A MODEL FOR HIGH ENERGY NUCLEON NUCLEON SCATTERING

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C O N T E N T S

* * *

ABSTRACT

In this thesis a model is set up for the discussion of nucleon nucleon scattering. Bethe-Salpeter type equations are written down for interactions between two nucleons, taking into account not only processes where a meson is exchanged by the nucleons but processes where a meson is emitted by one nucleon either to be reabsorbed or to be a free particle. The time dependence of these equations is eliminated for the spin singlet $J = 0$ case and the remaining equations interpreted by their similarity to the space part of the Klein-Gordon equation.

The solutions of the equations obtained, which are relevant to the scattering problem are found for the two cases where the interaction region is spherical and where the interaction region is spheroidal and the cross sections, both elastic and inelastic, are investigated, particularly at high energies. Energies near the threshold are also examined in detail.

In the course of solution it is noted that the wave function and cross sections depend critically on the singularities and branch points of the scattering amplitude; thus these singularities and branch points are examined. The appearance of similar singularities in scattering amplitudes investigated by the method of dispersion relations is discussed.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and to the best of my knowledge and belief, the thesis contains no material previously published or written by another person, except when due reference is made in the text of the thesis.

L. C. Johnston.

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Chapter 1.

INTRODUCTION

In this thesis a model will be presented for nucleon nucleon scattering at high energies. This is a subject on which only limited experimental data is available. At present, more detailed and accurate data is of use only to a theoretical physicist approaching the subject by considering a phenomenological potential; accurate calculations based on field theory are too complicated to carry out and if expressions for the scattering cross sections are required then drastic approximations must be made. This is the approach presented here, the aim being to achieve results which show the same general features as the cross sections determined by experiment, with detailed agreement in some aspects.

The data available is summarised in various sources. Results of experiments determining nucleon nucleon scattering cross sections have been gathered together by Fowler et al[1] and correlated with their own results to show how the total, elastic, and inelastic scattering cross sections vary with energy up to 3 Bev and also how the differential cross section changes in form with increasing energy. Since then many other sets of results have been added to the data available; MacGregor et al[2] gave a summary of the work done on all aspects of nucleon nucleon scattering up to 1960 and in this summary included a list of sources for data. The data available on elastic scattering up to 1964 is summarised by MacGregor et al[3] and elastic and inelastic scattering data between 12 and 27 Gev is summarised by Diddens et al[4]. The data available does not show the detailed behaviour of the cross sections with changing energy but nor is the model presented here accurate enough to be expected to predict detailed behaviour. The experimental results do, however, show the general behaviour and the model is accurate enough to predict this.

At high energies the differential cross section exhibits a marked peaking in the forward and backward directions, the peaking becoming greater with increasing energy so that when the kinetic energy of the incident nucleon is 2.75 Bev there is practically no scattering at all outside a region within 30 degrees of the forward and backward directions. The data may be fitted at any given energy by an optical model involving a spherical potential well by choosing the parameters of the well suitably[5,6] but as Green[7] has pointed out it is quite wrong for the scattering region to be spherical for high energies because a spherical region suffers a Fitzgerald contraction in the direction of motion turning it into a spheroidal region. This being so, results derived from a spherical interaction region as to the relative importance of the parameters involved in the potential will be wrong, possibly seriously at high energies. Clearly the flattening of the interaction region into a spheroid is responsible for some of the forward and backward peaking while the introduction of higher angular momentum states and spin-orbit coupling terms into the potential as they become energetically possible will account for the rest.

Another general feature of the cross sections which is apparent from the data available is that the total scattering cross section falls rapidly with energy till about 10 Mev when it becomes almost constant at about 25 millibarns. This has been interpreted by Jastrow[8] to indicate the presence of a hard core in the nucleons. The total scattering cross section then remains almost constant until the first meson emission threshold is reached when the total cross section rises sharply but the elastic cross section remains constant. This fact is explained by supposing that the elastic scattering is almost entirely confined to the surface of the hard core while the inelastic scattering is spread over the whole interaction region. That is, from about 10 Mev the elastic scattering is confined mostly to the surface of the hard core and as energy increases through the threshold no significant change in the elastic scattering occurs because the inelastic scattering, coming from a different region, does not interfere with it.

A particular feature of the inelastic scattering cross section is that as the threshold is passed it rises sharply from zero, smoothing off and becoming constant at about 25 millibarns. This feature may be expected intuitively and will be predicted by the model presented in this thesis.

Phenomenological potentials have been obtained by fitting scattering data at energies below the meson production threshold which give cross sections in close agreement with those obtained by experiment over the whole range of energy considered. Such potentials have been obtained by Gammel and Thaler[9], Hamada[10], and Lassila et al[11] (Hull and Breit). Recently a new type of phenomenological potential has been considered; in these the pion resonance states are considered to be new elementary particles. Such potentials are those obtained by Bryan and Scott[12], and Sawada et al[13,14], and are in close agreement with the potential suggested by Lassila et al.

Green and Sharma[15] have drawn attention to work which was presented to the American Physical Society in Cambridge in 1949. In this work nucleons were considered to be represented by five-vectors and the properties of the deuteron were studied. Green and Sharma have considered the w and ρ mesons (783 Mev and 763 Mev respectively) to be manifestations of five-vector fields and found the tensor, spin-spin, and spin-orbit potential generated by these particles or resonances. They have also studied the contribution of the η meson (549 Mev) to the isoscalar part of the potential together with the w meson potential and the contribution of the pi meson together with the ρ meson.

The value of a phenomenological potential is that when the parameters are such that a good fit is obtained with experimental data their values at a given energy indicate the relative importance at that energy of the interaction processes represented by the potential. However it is possible that in an exact theory a particular interaction may not give rise to a potential type term at all, in which case a phenomenological potential may be misleading. The model presented here indicates that the single meson interaction processes do give rise to potential type terms for elastic scattering but not for inelastic scattering.

Potentials which predict inelastic scattering have been obtained by Grishin et al[16] and by Brown[5]. These potentials contain complex parameters and are analogous to the optical potentials used for the nucleus. This optical model for the nucleon will be discussed later in connection with work done by Feshbach.

Another approach to the problem of nucleon nucleon scattering has been by using dispersion relations. Scotti and Wong[17] and Bryan et al[18] have used this approach, analysing the nucleon nucleon interaction in terms of one boson exchange processes and obtained phase shifts for nucleon nucleon scattering below the meson emission threshold.

In this thesis the interactions between two nucleons which involve only one meson being "in the air" at a given time are taken into account in detail while all other interaction processes are taken into account by representing the nucleon by a square potential well. This very simple potential is used so that the equations obtained are solvable and the effect of one meson interactions can be clearly seen. It cannot be hoped that the model will give reasonable results at energies higher than the threshold for emission of two mesons but it is expected to show exactly in what way the single meson emission affects the cross sections near the threshold. In fact the model predicts cross sections which behave in the expected way around the threshold but the inelastic scattering cross section does not arise in the way predicted by a complex potential. Thus while the optical model can give good results it could not be equivalent to any model which took into account exactly all the interaction processes involved.

Feshbach[19,20] has developed a theory of nuclear reactions from which all other descriptions such as the compound nucleus, the optical model, and the direct interaction model may be obtained by specialising or approximating in a particular way. This work has relevance to the model for nucleon nucleon scattering being presented here for the following reason. Feshbach shows that when several exit channels are possible after a nucleon has been shot at a nucleus the wave function for the system after elastic scattering is the solution of a set of coupled equations which reduce to a single equation which is of the Schrodinger type if a complicated term involving inverse operators is replaced by a potential. He shows that this term is complex and non local and replaces it by a complex potential to produce the optical model for the nucleus. The same procedure is presumably possible for nucleon nucleon scattering where different channels for inelastic scattering are possible. In the model presented here only one inelastic process is allowed and only two coupled equations are obtained. It is thus not necessary to replace the inverse operator by a potential function as these two equations can be solved almost exactly. It will be seen that while an optical potential for a nucleon may be deduced it is not necessary to use it in a model which seeks to describe single meson emission only, provided that the model is simple enough otherwise.

The results of inelastic scattering experiments show that the inelastic scattering cross section rises sharply from zero with increasing energy past the threshold. This sort of behaviour cannot be satisfactorily explained using a complex potential for if it were forced to fit the data its energy dependence would be highly artificial. It is much more natural that the inelastic cross section should arise as a term which is a function of energy increasing with increasing energy and zero at the threshold. This is the way it arises in the present model; the inelastic scattering terms arise from an integral around a cut in the complex plane, the length of the cut increasing with energy and being zero at the threshold.

As mentioned earlier, another approach to the problem of the nucleon nucleon interaction is through the theory of dispersion relations. The difficulty with this approach is that while it shows clearly the singularities of the scattering amplitude corresponding to the various thresholds and stable states the actual calculation of the scattering cross sections is very complicated, particularly above the meson emission threshold. The problem of two particle interactions has been considered in general by Mandelstam[21] while the particular case of nucleon nucleon scattering has been considered by Hara[22] and by Hsieh[23]. The singularities of the scattering amplitude revealed by the dispersion relation approach will be discussed in more detail in Chapter 7. The model presented here is simple enough for the scattering amplitude to be obtained explicitly, exhibiting the nature and position of the singularities, thereby throwing light on the structure of the scattering amplitude which would be obtained in an exact theory.

The model presented here, then, is aimed at describing two aspects of nucleon nucleon scattering. They are the effect of the flattening by the Fitzgerald contraction of the interaction region on the differential cross section and the effect of the single meson interaction on all the cross sections. To do this a Bethe Salpeter type equation is set up, which takes into account all types of single meson interaction, and a generalisation of a method used by Green and Biswas[24] is used to derive from it a pair of coupled differential equations, which hold in the centre of mass frame of the nucleons, involving the wave functions for a nucleon and a nucleon plus meson. In this derivation the instantaneous interaction approximation is used and, as the interactions other than those involving a single meson are to be approximated by a simple square well potential, only the spin singlet $J = 0$ case is considered. These coupled differential equations are then solved and the effect of the single meson interaction is analysed in detail. Solutions are given assuming both that the interaction region is spherical and spheroidal in the centre of mass frame and thus the effect of the flattening becomes clear. The analysis of the single meson interaction is best carried out in the case where a spherical well is used as the calculation is simpler and the important points are not obscured by calculational detail. The effect of the flattening of the interaction region is seen qualitatively only; the importance of the calculations made is that with a model which was detailed enough to justify accurate calculation a spheroidal well could be used and cross sections found almost exactly. The model presented here is not detailed enough to justify accurate numerical calculations of the differential cross section.

As mentioned before, this model allows the scattering amplitude to be found explicitly so that its singularities, branch points, and cuts can be analysed exactly and their effect on the cross sections seen clearly. This is discussed in the last chapter.

THE FIELD EQUATIONS AND THE ELIMINATION OF THEIR TIME DEPENDENCE

In this chapter field equations will be found for the scattering of two nucleons, based on relativistic field theory. This will be done making sufficient approximations to allow the equations to be solved but remaining close enough to the accurate field theory model to give an indication of how this model accounts for meson emission threshold phenomena.

Green and Biswas[24] have given a method of obtaining covariant solutions of the Bethe-Salpeter equation which is an equation for the nucleon interaction where the "ladder" approximation has been made. In the relevant part of their method Green and Biswas have also used the "instantaneous interaction" approximation. The present chapter is an extension of the Green Biswas method applied to a model which includes, not only ladder type interactions but also interactions where the emission and absorption of one meson by each nucleon is allowed. Thus, interactions associated with Feynman diagrams of the following types are taken into account.

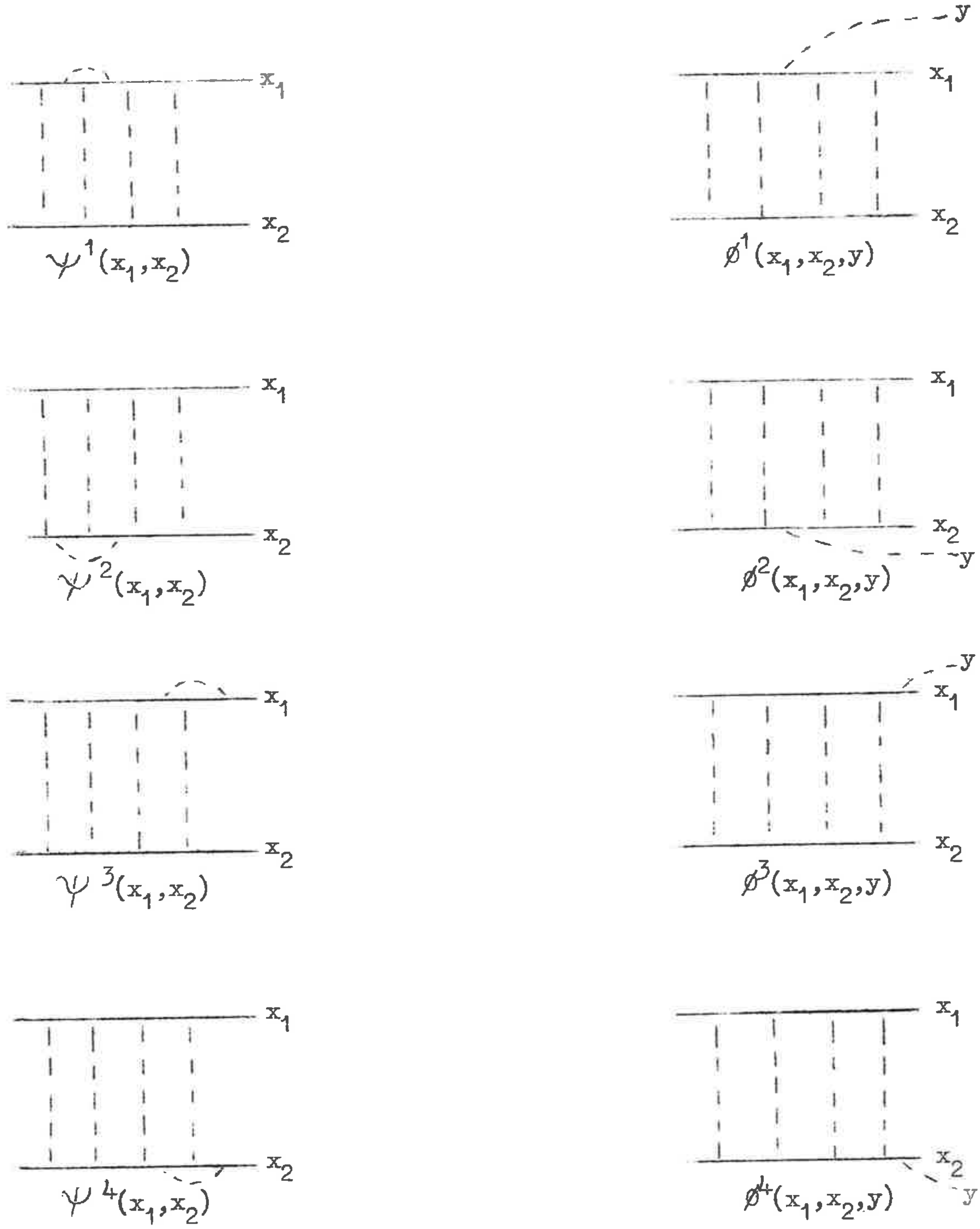


Figure 1

The full lines indicate nucleons and the dotted lines indicate mesons.

Using the rules derived by Feynman[25] for obtaining field equations from such diagrams the following field equations may be written down:

$$B\psi^1 = g^2 f(x) \gamma_5 (\psi^1 + \psi^3) \gamma_5 \qquad \text{(i)}$$
$$B\psi^2 = g^2 f(x) \gamma_5 (\psi^2 + \psi^4) \gamma_5 \qquad \text{(ii)}$$
$$B\psi^3 = g \gamma_5 (\phi_1^1 + \phi_1^3)(p_2 - M) \qquad \text{(iii)}$$
$$B\psi^4 = g (p_1 - M)(\phi_2^2 + \phi_2^4) \gamma_5 \qquad \text{(iv)} \quad .....2.1$$

$$KB\phi^1 = g^2 f(x) \gamma_5 K(\phi^1 + \phi^3) \gamma_5 \qquad \text{(v)}$$
$$KB\phi^2 = g^2 f(x) \gamma_5 K(\phi^2 + \phi^4) \gamma_5 \qquad \text{(vi)}$$
$$KB\phi^3 = g \gamma_5 (\psi^1 + \psi^3)(p_2 - M)\, \delta(x_1 - y) \qquad \text{(vii)}$$
$$KB\phi^4 = g (p_1 - M)(\psi^2 + \psi^4) \gamma_5\, \delta(x_2 - y) \qquad \text{(viii)} .....2.2$$

Here $B\,\Phi(x_1, x_2) = (p_1 - M)\,\Phi\,(p_2 - M)$,

and $K\,\Phi\,(y) = (p_y^{\,2} - \mu^2)\,\Phi$,

where μ is the meson mass and M is the nucleon mass.

Also $\phi_1 = \phi(x_1, x_2, x_1)$

and $\phi_2 = \phi(x_1, x_2, x_2)$.

Put $\psi^1 + \psi^3 = \psi$

and $\phi^1 + \phi^3 = \phi$.

Then

$$B\psi = g \gamma_5 \phi_1 (p_2 - M) + g^2 f(x) \gamma_5 \psi \gamma_5 \qquad ....2.3$$

and

$$KB\phi = g \gamma_5 \psi (p_2 - M)\, \delta(x_1 - y) + g^2 f(x) \gamma_5 K\phi \gamma_5 \qquad ......2.4$$

Equations 2.3 and 2.4 have been derived from (i), (iii), (v) and (vii) entirely so that any results obtained from 2.3 and 2.4 have analogous results which can be derived from (ii), (iv), (vi) and (viii).

The equations 2.3 and 2.4 are a generalization of the ordinary Bethe-Salpeter equation, which considers ladder type interactions only, to include inelastic scattering processes involving the emission of one meson. Green and Biswas have shown that the ordinary Bethe-Salpeter equation can be reduced, by restricting attention to the centre of mass frame of the two nucleons, to a Klein Gordon type equation for the spin singlet state and to a pair of coupled Klein Gordon type equations for the spin triplet state. In this model it is proposed to approximate the detailed spin-orbit and tensor interaction terms by a very simple scalar potential function. This being so, detailed analysis of such terms is wasted and after 2.17 attention is restricted to the spin singlet $j = 0$ state, for which the solution assumes its simplest form. Thus, all spin-dependent terms are neglected. With this approximation the pair of coupled equations 2.3 and 2.4 eventually lead to

$$\{\Box_x + a^2(x)\} \ \psi_e(x) \ = \ -g \ \psi_i(x,x)$$

and

$$g'(\Box_y + \mu^2) \{\Box_x + a^2(x)\} \ \psi_i(x,y) = g^* \ \psi_e(x) \ \delta^4(x-y)$$

where ψ_e and ψ_i are the wave functions of the nucleons which have or have not emitted a meson and $a^2(\underline{x})$ is the simple scalar potential function.

In 2.3 and 2.4 put

$$\chi = \gamma_5 \phi (p_2 - M)$$

then

$$L\chi = -\gamma_5 (p_1 - M) \phi (p_2^{\ 2} - M^2)$$

$$= -\gamma_5 B\phi (p_2 + M)$$

where

$$L\chi = (p_1 + M) \chi (p_2 + M) \ .$$

$$\therefore \qquad B\psi = g^2 f(x) \gamma_5 \psi \gamma_5 + g \chi_1 \qquad \ldots.2.5$$

$$\text{and} \qquad KL\chi = g^2 f(x) \gamma_5 K \chi \gamma_5 + g\psi(p_2^{\ 2} - M^2) \delta(x_1 - y) \qquad \ldots.2.6$$

In equations (2.5) and (2.6) make the substitutions

$$\psi = (p_1 + M) w + w(p_2 + M) + \theta$$

$$\chi = (p_1 - M)\Omega + \Omega(p_2 - M) + \Theta$$

where $\{w, \gamma_5\} = \{\Omega, \gamma_5\} = [\theta, \gamma_5] = [\Theta, \gamma_5] = 0$

together with a transformation to centre of mass coordinates

$$x = x_1 - x_2$$

and

$$X = x_1 + x_2$$

making

$$p_1 = p + i\nabla ,$$

$$p_2 = p - i\nabla ,$$

$$p_1^{\ 2} + p_2^{\ 2} = 2(p^2 - \Box) ,$$

and

$$p_1^{\ 2} - p_2^{\ 2} = 4ip. \nabla .$$

If y is now measured relative to the centre of mass frame the transformation to centre of mass coordinates, transforms $x_1 - y$ to $\frac{x}{2} - y$. The delta function, $\delta(x_1 - y)$ will be left untransformed until integrations need to be performed.

Noting the identities

$$(p_1 - M)\theta(p_2 - M) = \tfrac{1}{2}(p_1 - M)(p_1\theta + \theta p_2) + \tfrac{1}{2}(p_1\theta + \theta p_2)(p_2 - M)$$
$$- \tfrac{1}{2}(p_1^2 + p_2^2 - 2M^2)\theta \quad ,$$

$$(p_1 + M)\theta(p_2 + M) = \tfrac{1}{2}(p_1 + M)(p_1\theta + \theta p_2) + \tfrac{1}{2}(p_1\theta + \theta p_2)(p_2 + M)$$
$$- \tfrac{1}{2}(p_1^2 + p_2^2 - 2M^2)\theta \quad ,$$

$$(p_1^2 - M^2)w(p_2 - M) + (p_1 - M)w(p_2^2 - M^2)$$
$$= \tfrac{1}{2}(p_1^2 + p_2^2 - 2M^2)w(p_2 - M) + \tfrac{1}{2}(p_1^2 + p_2^2 - 2M^2)(p_1 - M)w$$
$$- \tfrac{1}{2}(p_1^2 - p_2^2)(p_1 w - w p_2) \quad ,$$

and $(p_1^2 - M^2)\Omega(p_2 + M) + (p_1 + M)\Omega(p_2^2 - M^2)$

$$= \tfrac{1}{2}(p_1^2 + p_2^2 - 2M^2)\Omega(p_2 + M) + \tfrac{1}{2}(p_1^2 + p_2^1 - 2M^2)(p_1 + M)\Omega$$
$$- \tfrac{1}{2}(p_1^2 - p_2^2)(p_1\Omega - \Omega p_2) \quad ,$$

equations (2.5) and (2.6) become

$$(p^2 - \Box - M^2)w(p_2 - M) + (p^2 - \Box - M^2)(p_1 - M)w - 2i\, p.\nabla\,(p_1 w - w p_2)$$
$$+ \tfrac{1}{2}(p_1 - M)(p_1\theta + \theta p_2) + \tfrac{1}{2}(p_1\theta + \theta p_2)(p_2 - M) - (p^2 - \Box - M^2)\theta$$
$$= - g^2 f(x)\left[(p_1 - M)w - w(p_2 - M) + \theta\right] + g\left[(p_1 - M)\Omega_1 + \Omega_1(p_2 - M) + \Theta_1\right] \qquad .....2.7$$

and

$$K\left[(p^2 - \Box - M^2)\Omega(p_2 + M) + (p^2 - \Box - M^2)(p_1 + M)\Omega\, 2ip.\nabla\,(p_1\Omega - \Omega p_2)\right.$$
$$\left. + \tfrac{1}{2}(p_1 + M)(p_1\Theta + \Theta p_2) + \tfrac{1}{2}(p_1\Theta + \Theta p_2)(p_2 + M) - (p^2 - \Box - M^2)\Theta\right]$$
$$= - g^2 f(x)\left[(p_1 + M)\Omega + \Omega(p_2 + M) + \Theta\right] + g\left[(p_1 + M)w + w(p_2 + M) + \Theta\right]$$
$$(p_2^2 - M^2)\delta(x_1 - y) \qquad .....2.8$$

Equations (13) and (14) are satisfied if

$$(\Box - p^2 + M^2)w = \tfrac{1}{2}(p_1\theta + \theta p_2) + g^2 f(x)\, w - g\Omega_1 \qquad .....2.9$$

$$(\Box - p^2 + M^2)\theta = 2ip.\nabla\,(p_1 w - w p_2) - g^2 f(x)\,\theta + g\Theta_1 - ig^2(\nabla f, w) \qquad .....2.10$$

$$K(\Box - p^2 + M^2)\Omega = K\tfrac{1}{2}(p_1\Theta + \Theta p_2) + g^2 f(x)\, K\Omega - gw(p_2^2 - M^2)\delta(x_1 - y) \qquad .....2.11$$

$$K(\Box - p^2 + M^2)\,\Theta = K2ip.\nabla(p_1\Omega - \Omega p_2) - g^2 f(x)\,K\,\Theta + g\theta(p_1 - M^2)\delta(x_1 - y) - ig^2[\nabla f, \Omega] \quad \ldots .2.12$$

For simplification put

$$N \doteq \Box - p^2 + M^2 \quad .$$

Then the equations become

$$Nw - g^2 f(x)\,w = \tfrac{1}{2}(p_1\,\theta + \theta\,p_2) - g\Omega_1 \quad \ldots .2.13$$

$$N\theta - g^2 f(x)\,\theta = 2ip.\nabla(p_1 w - wp_2) - ig^2[\nabla f, w] + g\Theta_1 \quad \ldots .2.14$$

$$N\Omega - g^2 f(x)\Omega = \tfrac{1}{2}(p_1\Theta + \Theta p_2) - \frac{1}{K}gw(p_2{}^2 - M^2)\delta(x_1 - y) \quad \ldots .2.15$$

$$N\Theta + g^2 f(x)\,\Theta = 2ip.\nabla(p_1\Omega - \Omega p_2) + \frac{1}{K}g\theta(p_2{}^2 - M^2)\delta(x_1 - y) - i\frac{g^2}{K}[\nabla f, \Omega] \quad \ldots .2.16$$

Now define Ω' and w' such that

$$Nw' + g^2 f(x)\;w' = 2p.\nabla w + g\Omega_1{}'$$

and

$$N\Omega' + g^2 (x)\Omega' = 2p.\nabla\Omega + \frac{g}{K}w'\;(x_1 - y)(p_2{}^2 - M^2)$$

and define ϕ and Φ such that

$$\phi = \theta - i(p_1 w' - w'\,p_2)$$

and

$$\Phi = \Theta - i\,(p_1\Omega' - \Omega' p_2)$$

The equations satisfied by ϕ , w , and w' are ,

$$\begin{aligned}
\{N + g^2 f(x)\}\,\phi &= \{N + g^2 f(x)\}\,\theta - i\{N + g^2 f(x)\}(p_1 w' - w'p_2)\\
&= 2ip.\nabla(p_1 w - wp_2) - ig^2[\nabla f, w] + g\Theta_1\\
&\quad - 2ip_1\,p.\nabla\Omega + 2i\,p.\nabla\Omega p_2 - ig(p_1\Omega_1{}' - \Omega_1{}'p_2)\\
&\quad + g^2\{\nabla f, w'\}\\
&= -ig^2[\nabla f(w + iw') - (w - iw')\nabla f] + g\Theta_1\\
&\quad -ig(p_1\Omega_1{}' - \Omega_1{}'p_2)\\
&= -ig^2[\nabla f(w + iw') - (w - iw')\nabla f] + g\Phi_1 \quad ,
\end{aligned}$$

$$\{N - g^2 f(x)\} w = \tfrac{1}{2}(p_1\Theta + \Theta p_2) - g\Omega_1$$
$$= \tfrac{1}{2}(p_1\phi + \phi p_2) - g\Omega_1 + \tfrac{1}{2}i\,(p_1{}^2 w' - w' p_2{}^2)$$
$$= \tfrac{1}{2}i\,[\nabla, \phi] + \tfrac{1}{2}\{p, \phi\} - 2p.\nabla w' - g\Omega_1 \quad ,$$

and

$$\{N + g^2 f(x)\} w' = 2p.\nabla w + g\Omega'_1$$

The equations satisfied by Φ , Ω, and Ω' are,

$$\{N + g^2 f(x)\}\,\Phi = 2ip.\nabla\,(p_1\Omega - \Omega p_2) + \frac{1}{K}\,g\Theta\,(p_2{}^2 - M^2)\,\delta(x_1 - y)$$
$$- i\,\frac{g^2}{K}\,[\nabla f , \Omega] - ip_1 2p.\nabla\Omega + i2p.\nabla\Omega p_2$$
$$- i\,\frac{g}{K}\,(p_1 w' - w' p_2)\,\delta(x_1 - y)(p_2{}^2 - M^2) + \frac{g^2}{K}\{\nabla f, \Omega'\}$$
$$= - i\,\frac{g^2}{K}[\nabla f(\Omega + i\Omega') - (\Omega - i\Omega')\nabla f\,]$$
$$+ \frac{g}{K}\,\Theta\,(p_2{}^2 - M^2)\,\delta(x_1 - y)$$
$$- \frac{ig}{K}\,(p_1 w' - w' p_2)(p_2{}^2 - M^2)\,\delta(x_1 - y)$$
$$= - i\,\frac{g^2}{K}[\nabla f(\Omega + i\Omega') - (\Omega - i\Omega')\nabla f\,]$$
$$+ \frac{g}{K}\,\phi\,(p_2{}^2 - M^2)\,\delta(x_1 - y) \quad ,$$

$$\{N - g^2 f(x)\}\,\Omega = \tfrac{1}{2}\,(p_1\,\Theta + \Theta\,p_2) - \frac{g}{K}\,w\,(p_2{}^2 - M^2)\,\delta(x_1 - y)$$
$$= \tfrac{1}{2}\,i\,[\nabla , \Phi] + \tfrac{1}{2}\{p , \Phi\} - 2p.\nabla\,\Omega' - \frac{g}{K}w'(p_2{}^2 - M^2)\delta(x_1 - y)$$
$$\{N + g^2 f(x)\}\,\Omega' = 2p.\nabla\Omega - \frac{g}{K}\,w'\,(p_2{}^2 - M^2)\,\delta(x_1 - y)$$

As was indicated earlier, at this point the discussion will be restricted to include only spin singlet $J = 0$ interactions.* Later it is intended to replace terms arising from these interactions by a simple potential function so that a more detailed analysis at this stage would be wasted. The simplification is made so that the resulting equations can be solved in closed form and the appearance of the meson emission threshold seen in the expression for the scattering cross section.

*For a more detailed consideration of the interaction terms see Reinfelds[28] or Biswas.[26]

This being so, solutions are sought which are even in γ_4 and x_4 so that the following substitutions are made:

$$
\begin{aligned}
w_a &= \tfrac{1}{2}(w + \gamma_4 w \gamma_4) \\
w_b &= \tfrac{1}{2}(w - \gamma_4 w \gamma_4) \\
w_a' &= \tfrac{1}{2}(w' + \gamma_4 w' \gamma_4) \\
w_b' &= \tfrac{1}{2}(w' - \gamma_4 w' \gamma_4) \\
\phi_a &= \tfrac{1}{2}(\gamma_4 \phi + \phi \gamma_4) \\
\phi_b &= \tfrac{1}{2}(\gamma_4 \phi - \phi \gamma_4) \qquad \ldots.2.17
\end{aligned}
$$

and similar substitutions for Ω, Ω', and Φ .

Then in the centre of mass frame with $\underline{p} = 0$ and $p_4 = E$ for the w's and ϕ , and $\underline{p} = 0$ and $p_4 = E'$ for the Ωs and Φ , (E' may be regarded as the energy operator of the nucleons which have emitted mesons).

$$
\begin{aligned}
\{N - g^2 f(x)\} w_a + 2E \nabla_4 w_a' &= E\phi_a - \gamma_4 \gamma . \nabla \phi_b - g\Omega_{1a} , \\
\{N + g^2 f(x)\} w_a' - 2E \nabla_4 w_a &= + g\Omega'_{1a} , \\
\{N - g^2 f(x)\} w_b + 2E \nabla_4 w_b' &= \nabla_4 \phi_b - g\Omega_{1b} , \\
\{N + g^2 f(x)\} w_b' - 2E \nabla_4 w_b &= + g\Omega'_{1b} \\
\{N + g^2 f(x)\} \phi_a &= 2f\nabla_4 w_a' - 2\gamma_4(\gamma . \nabla f) w_b' + g\,\Phi_{1a} , \\
\{N + g^2 f(x)\} \phi_b &= 2f\nabla_4 w_b - 2\gamma_4(\gamma . \nabla f) w_a + g\,\Phi_{1b} ,
\end{aligned}
$$

$$
\begin{aligned}
\{N' - g^2 f(x)\}\Omega_a + 2E'\nabla_4\Omega_a' &= E'\Phi_a - \gamma_4 \gamma . \nabla \Phi_b - \tfrac{g}{K} w_a (p_2{}^2 - M^2)\delta(x_1 - y) \\
\{N' + g^2 f(x)\}\Omega_a' - 2E'\nabla_4\Omega_a &= + \tfrac{g}{K} w_a'(p_2{}^2 - M^2)\delta(x_1 - y) \\
\{N' - g^2 f(x)\}\Omega_b + 2E'\nabla_4\Omega_b' &= \nabla_4 \Phi_b \qquad \tfrac{g}{K} w_b (p_2{}^2 - M^2)\delta(x_1 - y) \\
\{N' + g^2 f(x)\}\Omega_b' - 2E'\nabla_4\Omega_b &= \tfrac{g}{K} w_b' (p_2{}^2 - M^2)\delta(x_1 - y) \\
\{N' + g^2 f(x)\}\Phi_a &= \tfrac{g^2}{K}\left[2f\nabla_4\Omega_a' - 2\gamma_4(\gamma \nabla f)\Omega_b'\right] \\
&\quad + \tfrac{g}{K}\phi_a(p_2{}^2 - M^2)\delta(x_1 - y) \\
\{N' + g^2 f(x)\}\Phi_b &= \tfrac{g^2}{K}\left[2f\nabla_4\Omega_b - 2\gamma_4(\gamma \nabla f)\Omega_a\right] \\
&\quad + \tfrac{g}{K}\phi_b(p_2{}^2 - M^2)\delta(x_1 - y) \qquad \ldots.2.18
\end{aligned}
$$

In order to see the nature of the solution of these equations more easily the following change of variable is made:

$$w_a^+ = w_a + iw_a' ,$$
$$w_a^- = w_a - iw_a' ,$$
$$w_b^+ = w_b + iw_b' ,$$
$$w_b^- = w_b - iw_b' ,$$

and similar changes defining Ω_a^+, Ω_a^-, Ω_b^+, and Ω_b^-.

Then the equations become

$$\{N - g^2 f(x)\} w_a^+ - 2iE\nabla_4 w_a^+ = E\phi_a - \gamma_4\gamma.\nabla\phi_b - g\Omega_{1a}^+$$
$$\{N - g^2 f(x)\} w_a^- + 2iE\nabla_4 w_a^- = E\phi_a - \gamma_4\gamma.\nabla\phi_b - g\Omega_{1a}^-$$
$$\{N + g^2 f(x)\} w_b^+ - 2iE\nabla_4 w_b^+ = \nabla_4 \phi_b - g\Omega_{1b}^+$$
$$\{N + g^2 f(x)\} w_b^- + 2iE\nabla_4 w_b^- = \nabla_4 \phi_b - g\Omega_{1b}^-$$
$$\{N + g^2 f(x)\} \phi_a = 2f\nabla_4 w_a' - 2(\gamma_4\gamma.\nabla f)w_b' + g\,\Phi_{1a}$$
$$\{N + g^2 f(x)\} \phi_b = 2f\nabla_4 w_b - 2(\gamma_4\gamma.\nabla f)w_a + g\,\Phi_{1b}$$

$$\{N' - g^2 f(x)\}\Omega_a^+ - 2iE'\nabla_4\Omega_a^+ = E\Phi_a - \gamma_4\gamma.\nabla\Phi_b - \frac{g}{K} w_a^+(p_2^2 - M^2)\delta(x_1 - y)$$
$$\{N' - g^2 f(x)\}\Omega_a^- + 2iE'\nabla_4\Omega_a^- = E\Phi_a - \gamma_4\gamma.\nabla\Phi_a - \frac{g}{K} w_a^-(p_2^2 - M^2)\delta(x_1 - y)$$
$$\{N' + g^2 f(x)\}\Omega_b^+ - 2iE'\nabla_4\Omega_b^+ = \nabla_4\Phi_b - \frac{g}{K} w_b^+(p_2^2 - M^2)\delta(x_1 - y)$$
$$\{N' + g^2 f(x)\}\Phi_a = 2f\nabla_4\Omega_a' - 2(\gamma_4\gamma.\nabla f)\Omega_a' + \frac{g}{K}\phi_a(p_2^2 - M^2)\delta(x_1 - y)$$
$$\{N' + g^2 f(x)\}\Phi_a = 2f\nabla_4\Omega_b - 2(\gamma_4\gamma.\nabla f)\Omega_a + \frac{g}{K}\phi_b(p_2^2 - M^2)\delta(x_1 - y) \quad \ldots .2.19$$

Note that

$N = \Box + M^2 - E^2$, $N' = \Box + M^2 - E'^2$, $S^2 = M^2 - E^2 - \nabla^2$, $T^2 = M^2 - \nabla^2$,

$S'^2 - S^2 = E^2 - E'^2 = 2Ee - e^2$, and

$p_2^2 - M^2 = -(N + 2iE\nabla_4) = -(N - 2iE\nabla_4) - 4iE\nabla_4$.

The equations are now in a form from which the time dependence can easily be eliminated. The following solutions for the time dependence of the equations are substituted in order to find equations relating the coefficients of the time dependent parts of the solutions (i.e. the α's , β's , A's , and B's). These are functions of $\underline{x}$ and in the case of the A's and B's functions of the space components of y and $\underline{q}$.

$$w_a^{+} = (\alpha_1 + \alpha_2 \operatorname{sgn} x_4)e^{iT|x_4|}\, e^{iE_1 x_4} + (2ES)^{-1} \operatorname{sgn} x_4\, e^{iS|x_4|}\, (E\alpha - \gamma_4 \gamma . \nabla\beta)$$
$$w_a^{-} = (\alpha_1 - \alpha_2 \operatorname{sgn} x_4)e^{iT|x_4|}\, e^{-E_1 x_4} - (2ES)^{-1} \operatorname{sgn} x_4\, e^{iS|x_4|}\, (E\alpha - \gamma_4 \gamma . \nabla\beta)$$
$$w_b^{+} = (\beta_2 + \beta_1 \operatorname{sgn} x_4)e^{iT|x_4|}\, e^{iE_2 x_4} - (2iE)^{-1}\, e^{iS|x_4|}\beta$$
$$w_b^{-} = (-\beta_2 + \beta_1 \operatorname{sgn} x_4)e^{iT|x_4|}\, e^{-E_2 x_4} + (2iE)^{-1}\, e^{iS|x_4|}\beta$$
$$\phi_a = \alpha\, e^{iS|x_4|}$$
$$\phi_b = \beta\, e^{iS|x_4|}$$

$$\Omega_a^{+} = \int\Big[(A_1 + A_2 \operatorname{sgn} x_4)e^{iT|x_4|}\, e^{iE_1 x_4}\, e^{iq(y-x_1)} + (A_3 + A_4 \operatorname{sgn} x_4)\, e^{iS|x_4|}\, e^{iq(y-x_1)}\Big]\, dq$$
$$\Omega_a^{-} = \int\Big[(A_1 - A_2 \operatorname{sgn} x_4)e^{iT|x_4|}\, e^{-iE_1 x_4}\, e^{iq(y-x_1)} + (A_3 - A_4 \operatorname{sgn} x_4)\, e^{iS|x_4|}\, e^{iq(y-x_1)}\Big]\, dq$$
$$\Omega_b^{+} = \int\Big[(B_2 + B_1 \operatorname{sgn} x_4)e^{iT|x_4|}\, e^{\pm E_2 x_4}\, e^{iq(y-x_1)} + (B_3 + B_4 \operatorname{sgn} x_4)\, e^{iS|x_4|}\, e^{iq(y-x_1)}\Big]\, dq$$
$$\Omega_b^{-} = \int\Big[(-B_2 + B_1 \operatorname{sgn} x_4)e^{iT|x_4|}\, e^{-iE_2 x_4}\, e^{iq(y-x_1)} + (B_3 - B_4 \operatorname{sgn} x_4)\, e^{iS|x_4|}\, e^{iq(y-x_1)}\Big]\, dq$$
$$\Phi_a = \int (A + A' \operatorname{sgn} x_4)e^{iS|x_4|}\, e^{iq(y-x_1)}\, dq$$
$$\Phi_b = \int (B + B' \operatorname{sgn} x_4)e^{iS|x_4|}\, e^{iq(y-x_1)}\, dq \qquad \ldots.2.20$$

These solutions are analogous to those obtained by Green and Biswas for the ordinary Bethe-Salpeter equation, the main difference being the replacement of E by E_1 and E_2 in e^{iEx_4} . The E_1 and E_2 are in general different from E because they represent the energies of nucleons which may have emitted mesons. They are also different from each other as E_1 represents the energy of one nucleon and E_2 the other. This is the physical interpretation of the E_1 and E_2 but the reasons for supposing solutions of the type 2.20 are not only physical ones. The E_1 and E_2 need to be different from one another and from E so that solutions of this type may satisfy the equations at all. If $E = E_1 = E_2$ then the equations obtained for α_1 etc. by substituting 2.20 in 2.19 are inconsistent.

The w_a^+ etc. were needed to see what type of solution was to be expected and having found w_a^+ etc. it is now possible to write down corresponding expressions for w_a, w_a', etc.

They are:

$$w_a = \alpha_1 e^{iT|x_4|} \cos E_1 x_4 + i\alpha_2 \operatorname{sgn} x_4 e^{iT|x_4|} \sin E_1 x_4 ,$$

$$w_a' = \alpha_1 e^{iT|x_4|} \sin E_1 x_4 - i\alpha_2 \operatorname{sgn} x_4 e^{iT|x_4|} \cos E_1 x_4$$
$$+ (2iES)^{-1} \operatorname{sgn} x_4 e^{iS|x_4|} (E\alpha - \gamma_4 \gamma . \nabla \beta) ,$$

$$w_b = i\beta_2 e^{iT|x_4|} \sin E_2 x_4 + \beta_1 \operatorname{sgn} x_4 e^{iT|x_4|} \cos E_2 x_4 ,$$

$$w_b' = -i\beta_2 e^{iT x_4} \cos E_2 x_4 + \beta_1 \operatorname{sgn} x_4 e^{iT|x_4|} \sin E_2 x_4$$
$$+ (2E)^{-1} e^{iS|x_4|} ,$$

$$\Omega_a = \int \Big[A_1 e^{iT|x_4|} \cos E_1 x_4 e^{iq(y - x_1)}$$
$$+ iA_2 \operatorname{sgn} x_4 e^{iT|x_4|} \sin E_1 x_4 e^{iq(y - x_1)}$$
$$+ A_3 e^{iS|x_4|} e^{iq(y - x_1)} \Big] dq ,$$

$$\Omega_a' = \int \Big[A_1 e^{iT|x_4|} \sin E_1 x_4 e^{iq(y - x_1)}$$
$$- iA_2 \operatorname{sgn} x_4 e^{iT|x_4|} \cos E_1 x_4 e^{iq(y - x_1)}$$
$$- iA_4 \operatorname{sgn} x_4 e^{iS|x_4|} e^{iq(y - x_1)} \Big] dq ,$$

$$\Omega_b = \int \Big[iB_2 e^{iT|x_4|} \sin E_2 x_4 e^{iq(y - x_1)}$$
$$+ B_1 \operatorname{sgn} x_4 e^{iT|x_4|} \cos E_2 x_4 e^{iq(y - x_1)}$$
$$+ B_4 \operatorname{sgn} x_4 e^{iS|x_4|} e^{iq(y - x_1)} \Big] dq ,$$

and

$$\Omega_b' = \int \Big[-iB_2 e^{iT|x_4|} \cos E_2 x_4 e^{iq(y - x_1)}$$
$$+ B_1 \operatorname{sgn} x_4 e^{iT|x_4|} \sin E_2 x_4 e^{iq(y - x_1)}$$
$$-B_3 e^{iS|x_4|} e^{iq(y - x_1)} \Big] dq . \qquad \ldots .2.21$$

If these possible solutions of the equations are to satisfy them then the following relations must hold between the α's , β's , A's , and B's.*

$$(2e_1 T \operatorname{sgn} x_4 + e_1^2)(\alpha_1 + \alpha_2 \operatorname{sgn} x_4)e^{iT|x_4|} e^{i(E+e_1)x_4} = g\Omega^{+'}_{1a} \quad \text{....(i)}$$

$$2ES\alpha_2 + E\alpha - \gamma_4\gamma . \nabla \beta = 0 , \quad \text{......(ii)}$$

$$(T - V)\alpha_1 + (E + e_1)\alpha_2 = 0 , \quad \text{.....(iii)}$$

$$A_2 + A_4 = 0 , \quad \text{..... (iv)}$$

$$(T - V) A_1 + (E + e_1) A_2 + (S-V) A_3 = 0 , \quad \text{...... (v)}$$

$$\int(2e_1 T \operatorname{sgn} x_4 + e_1^2)(A_1 + A_2 \operatorname{sgn} x_4) e^{iq(y - x_1)} dq$$
$$= \frac{g}{K}\delta(x_1 - y)(e_1^2 + 4ET \operatorname{sgn} x_4 + 4Ee_1 + 4E^2 + 2e_1 T \operatorname{sgn} x_4)(\alpha_1+\alpha_2 \operatorname{sgn} x_4) \quad \text{..... (vi)}$$

and

$$(2ES \operatorname{sgn} x_4 - q_4 S \operatorname{sgn} x_4 + q_4^2/2)(A_3 + A_4 \operatorname{sgn} x_4)$$
$$= \int E(A + A' \operatorname{sgn} x_4) - \gamma_4\gamma . \nabla (B + B' \operatorname{sgn} x_4) e^{iq(y - x_1)} dq$$
$$- \frac{g}{K}\delta(x_1 - y) \operatorname{sgn} x_4 (E\alpha - \gamma_4\gamma . \nabla\beta)(1 - S/2E) \quad \text{.....(vii)}$$

$$\text{....2.22}$$

where $E_1 = E + e_1$.

At this point not all the time dependence has been made explicit; the $e^{iq(y - x_1)}$ has a factor $e^{iq_4(y_4 - x_{14})}$, the K operator contains a term $\partial^2/\partial y_4^2$, and the $\delta(x_1 - y)$ contains a factor $\delta(x_{14} - y_4)$. This time dependence may be eliminated by supposing that A_1 and A_2 both have a factor e^2/q_4^2 , where e is the meson energy, and no other q_4 dependence. Then K becomes $(-e^2 - \underline{p}_y^2 - \mu^2)$.

Equations 2.22 are obtained by substituting in the first pair of each set of six equations. The results of the substitutions for the other eight may be conjectured* and are not needed here.

Note that in (i) $\Omega^{+'}_{1a}$ is used to mean that part of Ω^{+}_{1a} which has $e^{iT|x_4|} e^{i(E+e_1)x_4}$ as a factor. The equivalent equation for $(\alpha_1 - \alpha_2 \operatorname{sgn} x_4)$ gives identical equations for α_1 and α_2 separately

*See appendix

and is thus omitted. Similarly the equivalent equation to (vi) from the equation for $A_1 - A_2 \operatorname{sgn} x_4$ gives (vi).

Equations (ii), (iii), (iv), and (v) are analogous to the equations obtained by Green and Biswas at this stage of their argument, in fact (ii) and (iii) are identical to two of their equations if $e_1 = e_2 = 0$. If a similar elimination process is carried out using the corresponding equations which arise from the differential equations for w_b, Ω_b, etc. then the twelve equations may be taken to define e_1 and e_2.

Equation (vii) gives the expressions for A_3 and A_4 needed to make Ω_a and Ω_a' solutions of their equations.

Equations (i) and (vi) are the ones which concern the present problem; the others need to be examined only to be certain that none of the quantities involved is over determined by inconsistent equations. Equations (i) and (vi) give a pair of coupled equations between the α's and the A's as follows:

Split (i) and (vi) into even and odd parts with respect to x_4 and put $a_1(x,y) = \int A_1(x,y)\, e^{i\underline{q}\cdot(\underline{y} - \underline{x}_1)}\, d\underline{q}$

and $a_2(x,y) = \int A_2(x,y)\, e^{i\underline{q}\cdot(\underline{y} - \underline{x}_1)}\, d\underline{q}$

where $A_1(x,y)\, e^2/q_4^2 = A_1$ and $A_2(x,y)\, e^2/q_4^2 = A_2$

Notation:

From this point in this chapter vectors x and y will be space vectors of three dimensions.

$$2e_1 T \alpha_1(x) + e_1^2 \alpha_2(x) = g a_2(x,x) ,$$

$$e_1^2 \alpha_1(x) + 2Te_1 \alpha_2(x) = g a_1(x,x) ,$$

$$2e_1 T\, a_1(x,y) + e_1^2\, a_2(x,y) = \frac{g}{K}\delta(x/2-y)\left[f_1\alpha_1 + f_2\alpha_2\right] ,$$

and $$e_1^2\, a_1(x,y) + 2e_1 T a_2(x,y) = \frac{g}{K}\delta(x/2-y)\left[f_2\alpha_1 + f_1\alpha_2\right]$$

where $$f_1 = 4ET + 2e_1 T$$

and $$f_2 = e_1^2 + 4Ee_1 + 4E^2 .$$

$$\therefore\ (4e_1T^2 - e_1^{\,3})\alpha_1(x) = 2T\, a_2(x,x) - e_1\, a_1(x,x) \quad ,$$
$$(4e_1T^2 - e_1^{\,3})\alpha_2(x) = -e_1\, a_2(x,x) + 2T\, a_1(x,x) \quad ,$$

$$(4e_1T^2 - e_1^{\,3})\, a_1(x,y) = \frac{g}{K}\delta(x/2-y)\ 2T(f_1\alpha_1 + f_2\alpha_2) - e_1(f_2\alpha_1 + f_1\alpha_2) \ ,$$
and $$(4e_1T^2 - e_1^{\,3})\, a_2(x,y) = \frac{g}{K}\delta(x/2-y)\ 2T(f_2\alpha_1 + f_1\alpha_2) - e_1(f_1\alpha_1 + f_2\alpha_2) \ .$$

$$\therefore\ (4e_1T^2 - e_1^{\,3})(\alpha_1(x) + \alpha_2(x)) = (2T - e_1)(a_1(x,x) + a_2(x,x)) \ ,$$
$$(4e_1T^2 - e_1^{\,3})(\alpha_1(x) - \alpha_2(x)) = (2T + e_1)(a_1(x,x) - a_2(x,x)) \ ,$$

$$(4e_1T^2 - e_1^{\,3})(a_1(x,y) + a_2(x,y)) = \frac{g}{K}\delta(x/2-y)(2T - e_1)(f_1\alpha_1(x) + f_2\alpha_2(x) + f_1\alpha_2(x) + f_2\alpha_1(x)) \ ,$$
and $$(4e_1T^2 - e_1^{\,3})(a_1(x,y) - a_2(x,y)) = \frac{g}{K}\delta(x/2-y)(2T + e_1)(f_1\alpha_1(x) + f_2\alpha_2(x) - f_1\alpha_2(x) - f_2\alpha_1(x)) \ .$$

$$\therefore\ (2T + e_1)(\alpha_1(x) + \alpha_2(x)) = \frac{g}{e_1}(a_1(x,x) + a_2(x,x)) \ ,$$
$$(2T - e_1)(\alpha_1(x) - \alpha_2(x)) = -\frac{g}{e_1}(a_1(x,x) - a_2(x,x)),$$

$$(2T + e_1)(a_1(x,y) + a_2(x,y)) = \frac{g}{e_1}\frac{1}{K}\delta(x/2-y)(f_1\alpha_1(x) + f_2\alpha_2(x) + f_1\alpha_2(x) + f_2\alpha_1(x)) \ ,$$
and $$(2T - e_1)(a_1(x,y) - a_2(x,y)) = \frac{g}{e_1}\ \frac{1}{K}\delta(x/2-y)(f_1\alpha_1(x) + f_2\alpha_2(x) - f_1\alpha_2(x) - f_2\alpha_1(x)) \ .$$

Consider

$$(4T^2 - e_1^{\,2})(\alpha_1(x) + \alpha_2(x)) = \frac{g}{e_1}\ (2T - e_1)(a_1(x,x) + a_2(x,x))$$
$$= \frac{g}{e_1}(2T + e_1)(a_1(x,x) + a_2(x,x) - 2g(a_1(x,x) + a_2(x,x))$$
$$= \frac{g^2}{e_1^{\,2}}\left[\frac{1}{K}\delta(x/2-y)\right]_{y=x/2}(f_1 + f_2)(\alpha_1(x) + \alpha_2(x) - 2g(a_1(x,x) + a_2(x,x))$$
$$= \frac{g^2}{e_1^{\,2}}\, h\, f_2\ (\alpha_1(x) + \alpha_2(x) + \frac{g^2}{e_1^{\,2}}\, h\ (2E + e_1)(2T + e_1)(\alpha_1(x) + \alpha_2(x))$$
$$- \frac{g^2 h}{e_1^{\,2}}(2E + e_1)(e_1)\ (\alpha_1(x) + \alpha_2(x))$$
$$- 2g\ (a_1(x,x) + a_2(x,x)) \quad .$$

$$\therefore \left[4T^2 - e_1^{\ 2} - \frac{g^2h}{e_1^{\ 2}}(e_1^{\ 2} + 4Ee_1 + 4E^2 - 2Ee_1 - e_1^{\ 2})\right](\alpha_1(x) + \alpha_2(x))$$
$$= \left[\frac{g^2h}{e_1^{\ 2}}(2E + e_1)\frac{g}{e_1}h - 2g\right]\left[a_1(x,x) + a_2(x,x)\right] .$$

$$\therefore \quad (T^2 - V^2)\psi_e(x) = g_1 \psi_i(x,x) .$$

and similarly

$$K(T^2 - V^2)\psi_i(x,y) = g_2 \psi_e(x)\,\delta(x/2 - y)$$

where $\psi_e(x) = \alpha_1(x) + \alpha_2(x)$

and $\psi_i(x,y) = a_1(x,y) + a_2(x,y)$.

Note that exactly similar equations could have been derived for $\alpha_1(x) - \alpha_2(x)$ and $a_1(x,y) - a_2(x,y)$.

Now that the original equations have been reduced to these equations for $\psi_e(x)$ and $\psi_i(x,x)$ their similarity to the Klein-Gordon equation reveals the nature of the functions $\psi_e(x)$ and $\psi_i(x,x)$. They are wave functions for the nucleon, and nucleon plus meson respectively. This assertion is borne out by the work of Reinfelds[28] and that of Green and Biswas[24] who, however, did not consider the possibility of real meson emission.

Since $T^2 = M^2 - \nabla^2$, the relativistic equations of which these are the space parts in their centre of mass frame may be written down. Calling x and y position four vectors again and redefining the functions slightly so that $x/2 - y$ becomes $x - y$

$$\left\{\Box_x + M^2 + f(x)\right\}\psi_e(x) = g_1\psi_i(x,x)$$

and

$$(\Box_y + \mu^2)\left\{\Box_x + M^2 + f(x)\right\}\psi_i(x,y) = g_2\psi_e(x)\,\delta(x - y) . \qquad \text{......2.23}$$

Here it is an approximation that the $f(x)$ in each equation is written the same and as a further approximation it will be supposed that the mass appearing in the second equation is renormalised while that in the first equation is not. This amounts to assuming that the complicated expression which appeared inside the curly bracket with $\Box_x + M^2$ is what has been called $f(x)$ minus δM^2 where δM^2 is the infinite function of M which must be subtracted to renormalise the mass.

The equations 2.23 will be solved in Chapters four and five with $f(x)$ replaced by a simple step function. In order to carry out this solution boundary conditions must be imposed. The conditions used arise from the fact that nucleon - nucleon scattering is being considered so that the wave function at infinity will be of the form of an incident plane wave plus an outgoing spherical wave and the fact that a causal solution is required. The causality condition is satisfied by incorporating an infinitesimal negative imaginary part with the masses at the appropriate stage of the solution.

THE PHYSICAL MEANING OF THE FIELD EQUATIONS AND THEIR DERIVATION FROM A LAGRANGIAN

In this chapter it is proposed to show that the equations of motion derived in Chapter 2 may be derived in an intuitive way from a Lagrangian. The chapter may be regarded either as this intuitive derivation or as a physical interpretation of the equations. Whatever way it is regarded, the fact that the field equations can be derived from a Lagrangian is important because it ensures the existence of a current, the conservation of particle density, momentum, and energy.

§1 Potential in a Relativistic Theory

The wave function of any free particle must satisfy the Klein-Gordon equation,

$$(\Box + M^2)\,\psi(x) = 0$$

in a relativistic theory. If the particle is not free but is interacting with another particle then some sort of perturbation or interaction must be introduced into this equation to represent the effect of the other particle. One way of doing this is equivalent to introducing a potential well in a Schrödinger equation in the non relativistic approximation, when $\psi(x)$ is put equal to $e^{iEt}\phi(\underline{x})$.

This effect is obtained by writing

$$\left\{\Box + m^2 - V(\underline{x})\right\}\psi(x) = 0 \quad . \qquad \ldots.3.2$$

This equation can also be derived, as shown in the second and fourth chapters, from field theory, and this derivation shows that $V(\underline{x})$ can be regarded as the expectation value of a scalar or pseudoscalar meson field, whose source is the other particle. The term, $V(\underline{x})$, has the interpretation of causing the effective rest mass of the particle to vary over space allowing different fractions of the total energy to be available as kinetic energy since

$$E^2 - \underline{p}^2 = m^2 \quad . \qquad \ldots 3.3$$

If the interaction depends on the distance between the particles then $V(\underline{x})$ is a function $V(r)$ of r only in the centre of mass frame of the two particles.

Equation 3.2 is not a relativistically covariant form and so cannot hold in more than one Lorentz frame. Thus a particular frame must be chosen and the only special frame involved with two particles is the centre of mass frame.

§2 The Relativistic Two-Body Problem

Consider a system of two particles of mass m. Suppose they are each surrounded by a spherical region of radius $2R$ which has the property that the other particle, on entering the region, has its effective rest mass reduced to "a" such that $a^2 = m^2 - V$. Suppose now that the positions of both particles are referred to their centre of mass frame of reference. Then the positions of the particles are $\underline{x}$ and $-\underline{x}$, their momenta $\underline{p}$ and $-\underline{p}$, and their total energies each the same. If $|\underline{x}| < R$ (fig.1) then both particles are within the interaction region of the other. Thus in the centre of mass frame the rest mass of either particle may be considered to be a function of position $a(\underline{x})$ such that

$$a^2(\underline{x}) = m^2 \quad \text{for} \quad r > R$$

$$\text{and} \qquad = a^2 \quad \text{for} \quad r < R \quad .$$

If the total energy of each particle is E then

$$E^2 = \underline{p}^2 + a^2(\underline{x})$$

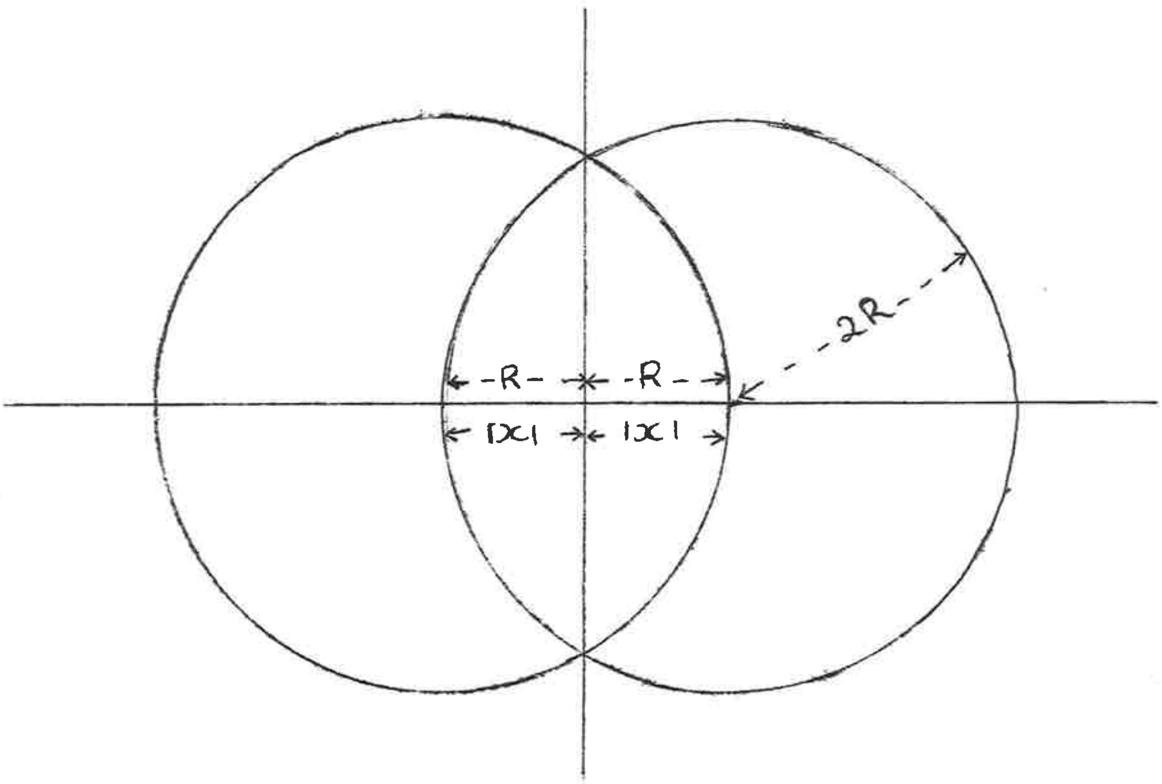


Figure 1.

and in the coordinate representation this equation becomes

$$\left\{\Box + a^2(x)\right\} \psi(x) = 0 \quad .$$

This shows that in the relativistic case, as in the non relativistic case, the equations of motion for the two body problem may be reduced to a single particle type equation by transforming to a centre of mass system of coordinates, and shows that for the physical situation considered here the single particle type equation is 3.2 as conjectured in §1, except that, here, the particular case of a square well potential is assumed.

§3 The Lagrangian

In this section the equations eventually derived in Chapter 2 will be derived from a Lagrangian based on the ideas of §1 and §2 of this chapter. They will be the equations of Chapter 2 except that the potential will be specialised to be a square well. This derivation is an intuitive one and is not based on exact field theory but does have the advantage that each term of the Lagrangian has an obvious physical interpretation and also has the virtue that the Lagrangian formalism ensures conservation of energy, momentum, and particle current. It also gives a method, using Noether's theorem, of finding the particle density.

A Lagrangian defining a field consisting of nucleons which may or may not have emitted a meson must contain two wave functions, $\psi_e(x)$ and $\psi_i(x,y)$ say, where $\psi_e(x)$ represents a nucleon, which has not emitted a meson, at x in space-time and $\psi_i(x,y)$ represents a nucleon at x in space-time, together with a meson which it has emitted, at y in space-time. Note that $\psi_e(x)$ represents not only those nucleons which have never emitted a meson but also those which have emitted and reabsorbed one in the course of a self energy process.

The coordinates of the Lagrangian will be referred to the centre of mass frame of the two particles and in this frame the model for the nucleon will be a particle moving in a frame of reference which has a region surrounding the origin in which the mass of the nucleon is reduced from m to a. So the mass of the nucleon is a function of $\underline{x}$, $a^2(r)$ such that

$$a^2(r) = m^2 \quad \text{for} \quad r > R$$

$$\text{and} \qquad\qquad = a^2 \quad \text{for} \quad r < R \quad .$$

It should be noted here that the Lagrangian is to take the interaction into account in two ways. First, it is taking into account exactly the interactions between the nucleons involving exchange or emission of one meson. Secondly, it is representing all other interactions, i.e. those involving more than one meson, by a square well potential. The presence of the potential well is essential to both types of interaction for without it the field equations would be those of one nucleon of constant mass which by conservation of energy and momentum could not emit real mesons. With the square well present, a nucleon approaching the well may emit a meson; it may reabsorb it again before reaching the well boundary but if it does not then the rest mass of the nucleon is suddenly changed allowing the meson to remain free without violation of energy momentum conservation. Also, a particle inside the well is required by the formula

$$E^2 = \underline{p}^2 + a^2$$

to use up less of its energy for rest mass so that it also may emit a free meson, without violation of energy momentum conservation. Thus it is only because the well is present that free mesons may be emitted at all. If the well were not square and had no sharp boundary then there would be continuous variation of the nucleon rest mass over all space, once again allowing the emission of free mesons.

The Lagrangian of the field of nucleons of this type is given by

$$\begin{aligned}\mathcal{L} = {} & \psi_e^*(x) \left\{ \Box_x + a^2(x) \right\} \psi_e(x)\, \delta^4(x-y) \\ & + g\, \psi_e^*(x)\, \psi_i(x,y)\, \delta^4(x-y) \\ & + g^*\, \psi_e(x)\, \psi_i^*(x,y)\, \delta^4(x-y) \\ & + g'\, \psi_i^*(x,y) \left\{ \Box_x + a^2(x) \right\} \left(\Box_y + \mu^2 \right) \psi_i(x,y) \\ & - g'\, \delta m^2\, \psi_i^*(x,y) \left(\Box_y + \mu^2 \right) \psi_i(x,y)\end{aligned}$$

where μ is the meson mass and g and g' are coupling constants.

This is the Lagrangian of a field of nucleons at x and mesons at y. To obtain the Lagrangian of the field of nucleons alone it is necessary to integrate with respect to y over all space time. It is more convenient here to integrate with respect to the space time interval $y - x$ which is an equivalent integration.

The change of variable

$$x' = x$$

$$y' = y - x$$

is made so that

$$\mathcal{L}(x,y) \longrightarrow \mathcal{L}(x',y')$$

and

$$L(x) = L(x') = \int \mathcal{L}(x',y')\, d^4y'$$

is the Lagrangian of a system of nucleons at x. Its physical interpretation is as follows.

The first term

$$\int \psi_e^*(x) \left\{ \Box_x + a^2(x) \right\} \psi_e(x)\, \delta^4(y')\, d^4y'$$
$$= \psi_e^*(x) \left\{ \Box_x + a^2(x) \right\} \psi_e(x)$$

represents "e" nucleons which are being propagated without interaction.

The second term

$$g \int \psi_e^*(x) \; \psi_i(x,y' + x) \; \delta^4(y') \; d^4y'$$
$$= \; g \, \psi_e^*(x) \; \psi_i(x,x)$$

represents the annihilation of an "i" nucleon together with a meson, both at x , and the creation of an "e" nucleon at x . This process would be the end of a self energy interaction where the virtual meson is being reabsorbed.

The third term which reduces to

$$g^* \; \psi_e(x) \; \psi_i^*(x,x)$$

represents the annihilation of an "e" nucleon at x and the creation of an "i" nucleon together with a meson at x . This process is the emission of a meson as either a real or virtual interaction.

The fourth term, before the integration over y' is carried out, represents the propagation without interaction of "i" nucleons and their associated mesons.

The fifth term,

$$g' \delta m^2 \, \psi_i^*(x,y)(\Box_y + \mu^2) \psi_i(x,y)$$

is a term introduced into the Lagrangian to help with mass renormalisation. If this term were not there the mass in the fourth term would remain as the "bare mass", there being no infinite mass to subtract off. The term may be interpreted as representing the propagation of the meson cloud which surrounds the nucleon due to the self energy processes.

§5 The Hermiticity of the Lagrangian

As mentioned earlier, one advantage of using a Lagrangian approach is that the Hermiticity of the Lagrangian ensures that the required conservation laws hold. The Lagrangian $L(x)$ taken here is clearly Hermitian in the first three terms but is Hermitian in the last term only if g' is taken to be pure imaginary. This is due to the fact that the relative time interval $y_4 - x_4$ must be regarded as pure imaginary. Wick[27] has shown that the wave function can be continued analytically to pure imaginary values of the relative time so that integration over the time interval is equally well carried out by rotating the integration path to lie along the imaginary axis. This procedure is often effectively adopted when evaluating integrals by Schwinger's and Feynman's methods.

Reinfelds[28] has adopted the same procedure in finding a current involving the Bethe-Salpeter amplitude. The density found reduces to $\psi\psi^*$ for free particles and is interpreted as the particle probability density for interacting particles.

The fact that g' is pure imaginary is discussed again in Chapter 6 when it is seen that it is necessarily so for the equation solutions to have a good physical interpretation.

§6 The Field Equations

If the principle of least action is applied to the Lagrangian $L(x)$ the field equations

$$\left\{\Box_x + a^2(x)\right\}\psi_e(x) = -g\,\psi_i(x,x)$$

and

$$g'\left\{\Box_x + a^2(x)\right\}\left(\Box_y + \mu^2\right)\psi_i(x,y) = -g^*\psi_e(x)\,\delta^4(x-y) + d\,\delta m^2\left(\Box_y + \mu^2\right)\psi_i(x,y)$$

are obtained. These are two coupled differential equations which may be interpreted physically to say that the source of "e" nucleons is an "i" nucleon and a meson both at the same point of space-time and that the source of "i" nucleons at x and mesons at y is an "e" nucleon at x. The $\delta^4(x-y)$ may be interpreted as a source function for the meson.

THE SOLUTION OF THE FIELD EQUATIONS

In this chapter it is proposed to find approximate solutions of the field equations in the special cases where the potential well is spherical or spheroidal. As will be seen, the approximations necessary do not change the analytic properties of the solution. Thus the advantage of the simple square well potential is seen to be that the analytic properties of the solution may be examined in detail. These illustrate some of the properties of scattering amplitudes, which have been the subject of much investigation in dispersion relation theory and show how they affect the scattering cross sections.

In the course of solution boundary conditions must be introduced. The conditions used are that the wave function at infinity should be an incident plane wave plus a scattered outgoing spherical wave, that the wave function and its derivatives should be continuous across the boundary of the well, and that the wave function should vanish on the surface of a hard core which is introduced in the centre of the potential well. As mentioned in the introduction, such a core must be present to account for sharp drop and flattening out of the total cross section as energy increases from zero to the meson emission threshold.

§1 The General Solution

In this section the solution will be found in such a form that it can easily be specialised to either of the cases mentioned above.

The field equations are

$$\left\{ \Box_x + a^2(\underline{x}) \right\} \Psi_e(x) = -g \Psi_i(x,x) \qquad \ldots \; 4.1$$

and

$$g'(\Box_y + \mu^2)\left\{\Box_x + a^2(x)\right\}\Psi_i(x,y) = -g^*\Psi_e(x)\,\delta^4(x - y) \quad \ldots\ldots 4.2$$

where the mass in 4.1 is unrenormalised.

Equation 4.2 may be solved for y to give

$$g'\left\{\Box_x + a^2(x)\right\}\Psi_i(x,y) = -g^*\int \Psi_e(x)\,\delta^4(x - y')\,H(y - y')\,dy'$$

where $H(x)$ is given by

$$\bar{H}(k) = \frac{1}{\mu^2 - k^2} ,$$

$\bar{H}(k)$ being the Fourier transform of $H(x)$ so that $H(x)$ is a solution of

$$(\Box_y + \mu^2)\,H(x) = \delta^4(x) .$$

$$\therefore \left\{\Box_x + a^2(x)\right\}\Psi_i(x,y) = -\frac{g^*}{g'}\Psi_e(x)\,H(y - x) .$$

$$\therefore \quad \Psi_i(x,y) = -\frac{g^*}{g'}\int \Psi_e(x')H(y-x')L(x-x')dx' \quad \ldots 4.3$$

where $L(x)$ is a solution of

$$\left\{\Box_x + a^2(\underline{x})\right\} L(x) = \delta^4(x) .$$

The function $a^2(\underline{x})$ in this equation arises in Chapter 2 as an approximation to a complicated function, the approximation being to make it a step function. If a different approximation is made at that stage and $a^2(x)$ is put equal to a constant a^2 then the expression for $\Psi_i(x,y)$ above is simpler because $L(x)$ is replaced by a function whose Fourier transform is simpler than that of $L(x)$. This approximation amounts to supposing that once a nucleon has emitted a meson it behaves as though no potential well exists but its rest mass is reduced to a over all space. An approximation of this sort is consistent with the original aim of the model which was to take into account processes where a nucleon can emit only one meson.

If the effective rest mass of a nucleon which had emitted a meson were to be a step function $a^2(\underline{x})$ then the restriction that it could not emit another would be an artificial one but if its effective rest mass is taken to be a constant then it is impossible for it to emit another meson consistent with the conservation of energy and momentum.

With this approximation then, $L(x)$ becomes $G(x)$ where

$$\overline{G}(k) = \frac{1}{a^2 - k^2}$$

and

$$\psi_i(x,y) = -\frac{g^*}{g'}\int \psi_e(x')H(y - x')G(x - x')\,dx' \quad \ldots\ 4.4$$

This function has a singularity at $y = x$ so that the R.H.S. of 4.1 is infinite. This is to be expected as the mass on the L.H.S. is unrenormalised. A function $\delta m^2\,\psi_e(x)$ may be subtracted from both sides so that the L.H.S. is written in the same way but the mass is renormalised. From the R.H.S. the function

$$\left\{ \frac{gg^*}{g'}\int H(y - x')\,G_0(x - x')\,dx' - R \right\} \psi_e(x)$$

is subtracted. It is chosen because the first part obviously subtracts out the infinite part of the R.H.S. and R is a constant to be found later which will ensure that the "right sized infinity" has been subtracted. It is evaluated using the condition that when $a = m$ (i.e. no well exists)

$$\left(\Box + m^2\right)\psi_e(x) = 0 \quad ,$$

m being the renormalised mass.

This renormalisation is important for without it a nucleon which stayed inside the well would simply be "recognised" by the model as a particle of rest mass a. As such it could not emit a meson. It is only because of this renormalisation that the particle is "recognised" by the model to have a true rest mass of M, making it possible, with conservation of energy and momentum, for a nucleon in this region of constant rest mass "a" to emit a meson.

Also, it must be checked that the quantity subtracted is in fact a function of m alone multiplied by $\psi_e(x)$. The check is made when R has been determined.*

4.1 has now become

$$\left\{\Box_x + a^2(\underline{x})\right\}\psi(x) = \frac{gg^*}{g'}\int\left\{\psi(x')\,H(x-x')\,G(x-x') - \psi(x)\,H(x-x')\,G_0(x-x')\right\}dx' + R\,\psi(x)$$
$$= \int K(x) - x')\psi(x')\,dx' \qquad \ldots .4.5$$

where *

$$\bar{K}(k) = \frac{gg^*}{g'}\int \bar{H}(n)\left\{\bar{G}(k-n) - \bar{G}(\ell - n)\right\} dn \ \ldots 4.6$$

and ℓ is a four vector such that

$$\ell^2 = m^2 .$$

In 4.5 $\psi(x)$ has been written for $\psi_e(x)$.

Equation 4.5 cannot be solved to give an explicit expression for $\psi(x)$ but the following method allows suitable approximations to be made to give an approximate solution.

Put $\qquad \psi(x) = \psi^i(x) + \psi^o(x)$

where $\qquad \psi^i(x) = \theta(\chi_0 - \chi)\psi(x) \qquad \ldots .4.7$

and

$$\psi^o(x) = \theta(\chi - \chi_0)\psi(x) , \qquad \ldots .4.8$$

where θ is a step function and $\chi = \chi_0$ is the boundary of the potential well in some suitable coordinate system.

Then

$$(\Box_x + a^2)\psi^i(x) = \left\{\Box_x + a^2(\underline{x})\right\}\psi^i(x)$$
$$= \theta(\chi_0 - \chi)\left\{\Box_x + a^2(\underline{x})\right\}\psi(x) - \psi(x)\nabla^2\theta(\chi_0 - \chi)$$
$$-2\nabla\psi(x)\nabla\theta(\chi_0 - \chi)$$
$$= \theta(\chi_0 - \chi)\int K(x-x')\psi(x')\,dx' + \sigma(x)$$

---

*See appendix.

where

$$\sigma(x) = -\psi(x)\nabla^2\theta(\chi_0 - \chi) - 2\nabla\psi(x).\nabla\theta(\chi_0 - \chi) \qquad \text{.... 4.9}$$

$$\therefore (\Box_x + a^2)\psi^i(x) = \int K(x - x')\psi^i(x')\,dx' + T(x) + \sigma(x) \qquad \text{..4.10}$$

where

$$\begin{aligned} T(x) &= \theta(\chi_0 - \chi)\int K(x - x')\,\psi(x')\,dx' - \int K(x - x')\,\psi^i(x')\,dx' \\ &= \int K(x - x')\,\psi(x')\left\{\theta(\chi_0 - \chi) - \theta(\chi_0 - \chi')\right\}dx' \qquad \text{...4.11} \end{aligned}$$

Similarly

$$\begin{aligned} (\Box_x + m^2)\psi^o(x) &= \left\{\Box_x + a^2(\underline{x})\right\}\psi^o(x) \\ &= \theta(\chi - \chi_0)\left\{\Box_x + a^2(x)\right\}\psi(x) \\ &\quad -\psi(x)\nabla^2\,\theta(\chi - \chi_0) \\ &\quad -2\nabla\psi(x)\,.\nabla\,\theta\,(\chi - \chi_0) \\ &= \theta(\chi - \chi_0)\int K(x - x')\,\psi(x')\,dx' + \quad '(x) \end{aligned}$$

where

$$\begin{aligned} \sigma'(x) &= -\psi(x)\,\nabla^2\,\theta\,(\chi - \chi_0) - 2\nabla\psi(x).\nabla\theta(\chi - \chi_0) \\ &= -\sigma(x) \qquad \text{... 4.12} \end{aligned}$$

$$\therefore (\Box + m^2)\psi^o(x) = \int K(x - x')\psi^o(x')\,dx' + T'(x) - \sigma(x)$$

where

$$\begin{aligned} T'(x) &= \theta(\chi - \chi_0)\int K(x - x')\,dx' - \int K(x - x')\psi^o(x')\,dx' \\ &= \int K(x - x')\,\psi(x')\left\{\theta(\chi - \chi_0) - \theta(\chi' - \chi_0)\right\}dx' \\ &= -T(x) \end{aligned}$$

$$\begin{aligned} \therefore (\Box + a^2)\psi^i(x) - \int K(x - x')\psi^i(x')\,dx' \\ &= -(\Box + m^2)\psi^o(x) + \int K(x - x')\psi^o(x')\,dx' \\ &= T(x) + \sigma(x) \qquad \text{...4.13} \end{aligned}$$

The two functions $T(x)$ and $\sigma(x)$ are easily interpreted physically. Together they form either a source term for $\psi^i(x)$ which represents completely the effect of $\psi^o(x)$ on the space inside the well or a source term for $\psi^{o}(x)$ which represents completely the effect of $\psi^{i}(x)$ on the space outside the well. Separately the two terms have a simple interpretation too:

$$T(x) = \int K(x - x')\psi(x')\left\{\theta(\chi_0 - \chi) - \theta(\chi_0 - \chi')\right\}dx' \;.$$

The term in curly brackets is zero if χ and χ' are on opposite sides of the boundary but is 1 or -1 otherwise. If $\chi < \chi_0$ the term is 1 and $T(x)$ is the source due to the annihilation of particles at points χ' outside the boundary and their creation inside. If $\chi > \chi_0$ the term is -1 and $T(x)$ is the source due to the annihilation of particles inside the boundary and their creation outside.

$$\sigma(x) = \psi(x)\nabla^2\theta(\chi - \chi_0) + 2\nabla\psi(x).\nabla\theta(\chi - \chi_0)$$

which is a singular "function" at $\chi = \chi_0$ and zero at all other points of space. It is a source function representing the direct "transmission" of particles through the boundary.

To find the scattering cross section ψ^{o} must be known and the ratio of the scattered density to the incident density found. The form of ψ^{o} is known and the cross section can be calculated by finding $T(x)$ and $\sigma(x)$ from it, solving for ψ^{i} and using the condition that ψ^{i} is finite at the origin or vanishes on the core. The form of ψ^{o} varies according to the shape of the potential region so at this point the discussion is specialised to particular cases.

In the spherical case ψ^{o} must be asymptotically of the form of an incident plane wave plus an outgoing spherical wave, and as the $J = 0$ case is being considered an approximation to ψ^{o} is

$$\psi^{o}(x) = \psi(r)\ e^{-iEt}\ \theta(r - R)$$

where

$$\psi(r) = (\sin|\underline{k}|r)/r + e^{i\,\underline{k}\,r}/r\ , \qquad \ldots .\ 4.14$$

the well boundary is $r = R$, and α is a function of energy which will determine the cross sections. $|\underline{k}|$ is the magnitude of the momentum of the particle in a region where its rest mass is m so that

$$E^{2} - \underline{k}^{2} = m^{2} \qquad \ldots .\ 4.15$$

This is an approximate expression for ψ^{o} because, while it does not satisfy 4.12,

$$(\Box + m^{2})\,\psi^{o}(x) = -\sigma''(x)$$

where $\sigma''(x)$ may be identified with $\sigma(x)$ and the remaining term in 4.12

$$\int K(x - x')\,\psi^{o}(x')\,dx' - T(x)$$

is small.

With this approximation then,

$$\begin{aligned}\sigma(x) &= -(\Box + m^{2})\,\psi^{o}(x)\\ &= \psi(r)\,e^{-iEt}\,\nabla^{2}\theta(r - R) + 2\,\nabla\psi(r).\nabla\theta(r - R)\,e^{-iEt}\\ &= e^{-iEt}\left[2\,\delta(r - R)\left(\psi_{r}(r) + \tfrac{1}{r}\psi(r)\right) + \psi(r)\,\delta'(r - R)\right]\end{aligned}$$

From this expression for $\psi^{o}(x)$ the source functions $T(x)$ and $\sigma(x)$ can be found.

$$\begin{aligned}T(x) + \sigma(x) &= -(\Box + m^{2})\,\psi^{o}(x) + \int K(x - x')\,\psi^{o}(x')\,dx'\\ &= \psi(r)\,e^{-iEt}\,\nabla^{2}\theta(r - R) + 2\,\nabla\psi^{o}(r).\nabla\theta(r-R)\,e^{-iEt}\\ &\qquad - \int K(x - x')\,\psi(r')\,\theta(r' - R)\,e^{-iEt}\,dx'\ .\end{aligned}$$

$\sigma(x)$ is the singular part of the expression so that

$$\sigma(x) = e^{-iEt}\left[2\delta(r-R)(\psi_r(r) + \tfrac{1}{r}\psi(r)) + \psi(r)\delta'(r-R)\right]$$

and

$$\bar{\sigma}(k) = \delta(k_4 - E)\int\Big[2\delta(r-R)(\psi_r(r) + \tfrac{1}{r}\psi(r)) + \psi(r)\delta'(r-R)\Big]\, e^{-i\underline{k}.\underline{x}}\, d\underline{x}$$

$$= \delta(k_4 - E)\int_0^\infty \Big[2(\psi_r(r) + \tfrac{1}{r}\psi(r))\delta(r-R) + \psi(r)\delta'(r-R)\Big]\frac{\sin|\underline{k}|r}{|\underline{k}|}\, r\, dr$$

$$= \delta(k_4 - E)\left[\frac{\sin|\underline{k}|R}{|\underline{k}|}(R\psi_r(R) + \psi(R)) - R\psi(R)\cos|\underline{k}|R\right] \quad \ldots.\ 4.16$$

Equation 4.2 for $\psi^i(x)$ is first considered exactly without making the approximations which were made for $\psi^o(x)$ (i.e. neglecting $T(x)$ and $K(x - x')$). The reason for this is that the effect of these terms on $\psi^i(x)$ is much greater than their effect on $\psi^o(x)$. This is best seen physically by noticing that both terms represent the effect of the emission and absorbtion of mesons, processes which are not expected to affect $\psi^o(x)$ greatly, especially the emission of real mesons which is impossible outside the potential well. It will also be seen mathematically that the incorporation of another small term in $\psi^o(x)$, as would be the result of employing a perturbation method, would not greatly affect α and hence the cross sections.

Thus from 4.12

$$(-k^2 + a^2)\bar{\psi}^i(k) = \bar{K}(k)\bar{\psi}^i(k) + \bar{T}(k) + \bar{\sigma}(k)$$

so that

$$\bar{\psi}^i(k) = \frac{\bar{\sigma}(k)}{F(k^2)} + \frac{\bar{T}(k)}{F(k^2)} \quad \ldots.\ 4.17$$

where

$$F(k^2) = -k^2 + a^2 - \bar{K}(k) \quad . \quad \ldots.\ 4.18$$

In 4.16 put

$$\bar{\phi}(|\underline{k}|, R) = \frac{\sin|\underline{k}|R}{|\underline{k}|F(\underline{k}^2)} \qquad \ldots. 4.19$$

where

$$F(\underline{k}^2) = F(E^2 - |\underline{k}|^2) \quad ;$$

then

$$\phi(r,R) = \int \bar{\phi}(|\underline{k}|, R)\, e^{i\underline{k}.\underline{x}}\, d\underline{k}$$

$$= \int_0^\infty \frac{\sin|\underline{k}|R}{F(\underline{k}^2)|\underline{k}|} \frac{\sin|\underline{k}|r}{r} |\underline{k}|\, d|\underline{k}|$$

$$= \frac{1}{r}\int_0^\infty \frac{\sin|\underline{k}|R \sin|\underline{k}|r}{F(k^2)}\, d|\underline{k}| \quad .$$

$$\therefore \phi(r, R) = -\frac{1}{4r}\int_0^\infty \frac{1}{F(k^2)}\left\{ e^{i|\underline{k}|(r+R)} + e^{-i|\underline{k}|(r+R)} - e^{i|\underline{k}|(r-R)} - e^{-i|\underline{k}|(r-R)}\right\} d|\underline{k}|$$

$$= -\frac{1}{4r}\int_{-\infty}^\infty \frac{1}{F(k^2)}\left\{ e^{i|\underline{k}|(r+R)} - e^{-i|\underline{k}|(r-R)}\right\} d|\underline{k}| \qquad \ldots. 4.20$$

From 4.16 and 4.17 putting $R\psi(R) = \Phi(R)$

$$\bar{\psi}^i(k) = \delta(k_4 - E)\left[\bar{\phi}(|\underline{k}|,R)\,\Phi_R(R) - \bar{\phi}_R(|\underline{k}|, R)\,\Phi(R)\right] + \frac{\bar{T}(k)}{F(k^2)} \quad .$$

$$\therefore \psi^i(x) = e^{-iEt}\; \phi(r,R)\,\Phi_R(R) - \phi_R(r,R)\,\Phi(R) + \frac{T(k)}{F(k^2)}\, e^{-ik.x}\, dk$$

$$\ldots.. 4.21$$

This expression for $\psi^{i}(x)$ depends upon the way in which the singular integrals in $\emptyset(r,R)$ and $\int \bar{T}(k)/F(k^2)\, e^{-ik.x}\, dk$ are evaluated. As will be seen in the next chapter, it is possible to choose the contours so that $\psi^{i}(x)$ is approximately zero for $r > R$, $\int \bar{T}(k)/F(k^2)\, e^{-ik.x}\, dk$ being small.

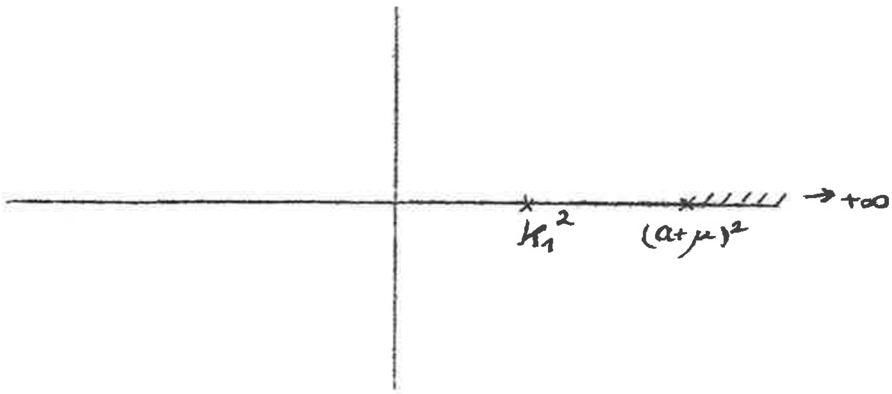
As is shown in the appendix, $F(k^2)$ has branch points at $k^2 = (a + \mu)^2$ and ∞ and has a zero at a point on the real line $k^2 = \chi_1^2$ say, such that $\chi_1^2 < (a + \mu)^2$. $F(k^2)$ has no other zeros, poles, or branch points and thus the only finite singular points of $1/F(k^2)$ are $k^2 = (a + \mu)^2$ and $k^2 = \chi_1^2$. $F(k^2) = (k^2 - \chi_1^2)\, R(k^2)$ such that $R(\chi_1^2) \neq 0$ so the pole of $1/F(k^2)$ is simple. The function $1/F(k^2)$ has a cut along the real axis between $k^2 = (a + \mu)^2$ and $k = +\infty$. (fig. 3)

Consider the $\underline{k}^2$ plane where $\underline{k}^2 = E^2 - k^2$. In this plane $1/F(k^2)$ has branch points at $E^2 - (a + \mu)^2$ and $-\infty$ and a pole at k_1^2 where $k_1^2 = E^2 - \chi_1^2$. There is a cut along the real axis between $E^2 - (a + \mu)^2$ and $-\infty$.

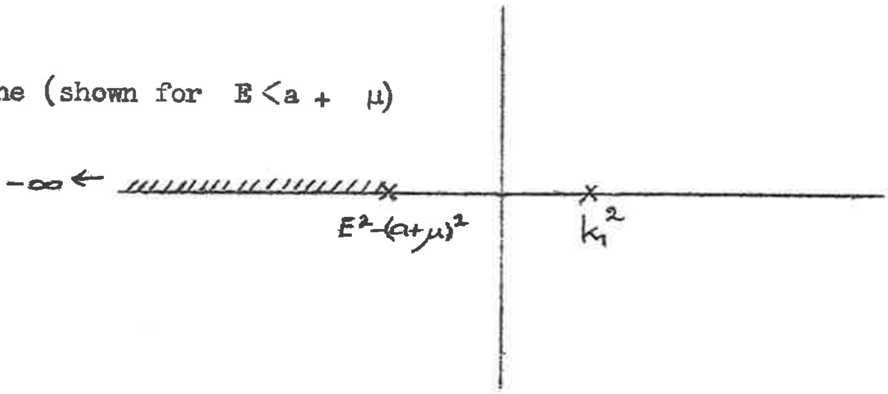
Consider the $|\underline{k}|$ plane. This plane may be divided into two halves, in each of which the whole range of the function is produced. The function has poles at $|\underline{k}| = \pm\sqrt{E^2 - \chi_1^2} = \pm k_1$ and branch points at $\pm i\infty$ and $\pm\sqrt{E^2 - (a + \mu)^2}$. The latter branch points are on the real line if $E > a + \mu$ but on the pure imaginary axis if $E < a + \mu$. That is, they are real if the energy is greater than the threshold energy but pure imaginary if it is less. If $E < a + \mu$ the cuts lie along the imaginary axis between $\pm i\sqrt{(a + \mu)^2 - E^2}$ and $\pm i\infty$, (fig. 4) but if $E > a + \mu$ they lie along the whole of the imaginary axis and the part of the real axis between 0 and $\pm\sqrt{E^2 - (a + \mu)^2}$. (fig. 5) In the latter case the rule of adding a small negative imaginary part to the mass of the particles to obtain causal solutions shows that the cuts do not "pinch"* and contours like those shown in fig. 5 are possible.

*See appendix.

(i) k^2 plane



(ii) $\underline{k}^2$ plane (shown for $E < a + \mu$)



(iii) $|\underline{k}|$ plane (shown for $E > a + \mu$)

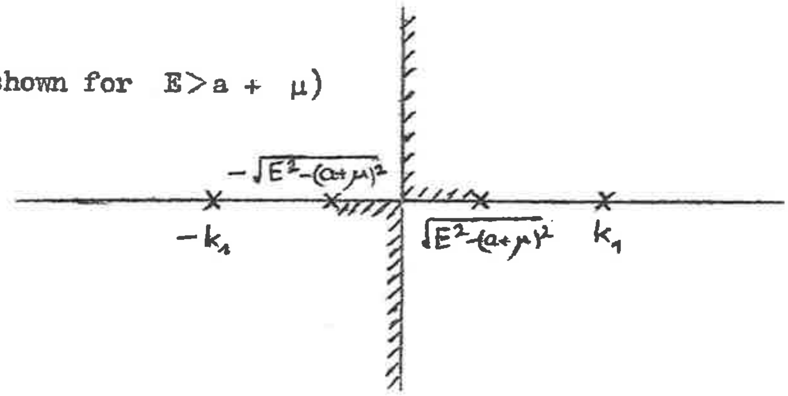


Figure 3.

The singular points and cuts of $1/F(k^2)$ on the k^2, $\underline{k}^2$, and $|\underline{k}|$ planes are shown.

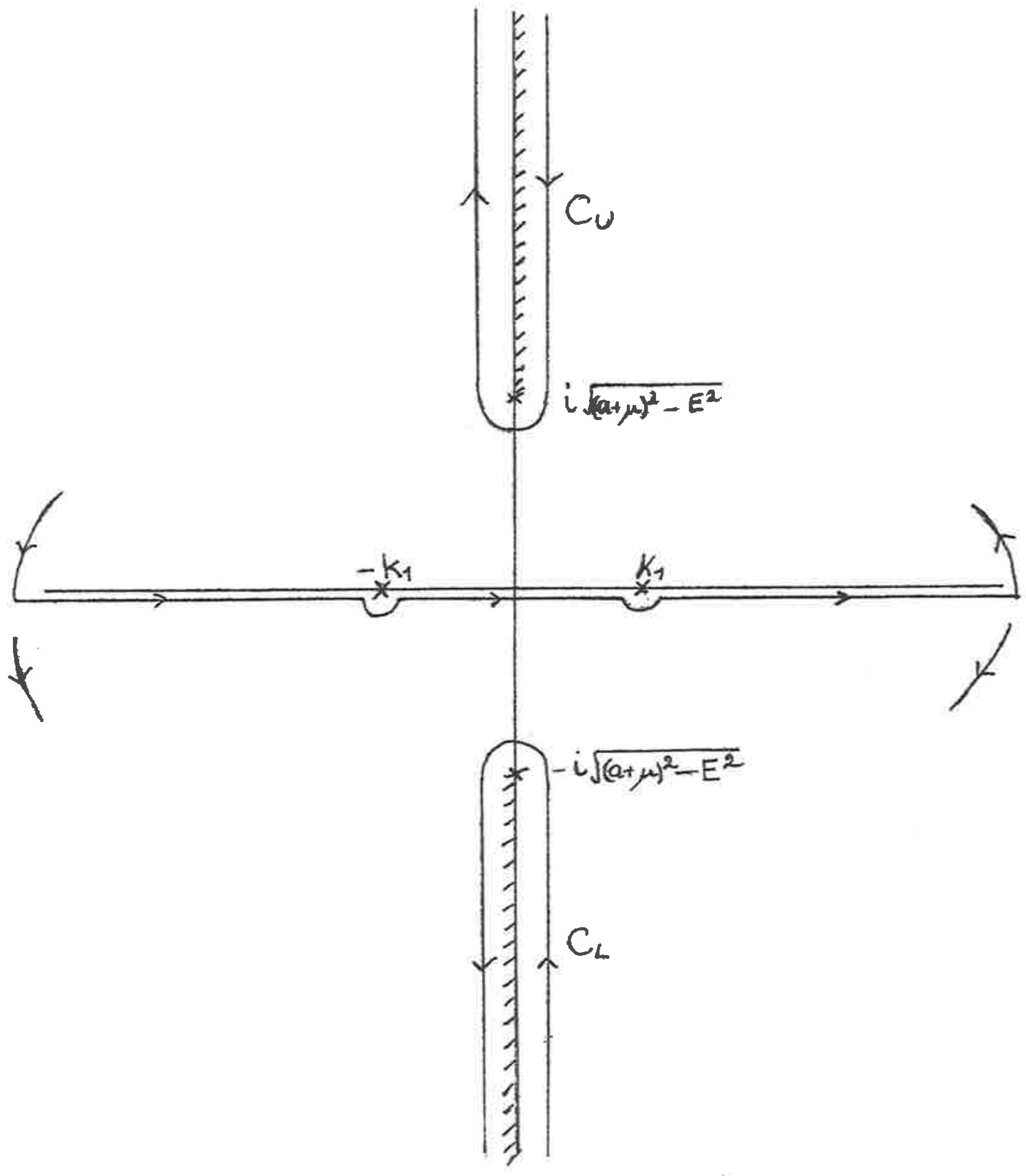


Figure 4.

Contours for $E < a + \mu$ are shown.

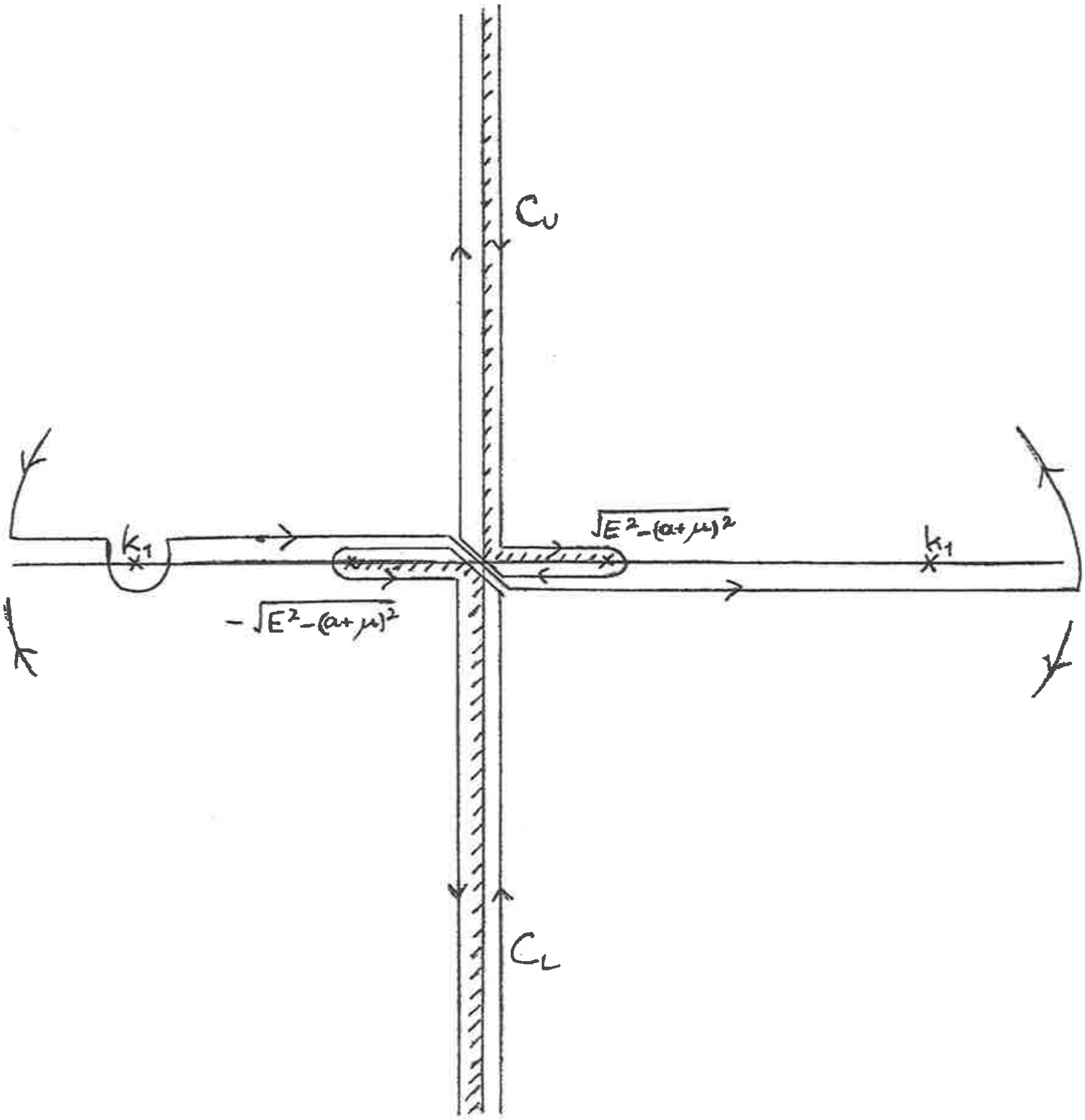


Figure 5.

Contours for $E > a + \mu$ are shown.

Chapter 5

PROPERTIES OF THE SOLUTION

In this chapter the difference between the solutions below and above the thresholds and for spheroidal and spherical wells will be discussed.

§1 The Solution Below the Threshold for a Spherical Well

From 4.20

$$\phi(r, R) = \frac{1}{4r}(I_2 - I_1)$$

where

$$I_1 = \frac{1}{F(\underline{k}^2)} \int_{-\infty}^{\infty} e^{-i|\underline{k}|(r+R)} \, d|\underline{k}|$$

and

$$I_2 = \frac{1}{F(\underline{k}^2)} \int_{-\infty}^{\infty} e^{-i|\underline{k}|(r-R)} \, d|\underline{k}| \quad .$$

I_1 and I_2 are evaluated by contour integration, completing an infinite semicircle in the lower half plane for I_1 since $r + R > 0$ always, and in the lower half plane for I_2 if $r > R$, but in the upper half plane for I_2 if $r < R$. The path along the real line may avoid the poles in any of four ways and the integral may be a combination of the results from these paths. The path which makes $\phi(r,R)$ approximately zero for $r > R$ is that shown in figure 4, for then the cases where the contour is completed in the lower half plane

$$I_1 = -\int_{C_L} \frac{1}{F(\underline{k}^2)} e^{-i|\underline{k}|(r+R)} \, d|\underline{k}|$$

and

$$I_2 = -\int_{C_L} \frac{1}{F(\underline{k}^2)} e^{-i|\underline{k}|(r-R)} \, d|\underline{k}| \quad ,$$

both of which are small, there being no poles within the contour.

$\therefore$ When $r<R$

$$\phi(r,R) = -\int_{C_L} \frac{1}{F(\underline{k}^2)} \frac{1}{2r} e^{-i\,\underline{k}\,r} \cos|\underline{k}|R \quad d|\underline{k}|$$

and

$$\psi^i(x) = e^{-iEt}\Big[\phi(r,R)\,\Phi_R(R) - \phi_R(r,R)\,\Phi(R)\Big] + \int \overline{T}(k) / F(k^2)\, e^{-i\,k.x}\, dk \qquad \ldots\ldots 5.1$$

By definition of $\psi^i(x)$ it is zero for $r>R$; it is not expected, however, that this will be exactly true for the $\psi^i(x)$ derived here because an approximation was made in obtaining $\sigma(x)$, $\psi^o(x)$ being assumed to be a function which only approximately satisfied 4.12. However, $\psi^i(x)$ given by 5.1 is expected to be approximately zero for $r>R$ which is true since $\phi(r,R)$ is small for $r>R$ and $\int \overline{T}(k)/F(k^2)\; e^{-i\,k.x}\, d\underline{k}$ can be seen to be small on both physical and mathematical grounds. Physically, the processes which it represents are expected to be of small consequence to either $\psi^i(x)$ or the cross sections. Mathematically,

$$\int \overline{T}(k)/F(k^2)\, e^{-ik.x}\, dk = e^{-iEt}\int \overline{K}(\underline{m}^2)\overline{\psi}(\underline{m})\Big\{\theta(r''-R)-\theta(r'-R)\Big\} 1/F(\underline{k}^2) e^{i\underline{m}.(\underline{x}''-\underline{x}')}\, e^{-i\underline{k}.(\underline{x}''-\underline{x})}\, d\underline{x}'\, d\underline{x}''\, d\underline{m}\, d\underline{k}$$

$$= \int_0^\infty d\,\underline{m} \int_0^\infty d\,\underline{k} \int d\underline{x}' \int d\underline{x}''\, \overline{K}(\underline{m}^2)\overline{\psi}(m)|\underline{m}||\underline{k}|\; 1/F(\underline{k}^2)$$

$$\Big\{\theta(r''-R) - \theta(r'-R)\Big\}$$

$$\sin|\underline{k}||\underline{x}''-\underline{x}|\, \sin|\underline{m}|\,|\underline{x}''-\underline{x}'|\Big\{1/(|\underline{x}''-\underline{x}||\underline{x}''-\underline{x}'|\;)\Big\}.$$

Change the variables of integration so that the integrations over $\underline{x}'$ and $\underline{x}''$ become $\int d(\underline{x}''-\underline{x})\int d(\underline{x}'-\underline{x}'')$ then the integral becomes

$$\int_0^\infty d|\underline{m}|\,d|\underline{k}|\,d|\underline{x}''-\underline{x}|\Big[\overline{K}(m^2)\overline{\psi}(m)|\underline{m}||\underline{k}|\Big\{\theta(r''-R)-\theta(r'-R)\Big\} \sin|\underline{k}||\underline{x}''-\underline{x}|\,\sin|m||\underline{x}''-\underline{x}'|\Big]\, d|\underline{x}'-\underline{x}''|$$

This integral may be transformed using contour integration methods by completing the contour from 0 to ∞ around an infinite quarter circle in either the upper or lower half plane so that the contribution from the curved part tends to zero. Then each integral from 0 to ∞ is equal to a sum of integrals from $\pm i\infty$ to 0 which are of the form $\int_0^\infty e^{-\alpha y}\,dy$ where α is positive. The only poles involved are those of $1/F(\underline{k}^2)$ which can be avoided and the only cuts are those of $1/F(\underline{k}^2)$ and $\bar{K}(\underline{m}^2)$. The cuts of $1/F(\underline{k}^2)$ are the same place on the $|\underline{k}|$ plane as the cuts of $\bar{K}(\underline{k}^2)$ so that the paths must be as shown in figure 6. Thus $\int \bar{T}(k)/F(k^2)\, e^{-ik.x}\,dk$ is small, a fact which justifies the approximate expression used for $\psi^o(x)$ in two ways. First it shows that this approximation does not significantly upset the property of $\psi^i(x)$ that it is zero for $r > R$ and second, it shows directly that the approximation amounts to the neglect of a small term, the neglected term being similar to $\int \bar{T}(k)/F(k^2)\, e^{-ik.x}\,dk$.

Approximately then,

$$\psi^i(x) = e^{-iEt}\left[\phi(r,R)\,\Phi_R(R) - \phi_R(r,R)\,\Phi(R)\right] \qquad \ldots 5.2$$

where

$$\phi(r,R) = \frac{1}{4r}(I_2 - I_1) \;.$$

When $r < R$, $\phi(r,R)$ changes its nature since the contour used in evaluating I_2 must be completed in the upper half plane thus including the poles of $1/F(\underline{k}^2)$ within the contour. Thus while

$$I_1 = -\int_{C_L} \frac{1}{F(\underline{k}^2)}\, e^{-i|\underline{k}|(r+R)}\, d|\underline{k}|$$

as before

$$I_2 = -\int_{C_U} \frac{1}{F(\underline{k}^2)}\, e^{-i|\underline{k}|(r-R)}\, d|\underline{k}| + 2\pi i(R_1 + R_2)$$

where R_1 and R_2 are the residues of $\frac{1}{F(\underline{k}^2)} e^{-i|\underline{k}|(r-R)}$ at the points $|\underline{k}| = \pm k_1$. That is

$$I_2 = -\int_{C_U} \frac{1}{F(\underline{k}^2)} e^{-i|\underline{k}|(r-R)} \, d|\underline{k}| + \frac{2\pi}{k_1 R(k_1{}^2)} \sin k_1(r-R) .$$

∴ For $r < R$

$$\phi(r,R) = \frac{1}{4r} \int_{C_U} \frac{1}{F(\underline{k}^2)} e^{-i|\underline{k}|(r-R)} \, d|\underline{k}| - \frac{1}{4r}\int_{C_L} \frac{1}{F(\underline{k}^2)} e^{-i|\underline{k}|(r+R)} d|\underline{k}$$

$$- \frac{1}{4r} \frac{2}{k_1 R(k_1{}^2)} \sin k_1 (r-R)$$

$$= \frac{1}{r} I_{C_U} - \frac{1}{2r} \frac{\pi}{k_1 R(k_1{}^2)} \sin k_1 (r-R)$$

where $I_{C_U} = -\frac{1}{4\pi} \int_{C_U} \frac{1}{F(\underline{k}^2)} e^{i|\underline{k}|R} \, 2i \sin |\underline{k}| r$.

The reason for the change in nature of $\phi(r,R)$ as r crosses the boundary $r = R$ is that the source function $\sigma(x)$ involved in $\phi(r,R)$ is singular on the boundary. The term $\int \bar{T}(k)/F(k^2) e^{-ik.x} dk$, however, involves no terms which are singular at $r = R$ and as can be seen by the above analysis of it it does not change its nature as r crosses the boundary. Thus where $r < R$ it is still small and $\psi^i(x)$ is still assumed to be given by 5.2.

The procedure from here is to use the condition that $\psi^i(x)$ must vanish on the core; i.e. where $r = \rho$, ρ being the core radius. This gives an expression for α from which the cross sections can be derived.

§2 The Solution Above the Threshold

There is no difference in the form of the solutions below and above the threshold. The solutions are different in nature, however, because the branch points of $1/F(k^2)$ are on the real line for $E > a + \mu$ and the contours C_L and C_U are those shown in figure 5. This has the consequence that I_{C_U} is no longer small, having contributions from integrals of the type $\int e^{iax}\,dx$ along the real axis. This is not the only way in which the movement of the branch points of $1/F(k^2)$ onto the real line affects the solution. The branch points of $F(\underline{k}^2)$ are also the branch points of $\overline{K}(\underline{k}^2)$ and the contours of figure 6 must be modified to those in figure 7 for the integrations over $|\underline{k}|$ and $|\underline{m}|$ in evaluating $\int \overline{T}(k)/F(k^2)e^{-ik.x}dk$ which is evaluated by integration around contours in the upper half plane and can no longer be neglected. Also the expression assumed for $\psi^{o}(x)$ will no longer be a good approximation. Thus new effects are introduced into the solution when $E > a + \mu$ but these will simply add a term to $\psi^{i}(x)$ which will be denoted by I. I is an integral around a cut on the complex plane, the significant part of which extends from the origin along the real line to $\sqrt{E^2 - (a + \mu)^2}$. The range of integration thus increases with E from zero when $E = a + \mu$.

The net result of these effects is that for energies above the threshold $\psi^{i}(x)$ changes its nature, the change being small at $E = a + \mu$ but increasing with energy above this. The effect of such a change on the cross sections will be examined in §4.

§3 The Spheroidal Well

In this section the model will be modified to have a spheroidal well instead of a spherical well. Green[7] has considered the problem of hard sphere scattering, both in a classical and wave mechanical context, where spheres are flattened into spheroids relative to one

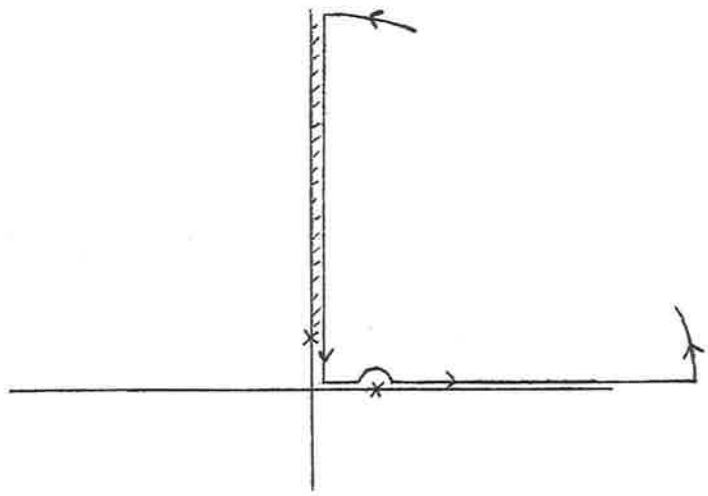


Figure 6.

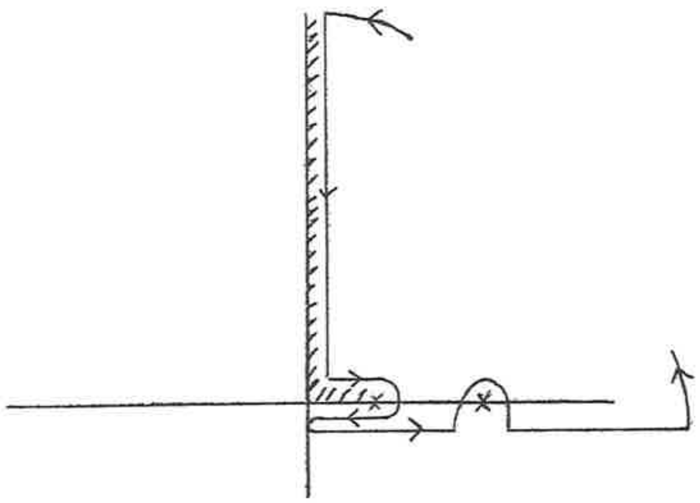


Figure 7.

another by the Fitzgerald contraction and concluded that the forward and backward peaking of the differential cross section for nucleon nucleon scattering at high energies is better explained by such a model than by a spherical model. It is therefore of interest to examine the consequences of assuming that a general potential region is flattened into a spheroid in the same way.

To consider the boundary of the potential region in the present model to be a spheroid would lead to straightforward calculations were it not for the fact that the hard core will also be flattened into a spheroid of the same eccentricity by the Fitzgerald contraction. The orthogonal spheroidal coordinate system* has surfaces of constant ξ as spheroids. If $\xi = \xi_0$ is the boundary of the well and this is a spheroid of eccentricity e then no other surface ξ = constant of the coordinate system has this same eccentricity. This means that the core cannot be represented by a surface ξ = constant. This difficulty is overcome in this case by the fact that if two spheroidal coordinate systems are chosen so that $\xi = \xi_b$ is the well boundary in one and $\xi' = \xi_c$ is the core boundary in the other then the spheroidal wave functions of the two systems are approximately orthogonal.* The difficulties would be more complex in the case of a well which was not square.

The calculation in the case of a spheroidal well follows similar lines to the spherical case. The notation and properties of spheroidal wave functions used in this chapter are explained in the appropriate section of the appendix.

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*See appendix.

Equation 4.12 is independent of coordinate system and well boundary. That is

$$(\Box + a^2)\,\psi^i(x) - \int K(x - x')\,\psi^i(x')\,dx' = T(x) + \mathcal{T}(x)$$

and

$$(\Box + m^2)\,\psi^o(x) - \int K(x - x')\,\psi^o(x')\,dx' = -T(x) - \mathcal{T}(x)\ .$$

In the calculation two systems of spheroidal coordinates will/ be used, (ξ, η), c and $(\bar{\xi}, \bar{\eta})$, q such that $\xi = \xi_b$ and $\bar{\xi} = \xi_c$ are the well and are boundaries respectively. Then

$$\psi^i(x) = \psi(x)\,\theta(\xi_b - \xi)$$

and

$$\psi^o(x) = \psi(x)\,\theta(\xi - \xi_b)\ .$$

With angle origins chosen suitably a plane wave with cylindrical symmetry may be expanded

$$e^{ik\xi\eta} = 2\sum_{\ell} i^{\ell}\, S_{\ell}(c, \eta)\, R_n(c, \xi)\, S_{\ell}(c, \cos\theta)\ ,$$

where

$$c = kd/2\ .$$

Note that ℓ in this expansion is not the angular momentum in the usual sense but for this simple model the $\ell = 0$ term only will be considered. The advantage of a spheroidal model is that $\ell = 0$ does not give a constant differential cross section but one which is peaked in the forward and backward directions. What is meant by a better model in this context is that lower order partial waves will give differential cross sections of the right shape.

Since the solution, $\psi(x)$, outside the well is to be of the form of a plane wave plus a wave which, at infinity, is an outgoing spherical wave, $\psi^o(x)$ is given by

$$\Psi^{o}(x) = \left\{ R_0(c,\xi) + \alpha\, R_0^*(c,\xi) \right\} S_0(c,\eta)\, e^{-iEt}\, \theta(\xi - \xi_b)$$

$$= \Psi(\xi)\, S_0(c,\eta)\, e^{-iEt}\, \theta(\xi - \xi_b) \quad \text{say} \quad . \qquad \ldots .5.3$$

Then

$$T(x) + \mathcal{T}(x) = -(\Box + m^2)\, \psi^{o}(x) + \int K(x - x')\, \psi^{o}(x')\, dx'$$

and

$$\mathcal{T}(x) = \left[\Psi(\xi) \nabla^2 \theta(\xi - \xi_b) + 2\nabla(\xi).\nabla\theta(\xi - \xi_b) \right] e^{-iEt} S_0(c,\eta) \quad .$$

$$\therefore\ \bar{\mathcal{T}}(k) = \delta(k_4 - E)\, \frac{4}{d^2} \int \left[\left\{ \frac{2\xi}{\xi^2 + \eta^2} \Psi(\xi) + \frac{2(1 + \xi^2)}{\xi^2 + \eta^2} \Psi_{\xi}(\xi) \right\} \delta(\xi - \xi_b) \right.$$

$$\left. + \frac{1 + \xi^2}{\xi^2 + \eta^2} \Psi(\xi)\, \delta'(\xi - \xi_b) \right] S_0(c,\eta)\, e^{-i\underline{k}.\underline{x}}\, d\underline{x} \quad \ldots 5.4$$

But

$$e^{-i\underline{k}.\underline{x}} = 2 \sum_{\ell} (-i)^{\ell}\, S_{\ell}(c, \cos\theta)\, S_{\ell}(c,\eta)\, R_{\ell}(c,\xi) \quad .$$

$$\therefore\ \bar{\mathcal{T}}(k) = \int_{-1}^{1} \int_0^{\infty} \mathcal{T}(x)\, e^{-ic\eta\xi} (\xi^2 + \eta^2)\, d\xi\, d\eta$$

$$= \delta(k_4 - E)\, \frac{8}{d^2} S_0(c, \cos\theta) \int_0^{\infty} d\xi \left[\left\{ 2\xi \Psi(\xi) \right.\right.$$

$$\left.\left. + 2(1 + \xi^2) \Psi_{\xi}(\xi) \right\} \delta(\xi - \xi_b) + (1 + \xi^2) \Psi(\xi) \delta'(\xi - \xi_b) \right] R_0(c,\xi)$$

$$= \delta(k_4 - E)\, \frac{8}{d^2} S_0(c, \cos\theta) \left[R_0(c,\xi_b) \Psi_{\xi}(\xi_b) - \Psi(\xi_b)\, R_{0\,\xi}(c,\xi_b) \right] (1 + \xi_b^2) \qquad \ldots .. \quad 5.5$$

Put

$$\bar{\phi}(c,\xi_b) = \frac{8}{d^2}\, S_0(c, \cos\theta)\, R_0(c,\xi_b)\, \frac{1}{F(\underline{k}^2)}$$

then

$$\phi(\xi,\xi_b) = \frac{8}{d^2} \int S_0(c, \cos\theta)\, R_0(c,\xi_b)\, \frac{1}{F(\underline{k}^2)}\, e^{i\underline{k}.\underline{x}}\, d\underline{k} \quad \ldots . \ 5.6$$

In finding the Fourier transform of σ the $e^{-i\underline{k}.x}$ was expanded in the (ξ, η), c coordinates to allow the integration to be carried out using the delta functions of σ. Here $\phi(\bar{\xi}, \xi_b)$ is to be found in terms of the $(\bar{\xi}, \bar{\eta})$, q coordinates as it is to be used to find Ψ^i which must be zero on the core boundary $\bar{\xi} = \xi_c$. In order that this condition may be applied easily $e^{i\underline{k}.\underline{x}}$ is here expanded in the $(\bar{\xi}, \bar{\eta})$, q coordinates.

$$\phi(\bar{\xi}, \xi_b) = \int_{-1}^{1} \int_{0}^{\infty} \frac{16}{d^2} S_0(c, \cos\theta)\, R_0(c, \xi_b) \frac{1}{F(\underline{k}^2)} S_0(q, \cos\theta)$$

$$S_0(q, \bar{\eta})\, R_0(q, \bar{\xi})\, \underline{k}^2\, d|\underline{k}|\, d(\cos\theta) \quad .$$

$$\simeq \int_0^{\infty} \frac{16}{d^2} R_0(c, \xi_b)\, R_0(q, \bar{\xi})\, S_0(q, \bar{\eta}) \frac{1}{F(\underline{k}^2)}\, \underline{k}^2\, d|\underline{k}|$$

$$= \frac{16}{d^2} \int_{-\infty}^{\infty} S_0(q, \bar{\eta}) \frac{1}{F(\underline{k}^2)}\, \underline{k}^2 \Big\{ R_0^-(c, \xi_b)\, R_0^-(q, \bar{\xi})$$

$$+ R_0^+(c, \xi_b)\, R_0^-(q, \bar{\xi}) \Big\}^* ,$$

and

$$\phi(\bar{\xi}, \xi_b) = \frac{16}{d^2} \int_{-\infty}^{\infty} S_0(q, \bar{\eta}) \frac{1}{F(k^2)}\, \underline{k}^2 \Big\{ R^-_{0\,\xi_b}(c, \xi_b)\, R_0^-(q, \bar{\xi})$$

$$+ R^+_{0\,\xi_b}(c, \xi_b)\, R_0^-(q, \bar{\xi})$$

since $S_\ell(c, \eta)$ and $S_m(q, \eta)$ are approximately orthonormal.

$$\bar{\Psi}^i(k) = \frac{\bar{J}(k)}{F(k^2)} + \frac{\bar{T}(k)}{F(k^2)}$$

$$\therefore \Psi^i(x) = e^{-iEt} (1 + \xi_b^2) \frac{8}{d^2} \Big[\phi(\bar{\xi}, \xi_b)\, \Psi_\xi(\xi_b)$$

$$- \phi_{\xi_b}(\bar{\xi}, \xi_b)\, \Psi(\xi_b) \Big] + \int \frac{\bar{T}(k)}{F(k^2)}\, e^{-ik.x}\, dk \quad \ldots 5.7$$

As in the spherical case the value of $\Psi^i(x)$ depends upon the way in which the singular integral for $\phi(\bar{\xi}, \xi_b)$ is evaluated.

*See equation 5.7a.

Note that

$$R_0(c,\xi) = \int_{-1}^{1} e^{ic\xi\eta}\, S_0(c,\eta)\, d\eta$$

$$= \int_{0}^{1} e^{ic\xi\eta}\, S_0(c,\eta)\, d\eta + \int_{0}^{1} e^{-ic\xi\eta}\, S_0(c,\eta)\, d\eta$$

$$= R_0^{+}(c,\xi) + R_0^{-}(c,\xi) \quad \text{say,} \qquad \ldots.\ 5.7a.$$

where $R_0^{+}(c,\xi) \rightarrow 0$ as $|\underline{k}| \rightarrow + i\infty$ and $R_0^{-}(c,\xi) \rightarrow 0$ as $k \rightarrow - i\infty$. Thus an integral involving R_0^{+} could have its contour completed as an infinite semicircle in the upper half plane and R_0^{-} in the lower half plane. This situation is complicated by the fact that there is a product of radial function in the integral to be evaluated.

Since the core and well boundary have the same eccentricity

$$\xi_b = \xi_c = \xi_0 \quad \text{say} \ .$$

Thus on the core the problem terms are those like

$$R_0^{+}(c,\xi_0)\, R_0^{-}(q,\xi_0) = \int_0^1 e^{ic\xi_0\eta}\, S_0(c,\eta)\, d\eta \int_0^1 e^{-iq\xi_0\eta}\, S_0(q,\eta) d\eta.$$

Since the angle functions achieve a sharp maximum at $\eta = 1$ this term is approximately equal to

$$e^{i\xi_0(c-q)} \left[\int_0^1 S_0(c,\eta)\, d\eta\right] \left[\int_0^1 S_0(q,\eta)\, d\eta\right]$$

and it is clear that the contour chosen to integrate it must be completed in the upper half plane.

It is also necessary to know the behaviour of $\phi(\xi,\xi_0)$ on the well boundary. This is more difficult as the well boundary is not given by $\bar{\xi}$ = constant. However as the scattering is strongly peaked in the $\eta = \pm 1$ directions the spheroid $\bar{\xi}$ = constant which coincides with $\xi = \xi_0$ at $\eta = \pm 1$ may be considered to approximate the well boundary. If $\bar{\xi} = \xi_1$ is such a surface then

$$q\xi_1 = c\xi_0 \ .$$

This means that a term like $R_0^+(c,\xi_b)\, R_0^-(q,\bar{\xi})$ must be completed in the lower half plane if $\bar{\xi}$ is outside the well boundary (i.e. when $\xi_1 q > \xi_0 c$) but in the upper half plane when $\bar{\xi}$ is inside the well boundary. The derivative with respect to ξ_b of $R_0^+(c,\xi_b)$ has the same behaviour as $R_0(c,\xi_b)$ as far as these contours are concerned.

The solution for $\psi^i(x)$ in the case of a spheroidal well is similar in form to that in the case of a spherical well, especially in its behaviour as energy crosses the threshold. The same three effects are introduced and these may be expected to have a similar influence on the scattering cross sections. The real differences between the spherical and spheroidal cases are in the differential cross sections of the partial waves which will be discussed in §5.

§4 The Threshold in the Cross Sections

The way in which the threshold affects the cross sections is clearly much the same for the spheroidal and spherical well cases so for this section the spherical case only will be considered.

For

$$\psi^o(x) = (1/r)\sin|k|r + (\alpha/r)\, e^{i|\underline{k}|r}$$

the following fractions are obtainable.

$$\frac{\text{particles elastically scattered}}{\text{particles incident}} = |2i\alpha|^2 = 4\alpha\bar{\alpha} ,$$

$$\frac{\text{outgoing particles}}{\text{ingoing particles}} = \left|\frac{\frac{1}{2i}+\alpha}{\frac{1}{2i}}\right|^2 = 1 + 4\alpha\bar{\alpha} - 4\,\mathrm{Im}(\alpha) ,$$

and $\therefore$

$$\frac{\text{particles absorbed}}{\text{ingoing particles}} = 4\,\mathrm{Im}(\alpha) - 4\alpha\bar{\alpha} .$$

If there is no inelastic scattering

$$\mathrm{Im}(\alpha) = \alpha\bar{\alpha}$$

and if there is inelastic scattering the fraction of particles either inelastically scattered (absorbed) or elastically scattered is

$$4\alpha\bar{\alpha} + 4\,\mathrm{Im}\,(\alpha) - 4\,\alpha\bar{\alpha} = 4\,\mathrm{Im}\,(\alpha) \quad .$$

Thus $4\,\mathrm{Im}(\alpha)$ is the fraction of particles scattered whether there is inelastic scattering or not.

From 5.1

$$\psi^{i}(x) = e^{-iEt}\left[\phi(r,R)\,\Phi_R(R) - \phi_R(r,R)\,\Phi\,(R)\right] + \int \bar{T}(k)/F(k^2)\ e^{-ik.x}\ dk \quad .$$

where

$$\phi(r,R) = \frac{1}{r}I_{C_U} - (C_1/k_1 r)\sin k_1(r - R) \quad ,$$

$$C_1 = \pi/2R(k_1{}^2)$$

and

$$I_{C_U} = -\frac{1}{4}\int_{C_U}\frac{1}{F(\underline{k}^2)}\ e^{i|\underline{k}|R}\ 2i\sin|\underline{k}|r\quad d|\underline{k}| \quad .$$

Putting $\int\left[\bar{T}(k)\,/\,F(k^2)\right]\ e^{-ik.x}\,dk = T\,e^{-iEt}$ the condition that $\psi^{i}(x)$ should vanish on the core boundary $r = \rho$ is

$$0 = \phi(\rho,R)\left[|\underline{k}|\cos|\underline{k}|R + i|\underline{k}|\alpha\ e^{i|\underline{k}|R}\right] - \phi_R(\rho,R)\left[\sin|\underline{k}|R + \alpha\ e^{i|\underline{k}|R}\right] + T$$

and $\therefore$

$$\alpha = \frac{\phi_R(\rho,R)\sin|\underline{k}|R - \phi(\rho,R)|\underline{k}|\cos|\underline{k}|R - T}{\left[i|\underline{k}|\,\phi(\rho,R) - \phi_R\,(\rho,R)\right]\ e^{i|\underline{k}|R}} \quad .$$

$$\phi_R(\rho, R) = \frac{1}{\rho} I_{RC_U} + (C_1/\rho) \cos k_1(\rho - R)$$

where

$$I_{RC_U} = \frac{1}{4} \int_{C_U} \frac{|\underline{k}|}{F(\underline{k}^2)} e^{i|\underline{k}|R} \, 2 \sin |\underline{k}| r \; d|\underline{k}| \quad .$$

Then

$$\alpha = -(A/B) \, e^{-i|\underline{k}|R}$$

where

$$A = \sin |\underline{k}| R \cos k_1(\rho - R) + (|k|/k_1) \cos|\underline{k}|R \sin k_1(\rho - R)$$
$$+ (1/C_1)(I_{RC_U} \sin |\underline{k}| R - |\underline{k}| I_{C_U} \cos |\underline{k}|R) - T$$

and

$$B = \cos k_1 (\rho - R) + i(|\underline{k}|/k_1) \sin k_1 (\rho - R)$$
$$- (1/C_1) (i|\underline{k}| I_{C_U} - I_{RC_U}) \quad .$$

Since T and I_{C_U} are approximately zero below the threshold, the expression for α there is

$$\alpha = \frac{\sin |\underline{k}|R \cos k_1(\rho - R) + (|\underline{k}|/k_1)\cos|\underline{k}|R \sin k_1(\rho - R)}{\cos k_1(\rho - R) + i(|\underline{k}|/k_1) \sin k_1(\rho - R)} e^{-i|\underline{k}|R}$$

which is exactly the expression obtained for α for a simple square well model, the radius and depth of the well being R and $m - k_1$ respectively, with a hard core of radius ρ . Below the threshold no inelastic scattering is expected; it is easily checked that this is predicted by the value for α given above since

$$\mathrm{Im}(\alpha) = \alpha \bar{\alpha} \quad .$$

Thus below the threshold the cross section curve will be just that obtained for this simple model; i.e. it will decrease with increase of energy and then level out to an almost constant value. This can be seen from the expression for α above for when $|\underline{k}|$ is small

$$\mathrm{Im}(\alpha) \doteq \sin^2(|\underline{k}|\,R)$$

and the total cross section is proportional to $\sin^2(|\underline{k}|R)/\underline{k}^2$ which is approximately equal to R^2.

When $|\underline{k}|$ is large enough for $|\underline{k}|$ to be approximately equal to k_1

$$\alpha \doteq \frac{\sin(|\underline{k}|R + |\underline{k}|\rho - |\underline{k}|R)}{1} e^{-i|\underline{k}|R} e^{-i|\underline{k}|(\rho - R)}$$

$$= \sin|\underline{k}|\rho \quad e^{-i|\underline{k}|\rho}$$

and

$$\mathrm{Im}(\alpha) \doteq \sin^2(|\underline{k}|\,\rho)$$

so that the total cross section is proportional to

$$\sin^2(|\underline{k}|\,\rho)/\underline{k}^2 \quad .$$

At the threshold it will certainly be the case that $|\underline{k}| \doteq k_1$ and it may be assumed that

$$A = \sin|\underline{k}|\rho + (1/C_1)(I_{RC_U}\sin|\underline{k}|R - |\underline{k}|\,I_{C_U}\cos|\underline{k}|R) - T$$

and

$$B = e^{i|\underline{k}|(\rho - R)} - (1/C_1)(i|\underline{k}|\;I_{C_U} - I_{RC_U})$$

at the threshold and above. Above the threshold I_{C_U}, I_{RC_U}, and T are not negligible and it is no longer the case that

$$\mathrm{Im}(\alpha) = \alpha\bar{\alpha} \quad .$$

This has the expected physical interpretation that inelastic scattering is now occurring since the total scattering cross section is not equal to the elastic scattering cross section.

It is of interest to calculate α approximately to see how $\mathrm{Im}(\alpha) - \alpha\bar{\alpha}$ behaves above the threshold.

First consider I_{C_U}

$$I_{C_U} = -\tfrac{1}{4}\int_{C_U} \frac{1}{F(\underline{k}^2)}\, e^{i|\underline{k}|R}\, 2i \sin|\underline{k}|\rho \quad d|\underline{k}|$$

$$\simeq -\tfrac{1}{4}\int_0^{\sqrt{E^2-(a+\mu)^2}} 2i\sin|\underline{k}|\rho\; e^{i|\underline{k}|R}\, 2i\,\mathrm{Im}\left(\frac{1}{F(k^2)}\right) d|k|$$

$$= \int_0^{\sqrt{E^2-(a+\mu)^2}} \sin|\underline{k}|\rho\; e^{i|\underline{k}|R}\; \mathrm{Im}\left(\frac{1}{F(k^2)}\right) d|\underline{k}| \quad .$$

From A.5 $\mathrm{Im}\left(\frac{1}{F(\underline{k}^2)}\right)$ is positive if β is chosen to be negative and from values derived from the Fortran programme given in the appendix it varies almost linearly with $|\underline{k}|$, at least for small $|\underline{k}|$. The quantity $\mathrm{Im}(\alpha) - \alpha\bar{\alpha}$ would be very complicated to evaluate exactly for all values of E and this simple model would not justify such calculations. However it is easy to calculate for values of E near to $a + \mu$. In this case $|\underline{k}|$ is small for the whole range of the integral. Supposing that $\mathrm{Im}\left(\frac{1}{F(k^2)}\right) = \nu\,|\underline{k}|$ and that E is near to $a + \mu$

$$I_{C_U} = \int_0^{\sqrt{E^2-(a+\mu)^2}} |\underline{k}|\rho\,\nu\,|\underline{k}| \quad d|\underline{k}|$$

$$= \tfrac{1}{3}\rho\nu\left[E^2 - (a+\mu)^2\right]^{3/2}$$

and

$$I_{RC_U} = i\int_0^{\sqrt{E^2-(a+\mu)^2}} |\underline{k}|^3 \rho\nu \; d|\underline{k}|$$

$$= \frac{i}{4} \rho\nu \left[E^2 - (a+\mu)^2\right]^2$$

From the expression for T on p.44, since $\bar{K}(\underline{m}^2)$ involves the same sort of imaginary part as $1/F(\underline{k}^2)$, it is clear that T will be of the order

$$\left[E^2 - (a+\mu)^2\right]^3$$

and is thus neglected. I_{RC_U} is included along with I_{C_U} as all the terms in $\mathrm{Im}(\alpha) - \alpha\bar{\alpha}$ involving I_{C_U} cancel out.

$$\alpha = -(A/B)\, e^{-i\,\underline{k}\,R}$$

where now

$$A = \sin|\underline{k}|\rho - (|\underline{k}|/C_1)\, I_{C_U} \cos|\underline{k}|R + (1/C_1)\, I_{RC_U} \sin|\underline{k}|R$$

$$= \sin|\underline{k}|\rho - I_1 \cos|\underline{k}|R + iI_2 \sin|\underline{k}|R \quad \text{say}$$

and

$$B = e^{i|\underline{k}|(\rho - R)} - i\, I_1 + i\, I_2 \;;$$

$$I_1 = (|\underline{k}|/C_1)\, I_{C_U}$$

and

$$I_2 = (1/C_1)\, I_{RC_U} \quad .$$

$$C_1 = \pi/2R(k_1^2)$$

$$\simeq \pi/2R(\underline{k}^2)$$

$$= \pi(E^2 - \underline{k}^2 - k_1^2)/2F(\underline{k}^2)$$

which is real and slowly varying at the threshold. Near the threshold, then, C_1 will be regarded as real even though it will have a small imaginary part just above the threshold.

$$\text{Im}(\alpha) = \text{Re}(A/B)\sin|\underline{k}|R - \text{Im}(A/B)\cos|\underline{k}|R$$

and

$$\alpha\bar{\alpha} = A\bar{A}/B\bar{B} \quad .$$

$$\text{Re}(A/B) = \Big[(\sin|\underline{k}|\rho - I_1\cos|\underline{k}|R)\cos|\underline{k}|(\rho - R) + I_2\sin|\underline{k}|R(\sin|\underline{k}|(\rho - R) - I_1 + I_2)\Big] / B\bar{B}$$

and

$$\text{Im}(A/B) = \Big[I_2\sin|\underline{k}|R\cos|\underline{k}|(\rho - R) - (\sin|\underline{k}|\rho - I_1\cos|\underline{k}|R)(\sin|\underline{k}|(\rho - R) - I_1 + I_2)\Big] / B\bar{B} \quad .$$

$\therefore$ neglecting I_1^2, I_2^2, and $I_1 I_2$

$$\text{Im}(\alpha) - \alpha\bar{\alpha} = \Big[\sin^2|\underline{k}|\rho - 2I_1\sin|\underline{k}|\rho\cos|\underline{k}|R + I_2\sin|\underline{k}|(\rho - R) - \sin^2|\underline{k}|\rho + 2I_1\sin|\underline{k}|\rho\cos|\underline{k}|R\Big] / B\bar{B}$$

$$= I_2\sin|\underline{k}|(\rho - R) / B\bar{B} \quad .$$

$$I_2 = (i/4C_1)\,\rho^{\nu}\left[E^2 - (a + \mu)^2\right]^2 \quad .$$

C_1 is shown to be negative by calculation of $F(k^2)$ using the programme given in the appendix and $R > \rho$.

$$\therefore \text{Im}(\alpha) - \alpha\bar{\alpha} > 0$$

and varies with energy like $\left[E^2 - (a + \mu)^2\right]^2$.

The cross section is obtained from $\text{Im}(\alpha)$ by dividing by $\underline{k}^2$ so that the inelastic scattering cross section varies like

$$\left[E^2 - (a + \mu)^2\right]^2 / \left[E^2 - m^2\right]$$

near the threshold. That is, the cross section curve is of quadratic form with a minimum of zero at the threshold. This is in accord with the experimental data available.

§5 The Differential Cross Section

For a spherical potential well the differential cross section in the $\ell = 0$ case calculated is a constant. Calculations have been made using a spheroidal potential well to make the model more realistic at relativistic energies and in the hope of obtaining a differential cross section having the desired peaking in forward and backward directions. This, of course, will not be achieved using the $\ell = 0$ solution given here but it is clearly possible to solve the equations for ℓ equal to other values in the same way. The $\ell = 0$ case gives an indication of the type of result expected but it is not considered worth making a detailed calculation for such a simple model as a square well model.

In the $\ell = 0$ case the differential cross section is $S_0(c,\eta)$ which in contrast to $P_0(\cos\theta)$ is not constant but peaked in the forward and backward direction*. ($S_0(c,\eta)$ is an even function of η.) If the well radius is taken to be the Compton wave length on the meson then $c = |\underline{k}|R$ varies from 0 to approximately 3 at the meson emission threshold. As c increases the peaking of $S_0(c,\eta)$ increases, similar behaviour being found for $S_n(c,\eta)$ for other values of n.

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*See appendix.

Thus increasing energy increases the forward and backward peaking in each partial wave separately.

SINGULARITIES AND THRESHOLDS

This chapter will be a general discussion of the nature of the solution, $\psi(x)$, of the field equations, found in Chapter 4 with particular reference to the influence of the singularities of $1/F(k^2)$ on the results.

§1 Poles of $1/F(k^2)$

The basic part of the solution, $\psi(x)$, inside the potential well is a term like $e^{-iEt}\frac{\sin k_1 r}{r}$ where $\kappa_1^2 = E^2 - k_1^2$ and κ_1^2 is the pole of $1/F(k^2)$. Thus it is a plane wave of momentum k_1 so that inside the well the nucleon behaves like a particle of rest mass κ_1. This value κ_1 is not equal to a, a fact which shows that the effective well depth is altered by the presence of self energy processes involving one meson and suggests that the original model representing all self energy interactions by a potential well was a good one.

The basic plane wave form of the solution inside the well is to be expected, indicating that poles of $1/F(k^2)$ at $k^2 = \kappa_1^2$ for κ_1 real are essential to a reasonable theory. This being so, the assertion made in Chapter 2 that g' is pure imaginary, thus allowing the Lagrangian to be Hermitian and have a Euclidean metric space time interval between the nucleon and emitted meson, is borne out. This is because $1/F(k^2)$ has no real poles if g' is not pure imaginary.

The optical model suggests that a complex pole of $1/F(k^2)$ could be interpreted to give a plane wave with complex momentum representing inelastic scattering or emission of mesons. This representation of inelastic scattering, however, is not realistic for a model which is to hold for all energies; such a complex pole would predict inelastic processes at all energies, an unrealistic situation as inelastic processes will be impossible below their energy threshold. The model of this thesis does not represent inelastic processes in this way but by the presence of a cut in the complex $|\underline{k}|$ plane of the function $1/F(k^2)$. This will now be discussed.

§2 Branch Points and Cuts of $1/F(k^2)$.

Apart from the part of the solution representing a plane wave inside the well there is term which is very small below the energy threshold for meson emission but becomes significant above the threshold and increases with energy increase. This term, then, is easily interpreted to represent the inelastic scattering processes. It is due to a cut in the complex $|\underline{k}|$ plane of the function $1/F(k^2)$ from $|\underline{k}| = \sqrt{E^2-(a+\mu)^2}$ to $i\infty$. For ease of evaluation of the integral around the cut it is drawn along the imaginary axis and real axis if necessary. It is the integral along the part of the cut on the real axis which is significant and this exists only if $E > a + \mu$. Thus the presence of a square root branch point at $k^2 =$ "square of threshold energy" and infinity is responsible for the prediction of inelastic scattering.

Apart from the poles and branch points mentioned, other singularities of $1/F(k^2)$ would be difficult to interpret so that it appears that a function like the $1/F(k^2)$ produced by the present model in its singularity is of the only suitable type for a model which is to predict interactions involving one meson.

§3 Thresholds and Anomalous Thresholds

It is well recognised that poles of the transition amplitude between one state of particles to another is associated with the existence of stable particles if they are on the physical sheet and it is conjectured that they are associated with unstable particles if they are on other sheets. The poles on unphysical sheets may migrate onto the physical sheet as parameters of the theory vary and to do this they must "come up" through a cut on the physical sheet. Thus the existence of unstable particles and hence of inelastic scattering may be expected to be associated with cuts of the physical sheet of the transition amplitude. The situation where two particles interact producing two other particles has been discussed in detail by Blanckenbecher et al. and by Mandelstam, while more general discussions have been given by Polkinghorne and Landshoff.

In the present model the existence of stable particles of effective rest mass $\mathcal{K}_1$ is demonstrated by the presence of a pole at $\mathcal{K} = \mathcal{K}_1$ in the complex $\mathcal{K}$ plane of $1/F(k^2)$, and the existence of a threshold causing the combination of nucleon plus meson to be unstable, is demonstrated by the presence of a branch point at $(a+\mu)^2$ and a cut from this point to $i\infty$. The work of Mandelstam is particularly interesting here in view of the fact that an expression derived by him for the imaginary part of the transition amplitude is similar in nature to part of $F(k^2)$. That is, it is of the form

$$\sqrt{A}\ \log\left\{(B+\sqrt{A})\,/\,(B-\sqrt{A})\right\} \qquad \ldots\ 5.1$$

where the branches of the log are taken as for $F(k^2)$ in this work.*

*See appendix for detailed nature of $F(k^2)$.

The properties of $F(k^2)$ for which this expression is responsible may be interpreted physically in terms of the meson emission threshold and an anomalous threshold.

Consider $F(k^2)$ for k real and such that

$$a - \mu < k < a + \mu \ ;$$

then the expression

$$\sqrt{\left\{k^2 - (a - \mu)^2\right\}\left\{k^2 - (a + \mu)^2\right\}} \equiv A$$

is real and the argument of the log in 5.1 is real. At first sight 5.1 may be expected to have square root branch points at the zeros of A ; that is at $k = a \pm \mu$. This however is not the case because 5.1 is single valued if the principal log function is taken.

$$\left[-\sqrt{A} \log\left\{(B - \sqrt{A})(B + \sqrt{A})\right\} = \sqrt{A} \log\left\{(B + \sqrt{A}) \, / \, (B - \sqrt{A})\right\} \right]$$

However, in $F(k^2)$ the log function takes principal values only for $k^2 < a^2 + \mu^2$. Therefore, at the zero of A , $k = a - \mu$, the log is a principal function and $F(k^2)$ has no branch point. At the zero of A , $k = a + \mu$, however, the log is not a principal function and $F(k^2)$ has a term equal to $2 \ i\sqrt{\left\{k^2 -(a + \mu)^2\right\}\left\{k^2 -(a - \mu)^2\right\}}$. This term has a branch point at $k = a + \mu$ so that $F(k^2)$ has a branch point there but not at $k = a - \mu$.

The point $k^2 = a^2 + \mu^2$ is an anomalous threshold. As k moves along the real line from $a - \mu$ towards $a + \mu$, if it is such that $k^2 < a^2 + \mu^2$ the functional form of $F(k^2)$ suggests that it has no branch points. This corresponds to the physical idea that when the energy of a nucleon is less than $\sqrt{a^2 + \mu^2}$ the nucleon is not recognisable as a composite nucleon plus meson. If k is such that $k^2 > a^2 + \mu^2$ the functional form of $F(k^2)$ makes it obvious that there is a branch point at $k = a + \mu$ corresponding to the physical idea that when the energy of a nucleon is greater than $\sqrt{a^2 + \mu^2}$ it is recognised as a stable combination of nucleon plus meson. When $E > a + \mu$ the combination becomes unstable and the nucleon may emit a meson.

§4 Extension to a Many Meson Theory

By anology with the present model it may be expected that in a model where many mesons were able to be emitted by a nucleon an equation like

$$\overline{\psi}^{\,i}(k)\, G(k^2) = \overline{\sigma}\,(k)$$

may arise where $G(k^2)$ has two poles between $k = a \pm \mu$ and square root branch points at $k = a + n\mu$, $n = 1, 2, \ldots$. In this case the cuts on the complex k plane would presumably join these points in pairs and the effect on the wave function inside the well would be to introduce an integral with respect to $|\underline{k}|$ around a cut from the origin to $\sqrt{E^2 - (a+\mu)^2}$ on the real line when energy reached the threshold for one meson emission, to change the range to $\sqrt{E^2 - (a + 2\mu)^2}$ to $\sqrt{E^2 - (a+\mu)^2}$ when energy reached the threshold for two meson emission, etc. It is possible that such a model would produce cross section curves of the right kind through these thresholds.

APPENDIX

§1 Notation and some Mathematical Results

Throughout this thesis integrations will be over the whole of the appropriate space unless limits are present. The space of integration will be indicated by the variables of integration.

The symbols x, y, k, ℓ, m, n, and p denote four vectors, and $k^2 = k_4^{\,2} - \underline{k}^2$ where $\underline{k}$ represents the space components of the four vector k. The symbols $\underline{x}$, $\underline{y}$, $\underline{k}$, $\underline{\ell}$, $\underline{m}$, $\underline{n}$, and $\underline{p}$ denote three vectors.

Mathematical results which will be used later in the appendix are as follows:

(a) $$\frac{1}{D_1 D_2 D_3} = 2\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \left\{ \frac{1}{\left[\alpha D_1 + \beta D_2 + (1-\alpha-\beta) D_3\right]^3} \right\} \quad \ldots A.$$

(b) $$\int \frac{dn}{(n^2 - A^2)^3} = \int\int_{-\infty}^{\infty} \frac{dn_4}{(n_4^{\,2} - \underline{n}^2 - A^2)^3} \; d\underline{n}$$

$$= \frac{3}{4}\, i\pi \frac{d\underline{n}}{(\underline{n}^2 + A^2)^{5/2}}$$

$$= \frac{\pi i}{4A^2} \quad \ldots A.$$

(c) $$\int_0^1 d\alpha \log(\alpha z - \alpha_i) = \alpha \log(\alpha z - \alpha_i)\Big|_0^1 - \int_0^1 d\alpha \frac{\alpha z}{\alpha z - \alpha_i}$$

$$= \log(z - \alpha_i) - \int_0^1 d\alpha \left\{ 1 + \frac{\alpha_i}{\alpha z - \alpha_i} \right\}$$

$$= \log(z - \alpha_i) - 1 - \frac{\alpha_i}{z} \log(\alpha z - \alpha_i)\Big|_0^1$$

$$= \log(z - \alpha_i) - 1 - \frac{\alpha_i}{z}\log(z - \alpha_i) + \frac{\alpha_i}{z}\log(-\alpha_i)$$

$$= \left(1 - \frac{\alpha_i}{z}\right)\log(z - \alpha_i) + \frac{\alpha_i}{z}\log(-\alpha_i) - 1 \quad . \qquad \ldots A.3$$

§2 The substitution of w_a, w_a', etc. into their equations. (P.20 of Chapter 2).

Only a sketch will be given of the substitutions as the detail adds nothing to the clarity of the work and the substitutions are straightforward.

Noting that $N - 2iE\nabla_4 = \left(\frac{\partial}{\partial t} - iE\right)^2 + T^2$ as $N = \Box + M^2 - E^2$, $S^2 = M^2 - E^2 - \nabla^2$, and $T^2 = M^2 - \nabla^2$, w_a^+ and w_a^- may be substituted into their equations. If they are to be solutions then by taking the regular parts of the equation the following relation must hold between the α s.

$$(2e_1 T \operatorname{sgn} x_4 + e_1^2)(\alpha_1 + \alpha_2 \operatorname{sgn} x_4)\, e^{iT|x_4|}\, e^{i(E + e_1)x_4} = g\Omega_{1a}^+$$

A similar relation is obtained from the equation for w_a^-; it is the same with $-x_4$ for x_4.

The singular parts of the two equations for w_a and w_a' each give separate relations,

$$2ES\alpha_2 + E\alpha - \gamma_4\gamma\cdot\nabla\beta = 0 \quad ,$$

and

$$(T-V)\alpha_1 + (E + e_1)\alpha_2 = 0 \quad .$$

Similar substitutions in equations for Ω_a^+, Ω_a^-, Ω_a, and Ω_a' yield

$$\int (2e_1 T \operatorname{sgn} x_4 + e_1^2)(A_1 + A_2 \operatorname{sgn} x_4)\, e^{iq(y - x_1)}\, dq$$
$$= \frac{g}{K}(x_1 - y)(e_1^2 + 4ET \operatorname{sgn} x_4 + 4Ee_1 + 4E^2$$
$$+ 2e_1 T \operatorname{sgn} x_4)(\alpha_1 + \alpha_2 \operatorname{sgn} x_4) \, ,$$
$$A_2 + A_4 = 0 \quad ,$$

and

$$(T-V)\ A_1 + (E + e_1)\ A_2 + (S-V)\ A_3 = 0 \quad .$$

There is also a relation which comes from the regular coefficients of $e^{iS\ x_4}$ in either the equation for $\Omega_a^{\ +}$ or $\Omega_a^{\ -}$.

$$(2ES\ \mathrm{sgn}\ x_4 - q_4\ S\ \mathrm{sgn}\ x_4 + q_4^{\ 2}/2)(A_3 + A_4\ \mathrm{sgn}\ x_4)$$
$$= \int \left\{ E(A + A'\ \mathrm{sgn}\ x_4) - \gamma_4 \gamma . \nabla (B + B'\ \mathrm{sgn}\ x_4) \right\} e^{iq(y - x_1)}\ dq$$
$$+ \frac{g}{K} \delta (x_1 - y) \left[- \frac{S}{2E}\ \mathrm{sgn}\ x_4\ (E\alpha - \gamma_4 \gamma . \nabla \beta) - \mathrm{sgn}\ x_4\ (E\alpha - \gamma_4 \gamma . \nabla \beta) \right.$$

This relation serves only to give values for A_3 and A_4 .

Similar relations can clearly be found from the equations for w_b etc. but these will turn out to be not of much interest as they serve only to find unwanted quantities. As is indicated in Chapter 2 the equations are consistent and give differential equations in the space variables for the unknown quantities.

§3 Mass Renormalisation

From equation 4.5

$$\left\{ \Box_x + a^2(\underline{x}) \right\} \psi(x) = \frac{gg^*}{g'} \int \left\{ \psi (x')H(x - x')\ G(x-x') \right.$$
$$- \psi(x)H(x - x')G_0(x - x')\ \ dx' + R\psi(x) \quad .$$

The Fourier transform of the R.H.S. is $\frac{gg^*}{g'} (I_1 - I_2) + R\overline{\psi}(k)$

where

$$I_1 = \iint \psi (x')H(x - x')\ G(x - x')\ e^{ik.x}\ dx\ dx'$$
$$= \iiiint\!\!\int \overline{\psi}(m)\ \overline{H}(n)\ \overline{G}(p) e^{-im.x'}\ e^{-in.(x-x')}\ e^{-ip.(x-x')}\ e^{ik.x}\ dx\ dx'\ dm\ dn\ dp$$
$$= \iiiint\!\!\int \overline{\psi}\ (m)\ \overline{H}(n)\ \overline{G}(p) e^{ix.(k-p-n)}\ e^{ix'(p+n-m)}\ dx\ dx'\ dm\ dn\ dp$$
$$= \iiint \overline{\psi}\ (m)\ \overline{H}(n)\ \overline{G}(k - n)\ e^{ix'.(k-m)}\ dx'\ dm\ dn$$
$$= \int \overline{\psi}(k)\ \overline{H}(n)\ \overline{G}\ (k - n)\ \ dn \quad ,$$

and

$$I_2 = \iiiint\!\!\int \overline{\Psi}(m)\overline{H}(n)\,\overline{G}_0(p)\, e^{ix.(k-p-m-n)}\, e^{ix'.(p+n)} dx\,dx'\,dm\,dn\,dp$$

$$= \int \overline{\Psi}(k)\,\overline{H}(n)\,\overline{G}_0(-n)\,dn \quad .$$

$\therefore$ The Fourier transform of $\left\{\Box_x + a^2(x)\right\}\Psi(x)$

$$= \overline{\Psi}(k)\left[\frac{gg^*}{g'}\int \overline{H}(n)\left\{\overline{G}(k-n) - \overline{G}_0(-n)\right\}dn + R\right] .$$

Now when $a = m$ $\quad (\Box + m^2)\Psi(x) = 0$ and $\Psi(x) = e^{\pm i\ell.x}$

where $\ell^2 = m^2$ (m = nucleon mass)

$$\therefore \overline{\Psi}(k) = \delta(k \pm \ell) \quad \text{when} \quad a = m .$$

$$\therefore \quad 0 = \left[\frac{gg^*}{g'}\int \overline{H}(n)\left\{\overline{G}_0(k-n) - \overline{G}_0(-n)\right\}dn + R\right]\delta(k \pm \ell) .$$

$$\therefore \quad 0 = \frac{gg^*}{g'}\ \overline{H}(n)\ \overline{G}_0(\ell - n) - \overline{G}_0(-n)\ dn + R \qquad (k \pm \quad)$$

$$\therefore \quad R = -\frac{gg^*}{g'}\int \overline{H}(n)\left\{\overline{G}_0(\ell - n) - \overline{G}_0(-n)\right\}dn$$

and the Fourier transform of $\left\{\Box_x + a^2(\underline{x})\right\}\Psi(x)$

$$= \overline{\Psi}(k)\left[\frac{gg^*}{g'}\int \overline{H}(n)\left\{\overline{G}(k-n) - \overline{G}_0(\ell - n)\right\}dn\right] .$$

$$\therefore \left\{\Box_x + a^2(\underline{x})\right\}\Psi(x) = \int K(x - x')\,\Psi(x')\,dx'$$

where $\quad \overline{K}(k) = \dfrac{gg^*}{g'}\displaystyle\int \overline{H}(n)\left\{\overline{G}(k-n) - \overline{G}_0(\ell - n)\right\}dn$.

A check must be made that the quantity subtracted from the R.H.S. of 4.4 is a function of m multiplied by $\Psi(x)$. The quantity subtracted is $\dfrac{gg^*}{g'}\Psi(x)\, I$,

where

$$I = \int H(x - x')\,G_0(x - x')\,dx' + \int \overline{H}(n)\left\{\overline{G}_0(\ell - n) - \overline{G}_0(-n)\right\}dn$$

$$= \int \overline{H}(n)\,\overline{G}_0(-n)\,dn + \int \overline{H}(n)\left\{\overline{G}_0(\ell - n) - \overline{G}_0(-n)\right\}dn$$

$$= \int \overline{H}(n)\,\overline{G}_0(\ell - n)\,dn$$

which is a function of m alone.

§4 Evaluation of $F(k^2)$

From equation 4.18

$$F(k^2) = -k^2 + a^2 - \overline{K}(k)$$

$$= -k^2 + a^2 - \frac{gg^*}{g'} I$$

where

$$I = \overline{H}(n)\left\{\overline{G}(k-n) - \overline{G}_0(\ell - n)\right\} \quad dn$$

$$= \int \frac{1}{\mu^2 - n^2}\left\{\frac{1}{a^2 - (k-n)^2} - \frac{1}{m^2 - (\ell - n)^2}\right\}$$

$$= \int \frac{2\ell.n - a^2 + k^2 - 2k.n}{[\mu^2 - n^2][a^2 - (k-n)^2][m^2 - (\ell - n)^2]} \quad dn \quad ,$$

using A.1

$$= 2\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int \frac{2n.(\ell - k) + k^2 - a^2 \quad dn}{\left\{\alpha\mu^2 - \alpha n^2 + \beta a^2 - \beta k^2 - \beta n^2 + 2\beta k.n + (1-\alpha-\beta)[m^2 - (\ell - n)^2]\right\}^3}$$

$$= 2\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int \frac{[2n.(\ell - k) + k^2 - a^2] \; dn}{[\alpha\mu^2 + \beta a^2 - \beta k^2 + 2\beta k.n + 2\ell.n(1-\alpha-\beta) - n^2]^3}$$

Put $n' = n - \beta k - \ell(1 - \alpha - \beta)$.

Then

$$I = -2\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int \frac{2[n' + \beta k + \ell(1-\alpha-\beta)] . (\ell - k) + k^2 - a^2}{[n'^2 - \{\beta k + \ell(1-\alpha-\beta)\}^2 + \beta k^2 - \beta a^2 - \alpha\mu^2]^3} \quad dn$$

This integrand divides into an even and an odd part, the odd part giving zero.

$$\therefore I = -2\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int \frac{2(\ell - k).[\beta k + \ell(1-\alpha-\beta)] + k^2 - a^2}{[n^2 - \{\beta k + \ell(1-\alpha-\beta)\}^2 - \alpha\mu^2 - \beta a^2 + \beta k^2]^3}$$

Using A.2

$$I = -\frac{i\pi}{2}\int_0^1 d\alpha \int_0^1 d\beta \left[\frac{2(\ell - k).[\beta k + \ell(1-\alpha-\beta)] + k^2 - a^2}{\{\beta k + \ell(1-\alpha-\beta)\}^2 + \alpha\mu^2 + \beta a^2 - \beta k^2}\right.$$

$$= \frac{i\pi}{2}\int_0^1 d\alpha \log\left[\left\{\beta k + \ell(1-\alpha-\beta)\right\}^2 + \alpha\mu^2 + \beta a^2 - \beta k^2\right]\Bigg|_0^{1-\alpha}$$

$$= \frac{i\pi}{2}\int_0^1 d\alpha \log\left[\frac{\alpha^2 k^2 - \alpha k^2 + \alpha\mu^2 - \alpha a^2 + a^2}{\alpha^2 m^2 - 2\alpha m^2 + \alpha\mu^2 + m^2}\right]$$

$$= \frac{i\pi}{2}\int_0^1 d\alpha \log\left[\frac{\alpha^2\frac{k^2}{m^2} - \alpha\left(\frac{k^2}{m^2} + \frac{a^2}{m^2} - \frac{2}{m^2}\right) + \frac{a^2}{m^2}}{(\alpha-1)^2 + \alpha\frac{\mu^2}{m^2}}\right] .$$

$$\therefore F(k^2) = -k^2 + a^2 - \frac{i\pi}{2}\frac{gg^*}{g'}\int_0^1 d\alpha \log\left[\frac{\alpha^2\frac{k^2}{m^2} - \alpha\left(\frac{k^2}{m^2} + \frac{a^2}{m^2} - \frac{2}{m^2}\right) + \frac{a^2}{m^2}}{(\alpha-1)^2 + \alpha\frac{2}{m^2}}\right]$$

A dimensional analysis of the Lagrangian gives the dimensions of $\frac{gg^*}{g'}$. Using units where $c = \hbar = 1$ mass is dimensionally $\frac{1}{L}$. Let the dimensions of ψ_e, ψ_i, g, and g' be W_1, W_2, G, and G' respectively. Since the dimensions of all terms of the Lagrangian must be the same

$$W_1^2 \; M^2 \; M^4 = G\, W_1 W_2 M^4 = G' \; W_2^2 \; M^4$$

where mass is of dimension $M = \frac{1}{L}$.

$$\therefore \quad W_1 = GW_2 \frac{1}{M^2} .$$

$$\therefore \quad G^2 W_2^2 = G' \, W_2^2 \; M^2 .$$

$$\therefore \quad \frac{G^2}{G'} = M^2 .$$

$\therefore \quad \frac{gg^*}{g'}$ has dimensions M^2.

Write $\qquad \frac{gg^*}{g'} = \frac{1}{\beta} m^2 \frac{2}{i\pi}$,

then β is a dimensionless constant.

$$F(k^2) = -k^2 + a^2 - \frac{1}{\beta}\ m^2 \int_0^1 d\alpha \log \left\{ \frac{\alpha^2 \frac{k^2}{m^2} - \alpha\left(\frac{k^2}{m^2} + \frac{a^2}{m^2} - \frac{\mu^2}{m^2}\right) + \frac{a^2}{m^2}}{(\alpha - 1)^2 + \alpha \frac{\mu^2}{m^2}} \right\}$$

Put $r = \frac{a}{m}$, $s = \frac{\mu}{m}$, $z = \frac{k}{m}$,

then

$$F(k^2) = F(z^2)$$
$$= m^2 \left[-z^2 + r^2 - \frac{1}{\beta} \int_0^1 d\alpha \log \left\{ \frac{\alpha^2 z^2 - \alpha(z^2 + r^2 - s^2) + r^2}{(\alpha - 1)^2 + \quad s^2} \right\} \right.$$

Put

$$I = \int_0^1 d\alpha \log \left\{ \frac{\alpha^2 z^2 - \alpha(z^2 + r^2 - s^2)}{(\alpha - 1)^2 + \alpha s^2} \right\}$$

$$= I_1 - I_2$$

where

$$I_1 = \int_0^1 d\alpha \log \left[\alpha^2 z^2 - \alpha(z^2 + r^2 - s^2) + r^2 \right]$$

and

$$I_2 = \int_0^1 d\alpha \log \left[(\alpha - 1)^2 + \alpha s^2 \right. .$$

$$I_2 = \int_0^1 d\alpha \log (\alpha - \alpha_1) + \int_0^1 d\alpha \log (\alpha - \alpha_2)$$

where

$$\alpha_1 , \alpha_2 = 1 - \frac{s^2}{2} \pm i \sqrt{s^2 - \frac{s^4}{4}} .$$

Using A.3

$$I_2 = (1 - \alpha_1) \log (1 - \alpha_1) + \alpha_1 \log (-\alpha_1)$$
$$+ (1 - \alpha_2) \log (1 - \alpha_2) + \alpha_2 \log (-\alpha_2) - 2$$
$$= \frac{s^2}{2} \log s^2 + (1 - \frac{s^2}{2}) \log 1 + i\sqrt{s^2 - \frac{s^4}{4}} \log \frac{(1-\alpha_2)(-\alpha_1)}{(1-\alpha_1)(-\alpha_2)} - 2$$
$$= \frac{s^2}{2} \log s^2 + i\sqrt{s^2 - \frac{s^4}{4}} \log \frac{1 - \alpha_1}{1 - \alpha_2} - 2$$

since $\alpha_1 \alpha_2 = 1$ and $(1 - \alpha_1)(1 - \alpha_2) = s^2$.

$$I_2 = \frac{s^2}{2} \log s^2 + i\sqrt{s^2 - \frac{s^4}{4}} \log \frac{\frac{s^2}{2} - i\sqrt{s^2 - \frac{s^4}{4}}}{\frac{s^2}{2} + i\sqrt{s^2 - \frac{s^4}{4}}} - 2$$

$$= \frac{s^2}{2} \log s^2 - i\sqrt{s^2 - \frac{s^4}{4}} \quad 2i \text{ arc tan } \frac{\sqrt{s^2 - \frac{s^4}{4}}}{\frac{s^2}{2}} - 2$$

$$= s^2 \log s + 2\sqrt{s^2 - \frac{s^4}{4}} \text{ arc tan } \frac{2\sqrt{1 - \frac{s^2}{4}}}{s} - 2 \ .$$

$$I_1 = \int_0^1 d\alpha \log \left\{ \alpha^2 z^2 - \alpha(z^2 + r^2 - s^2) + r^2 \right\}$$

$$= \int_0^1 d\alpha \quad (\alpha z - \alpha_1) + \int_0^1 d\alpha \, (\alpha z - \alpha_2)$$

where

$$\alpha_1 , \alpha_2 = \frac{z}{2} + \frac{r^2}{2z} - \frac{s^2}{2z} \pm \frac{1}{2z} \sqrt{\left\{z^2 - (r+s)^2\right\}\left\{z^2 - (r-s)^2\right\}} \ ,$$

$$\alpha_1 \alpha_2 = r^2 \ ,$$

and

$$(z - \alpha_1)(z - \alpha_2) = s^2 \ . \qquad \ldots\ldots \text{A.4.}$$

From A.3

$$I_1 = \left(1 - \frac{\alpha_1}{z}\right) \log (z - \alpha_1) + \frac{\alpha_1}{z} \log (-\alpha_1) + \left(1 - \frac{\alpha_2}{z}\right) \log (z - \alpha_2) + \frac{\alpha_2}{z} \log (-\alpha_2) - 2 \ .$$

From A.4 $(z - \alpha_1)$ and $(z - \alpha_2)$ have equal and opposite arguments.

$$\therefore \arg (z - \alpha_1) + \arg (z - \alpha_2) = \arg \ (z - \alpha_1)(z - \alpha_2) = 0$$

and

$$I_1 = \log s^2 - 2 + \left(\frac{z}{2} + \frac{r^2}{2z} - \frac{s^2}{2z}\right) \log \frac{(-\alpha_1)(-\alpha_2)}{(z - \alpha_1)(z - \alpha_2)} + iA_1(z)$$

$$+ \frac{1}{2z}\sqrt{\left\{z^2 - (r+s)^2\right\}\left\{z^2 - (r-s)^2\right\}} \left[\log \frac{(-\alpha_1)(z - \alpha_2)}{(-\alpha_2)(z - \alpha_1)} + i\, A_2(z) \right]$$

where

$$A_1(z) = \arg (-\alpha_1) + \arg (-\alpha_2) - \arg(z - \alpha_1) - \arg(z - \alpha_2)$$

$$- \arg\left\{\frac{\alpha_1 \alpha_2}{(z - \alpha_1)(z - \alpha_2)}\right\}$$

and

$$A_2(z) = \arg(-\alpha_1) + \arg(z - \alpha_2) - \arg(-\alpha_2) - \arg(z - \alpha_1)$$

$$- \arg \frac{(-\alpha_1)(z - \alpha_2)}{(-\alpha_2)(z - \alpha_1)} \quad .$$

Consider, first, $A_1(z)$.

Since $\arg(\alpha_1 \alpha_2) = \arg \ (z - \alpha_1)(z - \alpha_2) \quad = \quad 0$,

$$\arg \frac{\alpha_1 \alpha_2}{(z - \alpha_1)(z - \alpha_2)} = 0$$

and

$$A_1(z) = \arg \alpha_1 \alpha_2 - \arg (z - \alpha_1)(z - \alpha_2)$$

$$= 0 \quad .$$

Now consider $A_2(z)$, taking $r - s < z < r + s$.

(i) If $\qquad z^2 < r^2 - s^2$

$$(r^2 - s^2) - z^4 > 0 \quad .$$

$$\therefore (r^2 - s^2 - z^2)(r^2 - s^2 + z^2) > 0 \quad .$$

$$\therefore \left(-\frac{z}{2} - \frac{r^2}{2z} + \frac{s^2}{2z}\right)\left(\frac{z}{2} - \frac{r^2}{2z} + \frac{s^2}{2z}\right) > 0 \quad .$$

$\therefore$ the real parts of $(z - \alpha_1)$ and $(-\alpha_1)$ have the same sign.
Also the imaginary parts of $(z - \alpha_1)$ and $(-\alpha_1)$ are equal.

$$\therefore \ -\frac{\pi}{2} < \arg(-\alpha_1) - \arg(z - \alpha_1) < \frac{\pi}{2} \quad .$$

similarly $-\frac{\pi}{2} < \arg(-\alpha_2) - \arg(z - \alpha_2) < \frac{\pi}{2}$.

$$\therefore \ -\pi < \arg(-\alpha_1) - \arg(-\alpha_2) + \arg(z - \alpha_2)$$

$$- \arg(z - \alpha_1) < \pi \quad .$$

$$\therefore \ A_2(z) = 0 \quad .$$

(ii) If, instead, $z^2 > r^2 - s^2$ then the real parts of $(z - \alpha_1)$ and $(-\alpha_1)$ have opposite signs but the imaginary parts are still equal.

(ii)a. If $z^2 < r^2 + s^2$

then

$$z^2 - \left(\frac{r^2 - s^2}{z}\right)^2 < -z^2 - \left(\frac{r^2 - s^2}{z}\right)^2 + 2(r^2 + s^2) \quad .$$

$$\therefore \left(\frac{z}{2} + \frac{r^2}{2z} - \frac{s^2}{2z}\right)\left(\frac{z}{2} - \frac{r^2}{2z} + \frac{s^2}{2z}\right) < -\frac{1}{4}\left(z - \frac{(r+s)^2}{z}\right)\left(z - \frac{(r-s)^2}{z}\right) \quad .$$

$\therefore$ the product of the real parts of $-\alpha_1$ and $z - \alpha_2$ is less than the square of their common imaginary part.

In figure

$$CO \cdot OD < AC^2 \quad .$$

$$\therefore CO^2 + OD^2 + 2CO \cdot OD < CO^2 + AC^2 + OD^2 + BD^2 \quad .$$

$$\therefore CD^2 < AO^2 + BO^2 \quad .$$

$$\therefore A\hat{O}B < \frac{\pi}{2} \quad .$$

$$\therefore -\frac{\pi}{2} < \arg(-\alpha_1) - \arg(z - \alpha_1) < \frac{\pi}{2} \quad .$$

$\therefore$ In this case $A_2(z) = 0$.

(ii)b. If $z^2 > r^2 + s^2$ then the argument of (ii)a holds with the inequality signs reversed

$\therefore \arg(-\alpha_1) - \arg(z - \alpha_1)$ is either $> \frac{\pi}{2}$ or $< -\frac{\pi}{2}$.

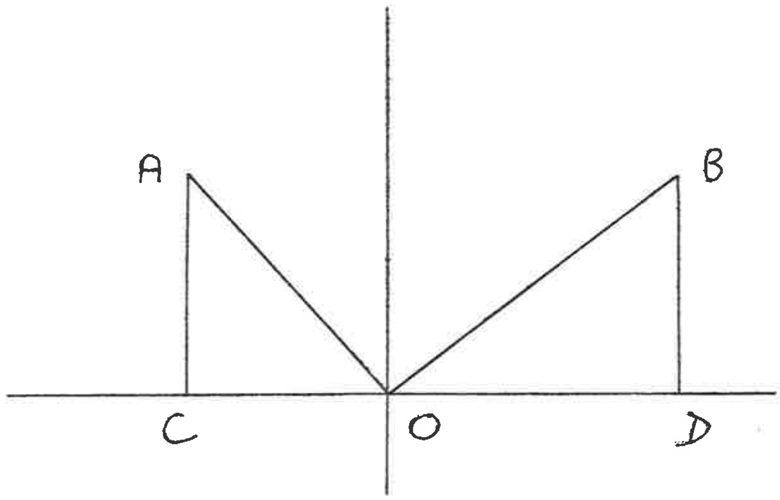

Figure 8.

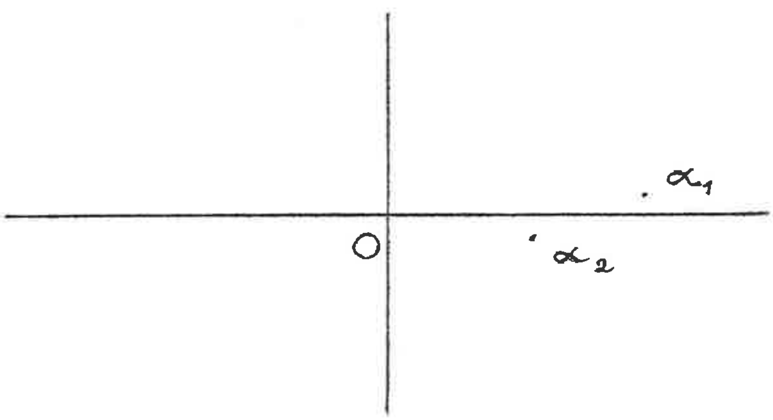

Figure 9.

$\therefore$ Since $\arg(-\alpha_1) = -\arg(-\alpha_2)$

and $\arg(z - \alpha_1) = -\arg(z - \alpha_2)$

$$A_2(z) = \pm 2\pi .$$

Hence $I = I_1 - I_2$

$$= (2 - s^2)\log s + \left(z + \frac{r^2}{z} - \frac{s^2}{z}\right)\log\frac{r}{s}$$

$$- 2\sqrt{s^2 - \frac{s^4}{4}}\ \arctan\frac{2\sqrt{1 - \frac{s^2}{4}}}{s}$$

$$+ \frac{1}{2z}\sqrt{\left\{z^2 - (r+s)^2\right\}\left\{z^2 - (r-s)^2\right\}}$$

$$\left[\log\frac{z^2 - r^2 + s^2 + \sqrt{\left\{z^2 - (r+s)^2\right\}\left\{z^2 - (r-s)^2\right\}}}{z^2 - r^2 + s^2 - \sqrt{\left\{z^2 - (r+s)^2\right\}\left\{z^2 - (r-s)^2\right\}}}\right.$$

$$\left. + \begin{Bmatrix} +2\pi \\ -2\pi \\ 0 \end{Bmatrix}\right] ,$$

the 0 being applicable if z is real so that $z^2 < r^2 + s^2$ and one of ± 2 being applicable if z is real and $z^2 > r^2 + s^2$.

In order to determine which of the $\pm$ signs is physically applicable to this situation it is necessary to go back to the original integral and consider what happens to the argument of the logarithm in the course of integration. This is determined by Feynman's rule of putting a small negative imaginary part on the masses to ensure a result corresponding to causal physical processes.

$$I = \int_0^1 d\alpha\ \log\left\{\frac{\alpha^2 k^2 - \alpha(k^2 + a^2 - \mu^2) + a^2}{(\alpha - 1)^2 m^2 + \alpha\mu^2}\right\} .$$

The denominator of the integrand may be neglected from this consideration as it is positive definite and its integral is well defined. Consider

$$\int_0^1 d\alpha\ \log\left[(\alpha k - \alpha_1)(\alpha k - \alpha_2)\right]$$

where replacing a by $a - i\Sigma$ and μ by $\mu - i\delta$

$$\alpha_1 , \alpha_2 = \frac{k}{2} + \frac{a^2}{2k} - \frac{\mu^2}{2k} - \frac{2ai\Sigma}{2k} + \frac{2\mu i\delta}{2k}$$

$$\pm \frac{1}{2}\sqrt{\left(k - \frac{(a+\mu)^2}{k} + 2\frac{i(a+\mu)(\Sigma+\delta)}{k}\right)\left(k - \frac{(a-\mu)^2}{k} + 2\frac{i(a-\mu)(\Sigma-\delta)}{k}\right)}$$

taking the + sign for α_1 .

Now

$$\sqrt{A+\Sigma} = \sqrt{A}\left(1 + \frac{\Sigma}{A}\right)^{\frac{1}{2}} \simeq \sqrt{A}\left(1 + \frac{\Sigma}{2A}\right) = \sqrt{A} + \frac{\Sigma}{2\sqrt{A}} .$$

Put $$\alpha_1', \alpha_2' = \frac{k}{2} + \frac{a^2}{2k} - \frac{\mu^2}{2k} \pm \sqrt{\left\{k - \frac{(a+\mu)^2}{k}\right\}\left\{k - \frac{(a-\mu)^2}{k}\right\}} ,$$

then

$$\alpha_1 , \alpha_2 = \alpha_1' , \alpha_2' - \frac{2ai\Sigma}{2k} + \frac{2\mu i\delta}{2k} \pm \frac{\gamma}{4\sqrt{m}}$$

where $$\sqrt{m} = \sqrt{\left\{k - \frac{(a+\mu)^2}{k}\right\}\left\{k - \frac{(a-\mu)^2}{k}\right\}}$$

and

$$\gamma = 2i\,(a+\mu)(\Sigma+\delta) + 2i\,(a-\mu)(\Sigma-\delta)$$
$$- \frac{2i}{k^2}(a^2-\mu^2)\quad (a-\mu)(\Sigma+\delta) + (a+\mu)(\Sigma-\delta)$$
$$= 4i\,(a\Sigma + \mu\delta) - \frac{4i}{k^2}(a^2-\mu^2)(a\Sigma - \mu\delta)$$

$$\alpha_1, \alpha_2 = \alpha_1' , \alpha_2' + \frac{ai\Sigma}{k}\left\{-1 \pm \frac{k}{\sqrt{m}} \pm \frac{\mu^2 - a^2}{k\sqrt{m}}\right\}$$
$$+ \frac{\mu i\delta}{k}\left\{1 \pm \frac{k}{\sqrt{m}} \pm \frac{a^2-\mu^2}{k\sqrt{m}}\right\}.$$

Now $$-1 + \frac{k^2 - a^2 + \mu^2}{\sqrt{m}} > 0$$

and $$1 + \frac{k^2 - a^2 + \mu^2}{\sqrt{m}} > 0$$

but

$$-1 - \frac{k^2 - a^2 + \mu^2}{\sqrt{\sim}} < 0$$

and

$$+1 - \frac{k^2 - a^2 + \mu^2}{\sqrt{\sim}} < 0$$

$\therefore$ α_1 has a positive imaginary part while α_2 has a negative imaginary part. Therefore α_1 and α_2 are placed as in figure 9 ..

Thus the imaginary part of I is $-\pi(\alpha_1 - \alpha_2)$

$$= -\pi i \sqrt{\sim}$$

and over the whole complex plane

$$F(k^2) = F(z^2)$$

$$= m^2 \Big[- z^2 + r^2 - \frac{1}{\beta} \Big\{ (2 - s^2) \log s + \left(z + \frac{r^2}{z} - \frac{s^2}{z}\right) \log \frac{r}{s}$$

$$- 2\sqrt{s^2 - \frac{s^4}{4}} \ \text{arc tan} \ \frac{2\sqrt{1 - \frac{s^2}{4}}}{s}$$

$$+ \frac{1}{2z}\sqrt{z^2 - (r + s)^2 \quad z^2 - (r - s)^2}$$

$$\left[\log \frac{z^2 - r^2 + s^2 + \sqrt{z^2 - (r - s)^2 \quad z^2 - (r + s)^2}}{z^2 - r^2 + s^2 - \sqrt{z^2 - (r - s)^2 \quad z^2 + (r + s)^2}} - 2\pi i \theta \right] \Big\} .$$

.... A.5.

where θ is zero on the left of a line through $z^2 = r^2 + s^2$ and is unity on the right of this line.

The function $F(k^2)$ has some interesting properties which are discussed in greater detail in Chapter 6. It cannot be written as a single expression if the principal value is to be taken for the logarithm but must be written as two functions which join smoothly along a line through the point $z^2 = r^2 + s^2$. The function is clearly analytic everywhere except possibly for the last term in 3.14. At first sight this would appear to have branch points at the zeros of the square roots; i.e. at $z^2 = (r + s)^2$ and $z^2 = (r - s)^2$. This, however, is not always the case for

$$-\sqrt{A} \log \frac{B - \sqrt{A}}{B + \sqrt{A}} = + \sqrt{A} \log \frac{B + \sqrt{A}}{B - \sqrt{A}} ,$$

and unless θ is unity no branch points are present.

If z is to the left of a line through the point $z^2 = r^2 + s^2$ the function is written so that no branch points are involved. Thus, the point $z^2 = (r - s)^2$ is not a branch point of $F(z^2)$. On the other hand the point $z^2 = (r + s)^2$ is to the right of the point $z^2 = r^2 + s^2$ and there is a term, $- 2\pi i \sqrt{\left\{z^2 - (r + s)^2\right\}\left\{z^2 - (r-s)^2\right\}}/z$, involved which means that $z^2 = (r + s)^2$ is a branch point of $F(z^2)$. The point $z^2 = r^2 + s^2$ has some special properties and physically these are interpreted to mean that $r^2 + s^2$ is the square of the anomolous threshold energy. This is discussed more fully in Chapter 4.

§5 The Orthogonal Spheroidal Coordinate System

In this section results will be derived which are used in §3 of Chapter 5. Some of the results have been set out by Flammer[32] but are included here for completeness.

The spheroidal coordinate system used is related to the rectangular cartesian coordinates by

$$x = (d/2)\left[(1 - \eta^2)(\xi^2 + 1)\right]^{\frac{1}{2}} \cos \phi \quad ,$$

$$y = (d/2)\left[(1 - \eta^2)(\xi^2 + 1)\right]^{\frac{1}{2}} \sin \phi \quad ,$$

and

$$z = (d/2)\, \xi \eta$$

with $-1 \leqslant \eta \leqslant 1$, $0 < \xi < \infty$, and $0 \leqslant \phi \leqslant 2\pi$.

The surfaces of constant ξ are oblate spheroids given by

$$\frac{r^2}{(d^2/4)(1 + \xi^2)} + \frac{z^2}{(d^2/4)\,\xi^2} = 1 \quad .$$

The eccentricity of this spheroid is given by

$$e = 1/\sqrt{1 + \xi^2}$$

which is independent of d . Thus in any oblate spheroidal coordinate system the eccentricity of a surface $\xi = \xi_0$ is determined by ξ_0 .

If the wave equation

$$(\nabla^2 + k^2)\psi = 0$$

is solved using this coordinate system separation of variables yields two equations which are analogous to Legendre's equation and Bessel's equation.

These are

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}S_\ell(c,\eta)\right]+\left[\lambda_\ell(c)+c^2\eta^2\right]S_\ell(c,\eta)=0$$

and

$$\frac{d}{d\xi}\left[(\xi^2+1)\frac{d}{d\xi}R_\ell(c,\xi)\right]-\left[\lambda_\ell(c)-c^2\xi^2\right]R_\ell(c,\xi)=0 \quad ,$$

where

$$c = kd/2$$

and $\lambda_\ell(c)$ is a separation constant.

The solutions of these are the spheroidal angle and radial functions respectively. The angle function equation has a cylindrically symmetrical solution, analogous to the ordinary Legendre function, which is written $S_\ell(c,\eta)$. The radial function equation has two independent solutions which can be written

$$R_\ell(c,\xi) = \int_{-1}^{1} e^{ic\xi\eta}\, S_\ell(c,\eta)\, d\eta$$

and

$$R^*_\ell(c,\xi) = \int_{-1}^{1} \frac{e^{-q(\eta-i\xi)}\, S_\ell(c,\eta)}{q(\eta-i\xi)}\, d\eta \, .$$

It is easily checked that the first is a solution and the second may be shown to be a solution by using the theorem on p.44 of Flammer[32].

From the angle function equation it can be shown that

$$\int_{-1}^{1} d\eta\, S_m(q,\eta)\, S_n(c,\eta)\left[\lambda_m(q)-\lambda_n(c)+q^2\eta^2-c^2\eta^2\right] = 0 \, .$$

Thus if $c = q$

$$\int_{-1}^{1} d\eta\, S_m(c,\eta)\, S_n(c,\eta) = 0$$

and the normalisation of the angle functions is chosen so that

$$\int_{-1}^{1} d\eta\left[S_\ell(c,\eta)\right]^2 = 1 \, .$$

This orthonormal property of the angle functions does not hold exactly if $c \neq q$ but it does hold approximately. The above equation gives

$$\int_{-1}^{1} d\eta \; S_m(q,\eta)\, S_n(c,\eta) = \frac{c^2 - q^2}{\lambda_m(q) - \lambda_n(c)} \int_{-1}^{1} \eta^2 S_m(q,\eta) S_n(c,\eta)\, d\eta \; .$$

Since the angle functions achieve a sharp maximum at $\eta^2 = 1$ the R. H. S. is approximately equal to

$$\frac{c^2 - q^2}{\lambda_m(q) - \lambda_n(c)} \int_{-1}^{1} S_m(q,\eta)\, S_n(c,\eta)\, d\eta \; .$$

Since $c^2 - q^2 \neq \lambda_m(q) - \lambda_n(c)$ if $c \neq q$

$$\int_{-1}^{1} S_m(q,\eta)\, S_n(c,\eta)\, d\eta \simeq 0 \; .$$

This result has been checked by numerical integration and found to be a very good approximation. The fact that the angle functions achieve a sharp maximum at $\eta^2 = 1$ may be checked by referring to the tables in Flammer[32].

Asymptotically $R(c,\xi)$ and $R^*(c,\xi)$ are like $\cos c\xi$ and $e^{iq\xi}/iq\xi$. Thus $R(c,\xi)$ contains an ingoing part and an outgoing part of equal amplitude while $R^*(c,\xi)$ is purely an outgoing wave. $R(c,\xi)$ is finite at $\xi = 0$ and ∞ and is thus the equivalent of a plane wave while $R^*(c,\xi)$ is asymtotically like e^{ikr}/r since $\xi \rightarrow r$ as $\xi \rightarrow \infty$ and is thus like an outgoing spherical wave.

The last result used in Chapter 5 is the expansion of the plane wave

$$e^{ikz\cos\theta} = e^{ic\xi\eta\cos\theta}$$

in radial and angle functions. The result is derived on p.48 of Flammer[32] and states that

$$e^{ic\xi\eta\cos\theta} = 2\sum_{\ell} i^{\ell}\, S_{\ell}(c,\eta)R_{n}(c,\xi)S_{\ell}(c,\cos\theta) \quad .$$

§6 Fortran Programmes

$F(k^2)$ may be investigated by evaluating it on a computer. In the course of the work for this thesis such an evaluation was carried out on a 1620 computer using the following Fortran pr gramme. The programmes given are for

$$-\beta\left[F(z^2)/m^2 + z^2 - r^2\right]$$

from which $F(z^2)$ is easily found. See equation A.5.

Programme for $F(z^2)$ for real values of z between $r - s$ and $r + s$.

```
  1 FORMAT(F8.4,1XF8.4,1XF8.4,1XF8.4,1XF8.4,1XF8.4)
  2 FORMAT(E12.6,1XE12.6,1XE12.6)
  3 ACCEPT1,R,S,ZMINZ,ZINTZ,RINT,RMAX
    PRINT1,R,S,ZMINZ,ZINTZ,RINT,RMAX
 12 Z=R-S
    PUNCH1,R,S
    GOTO5
  4 Z=Z+ZINTZ
  5 SS=S*S
    RR=R*R
    ZZ=Z*Z
    SQT=SQRTF(SS-(SS*SS)/4.)
    T=2.*SQT
    A=(2.-SS)*LOGF(S)+(Z+RR/Z-SS/Z)*LOGF(R/S)-2.*SQT*ATANF(T/SS)
    E=ZZ-RR-SS
    PI=3.1415927
    SCRT=SQRTF(((R+S)*(R+S)-ZZ)*(ZZ-(R-S)*(R-S)))
```

```
      IF(ZZ-RR-SS)7,7,8
    7 REB=(1./Z)*SCRT*ATANF((SCRT/(RR+SS-ZZ)))
      GOTO9
    8 REB=(1./Z)*SCRT*(ATANF((SCRT/(RR+SS-ZZ)))+PI)
    9 RIB=0.
      REFZ=A+REB
      RIFZ=RIB
      PUNCH2,Z,REFZ,RIFZ
      IF(Z-(R+S))4,6,6
    6 IF(R-RMAX)10,11,11
   10 R=R+RINT
      GOTO12
   11 STOP
      END
```

A similar programme may be used to evaluate $F(z^2)$ for real values of z above $R + S$.

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