual spaces $N'_\pm$ contain bases $\{\Psi^{(\pm)}_{\underline{k},\underline{\ell}}\}$ of eigendistributions of M satisfying

$$M\, \Psi^{(\pm)}_{\underline{k},\underline{\ell}} = m^{\pm}(\underline{k},\underline{\ell})\, \Psi^{\pm}_{\underline{k},\underline{\ell}} \quad \text{with} \quad m^{\pm} \gtrless 0 \,, \tag{12.5a}$$

$$\pm < \Psi^{(\pm)}_{\underline{k},\underline{\ell}} \,,\, \eta\, \Psi^{(\pm)}_{\underline{k}',\underline{\ell}'} > = \delta_{\underline{k},\underline{k}'}\, \delta(\underline{\ell} - \underline{\ell}')\,, \tag{12.5b}$$

$$< \Psi^{(+)}_{\underline{k},\underline{\ell}} \,,\, \eta\, \Psi^{(-)}_{\underline{k}',\underline{\ell}'} > = 0 \,. \tag{12.5c}$$

Now

$$\begin{aligned} M\, C\, \Psi^{(+)}_{\underline{k},\underline{\ell}} &= -\, C\, M\, \Psi^{(+)}_{\underline{k},\underline{\ell}} \\ &= -\, m^{+}(\underline{k},\underline{\ell})\, C\, \Psi^{(+)}_{\underline{k},\underline{\ell}} \,. \end{aligned} \tag{12.6}$$

Hence, $\Psi^{(+)}_{\underline{k},\underline{\ell}}$ is an eigendistribution of M corresponding to spectral value $m^+$ if and only if $C\, \Psi^{(+)}_{\underline{k},\underline{\ell}}$ is an eigendistribution corresponding to spectral value $-\, m^+$ .

In addition,

$$\begin{aligned} <C\Psi,\, \eta C\Phi> &= (C\Psi,\, KG^{[2]}\, C\Phi) + i(C\Psi,\, G^{[2]}\, C\Phi) \\ &= -(\Psi,\, KG^{[2]}\Phi) - i(\Psi,\, G^{[2]}\Phi) \\ &= -<\Psi,\, \eta\Phi> \,. \end{aligned} \tag{12.7}$$

Therefore, we may take the bases of eigendistributions to be $\{\Psi^{(+)}_{\underline{k},\underline{\ell}}\}$ and $\{C\Psi^{(+)}_{\underline{k},\underline{\ell}}\}$ .

From the basis distributions

$$\Psi_{\underline{k},\underline{\ell}} = \psi_{\underline{k},\underline{\ell}} \oplus \psi'_{\underline{k},\underline{\ell}}$$

and

$$C\Psi_{\underline{k},\underline{\ell}} = \psi_{\underline{k},\underline{\ell}} \oplus - \psi'_{\underline{k},\underline{\ell}}$$

let us construct

$$X_{\underline{k},\underline{\ell}} = \tfrac{1}{2}(\Psi_{\underline{k},\underline{\ell}} + C\Psi_{\underline{k},\underline{\ell}}) = \psi_{\underline{k},\underline{\ell}} \oplus \underline{0}$$

and

$$\Xi_{\underline{k},\underline{\ell}} = \tfrac{1}{2}K(\Psi_{\underline{k},\underline{\ell}} - C\Psi_{\underline{k},\underline{\ell}}) = - \psi'_{\underline{k},\underline{\ell}} \oplus \underline{0} .$$

Since $\{\Psi_{\underline{k},\underline{\ell}} , C\Psi_{\underline{k},\underline{\ell}}\}$ is a basis for $H^C$, every element $\Psi = \psi \oplus \psi'$ being expanded as

$$\psi \oplus \psi' = \Psi = \int [a(\underline{k},\underline{\ell}) + b(\underline{k},\underline{\ell})K]\Psi_{\underline{k},\underline{\ell}} + [a'(\underline{k},\underline{\ell}) + b'(\underline{k},\underline{\ell})K]C\Psi_{\underline{k},\underline{\ell}} \, d\sigma(\underline{k},\underline{\ell})$$

$$(a,b,a',b' \in R)$$

$$= \int [(a+a') + (b+b')K]X_{\underline{k},\underline{\ell}} + [(b-b') + (a-a')K]\Xi_{\underline{k},\underline{\ell}} \, d\sigma$$

$$= \int [a(\underline{k},\underline{\ell}) + a'(\underline{k},\underline{\ell})]\psi_{\underline{k},\underline{\ell}} + [b'(\underline{k},\underline{\ell}) - b(\underline{k},\underline{\ell})]\psi'_{\underline{k},\underline{\ell}} \, d\sigma$$

$$\oplus \int [b(\underline{k},\underline{\ell}) + b'(\underline{k},\underline{\ell})]\psi_{\underline{k},\underline{\ell}} + [a'(\underline{k},\underline{\ell}) - a(\underline{k},\underline{\ell})]\psi'_{\underline{k},\underline{\ell}} \, d\sigma$$

$\{\psi_{\underline{k},\underline{\ell}}, \psi'_{\underline{k},\underline{\ell}}\}$ is evidently a basis for the real Hilbert space $H$.

Now

$$\langle X_{\underline{k},\underline{\ell}} , G^{[2]} \Xi_{\underline{k}',\underline{\ell}'}\rangle = \tfrac{1}{2} \delta_{\underline{k},\underline{k}'} \delta(\underline{\ell} - \underline{\ell}') \tag{12.8}$$

and

$$\langle X_{\underline{k},\underline{\ell}} , G^{[2]} X_{\underline{k}',\underline{\ell}'}\rangle = \langle \Xi_{\underline{k},\underline{\ell}} , G^{[2]} \Xi_{\underline{k}',\underline{\ell}'}\rangle = 0 ,$$

which implies

$$(\psi_{\underline{k},\underline{\ell}} , G \psi'_{\underline{k}',\underline{\ell}'}) = (\psi'_{\underline{k},\underline{\ell}} , G \psi'_{\underline{k}',\underline{\ell}'}) = 0 \tag{12.9}$$

and

$$(\psi_{\underline{k},\underline{\ell}} , G \psi'_{\underline{k}',\underline{\ell}'}) = - \tfrac{1}{2} \delta_{\underline{k},\underline{\ell}'} \delta(\underline{\ell} - \underline{\ell}') . \tag{12.10}$$

Also,

$$\langle X_{\underline{k},\underline{\ell}}, \hat{H}^{[2]} X_{\underline{k}',\underline{\ell}'}\rangle = \tfrac{1}{4} \langle \Psi^{(+)}_{\underline{k},\underline{\ell}} + C\Psi^{(+)}_{\underline{k},\underline{\ell}}, \eta M(\Psi^{(+)}_{\underline{k}',\underline{\ell}'} + C\Psi^{(+)}_{\underline{k}',\underline{\ell}'})\rangle$$

$$= \tfrac{1}{2} m^{+}(\underline{k},\underline{\ell}) \quad \delta_{\underline{k},\underline{k}'} \; \delta(\underline{\ell} - \underline{\ell}') \; .$$

This implies

$$(\psi_{\underline{k},\underline{\ell}}, \hat{H} \psi_{\underline{k}',\underline{\ell}'}) = \tfrac{1}{2} m^{+}(\underline{k},\underline{\ell}) \, \delta_{\underline{k},\underline{k}'} \, \delta(\underline{\ell} - \underline{\ell}') \qquad (12.11)$$

Similarly,

$$(\psi'_{\underline{k},\underline{\ell}}, \hat{H} \psi'_{\underline{k}',\underline{\ell}'}) = \tfrac{1}{2} m^{+}(\underline{k},\underline{\ell}) \delta_{\underline{k},\underline{k}'} \, \delta(\underline{\ell} - \underline{\ell}')$$

and

$$(\psi_{\underline{k},\underline{\ell}}, \hat{H} \psi'_{\underline{k}',\underline{\ell}'}) = (\psi'_{\underline{k},\underline{\ell}}, \hat{H} \psi_{\underline{k}',\underline{\ell}'}) = 0 \; . \qquad (12.12)$$

By (12.9-12), the basis of distributions $\{2^{\frac{1}{2}} \psi_{\underline{k},\underline{\ell}}\} \cup \{2^{\frac{1}{2}} \psi'_{\underline{k},\underline{\ell}}\}$ satisfies all the properties required to verify proposition (12.4).

If $\hat{H}$ satisfies the conditions specified in proposition (12.4), it follows that the symplectic dynamics $\underline{\xi}(t) = \exp[-G\hat{H}t] \, \underline{\xi}(o)$ (12.13) may be unitarized.

<u>Proposition</u> (12.14): Let $\hat{H}$ be a self-adjoint operator on real Hilbert space $(H,(\cdot,\cdot))$, such that $\hat{H} = \varepsilon I \geq 0$, for some $\varepsilon > 0$. Then there exists a complex structure J on $H$, by which the symplectic dynamics (12.5) may be unitarized.

<u>Proof</u>: As in the proof of (12.4), let $M = \eta \hat{H}^{[2]}$, with $\eta = KG^{[2]}$. With $H_j$ and $E_j(s)$ defined as in theorem (5.9), define a linear operator $J^{[2]}$ by

$$H_j \ni \phi \rightarrow J^{[2]}\phi = - K[1 - 2E_j(0)]\phi \quad ; \quad j = 1,2 \; . \qquad (12.15)$$

Although $J^{[2]}$ is an operator on the complex space $H_c$, the real space $H \oplus \{0\} \approx H$ is $J^{[2]}$-invariant. To see this, recall that proposition (12.4) establishes a basis of eigendistributions

$$\{X_{\underline{k},\underline{\ell}} = \tfrac{1}{2}(\Psi^{(+)}_{\underline{k},\underline{\ell}} + C\Psi^{(+)}_{\underline{k},\underline{\ell}})\} \cup \{\Xi_{\underline{k},\underline{\ell}} = \tfrac{1}{2}K(\Psi^{(+)}_{\underline{k},\underline{\ell}} - C\Psi^{(+)}_{\underline{k},\underline{\ell}})\}$$

for $\mathcal{H} \oplus \{0\}$.

From the definition of X and $\Xi$, it is easily varified that

$$J^{[2]} X_{\underline{k},\underline{\ell}} = - \Xi_{\underline{k},\underline{\ell}} , \tag{12.16a}$$

and

$$J^{[2]} \Xi_{\underline{k},\underline{\ell}} = X_{\underline{k},\underline{\ell}} . \tag{12.16b}$$

We may define J to be the restriction of $J^{[2]}$ to $\mathcal{H} \oplus \{0\} \simeq \mathcal{H}$. Clearly,

$$J^{[2]2} = -I^{[2]}, \text{ so that } J^2 = -I . \tag{12.17}$$

$J^{[2]}$ is $\eta$-unitary, since

$$J^{[2]+} \ J^{[2]} = [1-2E_j(0)]K^+K\,[1-2E_j(0)] = I . \tag{12.18}$$

However, $<\cdot, KG^{[2]}\cdot>$ restricted to $\mathcal{H} \oplus 0$ is simply a multiple of the symplectic form on $\mathcal{H}$.

$$< \begin{pmatrix}\phi\\0\end{pmatrix}, KG^{[2]}\begin{pmatrix}\psi\\0\end{pmatrix}> = \left(\begin{pmatrix}\phi\\0\end{pmatrix}, \begin{pmatrix}0 & -G\\ G & 0\end{pmatrix}\begin{pmatrix}\psi\\0\end{pmatrix}\right) + i\left(\begin{pmatrix}\phi\\0\end{pmatrix}, \begin{pmatrix}G & 0\\ 0 & G\end{pmatrix}\begin{pmatrix}\psi\\0\end{pmatrix}\right)$$

$$= i\ (\phi, G\psi)$$

$$= i\ B(\phi,\psi) . \tag{12.19}$$

Therefore, from (12.18), J is symplectic on $\mathcal{H}$. To show that $- B(\phi,J\psi)$ is positive definite ,

$$- B(\phi,J\psi) = i <\begin{pmatrix}\phi\\0\end{pmatrix}, KG^{[2]}\ J^{[2]}\begin{pmatrix}\phi\\0\end{pmatrix}> , \text{ by } (12.19)$$

Now

$$i\ <X_{\underline{k},\underline{\ell}},\ KG^{[2]}\ J^{[2]}\ X_{\underline{k}',\underline{\ell}'}> = -i <X_{\underline{k},\underline{\ell}},\ KG^{[2]}\ \Xi_{\underline{k}',\underline{\ell}'}> , \text{ by } (12.16)$$

$$= <X_{\underline{k},\underline{\ell}},\ G^{[2]}\ \Xi_{\underline{k}',\underline{\ell}'}>$$

(since $<\cdot,\cdot>$ is a K-sesquilinear form)

$$= \tfrac{1}{2}\ \delta_{\underline{k},\underline{k}'}\ \delta(\underline{\ell}-\underline{\ell}') , \quad \text{by } (12.8) .$$

Similarly,

$$i < \Xi_{\underline{k},\underline{\ell}},\ KG^{[2]}\ J^{[2]}\ \Xi_{\underline{k}',\underline{\ell}'}> = \tfrac{1}{2}\ \delta_{\underline{k},\underline{k}'}\ \delta(\underline{\ell}-\underline{\ell}') .$$

To show that J commutes with $G\hat{H}$,

$$J^{[2]} G^{[2]} \hat{H}^{[2]} X_{\underline{k},\underline{\ell}} = - J^{[2]} KM X_{\underline{k},\underline{\ell}}$$

$$= - m^{+}(\underline{k},\underline{\ell}) J^{[2]} \Xi_{\underline{k},\underline{\ell}} , \quad \text{by (12.6)}$$

$$= - m^{+}(\underline{k},\underline{\ell}) X_{\underline{k},\underline{\ell}} , \quad \text{by (12.16)}$$

and

$$G^{[2]} \hat{H}^{[2]} J^{[2]} X_{\underline{k},\underline{\ell}} = - G^{[2]} \hat{H}^{[2]} \Xi_{\underline{k},\underline{\ell}} , \quad \text{by (12.18)}$$

$$= KM \Xi_{\underline{k},\underline{\ell}}$$

$$= m^{+}(\underline{k},\underline{\ell}) K^2 X_{\underline{k},\underline{\ell}} ,$$

$$= - m^{+}(\underline{k},\underline{\ell}) X_{\underline{k},\underline{\ell}}$$

$$= J^{[2]} G^{[2]} \hat{H}^{[2]} X_{\underline{k},\underline{\ell}} . \qquad (12.20)$$

Similarly,

$$J^{[2]} G^{[2]} \hat{H}^{[2]} \Xi_{\underline{k},\underline{\ell}} = G^{[2]} \hat{H}^{[2]} J^{[2]} \Xi_{\underline{k},\underline{\ell}} = - m^{+}(\underline{k},\underline{\ell}) \Xi_{\underline{k},\underline{\ell}} .$$

We have now verified that $J^{[2]}$, restricted to $\mathcal{H} \oplus \{0\}$, satisfies all the properties of a unitarizing complex structure for the symplectic dynamical group $\exp[-G\hat{H}t]$.

The above solution for J generalizes that found in Section 9 for finite dimensional systems, namely

$$J = [-G\hat{H} G\hat{H}]^{-\frac{1}{2}} G\hat{H} . \qquad (12.21)$$

The use of complex structure K on $\mathcal{H} \oplus \mathcal{H}$ is equivalent to the extension of $\mathcal{H}$ to the complex field by the formal square root i of -1. If $\underline{\phi}_1, \underline{\phi}_2 \in \mathcal{H}_{1,2}$,

$$[-G\hat{H}G\hat{H}]^{-\frac{1}{2}} G\hat{H}[\underline{\phi}_1 \oplus \underline{\phi}_2] = [-M^2]^{-\frac{1}{2}} iM[\underline{\phi}_1 \oplus \underline{\phi}_2] ,$$

(where $M = \eta\hat{H}$, $\eta = iG$)

$$= - i \int_{-\infty}^{\infty} \frac{t}{|t|} dE_1(t)\underline{\phi}_1 \oplus - i \int_{-\infty}^{\infty} \frac{t}{|t|} dE_2(t)\underline{\phi}_2$$

(using spectral theory in Krein space)

$$= - i\,[I - 2E_1(0)]\phi_1 \oplus - i\,[I - 2E_2(0)]\phi_2 \; , \qquad (12.22)$$

in agreement with (12.15).

Since completing this work, we have obtained a copy of Paneitz's publication [108], which achieves (12.21) using a different method. Paneitz notes that $-G\hat{H}$ is skew symmetric with respect to the positive inner product $B(\phi,-G\hat{H}\psi) = (\phi,\hat{H}\psi)$. The indefinite metric is thus avoided, since J of (12.21) is viewed as the orthogonal component of $G\hat{H}$ with respect to polar decomposition in an ordinary real Hilbert space. The advantage of using our approach, via spectral theory in Krein space, is that we are working with a metric which is not $\hat{H}$-dependent and this allows us to obtain an expression for J which is very similar to (11.24), which we obtained in the case of Fermi-Dirac statistics. The advantage of Paneitz's method is that by avoiding the indefinite metric, the mathematics is better established and easier to follow.

From (12.22), we see that the existence of J follows from the existence of the projection operator $E(o) = E_1(o) \oplus E_2(o)$, which separates the negative frequency states from the positive frequency states in a dynamically invariant fashion. When such an unambiguous separation is not possible, it is said that the Klein paradox is in vogue. The relationship between the Klein paradox and dynamical instability has been discussed by Fulling [109].

In fact, an expression similar to (12.21) was first derived by Ashtekar and Magnon [110] for an infinite dimensional system consisting of the Klein Gordon equation in stationary curved space-time. They arrived at this result via a different route, by imposing a condition which in the case of finite degrees of freedom, would be

$$H = \tfrac{1}{2}(\underline{z},\hat{H}\underline{z}) = \langle \underline{z}, -JG\hat{H}\underline{z}\rangle_1 \quad (\text{since } -JG\hat{H} \text{ is the Hamiltonian on } \mathcal{H}^{(1)})$$

$$= \alpha^{-2}(\underline{z},\hat{H}\underline{z}) + \alpha^{-2} i(\underline{z},GJG\hat{H}\underline{z}) \quad (\text{by } (8.16a)) \qquad (12.23)$$

Since (12.23) must be true for all $\underline{z} \in M$

$$\alpha = \sqrt{2}$$

and

$$(GJG\hat{H}) = -(GJG\hat{H})^T$$

$$= \hat{H}GGJ \quad \text{(since, from (8.16a), GJ is symmetric)}$$

$$= -\hat{H}J$$

$$= GG\hat{H}J$$

$\Rightarrow$ $[J,G\hat{H}] = 0$ , a condition which we formerly interpreted as unitarity of time evolution on $H^{(1)}$.

(12.23) does not imply that every classical value of the Hamiltonian H can be attained as an expectation value on the quantum mechanical single particle space. The right hand side is not an expectation value until $\underline{z}$ is normalized in $H^{(1)}$. For example, if we take $\hat{H} = \text{diag}[\omega,\omega]$, with $\omega > 0$, $H = \frac{1}{2}\,\omega(\underline{z},\underline{z})$ can take any non-negative value. However, we know from (12.21) that since the classical system executes single harmonic motion, there is only one single phonon energy $\omega$ (assuming $\hbar = 1$). When $\hat{H} = \omega I$, we know that the unique solution for J is J = G. Then $\langle\underline{z},\underline{z}'\rangle_1 = \frac{1}{2}(\underline{z},\underline{z}') - \frac{1}{2}\,i(\underline{z},G\underline{z}')$ (assuming $\alpha = \sqrt{2}$) and $\langle\underline{z},\underline{z}'\rangle_1 = \frac{1}{2}(\underline{z},\underline{z})$ .

Therefore, $\langle\underline{z},\underline{z}\rangle_1 = 1 \Rightarrow (\underline{z},\underline{z}) = 2$

The restriction $(\underline{z},\underline{z}) = 2$ implies $\frac{1}{2}(\underline{z},\hat{H}\underline{z}) = \frac{1}{2}\,\omega(\underline{z},\underline{z}) = \omega$, an expression which equates the classical Hamiltonian to the unique single phonon energy.

In the case of finite degrees of freedom, we have shown that unless $iG\hat{H}$ is similar to a Hermitean matrix, so that all frequencies are real, J does not exist. It is not immediately obvious that this result will generalize to infinite degrees of freedom. For example, one could envisage a case where the spectrum of $iG\hat{H}$ is real, except for a negligible set of complex frequencies. The remainder of this section will be devoted to proving that if the dynamics can be unitarized, then $iG\hat{H}$ is similar to a self adjoint operator, so that not even one of the infinite classical degrees of freedom can exhibit unstable dynamics.

Let G be a distinguished real skew symmetric complex structure on real space $(H,(\cdot,\cdot))$. This defines a symplectic form $B(\cdot,\cdot) = (\cdot,G\cdot)$ on $H$.

In the Segal quantization procedure, complex single particle space $H_c$ is constructed from real symplectic space $M = (H,B)$ by choosing a complex structure J on $H$ so that a prescribed one-parameter group C(t) of symplectic transformations on $M$ is a strongly continuous group of unitary transformations with respect to the complex inner product $\langle\cdot,\cdot\rangle_1$ of (8.16a). Then, by Stone's theorem, $C(t) = \exp(\hat{A}_c t)$, where $\hat{A}_c$ is skew adjoint on $H_c$. $\hat{A}_c$ will be denoted by $\hat{A}$ when considered as an operator on the real space $H$. Since $-GJ \geq 0$ and $-GJ$ has a bounded inverse $-JG$ on $H$, it must be true that $-GJ > \varepsilon I$ for some $\varepsilon > 0$. Hence, we may apply proposition (12.4) to $-GJ$. There must exist a basis

$$P = \{\psi_\alpha;\ \alpha \in S\} \cup \{\psi'_\alpha;\ \alpha \in S\}$$

of distributions such that

$$(\psi_\alpha, -GJ\ \psi_\beta) = (\psi'_\alpha, -GJ\ \psi'_\beta) = d(\alpha)\ \delta(\alpha-\beta) \text{ with } d > 0\ , \quad (12.24)$$

$$(\psi_\alpha, -GJ\ \psi'_\beta) = 0\ ,$$

$$(\psi'_\alpha,\ G\ \psi_\beta) = \delta(\alpha-\beta)$$

and

$$(\psi_\alpha,\ G\ \psi_\beta) = (\psi'_\alpha, G\ \psi'_\beta) = 0\ .$$

For the remainder of this section, it will be assumed that kernels of operators are specified in a certain chosen orthonormal basis P' of the same cardinality as P,

$$P' = \{\phi_\alpha;\ \alpha \in S\} \cup \{\phi'_\alpha;\ \alpha \in S\}\ .$$

$$(\phi'_\alpha,\phi'_\beta) = (\phi_\alpha,\phi_\beta) = \delta(\alpha-\beta)$$

$$(\phi_\alpha,\phi'_\beta) = 0\ .$$

Let us define a linear operator $G_o$ by

$$G_o\, \phi_\alpha = \phi'_\alpha \quad \text{and} \quad G_o\, \phi'_\alpha = -\, \phi_\alpha \ .$$

Since P' is an orthonormal basis, $G_o$ is the operator for which

$$(\phi_\alpha,\, G_o\, \phi_\beta) \;=\; (\phi'_\alpha,\, G_o\, \phi'_\beta) \;=\; 0$$

and

$$-(\phi_\alpha,\, G_o\, \phi_\beta) \;=\; (\phi'_\alpha,\, G_o\, \phi_\beta) \;=\; \delta(\alpha-\beta)\ .$$

One can define a linear transformation C by

$$C\phi_\alpha \;=\; \psi_\alpha \quad \text{and} \quad C\phi'_\alpha \;=\; \psi'_\alpha \ .$$

Then

$$C^T G C \;=\; G_o \quad (C^T \text{ is the transpose of } C) \tag{12.25}$$

and

$$-C^T GJC = D^{[2]} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}$$, with D positive definite diagonal in the basis P'. (12.26)

From (12.25),

$$C^T G = G_o\, C^{-1} \tag{12.27}$$

Substituting this in (12.26),

$$-G_o\, C^{-1}\, JC = D^{[2]}\ ,$$

implying

$$C^{-1}\, JC = G_o\, D^{[2]} = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} .$$

Then

$$J^2 = -\, I \Rightarrow D^2 = I$$

$$\Rightarrow D = I \quad \text{and} \quad D^{[2]} = I^{[2]}\ .$$

Now (12,26) implies

$$-\, GJ = C^{-1^T}\, C^{-1}\ . \tag{12.28}$$

Since GJ is bounded, $C^{-1}$ is bounded.

By (12.25),

$$C = GC^{-1^T}\, G_o\ ,$$

which must also be bounded.

Define $H_c'$ to be $H$ with complex structure $G_o = C^{-1} JC$ and with inner product

$$<\cdot,\cdot>_1' = \tfrac{1}{2}(\cdot,\cdot) - \tfrac{1}{2} i (\cdot, G_o \cdot) .$$

Let

$$\hat{A}' = C^{-1} \hat{A} C .$$

$\hat{A}_c'$ will denote $\hat{A}'$ as an operator on complex space $H_c'$ .

Lemma (12.29): $\quad \text{dom } \hat{A}' = \text{dom } \hat{A}_c'$ .

This follows from $<\psi,\psi>_1' = (\psi,\psi)$ .

Lemma (12.30): $\quad \psi \in \text{dom } \hat{A}_c \Leftrightarrow C^{-1} \psi \in \text{dom } \hat{A}_c'$

This follows from $<\hat{A}_c \psi, \hat{A}_c \psi>_1 < \infty \Leftrightarrow -(\hat{A}\psi, GJ \hat{A}\psi) < \infty$

$$\Leftrightarrow (\hat{A}\psi, {C^{-1}}^T C^{-1} \hat{A}\psi) < \infty$$

$$\Leftrightarrow (C^{-1} \hat{A} C C^{-1}\psi, C^{-1} \hat{A} C C^{-1}\psi) < \infty$$

Lemma (12.31): $\quad \hat{A}_c'$ is skew adjoint $\Leftrightarrow$ $\hat{A}_c$ is skew adjoint.

Proof: First note that

$$<\phi,\hat{A}\psi>_1 = (\phi, {C^{-1}}^T C^{-1} \hat{A}\psi) - i (\phi, G\hat{A}\psi)$$

$$= <C^{-1}\phi, \hat{A}' C^{-1}\psi>_1' \quad (\text{since } G = {C^{-1}}^T G_o C^{-1}) .$$

Then, using lemma (12.30), it follows that the skew adjointness of $\hat{A}_c$ is equivalent to the skew adjointness of $\hat{A}_c'$ .

Lemma (12.32): $\quad \hat{A}_c'$ is skew adjoint $\Leftrightarrow$ $\hat{A}'$ is skew adjoint and $G_o \hat{A}'$ is self adjoint on $H$.

Proof ($\Rightarrow$): If $\hat{A}_c'$ is skew adjoint, it can be seen, after taking real and imaginary parts of $<\phi, \hat{A}'\psi>_1'$, that $\hat{A}'$ is skew Hermitean and $G_o \hat{A}'$ is Hermitean on $H$. To prove that $\hat{A}'$ is skew-adjoint on $H$, we now need only to prove that $\text{dom } (\hat{A}')^T \subseteq \text{dom } \hat{A}'$.

Suppose $\phi \in \text{dom } (\hat{A}')^T$. According to the Frechet-Riesz theorem, this is equivalent to $\psi \to (\phi, \hat{A}'\psi)$ being continuous.

This implies that

$$\psi \to (\phi, G_o \hat{A}'\psi) \text{ is continuous, since } G_o \text{ is bounded },$$

$$\Rightarrow \quad \psi \to (\phi, \hat{A}'\psi) - i(\phi, G_o \hat{A}'\psi) \text{ is continuous },$$

$$\Rightarrow \quad \phi \in \text{dom } (\hat{A}'_c)^\dagger .$$

However, dom $(\hat{A}'_c)^\dagger$ = dom $\hat{A}'_c$ , since $\hat{A}'_c$ is skew adjoint, by assumption and dom $\hat{A}'_c$ = dom $\hat{A}'$ , by lemma (12.29).

Therefore, dom $(\hat{A}')^T \subseteq$ dom $\hat{A}'$, as required, and so $\hat{A}'$ must be skew adjoint.

Now to prove that $G_o \hat{A}'$ is self adjoint, we shall use the following lemma:

<u>Lemma</u> (12.33): Assume that $\hat{A}'_c$ is skew adjoint. Then

$$\psi \in \text{dom } \hat{A}' \Rightarrow G_o \psi \in \text{dom } \hat{A}'$$

and

$$\hat{A}' G_o \psi = G_o \hat{A}' \psi .$$

<u>Proof</u>: Assume that $\hat{A}'_c$ is skew adjoint.

This implies, by taking real and imaginary parts of the inner product in $H_c$, that for all $\phi \in$ dom $\hat{A}'$ (= dom $\hat{A}'_c$, by lemma (12.29)),

$$(\psi, \hat{A}'\phi) = - (\hat{A}'\psi, \phi) \tag{12.34a}$$

and

$$(\psi, G_o \hat{A}'\phi) = - (\hat{A}'\psi, G_o \phi) . \tag{12.34b}$$

Since $G_o$ is a skew symmetric complex structure and is therefore orthogonal on $H$, (12.34) may be replaced by

$$\forall \phi \in \text{dom } \hat{A}', (G_o \psi, G_o \hat{A}' \phi) = - (G_o \hat{A}' \psi, G_o \phi)$$

and

$$- (G_o \psi, \hat{A}' \phi) = (G_o \hat{A}' \psi, \phi) .$$

Since $\hat{A}'_c$ is skew adjoint, this implies that

$$G_o \psi \in \text{dom } \hat{A}'_c \ (= \text{dom } \hat{A}')$$

and

$$- \hat{A}' G_o \psi = - G_o \hat{A}' \psi .$$

Now we shall use lemma (12.33) to prove that

$$\text{dom } (G_o \hat{A}')^T \subseteq \text{dom } (G_o \hat{A}') ,$$

from which it follows that the Hermitean operator $G_o \hat{A}'$ is self adjoint.

$$\forall \psi \in \text{dom } G_o \hat{A}', (\phi, G_o \hat{A}' \psi) = - (G_o \phi, \hat{A}' \psi) .$$

Suppose $\phi \in \text{dom } (G_o \hat{A}')^T$ .

That is,

$$\exists \phi', \forall \psi \in \text{dom } (G_o \hat{A}'), (\phi, G_o \hat{A}' \psi) = (\phi', \psi)$$

$$\Rightarrow \exists \phi' \quad \forall \psi \in \text{dom } (G_o \hat{A}'), (- G_o \phi, \hat{A}' \psi) = (\phi', \psi)$$

$\Rightarrow \exists \ \phi', \forall \psi \in \text{dom } \hat{A}', (- G_o \phi, \hat{A}' \psi) = (\phi', \psi)$ (since $\text{dom } G_o \hat{A}' \supseteq \text{dom } \hat{A}'$)

$\Rightarrow G_o \psi \in \text{dom } \hat{A}'$ and $\phi' = - \hat{A}' (- G_o \phi)$ (since we have already proven that $\hat{A}'$ is skew adjoint)

$\Rightarrow \phi (= - G_o G_o \phi) \in \text{dom } \hat{A}'$ (by lemma (12.33)) .

and $\phi' = \hat{A}' G_o \phi$

$= G_o \hat{A}' \phi$ (by lemma (12.33)) .

Therefore, $\text{dom } (G_o \hat{A}')^T = \text{dom } (G_o \hat{A}')$ and $G_o \hat{A}'$ is Hermitean, and therefore is self adjoint.

We have now proved lemma (12.32) ($\Rightarrow$). The converse is obvious.

<u>Proposition</u> (12.35): Suppose that there exists a complex structure J for which a given one parameter symplectic group C(t) is a continuous one parameter unitary group on $H_c = (H, J, <\cdot,\cdot>_1)$. Then $C(t) = \exp[\hat{A} t]$,

with $\hat{A}$ similar to a skew adjoint operator on $\mathcal{H}$ and $G\,\hat{A}$ is a self adjoint operator on $\mathcal{H}$.

Proof: By Stone's theorem, $C(t) = \exp[\hat{A}_c\, t]$, with $\hat{A}_c$ skew adjoint on $\mathcal{H}_c$. Let $\hat{A}$ denote $\hat{A}_c$, considered as an operator on $\mathcal{H}$.

Consider $\hat{A}' = C^{-1}\,\hat{A}\,C$, with C as in (12.28). By lemmas (12.31) and (12.32), $\hat{A}'$ is skew adjoint on $\mathcal{H}$ and $G_o\,\hat{A}'$ is self adjoint on $\mathcal{H}$. Now

$$\begin{aligned} G\hat{A} &= G\;C\;\hat{A}'\;C^{-1} \\ &= {C^{-1}}^{T}\;G_o\;\hat{A}'\;C^{-1}\;, \text{ by (12.25)} . \end{aligned}$$

Since $C^{-1}$ is bounded, $G\hat{A}$ is seen to be self-adjoint.

Proposition (12.35) establishes that unitarizable linear symplectic dynamics must be a Hamiltonian flow $\underline{\xi}(t) = e^{-G\hat{H}t}\,\underline{\xi}(o)$, generated by the phase function $H = \frac{1}{2}(\underline{\xi}, \hat{H}\,\underline{\xi})$, with H a real self adjoint operator and that $G\hat{H}$ must be similar to a skew adjoint operator. This highlights the necessary stability of the classical dynamics. In Section 14, it will be shown how this restricts the choice of gauge in the Bosonized version of the Schwinger model.

Recently, the Segal quantization procedure has been extended to some time dependent linear dynamical systems and to some non-linear systems which allow a unique characterization of an Hermitean structure on the real symplectic manifold [111]. Existence theorems in these cases have been proven only after imposing some conditions which ensure stability of the classical dynamics. It seems that stability is a necessary condition for the existence of a canonical Hermitean structure on the real solution manifolds of these more complicated dynamical systems.

## SECTION 13 - ALGEBRAIC QUANTIZATION WITH INDEFINITE METRIC

From Sections 9 and 12, we know that unstable classical symplectic dynamics can not be unitarized. That is, if $G\hat{H}$ is not similar to a skew adjoint operator, there is no complex structure J which satisfies the conditions (9.2a-d) simultaneously. It is therefore pertinent to ask whether the Segal scheme of quantization could be extended by relaxing one of these four requirements. Various attempts have already been made to develop a quantum mechanics in which the state space is not a Hilbert space but an indefinite inner product space. It was found long ago by Gupta and Bleuler [112] that a covariant formulation of quantum electrodynamics required the introduction of a negative metric for the unphysical longitudinal and time-like photons. Strocchi [113] has shown in general that covariant locally gauge invariant fields must be represented on an indefinite inner product space. In addition, a covariant wave equation with minimal coupling to an external field leads to an instability which requires formal mode space to be indefinite, in the manner of Section 7 (see [114,115]).

In this section, in order to investigate the possible immersion of indefinite metric theories in the algebraic quantization scheme, we study the existence of a complex structure J satisfying (9.2a,b,d) without the positivity requirement (9.2c) and with finite degrees of freedom.

From proposition (9.8), if -GJ is positive definite, the symplectic complex structure J belongs to a unique symplectic conjugacy class. It has been proven by Tolimieri [116] and by Rossi [117] that for each signature of -GJ, there is a separate unique conjugacy class containing J. We proved this independently in [III], using the theory of symplectic canonical forms, presented in Section 1 of this thesis.

<u>Proposition</u> (13.1): Every symplectic complex structure J lies in the same Sp(2N,R) conjugacy class as a matrix of the form

$$S = \pm G^{(2)} \oplus_s \pm G^{(2)} \ldots \oplus_s \pm G^{(2)}$$ , with $\oplus_s$ denoting a symplectic direct sum.

<u>Proof</u>: Since $J^2 = -I$ and $J^T GJ = G$, GJ is symmetric. Therefore, J belongs to the symplectic algebra Sp(2N,R) as well as the group Sp(2N,R). Since $J^2 = -I$, J has eigenvalues ±i. Hence, by (1.16) and (1.17), J is conjugate to a symplectic direct sum

$$S = S_1 \oplus_s S_2 \oplus_s \cdots \oplus_s S_m \tag{13.2}$$

where for each $r = 1,\ldots,m$, $S_r = -G^{(2k)}\hat{K}_j^{(2k)}(\rho,1)$ with $j = 5$ and k odd or $j = 6$ and k even.

Since the symplectic direct sum (13.2) is obtained from the ordinary direct sum by a re-ordering of the basis

$$(\underline{q}_1,\underline{p}_1,\underline{q}_2,\underline{p}_2,\ldots,\underline{q}_m,\underline{p}_m) \to (\underline{q}_1,\ldots,\underline{q}_m,\underline{p}_1,\ldots,\underline{p}_m) ,$$

$$S_1 \oplus_s S_2 = M^{-1}[S_1 \oplus S_2]M \quad \text{with} \quad M = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{13.3}$$

and $$S^2 = S_1^2 \oplus_s S_2^2 \oplus_s \cdots \oplus_s S_m^2 . \tag{13.4}$$

$$= -I \quad , \quad \text{since S is conjugate to J.} \tag{13.5}$$

From (13.3-5), $S_r^2 = -I$ for $r = 1,\ldots,m$ .

Now if $S_r = -G\hat{K}_6$ , $S_r^2$ has lower left block of the form

$$-2\rho \begin{bmatrix} & & & & -1 \\ & & & 1 & \\ & & -1 & & \\ & \cdot^{\cdot^{\cdot}} & & & \\ -1 & & & & \\ 1 & & & & \end{bmatrix}$$

so that $S_r^2$ can not be -I.

Alternatively, if $S_r = -G\,\hat{K}_5$, the upper left block of $S_r^2$ is

$$\begin{bmatrix} -1 & 0 & 1 & & & \\ & -1 & 0 & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ & & & & \cdot & 0 \\ & & & & & -1 \end{bmatrix}$$

Therefore, $S_r^2 = -I$ only if $S_r = -G\,\hat{K}_5^{(2)}(\rho,1)$, with $\rho = \pm 1$.

$\hat{K}_5^{(2)}(\rho,1)$ (here, the unit signifies unit frequency) is nothing more than $-\rho\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

Hence, $S_r = \pm\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ for all r and proposition (13.1) is proven.

A change of order in the symplectic basis in (13.2) then ensures that the components $+G^{(2)}$ appear first, followed by a string of components $-G^{(2)}$. As a result, there exists a symplectic transformation C such that

$$C^{-1}JC = G^{(2)} \oplus_s \ldots \oplus_s G^{(2)} \oplus_s - G^{(2)} \oplus_s \ldots \oplus_s -G^{(2)} .$$

$$= \begin{bmatrix} 0 & 0 & -I^{(a)} & 0 \\ 0 & 0 & 0 & I^{(b)} \\ I^{(a)} & 0 & 0 & 0 \\ 0 & -I^{(b)} & 0 & 0 \end{bmatrix} \qquad (13.6)$$

The canonical form for J given in (13.6) will be signified by G(a,b) with a + b = N. If a symplectic transformation C commutes with G(a,b) then C is unitary on the complex inner product space

$$\mathcal{H}^{(1)} = (R^{2N}, J, \langle\cdot,\cdot\rangle_1), \quad \text{where } J = G(a,b)$$

$$\text{and } \langle\cdot,\cdot\rangle_1 = -(\cdot, GJ\cdot) - i(\cdot, G\cdot) \qquad (13.7)$$

$$-GJ = -GG(a,b) = \text{diag.}[I^{(a)}, -I^{(b)}, I^{(a)}, -I^{(b)}] \qquad (13.8)$$

With J = G(a,b), the first N real basis vectors of $\mathcal{M}$ span the whole complex space $\mathcal{H}^{(1)}$

$$\underline{e}_{N+j} = \begin{cases} J\,\underline{e}_j & \text{for } j = 1,\ldots,a \\ -J\,\underline{e}_j & \text{for } j = a+1,\ldots,N \end{cases} \qquad (13.9)$$

In the general case that J is conjugate to G(a,b),

$$C^{-1}JC = G(a,b) \quad \text{and} \quad C^{T}GC = G\ , \tag{13.10}$$

$$\Rightarrow \; C^{T}(-GJ)C = -GG(a,b)\ , \quad \text{by (1.14-15)} \tag{13.11}$$

Therefore, if -GJ has signature (2a,2b), J is conjugate to G(a,b) and by (13.7-11), the group of real symplectic transformations which commute with J, is isomorphic to the group U(a,b) which preserves a complex inner product $\langle\cdot,\cdot\rangle_1$ of signature (a,b). The subgroup $G_J \simeq U(N)$ of Sp(2N,R), which appears in the chain (9.13), may be replaced more generally by U(a,b) with a + b = N.

Definition (13.12): A one-parameter symplectic group $C(t) = \exp(-G\hat{H}t)$ has a pseudo-unitarizing complex structure J if J is a symplectic complex structure and J commutes with $G\hat{H}$.

Proposition (13.13): $C(t) = \exp(-G\hat{H}t)$ has a pseudo-unitarizing complex structure J if and only if for some C ∈ Sp(2N,R).

$$C^{T}\hat{H}C = \hat{H}_o = \begin{pmatrix} A & LK \\ -KL & KAK \end{pmatrix}$$

with $A = A^{T}$, $L = -L^{T}$ and $K = \text{diag.}[-I^{(a)}, I^{(b)}]$ with a + b = N.

Proof: Since J is a symplectic complex structure, by (13.6), there exists C ∈ Sp(2N,R) such that $C^{-1}JC = G(a,b) = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}$, with K defined as above.

$$\text{Now} \quad [J,G\hat{H}] = 0 \Leftrightarrow [CG(a,b)C^{-1},\ G\hat{H}] = 0$$

$$\Leftrightarrow [G(a,b),\ C^{-1}G\hat{H}C] = 0$$

$$\Leftrightarrow [G(a,b),\ G\,C^{T}\hat{H}C] = 0$$

$$\Leftrightarrow [G(a,b),\ C^{T}\hat{H}C] = 0 \quad (\text{since } [G(a,b),G] = 0) \tag{13.14}$$

The symmetric matrix $C^{T}\hat{H}C$ must have the structure $\begin{pmatrix} A & B \\ B^{T} & F \end{pmatrix}$ with $A - A^{T} = F - F^{T} = 0$.

Then (13.14) is equivalent to $-BK = K\,B^T \; (= (BK)^T)$ (13.15a)

and $AK = K\,F$ (13.15b)

Since K is an involution, F = KAK and B = LK, where L = BK is skew symmetric, by (13.15a).

Conversely, if for some $C \in Sp(2N,R)$, $C^T\hat{H}C$ has the form given in proposition (13.13),

$$[GC^T\hat{H}C,\; G(a,b)] = 0$$

$$\Rightarrow \; [G\hat{H},\; CG(a,b)C^{-1}] = 0$$

$$\Rightarrow \; CG(a,b)C^{-1} \text{ is a pseudo-unitarizing complex structure for } C(t) = \exp(-G\hat{H}t).$$

Even after allowing $H^{(1)}$ to have indefinite metric, it soon becomes apparent that a pseudo-unitarizing complex structure does not always exist. For example, in one degree of freedom every symplectic complex structure J is conjugate to $G(1,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = G^{(2)}$ or to $G(0,1) = -G^{(2)}$. Therefore, it is clear, from the argument used in the proof of proposition (9.10), that if $[J, G\hat{H}] = 0$, $\hat{H}$ must either be trivial ($\hat{H} = 0$) or must belong to the harmonic oscillator class. Neither the single repulsive oscillator nor the single free particle admit pseudo-unitarization. In fact, we shall prove that each elementary divisor $(s - i\,s_j)^{N_j}$ associated with an imaginary or zero frequency $i\,s_j$ must occur an even number of times before the symplectic dynamics in N degrees of freedom can be pseudo-unitarized.

<u>Proposition</u> (13.16): Suppose that a pseudo-unitarizing complex structure exists for $C(t) = \exp(-G\hat{H}t)$. Then, for any eigenvalue $i\,s_j$, of $iG\hat{H}$ with $s_j \in R$, each elementary divisor $(s - i\,s_j)^{N_j}$, for fixed $N_j$, must occur an even number of times.

Proposition (13.16) will be established after a sequence of preliminary results. Since C(t) can be pseudo-unitarized, $\hat{H}$ can be reduced, by a symplectic change of basis, to the canonical form $\hat{H}_o$ of proposition (13.13). Since elementary divisors of $iG\hat{H} - sI$ are invariant under symplectic transformations, we need only establish proposition (13.16) for the canonical form $\hat{H}_o$.

<u>Theorem</u> (13.17): (Jordan canonical form theorem).

For each complex $N \times N$ matrix F, there exists a uniquely determined set of ordered triples $(s_j, N_j, M_j)$, such that the elementary divisors of $F - sI$ are $(s - s_j)^{N_j}$, each occurring $M_j$ times. For each elementary divisor $(s - s_j)^{N_j}$, there exist $M_j\ N_j$ linearly independent root vectors $\underline{e}_{k,\ell}$; $k = 1,\ldots,M_j$ ; $\ell = 1,\ldots,N_j$ such that

$$\text{(a)} \quad (F - s_j I)\underline{e}_{k,\ell} = \begin{cases} \underline{o} & \text{if } \ell = N_j \\ \underline{e}_{k,\ell+1} & \text{if } \ell < N_j \end{cases}$$

and (b) $\underline{e}_{k,1} \in \text{ran.}(F - s_j I)$

Now we shall consider $F = iG\hat{H}_o$, with $\hat{H}_o$ specified in (13.13). Let $(s - is_j)^{N_j}$, with $s_j \in R$, be an elementary divisor of $F - sI$. There exists a set of $M_j\ N_j$ root vectors $\underline{e}_{k,\ell}$ satisfying the properties given in (13.17). Let $V$ be the root space spanned by the vectors $\underline{e}_{k,\ell}$. Let $\Gamma$ be the conjugation operator on $C^{2N}$ defined by

$$\Gamma \begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix} = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix} \begin{pmatrix} \underline{u}^* \\ \underline{v}^* \end{pmatrix} \quad \text{(K as in (13.13))} \qquad (13.18)$$

<u>Lemma</u> (13.19): $\Gamma$ has no characteristic values $\alpha \in C$ such that $\Gamma\ \underline{\omega} = \alpha\ \underline{\omega}$ , with $\underline{\omega}$ a non-null vector in $C^{2N}$.

<u>Proof</u>: Suppose $\Gamma \begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix} = \alpha \begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix}$ with $\alpha \in C$ .

$$\Rightarrow \quad K\ \underline{v}^* = \alpha\ \underline{u}$$

and $\quad -K\,\underline{u}^* = \alpha\,\underline{v}$

$$\Rightarrow \quad \underline{v}^* = \alpha\,K\,\underline{u} \qquad (\text{since } K^2 = I)$$

and $\quad \underline{u} = -\alpha^*\,K\,\underline{v}^*$

$$\Rightarrow \quad \underline{v}^* = -|\alpha|^2\,\underline{v}^*$$

$$\Rightarrow \quad \underline{v} = \underline{0}$$

$$\Rightarrow \quad \underline{v} = \underline{0} \quad \text{and} \quad \underline{u} = \underline{0}\ .$$

Lemma (13.20): $\quad (iG\hat{H}_o - i\,s_jI)\Gamma\,\underline{e}_{k,\ell} = \begin{cases} \underline{0} & \text{if } \ell = N_j \\ \Gamma\,\underline{e}_{k,\ell+1} & \text{if } \ell < N_j \end{cases}$

Proof: $\quad (iG\hat{H} - i\,s_jI)\Gamma\,\underline{e}_{k,\ell} = (iG\hat{H}_o - i\,s_jI)\begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}\underline{e}^*_{k,\ell}$

$$= \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}(iG\hat{H}_o - i\,s_jI)\,\underline{e}^*_{k,\ell} \quad \left(\text{since } G\hat{H}_o \text{ commutes with } \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}\right).$$

$$= -\begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}[(iG\hat{H}_o - i\,s_jI)\underline{e}_{k,\ell}]^*$$

$$= \begin{cases} \underline{0} & \text{if } \ell = N_j \\ \Gamma\,\underline{e}_{k,\ell+1} & \text{if } \ell < N_j \end{cases}$$

Lemma (13.21): $\quad \{\underline{e}_{k,\ell}\}_{\ell=1,\ldots,N_j} \cup \{\Gamma\,\underline{e}_{k,\ell}\}_{\ell=1,\ldots,N_j}$, is a set of $2N_j$ linearly independent vectors.

Proof: $\quad$ Suppose $\underline{0} = \sum_{\ell=1}^{N_j} \alpha_\ell\,\underline{e}_\ell + \beta_\ell\,\Gamma\,\underline{e}_\ell$
(temporarily neglecting the suffix k). (13.22)

Applying $(iG\hat{H}_o - i\,s_jI)^{N_j-1}$ to each side of (13.22), we obtain

$$\underline{0} = \alpha_1\,\underline{e}_{N_j} + \beta_1\,\Gamma\,\underline{e}_{N_j}$$

$$\Rightarrow \quad \alpha_1 = \beta_1 = 0\ , \quad \text{by lemma (13.19).}$$

Assuming $\alpha_1 = \beta_1 = 0$, applying $(iG\hat{H}_o - i\,s_j I)^{N_j-2}$ to each side of (13.22), we similarly obtain $\alpha_2 = \beta_2 = 0$. Continuing this recursive attack, we can show $\alpha_\ell = \beta_\ell = 0$ for all $\ell$. Therefore, the vectors $\{\underline{e}_\ell\}$ and $\{\Gamma\underline{e}_\ell\}$ are linearly independent.

Now let $\mathcal{U}_1 = V \cap (\underline{e}_{1,1},\ldots,\underline{e}_{1,N_j}\ ;\ \Gamma\underline{e}_{1,1},\ldots,\Gamma\underline{e}_{1,N_j})^\perp$. Since, by theorem (13.17), $N_j$ and $M_j$ are uniquely determined, $iG\hat{H}_o - s_j I$, restricted to $\mathcal{U}_1$, must have $M_j-2$ elementary divisors $(s - i\,s_j)^{N_j}$.

__Lemma__ (13.23): $\mathcal{U}_1$ is $\Gamma$-invariant.

__Proof__: Let $\underline{\omega} = \begin{pmatrix}\underline{u}\\ \underline{v}\end{pmatrix} \in \mathcal{U}_1$. Since $\underline{\omega} \in V$, $\Gamma\underline{\omega} \in V$, by lemma (13.20).

Since $\underline{\omega} \in \mathcal{U}_1$, $\underline{\omega}$ is orthogonal to $\underline{e}_{1,\ell}$ and $\Gamma\underline{e}_{1,\ell}$ for all $\ell$.

Now $(\Gamma\underline{\omega}, \underline{e}_{1,\ell}) = (G(a,b)\begin{pmatrix}\underline{u}^*\\ \underline{v}^*\end{pmatrix}, \underline{e}_{1,\ell})$ (with $G(a,b) = \begin{pmatrix}0 & K\\ -K & 0\end{pmatrix}$)

$$= -(\begin{pmatrix}\underline{u}^*\\ \underline{v}^*\end{pmatrix}, G(a,b)\,\underline{e}_{1,\ell})$$

$$= -(\begin{pmatrix}\underline{u}\\ \underline{v}\end{pmatrix}, \Gamma\underline{e}_{1,\ell})^*$$

$$= -(\underline{\omega}, \Gamma\underline{e}_{1,\ell})^*$$

$$= 0$$

Similarly, $(\Gamma\,\underline{\omega}, \Gamma\underline{e}_{1,\ell}) = 0$

Therefore, $\Gamma\,\underline{\omega} \in (\underline{e}_{1,1},\ldots,\underline{e}_{1,N_j}\ ;\ \Gamma\underline{e}_{1,1},\ldots,\Gamma\underline{e}_{1,N_j})^\perp$

and since also $\Gamma\underline{\omega} \in V$, $\Gamma\underline{\omega} \in \mathcal{U}_1$.

Since $\underline{\omega}$ was an arbitrary element of $\mathcal{U}_1$, $\mathcal{U}_1$ is $\Gamma$-invariant.

Since $\mathcal{U}_1$ is $\Gamma$-invariant, we can reapply lemmas (13.19-21) to extract two sets of linearly independent Jordan root vectors $\underline{e}_{2,\ell}$ and $\Gamma\underline{e}_{2,\ell}$, $\ell = 1,\ldots,N_j$ corresponding to a pair of elementary divisors $(s - i\,s_j)^{N_j}$.

We can recursively define $\mathcal{U}r = \mathcal{U}r\text{-}1 \cap (\underline{e}r,1,\ldots,\underline{e}r,N_j;\ \Gamma\underline{e}r,1,\ldots,\ \underline{e}r,N_j)^{\perp}$ and at all iterations, $\mathcal{U}r$ is either empty or it contains at least two sets of linearly independent Jordan root vectors, corresponding to a pair of elementary divisors $(s - i\, s_j)^{N_j}$. This establishes proposition (13.16).

For every dynamical group $\exp(-G\hat{H}t)$ which is rendered pseudo-unitary by some complex structure, we shall construct one such complex structure explicitly. Unlike the positive metric case, if a unitarizing complex structure exists, it can not be expected to be unique. For example, if $\hat{H}$ is the $2N \times 2N$ unit matrix, $-G\hat{H} = -G$. For all a and b such that $a+b = N$, the symplectic complex structure $G(a,b)$ commutes with G. Therefore, we may choose $J = G(a,b)$, resulting in the metric in $H^{(1)}$ having any signature (a,b). This is a real detraction from allowing an indefinite metric to enter algebraic quantization, since the representation of the C.C.R. no longer follows uniquely from the classical dynamics. Nevertheless, we are able to explicitly construct one of the pseudo-unitarizing complex structures, whenever one exists. Once again, existence of J is a property which is determined by the canonical class of $\hat{H}$ and so the whole task may be completed by recourse to a suitable set of canonical forms.

Proposition (13.24): Suppose that J is a symplectic complex structure which commutes with $G\hat{H}$. If $\hat{H}_1 = C^T\hat{H}C$, with $C \in Sp(2N,R)$, then $J_1 = C^{-1}JC$ is a symplectic complex structure which commutes with $G\hat{H}_1$.

Proof: As for proposition (9.5).

When looking for the full set of Hamiltonians for which J exists, it is natural to enquire whether the condition given in proposition (13.16) is sufficient as well as necessary. The following result will help to produce counter examples to this conjecture.

Proposition (13.25): Suppose that a pseudo-unitarizing complex structure exists for $C(t) = \exp(-G\hat{H}t)$. Then sig.$\hat{H}$ = (0,0) (modulo 2).

Proof: Suppose that a pseudo-unitarizing complex structure exists for $C(t) = \exp(-G\hat{H}t)$. Then, by proposition (13.13), there exists $C \in Sp(2N,R)$ such that $C^T\hat{H}C = \begin{pmatrix} A & LK \\ -KL & KAK \end{pmatrix}$ with $A = A^T$, $L = -L^T$ and $K = \text{diag.}[-I^{(a)}, I^{(b)}]$ with $a + b = N$.

$$C^T\hat{H}C = \begin{pmatrix} I & 0 \\ 0 & K \end{pmatrix}\begin{pmatrix} A & L \\ -L & A \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & K \end{pmatrix}$$

Since $\begin{pmatrix} I & 0 \\ 0 & K \end{pmatrix}$ is an involution, $\hat{H}$ must have the same signature as $\begin{pmatrix} A & L \\ -L & A \end{pmatrix}$.

This implies that $\hat{H}$ has the same signature as

$$P^{-1}\begin{pmatrix} A & L \\ -L & A \end{pmatrix}P = \begin{pmatrix} A-iL & 0 \\ 0 & A+iL \end{pmatrix} \quad \left(\text{with } P = 2^{-\frac{1}{2}}\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix} = P^{\dagger^{-1}}\right)$$

Therefore, $\text{sig.}\hat{H} = \text{sig.}(A-iL) + \text{sig.}(A+iL)$

$= \text{sig.}(A+iL)^* + \text{sig.}(A+iL)$

$= 2\,\text{sig.}(A+iL)$ (since $A+iL$ is Hermitean).

Hence, both the positive and negative eigenspaces of $\hat{H}$ must be even dimensional.

As an application of proposition (13.25), we recall from table 1.18 that $\hat{K}_3^{(2k)}$ has signature (k-1,k-1). Therefore, if $\hat{H}$ belongs to the same canonical class as $\hat{K}_3^{(2k)}$, with k even, $\exp(-G\hat{H}t)$ can not be pseudo-unitarized, even though the elementary divisors of $iG\hat{H} - sI$ occur in pairs $s^k, s^k$. This provides a counter example for the converse of (13.16). However, in the simplest case with k odd, $k = 1$, $\hat{K}_3^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $-G\hat{K}_3^{(2)}$ commutes with every symplectic complex structure. With $k = 3$, the conditions $J = -I$, $J^T G J = G$ and $[J, GK_3^{(6)}] = 0$ yield a general solution for the $6 \times 6$ matrix J:

$$
J = \begin{bmatrix}
\alpha & 0 & \beta & \gamma & 0 & -\delta \\
0 & \alpha & 0 & 0 & \delta & 0 \\
0 & 0 & \alpha & -\delta & 0 & 0 \\
0 & 0 & \varepsilon & -\alpha & 0 & 0 \\
0 & -\varepsilon & 0 & 0 & -\alpha & 0 \\
\varepsilon & 0 & \mu & -\beta & 0 & -\alpha
\end{bmatrix}
$$

with $\delta \neq 0$

$\varepsilon = (1+\alpha^2)/\delta$

$\mu = 2\alpha\beta/\delta + \gamma(1+\alpha^2)/\delta^2$

$\alpha,\beta,\gamma,\delta,\varepsilon,\mu \in R$ . (13.26)

The example

$$
J = \left[\begin{array}{ccccccc}
 & & & \cdot & 1 & 0 & -1 \\
 & & & \cdot & 0 & 1 & 0 \\
 & & & \cdot & -1 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 1 & \cdot & & & \\
0 & -1 & 0 & \cdot & & & \\
1 & 0 & 1 & \cdot & & &
\end{array}\right]
$$

extends to a pseudo-unitarizing complex structure for the general case $\hat{K}_3^{(2k)}$ with k odd:

$$
J = \left[\begin{array}{ccccccccccccccc}
 & & & & & & & \cdot & & & & & 1 & 0 & -1 \\
 & & & & & & & \cdot & & & & -1 & 0 & 1 & \\
 & & & & & & & \cdot & & & \cdot & \cdot & -1 & & \\
 & & & & & & & \cdot & & -1 & \cdot & \cdot & & & \\
 & & & & & & & \cdot & 1 & 0 & \cdot & & & & \\
 & & & & & & & \cdot & 0 & 1 & & & & & \\
 & & & & & & & \cdot & -1 & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & 1 & \cdot & & & & & & & \\
 & & & & & -1 & 0 & \cdot & & & & & & & \\
 & & & & \cdot & 0 & 1 & \cdot & & & & & & & \\
 & & & \cdot & \cdot & -1 & & \cdot & & & & & & & \\
 & & \cdot & \cdot & \cdot & & & \cdot & & & & & & & \\
 & -1 & 0 & -1 & & & & \cdot & & & & & & & \\
1 & 0 & 1 & & & & & \cdot & & & & & & &
\end{array}\right] \qquad (13.27)
$$

It can be verified that $J^2 = -I$, $J^T GJ = G$ and $[J, G\,\hat{K}_3^{(2k)}] = 0$, as required.

As another example, let $\hat{H} = \hat{K}_4^{(2k)}(\rho_1) \oplus_s \hat{K}_4^{(2k)}(\rho_2)$. $iG\hat{H} - sI$ has a pair of elementary divisors $s^{2k}, s^{2k}$ but sig.$\hat{H}$ may be varied by varying $\rho_1$ (= ±1) and $\rho_2$ (= ±1). Since ker $\hat{K}_4^{(2k)}(\rho_1)$ is one dimensional, sig.$\hat{K}_4^{(2k)}(\rho_1)$ = (0,1) or (1,0) (modulo 2). We recall from Section 1, that

$\hat{K}_4^{(2k)}(-\rho_1)$ may be obtained from $\hat{K}_4^{(2k)}(\rho_1)$ by a signature - reversing transformation $\hat{K}_4^{(2k)}(-\rho_1) = - \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \hat{K}_4^{(2k)}(\rho_1) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. Therefore,

$\text{sig}.\hat{K}_4^{(2k)}(-\rho_1) = (1,1) - \text{sig}.\hat{K}_4^{(2k)}(\rho_1)$ (modulo 2). This implies that if

$\rho_2 = -\rho_1$,

$$\text{sig}.(\hat{K}_4^{(2k)}(\rho_1) \oplus_s \hat{K}_4^{(2k)}(\rho_2)) = (1,1) \quad \text{(modulo 2)}.$$

Therefore, by proposition (13.25), if $\hat{H}$ belongs to the same canonical class as $\hat{K}_4^{(2k)}(\rho_1) \oplus_s \hat{K}_4^{(2k)}(\rho_2)$, $\exp(-G\hat{H}t)$ can not be pseudo-unitarized unless $\rho_1 = \rho_2$. It is not good enough that the elementary divisors occur in pairs $s^{2k}, s^{2k}$. The other invariants $\rho_1$ and $\rho_2$ on the associated principal sub-spaces must also be equal. When $\rho_1 = \rho_2$, one solution for the unitarizing complex structure is

$$J = \pm \begin{pmatrix} G^{(2k)} & 0 \\ 0 & G^{(2k)} \end{pmatrix} \tag{13.27a}$$

Up to this point, we have found the most general symplectic dynamical system which can not be pseudo-unitarized, namely that generated by $\hat{H}$ which contains an odd length chain $\hat{K}_1^{(2k)}(ai) \oplus_s \hat{K}_1^{(2k)}(ai) \oplus_s \ldots \oplus_s \hat{K}_1^{(2k)}(ai)$ or $\hat{K}_4(\rho) \oplus_s \ldots \oplus_s \hat{K}_4(\rho)$ in its canonical form. Conversely, the most general possible criterion for the existence of J is given in the following:

<u>Proposition</u> (13.28): If, in the reduction of $\hat{H}$ to canonical form, the only indecomposable canonical submatrices which correspond to frequencies $i\, s_j$ ($s_j \in R$) and which occur an odd number of times are $\hat{K}_3^{(2k)}$ with k odd, then a pseudo-unitarizing complex structure exists for $\exp(-G\hat{H}t)$.

<u>Proof</u>: If $\hat{H}$ satisfies the conditions of proposition (13.28), then $\hat{H}$ can be reduced, by a symplectic transformation, to a canonical form $\hat{K}$ which is a symplectic direct sum of components $\hat{K}_1^{(2k)}(ai) \oplus_s \hat{K}_1^{(2k)}(ai)$, $\hat{K}_2^{(4k)}(b+ai)$, $\hat{K}_3^{(4k)} \oplus_s \hat{K}_3^{(4k)}$, $\hat{K}_3^{(4k+2)}$, $\hat{K}_4^{(2k)}(\rho) \oplus_s \hat{K}_4^{(2k)}(\rho)$, $\hat{K}_5^{(4k+2)}(\rho,b)$ and $\hat{K}_6^{(4k)}(\rho,b)$. On a $G\hat{H}$-invariant subspace on which the restriction of $\hat{H}$ is $\hat{K}_2^{(4k)}(b+ai)$,

one solution for J is

$$J = \pm\text{diag.}[G^{(2)}, G^{(2)}, \ldots, G^{(2)}] \tag{13.29}$$

In fact, the case $\hat{H} = \hat{K}_2^{(4k)}(b+ai)$, for which $iG\hat{H} - sI$ has elementary divisors $(s + b + ai)^k$, $(s + b - ai)^k$, $(s - b + ai)^k$ and $(s - b - ai)^k$, reduces to the canonical class of $\hat{H} = \hat{K}_1^{(2k)} \oplus_s \hat{K}_1^{(2k)}$, when $b = 0$. The canonical form $\hat{K}_1^{(4k)}(ai) \oplus_s \hat{K}_1^{(4k)}(ai)$ reduces directly to $\hat{K}_3^{(4k)} \oplus_s \hat{K}_3^{(4k)}$, when $a = 0$. Therefore, the latter example admits a complex structure $J = \pm\text{diag.}[G^{(4k)}, G^{(4k)}]$ deduced from (13.28). For the special case $\hat{H} = \hat{K}_3^{(2k)}$ with k odd, a solution for J is given in (13.27). For the case $\hat{H} = \hat{K}_4^{(2k)}(\rho) \oplus_s \hat{K}_4^{(2k)}(\rho)$, a solution for J is given in (13.27a). In the cases $\hat{H} = \hat{K}_5^{(4k+2)}(\rho,b)$ and $\hat{H} = \hat{K}_6^{(4k)}(\rho,b)$, we have obtained a solution for J by first transforming $\hat{H}$ to the canonical form of proposition (13.13).

$$C^T\hat{H}C = \hat{H}_o \ ; \ C^TGC = G$$

With $\hat{H}$ in this canonical form $\hat{H}_o$, a solution for J is $G = \pm G(a,b)$. Then, after transforming back to the original basis, proposition (13.24) assures us that a pseudo-unitarizing complex structure is given by

$$J = \pm C^{-1}\, G(a,b)C \ .$$

Our results are that with $\hat{H} = \hat{K}_5^{(4k+2)}(\rho,b)$, one solution is

$$J = \left[\begin{array}{ccccc:ccccc} & & & & & & & & & -b \\ & & & & & & & & b & \\ & & & & & & & -b & & \\ & & & & & & \cdot^{\cdot^{\cdot}} & & & \\ & & & & & b & & & & \\ & & & & & -b & & & & \\ \hdashline & & & & b^{-1} & & & & & \\ & & & -b^{-1} & & & & & & \\ & & \cdot^{\cdot^{\cdot}} & & & & & & & \\ & -b^{-1} & & & & & & & & \\ b^{-1} & & & & & & & & & \end{array}\right] \tag{13.30}$$

and with $\hat{H} = \hat{K}_6^{(4k)}(\rho,b)$, one solution is

$$J = \begin{bmatrix} 0 & -b & & & & & & & & \\ b^{-1} & 0 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & 0 & -b & & & & & \\ & & & b^{-1} & 0 & & & & & \\ & & & & & 0 & -b^{-1} & & & \\ & & & & & b & 0 & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & 0 & -b^{-1} \\ & & & & & & & & b & 0 \end{bmatrix} \tag{13.31}$$

If $\hat{H}$ is a symplectic direct sum of canonical matrices of the type listed in proposition (13.28), then J may be taken to be a symplectic direct sum of the solutions, listed above, for each contributing $G\hat{H}$-invariant subspace. If $\hat{H}$ belongs to the same canonical class as such a symplect direct sum, then a solution J may be constructed with the assistance of proposition (13.24).

Given a complex (single particle) Hilbert space $H^{(1)}$, the rigorous construction of both Fock space $F(H^{(1)})$ over $H^{(1)}$ and the natural creation and annihilation on $F(H^{(1)})$ were devised by Cook [37]. The Segal procedure for quantization of linear systems, which we have now fully investigated, involves finding a complex structure J on the classical real linear symplectic space $M$ so that the classical symplectic dynamics C(t) on $M$, is unitary with respect to the associated J-sesquilinear form $<\cdot,\cdot>_1$. The complex space $(M, J, <\cdot,\cdot>_1)$ then becomes $H^{(1)}$ in the Cook construction. Recently, Mintchev [118] has extended the Cook construction to the case that $H^{(1)}$ is an indefinite inner product space. The results which we have presented so far in this section, may be regarded as rigorous construction of a complex indefinite inner product space $H^{(1)}$ with pseudo-unitary dynamics, which was the starting point of Mintchev's work. Given an indefinite inner product $<\phi,\psi>_1 = (\phi, \eta\psi)$, with $\eta$ a self-adjoint contraction on complex Hilbert space $(H, (\cdot,\cdot))$, define $H^{(1)} = (H, <\cdot,\cdot>_1)$. Fock space $F(H^{(1)})$ with indefinite inner product $<\cdot,\cdot>$ is constructed as in (8.1).

Annihilation operators $a(\phi)$ and creation operators $a^+(\psi)$ are defined as in (8.2). Field operators $\Phi(\phi)$ are defined as in (8.9), except that the symbol $a^\dagger(\psi)$ is replaced by $a^+(\psi)$, which is the pseudo-adjoint, rather than the Hilbert space adjoint of $a(\psi)$. We write the finite particle subspace as

$$\mathcal{F}_o = \{\phi \in \mathcal{F}(\mathcal{H}^{(1)}) \;;\; \exists\, n \in Z \;,\; \phi = (\phi^{(o)}, \phi^{(1)}, \ldots, \phi^{(m)}, \ldots) \text{ with } \phi^{(m)} = \underline{0} \text{ for all } m > n\}$$

Theorem (13.32): (Mintchev [118])

(a) $\Phi(\phi)$ is closable for all $\phi \in \mathcal{H}^{(1)}$.

(b) $\mathcal{F}_o$ is a set of analytic vectors for $\Phi(\phi)$, for all $\phi \in \mathcal{H}^{(1)}$.

(c) If $\{\phi_k\} \subset \mathcal{H}^{(1)}$ and $\underset{k\to\infty}{\text{s-lim}}\ \phi_k = \phi$,

then $\underset{k\to\infty}{\text{s-lim}}\ \Phi(\phi_k)\psi = \Phi(\phi)\psi$ for all $\psi \in \mathcal{F}_o$.

(d) The vacuum vector $\psi_o$ is cyclic with respect to $\{\Phi(\phi);\ \phi \in \mathcal{H}^{(1)}\}$.

(e) For all $\chi \in \mathcal{F}_o$ and $\phi, \psi \in \mathcal{H}^{(1)}$, $[\Phi(\phi),\Phi(\psi)]\chi = i\ \mathrm{Im} \langle\phi,\psi\rangle_1 \chi$ and $[a(\phi),a^+(\psi)]\chi = \langle\phi,\psi\rangle_1 \chi$.

Since the operators $\Phi(\psi)$ satisfy the C.C.R. on $\mathcal{F}_o$, which is a subspace of the space of analytic vectors for $\Phi(\psi)$, just as in the case of positive metric discussed at the beginning of Section 9,

$$a^+(\underline{\xi}_1(t)) \ldots a^+(\underline{\xi}_n(t))\psi_o = e^{-i(H+\alpha)t}\, a^+(\underline{\xi}_1(0)) \ldots a^+(\underline{\xi}_n(0))\psi_o, \tag{13.33}$$

where

$$H = \tfrac{1}{2}\, \underline{Z}^T \hat{H} \underline{Z} \;;\; Z_\mu = \sum_{\nu=1}^{2N} G_{\mu\nu}\, \Phi(\underline{e}_\nu) \qquad (Z_\mu \text{ obey the C.C.R.}) \tag{13.34}$$

and in (13.33) it is assumed that $H\,\psi_o = \alpha\,\psi_o$ with $\alpha \in R$. This assumption is valid in the case that $\hat{H}$ is reduced to the canonical form $\hat{H}_o$ of proposition (13.13) and $J = G(a,b)$. In this case,

$$H\,\psi_o = \tfrac{1}{2} \sum_{\mu,\nu=1}^{2N} \Phi(\underline{e}_\mu)\, \hat{Y}_{\mu\nu}\, \Phi(\underline{e}_\nu)\, \psi_o\,, \tag{13.35}$$

where $\hat{Y} = G^T \hat{H}_o G = \begin{pmatrix} KAK & KB \\ -BK & A \end{pmatrix}$ ; $A = A^T$ , $B = -B^T$

$$K = \text{diag.}[\rho_1,\ldots,\rho_N]$$

$$\text{with } \rho_j = \begin{cases} -1 & \text{if } j \le a \\ +1 & \text{if } j > a \end{cases}$$

Since $a(J\,\underline{\xi}) = -i\,a(\underline{\xi})$ and $J = \mathcal{G}(a,b)$, we have

$$a(\underline{e}_{j+N}) = \begin{cases} a(J\underline{e}_j) = -i\,a(\underline{e}_j) & \text{if } 1 \le j \le a \\ a(-J\underline{e}_j) = i\,a(\underline{e}_j) & \text{if } a+1 \le j \le N \end{cases} \tag{13.36}$$

Combining (13.36) with (8.9), we obtain

$$\Phi(\underline{e}_j) = 2^{-\frac{1}{2}}[a(\underline{e}_j) + a^+(\underline{e}_j)]$$

$$\Phi(\underline{e}_{j+N}) = 2^{-\frac{1}{2}}[i\,\rho_j\,a(\underline{e}_j) - i\,\rho_j\,a^+(\underline{e}_j)]$$

$$\text{for all } j \le N\,. \tag{13.37a}$$

We shall write (13.37), in a condensed notation, as

$$\Phi = P'\,\underline{\alpha} \tag{13.37b}$$

with $\alpha_j = a(\underline{e}_j)$ , $\alpha_{j+N} = a^+(\underline{e}_j)$ , $\Phi_\mu = \Phi(\underline{e}_\mu)$ ,

and $P^1 = 2^{-\frac{1}{2}}\begin{pmatrix} I & I \\ iK & -iK \end{pmatrix}$ .

Substituting (13.37) into (13.35), we obtain

$$H\,\psi_o = \tfrac{1}{2}\,\underline{\Phi}^T\,\hat{Y}\,\underline{\Phi}\,\psi_o$$

$$= \tfrac{1}{2}\,\underline{\Phi}^+\,\hat{Y}\,\underline{\Phi}\,\psi_o \quad (\text{since } \Phi^+_\mu = \Phi_\mu \text{ on } F_o)\,.$$

$$= \tfrac{1}{2}\,(P'\underline{\alpha})^+\,Y(P'\underline{\alpha})\,\psi_o$$

$$= \tfrac{1}{2}\,\underline{\alpha}^+\,P'^\dagger\,\hat{Y}\,P'\underline{\alpha}\,\psi_o \quad (\text{where } \underline{\alpha}^+ = (\alpha^+_1,\ldots,\alpha^+_N,\alpha_1,\ldots,\alpha_N)),$$

(13.38)

Now
$$P'^\dagger\,\hat{Y}\,P' = \begin{pmatrix} K(A+iB)K & 0 \\ 0 & K(A-iB)K \end{pmatrix} \tag{13.39}$$

Therefore, from (13.38),

$$H\,\psi_o = \tfrac{1}{2}\sum_{j,k=1}^{N} (K(A-iB)K)_{jk}\;a(\underline{e}_j)\;a(\underline{e}_k)^+\,\psi_o$$

$$= \tfrac{1}{2} \sum_{j,k=1}^{N} (K(A-iB)K)_{jk} \quad \langle \underline{e}_j, \underline{e}_k \rangle_1 \; \psi_o$$

$$= \alpha \, \psi_o \quad , \text{ where } \alpha = -\tfrac{1}{2} \sum_{j=1}^{N} P_j \, A_{jj} = -\tfrac{1}{2} \, \mathrm{Tr}(KA) \qquad (13.40)$$

It will become evident that complex indefinite inner product space $\mathcal{H}^{(1)}$ is a rigorous version of the heuristic single mode space, which, by proposition (7.33), must have indefinite inner product when the classical dynamics is unstable. To clarify this point, we shall continue to consider a classical system with finite degrees of freedom. Proposition (13.13) then allows us to assume $\hat{H} = \hat{H}_o$ and $G = G(a,b)$, which can always be achieved by a symplectic transformation, given that a pseudo-unitarizing complex structure exists. The C.C.R. may then be expressed

$$[a_j, \, a_k^+] \;=\; - \, \rho_j \, \delta_{jk} \qquad \text{(from (13.32c)}$$

$$\text{and} \quad [a_j, \, a_k] \;=\; 0 \qquad (13.41)$$

(here and in the following, $a_j$ is an abbreviation for $a(\underline{e}_j)$).

To find the single mode eigenstates for H, suppose that $H \, \phi = \omega \, \phi$, with $\omega \in C$ and $\phi = \sum_{j=1}^{N} \beta_j \, a_j^+ \, \psi_o$

Then $[H, \sum_{j=1}^{N} \beta_j \, a_j^+]\psi_o = (\omega - \alpha)\phi$ .

By (13.41) and (13.38-39), this is equivalent to

$$- \sum_{r,s=1}^{N} \rho_r (A+iL)_{rs} \, \beta_s \, a_r^+ \, \psi_o = \sum_{r=1}^{N} (\omega - \alpha) \, \beta_r \, a_r^+ \, \psi_o$$

$\Leftrightarrow \quad - K(A+iL)\underline{\beta} = (\omega - \alpha)\underline{\beta}$ (since the single mode states $a_r^+ \, \psi_o$ are linearly independent) (13.42)

That is, the vector $\underline{\beta}$ with N components $\beta_j$ is an eigenvector of -K(A+iL) corresponding to eigenvalue $\omega - \alpha$. To relate these values to the classical frequencies, notice that

$$i G\hat{H}_o = \begin{pmatrix} iKL & -iKAK \\ iA & iLK \end{pmatrix},$$

which is similar to $\begin{pmatrix} I & -iK \\ I & iK \end{pmatrix} iG\hat{H}_o \begin{pmatrix} I & -iK \\ I & iK \end{pmatrix}^{-1}$

$$= \tfrac{1}{2}\begin{pmatrix} I & -iK \\ I & iK \end{pmatrix} iG\hat{H}_o \begin{pmatrix} I & I \\ iK & -iK \end{pmatrix}$$

$$= \begin{pmatrix} K(A-iL) & 0 \\ 0 & -K(A+iL) \end{pmatrix}$$

Therefore, the classical frequencies include both the single mode energies $\omega - \alpha$, which are eigenvalues of $-K(A+iL)$ and also the values $-(\omega-\alpha)^*$, which are eigenvalues of $K(A-iL)$. In the most familiar case, $\hat{H}$ is positive definite and by theorem (1.26), $\hat{H}$ can be reduced to $\text{diag.}[s_1,\ldots,s_N,s_1,\ldots,s_N]$. This conforms to the canonical matrix $\hat{H}_o$ of proposition (13.13). We may take $A = \text{diag.}[s_1,\ldots,s_N]$, $B = 0$ and $K = \text{diag.}[-I^{(a)},I^{(b)}]$, with a arbitrary. The choice $a,b \neq N$ and $J = G(a,b)$ leads to an indefinite inner product space $H^{(1)}$. However, it is customary to construct a positive definite inner product space whenever possible. This choice, $K = -I$, leads to a set of single particle energies which includes the eigenvalues $s_j$ of $-K(A+iL)$ but not the negative eigenvalues $-s_j$ of $iG\hat{H}$.

For the purposes of this section, we are most interested in the situation when $H^{(1)}$ may have an indefinite inner product but may not have a definite inner product. For example, consider the case that $iG\hat{H}$ has simple complex eigenvalues $\pm b \pm ai$. $\hat{H}$ belongs to the same canonical orbit as

$$\hat{K}_2^{(4)} = \begin{bmatrix} 0 & 0 & -a & -b \\ 0 & 0 & b & -a \\ -a & b & 0 & 0 \\ -b & -a & 0 & 0 \end{bmatrix}$$

By applying the symplectic transformation $C = 2^{-\frac{1}{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$

$\hat{K}_2^{(4)}$ transforms to $C^T \hat{K}_2^{(4)} C = \begin{pmatrix} A & 0 \\ 0 & KAK \end{pmatrix} = \hat{H}_o$, with $A = \begin{pmatrix} -b & a \\ a & -b \end{pmatrix} = A^T$

and $K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then according to (13.42), relative to the vacuum, the single mode energies are the eigenvalues of -KA, namely $-b \pm ai$. The most general single mode state corresponding to $E = -b \pm ai$ is

$$\psi_E = \gamma(a_1^+ \psi_o + i\, a_2^+ \psi_o), \text{ with } \gamma \in C. \tag{13.43a}$$

The most general single mode state corresponding to $E^* = -b - ai$ is

$$\psi_{E^*} = \beta(a_1^+ \psi_o - i\, a_2^+ \psi_o)\ , \text{ with } \beta \in C. \tag{13.43b}$$

From the commutation relations (13.32e),

$$\langle\psi_E,\psi_E\rangle = \langle\psi_{E^*},\psi_{E^*}\rangle = 0$$

$$\text{and} \qquad \langle\psi_{E^*},\psi_E\rangle = 2\beta^*\gamma\ . \tag{13.44}$$

From our discussion of heuristic mode space, in Section 7, we know that the indefinite metric blossoms not only when the classical frequencies are complex, but also when real classical frequencies are non-simple. For example, if $\hat{H}$ belongs to the same canonical orbit as

$$\hat{K}_6^{(4)}(1,b) = \begin{bmatrix} b^{-2} & 0 & 0 & 1 \\ 0 & 1 & -b^2 & 0 \\ 0 & -b^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

then $\hat{H}$ also belongs to the same canonical orbit as

$$\hat{H}_o = \begin{pmatrix} A & 0 \\ 0 & KAK \end{pmatrix} \text{ with } A = \tfrac{1}{2}\begin{pmatrix} 2b+b^{-1} & -b^{-1} \\ -b^{-1} & -2b+b^{-1} \end{pmatrix}$$

$$\text{and} \quad K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{13.45}$$

$\hat{H}_o = C^T \hat{K}_6^{(4)} C$, where

$$C = 2^{-\frac{1}{2}} \begin{bmatrix} 0 & 0 & b^{\frac{1}{2}} & b^{\frac{1}{2}} \\ b^{-\frac{1}{2}} & -b^{-\frac{1}{2}} & 0 & 0 \\ -b^{-\frac{1}{2}} & -b^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & b^{\frac{1}{2}} & -b^{\frac{1}{2}} \end{bmatrix} \in Sp(4,R)$$

Then, according to (13.42), the sole eigenvalue b of -KA is the only single mode energy and the most general single mode stationary state is

$$\psi_b = \gamma(a_1^+ \psi_o + a_2^+ \psi_o), \text{ with } \gamma \in C. \tag{13.46}$$

We can find another single mode state $\psi_o$, which is a principal vector for $H - \alpha$ and satisfies

$$(H - \alpha - b)\, \psi_D = \psi_b \; . \tag{13.47a}$$

and

$$(H - \alpha - b)^2 \psi_D = 0 \; . \tag{13.47b}$$

From (13.40) and (13.45), the vacuum energy $\alpha$ is equal to b. One solution for (13.47) is

$$\psi_D = \gamma b(a_1^+ \psi_o - a_2^+ \psi_o). \tag{13.48}$$

From the commutation relations (13.32c),

$$\langle\psi_b,\psi_b\rangle = 0 \; , \tag{13.49a}$$

$$\langle\psi_D,\psi_D\rangle = 0 \; , \tag{13.49b}$$

and

$$\langle\psi_D,\psi_b\rangle = 2|\gamma|^2 b \; . \tag{13.49c}$$

Because of (13.47a) and (13.49b), $\psi_D$ is often called the "dipole ghost".

So far, in this section, we have shown how a single mode space with indefinite metric can be constructed rigorously by extending the mathematics of algebraic quantization. The physical interpretation of the indefinite metric is a separate problem. This interpretation problem is no different from that which has prevailed over the last two decades, in the context of heuristic quantization [73]. It is generally agreed (e.g. [119]) that if the indefinite metric is to be given a probabilistic interpretation, the dynamically invariant single mode space $\mathcal{H}^{(1)}$ must be decomposable into a direct sum $\mathcal{H}^{(1)} = \mathcal{H}_p \oplus \mathcal{H}_n$, with each element $\psi_p$ of $\mathcal{H}_p$

having a non-negative norm $<\psi_p,\psi_p>_1$ . $H_p$ represents the physical states of the system and $H_n$ represents the non-physical states. Mathematically, there are many ways in which an indefinite inner product space can be decomposed in this way. For example, if $H^{(1)}$ is two dimensional, $<\psi_1,\psi_1>_1 = 1, <\psi_1,\psi_2>_1 = 0$ and $<\psi_2,\psi_2>_1 = -1$, $H_p$ could be taken to be all scalar multiples of $\psi_1$ but could just as well be taken to be all scalar multiples of $\psi_1 + \frac{1}{2}\psi_2$. Since $<\psi_1 + \frac{1}{2}\psi_2, \psi_1 + \frac{1}{2}\psi_2>_1 = \frac{3}{4}$ , $H_p$ would have positive metric after either choice. In practice, $H_p$ is determined by the physics of the system which is represented. For example, in the Gupta-Bleuler covariant formulation of quantum electrodynamics, $H_p$ includes the states representing transversely polarized photons, while $H_n$ includes any state involving logitudinal or time-like photons. According to Ascoli and Minardi [119], the minimal requirement for an indefinite inner product $H^{(1)}$, with pseudo-unitary Hamiltonian dynamics, to have a probabilistic interpretation, is that if $\psi(o)$ belongs to $H_p$, then for all t,

$$\psi(t) = e^{-iHt}\,\psi(o) = \psi_p(t) + \psi_n(t) \quad , \tag{13.50}$$

with $$\psi_{p,n} \in H_{p,n} \text{ and } <\psi_p(t),\psi_n(t)>_1 = <\psi_n(t),\psi_n(t)>_1 = 0 \tag{13.51}$$

(13.51 ensures that $\psi_n(t)$ in no way contributes to the norm of $\psi(t)$. However, it can easily be checked that these conditions can not be met either for the case of complex frequencies or for the dipole ghost which we have just examined. In the former case, $H^{(1)}$ is two dimensional, so that $H_p$ must be one dimensional, spanned by a single state vector $\psi_1 = \xi\,\psi_E + \zeta\,\psi_{E^*}$ , for some $\xi, \zeta \in C$. Assuming $\psi(o) = \psi_1$ ,

$$\psi(t) = \xi\, e^{-iEt}\,\psi_E + \zeta\, e^{-iE^*t}\,\psi_{E^*} \text{ , with } E = b + ai.$$

From (13.50), $$\psi_n(t) = \psi(t) - \mu\,\psi_1 \tag{13.52}$$

since $\mu\,\psi_1$, with $\mu \in C$, is an arbitrary element of $H_p$.

The condition $<\psi_n,\psi_n>_1 = 0$ implies

$$0 = (1 + |\mu|^2)\langle\psi_1,\psi_1\rangle_1 - 2\,\mathrm{Re}(\mu\,\langle\psi(t),\psi_1\rangle_1) \qquad (13.53)$$

From the condition $\langle\psi_n,\psi_p\rangle_1 = 0$, $\langle\psi(t),\psi_1\rangle_1 = \langle\mu\,\psi_1,\psi_1\rangle$, so that (13.53) becomes

$$(1 - |\mu|^2)\,\langle\psi_1,\psi_1\rangle_1 = 0 \qquad (13.54)$$

Unless every physical state in $H_p$ is to have zero norm, $\langle\psi_1,\psi_1\rangle_1 \neq 0$.

Therefore, (13.54) implies $|\mu| = 1$. (13.55)

However, the requirement $\langle\psi_p,\psi_n\rangle_1 = 0$ implies $\langle\psi_1,\psi_n\rangle_1 = 0$,

so that by (13.52), $\langle\psi_1,\psi(t)\rangle_1 - \mu\langle\psi_1,\psi_1\rangle_1 = 0$

$$\Rightarrow \mu = \langle\psi_1,\psi(t)\rangle_1 \,/\, \langle\psi_1,\psi_1\rangle_1$$

$$= \mathrm{Re}(2\beta^*\gamma\,\zeta^*\,\xi\, e^{-ibt}e^{at} + 2\,\beta\,\gamma^*\,\zeta\,\xi^*\, e^{-ibt}e^{-at})/\langle\psi_1,\psi_1\rangle_1\,,$$

using (13.44) (13.56)

This exponential increase in the amplitude of $\mu$ is impossible to reconcile with $|\mu| = 1$. This shows that in the case of simple complex frequencies, the usual requirements of physical interpretability can not be met.

In the dipole ghost situation of (13.47), $H^{(1)}$ is again two dimensional, so that $H_p$ must consist of scalar multiples of a single state $\psi_1 = \xi\,\psi_b + \zeta\,\psi_D$. Recalling that in (13.47), $\alpha = b$, we have

$$H\,\psi_b = 2b\,\psi_b$$

$$e^{-iHt}\,\psi_b = e^{-2ibt}\,\psi_b \qquad (13.57a)$$

$$H\,\psi_D = \psi_b + 2b\,\psi_D$$

$$H^2\psi_D = 4b\,\psi_b + (2b)^2\,\psi_D$$

$$H^n\psi_D = n(2b)^{n-1}\,\psi_b + (2b)^n\,\psi_D$$

$$e^{-iHt}\,\psi_D = e^{-2ibt}\,\psi_D - it\,e^{-2ibt}\psi_b \qquad (13.57b)$$

Assuming $\psi(o) = \psi_1$ and using (13.57),

$$\psi(t) = \xi\, e^{-2ibt}\psi_b + \zeta\, e^{-2ibt}\,\psi_D - it\,\zeta\, e^{-2ibt}\,\psi_b$$

$$= \mu\,\psi_1 + \psi_n(t),$$ since every element of $H_p$ has the form $\mu\,\psi_1$ with $\mu \in C$.

Therefore, $$\psi_n(t) = \psi(t) - \mu\,\psi_1 = (e^{-2ibt} - \mu)\psi_1 - it\,\zeta\, e^{-2ibt}\,\psi_b \qquad (13.58)$$

Hence, $$<\psi_n(t),\psi_p(t)>_1 = \mu(e^{2ibt} - \mu^*)<\psi_1,\psi_1>_1 + \mu\, it\,\zeta^*\, e^{2ibt}<\psi_b,\psi_1>_1$$

$$= 2|\gamma|^2 b(\xi^*\zeta + \xi\zeta^*) + 2\,\mu\, it|\zeta|^2|\gamma|^2 b e^{2ibt},$$

by (13.49).

From (13.49), the requirement $<\psi_n,\psi_p> = 0$ implies $\zeta = 0$. Therefore, an arbitrary physical state $\mu\,\psi_1$ has the form $\mu\,\xi\,\psi_b$, which has vanishing norm, by (13.49a). Therefore, this norm could not be interpretable as a probability. In Heisenberg's treatment of the Lee model [120], the dipole ghost occurs in two degrees of freedom of an interacting field. The Hamiltonian can be unambiguously partitioned into a free term and an interaction term. This leads to the definition of an S-operator which maps incoming physical states onto outgoing physical states of the same norm. Thus, in the words of Heisenberg, although a physical interpretation of the local behaviour in terms of probabilities can not be given, it is conceivable that such a model might be adequate for a scattering experiment.

Recently, a quadratic Hamiltonian with complex frequencies was considered by Englent [121], in connection with the radiation-damped oscillator. Classically, a charged harmonic oscillator is damped as it emits electro-magnetic radiation. The motion of the oscillator is then described by the Abraham-Lorentz equation [121].

$$m\,\ddot{q} = -k\,q + m\,\tau\,\dddot{q} \quad ; \quad \tau = \frac{2}{3}\,\frac{e^2}{mc^3} \qquad (13.59)$$

with e and m the charge and mass.

The general solution of (13.59) is given by

$$q(t) = b_1 e^{-\gamma t} \cos\omega t + b_2 e^{-\gamma t} \sin\omega t + b_3 e^{\Gamma t} \tag{13.60}$$

where $\Gamma$, $(-\gamma \pm i\omega)$ are the three solutions $\Omega_{1,2,3}$ of the characteristic equation $m\,\Omega^2 = -k + m\,\tau\,\Omega^3$.

In fact, $\gamma$, $\omega$ and $\Gamma$ are all positive. Usually, the general solution (13.60) to (13.59) is restricted by assuming $b_3 = 0$, so that the classical stable damped oscillator can be modelled. However, Englert was interested to see whether the general unstable system (13.60) could be retained and then restricted after quantization. To this end, the corresponding negative frequencies were introduced by including an extra degree of freedom in which the dynamics is that obtained from (13.59) by time reversal. The system was then able to be described by a quadratic Boson Hamiltonian with complex frequencies. However, to proceed further would be to walk on shaky ground, since there is still considerable debate on whether dissipative systems can be quantized using Hamiltonian techniques. The Hamiltonian can be interpreted as the generator of the motion but not as the energy. Even for conservative systems, the classical equations of motion can be generated by a Lagrangian which is not of the form T - V, with T and V the kinetic and potential energy respectively. It is known that after such a choice of Lagrangian, canonical quantization leads to inconsistencies [VII,39b,122]. For example, the choice of Lagrangian $L = \frac{1}{2}(\dot{q}_1^2 - q_1^2) - \frac{1}{2}(\dot{q}_2^2 - q_2^2)$ for the isotropic 2-dimensional oscillator leads to the angular momentum $L_z = q_1\dot{q}_2 - q_2\dot{q}_1$ $(= -q_1p_2 - q_2p_1)$ having a continuous spectrum [123].

In this section, we have exposed the following disadvantages in allowing the indefinite metric to enter algebraic quantization:

(A) When the requirement of positive metric is discarded, the unitarizing complex structure, when it exists, is no longer uniquely determined by the classical dynamics and neither is the signature of the metric.

(B) Any advantage of allowing the metric to be indefinite is limited by the fact that certain simple classical systems, such as the free particle in one dimension, still can not be (pseudo)-unitarized.

(c) When an unstable system can be pseudo-unitarized, the local behaviour can not be interpreted in terms of probabilities.

To this list, we may add another problem unearthed by Araki [124]:

(D) In order to analyze a Hamiltonian H on Fock space with indefinite metric, we close H in the topology determined by some chosen constructed positive metric. However, the spectrum of the closure $\overline{H}$ depends, in a dramatic way, on the choice of topology.

Since the free particle can already be handled quite easily in quantum mechanics, regardless of problem (B), some further comments must be made on the relationship between algebraic and other techniques of quantization. As discussed in Section 3, up to unitary equivalence, there is only one irreducible Hilbert space representation of the C.C.R. with finite degrees of freedom and except for an additive constant, there is a unique prescription for the quantum mechanical analogue of a quadratic Hamiltonian. From theorem (3.16) and other results of Section 3, we have a clear indication that the spectrum of a first quantized quadratic Hamiltonian is discrete only when the Hamiltonian belongs to the same canonical class as a set of independent harmonic oscillators. The Segal procedure, applied to a classical harmonic oscillator, produces a one-dimensional complex single particle space $\mathcal{H}^{(1)}$. The single creation operator $a^{\dagger}(\underline{e}_1)$ on Fock space $\mathcal{F}(\mathcal{H}^{(1)})$ is the ladder operator which effects the elementary excitation between two neighbours of the equally spaced energy levels. If, on the other hand, we were unaware of proposition (9.10) and we attempted to apply the Segal procedure to a classical free particle, once again the single classical degree of freedom would be expected to lead to a one dimensional complex single particle space $H^{(1)}$. However, a single independent creation operator could not be associated with the elementary excitations of the first quantized system, since the quantized free particle Hamiltonian has a continuous spectrum. Therefore, proposition (9.10) guarantees that such interpretation problems do not arise and that the Segal-Cook creation operators always effect elementary excitation of the unitarized dynamical system. On the other hand, it is common practice to treat Schrödinger single particle wave functions as belonging to the single particle subspace of a field which has a many-particle interpretation. This larger space could be achieved by the Segal-Cook procedure applied to $L^2(R)$, considered as a real space, with each wave function decomposed into its real and imaginary parts. The dynamics generated by $H = \frac{1}{2}P^2 = -\frac{1}{2} h^2 \frac{\partial^2}{\partial x^2}$ could be considered to be an example of real symplectic dynamics which is trivially unitarizable. Given any one parameter strongly continuous unitary group on $L^2(R)$, there is no mathematical obstruction to applying the Segal procedure to achieve a second quantized system. This procedure has particular significance when the original classical system is a free particle, since elementary particle theory is a sub-branch of elementary field theory. However, it is not usual practice to apply second quantization, for example, to a first quantized harmonic oscillator, achieving a collection of independent creation operators $a^{\dagger}(n)$ which create oscillators in the n'th excited state (n = 0,1,2,...).

Given a first quantized system, we have a well-defined second quantized system, if required. However, if we begin with an infinite dimensional classical symplectic dynamics, first quantization involves a choice of one of a myriad of inequivalent representations of the C.C.R. If the dynamics is unstable, unitarization is not possible and we can no longer use the Segal procedure to select an appropriate representation.

# CHAPTER III - APPLICATIONS AND OUTLOOK

## SECTION 14 - INVARIANT WAVE EQUATIONS

Perhaps the first indication that linear systems might be a source of inconsistency in quantum mechanics was the difficulty of interpretation of the Klein-Gordon equation. Within the space of solutions of the classical Klein-Gordon equation, the subspace of positive energy solutions is dynamically invariant. However, it was soon realized that if an innocuous-looking scalar potential v(x) were allowed to act externally in the manner

$$(\Box + m^2 - v(x))\ \phi(x) = 0, \quad (\Box = \partial^\mu \partial_\mu = \partial_t^2 - \nabla^2), \tag{14.1}$$

dynamically invariant separation of negative energy solutions from positive energy solutions may no longer be possible. The Klein paradox is that the negative energy density then requires an interpretation, even when the scalar field is chargeless, so that the associated indefinite energy norm (see e.g. [125]) can not even be interpreted as a charge density.

Research on the external field problem gained further impetus in 1940, with the discovery by Schiff, Snyder and Weinberg [126] that the scalar field is unstable when certain external scalar potentials are introduced. It seemed strange at that time that the instability problem did not occur for the spin ½ field.

Having applied the Fourier transformation to the scalar field $\phi(x) = \sum_{\underline{k}} a_{\underline{k}}(t)\ e^{i\underline{k}\cdot\underline{x}}$ in interaction with an external static scalar potential $v(\underline{x},t) = v(\underline{x},0)$, Schroer and Swieca [33] found that the function $a_{\underline{k}}(t)$ is exponentially unstable for some values of $\underline{k}$ and that the heuristic Fock space necessarily has an indefinite inner product.

The general results which we have presented in chapter two of this thesis apply to general quadratic Hamiltonians. As an illustration of the utility of these results as diagnostic tools, we shall quickly deduce all of the phenomena of the scalar field which we have mentioned above. When v(x) is allowed to exceed $m^2$, imaginary frequencies enter the frequency spectrum of the system, just as in the case of the Klein-Gordon field with imaginary mass. Therefore,

(a) since the classical Klein-Gordon system may be viewed as a classical Hamiltonian system, the frequencies must occur in complex conjugate pairs. Therefore, the presence of imaginary frequencies implies that the classical dynamics is unstable.

(b) since the classical dynamics is unstable, the Segal-Cook construction can not produce a Fock space with definite metric and unitary dynamics (proposition (12.35)).

(c) a dynamically invariant separation of negative energy solutions from positive energy solutions is not possible. Otherwise, we could construct a unitarizing complex structure of the form (12.22) and the Segal-Cook construction would produce a Fock space with positive definite metric and unitary dynamics.

(d) the heuristic mode space for finite dimensional unstable sub-systems must have indefinite metric (proposition (7.33)).

Since a quadratic Fermion Hamiltonian can always be diagonalized by a C.A.R. Bogoliubov transformation, which is a unitary transformation, the corresponding classical dynamics, discussed in Sections 8 and 10, is always stable. Therefore, the problems (a-d) above do not arise when an external classical potential interacts with a spin ½ field.

Since the discovery by Velo and Zwanziger that quantized fields of spin greater than one are a-causal, causality has been an important consideration in the study of general invariant-linear wave equations [127]. The general invariant linear wave equation can always be expressed as a first order Gel'fand-Yaglom equation [128]:

$$[\beta^{\mu}\partial\mu + m]\ \psi_{\alpha}(x) + v^{\beta}_{.\alpha}\ \psi_{\beta}(x) = 0\ . \tag{14.2}$$

It turns out that a-causality can be diagnosed at the classical level. If the equation (14.2) leads to causal propagation, it must at least be hyperbolic, so that there is an upper bound to the speed of propagation [128]. As pointed out by Wightman (see the discussion following [127]), the hyperbolicity of (14.2) is equivalent to the boundedness of the imaginary part of the frequency spectrum. If the free field defines an integral representation of the Poincaré group, then the system should be quantized according to Bose statistics. It is then necessary to view time evolution of $\psi_{\beta}(\underline{x},t)$ as a one parameter group of symplectic transformations. The symplectic time evolution defines a one parameter group of Bogoliubov transformations $a(\psi_{\beta}(\underline{x},0)) \to a(\psi_{\beta}(\underline{x},t))$. The generator of this group of Bogoliubov transformations may be represented by a matrix $\hat{I}\ \mathcal{D}_1 = \begin{pmatrix} A & B \\ -B^* & -A^* \end{pmatrix}$, with A Hermitean and B skew symmetric on single particle space. Then $\hat{I}\ \mathcal{D}_1$ is the sum of an Hermitean term $\begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$ and a skew Hermitean term $\begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}$. In order that the imaginary part of the frequency spectrum (i.e. the spectrum of $\hat{I}\ \mathcal{D}$) is bounded, it is sufficient that B is bounded. In the case that B is bounded but not in the Hilbert-Schmidt class, (14.1) is a hyperbolic equation but the corresponding quantum field dynamics will not be unitarily implementable in the Fock representation.

Another major source of interest in the general quadratic Hamiltonian has been the higher order wave equation. In the early 1950's, it was proposed by several people that higher order wave equations might resolve some of the divergence difficulties of quantum field theory. If we begin, as Pais and Uhlenbeck did, with a non-localized action [32]:

$$S = \int L\,dt$$

$$L(x) = \int \phi(x)\ \varepsilon(x-x^1)\ \phi(x^1)\ d_4x^1 \qquad (14.3)$$

$$\varepsilon(x) = (2\pi)^{-4} \int F(-k^2)\ \exp(ik_\mu x^\mu) d_4k \quad ,$$

$(k^2 = k^\mu k\mu$ and $F$ is a polynomial),

the principle of stationary action leads to a higher order equation

$$F(\Box) = 0 \qquad (14.4)$$

After reducing (14.4) to a set of first order equations by the introduction of subsidiary conditions, Pais and Uhlenbeck found that the finite dimensional dynamical subsystems were often described by strange quadratic Hamiltonians which do not belong to the harmonic oscillator class. As discussed in Section 3, this typically leads to a quantum mechanical Hamiltonian which is unbounded from below. In addition, Pais and Uhlenbeck showed that it was difficult to reconcile non-locality with micro-causality and since that time, the adjective "non-local" has become synonymous with "a-causal". Nevertheless, several people have found good reasons to persist with non-local models. Firstly, at the microscopic level, strongly interacting particles are believed to be mutually penetrating objects (e.g. [129]). At the macroscopic level, Bell [130] reports on recent experiments which indicate that the non-local predictions of quantum mechanics, which were unacceptable to Einstein, Podolski and Rosen, may be a fact of life.

Perhaps one of the simplest examples of (14.4) is the equation

$$\Box^2\phi(x) = 0 \quad (\Box = \partial_t^2 - \Delta, \text{ with } \Delta \text{ the Laplacian operator } \Delta^2) \qquad (14.5)$$

This is the equation satisfied, in the Landau gauge, by each component of the 4-potential for the free electromagnetic field [131]. To see that (14.5) is grossly non-local, note that it is equivalent to

$$\Box\ \phi(x) = j(x), \qquad (14.6)$$

where the non-localized source term j(x) is an arbitrary solution of the Klein-Gordon equation. The equation (14.5) is macroscopically a-causal, since its solutions include not only the Klein-Gordon field, but also the harmonic field [132], which results in a Wightman 2-point function which is proportional to $(x-y)^2 = (x-y)^\mu (x-y)_\mu$, resulting in non-vanishing commutators $[\phi(x),\phi(y)]$, with x-y space-like.

Serious interest has been shown in equation (14.5) as a model of the sub-hadronic gluon field [132,133]. A fundamental Green's function for the operator $-\Delta^2$ is $r = (\underline{x}.\underline{x})^{\frac{1}{2}}$.

$$-\frac{1}{8\pi}\,\Delta^2\, r = \delta(x) \tag{14.7}$$

Therefore, (14.5) is a very simple example of a system with confining static potential in the presence of a point source. Following the programme of heuristic quantization for the system (14.6), Narnhofer and Thirring [133] found the formal Hamiltonian to be

$$H = \sum_{\underline{k}} H_{\underline{k}} \,,$$

$$\text{with} \quad H_{\underline{k}} = \tfrac{1}{2}\,\underline{\alpha}^\dagger(\underline{k})\; \mathcal{D}_1(\underline{k})\; \underline{\alpha}(\underline{k}) \quad ; \quad \underline{\alpha}^\dagger(\underline{k}) = (a_1(\underline{k})^\dagger, a_2(\underline{k})^\dagger, a_1(\underline{k}), a_2(\underline{k})) \tag{14.8}$$

$$\text{with } \mathcal{D}_1(\underline{k}) = \begin{bmatrix} (4k_o)^{-1} & k_o - i(4k_o)^{-1} & (4k_o)^{-1} & i(4k_o)^{-1} \\ k_o + i(4k_o)^{-1} & (4k_o)^{-1} & i(4k_o)^{-1} & -(4k_o)^{-1} \\ (4k_o)^{-1} & -i(4k_o)^{-1} & (4k_o)^{-1} & k_o + i(4k_o)^{-1} \\ -i(4k_o)^{-1} & -(4k_o)^{-1} & k_o - i(4k_o)^{-1} & (4k_o)^{-1} \end{bmatrix},$$

$$\text{with } k_o = |\underline{k}|$$

In (14.8), the operators $a_j(\underline{k})$ are ordinary Boson annihilation operators:

$$[a_j(\underline{k}), a_\ell(\underline{k}')^\dagger] = \delta_{j\ell}\;\delta(\underline{k}-\underline{k}') \tag{14.9}$$

Therefore, according to the scheme of proposition (4.49), the corresponding classical Hamiltonian is

$$H = \sum_{\underline{k}} \tfrac{1}{2}\, \underline{z}(\underline{k})^T\, \hat{H}(\underline{k})\, \underline{z}(\underline{k}) \quad ;$$

$$\hat{H}(\underline{k}) = \begin{bmatrix} (2k_o)^{-1} & k_o & 0 & (2k_o)^{-1} \\ k_o & 0 & 0 & 0 \\ 0 & 0 & 0 & k_o \\ (2k_o)^{-1} & 0 & k_o & (2k_o)^{-1} \end{bmatrix} \qquad (14.10)$$

This Hamiltonian can be classified according to the scheme of Section 1. With $\hat{H} = \hat{H}(\underline{k})$, the elementary divisors of $iG\hat{H} - sI$ are $(s \pm k_o)^2$. Therefore, $\hat{H}$ must belong to one of the two canonical orbits with canonical form $\hat{K}_6^{(4)}(\rho)$, with $\rho = \pm 1$. To determine $\rho$, we must find the signature of $\hat{H}_N$, given that $-G\hat{H}_N$ is the nilpotent part in the Jordan decomposition of $-G\hat{H}$. The matrix $\hat{H}_N$ is given by

$$\hat{H}_N = \text{diag.}[(2k_o)^{-1}, 0, 0, (2k_o)^{-1}] \;, \qquad (14.11)$$

which is verified simply by checking that $-G\hat{H}_N$ is nilpotent and that $-G\hat{H}_N$ commutes with $-G\hat{H}$. Therefore, $\hat{H}_N$ is positive semi-definite and $\rho = +1$. The determination of the canonical orbit of $\hat{H}_{\underline{k}}$ immediately leads to further information on the quantum mechanical system (14.5). The quadratic elementary divisors and real frequencies $k_o$ lead to linear instability in the evolution of field operators $\phi(\underline{x},t)$. Since the elementary divisors are not linear, algebraic unitarization of the classical dynamics is not possible. However, there exists a complex structure which enables the classical dynamics to be pseudo-unitary, since all frequencies are real and non-vanishing. As pointed out by Narnhofer and Thirring, the one parameter group of Bogoliubov transformations generated by (14.8) is not unitarily implementable on Fock space. Equivalently, the formal Hamiltonian (14.8) can not be closed and then extended to a self-adjoint operator. Despite all these difficulties, Thirring and Narnhofer have provided a novel suggestion as to how consistency might be regained. Namely, if (14.5) is to be a gluon field, then other interaction terms must be introduced, since the gluon field is not isolated. Just as in the case of the external field problem, in which a healthy free field develops instability when an interaction is introduced, a pathological free field may be stabilized by introducing an interaction. An initial exploratory model, in which the extra field is represented by a harmonic oscillator, has shown some success. Since a fully interacting Hamiltonian H may be decomposed into a "free" term $H_o$ of the form (14.8) and an interaction term, the S-operator may be defined from a counterpart of $e^{iHt}\, e^{-iH_ot}$ and this can be unitary, even though the original Fock space has indefinite metric. The process of extracting a unitary S-matrix out of a pseudo-unitary dynamical system has recently been further developed by Demuth [134].

The system (14.5) may be modified by inserting a Boson mass,

$$(\Box + m^2)^2\, \psi(x) = 0 \tag{14.12}$$

This was one of the earliest higher order wave equations, originally proposed by Bhabha [135]. The static potential is then proportional to $e^{\pm mr}$.

The method of quadratic canonical forms also applies to some models involving both Bose and Fermi fields in interaction. For example, in the Schwinger model for quantum electrodynamics with massless Fermions in 1 + 1 dimensions, the Fermi field contributes to the interaction term in the Hamiltonian via a current $j^\mu(x)$. We may express $j^\mu(x)$ as $\delta^\mu\phi(x)$ for some scalar massless Boson field $\phi$ and thereafter express the Hamiltonian purely in terms of Boson operators, in accordance with Krönig's identity for the neutrino theory of light in 1 + 1 dimensions [136]. The form of the Hamiltonian partly depends on the initial choice of gauge. In the Fermi and Landau gauges, the Lagrangians provided by Hurst and Carey [137] lead to the Hamiltonians

$$\text{(Fermi)} \qquad H_F = \int_{-\infty}^{\infty} \left(K - \tfrac{1}{2}(\pi_o)^2\right) dx_1 \tag{14.13a}$$

$$\text{(Landau)} \qquad H_L = \int_{-\infty}^{\infty} K\, dx_1\ , \tag{14.13b}$$

where

$$K = \tfrac{1}{2}[\pi_1^2 + (\pi_\phi - gA^0)^2 + (\phi,_1 - gA)^2 - g^2(A_0)^2] \quad ,$$

with the variables $\pi_0$, $\pi_1$, $\pi_\phi$ conjugate to $A^0$, $A^1$ and $\phi$ respectively and $g = e^2/\pi$ .

Now apply the Fourier decomposition (for $t = x^0 = 0$)

$$A^\mu(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{i\underline{kx}} \hat{A}^\mu(\underline{k}) \, d\underline{k} \tag{14.14}$$

$$\phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{i\underline{kx}} \hat{\phi}(\underline{k}) \, d\underline{k}$$

$$\Pi_\mu(x) = (2\pi)^{-\frac{1}{2}} \int_{\infty}^{\infty} e^{-i\underline{kx}} \hat{\Pi}_\mu(\underline{k}) \, d\underline{k}$$

$$\Pi_\phi(x) = (2\pi)^{-\frac{1}{2}} \int_{\infty}^{\infty} e^{-i\underline{kx}} \hat{\Pi}_\phi(\underline{k}) \, d\underline{k} \, .$$

It is well known that in the quantum theory, the Fourier transformed generalised operators maintain the canonical commutation relations, including

$$[\hat{A}_\mu(\underline{k}) \, , \, \hat{\Pi}^\nu(\underline{k}')] = i \, \delta_\mu^{\ \nu} \, \delta(\underline{k} - \underline{k}') \, .$$

In the corresponding classical theory, the Fourier transforms of conjugate variables remain conjugate. In the classical theory, all variables are real so that

$$\hat{A}^\mu(-\underline{k}) = \hat{A}^\mu(\underline{k})^* \quad \text{and so on for } \hat{\phi}, \hat{\pi}_\phi, \hat{\pi}_\mu \, . \tag{14.15}$$

From (14.14-15), $H_F$ may be expressed as

$$H_F = \int_0^\infty H_F(\underline{k}) \, d\underline{k} \, , \tag{14.16}$$

where
$$H_F(\underline{k}) = -|\hat{\pi}_0(\underline{k})|^2 + |\hat{\pi}_1(\underline{k})|^2 + |\hat{\pi}_\phi(\underline{k})|^2$$
$$+ g^2 |\hat{A}^0(\underline{k})|^2 + g^2 |\hat{A}^1(\underline{k})|^2 + k^2 |\hat{\phi}(\underline{k})|^2$$
$$- 2g \operatorname{Re}[\hat{\pi}_2(\underline{k}) \hat{A}^0(\underline{k})] + k\operatorname{Im}[\hat{\pi}(\underline{k}) \hat{A}^0(\underline{k})]$$
$$+ k\operatorname{Im}[\hat{\pi}_0(\underline{k}) \hat{A}^1(\underline{k})] + kg \operatorname{Im}[\hat{A}^2(\underline{k}) \hat{A}^{1*}(\underline{k})] .$$

$H_F(\underline{k})$ may be expressed in terms of six real variables, the real and imaginary parts of $\hat{A}^\mu$, $\hat{\phi}$ and their conjugate variables. Hence, the Schwinger model may be analysed in terms of subsystems of degree six. On each subsystem, the Hamiltonian is given by $\frac{1}{2} \underline{z}^T \hat{H}_F(\underline{k}) \underline{z}$, with $\hat{H}_F(\underline{k})$ a symmetric matrix. However, it turns out that in thecase of Fermi gauge, the eigenvalues of $i\,G\,\hat{H}_F(\underline{k})$ are not all real. Indeed, explicit solution of the classical dynamics [137] uncovers an exponential instability. Due to the presence of non-real frequencies, by proposition (12.35), unitarization of the classical dynamics is not possible. On the other hand, after choosing the Landau gauge, the classical dynamics is stable. This supports the surprising result of Carey and Hurst [137] that the existence of a time invariant vacuum representation, exhibiting unitary time evolution for the Schwinger model, depends on the choice of gauge. There are three main philosophical viewpoints that can be taken, following this result. The first viewpoint is that Bosonization is possible only in 1 + 1 dimensions, so that this apparent violation of gauge invariance is only an artifact of an artificial situation. Indeed, there are obstructions in extending Krönig's identity to four dimensional space-time [138]. Therefore, there is no Bosonized Schwinger-type model which threatens gauge invariance in 3 + 1 dimensions. The second viewpoint is that there are as yet undetermined preferred non-Fock representations for the C.C.R. which will allow algebraic quantization to cover unstable classical dynamics, thereby allowing quantization in any gauge. The third, more radical viewpoint is that the electromagnetic 4 potential is a significant observable and that there is a preferred gauge in nature, which will eventually be determined, along with the fine structure constant, by consistency and boundedness requirements [139].

## SECTION 15 - THE ISING MODEL

The two dimensional Ising model of ferro-magnetic materials with nearest neighbour interactions on a toroidal lattice, is the simplest example in classical statistical mechanics of a model which exhibits a phase transition. The properties of this model result from a transfer matrix which may be viewed as a quadratic element of a Clifford algebra. Hence, we make contact with the theory of quadratic Fermion Hamiltonians, as discussed in Section 10. The ideas behind the algebraic approach to the Ising model, due originally to Onsager and Kaufmann [140], can be found in the review article by Schultz, Mattis and Lieb [141], which is now widely recognized as a standard reference.

Lattice points are labelled $(i,j)$; $i = 1,\ldots,L$ and $j = 1,\ldots,N$. To each point $(i,j)$, there is associated a value $\sigma(i,j) = \pm 1$ for the classical spin. Neighbouring pairs contribute energies $-J_x\, \sigma(i,j)\, \sigma(i+1,j)$ or $-J_y\, \sigma(i,j)\, \sigma(i,j+1)$. To wrap the lattice on a torus is equivalent to enforcing the boundary conditions $\sigma(i+L,j) = \sigma(i,j) = \sigma(i,j+N)$. Classical observables f are functions f of all the values $\sigma(i,j)$. The Gibbs state is a linear functional that maps classical observables f to their expectation values $\langle f \rangle$.

$$\langle f \rangle = Z^{-1} \sum_{\substack{\sigma(i,j)=\pm 1 \\ i=1,\ldots,L \\ j=1,\ldots,N}} f(\sigma(i,j)) \exp - \beta H(\sigma(i,j)) , \qquad (15.1)$$

where $\beta = 1/kt$, with k the Boltzmann constant and T the absolute temperature. In the case of toroidal boundary conditions,

$$H = - \sum_{\substack{i=1,\ldots,L \\ j=1,\ldots,N}} J_x\, \sigma(i,j)\, \sigma(i+1,j) + J_y\, \sigma(i,j)\, \sigma(i,j+1) \qquad (15.2)$$

The partition function Z is given by

$$Z = \sum_{\substack{\sigma(i,j)= 1 \\ i=1,\ldots,L \\ j=1,\ldots,N}} \exp - \beta H(\sigma(i,j)) \qquad (15.3)$$

It is fruitful to re-express the partition function as

$$Z = \text{Tr } V^N \quad , \tag{15.4}$$

where the transfer matrix V is of order $2^L \times 2^L$. V may be modified by an arbitrary similarity transformation and various closely related choices may be found in the literature. An important step in the algebraic approach is to expand V in terms of the usual matrix representatives of the generators $\sigma_k^\alpha$ of the Paulion algebra $A_L$.

$$\sigma_k^\alpha = I \otimes I \otimes \ldots \otimes I \otimes \underset{\substack{\uparrow \\ \text{k'th place}}}{\sigma^\alpha} \otimes I \otimes \ldots \otimes I \quad \text{for } k = 1,\ldots,L ,$$

$\sigma^\alpha$ being the Pauli spin matrices with $\alpha = x,y$ or $z$ and $(\sigma^\alpha)^2 = I$. For example, in the formulation of Schultz, Mattis and Lieb [141],

$$V = V_2^{\frac{1}{2}} \; V_1 \; V_2^{\frac{1}{2}} \quad , \tag{15.5}$$

where $$V_1 = (2 \sinh 2 \beta J_x)^{\frac{1}{2}} \exp(\beta J_x{}^* \sum_{k=1}^{L} \sigma_k{}^x) \quad , \tag{15.6}$$

with $J_x^*$ defined by $\tanh \beta J_x^* = \exp(-2 \beta J_x)$

and $$V_2 = \exp(\beta J_y \sum_{k=1}^{L} \sigma_k{}^z \sigma_{k+1}^z) \tag{15.7}$$

In (15.5), V is chosen to be an Hermitean matrix rather than a non-Hermitean matrix $V_1 V_2$ or $V_2 V_1$. To illustrate how the matrix V effects a transfer, we note that the two point correlation functions are given by (e.g. [142]):

$$\langle\sigma(i,j)\; \sigma(i+r,j+s)\rangle = (\text{Tr } V^N)^{-1} \; \text{Tr}(V^N \, V^{-r} \, \hat{\sigma}_1 \, V^r \, \hat{\sigma}_{1+s}) \quad , \tag{15.8}$$

with $\hat{\sigma}_j = \cosh(\beta J_x^*)\sigma_j{}^x - i \sinh(\beta J_x^*)\sigma_j{}^y$ .

According to Lewis and Sisson [143], for any $F \in A_L$, there exists a classical observable f such that

$$\langle f\rangle = \chi(F) = (\text{Tr } V^N)^{-1} \; \text{Tr } V^N \, F \tag{15.9}$$

From (15.8), the classical observables $\sigma(i,j)\;\sigma(i+r,j+s)$ correspond to

$$V^{-r} \; \hat{\sigma}_1 \; V^r \; \hat{\sigma}_{1+s} \in A_L \quad . \tag{15.10}$$

Alternatively, we may view classical observables as corresponding to elements of a Clifford algebra, obtained by applying the Jordan-Wigner transformation $\eta$ defined by (e.g. [141]) :

$$\eta(\sigma_1{}^x) = Z(\underline{e}_1) \ ,$$

$$\eta(\sigma_1{}^y) = Z(\underline{e}_{1+L}) \ ,$$

$$\eta(\sigma_k{}^x) = \prod_{j=1}^{k-1} (-i\, Z(\underline{e}_j)\, Z(\underline{e}_{j+L}))\, Z(\underline{e}_k) \ ,$$

and

$$\eta(\sigma_k{}^y) = \prod_{j=1}^{k-1} (-i\, Z(\underline{e}_j)\, Z(\underline{e}_{j+L}))\, Z(\underline{e}_{k+L}) \quad \text{for } k = 2,\ldots,L \ . \tag{15.11}$$

$$[Z(\underline{e}_\mu),\, Z(\underline{e}_\nu)]_+ = 2\, \delta_{\mu\nu}\, I \quad \text{for } \mu,\nu = 1,\ldots,2L \tag{15.12}$$

The set $\{\underline{e}_k, \underline{e}_{k+L}\}$ is a basis for a real space $V \oplus V$ of dimension 2L. With $V \oplus V$ equipped with the Euclidean inner product, the set $\{Z(\underline{e}_\mu)\}_{\mu=1,\ldots,2L}$ generates the Clifford algebra $U(V \oplus V)$ over $V \oplus V$. For an infinite lattice, the relations (15.12) are defined as in (10.2), except that in this section it is more convenient to consider generators $Z(\underline{e}_\mu)$ which, compared to those used in Section 10, have been scaled up by a factor of $\left(\frac{\hbar}{2}\right)^{-\frac{1}{2}}$. With $\underline{\psi} = \underline{u} \oplus \underline{v}$ and $\underline{\psi}' = \underline{u}' \oplus \underline{v}'$ ,

$$[Z(\underline{\psi}),\, Z(\underline{\psi}')]_+ = 2\, S(\underline{u},\underline{u}') + 2\, S(\underline{v},\underline{v}') \tag{15.13}$$

for all $\underline{u},\underline{v},\underline{u}',\underline{v}' \in V$ (a real Hilbert space with inner product S). It will later be convenient to abbreviate the right hand side of (15.13) to $2\, S(\psi,\psi')$.

In the algebraic treatment of Onsager and Kaufmann [140], log $V_1$ and log $V_2$ were shown to be quadratic elements of the Clifford algebra. However, with V defined as in (15.5), as a product of non-commuting matrices of large order, the calculation of log V by the Baker-Campbell-Hausdorff technique appeared to be a daunting task. Fortunately, by inspection, a canonical transformation was found which simultaneously reduced $V_1$ and $V_2$

to a direct sum of quadratic combinations of only four Clifford generators. Next, it was observed that log $V_1$ and log $V_2$ are linear combinations of matrices which generate a faithful representation of the Lie algebra su(2). Log V, as an element of su(2), could then be determined in a simpler representation, for example that generated by the Pauli spin matrices. After transforming back to the original representation, log V is given explicitly as a quadratic element of the Clifford algebra. For example, from the description by Hurst and Green [144], we could replace $V_1 V_2$ by $V_1^{\frac{1}{2}} V_2 V_1^{\frac{1}{2}}$ and obtain

$$Z = \mathrm{Tr}(V_-)^N + \mathrm{Tr}(V_+)^N , \tag{15.14}$$

where
$$V_- = \exp A_- , \tag{15.15}$$

$$A_- = \bigoplus_{r=1}^{L} Ar_- \; ; \quad Ar_- = \tfrac{1}{2}\, i\, \underline{\Gamma}^T(r)\, \hat{A}r_-\, \underline{\Gamma}(r) \tag{15.16}$$

$$\underline{\Gamma}^T(r) = [\Gamma(\underline{e}_r), \Gamma(\underline{e}_{L-r}), \Gamma(\underline{e}_{r+L}), \Gamma(\underline{e}_{21-r})] , \tag{15.17}$$

with $\Gamma(\underline{e}_r)$ obeying the same relations as those obeyed by $Z(\underline{e}_r)$, namely (15.12),

and
$$\hat{A}_{r-} = \log(a+(a^2-1)^{\frac{1}{2}}) \begin{bmatrix} 0 & -b_2 & b_1 & 0 \\ b_2 & 0 & 0 & b_1 \\ -b_1 & 0 & 0 & b_2 \\ 0 & -b_1 & -b_2 & 0 \end{bmatrix} \tag{15.18}$$

with
$$\begin{aligned} a = {} & \cosh \tfrac{1}{2}\beta J_y^*(\cosh \tfrac{1}{2}\beta J_y^* \cosh \beta J_x - \cos\theta_r \sinh \beta J_x \sinh \tfrac{1}{2}\beta J_y^*) \\ & + \sinh \tfrac{1}{2}\beta J_y^*(\sinh \tfrac{1}{2}\beta J_y^* \cosh \beta J_x - \cos\theta_r \sinh \beta J_x \cosh \tfrac{1}{2}\beta J_y^*) \end{aligned} \tag{15.19}$$

$$\begin{aligned} b_1 = {} & \cosh \tfrac{1}{2}\beta J_y^*(-\sinh \tfrac{1}{2}\beta J_y^* \cosh \beta J_x + \cos\theta_r \sinh \beta J_x \cosh \tfrac{1}{2}\beta J_y^*) \\ & + \sinh \tfrac{1}{2}\beta J_y^*(-\cosh \tfrac{1}{2}\beta J_y^* \cosh \beta J_x + \cos\theta_r \sinh \beta J_x \sinh \tfrac{1}{2}\beta J_y^*) \end{aligned} \tag{15.20}$$

and
$$\begin{aligned} b_2 = {} & \cosh \tfrac{1}{2}\beta J_y^*(\sin\theta_r \sinh\beta J_x \cosh \tfrac{1}{2}\beta J_y^*) \\ & - \sinh \tfrac{1}{2}\beta J_y^*(\sin\theta_r \sinh\beta J_x \sinh \tfrac{1}{2}\beta J_y^*) \end{aligned} \tag{15.21}$$

Explicit solution of the Ising model amounts to diagonalization of $V_-$ . $V_+$ can then be diagonalized in the same way and (15.4) can be evaluated. Diagonalization of $V_-$ is readily achieved using the explicit reduction of the quadratic Clifford operator $A\bar{r}$ to canonical form by the technique of Section 10. (10-31b) gives an orthogonal matrix C such that

$$C^{-1}\hat{A}r_- \, C = (a^2-1)^{\frac{1}{2}} \log(a+(a^2-1)^{\frac{1}{2}}) \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} = \hat{A}'r_- \qquad (15.22)$$

Since every orthogonal transformation C in finite dimensions is unitarily implementable as an automorphism of the Clifford algebra, $\underline{\Gamma}^T \hat{A}r_- \underline{\Gamma}$ is unitarily equivalent to $\underline{\Gamma}^T \hat{A}'r_- \underline{\Gamma}$ . In terms of Fermion annihilation operators $a_r = \frac{1}{2}(\Gamma(\underline{e}_r) + i\,\Gamma(\underline{e}_{r+L}))$, $\underline{\Gamma}^T \hat{A}'r_- \underline{\Gamma}$ is $(a^2-1)^{\frac{1}{2}} \log(a+(a^2-1)^{\frac{1}{2}})(a_r^\dagger a_r - a_r a_r^\dagger + a_{L-r}^\dagger a_{L-r} - a_{L-r} a_{L-r}^\dagger)$, which has eigenvalues $(a^2-1)^{\frac{1}{2}} \log(a+(a^2-1)^{\frac{1}{2}})(\pm 1 \pm 1)$.

The Gibbs state <f> may be equated with a state ω on $\overline{u}_L$, according to

$$\langle f \rangle = \chi(F) = \omega(\eta(F)) \,, \qquad (15.23)$$

with χ(F) and η defined in (15.9) and (15.11) respectively. Pirogov [145] and Lewis and Sisson [143] have emphasized that in the thermodynamic limit (N,L→∞), ω is a quasi-free state on the Clifford C* algebra $\overline{U}$. That is, ω is even, vanishing on odd elements of $\overline{U}$, whilst on the even subalgebra of $\overline{U}$, $\omega(Z(\underline{\psi}_1)\ldots Z(\underline{\psi}_{2p}))$ is the Pfaffian determinant of the array of two-point functions $\omega(Z(\underline{\psi}_i)\ Z(\underline{\psi}_j))$. The abstract theory of quasi-free states has been cemented by the Marseilles group [106] and from this, the Ising model can be seen in a new light. The quasi-free two-point functions must have the form (e.g. [146]):

$$\omega(Z(\underline{\psi})\ Z(\underline{\psi}')) = S(\underline{\psi},\underline{\psi}') - i\ S(\underline{\psi},J\underline{\psi}') \,, \qquad (15.24)$$

where J is a skew-adjoint contraction operator, known as the covariance operator. To each inverse temperature β, there corresponds a quasi-free state $\omega_\beta$, determined by covariance operator $J_\beta$. Pirogov [145] showed that above the critical temperatore $T_c = (k\ \beta_c)^{-1}$, the covariance operator is a complex structure, $J^2 = -I$. When the covariance operator is a complex structure, the corresponding quasi-free state is said to be Fock-type. In the Fock-Cook representation, the fundamental Fock state is realized as

$$\omega_o(Z(\underline{\psi})\ Z(\underline{\psi}')) = \langle\psi_o,\ \Phi(\underline{\psi})\ \Phi(\underline{\psi}')\ \psi_o\rangle \quad ,$$

where $\Phi(\underline{\psi})$ are the Segal operators

$$\Phi(\underline{\psi}) = a(\underline{\psi}) + a^\dagger(\underline{\psi}) \ ,$$

with $a(\underline{\psi})$ and $a^\dagger(\underline{\psi})$ Fermion construction operators on Fock space $F(H^{(1)})$ over $H^{(1)}$. $H^{(1)}$ is the space $V \oplus V$, considered as a complex space with complex structure $G = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ on $V \oplus V$ and inner product $\langle\underline{\psi},\underline{\psi}'\rangle_1 = S(\underline{\psi},\underline{\psi}') - i\ S(\underline{\psi},G\underline{\psi}')$, while $\psi_o$ is the vacuum such that $a(\underline{\psi})\psi_o = 0$. By applying a canonical transformation $\Phi'(\underline{e}_\mu) = \sum_{\nu=1}^{2L} C_\mu^{\ \nu}\ \Phi(\underline{e}_\nu)$, or $\Phi'(\underline{\psi}) = \Phi(C^{-1}\underline{\psi})$, with $\underline{\psi}$ contravariant vectors and C orthogonal, we realize a new Fock-type state $\omega'(Z(\underline{\psi})\ Z(\underline{\psi}')) = \langle\psi_o,\ \Phi'(\underline{\psi})\ \Phi'(\underline{\psi}')\ \psi_o\rangle$ , which has covariance operator $CGC^{-1}$. Since any skew-symmetric complex structure must have this form, it is now evident that every Fock-type state can be achieved from the Fock state $\omega_o$ by a canonical transformation, a fact known by Manuceau and Verbeure [147]. However, the canonical transformation C may be quite general and therefore, as discussed in Section 6, in the thermodynamic limit, with $V$ infinite dimensional, it need not necessarily be unitarily implementable. In fact, it was demonstrated by Lewis and Sisson [143] that $\omega_\beta'$ is obtained from $\omega_\beta$ by a non-implementable canonical transformation, whenever $\beta \neq \beta'$.

In [143], it is shown that $J_\beta = CGC^{-1}$, with $G = J_o$ (the infinite temperature solution) and C the transformation of (15.22) which reduces the temperature dependent matrix $\hat{A}r_-$ to canonical form. We see that $J_\beta$ is the very object obtained in proposition (11.8) as the unique unitarizing complex structure for the classical orthogonal dynamics generated by $\hat{A}r_-$ . This must come as no surprise to one who recognises the similarity in the mathematical structures of Euclidean quantum field theory and classical statistical mechanics. There is a popular notion that the transformation $\beta \to t$ changes the Ising model to a Euclidean lattice field theory and that a further transformation $t \to it$ changes the Euclidean lattice model to a space-time lattice model. The combined effect would be to change the quadratic logarithm A of the transfer matrix V from an Hermitean matrix to a skew Hermitean matrix which can be considered as the generator of unitary time evolution in the corresponding quantum mechanical lattice system. Lebowitz [148] has shown that above the critical temperature, there is a unique translation-invariant classical equilibrium state. This must determine a unique state $\omega$ on the Clifford algebra $\overline{\mathcal{U}}$, via the scheme (15.23). The translation invariance of $\omega$ is expressed, from (15.10), as $\omega(V^{-r} F V^{r}) = \omega(F)$. This corresponds very closely to the time-invariance of Segal's unique vacuum state, which was discussed in Section 8. From proposition (11.20), we know that the complex structure $J_\beta = CGC^{-1}$ may be alternatively characterized as the orthogonal component in the polar decomposition of $\hat{A} - (= \bigoplus_r \hat{A}r-)$. If we use the operators $\Gamma(\underline{e}_\mu)$ of (15.17) as the generators of $\overline{\mathcal{U}}$, then the columns of $\hat{A}r-$ are already mutually orthogonal and each column has norm $(a^2-1)\log(a+(a^2-1)^{\frac{1}{2}})$, which is verified by making use of the identity $a^2-1 = b_1^2 + b_2^2$, which follows from (15.19-21). Therefore, J is given simply as the direct sum $\bigoplus_r J_r$, with

$$J_r = [(a^2-1)\log(a+(a^2-1)^{\frac{1}{2}}]^{-1}\hat{A}r- \qquad (15.24)$$

The orthogonal component $J_r$ in the polar decomposition of $\hat{A}r-$ will be a complex structure provided $\hat{A}r-$ is non-singular. When the system develops a soft mode, one of the frequencies vanishes, $\hat{A}r-$ becomes singular and $J_r$ can no longer be used in the construction of a covariance operator for a Fock-type state. From (15.18), $Ar-$ becomes singular only if $b_1 = 0$ or $b_2 = 0$.

By (15.21), $b_2 = 0$ implies

$$\sin\theta_r \sinh\beta J_x \cosh\beta J_y^* = 0 ,$$ which is not satisfied for any finite temperature.

By (15.20), $b_1 = 0$ implies

$$\cos\theta_r = \coth\beta J_x \tanh\beta J_y^* \tag{15.25}$$

In the thermodynamic limit, the variable $\theta_r = \frac{2\pi r}{N}$ becomes continuous. Therefore, (15.25) will first have a solution when $\coth\beta J_x \tanh\beta J_y^*$ reduces to 1. Since $\tanh\beta J_y^* = e^{-2\beta J_y}$, this condition may be written

$$\tanh\beta J_x = e^{-2\beta J_y} .$$

Therefore,
$$\begin{aligned}\sinh^2 2\beta J_x \sinh^2 2\beta J_y &= 4\sinh^2\beta J_x \cosh^2\beta J_x \sinh^2 2\beta J_y \\ &= 4\frac{\coth^2\beta J_x}{(1-\coth^2\beta J_x)^2}\tfrac{1}{4}(e^{4\beta J_y}-2+e^{-4\beta J_y}) \\ &= \frac{e^{8\beta J_y}-2e^{4\beta J_y}+1}{(1-e^{4\beta J_y})^2} \\ &= 1 \end{aligned} \tag{15.26}$$

Hence, we recover the Kramers-Wannier result [149] for the critical temperature $\beta_c$.

Kuik [150] has shown that below the critical temperature, the quasi-free state $\omega_\beta$ is no longer a Fock-type state. Kuik's direct calculation has shown that the covariance operator $J_\beta$ has a kernel of odd dimension when $\beta > \beta_c$. From a result of Manuceau and Verbeure [151], this is equivalent to the state $\omega_\beta$ being non-primary. The G.N.S. representation of $\overline{\mathcal{U}}$, constructed from $\omega_\beta$, is reducible. The covariance operator $J_\beta$ can not be constructed from $V_-$ using the fore-going theory of quadratic Hamiltonians. Instead, $\omega_\beta$ is now a mixture of two states

$$\omega_\beta = \tfrac{1}{2}\,\omega_\beta^+ + \tfrac{1}{2}\,\omega_\beta^- \tag{15.25}$$

and it is evident from [150] that $\omega_\beta^\pm$ do not vanish on odd elements of $\overline{\mathcal{U}}$.

The two-fold mixture (15.25) corresponds to the two pure thermodynamic phases which can occur below the critical temperature.

## SECTION 16 - STATISTICAL SYSTEMS OF BOSONS

The correspondence between mixed quasi-free states and mixtures of thermodynamic phases, which was discussed at the end of Section 15, also applies to Boson systems. In fact, the concept of the quasi-free state was first introduced by Robinson [152], in connection with the free Boson gas. As noted by Lewis and Sisson [143], it is apparent from the structure of the quasi-free gauge-invariant equilibrium state for the free Boson gas [153,154] that it can be decomposed into non-gauge invariant primary states below the critical temperature.

In Section 15, we showed that the critical temperature of the Ising model is just the temperature at which a soft mode first appears. The soft mode theory of phase transitions also applies to systems of Bosons, for example in the theory of crystal vibrations [155]. For those systems which can be described by a quadratic self-adjoint combination H of Boson creation and annihilation operators, the frequencies may be defined as in (4.54). If H is bounded from below, then H can be transformed linearly to a system of independent harmonic oscillators and free particles.

$$H = \sum_{j=1}^{N} H_j , \tag{16.1}$$

with

$$H_j = \tfrac{1}{2}(P_j^2 + \omega_j^2 Q_j^2) \; ; \quad \omega_j \in R \text{ (possibly zero)} \tag{16.2}$$

$$Q_j = 2^{-\frac{1}{2}}(C_j + C_j^\dagger)$$

$$P_j = -2^{-\frac{1}{2}} i (C_j - C_j^\dagger) ,$$

with $C_j$ and $C_j^\dagger$ Boson construction operators obtained by a Bogoliubov transformation. When one of the frequencies $\omega_j$ becomes zero, H adopts a higher symmetry group [156] and the corresponding classical motion becomes unbounded.

In Section 14, we discussed the quadratic Boson Hamiltonians which arise from Bosonization of the Schwinger model, which consists of a Fermion field interacting with a Boson field. Similarly, the purely Fermionic Bardeen-Cooper-Schrieffer Hamiltonian for a superconducting fluid may be effectively Bosonized. The B.C.S. Hamiltonian is quartic in a set of Fermion operators $a_{\underline{\sigma}}(\underline{k})$ and $a^{\dagger}_{\underline{\sigma}}(\underline{k})$, with $\underline{\sigma}$ labelling charge and spin and $\underline{k}$ labelling 3-momentum. However, the phonon-mediated forces result in the fourth order terms being constructed purely from operators which create and annihilate Cooper pairs. Kato and Mugibayashi [47] showed that if the Cooper pairs are replaced by Bosons, then the resulting quadratic Boson Hamiltonian has, in the thermodynamic limit, the same spectrum as that of the B.C.S. Hamiltonian, so that the qualitative explanation of superconductivity can still be given. However, we should not forget that the B.C.S. model is qualitative in nature, since it neglects the Coulombic force and even then, it retains only the interaction between Cooper pairs. Kato and Mugibayashi argued that some minor adjustment of the equivalent quadratic Boson Hamiltonian could result in a finite number of the frequencies in (16.2) vanishing, thereby contributing a continuous part to the spectrum, which also has a discrete part with a gap structure, which can be calculated after diagonalizing the discrete part of H by a Bogoliubov transformation.

Following the success of the B.C.S. model, considerable effort went into investigating the application of similar techniques to many-body nuclear systems. In the paper of Thouless [54], an approximate ground state was chosen to be the solution to the Hartree-Fock equations which minimized the expectation value of the energy. Then, using the independent particle model, low-lying excited states were viewed as Fermion-hole pairs. The Green's function for the system was calculated as the kernel of a time evolution operator which connected Fermion-hole pairs as if they were elementary objects. The underlying structure of the single Fermion propagator was to some extent neglected. This corresponds to adopting Cooper pairs as the elementary objects in the B.C.S. model. From the approximations used by Thouless, the energy eigenvalues $E_n$ were solutions to

$$\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} = E_n \begin{pmatrix} X_n \\ -Y_n \end{pmatrix} \quad \text{(equation 13 of [54])} \tag{16.3}$$

with A Hermitean and B symmetric.

We note that $\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}$ has the structure of a Boson dynamical matrix $\mathcal{D}_1$ and that (16.3) gives $E_n$ as an eigenvalue of $\hat{I}\,\mathcal{D}_1$, with $\hat{I} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$.
Hence, each excitation energy $E_n$ is a frequency for a system with quadratic Boson Hamiltonian $H = \frac{1}{2}\,\underline{\alpha}^\dagger\,\mathcal{D}_1\,\underline{\alpha}$. For such a system, the frequencies are identical to the single quasi-Boson energies, since with $\mathcal{D}_1 > 0$, there exists a Bogoliubov transformation which reduces H to $\sum_j E_j\, C_j^\dagger\, C_j$ + scalar. This gives us insight into the nature of Thouless's approximations. These approximations lead to the equation (16.3) which could have been obtained by replacing Fermion-hole pairs by Bosons. Suppose that $a_{j+}$ annihilates a Fermion of type j and that $a_{k-}$ annihilates a hole of type k. Then

$$[a_{k-}\, a_{j+},\, a_{j+}\, a_{k-}] = I + a_{k-}^\dagger\, a_{k-} - a_{j+}^\dagger\, a_{j+} \tag{16.4}$$

Since the charge operator $a_{k-}^\dagger\, a_{k-} - a_{j+}^\dagger\, a_{j+}$ vanishes on all states obtained from the ground state by pair production, the right hand side of (16.4) reduces to I, as it should if $a_{k-}\, a_{j+}$ is to be replaced by a Boson construction operator. Further problems arise when the commutator $[a_{k-}\, a_{j+},\, a_{\ell+}^\dagger\, a_{m-}^\dagger]$ is considered in the general case when it may occur that exactly one of the differences k-m and j-$\ell$ vanishes. For example, if $k \neq m$ and $j = \ell$, then

$$[a_{k-}\, a_{j+},\, a_{\ell+}^\dagger\, a_{m-}^\dagger] = a_{k-}\, a_{m-} \tag{16.5}$$

The replacement of each different Fermion-hole pair by a different Boson would involve the neglect of the right hand side of (16.5). To attempt to rectify the construction of the Boson operators, one would, in effect, confront the problems of the neutrino theory of light [138]. In the B.C.S. model, the replacement of each Cooper pair by a Boson leads, in the thermodynamic limit, to the correct spectrum of the Hamiltonian for a superconducting fluid. However, the thermodynamic limit will not be realistic in nuclear calculations, since natural nuclei contain no more than 200 nucleons.

Thouless's paper has a special place in the history of the generalized Bogoliubov transformation, since, to the best of our knowledge, it contained the first proof, at least in the physical literature, that (16.3) has a complete set of eigen-solutions when $D_1$ is positive definite. Hence, even though this result was not at first specifically related to quadratic Boson Hamiltonians, it contained all the mathematics necessary to convince one that all positive definite quadratic Boson Hamiltonians can be diagonalized. The condition $D_1 \geq 0$ was interpreted by Thouless to be the condition that a chosen solution of the Hartree-Fock equations minimized the energy expectation value and was therefore a good approximation to the ground state wave function. The possibility of a frequency being zero was interpreted as a possible ambiguity in the choice of the trial ground state. The convergence of a non-zero real frequency $\omega_j$ to zero at some particular set of values of the system's parameters was viewed as a type of phase transition occurring at the point where the trial ground state wave function became inadequate and no longer minimized the energy functional.

## SECTION 17 - CONCLUSIONS AND OUTLOOK

It is now common knowledge that although any quadratic Fermion Hamiltonian can be reduced to a linear combination of quasi-particle number operators by a Bogoliubov transformation, the same is not true of an arbitrary quadratic Boson Hamiltonian. Nevertheless, there is a one to one correspondence between quadratic Boson Hamiltonians and quadratic functions on classical phase space. The full set of equivalence classes of the former can be deduced from that of the latter. The diagonalizability of a positive definite quadratic Boson Hamiltonian follows from a much older equivalent result in classical mechanics. In turn, these results can be extended to infinite degrees of freedom by using spectral theory in Krein space.

We have shown that the full set of invariants which determines the canonical form of an arbitrary classical quadratic Hamiltonian is a valuable diagnostic tool. We have been able to relate stability, unitarizability and pseudo-unitarizability of a classical symplectic dynamical system with the causal properties of the corresponding heuristically quantized system, together with the spectral type of the canonically quantized Hamiltonian. With finite degrees of freedom, the frequencies of the system (i.e. the eigenvalues of the dynamical generator $i G\hat{H}$) may take any complex value and $i G\hat{H}$ may have any (possibly non-diagonal) Jordan canonical form. Assuming a formal mode expansion (7.1b), we have shown that the Boson commutation relations, among an independent set of mode operators, are incompatible with the canonical commutation relations among canonical variables, unless $i G\hat{H}$ is similar to an Hermitean matrix. We have proven an analogous result in rigorous algebraic quantization, namely that Segal unitarization of the dynamics is possible if and only if $i G\hat{H}$ is similar to an Hermitean matrix. This may be interpreted as saying that a quantum mechanical particle picture can not be drawn from finite dimensional classical dynamics unless the quantized Hamiltonian has discrete spectrum, the particle picture then being related to the elementary excitations of the quantized system. The latter interpretation is valid, provided it can be shown that the quantized Hamiltonian has continuous spectrum unless $i\hat{GH}$ is similar to an Hermitean matrix. In fact, we have demonstrated that the spectrum in question is continuous whenever $i\hat{GH}$ has a non-real eigenvalue. In the case of multiple real resonant frequencies, it is known that an algebraic multiplicity of two leads to a continuous spectrum and it remains a conjecture that the same is true for higher multiplicities. If we were to prove this conjecture, we would establish that the Segal quantization procedure is consistent for the widest possible class of finite dimensional linear systems for which the quantized Hamiltonian has a purely discrete spectrum. In this sense, the Segal quantization procedure is universally applicable as it could not be expected to produce a quantum mechanical Hamiltonian with continuous spectrum, except by taking a direct integral, in the case of infinite degrees of freedom.

If the classical dynamics is taken to be a one parameter group of orthogonal transformations on a real Hilbert space $V$, then there will always exist a complex structure J on $V$ which will enable the dynamics to be unitarized. If there are no non-trivial dynamically invariant vectors in $V$, then -J is given uniquely as the orthogonal component in the polar decomposition of the dynamical generator $\hat{A}$. A similar result holds for symplectic dynamics. When the classical Hamiltonian is positive definite, J is given uniquely as the pseudo-orthogonal component in the polar decomposition of the dynamical generator, using spectral theory in Krein space with indefinite metric $iB(\cdot,\cdot)$, B being the real symplectic form. This result extends to infinite degrees of freedom. J can be identified with $-i(I-2E(0))$, E(0) being the projection onto the maximal dynamically invariant subspace on which $i\hat{GH}$ is negative definite.

In the case that the classical dynamics on $V$ is unstable, even in one of an infinite number of subsystems, there does not exist a complex structure J on $V$ which enables the dynamics to be unitarized. However, there may exist a complex structure which enables the dynamics to be pseudo-unitary with respect to an indefinite complex inner product. We have shown how to test whether a dynamical system of finite degrees of freedom can be pseudo-unitarized. Some dynamical systems can not even be pseudo-unitarized and even then, when this task can be achieved, the interpretability of the indefinite metric is problematical.

In this thesis, a number of gaps in the theory of quadratic Hamiltonians have been identified and repaired. It is hoped that the reader will recognize some approximation to completeness in this task. However, to claim that this is the end of the story would be either naive or dishonest. As in any area of mathematical physics, it is easy for the author to identify some problems which are still outstanding, at least in his own mind and which deserve more of his attention in the future. Firstly, we are still a long way short of a full set of canonical forms for infinite dimensional dynamical systems under the action of the group of unitarily implementable canonical transformations. Some relevant results, known at this time, include

(a) If a Boson dynamical matrix $D_1$ is strictly positive, then $D_1$ can be diagonalized by a possibly non-implementable Bogoliubov transformation.

(b) There are extra conditions on $D_1$ given by Kuliev and Dadashev [74-75] which are sufficient to ensure that the diagonalizing transformation is implementable.

(c) There are conditions given in proposition (6.23) which are necessary for a dynamical matrix to be reducible to that of a free field by a unitarily implementable Bogoliubov transformation.

If we begin with the Fock representation of the C.C.R. or C.A.R. and apply a non-implementable Bogoliubov transformation, we obtain an inequivalent Fock-type quasi-free representation [147]. However, there can not, in the same irreducible representative space, be a vacuum vector for each of the two inequivalent sets of annihilation operators [38,77]. We can construct a reducible representation in which both vacua exist. Next, we consider a one parameter group of non-implementable Bogoliubov transformations, such as that generated by the formal Hamiltonian of Thirring and Narnhofer [133]. So that the vacuum $\psi_o(t)$ exists at each time t, a direct integral representation would be necessary. The types of reducible representations which could arise from a general non-implementable dynamical flow have not yet been classified.

In Section 6, we quoted the result of Carey, Hurst and O'Brien [81] that within the group of charge conserving implementable C.A.R. Bogoliubov transformations $\mathcal{T}$, the connected components are distinguished by vacuum charge, which is given by dim ker $S_1$ - dim ker $S_2$, $S_1$ and $S_2$ defined as in

$$\mathcal{T} = \begin{pmatrix} T_1 & T_2 \\ T_2^* & T_1^* \end{pmatrix} \; ; \; T_1 = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} .$$

For a strongly, but not uniformly, continuous one parameter group of implementable C.A.R. Bogoliubov transformations $\mathcal{T}(t)$, the charge of the vacuum may change discontinuously in time. It is not yet known whether this phenomenon has a physical application. On the other hand, even if the generation of vacuum charge may be ruled out, one may still consider spontaneous pair production, which conserves charge. In the case of simple Boson fields, for example, the Klein Gordon field, the insertion of a background space-time metric appropriate to the existence of singularities or to an open universe, seems to lead to the same difficulties which we previously discussed under the heading of the external field problem. The generation of particles in a cosmological vacuum has now become an accepted topic in physics [157].

An example of a one-parameter group of implementable C.C.R. Bogoliubov transformations, whose generator has infinite Hilbert-Schmidt norm in the number-increasing block, was given by Klein in remark 3 of [158]. In the case of the C.A.R., such transformations bridged the distinct connected components of the group of implementable charge-conserving Bogoliubov transformations [81]. Klein's example now provides some motivation for studying the topological structure of the group of implementable charge-conserving C.C.R. Bogoliubov transformations, appropriate to, for example, a charged Klein-Gordon field. In the case of the C.C.R., dim ker $S_1$ - dim ker $S_2$ could not be a useful topological index, since, as discussed in Section 6, $T_1$ must be non-singular.

This thesis has concentrated on linear systems but efforts are now being made to extend rigorous algebraic quantization to non-linear symplectic manifolds. So far, the Segal method has been extended to non-linear dynamical systems which are locally stable [111]. Quantization must eventually incorporate locally unstable systems which are commonly observed in nature. As this is achieved, we must clarify the relationship between the algebraic approach and other rigorous techniques of quantization, most notably geometric quantization (e.g. [159]), the functional integral approach (e.g. [160]) and homological techniques (e.g. [161]).

There is an ancient philosophical question, which, in the author's opinion, is still not answered by physics. That question is whether the corpuscular description can be universally applied to physical fields or whether there are physical phenomena which can be described only at the continuum level. If the latter is true, for example in the description of non-localized hadronic structure, then we must ask whether there is an appropriate algebraic scheme which generalizes the C.C.R. (e.g. [VII]). If the C.C.R. remain appropriate, then the question remains whether the Segal algebraic scheme may be extended to locally unstable systems, by selecting as yet unfamiliar representations of the C.C.R. It has not yet been shown that none of the representations of the C.C.R. which lack a particle picture can have an alternative physical interpretation.

**PUBLICATIONS ASSOCIATED WITH THIS THESIS**


I P. Broadbridge, "Normal Forms for Classical and Boson Systems", Physica A 99, 494-512 (1979).

II P. Broadbridge and C.A. Hurst, "Canonical Forms for Quadratic Hamiltonians", Physica A 108, 39-62 (1981).

III P. Broadbridge and C.A. Hurst, "Existence of a Complex Structure for Quadratic Hamiltonians?", Annals Phys. 131, 104-117 (1981).

IV P. Broadbridge, "Problems in the Quantization of Quadratic Hamiltonians", Hadronic Jour. 4, 899-948 (1981).

V P. Broadbridge and C.A. Hurst, "Fermi-Dirac Quantization of Linear Systems", Annals Phys. 137, 86-103 (1981).

VI P. Broadbridge, "Existence Theorems for Segal Quantization via Spectral Theory in Krein Space", to appear, Jour. Austral. Math. Soc., Series B, 1982.

VII P. Broadbridge, "Quantization, Non-Locality and Lie-admissible Formulations", Proc. 1st International Conference on Non-Potential Interactions.

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