"APPLICATION OF THE METHOD OF CONSTANT DEFLECTION CONTOUR LINES TO ELASTIC PLATE AND SHELL PROBLEMS."

by

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SUMMARY

The work presented in this thesis is an application of the method of constant deflection lines to the analysis of thin walled structures, plates and shells in particular. The analysis is valid only when the boundary of the plate or shell is subjected to a combination of clamped or simply supported boundary conditions. The work is given in two parts. Part One deals with the method of constant deflection lines as applied to plate analysis, while Part Two deals with the method as applied to shallow shell analysis.

The First Chapter is an introduction to the substance of the thesis. Part One begins with the Second Chapter where a brief resumé of the method of constant deflection lines for the analysis of an elastic plate is first presented. The governing differential equations for the following cases are then given:

(a) the bending of elastic plates;

(b) the buckling of elastic plates;

(c) the transverse vibration of elastic plates;

(d) the large amplitude deformation of elastic plates;

(e) the large amplitude vibration of elastic plates.

In the following chapters the analysis is further substantiated by obtaining the precise form for the lines of constant deflection.

The Third Chapter deals with the derivation of the exact equation of the lines of constant deflection for the bending of an elastic plate under various loading and edge conditions. This analysis is then illustrated by considering the following cases:

(a) a uniformly loaded, clamped, semi-circular plate;

(b) a uniformly loaded, simply supported, elliptic plate;

(c) a uniformly loaded, simply supported, equilateral triangular plate;

(d) a uniformly loaded circular plate clamped on one edge portion and simply supported on the remainder;

(e) an arbitrary loaded, clamped, cardiodal plate.

The Fourth Chapter deals with the buckling of hydrostatically compressed plates. The exact equation for the lines of constant deflection is obtained. As an illustration of the procedure, the buckling of a hydrostatically compressed equilateral triangular plate with either clamped or simply supported edges is discussed.

Chapter Five deals with the transverse vibration of elastic plates. The free vibration problem is first considered, and the exact equation for the lines of constant deflection is obtained. The transverse vibration of plates with hydrostatic edge loading is then discussed. It is shown that the deflection contours are independent of the magnitude of the applied edge load. To illustrate this analysis the following problems are considered:

(a) the transverse, free, vibration of an equilateral triangular plate when its edges are either clamped or simply supported;

(b) the transverse vibration of an elliptic plate with hydrostatic edge loading, when its edges are either clamped or simply supported.

Chapter Six is devoted to the large amplitude analysis of elastic plates. Here the large amplitude deformation, and the large amplitude vibration problems are considered. It is shown that in both cases the lines of constant deflection may be taken as for the corresponding plate problem under small deflection theory. The large amplitude deformation, and the large amplitude vibration of an elliptic plate is then discussed.

The second part starts with Chapter Seven where the method of constant deflection lines is applied to shallow shell analysis. Here it is shown that the deflection contours may be taken as for the corresponding flat plate problem. To illustrate this analysis the bending of a shallow dome of non zero Gaussian curvature is discussed.

The Eighth Chapter deals with the method as applied to the transverse vibration, and the stability of shallow shells. The precise form of the deflection contours is obtained, and as an illustration the transverse vibration of a shallow dome of non-zero Gaussian curvature is discussed.

In the last chapter some concluding remarks are presented, and an indication of a possible extension of the method is given.

The numerical calculations presented were performed using the C.D.C.6400 at the computing centre of the University of Adelaide.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University.

To the best of my knowledge and belief, the thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

Rhys Jones

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CHAPTER I

INTRODUCTION

The mathematical problems associated with thin elastic plates and shells, belong with the many problems from mechanics which have interested eminent mathematicians, and from which mathematical ideas of fundamental importance have originated. These problems have numerous practical applications. Indeed a considerable part of the recent interest in the study of plates and shells is due to the need for solutions to those problems which arise in the design of thin walled structures used in aircraft. The solutions to boundary-value problems of this type are of considerable technological importance, in particular to the constructional industry where variously shaped plates and shells are encountered. These problems have demanded considerable attention and, in the course of time different approximate methods of solution have been developed which deal effectively with many of these situations. Yet many problems are still awaiting solution.

Perhaps the most widely used methods for obtaining approximate solutions of Plates and Shells are the direct methods of variational calculus, for example the Rayliegh-Ritz, Galerkin, and Kantorovich methods. However these methods have several shortcomings, two of which are the strong dependence of the solution upon the initial choice of approximating functions, and the difficulty in satisfying the boundary conditions for an arbitrary shaped plate or shell. Of the other methods for finding solutions to boundary value problems in Plates and Shells, many proceed on the basis that the solution is regular. The rate of convergence of these methods is then related to the degree of regularity of the actual solution, i.e. to the continuity of successive derivatives, which depend upon the smoothness of the boundary contour and the boundary conditions. Furthermore, the designer requires accurate numerical values for the bending moments (involving second derivatives of the deflection) and for the normal forces (involving third derivatives), whereas physical considerations demand only that the first derivatives be continuous. Therefore a poor choice of the approximating function for the deflection can easily lead to severe computational difficulties.

An entirely new[1] treatment of the bending, buckling and vibration of thin elastic plates has recently been proposed by Mazumdar [35,36,37]. This method involves the so called "Lines of Equal Deflection", i.e., contour lines which are obtained by intersecting the bent plate with planes parallel to the original unbent plane of the plate.

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[1]Although this method had not previously been used to solve plate problems, Polyaand Szëgo [50] had used the concept of "Lines of Equal Deflection" to determine several important results concerning the shape of the deflected mod for the buckling, and vibration of thin elastic plates.

This will be described in detail in chapter II. The differential equation for the deflection $w(x,y)$ in accordance with this method, is reduced to a third order ordinary differential equation with three conditions to satisfy. Consequently, in principle, we must expect to obtain the exact solution of the plate problem by this method. In cases wher its precise solution is difficult to obtain we may use the various numerical techniques for finding the solution of this differential equation.

In this thesis attention is first focused upon obtaining the true equation for the lines of constant deflection for the bending, buckling, and vibration of thin elastic plates. Subsequently we extend the analysis to include the large amplitude deformation and vibration of thin elastic plates, and the bending, vibration and stability of thin, elastic, shallow shells of arbitrary shape. It is further shown that the method of constant deflection contours is a powerful tool for the investigation of those problems of elastic plates and shells which could not be solved by conventional methods because of the difficulty o mathematical treatment.

PART I
CHAPTER II

THE METHOD OF LINES OF CONSTANT DEFLECTION FOR PLATE ANALYSIS

2.1 FUNDAMENTAL CONCEPTS.

Recently a new method has been developed for dealing with a large class of boundary value problems associated with the bending, buckling,and vibration of elastic plates of arbitrary shape. This method involves the concept of "Lines of Equal Deflection". Up till now it was made clear that the accuracy of the solution of the particular plate problem depended mainly on the rational assumption of the shape of the deflected surface generating the deflection contours of the plate. Consequently attention was focused upon a possible approximation for such deflection contours.

However this method was, till now, incomplete in the sense that the differential equation determining the lines of equal deflection for an arbitrary shaped plate, under diverse loading and boundary conditions, was yet to be obtained.

The following chapter shows how the exact equation of these contour lines can be obtained for a plate of arbitrary shape. As illustrations of this procedure the method has been applied to the following cases:

i) a uniformly loaded, semicircular plate clamped along its boundary;

ii) a uniformly loaded circular plate, clamped on part of its boundary and simply supported on the remainder;

iii) a uniformly loaded, elliptical plate, simply supported along its boundary;

iv) a clamped, cardiodal plate under arbitrary non uniform loading.

For simplicity, analysis will be confined to a simply connected plate which occupies a finite region, and is bounded by a piecewise smooth contour. The treatment of a multiply connected plate is, in general, more involved, but involves no essential difficulties.

The theory which is developed here is based upon the following assumptions:

i) the plate consists of homogeneous isotropic material;

ii) the material obeys Hooke's Law;

iii) normals to the middle plane before bending are deformed into normals of the middle plane after bending;

iv) the middle surface remains unstrained after bending;

v) the deflection of the plate is small in comparison with its thickness, h;

vi) the thickness, h, of the plate is small in comparison with its other dimensions.

The plane parallel to the faces of the plate and bisecting the thickness of the plate, in the undeformed

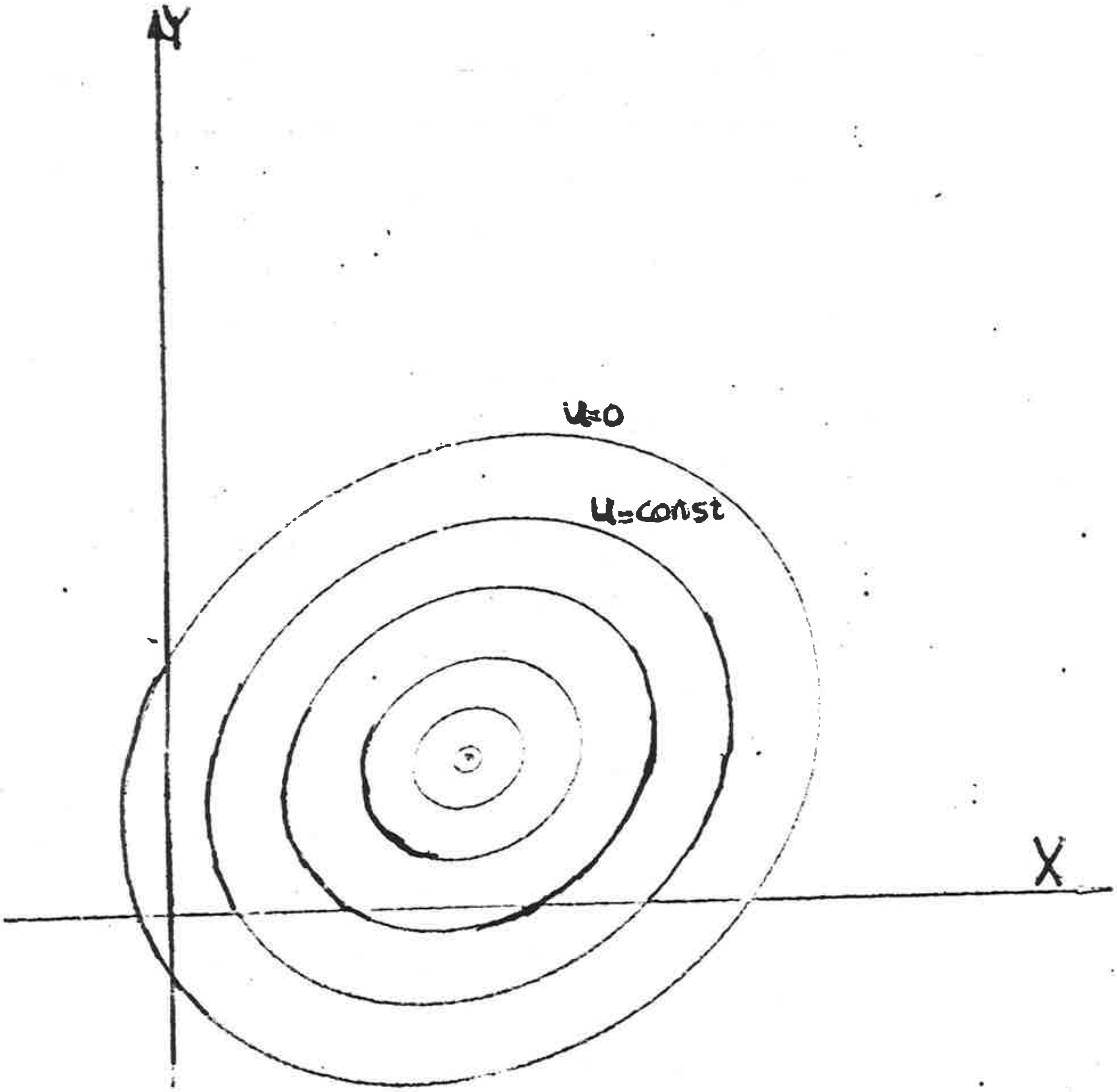


Fig1.

state, is called the middle surface of the plate. We choose coordinate axes so that the x and y axes are in the middle plane of the plate, and the z axis is perpendicular to the middle plane of the plate.

2.2 DERIVATION OF THE EQUATIONS OF EQUILIBRIUM FOR THE BENDING ANALYSIS OF THIN ELASTIC PLATES

When an elastic plate, subject to any combination of clamped and simply supported boundary conditions is bent under the action of an external load, the corresponding deflected surface can be described by a family of curves which are called the "Lines of Equal Deflection". These deflection contours may be obtained by considering the projection onto the XOY-plane of the intersections between the deflected surface $z = w(x,y)$, and the parallels $z = \text{const.}$ We will denote the family of such level curves by

$$u(x,y) = \text{const.} \qquad (2.2.1)$$

If the boundary of the plate, assumed to be a simple closed curve C, does not move in the direction perpendicular to the plane of the plate [this case corresponds to elastically supported edges], then clearly it must belong to the family of lines of equal deflection, and we may, without loss in generality, consider that $u=0$ on the boundary (Fig.1.). It is clear that the lines of equal deflection form a system of non intersecting closed curves starting with the closed boundary C as one of the

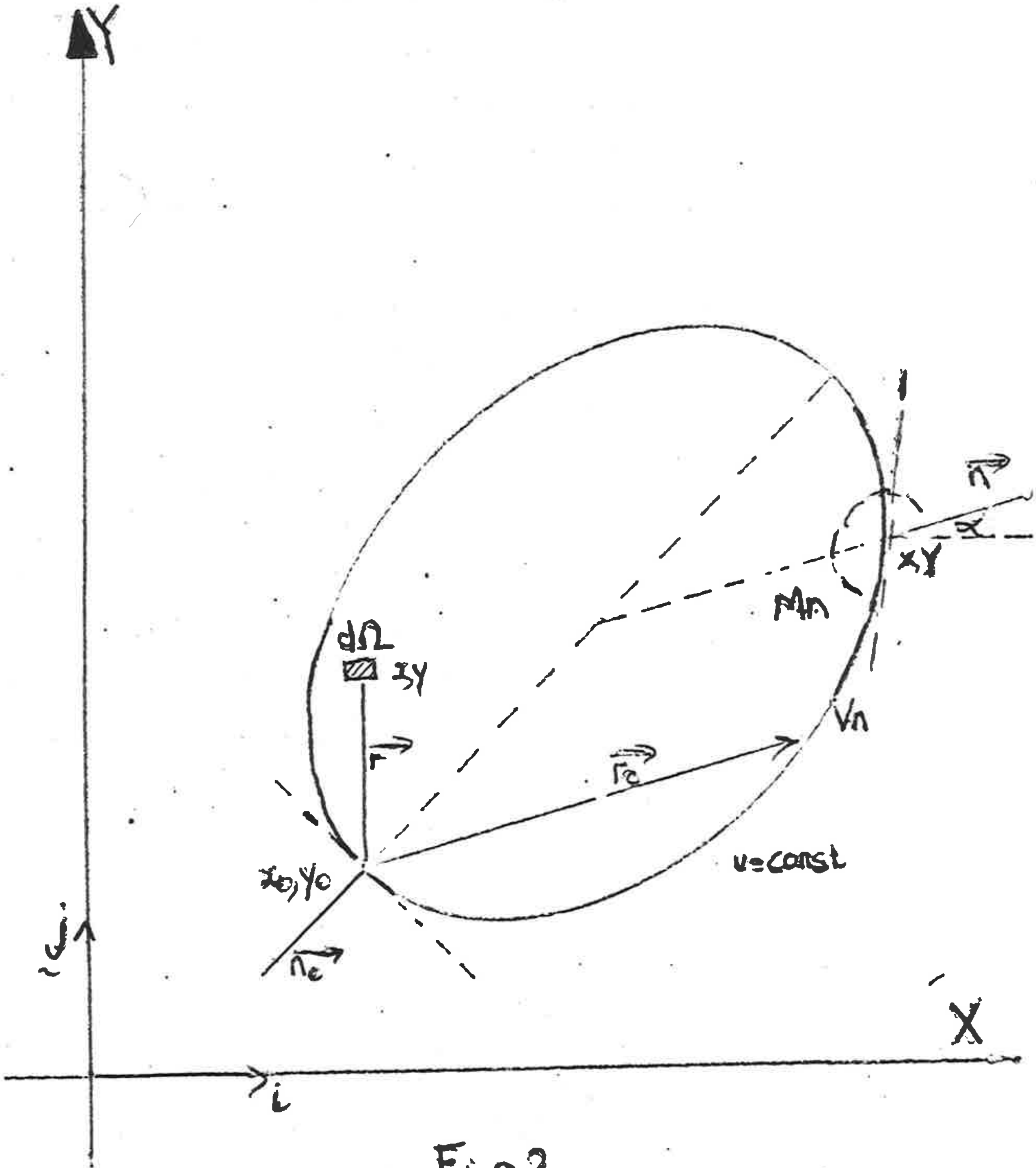


Fig2

lines. We denote the family of curves $u = \text{const.}$ by C_u, $0 \leqslant u \leqslant u^*$, so that $C_0 = C$ is the boundary of the plate and C_{u^*} coincides with the point(s) at which the maximum $u = u^*$ is obtained. Here we have assumed the value of u increases as we go towards the centre of the region.

For the symmetrically loaded, circular plate with clamped or simply supported boundary the curves C_u are known to be a family of concentric circles with centre at the origin, while for the uniformly loaded, clamped, elliptic plate they are known to be a family of similar ellipses.

Consider the equilibrium of an element of the plate bounded by any line of equal deflection C_u. Let (x_0,y_0) be any fixed point on the contour C_u, and $\underset{\sim}{n}$, $\underset{\sim}{n}_0$ denote the unit normal vectors to the curve C_u at an arbitrary point (x,y), and the fixed point (x_0,y_0) respectively, and let $\underset{\sim}{r}$, $\underset{\sim}{r}_0$ denote the position vectors from the fixed point (x_0,y_0) to any arbitrary point inside, and on the contour C_u, respectively (see Fig.2.). We thus have the relationships

$$\begin{aligned}
\underset{\sim}{r} &= (x-x_0)\underset{\sim}{i} + (y-y_0)\underset{\sim}{j} \\
\underset{\sim}{r}_0 &= (x-x_0)\underset{\sim}{i} + (y-y_0)\underset{\sim}{j}\Big|_{C_u} \\
\underset{\sim}{n} &= \left.\frac{u_x\underset{\sim}{i} + u_y\underset{\sim}{j}}{\sqrt{u_x^2 + u_y^2}}\right|_{C_u} \\
\underset{\sim}{n}_0 &= \left.\frac{u_x\underset{\sim}{i} + u_y\underset{\sim}{j}}{\sqrt{u_x^2 + u_y^2}}\right|_{(x_0,y_0)} .
\end{aligned} \tag{2.2.2}$$

The conditions for the equilibrium of an element of plate, bounded by any contour line C_u, require that the sum of the moments about the tangent line to the curve C_u at any point (x_o, y_o) of all the forces acting on the element and the sum of all the forces normal to the plane XOY vanish. We do not require that the sum of the moments about the normal to the curve C_u should vanish for the same reason that we do not seek the vanishing of the twisting moment along the free edge of any plate. Thus we finally obtain

$$\Sigma M = \underset{\sim}{n}_o \oint_{C_u} M_n \underset{\sim}{n} ds + \underset{\sim}{n}_o \oint_{C_u} V_n \underset{\sim}{r}_o ds - \underset{\sim}{n}_o \iint_{\Omega_u} q\underset{\sim}{r} d\Omega$$

$$= 0, \qquad (2.2.3)$$

and

$$\Sigma Z = \oint_{C_u} V_n ds = \iint_{\Omega_u} q d\Omega$$

$$= 0, \qquad (2.2.4)$$

where the contour integrals are taken around the closed path $u(x,y) = \text{const.}$, and the double integrals are taken over the area bounded by the closed contour $u(x,y) = \text{const.}$ If we now calculate the expressions for the bending moment M_n, and the transverse reactive force V_n which contains the shearing force Q_n, and the edge rate of change of the twisting moment M_{nt} along the contour C_u, we obtain

$$\underset{\sim}{n}_o \frac{d^2w}{du^2} \oint_{C_u} P\underset{\sim}{n}ds + \underset{\sim}{n}_o \frac{dw}{du} \oint_{C_u} Q\underset{\sim}{n}ds + \underset{\sim}{n}_o \frac{d^3w}{du^3} \oint R\underset{\sim}{r}_o ds$$

$$+ \underset{\sim}{n}_o \frac{d^2w}{du^2} \oint_{C_u} F\underset{\sim}{r}_o ds + \underset{\sim}{n}_o \frac{dw}{du} \oint_{C_u} G\underset{\sim}{r}_o ds - \underset{\sim}{n}_o \iint_{\Omega_u} q\underset{\sim}{r}d\Omega$$

$$= 0, \qquad (2.2.5)$$

and

$$\frac{d^3w}{du^3} \oint_{C_u} R ds + \frac{d^2w}{du^2} \oint_{C_u} F ds + \frac{dw}{du} \oint_{C_u} G ds - \iint_{\Omega_u} q d\Omega$$

$$= 0, \qquad (2.2.6)$$

where we have made use of the well known relations

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) = -D\left[\frac{d^2w}{du^2}(u_x^2+\mu u_y^2) + \frac{dw}{du}(u_{xx}+\mu u_{yy})\right],$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right) = -D\left[\frac{d^2w}{du^2}(u_y^2+\mu u_x^2) + \frac{dw}{du}(u_{yy}+\mu u_{xx})\right],$$

$$M_{xy} = -M_{yx} = D(1-\mu) \frac{\partial^2 w}{\partial x \partial y} = D(1-\mu) \left[\frac{d^2w}{du^2}u_x u_y + \frac{dw}{du} u_{xy}\right],$$

$$Q_x = -D \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -D\left[\frac{d^3w}{du^3}(u_x^3+u_y^2u_x) + \frac{d^2w}{du^2}(3u_{xx}u_x + 2u_{xy}u_y+u_{yy}u_x) + \frac{dw}{du}(u_{xxx}+u_{yyy})\right],$$

$$Q_y = -D \frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -D\left[\frac{d^3w}{du^3}(u_y^3+u_x^2u_y) + \frac{d^2w}{du^2}(3u_{yy}u_y + 2u_{xy}u_x+u_{xx}u_y) + \frac{dw}{du}(u_{yyy}+u_{xxy})\right],$$

$$M_n = M_x\cos^2\alpha + M_y\sin^2\alpha - 2M_{xy}\sin\alpha\cos\alpha = P\frac{d^2w}{du^2} + Q\frac{dw}{du},$$

$$M_{nt} = M_{xy}(\cos^2\alpha-\sin^2\alpha)+(M_x-M_y)\sin\alpha\cos\alpha = H\frac{dw}{du},$$

$$M_t = M_x\sin^2\alpha+M_y\cos^2\alpha+2M_{xy}\sin\alpha\cos\alpha = P'\frac{d^2w}{du^2} + Q'\frac{dw}{du},$$

$$Q_n = Q_x\cos\alpha+Q_y\sin\alpha,$$

$$V_n = Q_n - \frac{\partial M_{nt}}{\partial s} = R\frac{d^3w}{du^3} + F\frac{d^2w}{du^2} + G\frac{dw}{du}, \qquad (2.2.7)$$

and the transformation relationships

$$\frac{\partial w}{\partial x} = \frac{dw}{du}\frac{\partial u}{\partial x} = \frac{dw}{du}u_x$$

$$\frac{\partial w}{\partial y} = \frac{dw}{du}\frac{\partial u}{\partial y} = \frac{dw}{du}u_y \qquad (2.2.8)$$

$$\frac{\partial^2 w}{\partial x\partial y} = \frac{dw}{du}\frac{\partial^2 u}{\partial x\partial y} + \frac{d^2w}{du^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$
$$= \frac{dw}{du}u_{xy} + \frac{d^2w}{du}u_xu_y,$$

and have used the fact that w and its derivatives with respect to u are constant on the contour u = const.

Here P, Q, P', Q', etc. are given by the following expressions which involve u and its partial derivatives

$$P = -Dt,$$

$$Q = -\frac{D}{t}[u_{xx}u_x^2+u_{yy}u_y^2+\mu u_{yy}u_x^2+\mu u_{xx}u_y^2+2(1-\mu)u_{xy}u_xu_y],$$

$$P' = -D\mu t,$$

$$Q' = -\frac{D}{t}[u_{xx}u_y^2+u_{yy}u_x^2+\mu u_{xx}u_x^2+\mu u_{yy}u_y^2-2(1-\mu)u_{xy}u_xu_y],$$

$$R = -Dt^{\frac{3}{2}},$$

$$F = -\frac{D}{t^{\frac{1}{2}}}[3u_{xx}u_x^2+3u_{yy}u_y^2+u_{xx}u_y^2+u_{yy}u_x^2+4u_{xy}u_xu_y],$$

$$G = -\frac{D}{t^{\frac{3}{2}}}[u_{xxx}u_x^3+u_{yyy}u_y^3+(1-\mu)(u_{xxx}u_xu_y^2+u_{yyy}u_x^2u_y$$

$$+ u_{xyy}u_x^3+u_{xxy}u_y^3)+(2\mu-1)(u_{xyy}u_xu_y^2+u_{xxy}u_x^2u_y)$$
$$- 2(1-\mu)u_{xy}(u_xu_yu_{xx}-u_y^2u_{xy}-u_x^2u_{xy}+u_xu_yu_{yy})$$
$$+ (1-\mu)(u_{xx}-u_{yy})(u_{xx}u_y^2-u_{yy}u_x^2)]$$
$$+ \frac{2D(1-\mu)}{t^{\frac{5}{2}}}[u_{xy}(u_x^2-u_y^2)-u_xu_y(u_{xx}-u_{yy})]^2,$$

$$H = \frac{D(1-\mu)}{t}[u_{xy}u_x^2-u_{xy}u_y^2-u_{xx}u_xu_y+u_{yy}u_xu_y],$$
$$t = u_x^2+u_y^2,$$
$$\cos\alpha = \frac{dy}{ds},\ \sin\alpha = -\frac{dx}{ds},\ \frac{dy}{dx} = -\frac{u_x}{u_y}, \qquad (2.2.9)$$

where

$D = Eh^3/(12(1-\mu^2))$ is the flexural rigidity of the plate,

E is Young's modulus,

μ Poisson's ratio,

h the thickness of the plate.

Since it is not always convenient to use cartesian coordinates our first aim is to obtain the expressions for $P, Q, R, F, G,$ and t in a general orthogonal curvilinear coordinate system. In polar coordinates these become

$$P = -Dt,$$
$$Q = -\frac{D}{t}\Big[u_{rr}u_r^2 + \frac{u_ru_\theta^2}{r^3} + \frac{1}{r^4}u_{\theta\theta}u_\theta^2 + \mu\frac{u_r^3}{r}$$
$$+ \mu\frac{u_{\theta\theta}u_r^2}{r^2} + \mu\frac{u_{rr}u_\theta^2}{r^2} - 2(1-\mu)\Big[\frac{u_{r\theta}u_ru_\theta}{r^2} - \frac{u_ru_\theta^2}{r^3}\Big]\Big],$$
$$R = -Dt^{\frac{3}{2}},$$
$$F = \frac{-D}{t^{\frac{1}{2}}}\Big[3u_r^2u_{rr} + \frac{u_r^3}{r} + \frac{u_r^2u_{\theta\theta}}{r^2} + 4\frac{u_{r\theta}u_ru_\theta}{r^2}$$
$$+ \frac{u_\theta^2u_r}{r^3} + 3\frac{u_{\theta\theta}u_\theta^2}{r^4} + \frac{u_\theta u_{rr}}{r^2}\Big],$$

$$G = \frac{2D(1-\mu)}{t^{\frac{5}{2}}} \Bigg[\frac{u_{r\theta} u_r^2}{r} - \frac{1\; u_{r\theta} u_\theta^2}{r^3} + \frac{u_\theta^3}{r^4}$$

$$+ \frac{u_{\theta\theta} u_r u_\theta}{r^3} - \frac{u_r u_\theta u_{rr}}{r} \Bigg]^2 - \frac{D(1-\mu)}{t^{\frac{3}{2}}} \Bigg[$$

$$- \frac{u_{r\theta} u_\theta u_{rr} u_r}{r^2} - \frac{u_r^2 u_{rr} u_{\theta\theta}}{r^2} - \frac{u_r^2 u_\theta u_{rr\theta}}{r^2}$$

$$+ \frac{u_{\theta\theta\theta} u_r^2 u_\theta}{r^4} + \frac{u_{\theta\theta}^2 u_r^2}{r^4} + \frac{u_{\theta\theta} u_{r\theta} u_r u_\theta}{r^4}$$

$$+ \frac{u_r^3 u_{r\theta\theta}}{r^2} + 2 \frac{u_{r\theta}^2 u_r^2}{r^2} - \frac{u_r u_\theta^2 u_{r\theta\theta}}{r^4}$$

$$-2 \frac{u_{\theta\theta} u_\theta u_{r\theta} u_r}{r^4} + 3 \frac{u_{\theta\theta} u_\theta^2 u_r}{r^5} + \frac{u_{rr}^2 u_\theta^2}{r^2}$$

$$+ \frac{u_r u_{r\theta} u_{rr} u_\theta}{r^2} \qquad \frac{u_{rrr} u_r u_\theta^2}{r^2} \qquad \frac{u_{r\theta\theta} u_r u_\theta^2}{r^4}$$

$$- \frac{u_{\theta\theta} u_{rr} u_\theta^2}{r^4} - \frac{u_{\theta\theta} u_{r\theta} u_r u_\theta}{r^4} - 2 \frac{u_{rr} u_{r\theta} u_\theta}{r^2}$$

$$- \frac{u_r^2 u_{rr\theta} u_\theta}{r^2} + \frac{u_{rr\theta} u_\theta^3}{r^4} + 2 \frac{u_{r\theta}^2 u_\theta^2}{r^4}$$

$$- 6 \frac{u_{r\theta} u_\theta^3}{r^5} - \frac{u_r u_\theta^2 u_{rr}}{r^3} + 3 \frac{u_{\theta\theta} u_r u_\theta^2}{r^5}$$

$$+ \frac{u_r^2 u_{r\theta} u_\theta}{r^3} + 4 \frac{u_\theta^4}{r^6} \Bigg]$$

$$- \frac{D}{t^{\frac{1}{2}}} \Bigg[u_r u_{rrr} + \frac{u_{rr} u_r}{r} - \frac{u_r^2}{r^2} + \frac{u_{r\theta\theta} u_r}{r^2}$$

$$- 2 \frac{u_{\theta\theta} u_r}{r^3} + \frac{u_{rr\theta} u_\theta}{r^2} + \frac{u_{r\theta} u_\theta}{r^3} + \frac{u_{\theta\theta\theta} u_\theta}{r^4} \Bigg],$$

$$t = u_r^2 + \frac{u_\theta^2}{r^2} . \qquad (2.2.10)$$

However for a general system of curvilinear coordinates these expressions become complex and unwieldy. So we proceed by transforming the line integrals $\oint Rds$, $\oint Fds$, $\oint Gds$, etc. into double integral form. Making use of Green's theorem these integrals become

$$\oint_{C_u} Rds = D \iint_{\Omega_u} (t\, \nabla^2 u + \underset{\sim}{\nabla} u \cdot \underset{\sim}{\nabla} t)\, d\Omega,$$

$$\oint_{C_u} Fds = \frac{D}{2} \iint_{\Omega_u} (\nabla^4 u^2 - 2u\nabla^4 u)\, d\Omega\ ,$$

$$\oint_{C_u} Gds = D \iint_{\Omega_u} (\nabla^4 u d\Omega - \oint_{C_u} \frac{\partial H}{\partial S}\, ds, \qquad (2.2.11)$$

$$ds^2 = h_1^2 d\xi^2 + h_2^2\, d\eta^2,$$

$$t = \frac{1}{h_1^2}\left(\frac{\partial u}{\partial \xi}\right)^2 + \frac{1}{h_2^2}\left(\frac{\partial u}{\partial \eta}\right)^2,$$

$$\underset{\sim}{\nabla} = \frac{1}{h_1}\frac{\partial}{\partial \xi}\underset{\sim}{e}_1 + \frac{1}{h_2}\frac{\partial}{\partial \eta}\underset{\sim}{e}_2,$$

$$\nabla^2 = \frac{1}{h_1 h_2}\frac{\partial}{\partial \xi}\left(\frac{h_2}{h_1}\frac{\partial}{d\xi}\right) + \frac{1}{h_1 h_2}\frac{\partial}{\partial \eta}\left(\frac{h_1}{h_2}\frac{\partial}{\partial \eta}\right),$$

where ξ, η are orthogonal curvilinear coordinates, h_1 and h_2 are the metric tensors associated with ξ and η, and $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$ are the corresponding unit base vectors. Consequently these equations are now in a form which may be readily computed for any given system of orthogonal curvilinear coordinates.

The differential operator L, where $L[w]$ is defined by

$$L[w] = \frac{d^3w}{du^3}\oint_{C_u} R ds + \frac{d^2w}{du^2}\oint_{C_u} F ds + \frac{dw}{du}\oint_{C_u} G ds, \qquad (2.2.12)$$

will frequently occur in our further analysis. By replacing the coefficients $\oint R ds$, $\oint F ds$ and $\oint G ds$ by their corresponding double integral form $L[w]$ becomes

$$L[w] = \frac{d^3w}{du^3}\iint_{\Omega_u} D[t\nabla^2 u + \underset{\sim}{\nabla} u \cdot \underset{\sim}{\nabla} t] d\Omega + \frac{d^2w}{du^2}\iint_{\Omega_u} \frac{D}{2}[\nabla^4 u^2 - 2u\nabla^4 u] d\Omega$$

$$+ \frac{dw}{du}\left[\iint_{\Omega_u} D\nabla^4 u d\Omega - \oint_{C_u} \frac{\partial H}{\partial s} ds\right], \qquad (2.2.13)$$

which is a more convenient form than (2.2.12). The contribution of the term

$$-\oint_{C_u} \frac{\partial H}{\partial s} ds \qquad (2.2.14)$$

is particularly interesting, since it vanishes along all smooth closed curves with a continuous outward normal. And so for all smooth closed curves in the interior of the plate we must have

$$\oint_{C_u} \frac{\partial H}{\partial s} ds = 0 . \qquad (2.2.15)$$

However if the contour C_u has a corner point this integral may not vanish.

In the classical formulation of plate theory if one neglects the contribution due to the edge rate of change of the twisting moment, then there arise concentrated reactions at the corners of simply supported polygonal plates. By incorporating the edge rate of change of the twisting moment in our formulation of plate theory we attempt to overcome this incongruous result.

If however we neglect this term, then equation (2.2.6) can be expressed as

$$L[w] - \iint_{\Omega_u} q \, d\Omega = 0 , \qquad (2.2.16)$$

where

$$L[w] = \frac{d^3w}{du^3} \iint_{\Omega_u} D[t\nabla^2 u + \underset{\sim}{\nabla} u \cdot \underset{\sim}{\nabla} t] d\Omega$$

$$+ \frac{d^2w}{du^2} \iint_{\Omega_u} \frac{D}{2} [\nabla^4 u^2 - 2u\nabla^4 u] d\Omega$$

$$+ \frac{dw}{du} \iint_{\Omega_u} D\nabla^4 u \, d\Omega . \qquad (2.2.17)$$

This form for $L[w]$ may also be written[1] as

$$L[w] = \iint_{\Omega_u} D\nabla^4 w \, d\Omega , \qquad (2.2.18)$$

so that equation (2.2.16) now becomes

$$\iint_{\Omega_u} [D\nabla^4 w - q] d\Omega = 0 . \qquad (2.2.19)$$

---

[1] See appendix 1

And so if w satisfies the classical differential equation for the bending of a thin elastic plate, viz

$$D\nabla^4 w = q\,, \qquad (2.2.20)$$

then equation (2.2.19) is automatically satisfied. Thus the formulation of this problem via the classical theory, and the method of constant deflection lines must yield identical solutions when the term $\oint \frac{\partial H}{\partial s}\, ds$ vanishes.

This is the case whenever the contour C_u has no corner points.

Perhaps one of the most important properties of the constant deflection lines, C_u, is their non uniqueness in mathematical form. For example, if for a particular plate problem the lines of constant deflection are known to be

$$u_1(x,y) = \text{const.}, \qquad (2.2.21)$$

then any constant multiple, or integer power of u_1 may also be considered as an appropriate form for the lines of constant deflection. This may best be illustrated by considering the problem of the bending of a uniformly loaded, clamped, circular plate. The deflection contours are known to be a family of concentric circles with centre at the origin, so that the general equation for the equal deflection lines is:

$$u_1(x,y) = a^2 - x^2 - y^2, \qquad (2.2.22).$$

However the solution to this problem is well known and is

$$w(x,y) = \frac{q}{64D}(a^2-x^2-y^2)^2, \tag{2.2.23}$$

and so in accordance with section 2.1 the equation of the lines of equal deflection may be taken as

$$u_2(x,y) = (a^2-x^2-y^2)^2 = \text{const.} \tag{2.2.24}$$

Thus we have obtained two equations u_1 = const., and u_2 = const. for the lines of constant deflection. Indeed any constant multiple of u_1 or u_2 will also describe a family of concentric circles with centre at the origin, and so may be taken as a suitable form for the lines of constant deflection.

2.3 DERIVATION OF THE EQUATIONS OF EQUILIBRIUM FOR THE BUCKLING ANALYSIS OF THIN ELASTIC PLATES

We will assume here that the forces are applied in the plane of the plate and at its edges such that a plane stress system is induced in the plate. If in particular, large compressive stresses are developed in the plate the plane state may become unstable and the plate may bend or buckle. We will further assume that the deflection $w(x,y)$ is always small, and that small bending of the plate does not affect the plane stresses set up by the forces applied at the edges.

Consider a laterally loaded, thin, elastic plate subject to a combination of compressive and shearing forces applied to the middle plate at its edges The deflected form maintained by the plate in a state of neutral equilibrium may be described by a family of lines of

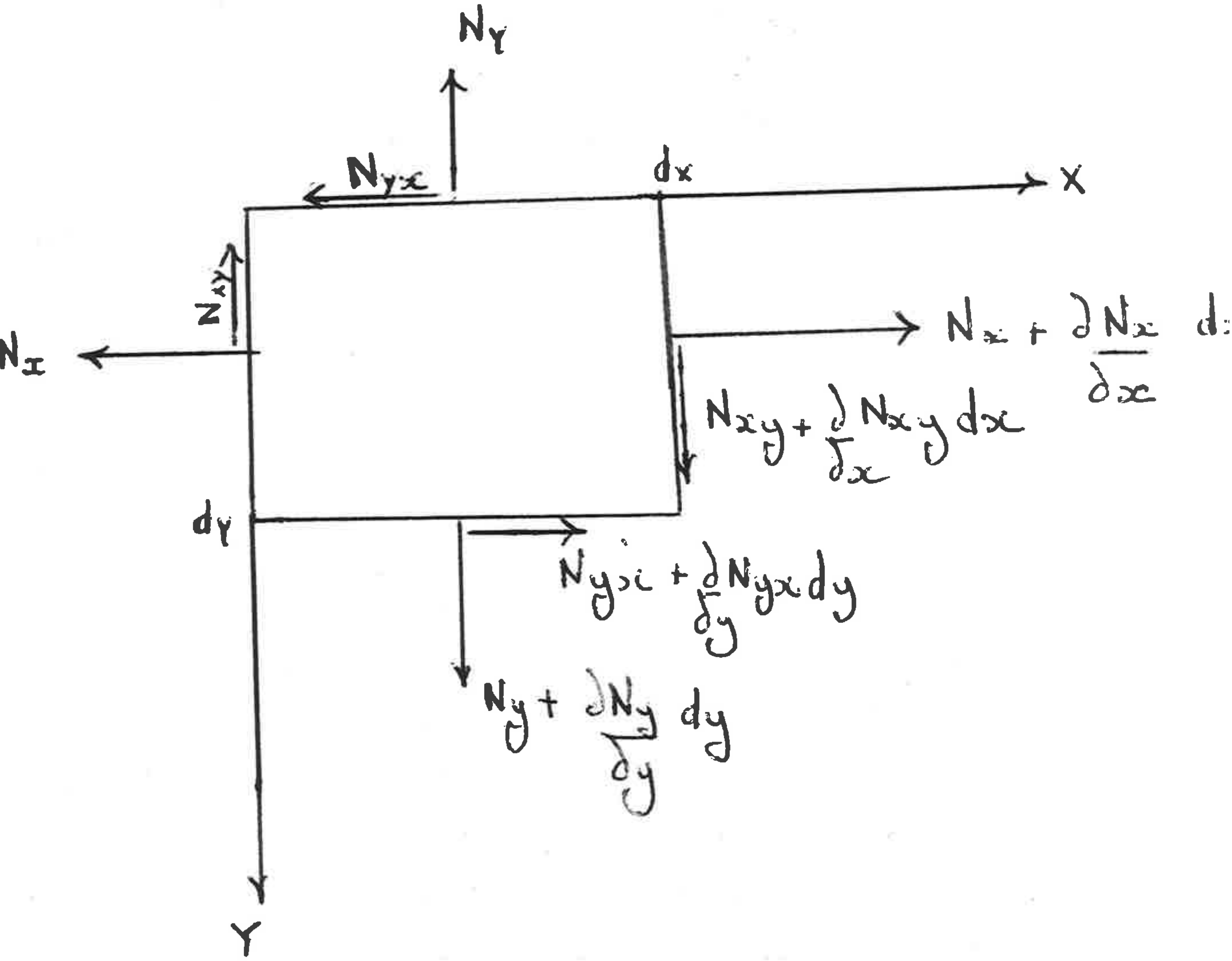


Fig 3.

constant deflection, which arise in the same manner as described in section 2.1.

Consider the equilibrium of an element of the plate bounded by any line of constant deflection C_u. Then proceeding as in section 2.2 we obtain the following moment, and force equation, viz.

$$\Sigma M = \underset{\sim}{n}_o \oint_{C_u} M_n \underset{\sim}{n}\, ds + \underset{\sim}{n}_o \oint_{C_u} V_n \underset{\sim}{r}_o ds - \underset{\sim}{n}_o \iint_{\Omega_u} [$$

$$q + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}] \underset{\sim}{r}\, d\Omega$$

$$= 0, \qquad (2.3.1)$$

and

$$\Sigma Z = \oint_{C_u} V_n ds - \iint_{\Omega_u} [q + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}]$$

$$d\Omega = 0, \qquad (2.3.2)$$

where the term

$$[N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial y \partial x}] d\Omega$$

represents the net downward contribution of the inplane forces N_x, N_y, and N_{xy} acting on a small element of area $d\Omega$ lying entirely within the closed curve C_u (see Fig.3.). Here the contour integration is taken around the closed curve C_u, and the double integration is taken over the area enclosed by the curve C_u.

Substituting into equations (2.3.1), and (2.3.2) the expressions for M_n, V_n, and Q_n we obtain

$$n_o \frac{d^2w}{du^2} \oint_{C_u} P\underset{\sim}{n} ds + \underset{\sim}{n}_o \frac{dw}{du} \oint_{C_u} Q\underset{\sim}{n} ds + \underset{\sim}{n}_o \frac{d^3w}{du^3} \oint_{C_u} R\underset{\sim}{r}_o ds$$

$$+ \underset{\sim}{n}_o \frac{d^2w}{du^2} \oint_{C_u} F\underset{\sim}{r}_o ds + \underset{\sim}{n}_o \frac{dw}{du} \oint_{C_u} G\underset{\sim}{r}_o ds - \underset{\sim}{n}_o \iint_{\Omega_u} [q$$

$$+ N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}] \underset{\sim}{r} d\Omega$$

$$= 0 , \tag{2.3.3}$$

and

$$\frac{d^3w}{du^3} \oint_{C_u} R ds + \frac{d^2w}{du^2} \oint_{C_u} F ds + \frac{dw}{du} \oint_{C_u} G ds$$

$$- \iint_{\Omega_u} [q + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}] d\Omega$$

$$= 0. \tag{2.3.4}$$

Here the term $N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}$ appearing in both equations (2.3.3) and (2.3.4) may be written as

$$N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}$$

$$= K \frac{d^2w}{du^2} + L \frac{dw}{du}, \tag{2.3.5}$$

where

$$K = N_x u_x^2 + N_y u_y^2 + 2N_x u_x u_y , \tag{2.3.6}$$

and

$$L = N_x u_{xx} + N_y u_{yy} + 2N_{xy} u_{xy}. \tag{2.3.7}$$

A special class of problems is obtained from equations (2.3.3), and (2.3.4) by assuming $q \equiv 0$. In other words it is assumed that there are no lateral loads to cause bending. In addition we will always take homogeneous boundary conditions for w.

For the sake of simplicity we will also assume that the horizontal forces are normal and compressive forces, which depend linearly on a factor of proportionality. Under these circumstances it is clear that $w=0$ is always a solution to (2.3.4), and (2.3.3), since $w(x,y)$ is assumed to satisfy homogeneous boundary conditions. This is also the unique solution for $w(x,y)$ when the applied compressive forces are small enough. However there is always a critical value of the compressive forces at which the plane stress state becomes unstable and the plate bends, or in engineering terminology "buckles". Mathematically this means that a bifurcation of the solution takes place for this critical value, and there are solutions for which w is not identically zero. We will be interested in investigating the lowest critical value for which buckling just begins.

Applying these assumptions to equation (2.3.3) and (2.3.4) we obtain

$$\underset{\sim}{n}_0 \frac{d^2w}{du^2} \oint_{C_u} P\underset{\sim}{n} ds + \underset{\sim}{n}_0 \frac{dw}{du} \oint_{C_u} Q\underset{\sim}{n} ds + \underset{\sim}{n}_0 \frac{d^3w}{du^3} \oint_{C_u} R\underset{\sim}{r}_0 ds$$
$$+ \underset{\sim}{n}_0 \frac{d^2w}{du^2} \oint_{C_u} F\underset{\sim}{r} s ds + \underset{\sim}{n}_0 \frac{dw}{du} \oint_{C_u} G\underset{\sim}{r} ds$$

$$+ \underset{\sim}{n}_o \iint_{\Omega_u} N\nabla^2 w \underset{\sim}{r} d\Omega = 0, \tag{2.3.8}$$

and

$$\frac{d^3w}{du^3} \oint_{C_u} R ds + \frac{d^2w}{du^2} \oint_{C_u} F ds + \frac{dw}{du} \oint_{C_u} G ds$$

$$+ \iint_{\Omega_u} N\nabla^2 w \, d\Omega = 0, \tag{2.3.9}$$

where we have assumed that the plate is subjected to normal compressive forces $(-N)$ per unit length uniformly distributed around its edges. Equation (2.3.9) may be further simplified by applying Green's theorem to $\iint_{\Omega_u} N\nabla^2 w d\Omega$, i.e.

$$\iint_{\Omega_u} N\nabla^2 w d\Omega = -N \frac{dw}{du} \oint_{C_u} \sqrt{t} ds \tag{2.3.10}$$

and on substituting in (2.3.9) an ordinary differential equation in w is obtained, viz.

$$\frac{d^3w}{du^3} \oint_{C_u} R ds + \frac{d^2w}{du^2} \oint_{C_u} F ds + \frac{dw}{du} \oint_{C_u} G ds$$

$$- N \frac{dw}{du} \oint_{C_u} \sqrt{t} \, ds = 0. \tag{2.3.11}$$

Our problem is thus reduced to finding the solution for W which satisfies two differential equations (2.3.8) and (2.3.11), and subsequently to obtain the lowest value of the critical load N_{crt}/D. However it must be noted that when $u = u(x,y)$ is the correct form for the lines of constant deflection then the moment equation (2.3.8) and the force equation (2.3.9) will be identical.

2.4 DERIVATION OF THE EQUATIONS GOVERNING THE TRANSVERSE VIBRATION OF THIN ELASTIC PLATES

Consider a thin, elastic, plate of homogeneous isotropic material loaded by compressive or tensile forces acting in the middle plane of the plate. The problem is two-fold, the determination of the lines of constant deflection, and the subsequent time dependent deflection field. The present analysis is applicable strictly to the case of a "thin plate", in which higher modes of vibration are of secondary importance.

When the plate vibrates in a normal mode the deflected form maintained by the plate any any instant τ may be described by a family of constant deflection lines. If $w(x,y,\tau)$ denotes the transverse displacement of a point in the middle surface of the plate, then w is a function of spatial coordinates (x,y), and time τ. So it is possible to write

$$w = W(x,y) \cos (\omega\tau + \varepsilon), \qquad (2.4.1)$$

where $\cos (\omega\tau + \varepsilon)$ is the normal coordinate, ω is the angular frequency, and W is the amplitude of the vibrating plate, which determines the form of the deflected surface. Consequently W can be expressed as a suitable function of u, where $u(x,y)$ = const. is the equation of the lines of equal deflection.

Consider a portion of the plate bounded by a closed contour C_u at any instant τ. We now require the sum of the moments about the tangent line to the curve $u(x,y)$ = const. at any point (x_0, y_0) of all the forces acting on the element, and the sum of all the forces normal to the plane of the plate to vanish. Thus we obtain the following dynamical equations

$$\underset{\sim}{n}_0 \oint_{C_u} M_n \underset{\sim}{n} ds + \underset{\sim}{n}_0 \oint_{C_u} V_n \underset{\sim}{r}_0 ds - \underset{\sim}{n}_0 \iint_{\Omega_u} [q - \rho h \frac{\partial^2 w}{\partial \tau^2}$$

$$+ N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}] \underset{\sim}{r} d\Omega$$

$$= 0, \tag{2.4.2}$$

and

$$\oint_{C_u} V_n ds - \iint_{\Omega_u} [q - \rho h \frac{\partial^2 w}{\partial \tau^2} + N_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

$$+ 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}] d\Omega$$

$$= 0, \tag{2.4.3}$$

where we have considered the combined effects of lateral loading, and inplane loading as well as transverse vibration.

If we are primarily interested in the free vibration of plates without any inplane or transverse loading, then equations (2.4.2), and (2.4.3) become

$$\underset{\sim}{n}_o \oint_{C_u} M_n \underset{\sim}{n} ds + \underset{\sim}{n}_o \oint_{C_u} V_n \underset{\sim}{r}_o ds + \underset{\sim}{n}_o \iint_{\Omega_u} \rho h \frac{\partial^2 w}{\partial \tau^2} \underset{\sim}{r} d\Omega$$

$$= 0, \tag{2.4.4}$$

and

$$\oint_{C_u} V_n ds + \iint_{\Omega_u} \rho h \frac{\partial^2 w}{\partial \tau^2} d\Omega = 0, \tag{2.4.5}$$

where the double integral in (2.4.5) represents the inertial force due to the vertical acceleration, ρ being the mass per unit area of the plate. If we now substitute in the expressions for the bending moment M_n, and the transverse force V_n, then after the factor $\cos(\omega\tau+\varepsilon)$ has been cancelled we obtain

$$\underset{\sim}{n}_o \frac{d^2W}{du^2} \oint_{C_u} P\underset{\sim}{n} ds + \underset{\sim}{n}_o \frac{dW}{du} \oint_{C_u} Q\underset{\sim}{n} ds + n_o \frac{d^3W}{du^3} \oint_{C_u} R\underset{\sim}{r}_o ds$$

$$+ \underset{\sim}{n}_o \frac{d^2W}{du^2} \oint_{C_u} F\underset{\sim}{r}_o ds + \underset{\sim}{n}_o \frac{dW}{du} \oint_{C_u} G\underset{\sim}{r}_o ds$$

$$- \iint_{\Omega_u} \rho h \omega^2 W \underset{\sim}{r} d\Omega = 0, \tag{2.4.6}$$

and

$$\frac{d^3W}{du^3} \oint_{C_u} R ds + \frac{d^2W}{du^2} \oint_{C_u} F ds + \frac{dW}{du} \oint_{C_u} G ds - \rho h \omega^2 \iint_{\Omega_u} W d\Omega$$

$$= 0. \tag{2.4.7}$$

where R,F,G,P, and Q are the same expressions involving u and its partial derivatives as given in section 2.1.

Thus W must satisfy two integro-differential equations (2.4.6) and (2.4.7). However if $u = u(x,y)$ is of the correct form for lines of constant deflection these equations will be identical.

Alternatively if we are interested in the transverse vibration of thin elastic plates, including the effects of inplane forces, then the governing equation will have the form

$$\underset{\sim}{n}_o \oint_{C_u} M_n \underset{\sim}{n} ds + \underset{\sim}{n}_o \oint_{C_u} V_n \underset{\sim}{r}_o ds$$

$$+ \underset{\sim}{n}_o \iint_{\Omega_u} [\rho h \frac{\partial^2 w}{\partial \tau^2} - N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} - 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}]$$

$$\cdot \underset{\sim}{r} d\Omega = 0, \qquad (2.4.8)$$

and

$$\oint_{C_u} V_n ds + \iint_{\Omega_u} [\rho h \frac{\partial^2 w}{\partial \tau^2} - N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2}$$

$$- 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}] d\Omega$$

$$= 0. \qquad (2.4.9)$$

And so upon substituting in equations (2.4.1) and (2.4.2) the expressions for M_n, V_n, and Q_n, and after cancelling the factor $\cos(\omega\tau + \varepsilon)$ we obtain

$$\underset{\sim}{n}_o \frac{d^2W}{du^2} \oint_{C_u} P\underset{\sim}{n}ds + \underset{\sim}{n}_o \frac{dW}{du} \oint_{C_u} Q\underset{\sim}{n}ds +$$

$$\underset{\sim}{n}_o \frac{d^3W}{du^3} \oint_{C_u} R\underset{\sim}{r}_o ds + \underset{\sim}{n}_o \frac{d^2W}{du^2} \oint_{C_u} F\underset{\sim}{r}_o ds +$$

$$\underset{\sim}{n}_o \frac{dW}{du} \oint_{C_u} G\underset{\sim}{r}_o ds - \underset{\sim}{n}_o \iint_{\Omega_u} [\rho h\omega^2 W + N_x \frac{\partial^2 W}{\partial x^2}$$

$$+ N_y \frac{\partial^2 W}{\partial y^2} + 2N_{xy} \frac{\partial^2 W}{\partial x \partial y}]\underset{\sim}{r}d\Omega = 0, \qquad (2.4.10)$$

and

$$\frac{d^3W}{du^3} \oint_{C_u} Rds + \frac{d^2W}{du^2} \oint_{C_u} Fds + \frac{dW}{du} \oint_{C_u} Gds$$

$$- \iint_{\Omega_u} [N_x \frac{\partial^2 W}{\partial x^2} + N_y \frac{\partial^2 W}{\partial y^2} + 2N_{xy} \frac{\partial^2 W}{\partial x \partial y} + \rho h\omega^2 W]d\Omega$$

$$= 0. \qquad (2.4.11)$$

Thus our problem is again reduced to finding the solution W which satisfies the two integro-differential equations (2.4.10) and (2.4.11), which must be identical when $u = u(x,y)$ is the correct form for the lines of constant deflection.

2.5 DERIVATION OF THE NEW EQUATIONS OF EQUILIBRIUM FOR THE LARGE DEFORMATION OF THIN ELASTIC PLATES

An extension of the linear theory, for the bending of thin elastic plates, was developed by Von Karman [59], in which the squares of the slopes of the middle surface were not neglected as in the case of the linear theory. At the same time the hypothesis that the stresses in the middle surface are zero, was rejected. The resulting equations are the following pair of fourth order, non linear, differential equations for two functions, the deflection $w(x,y)$, and a function $\phi(x,y)$ from which the stresses in the middle surface, called "membrane stresses", are derived:

$$D\nabla^4 w = q + h\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \phi}{\partial x^2} - \frac{2\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right] \quad (2.5.1)$$

and

$$\nabla^4 \phi = E\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right], \quad (2.5.2)$$

where h is the thickness of the plate, and E is a material constant. The stresses σ_x', σ_y', and τ_{xy} in the middle surface are obtained from ϕ by the formulae

$$\sigma_x' = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y' = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}. \quad (2.5.3)$$

Proceeding via the method of constant deflection lines, we obtain the following force equation:

$$\frac{d^3 w}{du^3}\oint_{C_u} R\,ds + \frac{d^2 w}{du^2}\oint_{C_u} F\,ds + \frac{dw}{du}\oint_{C_u} G\,ds - \iint_{\Omega_u}\Big[q$$

$$+ h\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \phi}{\partial x^2} - 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \phi}{\partial y \partial x}\right]d\Omega$$

$$= 0, \quad (2.5.4)$$

and a corresponding moment equation, viz.

$$\underset{\sim}{n}_o \frac{d^2w}{du^2} \oint_{C_u} P\underset{\sim}{n}ds + \underset{\sim}{n}_o \frac{dw}{du} \oint_{C_u} Q\underset{\sim}{n}ds +$$

$$\underset{\sim}{n}_o \frac{d^3w}{du^3} \oint_{C_u} R\underset{\sim}{r}_o ds + \underset{\sim}{n}_o \frac{d^2w}{du^2} \oint_{C_u} F\underset{\sim}{r}_o ds +$$

$$\underset{\sim}{n}_o \frac{dw}{du} \oint_{C_u} G\underset{\sim}{r}_o ds - \underset{\sim}{n}_o \iint_{\Omega_u} [q +$$

$$h\left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 \phi}{\partial x^2} - 2 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 \phi}{\partial x \partial y}\right)]$$

$$\underset{\sim}{r} d\Omega = 0, \qquad (2.5.5)$$

together with equation (2.5.2).

In order to solve for ϕ and w we must first obtain the true equation for the lines of constant deflection $u(x,y) = \text{const.}$ This may be achieved from the requirement that if $u(x,y)$ is correct, then the force equation (2.5.4), and the moment equation (2.5.5) must both be identically satisfied. However the solution of these equations and the derivation of $u(x,y)$ is at best extremely difficult.

An interesting method for analyzing the large amplitude deflection of plates was presented by Berger [2] in his doctoral dissertation (1956). Essentially the method is based upon the neglect of the second invariant of the middle surface strains in the expression corresponding to the total potential energy of the system.

Application of a variational technique to this simplified enery expression yields an approximate differential equation for the equilibrium of the plate. For the several cases of static loading of an initially flat plate investigated by Berger the resulting approximate equilibrium equations are still non linear, but can be decoupled in such a manner that they may be readily solved. Although no complete explanation of the validity of this method has been offered, the stresses and deflections obtained for both rectangular and circular plates agree with those found from the more precise analysis. Indeed Berger's analysis appears to be widely accepted, and has been applied to shell theory, and to the theory of anisotropic plates.

Adopting Berger's approximation, where the second invariant of the strain tensor has been neglected, the following differential equations occur:

$$\nabla^4 w - \alpha^2 \nabla^2 w = \frac{q}{D} , \qquad (2.5.6)$$

and

$$\begin{aligned} e &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ &= \frac{\alpha^2 h^2}{12} \\ &= \text{const.}, \end{aligned} \qquad (2.5.7)$$

where e is the first invariant of the strain tensor and α^2 is a normalized constant of integration.

Comparing equations (2.5.1), and (2.5.6) we note that the term $\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}$ was replaced by $\alpha^2\nabla^2 w$, so that formulation of the problem by the method of constant deflection lines yields

$$\frac{d^3w}{du^3}\oint_{C_u} R\,ds + \frac{d^2w}{du^2}\oint_{C_u} F\,ds + \frac{dw}{du}\oint_{C_u} G\,ds$$

$$+ \alpha^2 D\frac{dw}{du}\oint_{C_u}\sqrt{t}\,ds - \iint_{\Omega_u} q\,d\Omega$$

$$= 0, \tag{2.5.8}$$

and

$$\underset{\sim}{n}_o\frac{d^2w}{du^2}\oint_{C_u} P\underset{\sim}{n}\,ds + \underset{\sim}{n}_o\frac{dw}{du}\oint_{C_u} Q\underset{\sim}{n}\,ds + \underset{\sim}{n}_o\frac{d^3w}{du^3}\oint_{C_u} R\underset{\sim}{r}_o\,ds$$

$$+ \underset{\sim}{n}_o\frac{d^2w}{du^2}\oint_{C_u} F\underset{\sim}{r}_o\,ds + \underset{\sim}{n}_o\frac{dw}{du}\oint_{C_u} G\underset{\sim}{r}_o\,ds - \underset{\sim}{n}_o\iint_{\Omega_u}[q-D\alpha^2\nabla^2 w]$$

$$\underset{\sim}{r}\,d\Omega = 0, \tag{2.5.9}$$

together with the requirement that

$$e = \frac{\alpha^2 h^2}{12}$$

$$= \text{const.} \tag{2.5.10}$$

It is particularly interesting to note that equations (2.5.9), and (2.5.8) are identical to equations (2.3.4), and (2.3.3) of section 2.3 for the bending of a thin elastic plate, subject to hydrostatic edge loading

$$N_x = N_y = \alpha^2 D, \quad N_{xy} = 0 \tag{2.5.11}$$

and transverse load $q(x,y)$.

2.6 DERIVATION OF THE NEW EQUATIONS FOR THE LARGE AMPLITUDE VIBRATION OF THIN ELASTIC PLATES

The large amplitude vibration of thin elastic plates is governed by two, coupled, non linear, partial differential equations, which were initially studied by Herrman [20]. The general solutions of these equations is unknown, but first approximations to these solutions have been obtained by Chu and Herrman [19], and Yamaki [66]. As for the large amplitude deflection of plates, the theory of the large amplitude vibration of plates has become increasingly important, and several attempts have been made to obtain a simpler theory.

In 1960 Nash and Modeer[44] extended Berger's equations to the dynamical problem by adding the transverse inertia terms, and obtained results comparable to those of Chu and Herrman [19] for a rectangular plate with simply supported edges. Subsequently in 1963 Wah [61] again simplified the approach, considering the large amplitude vibration of a rectangular plate with three different boundary conditions, and in 1964 examined the large amplitude vibrations of circular plates [62].

Assuming that the deflection surface $w(x,y,\tau)$ is represented by

$$\begin{aligned} w(x,y,\tau) &= W(x,y)\ \theta(\tau) \\ &= W(u)\ \theta(\tau), \qquad (2.6.1) \end{aligned}$$

then when the transverse inertia terms are considered equations (2.4.1), and (2.4.2) become

$$\underset{\sim}{n}_o \frac{d^2W}{du^2} \oint_{C_u} P\underset{\sim}{n}ds + \underset{\sim}{n}_o \frac{dW}{du} \oint_{C_u} Q\underset{\sim}{n}ds +$$

$$\underset{\sim}{n}_o \frac{d^3W}{du^3} \oint_{C_u} R\underset{\sim}{r}_o ds + \underset{\sim}{n}_o \frac{d^2W}{du^2} \oint_{C_u} F\underset{\sim}{r}_o ds$$

$$+ \underset{\sim}{n}_o \frac{dW}{du} \oint_{C_u} G\underset{\sim}{r}_o ds - \underset{\sim}{n}_o \iint_{\Omega_u} [\alpha^2 D\nabla^2 w$$

$$- \frac{\rho hW}{\theta} \frac{d\theta^2}{d\tau^2}]\underset{\sim}{r}d\Omega = 0, \tag{2.6.2}$$

and

$$\frac{d^3W}{du^3} \oint_{C_u} Rds + \frac{d^2W}{du^2} \oint_{C_u} Fds + \frac{dW}{du} \oint_{C_u} Gds - \iint_{\Omega_u} [\alpha^2 D\nabla^2 W$$

$$- \frac{\rho hW}{\theta} \frac{d^2\theta}{d\tau^2}]d\Omega = 0, \tag{2.6.3}$$

together with the condition that

$$e = \frac{\alpha^2 h^2}{12}. \tag{2.6.4}$$

Here we have made use of the relations

$$\frac{\partial w}{\partial x} = \frac{dW}{du} u_x \theta, \quad \frac{\partial w}{\partial y} = \frac{dW}{du} u_y \theta, \quad \text{etc.} \tag{2.6.5}$$

At any instant τ these equations are identical to those obtained in section 2.4 for the transverse vibration of plates with inplane forces $N_x = N_y = \alpha^2 D$, $N_{xy} = 0$.

CHAPTER III

BENDING OF THIN ELASTIC PLATES

3.1 DERIVATION OF THE LINES OF CONSTANT DEFLECTION.

In this section a method for determining the lines of constant deflection $u(x,y) = \text{const.}$ will be discussed. If we substitute the expressions for the vector quantities $\underset{\sim}{n}_o$, $\underset{\sim}{n}$, $\underset{\sim}{r}_o$, $\underset{\sim}{r}$ as given by equation (2.2.2) into equation (2.2.5), and make use of equation (2.2.6), then after simplification we finally obtain, instead of (2.2.5) the following ordinary differential equation in $w(u)$:

$$A_1 \frac{d^3w}{du^3} + A_2 \frac{d^2w}{du^2} + A_3 \frac{dw}{du} - A_4 = 0, \qquad (3.1.1)$$

where

$$A_1 = \oint_{C_u} (\lambda_1 x + \lambda_2 y) R ds$$

$$A_2 = \oint_{C_u} [(\lambda_1 x + \lambda_2 y) F + (\lambda_2 u_x + \lambda_2 u_y) \frac{P}{\sqrt{t}}] ds \qquad (3.1.2)$$

$$A_3 = \oint_{C_u} [(\lambda_1 u_x + \lambda_2 u_y) \frac{Q}{\sqrt{t}} + (\lambda_1 x + \lambda_2 y) G] ds$$

$$A_4 = \iint_{\Omega_u} (\lambda_1 x + \lambda_2 y) q d\Omega,$$

and λ_1, λ_2 are constants given by

$$\lambda_1 = \left(\frac{u_x}{\sqrt{t}}\right)_o$$

$$\lambda_2 = \left(\frac{u_y}{\sqrt{t}}\right)_o, \qquad (3.1.3)$$

where $(\ )_0$ indicates that the expression is evaluated at the fixed point (x_0, y_0).

Our problem is thus reduced to finding the solution for w which satisfies two third order ordinary differential equation (2.2.6) and (3.1.1). It is to be noted that when $u = u(x,y)$ is an exact equation for a line of equal deflection then the moment equation (2.2.5) and the force equation (2.2.6) are identical. This can easily be verified for the simplest case of a uniformly loaded circular plate of radius a by considering $u = a^2-x^2-y^2$ as the exact equation for a line of equal deflection, the origin of coordinates being at the centre of the plate. Thus we must expect the following relationships between the coefficients to exist, viz.

$$\frac{A_1}{\oint R ds} = \frac{A_2}{\oint F ds} = \frac{A_3}{\oint G ds} = \frac{A_4}{\iint q d\Omega} \tag{3.1.4}$$

And so denoting

$$\kappa_1(c) = \frac{\oint_{C_u} xR ds}{\oint_{C_u} R ds}, \quad \kappa_2(c) = \frac{\oint_{C_u} yR ds}{\oint_{C_u} R ds}, \tag{3.1.5}$$

where in general $\kappa_1(c)$ and $\kappa_2(c)$ are functions of c, such that $u(x,y) = c$ is the equation for C_u, then the first and the last relationships in (3.1.4) gives

$$\lambda_1 \iint_{\Omega_u} q(x-\kappa_1(c))d\Omega + \lambda_2 \iint_{\Omega_u} q(y-\kappa_2(c))d\Omega = 0. \qquad (3.1.6)$$

However since the double integrals appearing in (3.1.6) are independent of the fixed point (x_0,y_0), and since the coefficients λ_1 and λ_2 depend only on (x_0,y_0) we have

$$\iint_{\Omega_u} q(x-\kappa_1(c))d\Omega = 0, \qquad (3.1.7)$$

and

$$\iint_{\Omega_u} q(y-\kappa_2(c))d\Omega = 0. \qquad (3.1.8)$$

As explained in [37] equation (3.1.7) and (3.1.8) may be expressed as

$$-\int_{u^*}^{u} \oint_{C_{u_0}} \frac{q(x-\kappa_1(c))}{\sqrt{t}} ds\, du_0 = 0, \qquad (3.1.9)$$

and

$$-\int_{u^*}^{u} \oint_{C_{u_0}} \frac{q(y-\kappa_2(c))}{\sqrt{t}} ds\, du_0 = 0, \qquad (3.1.10)$$

where u^* denotes the maximum value of u and C_{u_0} denotes an arbitrary curve $u = u_0$ inside the domain under consideration. So that in order that (3.1.9) and (3.1.10) are satisfied for all admissable values of u in $[0,u^*]$ we require

$$\oint_{C_u} q \, \frac{(x-\kappa_1(u))}{\sqrt{t}} ds = 0, \qquad (3.1.11)$$

and

$$\oint_{C_u} q \, \frac{(y-\kappa_2(u))}{\sqrt{t}} ds = 0. \qquad (3.1.12)$$

3.2 ANALYSIS FOR CONSTANT LOAD

If the load is uniform then since on each curve C_u, $t = u_x^2 + u_y^2$ is a finite, and strictly positive quantity it follows from (3.1.11) and (3.1.5) that

$$\oint_{C_u} (x-\kappa_1(u))/\sqrt{t} \; ds = 0, \qquad (3.2.1)$$

and similarly from (3.1.12), and (3.1.5) that

$$\oint_{C_u} (y-\kappa_2(u))/\sqrt{t} \; ds = 0, \qquad (3.2.2)$$

so that multiplying (3.2.2) by i and adding to (3.2.1); and noting that $ds = \dfrac{u_y + iu_x}{\sqrt{t}} dz$ we obtain

$$\oint_{C_u} \frac{(z-\xi)(u_y + iu_x)}{t} dz = 0, \qquad (3.2.3)$$

where

$$z = x + iy \quad \text{and} \quad \xi = \kappa_1 + i\kappa_2. \qquad (3.2.4)$$

Consider the following three cases: clamped, simply supported, and partly clamped and partly simply supported plates.

(a) CLAMPED EDGE. Along a clamped edge, we have only the geometrical boundary conditions viz.

$$\text{(i)}\quad u\Big|_{\Gamma} = 0, \quad \text{(ii)}\quad w\Big|_{u=0} = 0 \quad \text{(iii)}\quad \frac{\partial w}{\partial n} = \sqrt{t}\,\frac{dw}{du}\Big|_{u=0} = 0. \tag{3.2.5}$$

We note that equation (3.2.3) will be identically satisfied if we assume the integrand, or since t is a bounded function throughout the region of the plate, $(z-\xi)(u_y + iu_x)$ be an analytic expression. Due to the analyticity of the above expression it follows that

$$\nabla^2[(x-\kappa_1)u_x+(y-\kappa_2)u_y] = 0 \tag{3.2.6}$$

and

$$\nabla^2[(x-\kappa_1)u_y-(y-\kappa_2)u_x] = 0 \tag{3.2.7}$$

where the expressions in brackets are first evaluated along the contour $u(x,y)$ = const.. Making the coordinate transformations

$$\left.\begin{aligned} x - \kappa_1 &= X \\ y - \kappa_2 &= Y \end{aligned}\right\} \tag{3.2.8}$$

equations (3.2.6) and (3.2.7) transform to

$$\nabla^2(Xu_X + Yu_Y) = 0 \tag{3.2.9}$$

and

$$\nabla^2(Xu_Y - Yu_X) = 0 \tag{3.2.10}$$

or in terms of polar coordinates (r,θ) given by

$$\left.\begin{aligned} X &= r\cos\theta \\ Y &= r\sin\theta \end{aligned}\right\} \tag{3.2.11}$$

we have

$$\nabla^2\, r\, u_r = 0 \tag{3.2.12}$$

and

$$\nabla^2 u_\theta = 0 . \qquad (3.2.13)$$

Thus we obtain a pair of linear partial differential equations for the determination of the lines of equal deflection for a uniformly loaded, clamped plate. It has been mentioned earlier that the equation of a line of equal deflection is not unique, and consequently we will be interested here to obtain any function $u(r,\theta)$ which will satisfy either one or both of the above two equations subject to the condition that $u(r,\theta)$ vanishes on the boundary of the plate. This can easily be achieved if we assume

$$u(r,\theta) = H(r,\theta) + cr^2 \qquad (3.2.14)$$

where $H(r,\theta)$ is a harmonic function in r and θ, and c is a constant. That this form of u satisfies one of the equations (3.2.12) and (3.2.13) is apparent, from the fact that

$$ru_r = r\frac{\partial H(r,\theta)}{\partial r} - 2H(r,\theta) + 2u . \qquad (3.2.15)$$

And since both $r\frac{\partial H(r,\theta)}{\partial r}$ and $H(r,\theta)$ are harmonic functions and u is a numerical constant on the line of equal deflection, equation (3.2.12) is thus obviously satisfied. Further, since $u = \text{const.}$ represents the family of lines of equal deflection, the arbitrary constant appearing in the second term in (3.2.14) can be taken, without loss of generality, in such a way that we can get

$$\nabla^2 u(r,\theta) = -2 \qquad (3.2.16)$$

and obviously $u=0$ on the boundary of the plate. Thus our problem of finding the function defining the constant deflection contours of a uniformly loaded, clamped plate, bears an interesting analogy to Prandtl's stress function Ψ in the torsion problem of a beam of arbitrary solid cross-section. Hence we conclude that the lines of equal deflection $u(x,y) = \text{const.}$ for a uniformly loaded, clamped plate are similar to the lines of shearing stress $\Psi(x,y) = \text{const.}$ in the torsion problem for a beam having the same cross-section as that of the plate boundary.

(b) SIMPLY SUPPORTED EDGE. Along a boundary which is simply supported, we have the following conditions

$$\text{(i)} \quad u\big|_C = 0, \quad \text{(ii)} \quad w\big|_{u=0} = 0,$$
$$\text{(iii)} \quad M_n = P\frac{d^2w}{du^2} + Q\frac{dw}{du}\bigg|_{u=0} = 0, \quad (3.2.17)$$

where P and Q are functions of u and its derivatives.

Generally it is almost impossible in the simply-supported case to find the exact functions u and w such that they satisfy the mechanical boundary conditions (iii) above. Therefore in this case we will proceed as follows.

Remembering that the derivatives $\frac{dw}{du}$ and $\frac{d^2w}{du^2}$ are constants (may be zero) on the boundary $u=0$, in order to fulfil condition(iii) above, we demand that u and w must satisfy

$$\text{(i)} \quad Q\big|_{u=0} = 0, \quad \text{(ii)} \quad \frac{d^2w}{du^2}\bigg|_{u=0} = 0. \quad (3.2.18)$$

Further, since u must also satisfy the boundary condition (i) of (3.2.17), we will seek $u(x,y)$ of the form

$$u = UV, \tag{3.2.19}$$

such that both U and V are functions of spatial coordinates and $u = U$ is the equation for the lines of equal deflection of the same plate when the edges are clamped. Remembering that U and V are both bounded functions throughout the plate, we deduce[1] that V also satisfies the same equation as for U, viz.

$$\nabla^2 \left.\begin{matrix} U(r,\theta) \\ V(r,\theta) \end{matrix}\right\} = -2, \tag{3.2.20}$$

such that

$$\text{(i)}\quad U\big|_C = 0, \quad \text{(ii)}\quad Q\big|_{u=0} = 0. \tag{3.2.21}$$

The first condition will determine U uniquely and consequently through the use of the second condition, V can be determined and hence the function u. Once the function u is known, the differential equation (2.2.6) for deflection can be solved with conditions (ii) of (3.2.17) and (3.2.18) and the condition required at the centre.

(c) SIMPLY SUPPORTED - CLAMPED EDGE. In case when the plate is simply supported along certain parts of its edge Γ_1 and clamped on the remainder $\Gamma - \Gamma_1$ we have the following boundary conditions

[1] See appendix 2.

$$(i)\ \ u\big|_{\Gamma} = 0,\quad (ii)\ \ w\big|_{u=0} = 0,\quad (iii)\ P\,\frac{d^2w}{du^2} + Q\,\frac{dw}{du}\bigg|_{\Gamma_1} = 0,$$

$$(iv)\ \ \sqrt{t}\,\frac{dw}{du}\bigg|_{\Gamma-\Gamma_1} = 0. \qquad (3.2.22)$$

This includes the separate cases where the whole of the boundary is simply supported, i.e., $\Gamma \equiv \Gamma_1$, and also where the whole of the boundary is clamped, i.e. $\Gamma_1 \equiv 0$.

Small deflection plate theory requires the satisfaction of two boundary conditions for w and we note that the simply supported and clamped edge portions have one condition in common with them, that is, zero deflections. Thus, we can obtain the required condition from that for a completely simply supported plate, merely by applying suitable distribution of moments around the required portions of the edge $\Gamma-\Gamma_1$ so as to make the resultant slopes there zero. Hence, as in the case (b) above for a simply supported edge, we shall seek $u(x,y)$ in the form of the equation (3.2.19) such that both U and V appearing therein satisfy (3.2.20) subject to the following requirements

$$(i)\ \ U\big|_{\Gamma} = 0,\quad (ii)\ \ Q\big|_{\Gamma_1} = 0,\quad (iii)\ \ \sqrt{t}\,\frac{dw}{du}\bigg|_{\Gamma-\Gamma_1} = 0 \ \ \text{i.e.,}$$

$$V\big|_{\Gamma-\Gamma_1} = 0, \qquad (3.2.23)$$

Condition (i) above uniquely determines U and consequently, using conditions (ii) and (iii) V is

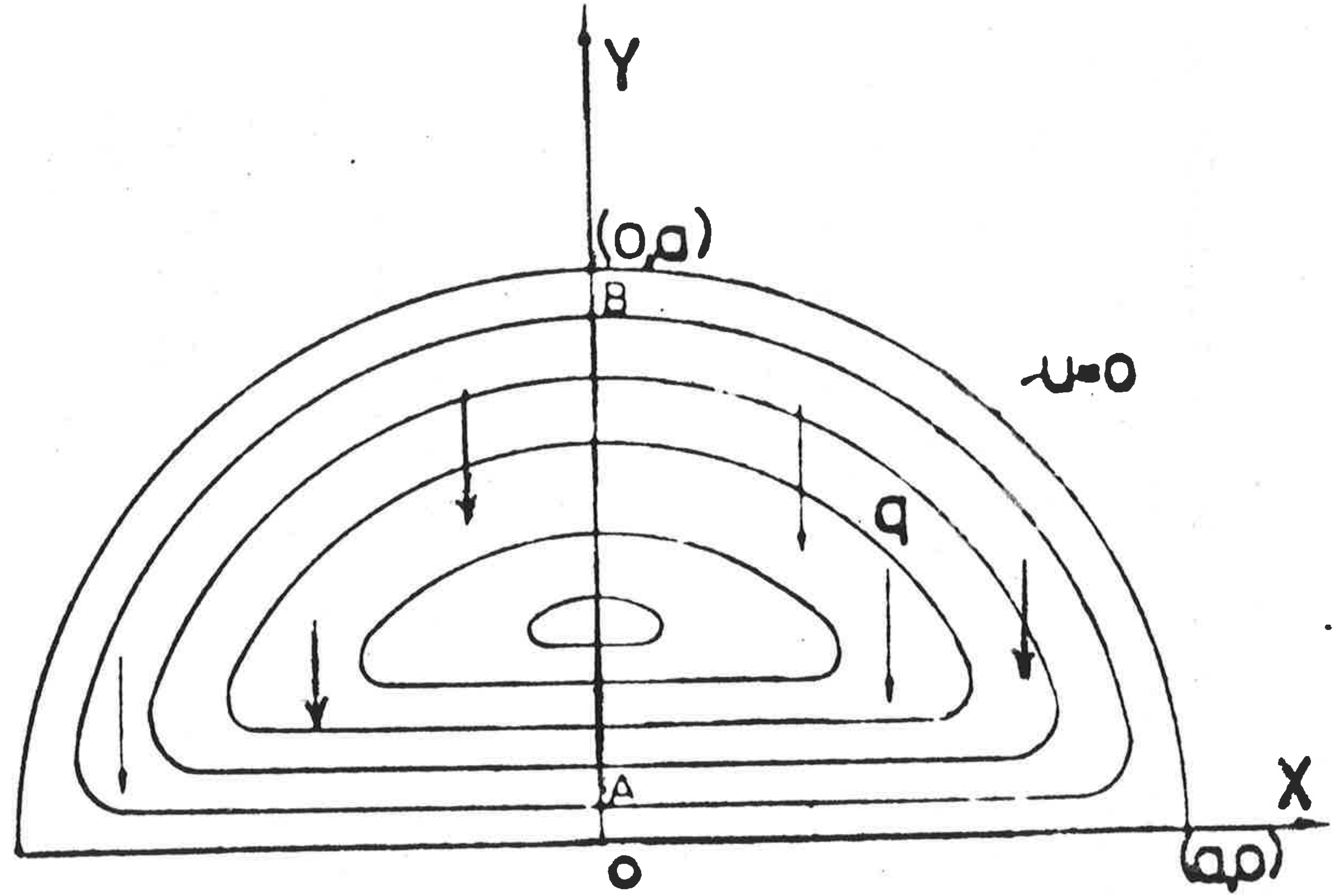


Fig 4 u=const.

determined and hence u and w. In order to make the procedure outlined above clearer, several problems will be discussed in the next section.

3.3 THE BENDING OF A UNIFORMLY LOADED, SEMICIRCULAR PLATE

As an illustration of the procedure, let us discuss at first the case of a uniformly loaded semicircular plate clamped along its edges. This problem is chosen as a model for the purpose of analysis as well as to demonstrate the fact that when the true equation of a line of equal deflection is known, one can expect to obtain a more exact solution than given in an earlier paper [38] where an approximate equation of lines of equal deflection was considered.

If the boundaries of the semicircular plate are given by

$$r = a, \ -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2} \tag{3.3.1}$$

then the solution of (3.2.16) subject to the condition that $u(r,\theta) = 0$ on the boundaries of the plate can be obtained by the same way as the stress function Ψ for a bar of semicircular cross-section is obtained. This is given by [56]

$$\Psi = \frac{1}{2}\left[-r^2(1 + \cos 2\theta) + \frac{16a^2}{\pi^2} \sum_{n=1,3,5,\ldots} (-1)^{\frac{n+1}{2}} \left[\frac{r}{a}\right]^n \frac{\cos n\theta}{n(n^2-4)}\right] \tag{3.3.2}$$

Therefore we can, without loss of generality, consider the expression of $u(r,\theta) = \text{const.}$ as

$$u(r,\theta) = \left[-r^2(1+\cos 2\theta) + \frac{16a^2}{\pi^2} \sum_{n=1,3,5\ldots} (-1)^{\frac{n+1}{2}} \left[\frac{r}{a}\right]^n \frac{\cos n\theta}{n(n^2-4)}\right] \tag{3.3.3}$$

This represents now the true equation of lines of equal deflection for a semicircular plate under uniform loading. Using this form for the function u, we can now proceed to solve (2.2.6). Since we are using the polar coordinates, it is desirable to use the expressions for P,Q,R,F, etc. given by equations (2.2.10).

As we see, the differential equation (2.2.6) for the deflection w contains coefficients in the form of contour integrals which are, in the present case, difficult to evaluate analytically. Consequently, we must attempt to solve the equation (2.2.6) by numerical methods. This we will attempt to do by denoting

$$\begin{aligned} \oint_{C_u} R\,ds &= \overline{R}s \\ \oint_{C_u} F\,ds &= \overline{F}s \\ \oint_{C_u} G\,ds &= \overline{G}s \\ \iint_{\Omega_u} q\,d\Omega &= \oint_{C_u} T\,ds = \overline{T}s \end{aligned} \tag{3.3.4}$$

where, $\overline{R},\overline{F},\overline{G},\overline{T}$ denote the mean values of R,F,G,T on the contour u = const. with perimeter s, and are evaluated by using the same technique as mentioned in [35]. Under general conditions, there exists a unique solution of (2.2.6). However, the analytical determination of $w(u)$ is quite another story from that of its existence and usually offers what are at present insurmountable problems specially when the plate has an arbitrary shape. Hence the approach here is through numerical analysis, i.e., we attempt to find the solution of the differential equation for $w(u)$ in the following power series form containing arbitrary parameters c_j

$$w = \sum_{j=0}^{m} c_j u^j \qquad (3.3.5)$$

with boundary conditions

$$w\Big|_{u=0} = 0, \quad \frac{dw}{du}\Big|_{u=0} = 0 \qquad (3.3.6)$$

From (3.3.6) it is clear that $c_0 = c_1 = 0$ in (3.3.5). Hence, we seek the solution for w as

$$w = \sum_{j=2}^{m} c_j u^j \qquad (3.3.7)$$

With w in this form when substituted in (2.2.6), we get an algebraic equation in c_j's which are determined by the so-called "Collocation Method" i.e., giving successively m values for u between $u=0$ and the maximum value of u

which is found from (3.3.3) to be $u = 0.3905a^2$ occurring at $r = 0.4802a$. With these m values for u, the matrix equations for the coefficients c_j are established. The values of the coefficients c_j thus obtained are tabulated in Table 1 for various values of m. Also tabulated is the value of α where

$$w_{max} = \alpha \frac{qa^4}{D} \tag{3.3.8}$$

For the sake of comparison, the difference between our results and those obtained previously in [6,17] and [38] are also shown in Table II.

TABLE I

	n=4	n=6	n=8	n=10	n=12
c_2	0.0103	0.0103	0.0103	0.0103	0.0103
c_3	.0018	.0027	.0025	.0028	.0026
c_4	.0089	.0117	.0196	.0216	.0325
c_5		-.0264	-.1027	-.1779	-.3914
c_6		.0519	.4218	.1206	.3666
c_7			-.8779	-.5126	-2.3327
c_8			.8342	1.3787	1.0268
c_9				-2.1090	-3.0634
c_{10}				1.4312	5.9334
c_{11}					-6.7366
c_{12}					3.4267

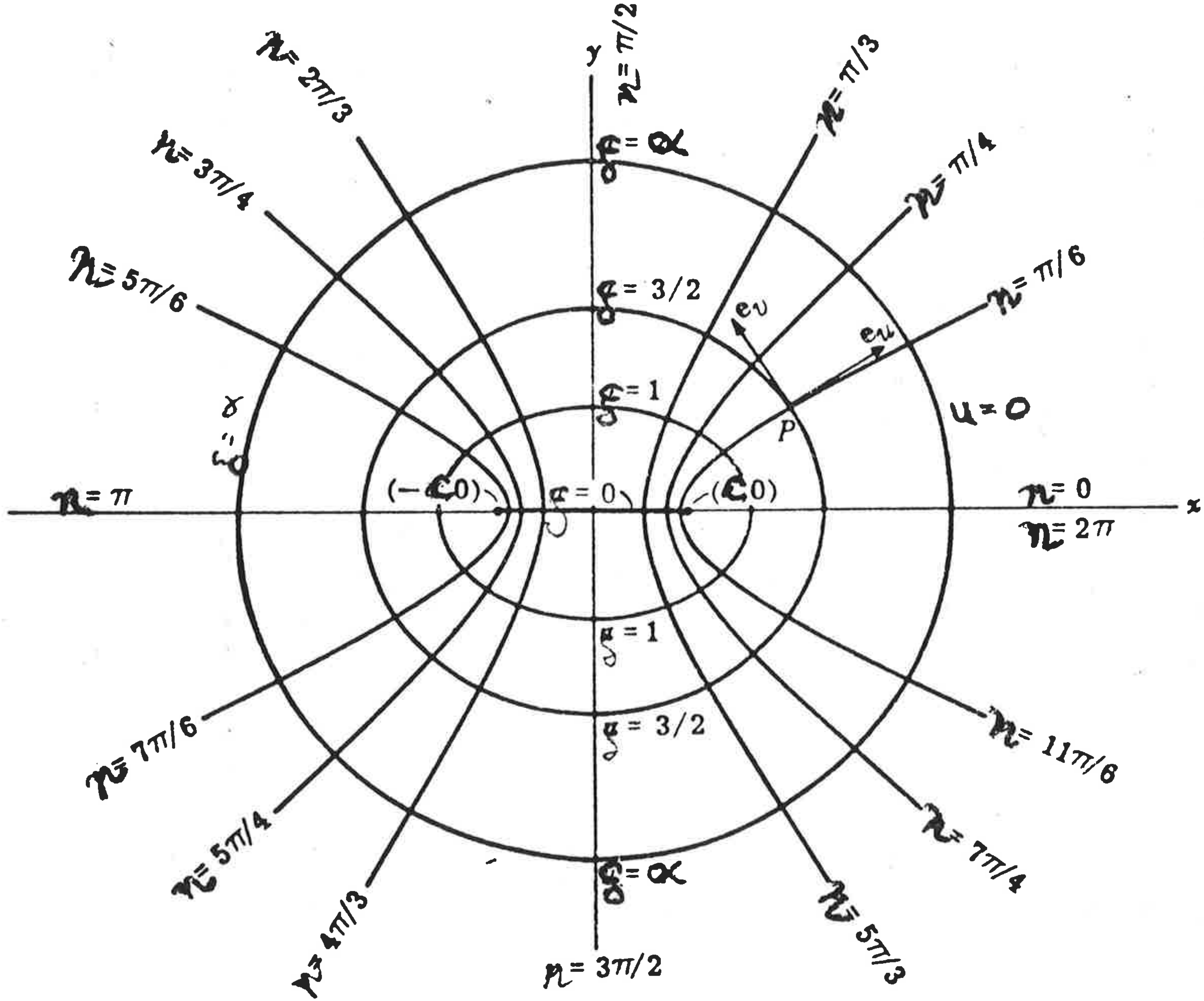
TABLE II

	α_{max}	difference from that given in [6,17]	difference from that given in [38]
n=4	.001891	6.5%	10%
n=6	.001949	3.5%	13.4%
n=8	.001983	1.7%	15.1%
n=10	.001994	1.3%	15.7%
n=12	.001998	1%	16.1%

From the above two tables it is apparent that if the number of terms is carefully chosen, very accurate results will be obtained. In fact, when only three terms in the series for w are considered, one obtains a very good result. Further, on the basis of the numerical results in these tables, it can be concluded that the convergence of the series is fairly rapid.

3.4 THE BENDING OF A UNIFORMLY LOADED, SIMPLY SUPPORTED ELLIPTIC PLATE.

As an illustration of this procedure for the simply supported case let us first consider the problem of simply supported elliptic plate. Although the exact solution to the problem of the bending of a uniformly loaded, thin, elliptic plate has been known for a long time, until recently [39] only an approximate solution was known for the case


y
x
η = π/2
η = π/3
η = π/4
η = π/6
η = 0
η = 2π
η = 2π/3
η = 3π/4
η = 5π/6
η = π
η = 7π/6
η = 5π/4
η = 4π/3
η = 3π/2
η = 5π/3
η = 7π/4
η = 11π/6
ξ = α
ξ = 3/2
ξ = 1
ξ = 0
ξ = α
ξ = 3/2
ξ = 1
ξ = α
u = 0
eυ
eu
P
(−c, 0)
(c, 0)

Fig

when the boundary of the plate was simply supported.

Consider the plate as shown in figure 5 where the boundary of the plate is denoted by $\xi=\alpha$, and the centre by $\xi=0$, $\eta = \pi/2$, where (ξ,η) are elliptic coordinates defined by

$$\begin{aligned} x &= c \cosh \xi \cos \eta \\ y &= c \sinh \xi \sin \eta. \end{aligned} \tag{3.4.1}$$

In accordance with case (b) of section 3.2 we seek the equation of the lines of equal deflection in the form

$$u(\xi,\eta) = U(\xi,\eta)\, V(\xi,\eta), \tag{3.4.2}$$

where $U(\xi,\eta)$ and $V(\xi,\eta)$ satisfy

$$\nabla^2 U = -2, \tag{3.4.3}$$

and

$$\nabla^2 V = -2, \tag{3.4.4}$$

subject to the boundary conditions

$$\begin{aligned} \text{i)}\quad & U \big/_C = 0, \\ \text{ii)}\quad & Q \big/_C = 0. \end{aligned} \tag{3.4.5}$$

In elliptical coordinates equations (3.4.3), and (3.4.4) transform to

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} = -\frac{c}{2}(\cosh 2\xi - \cos 2\eta), \tag{3.4.6}$$

and

$$\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} = -\frac{c}{2}(\cosh 2\xi - \cos 2\eta), \tag{3.4.7}$$

while the boundary conditions (3.4.5) i), and ii) become

$$\text{i)} \quad U \Big/_{\xi=\alpha} = 0, \qquad (3.4.8)$$

$$\text{ii)} \quad \frac{\partial^2 UV}{\partial \xi^2} - \frac{(1-\mu)\sinh 2\alpha}{\cosh 2\alpha - \cos 2\eta}\,\frac{\partial UV}{\partial \xi} \Big/_{\xi=\alpha} = 0.$$

This has solution

$$U = B_0(\cosh 2\alpha - \cosh 2\xi)(\cosh 2\alpha - \cos 2\eta), \qquad (3.4.9)$$

and

$$V = B_1(-4 + 6\cosh^2 2\alpha - 2\cosh 2\alpha \cosh 2\xi)(-4 + 6\cosh^2 2\alpha - 2\cosh 2\alpha \cos 2\eta) + B_2(\cosh 2\xi + \cos 2\eta) + B_3(2\cosh 2\alpha \cosh 2\xi + 1)(2\cosh 2\alpha \cos 2\eta + 1) + B_4, \qquad (3.4.10)$$

where B_0, B_1, B_2, B_3 and B_4 are constant coefficients chosen so as to satisfy equations (3.4.6), (3.4.7), and the boundary condition (3.4.8) (ii). Coupling these expressions we obtain

$$u = UV = (\cosh 2\xi - \cosh 2\alpha)(\cos 2\eta - \cosh 2\alpha)[D_0(-4 + 6\cosh^2 2\alpha - 2\cosh 2\alpha \cosh 2\xi)(-4 + 6\cosh^2 2\alpha - 2\cosh 2\alpha \cos 2\eta) + 2D_1(\cosh 2\xi + \cos 2\eta) + D_2(1+2\cosh 2\alpha \cosh 2\xi)(1+2\cosh 2\alpha \cos 2\eta) + D_3], \qquad (3.4.11)$$

where D_0, D_1, D_2 and D_3 are constants. However since the family lines of constant deflection $u(\xi,\eta) = \text{const.}$ are unaffected by multiplication by an arbitrary constant, we may consider the coefficients D_0, D_1, D_2, and D_3 as unknown quantities to be determined from the boundary condition (3.4.8) ii) and the force equation (2.2.6).

Upon substituting for $u(\xi,\eta)$ into equation (3.4.8) ii) we obtain

$$\begin{aligned}
&16D_0(1-\mu)(3\cosh^2 2\alpha-2)\sinh^4 2\alpha \\
&\quad + D_1\{8\cosh^3 2\alpha - 16\cosh 2\alpha\cosh 4\alpha + 4\cosh 2\alpha + 2(1-\mu)\sinh 2\alpha\sinh 4\alpha\} \\
&\quad -2D_2(1-\mu)(\cosh 4\alpha\sinh 2\alpha - 2\cosh 2\alpha\sinh 4\alpha)\sinh 2\alpha \\
&\quad +2D_3\{2\cosh^2 2\alpha - (1-\mu)\sinh^2 2\alpha\} \\
&\quad +[-16D_0(1-\mu)\sinh^4 2\alpha\cosh 2\alpha + 4D_2\{4\cosh 4\alpha - 4\cosh^2 2\alpha + (1-\mu)\sinh^2 2\alpha\} \\
&\quad -D_1\{12\cosh 2\alpha\cosh^2 4\alpha + 2(1-\mu)(\cosh 4\alpha\sinh 2\alpha - 2\cosh 2\alpha\sinh 4\alpha)\sinh 4\alpha\} \\
&\quad -4D_3\cosh 2\alpha]\cos 2\eta \\
&\quad + 4\{D_1\cosh 2\alpha + 3D_2\cosh^2 2\alpha\cosh 4\alpha\}\cos 4\eta = 0\,.
\end{aligned} \tag{3.4.12}$$

Since the equation (3.4.12) is true for all values of η, equating the constant member, and the coefficients of $\cos 2\eta$ and $\cos 4\eta$ equal to zero, we obtain

$$\begin{aligned}
&D_0(1-\mu)(3\cosh 12\alpha - 14\cosh 8\alpha + 29\cosh 4\alpha - 18) + 2D_1(1+\mu)\cosh 2\alpha - (5+\mu)\cosh 6\alpha\} \\
&\quad + D_2(1-\mu)(\cosh 8\alpha + 2\cosh 4\alpha - 3) + 2D_3\{(1+\mu)\cosh 4\alpha + (3-\mu)\} = 0
\end{aligned} \tag{3.4.13}$$

$$-D_0(1-\mu)(\cosh 10\alpha - 3\cosh 6\alpha + 2\cosh 2\alpha) + D_1\{8\cosh 4\alpha - 8 + 2(1-\mu)(\cosh 4\alpha - 1)\}$$

$$+ D_2\{(1-\mu)(\sinh 6\alpha + 3\sinh 2\alpha)\sinh 4\alpha - 3(\cosh 10\alpha + \cosh 6\alpha + 2\cosh 2\alpha)\} - 4D_3 \cosh 2\alpha = 0 \qquad (3.4.14)$$

$$D_1 + 3D_2 \cosh 4\alpha \cosh 2\alpha = 0. \qquad (3.4.15)$$

The coefficients D_0, D_2, and D_3 may now be determined in terms of D_1.

Our problem now reduces to finding the form of the deflected surface $w(u)$ which satisfies the force equation (2.2.6), viz.

$$\frac{d^3w}{du^3}\oint_{C_u} R\,ds + \frac{d^2w}{du^2}\oint_{C_u} F\,ds + \frac{dw}{du}\oint_{C_u} G\,ds - \iint_{\Omega_u} q\,d\Omega = 0, \qquad (3.4.16)$$

subject to the boundary conditions:

$$\text{i)}\quad W\big|_{u=o} = 0,$$

$$\text{ii)}\quad \left.\frac{d^2w}{du^2}\right|_{u=o} = 0. \qquad (3.4.17)$$

In order to solve the differential equation (3.4.16) we first transform the third line integral in equation (3.9.16) to a double integral, using Green's theorem. This yields

$$\oint_{C_u} G\,ds = D\iint_{\Omega_u} \nabla^4 u\,d\Omega - D\oint_{C_u} \frac{\partial H}{\partial s}\,ds. \qquad (3.4.18)$$

However since the contour integral appearing in the right hand side of (3.4.18) vanishes along the smooth closed curve C_u the differential equation (3.4.16) reduces to

$$\frac{d^3w}{du^3}\oint_{C_u} R\,ds + \frac{d^2w}{du^2}\oint_{C_u} F\,ds + \frac{dw}{du}\iint_{\Omega_u} D\nabla^4 u\,d\Omega - \iint_{\Omega_u} q\,d\Omega = 0. \tag{3.4.19}$$

But since the form of u obtained has the property that

$$\nabla^4 u(\xi,\eta) = \frac{256}{c^4}\,[6D_0 \cosh 4\alpha \cosh 2\alpha - 2D_1] \tag{3.4.20}$$

a close look at equation (3.4.19) and the boundary conditions (3.4.17), reveals that if we determine D_1 from the relationship

$$6D_0 \cosh 2\alpha \cosh 4\alpha - 2D_1 = \frac{c^4 q}{256 \cosh 2\alpha D} \tag{3.4.21}$$

then the solution for w must be

$$w = u. \tag{3.4.22}$$

Solving equations (3.4.13), (3.4.14), (3.4.15), and (3.4.21) for D_0, D_1, D_2 and D_3 yields

$$D_0 = p(R-r_0)/3R \cosh 2\alpha \cosh 4\alpha \tag{3.4.23}$$

$$D_1 = -\frac{pr_0}{R} \tag{3.4.24}$$

$$D_2 = \frac{pr_0}{3R \cosh 2\alpha \cosh 4\alpha} \tag{3.4.25}$$

$$D_3 = \frac{pR_1}{6R\{(1+\mu)\cosh 4\alpha + 3-\mu\}\cosh 2\alpha \cosh 4\alpha} \tag{3.4.26}$$

where

$$p = \frac{qc^4}{D\ 512 \cosh 2\alpha}$$

$$r_0 = 2(1-\mu)(3 \cosh 12\alpha - 14 \cosh 8\alpha + 29 \cosh 4\alpha - 18) \times \cosh 2\alpha - (1-\mu)(\cosh 10\alpha - 3 \cosh 6\alpha + 2 \cosh 2\alpha) \times \{(1+\mu)\cosh 4\alpha + 3-\mu\} \tag{3.4.27}$$

$$r_1 = 4\{(1+\mu)\cosh 2\alpha - (5+\mu)\cosh 6\alpha \cosh 2\alpha + \quad (3.4.28)$$
$$\{8 \cosh 4\alpha - 8 + 2(1+\mu)(\cosh 4\alpha - 1) \text{ X}$$
$$\{(1+\mu)\cosh 4\alpha + 3-\mu\}.$$

$$r_2 = 2(1-\mu)(\cosh 8\alpha + 2 \cosh 4\alpha - 3)\cosh 2\alpha + \quad (3.4.29)$$
$$\{(1-\mu)(\sinh 6\alpha + 3 \sinh 2\alpha)\sinh 4\alpha$$
$$-3(\cosh 10\alpha + \cosh 6\alpha + 2 \cosh 2\alpha) \text{ X}$$
$$\{(1+\mu)\cosh 4\alpha + 3-\mu\}$$

$$r_0 + 3r_1 \cosh 2\alpha \cosh 4\alpha - r_2 = R \quad (3.4.30)$$

and

$$R_1 = 6r_0\{(1+\mu)\cosh 2\alpha - (5+\mu)\cosh 6\alpha\} \cosh 2\alpha \cosh 4\alpha - \quad (3.4.31)$$
$$(1-\mu)(\cosh 8\alpha + 2 \cosh 4\alpha - 3)r_0 - (1-\mu) \text{ X}$$
$$(3 \cosh 12\alpha - 14 \cosh 8\alpha + 29 \cosh 4\alpha - 18)(R-r_0).$$

If we now substitute the values for $D_0, D_1, D_2,$ and D_3 into equation (3.4.11), the expression for the deformed middle surface of the plate is given by

$$W = \frac{p(R-r_0)}{3R \cosh 2\alpha \cosh 4\alpha}(3 \cosh 2\alpha \cosh 4\alpha -$$
$$4 \cosh 4\alpha \cosh 2\xi +$$
$$\cosh 2\alpha \cosh 4\xi) \text{ X}$$
$$(3 \cosh 2\alpha \cosh 4\alpha - 4 \cosh 4\alpha$$
$$\cos 2\eta + \cosh 2\alpha \cos 4\eta)$$
$$- \frac{2pr_0}{R}(\cosh 2\alpha - \cosh 2\xi)(\cosh 2\alpha - \cos 2\eta) \text{ X}$$
$$(\cosh 2\xi + \cos 2\eta)$$
$$+ \frac{pr_0}{3R \cosh 2\alpha \cosh 4\alpha}(\cosh 4\alpha \cosh 2\xi - \cosh 2\alpha \cosh 4\xi)$$
$$(\cosh 2\alpha - \cos 2\eta) \text{ X}$$
$$(1+2 \cosh 2\alpha \cos 2\eta)$$

$$+ \frac{pR_1}{6R\{(1+\mu)\cosh 4\alpha+3-\mu\}\cosh 2\alpha \cosh 4\alpha}(\cosh 2\xi - \cosh 2\alpha)(\cosh 2\alpha - \cos 2\eta). \quad (3.4.32)$$

Let us now calculate the deflection at the centre of the plate $(\xi=0, \ \eta = \pi/2)$ for various values of the aspect ratio a/b and with $\mu = .3$. Assuming $a/b > 1$ we represent the deflection at the centre by the formulae

$$W\Big|_{\substack{\xi=0 \\ \eta=\pi/2}} = \gamma \frac{qb^4}{Eh^3} . \quad (3.4.33)$$

The numerical values of γ so obtained are shown in Table 3 along with those given by Galerkin [10] and Mazumdar [35].

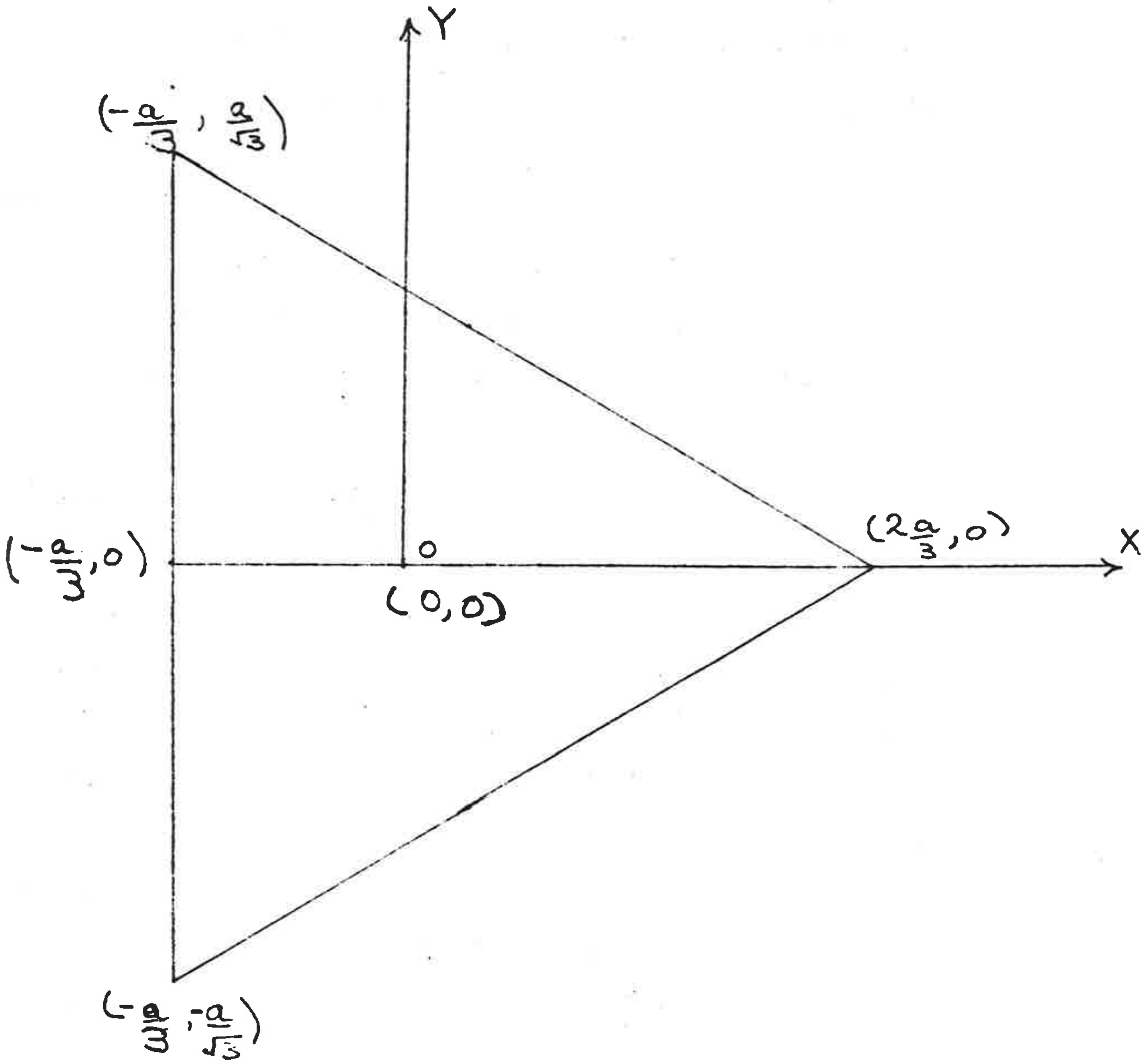
TABLE 3

a/b	1	1.1	1.2	1.3	1.4	1.5	2	∞
γ	.6957	.8316	.9559	1.0688	1.1765	1.2656	1.575	2.279
[10]	.70	.83	.96	1.07	1.17	1.26	1.58	2.28
[35]	.70	.83	.95	1.07	1.16	1.24	1.51	1.86

From Table 3 we see that for moderate aspect ratios, $a/b \leqslant 2$, the deflection contours obtained above may be closely approximated by a family of similar and similarly situated ellipses, whose general equation is

$$u(x,y) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} = \text{const.} \quad (3.4.34)$$

However for high aspect ratios $a/b > 2$ this result is no longer true.



Y
$(-\frac{a}{3}, \frac{a}{\sqrt{3}})$
$(-\frac{a}{3}, 0)$
0
$(0,0)$
$(2\frac{a}{3}, 0)$
X
$(-\frac{a}{3}, -\frac{a}{\sqrt{3}})$

Fig 6

Since W given by equation (3.4.32) satisfies the differential equation (2.2.6) it is clear that W represents an exact solution for the bending of a uniformly loaded elliptic plate with a simply supported edge, and is identical to the solution obtained in [39].

3.5 THE BENDING OF A SIMPLY SUPPORTED, UNIFORMLY LOADED, EQUILATERAL TRIANGULAR PLATE

To further illustrate this procedure, let us consider the bending of a uniformly loaded, simply supported equilateral triangular plate where the geometry of the plate is as shown in figure 6 and where the equation of the boundary is given by

$$x^3 - 3xy^2 - ax^2 - ay^2 + \frac{4a^3}{27} = 0. \qquad (3.5.1)$$

In accordance with case (b) we again seek the equation for the lines of constant deflection in the form

$$u(x,y) = U(x,y)\, V(x,y), \qquad (3.5.2)$$

such that $U(x,y)$ and $V(x,y)$ satisfy

$$\nabla^2 U = -2, \qquad (3.5.3)$$

and

$$\nabla^2 V = -2, \qquad (3.5.4)$$

subject to the boundary conditions

$$\left.\begin{array}{ll} \text{i)} & U\big|_C = 0, \\ \text{ii)} & Q\big|_C = 0. \end{array}\right\} \qquad (3.5.5)$$

This has solution

$$U = \frac{1}{2a}\,(x^3-3xy^2-ax^2-ay^2 + \frac{4a^3}{27}), \tag{3.5.6}$$

and

$$V = \tfrac{1}{2}\left(\frac{4a^2}{9} - x^2 - y^2\right), \tag{3.5.7}$$

so that coupling these expressions we obtain

$$\begin{aligned} u &= U(x,y)V(x,y) \\ &= \frac{1}{4a}\,(x^3-3xy^2-ax^2-ay^2+\frac{4a^3}{27})(\frac{4a^2}{9} - x^2 - y^2). \end{aligned} \tag{3.5.8}$$

Our problem thus reduces to finding the form of the deflected surface $w(u)$ which satisfies the force equation (2.2.6) subject to the boundary conditions

$$\begin{aligned} &\text{i)} \quad w\Big|_{u=0} = 0, \\ &\text{ii)} \quad \frac{d^2w}{du^2}\Big|_{u=0} = 0. \end{aligned} \tag{3.5.9}$$

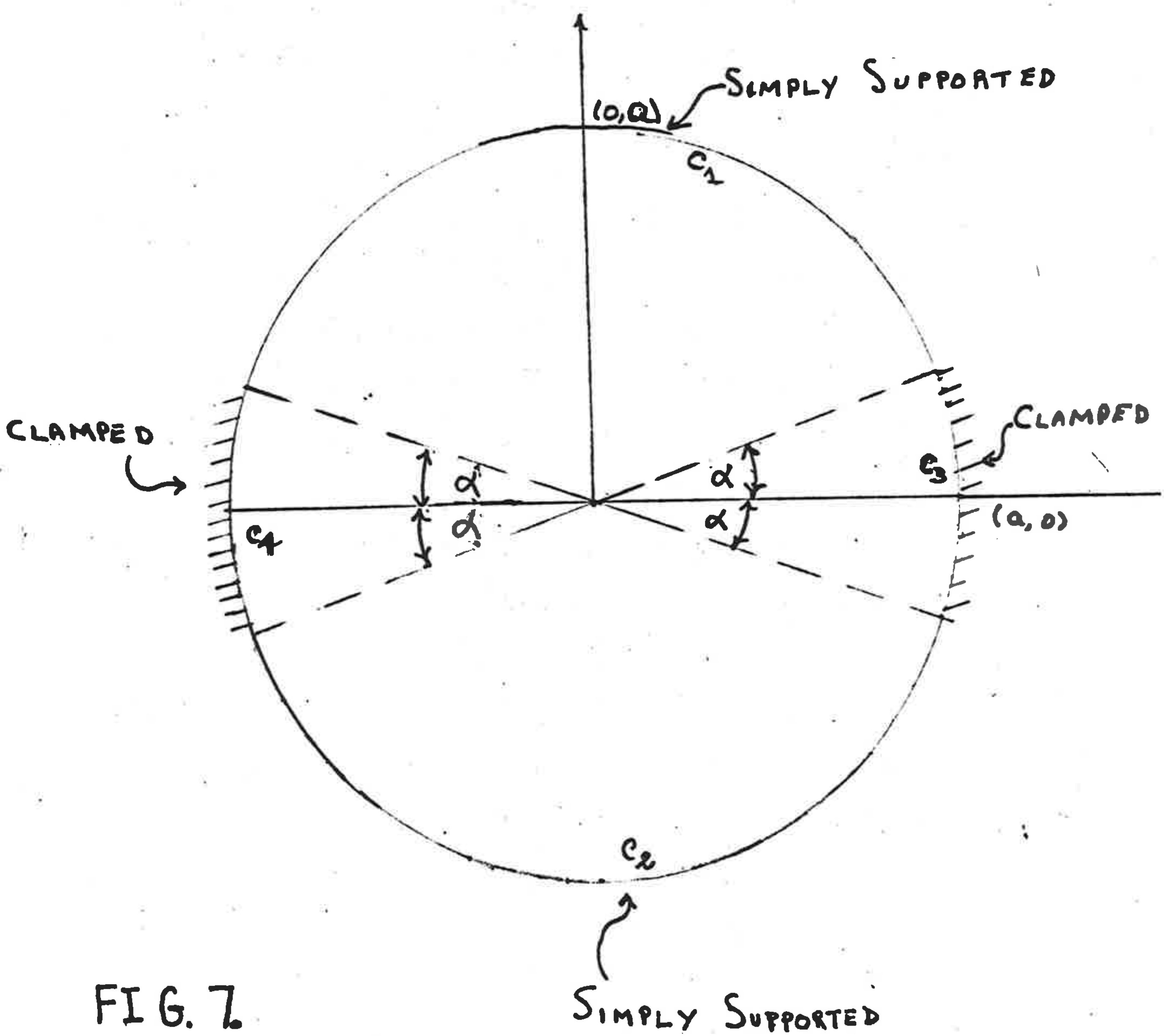
However since

$$\nabla^4 u = 16, \tag{3.5.10}$$

proceeding as in section 3.4 we obtain as the solution

$$\begin{aligned} w &= \frac{u}{16D} \\ &= \frac{1}{64aD}\left(x^3-3xy^2-ax^2-ay^2 + \frac{4a^3}{27}\right)\left(\frac{4a^2}{9} - x^2 - y^2\right), \end{aligned} \tag{3.5.11}$$

which coincides with the exact solution as obtained by Woinowsky-Krieger [65].



SIMPLY SUPPORTED
(0,a)
C_1
CLAMPED
CLAMPED
C_3
(a,0)
C_4
α
α
α
α
C_2
SIMPLY SUPPORTED

FIG. 7.

3.6 UNIFORMLY LOADED CIRCULAR PLATE SIMPLY SUPPORTED ON ONE EDGE PORTION AND CLAMPED ON THE REMAINDER

Let us now consider as an illustration of the procedure outlined in section 3.2, for a partly clamped - partly simply supported plate, the more involved case of a circular plate clamped on part of its boundary and simply supported on the remainder. This problem was first considered by Nowacki and Olesiak [47] and more recently by Conway and Farnham [4], Kerr and Stàhl [24], and Leissa and Claussen [29]. However no exact solution has been found.

We consider the plate clamped over two symmetrically disposed portions of the edge subtending an angle 2α at the plate centre, and simply supported over the remaining edge as in Fig.7.

In accordance with the case (c) in section 3.3 above, we will write the equation of lines of equal deflection $u(r,\theta)$ as

$$u = UV, \tag{3.6.1}$$

where both U and V are functions of r and θ, and are solutions of

$$\text{(i)}\quad \nabla^2 U = -2, \tag{3.6.2}$$

$$\text{(ii)}\quad \nabla^2 V = -2,$$

subject to

$$\text{(i)}\quad U\big|_{r=a} = 0,\quad \text{(ii)}\quad Q\big|_{c_1+c_2} = 0,\quad \text{(iii)}\quad V\big|_{c_3+c_4} = 0. \tag{3.6.3}$$

This clearly yields

$$U = \tfrac{1}{2}(a^2-r^2). \qquad (3.6.4)$$

In view of the symmetry with respect to the coordinate axes, we seek V in the form

$$V = \tfrac{1}{2}[-r^2 + \sum_{n=0}^{\infty} B_n\left(\frac{r}{a}\right)^{2n}\cos 2n\theta]. \qquad (3.6.5)$$

The form of the second boundary condition of (3.6.3) now reduces to

$$(1+\mu)V(a,\theta) + 2a\, V_r(a,\theta)/_{c_1+c_2} = 0. \qquad (3.6.6)$$

On the basis of (3.6.6) together with the condition (iii) of (3.6.3) we have

$$\sum_{n=0}^{\infty} \frac{4n+1+\mu}{5+\mu} B_n \cos 2n\theta = a^2 \quad \text{on} \quad c_1+c_2, \qquad (3.6.7)$$

and

$$\sum_{n=0}^{\infty} B_n \cos 2n\theta = a^2 \quad \text{on} \quad c_3+c_4, \qquad (3.6.8)$$

In order to solve for the unknown coefficients B_n, we perform the integrations on a^2 and $a^2\cos m\theta$ over the interval $\left(0,\frac{\pi}{2}\right)$.

This yields the following system of equations

$$B_0[\alpha+\left(\frac{\pi}{2} - \alpha\right)\left(\frac{1+\mu}{5+\mu}\right)] + \sum_{n=1}^{\infty} B_n[\int_0^{\alpha}\cos 2n\theta d\theta + \left(\frac{4n+1+\mu}{5+\mu}\right) \times \int_{\alpha}^{\frac{\pi}{2}}\cos 2n\theta d\theta] = \frac{\pi}{2}a^2, \qquad (3.6.9)$$

and

$$B_o\left[\int_0^{\alpha}\cos m\theta d\theta + \left(\frac{1+\mu}{5+\mu}\right)\int_{\alpha}^{\frac{\pi}{2}}\cos m\theta d\theta\right] + \sum_{n=1}^{\infty} B_n\left[\int_0^{\alpha}\cos m\theta\cos 2n\theta d\theta\right.$$

$$\left. + \left(\frac{4n+1+\mu}{5+\mu}\right)\int_{\alpha}^{\frac{\pi}{2}}\cos m\theta \cos 2n\theta d\theta\right] = \frac{(-1)^m}{m}\, a^2, \qquad (3.6.10)$$

where $m = 1,3,5,\ldots$

Denoting

$$I_p = \int_0^{\alpha}\cos p\theta d\theta, \quad J_p = \int_{\alpha}^{\frac{\pi}{2}}\cos p\theta d\theta,$$

$$I_{p,q} = \int_0^{\alpha}\cos p\theta \cos q\theta d\theta, \quad J_{p,q} = \int_{\alpha}^{\frac{\pi}{2}}\cos p\theta \cos q\theta d\theta, \qquad (3.6.11)$$

the above system of equations obtained may be written in the form

$$B_o\left[\alpha+\left(\frac{\pi}{2} - \alpha\right)\left(\frac{1+\mu}{5+\mu}\right)\right] + \sum_{n=1}^{\infty} B_n\left[I_{2n} + \frac{4n+1+\mu}{5+\mu}\, J_{2n}\right] = \frac{\pi}{2}a^2, \qquad (3.6.12)$$

and

$$B_o\left[I_m + \frac{1+\mu}{5+\mu}\, J_m\right] + \sum_{n=1}^{\infty} B_n\left[I_{m,2n} + \frac{4n+1+\mu}{5+\mu}\, J_{m,2n}\right] = \frac{(-1)^m}{m}\, a^2, \qquad (3.6.13)$$

In order to solve the infinite system of equations given by (3.6.12) and (3.6.13) in an infinite set of unknowns B_n, we use the so-called method of reduction [22], i.e., by means of a passage to the limit in the solution of the finite system obtained from the given infinite system by discarding all equations and unknowns commencing with a

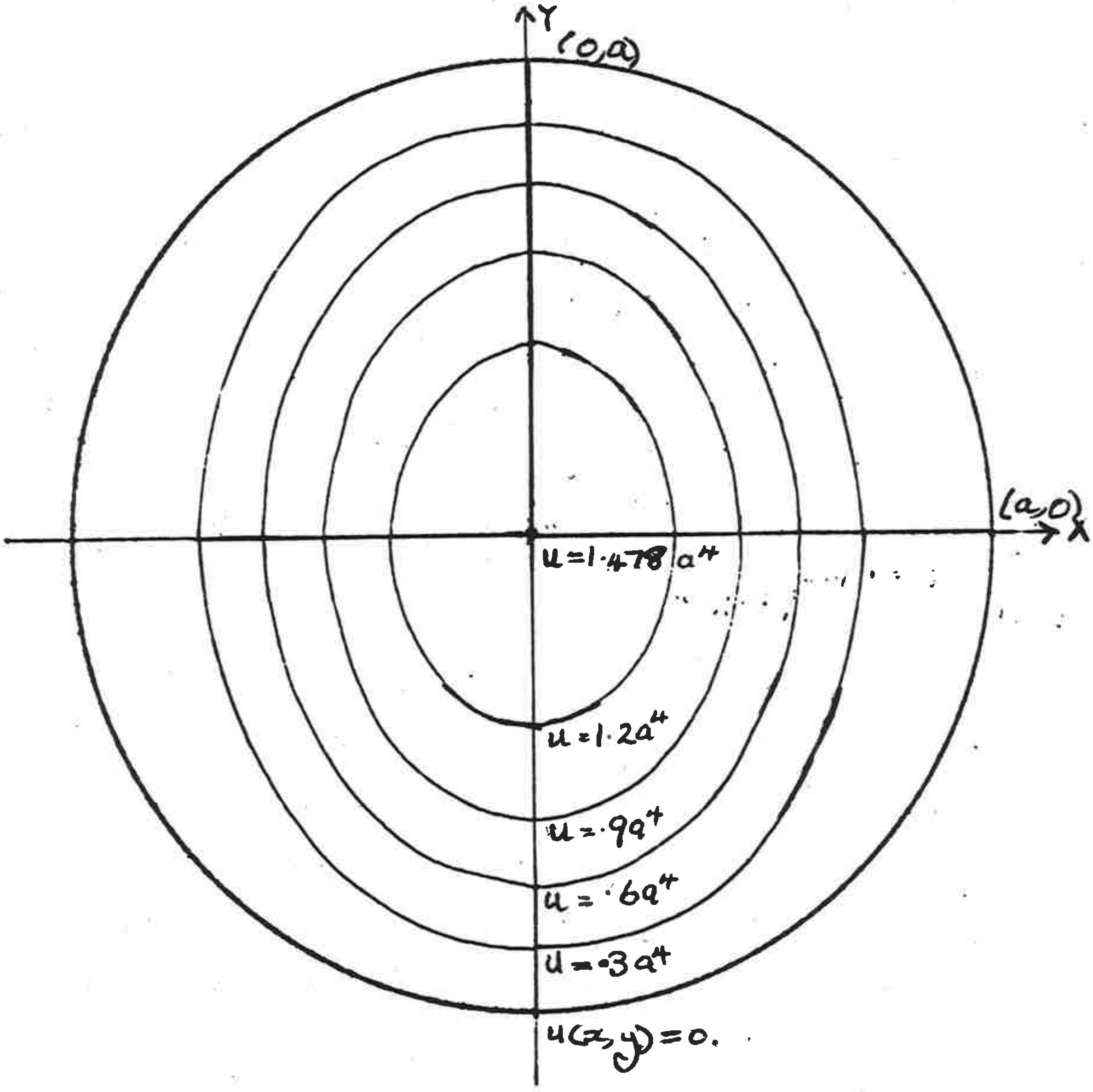


FIG. 8

certain one. It is found that the principal solution obtained from eight equations with eight number of unknowns gives a fairly good approximate solution. The calculations were carried out using machine computation for several values of the semi-clamped angle α.

The constant deflection contours

$$u(r,\theta) = (a^2-r^2)\left(\sum_{n=0}^{\infty} B_n\left(\frac{r}{a}\right)^{2n}\cos 2n\theta - r^2\right) = \text{const.} \tag{3.6.14}$$

are then plotted in Figure 8 for $\alpha = 45°$.

Having obtained the expression for $u(r,\theta)$, we can now proceed to determining the deflection $w(u)$ from the third order ordinary differential equation (2.2.6) subject to

$$\text{(i)}\quad w|_{u=o} = 0, \quad \text{(ii)}\quad \left.\frac{d^2w}{du^2}\right|_{c_1+c_2} = 0, \quad \text{(iii)}\quad \sqrt{t}\left.\frac{dw}{du}\right|_{c_3+c_4} = 0, \tag{3.6.15}$$

and that $\frac{dw}{du}$ is finite on c_3+c_4 as well as at the centre which is obviously the origin of coordinates at which $u = B_0a^2$.

In order to solve the differential equation (2.2.6) we first transform the line integral $\oint G ds$ to a double integral, using Green's Theorem. This yields,

$$\oint_{C_u} G ds = D\iint_{\Omega_u} \nabla^4 u d\Omega - D\oint_{C_u} \frac{\partial H}{\partial s}\, ds, \tag{3.6.16}$$

since the contour integral appearing in the right hand side of (3.6.16) vanishes along the closed smooth curve given by

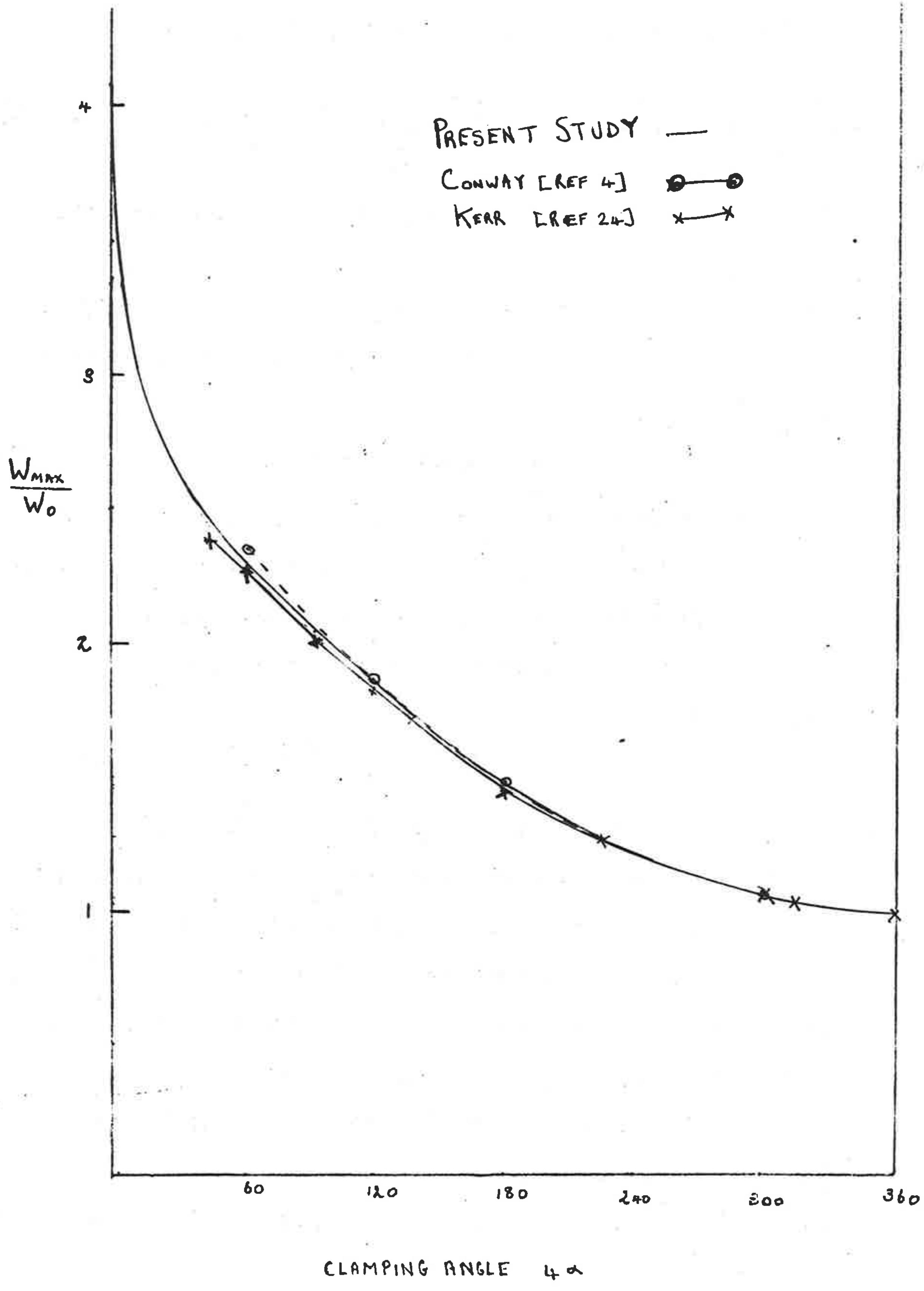

PRESENT STUDY
CONWAY [REF 4]
KERR [REF 24]
4
3
2
1
$\frac{W_{MAX}}{W_0}$
60
120
180
240
300
360
CLAMPING ANGLE 4α

FIG. 9.

(3.6.14), the differential equation (2.2.6) reduces to

$$\frac{d^3w}{du^3} \oint_{C_u} R ds + \frac{d^2w}{du^2} \oint_{C_u} F ds + D \frac{dw}{du} \iint_{\Omega_u} \nabla^4 u d\Omega - \iint_{\Omega_u} q d\Omega = 0. \quad (3.6.17)$$

But since the form of u obtained in (3.6.14) has the property

$$\nabla^4 u(r,\theta) = 64, \quad (3.6.18)$$

a close look of the equation (3.6.17) and boundary conditions (3.6.15) reveals that

$$w = \frac{q}{64D} u, \quad (3.6.19)$$

must be the solution. It is interesting to note that Conway and Farnham [4] assumed precisely the same form of w in their alternate method of analysis.

The maximum deflection occurs at the centre $(r=0)$ and its value is

$$w_{max} = \frac{q}{64D} u_{max} = \frac{qa^2}{64D} B_0. \quad (3.6.20)$$

The numerical values of w_{max} for various values of angle α and $\mu=0.3$ were computed and are displayed in the following table as well as in Fig.9. where, for the sake of comparisons, the results obtained in [4,24] are also included. Here $w_0 = \frac{qa^4}{64D}$ is the maximum deflection for a completely clamped plate.

From Table 4, we also obtain the exact results for the known cases when the plate is completely clamped $(\alpha = 90)$ and when the plate is completely simply supported $(\alpha = 0)$.

TABLE 4

Clamping angle 4α, in degrees	$\frac{W_{max}}{W_0}$ Present Study	According to Ref.24	According to Ref.4
0	4.077	-	-
4	3.371	3.257	-
45	2.455	2.406	-
60	-	2.259	2.342
90	2.035	2.0032	-
120	-	1.7984	1.843
135	1.724	1.702	-
180	1.478	1.465	1.504
225	1.283	1.274	-
240	-	1.222	1.229
276	1.136	-	-
300	-	1.062	1.062
315	1.039	1.037	-
360	1.000	1.000	.998

3.6.1. EXPERIMENTAL VERIFICATION USING SHADOW MOIRE METHOD

In order to investigate in more detail, an attempt was made to observe the constant deflection lines of circular plates under bending by using the shadow moire technique. The method is based on the following principle: If a grating is placed parallel to the plate before bending and its shadow cast on the plate by a collimated illumination, the interference of the shadow with the grating itself will create moire fringes which can be interpreted as lines of constant deflection. Indeed if the grating pitch is p and the incident angle of the light is i, it

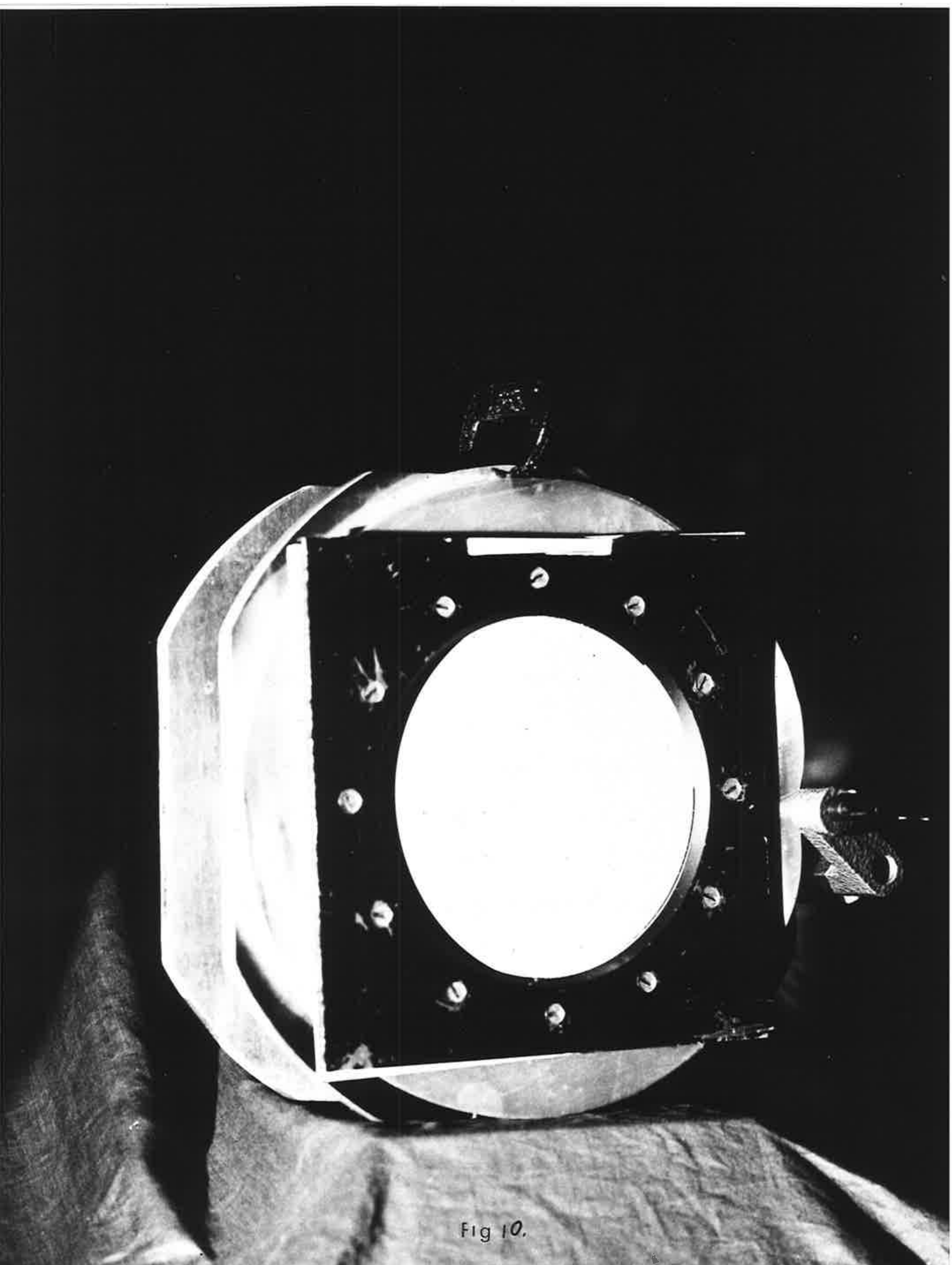


Fig 10.

can be shown [55] that each moire fringe represents a constant deflection given by

$$w = \frac{p}{\tan i}, \tag{3.6.22}$$

The schematic of shadow moire method is shown in Fig. 10.

In this study a series of models with varying angle α were constructed and tested under uniform pressure. The plates were made of plexiglas and were 6" in diameter and 1/4" thick. Fixed-end boundary condition was simulated by clamping the plate between two bolted aluminium plates whereas the simply-supported condition was simulated by machining a narrow vee-groove on both sides of the plate so that it would not take much bending moment. The remaining thickness at the groove was about 1/16". The plates were mounted on an apparatus such that vacuum can be exerted from the backside of the plate. A grating of about 80 lpi was placed in front of the plate; and a nearly parallel light source (coming from a long-focal-length slide projector situated at a large distance away) was incident on the grating-plate assembly at an angle of about 45°. The combined value of p/tan i was calibrated using a known wedge "deflection" cut into the mounting plate. In this way, it was not necessary to measure p and i individually. A set of six selected moire pictures obtained this way is shown in Fig.11. It is seen, as expected, that the fully clamped as well as the fully

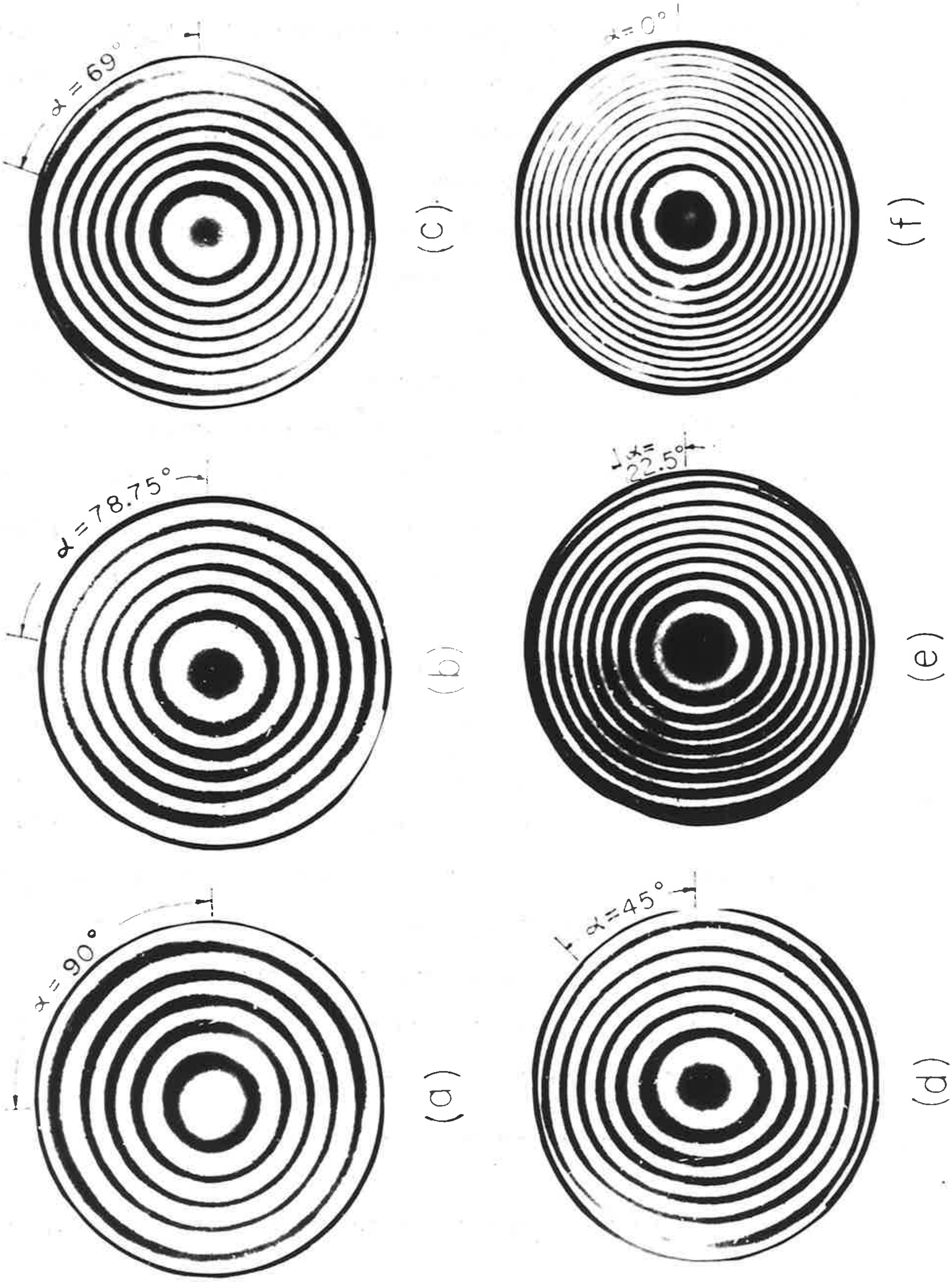


Fig 11.

simply-supported plates have concentrated circles as lines of constant deflection. As α starts to decrease the circles tend to be elongated along the direction of the axis of simply-supported boundary. A direct comparison may be made between the theoretical constant deflection lines shown in Fig.11. and the experimental ones shown in Fig.11 (d). Qualitative similarities do exist between these two figures. However, quantitative comparison is not possible because of the difficulty of simulating simply-supported boundary condition. Although the plate was very thin at the grooved boundary, it nevertheless had bending rigidity. While the experimental result of the fully-clamped plate agrees fairly well with the theoretical predication, the experimentally measured deflection of fully simply-supported plate fell well below that predicated by theory. The consequence of this is that the moire circles are less elongated along the axis of simply-supported boundary than they should have been. This fact is evident from the comparison of constant deflection curves in Fig.8. and Fig.11. However the general feature of the constant deflection lines obtained by shadow moire method do tend to support strongly the theoretical predications offered by the analytical method.

3.7 ANALYSIS FOR NON UNIFORM LOAD

When the plate is under non uniform loading, the constant deflection contours will clearly be different than when it is under uniform loading. In this case the procedure may be exemplified as follows. Since equations (3.1.11) and (3.1.12) still hold, and assuming as before that throughout the region of the plate $t = u_x^2 + u_y^2$ is a finite quantity, we assume as in the case of uniform loading that $(z-\xi)q(u_y+iu_x)$ is an analytic expression. Hence we have

$$\nabla^2[\{(x-\kappa_1(u))u_x+(y-\kappa_2(u))u_y\}q] = 0, \qquad (3.7.1)$$

and

$$\nabla^2[\{(x-\kappa_1(u))u_y-(y-\kappa_2(u))u_x\}q] = 0 \qquad (3.7.2)$$

where the expressions in brackets are first evaluated along the contour $u(x,y) = c$. Making the coordinate transformation

$$x - \kappa_1 = X, \qquad (3.7.3)$$
$$y - \kappa_2 = Y.$$

equations (6.11) and (6.12) transform to

$$\nabla^2[(Xu_X + Yu_Y)q] = 0, \qquad (3.7.4)$$

and

$$\nabla^2[(Xu_Y - Yu_X)q] = 0, \qquad (3.7.5)$$

or, in terms of polar coordinates (r,θ) given by (3.2.11) we have

$$\nabla^2[ru_r q(r,\theta)] = 0, \qquad (3.7.6)$$

and

$$\nabla^2[u_\theta q(r,\theta)] = 0. \qquad (3.7.7)$$

It can easily be seen that for constant loading these equations will yield the same solution for $u(x,y)$ as has previously been obtained.

Although under general conditions, there exists a unique solution, its determination usually offers insurmountable difficulties, especially when the plate has an arbitrary shape. Hence the approach here is to devise a technique for obtaining an approximate solution. This is done by using the method of successive approximations, according to which the kth approximation u_k is obtained from the (k-1)th approximation u_{k-1}. We then formulate its general solution satisfying the boundary conditions.

It is obvious that the rate of convergence of the whole process of successive approximation will depend greatly upon how the first approximation is chosen. Taking into account the above fact, we will outline here three alternative ways of choosing $u_1(x,y)$, which have been found to be quite successful. They are as follows.

(a) If a particular solution to either (3.7.4), or (3.7.5) can be found, then we can take that solution as a first approximation $u_1(x,y)$.

(b) If the boundary of the plate is given by the equation $g(x,y) = 0$, and the function g is different from zero within the region of the plate, then we can choose as a first approximation

$$u_1(x,y) = g(x,y). \tag{3.7.8}$$

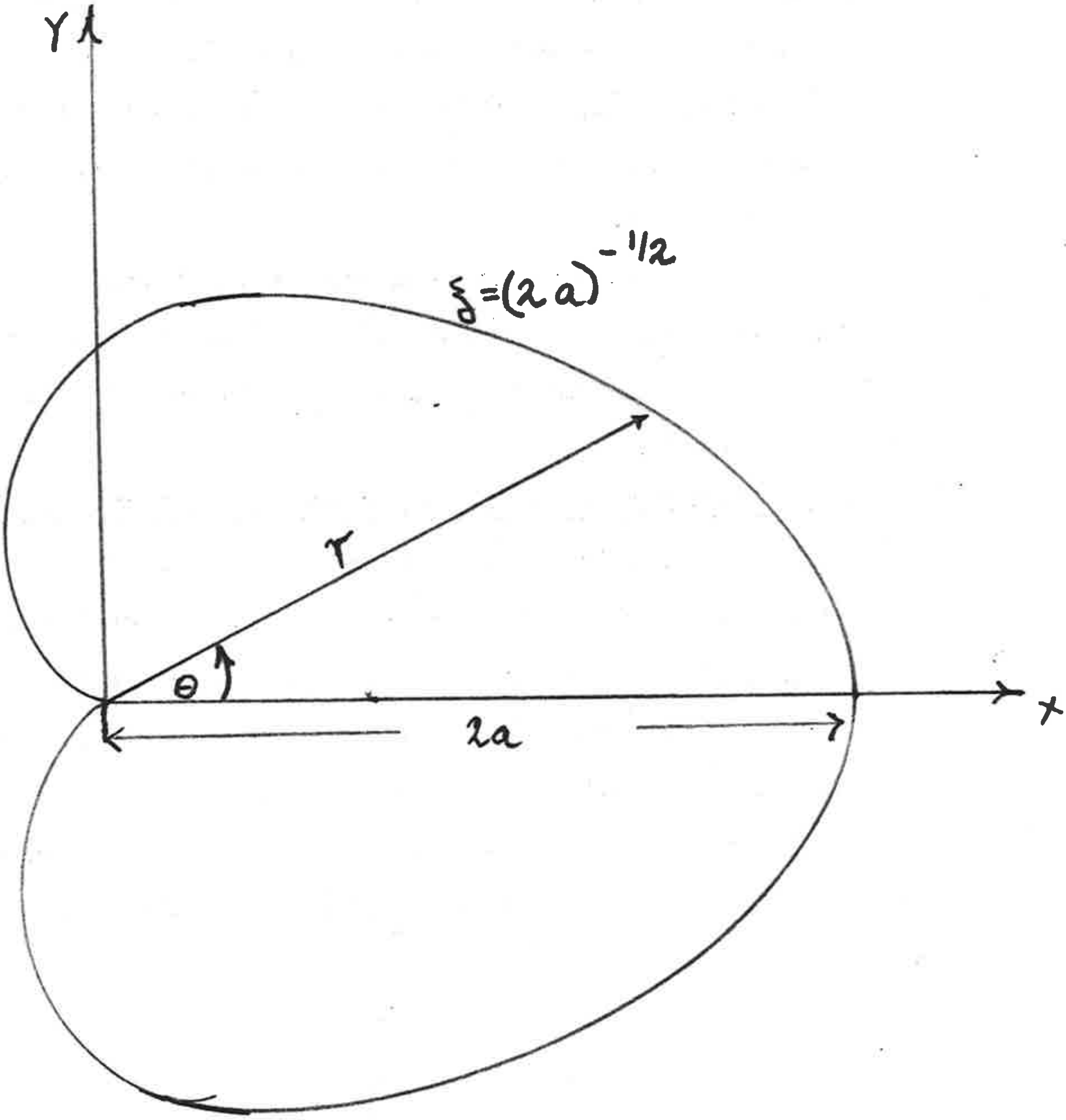


Fig. 12.

(c) If the loading in the plate does not vary in sign, then we can choose as a first approximation the same function $u(x,y)$ which would have been obtained for uniform loading, derived by the technique of the section 3.2.

It is to be mentioned here that if one speaks not of the exception, but of the rule then the considerations adduced above for choosing $u_1(x,y)$ appear sufficiently convincing.

3.8 THE BENDING OF AN ARBITRARY LOADED CARDIODAL PLATE

As an illustration of the procedure explained in the last section let us consider the problem of a clamped, cardiodal plate acted upon by an arbitrarily distributed load. With the geometry of the plate as shown in figure 12. the equation of the boundary is given by

$$(2a)^{-\frac{1}{2}} - \xi = 0, \tag{3.8.1}$$

where (ξ,η) are a system of orthogonal curvilinear coordinates defined by

$$\begin{aligned} \xi &= r^{-\frac{1}{2}}\cos\theta/2 \\ \eta &= r^{-\frac{1}{2}}\sin\theta/2. \end{aligned} \tag{3.8.2}$$

If we assume that $q(\xi,\eta)$ can be represented as power series in ξ and η, viz.

$$q(\xi,\eta) = \sum_{n,m=0}^{\infty} q_{mn}\xi^m\eta^n, \tag{3.8.3}$$

then, considering only the first two terms we obtain

$$q(\xi,\eta) = q_{00} + q_{-10}\,\xi^{-1} \qquad (3.8.4)$$
$$= \alpha + \beta\xi^{-1} (\text{say}).$$

In accordance with case b) of section 3.7 we now take $u(\xi,\eta)$ of the form

$$u(\xi,\eta) = (2a)^{-\frac{1}{2}} - \xi, \qquad (3.8.5)$$

so that upon substituting for u into equation (2.2.6), and using the usual mean value technique we obtain

$$-\frac{3}{16}\xi^{9}\frac{d^{3}w}{d\xi^{3}} - \frac{9}{8}\frac{d^{2}w}{d\xi^{2}} = \frac{\alpha}{\xi^{2}} + \frac{\beta}{2\xi^{3}}, \qquad (3.8.6)$$

where the boundary conditions to be imposed on w are

$$\text{i)}\quad w\Big|_{\xi=(2a)^{-\frac{1}{2}}} = 0,$$
$$\qquad (3.8.7)$$
$$\text{ii)}\quad \frac{dw}{d\xi}\Big|_{\xi=(2a)^{-\frac{1}{2}}} = 0.$$

If we now seek the solution for w in the following power series form involving unknown coefficients C_j, viz.

$$w = ((2a)^{-\frac{1}{2}} - \xi)^{2}\sum_{j=0}^{\infty} C_j\,\xi^{j}, \qquad (3.8.8)$$

then upon substituting into equation (3.8.6) and comparing the coefficients of ξ^{j} we obtain a system of algebraic equations for determining unknown coefficients C_j. In particular considering the first three terms in the series expansion of the residue we obtain

$$w = \frac{\alpha 8a}{216}\left((2a)^{-\frac{1}{2}}-\xi\right)^{2}\left(\xi^{-8} + 2\xi^{-7}(2a)^{\frac{1}{2}} + 6a\xi^{-6}\right) +$$
$$\frac{\beta 16a}{1350}\left((2a)^{-\frac{1}{2}}-\xi\right)^{2}\left(\xi^{-9} + 2\xi^{-8}(2a)^{\frac{1}{2}} + 6a\xi^{-7} + 4\xi^{-6}(2a)^{\frac{3}{2}}\right). \qquad (3.8.9)$$

For comparison let us consider the particular case when $\beta=0$ and the load is uniform. Equation (3.8.9) now gives a maximum deflection of

$$(\bar{w})_{max} = .031a^4, \qquad (3.8.10)$$

occurring at $r = .952a$, as against $(w)_{max} = .029a^4$ occurring at $r = .902a$ as given in [53]. This shows that the obtained result is only about 7% greater than the previously established results. However, we can always improve the accuracy of the solution by increasing the number of terms in (3.8.8).

CHAPTER IV

BUCKLING OF THIN ELASTIC PLATES

4.1 DERIVATION OF THE CORRECT FORM OF THE LINES OF CONSTANT DEFLECTION

The analysis of the buckling of thin, elastic plates as described in section 2.3 will be substantiated in this chapter by finding the exact form of the deflection contours under various edge conditions.

Let us first consider the buckling of a thin elastic plate subject to hydrostatic edge loading $N_x = N_y = -N$, $N_{xy} = 0$. If we now substitute the expressions for the vector quantities $\underset{\sim}{n}$, $\underset{\sim}{n}_o$, r, and $\underset{\sim}{r}_o$ in equation (2.3.8) and make use of equation (2.3.9), then after simplification equation (2.3.8) will be replaced by

$$A_1 \frac{d^3w}{du^3} + A_2 \frac{d^2w}{du^2} + A_3 \frac{dw}{du} + A_5 = 0, \tag{4.1.1}$$

where A_1, A_2 and A_3 are as given in section 3.1 and

$$A_5 = \lambda_1 \iint\limits_{\Omega_u} xN\nabla^2 w d\Omega + \lambda_2 \iint\limits_{\Omega_u} yN\nabla^2 w d\Omega. \tag{4.1.2}$$

Therefore as in section 3.2, we require the following relationships between the coefficients of equations (4.1.1) and (2.3.8) to exist, viz.

$$\frac{A_1}{\oint\limits_{C_u} R ds} = \frac{A_2}{\oint\limits_{C_u} F ds} = \frac{A_3}{\oint\limits_{C_u} G ds} = \frac{A_5}{\iint\limits_{\Omega_u} N\nabla^2 w d\Omega} \tag{4.1.3}$$

From the first and the last relationships in (4.1.3) we obtain

$$\frac{\lambda_1 \oint_{C_u} x\, R ds + \lambda_2 \oint_{C_u} y\, R ds}{\oint_{C_u} R ds} = \frac{\lambda_1 \iint_{\Omega_u} x N \nabla^2 w d\Omega + \lambda_2 \iint_{\Omega_u} y N \nabla^2 w d\Omega}{\iint_{\Omega_u} N \nabla^2 w dw}, \tag{4.1.4}$$

so that denoting

$$\kappa_1(c) = \frac{\oint_{C_u} xRds}{\oint_{C_u} Rds}, \qquad \kappa_2(c) = \frac{\oint_{C_u} yRds}{\oint_{C_u} Rds}, \tag{4.1.5}$$

and noting that

$$\nabla^2 w = \nabla^2 u \frac{dw}{du} + t \frac{d^2 w}{du^2}, \tag{4.1.6}$$

we finally arrive at

$$\lambda_1 \iint_{\Omega_u} (x - \kappa_1(c)) [\nabla^2 u \frac{dw}{du} + t \frac{d^2 w}{du^2}] d\Omega + \lambda_2 \iint_{\Omega_u} (y - \kappa_2(c)) [\nabla^2 u \frac{dw}{du} + t \frac{d^2 w}{du^2}] d\Omega = 0. \tag{4.1.7}$$

However since the double integrals appearing in (4.1.7) are independent of the fixed point (x_0, y_0), and since λ_1, and λ_2 depend only on (x_0, y_0) we have

$$\iint_{\Omega_u} (x-\kappa_1(c))[\nabla^2 u \frac{dw}{du} + t \frac{d^2w}{du^2}] d\Omega$$

$$= 0, \qquad (4.1.8)$$

and

$$\iint_{\Omega_u} (y-\kappa_2(c))[\nabla^2 u \frac{dw}{du} + t \frac{d^2w}{du^2}] d\Omega$$

$$= 0. \qquad (4.1.9)$$

Multiplying equation (4.1.9) by $i = \sqrt{-1}$, and adding to (4.1.8) we obtain

$$\iint_{\Omega_u} (z-\xi)\left(\nabla^2 u \frac{dw}{du} + t \frac{d^2w}{du^2}\right) d\Omega = 0, \qquad (4.1.10)$$

which may also be expressed as

$$-\int_{U^*}^{u} \left\{\left(\frac{dw}{du}\right)_0 \oint_{C_{u_0}} \frac{(z-\xi)\nabla^2 u}{\sqrt{t}} ds + \left(\frac{d^2w}{du^2}\right)_2 \oint_{C_{u_0}} (z-\xi)\sqrt{t} ds\right\} du_0$$

$$= 0, \qquad (4.1.11)$$

where C_{u_0} is an arbitrary curve $u=u_0$ inside the domain under consideration and $()_0$ signifies that the quantity has been evaluated on $u=u_0$.

However since equation (4.1.11) must vanish for all admissible values of u contained in $[o,u^*]$ we require

$$\frac{dw}{du}\oint_{C_u} \frac{(z-\xi)\nabla^2 u}{\sqrt{t}}\, ds + \frac{dw}{du}\oint_{C_u} (z-\xi)\sqrt{t}\, ds = 0. \qquad (4.1.12)$$

Using Green's theorem the above equation reduces[1] to

$$\frac{d}{du}\left[\frac{dw}{du}\oint_{C_u} (z-\xi)\sqrt{t}\, ds\right] = 0, \qquad (4.1.13)$$

which will be satisfied if we demand that

$$\oint_{C_u} (z-\xi)\sqrt{t}\, ds = 0. \qquad (4.1.14)$$

However equation (4.1.14) is the same type of equation for $u(x,y)$ as obtained in section (3.2) for the bending of a uniformly loaded, thin, elastic plate. Thus the lines of constant deflection for the non nodal buckling mode of a hydrostatically compressed thin elastic plate must coincide with the lines of constant deflection for the same plate subject to uniform loading, and with identical boundary conditions. Furthermore, it follows from [50] that we may take the first buckling mode to be the non nodal mode.

Thus the determination of the exact form of the deflection contours for the buckling of a thin, elastic plate subject to hydrostatic edge loading follows from the analysis given in section 3.2 for determining the lines of constant deflection for the bending of a uniformly loaded plate.

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[1] See appendix 3

4.2 THE BUCKLING OF AN EQUILATERAL TRIANGULAR PLATE

As an illustration of the method let us consider the buckling of an equilateral triangular plate subject to hydrostatic edge loading, -N, where the edges of the plate are either clamped or simply supported. The value of the critical load obtained for the simply supported case may be compared with the solution obtained by Woinowsky-Kreiger [65.], while for the clamped plate a comparison can be made with an approximate value for the critical load given by Yoshiki and Kawai [67].

4.2.1 THE EDGES OF THE PLATE ARE CLAMPED

Consider the plate as described in figure 6, where the edges of the plate are clamped and where the equation of the boundary is given by

$$x^3 - 3xy^2 - ax^2 - ay^2 + \frac{4a^3}{27} = 0. \qquad (4.2.1)$$

We now determine $u(x,y)$ in accordance with the analysis given in chapter 3. Thus $u(x,y)$ may be taken as the solution to

$$\nabla^2 u = -2 \qquad (4.2.2)$$

subject to

$$u\Big|_C = 0, \qquad (4.2.3)$$

which gives as the equation of the lines of constant deflection

$$u(x,y) = x^3 - 3xy^2 - ax^2 - ay^2 + \frac{4a^3}{27} = \text{const.} \qquad (4.2.4)$$

Here $0 \leqslant u \leqslant \frac{4a^3}{27}$.

With this $u(x,y)$ the deflected surface $w(u)$ must satisfy equation (2.3.11) subject to the clamping conditions

$$\text{i)}\quad W\Big|_{u=0} = 0,$$
$$\text{ii)}\quad \frac{dw}{du}\Big|_{u=0} = 0, \qquad (4.2.5)$$
$$\text{iii)}\quad \sqrt{t}\,\frac{dw}{du}\Big|_{u=\frac{4a^3}{27}} = 0.$$

Unfortunately the line integrals involved in equation (2.3.11) are difficult to evaluate analytically, and so we proceed numerically denoting

$$\oint_{C_u} R\,ds = \hat{R}S,$$
$$\oint_{C_u} F\,ds = \hat{F}S,$$
$$\oint_{C_u} G\,ds = \hat{G}S,$$

where $\hat{R}$, $\hat{F}$, and $\hat{G}$ are the mean values of R, F, and G evaluated on the contour $u(x,y) = \text{const.}$ at the points

$$y = 0,\ x = \frac{a}{3} + \frac{2a}{3}\cos\left(1/3\cos^{-1}\frac{27}{2a^3}\left(u - \frac{2a^3}{27}\right)\right).$$

If we now seek $w(u)$ in the following power series form containing unknown parameters c_j, viz.

$$w(u) = \sum_{j=0}^{m} c_j u^j. \qquad (4.2.7)$$

Then from (4.2.5) it is clear that $c_1=c_0=0$, and we seek the solution for w as

$$w(u) = \sum_{j=2}^{m} c_j u^j. \qquad (4.2.8)$$

Consequently if we now substitute for w in equation (2.3.11) and minimize the residue with respect to coordinate functions u^j, the integration being performed numerically using Simpson's rule, we obtain

$$\sum_{j=2}^{m} \int_0^{\sqrt{\frac{4a^3}{27}}} [\oint_{C_u} R ds\ j(j-1)(j-2)u^{i+j-3} + \oint_{C_u} F ds\ j(j-1)u^{i+j-2}$$

$$+ \oint_{C_u} G ds\ ju^{i+j-1}c_j - N \oint_{C_u} \sqrt{t}\ ju^{i+j+1}]\, c_j du$$

$$= 0, \quad \text{where} \quad 2 \leqslant i \leqslant m, \qquad (4.3.9)$$

which represents an eigenvalue problem for N_{cr}/D. This eigenvalue problem is then solved using the usual numerical techniques.

The values of N_{cr}/D so obtained are listed in Table 5 for various values of m along with the corresponding results of Yoshiki and Kawai [67], who used Galerkin's technique to obtain a first approximation for the critical load.

TABLE 5

m	1	2	3	[67]
α	128.2	129.6	129.9	132.0
β	12.99	13.13	13.13	13.37

Here α and β are defined by

$$\frac{N_{cr}}{D} = \frac{\alpha}{a^2} = \frac{\beta\pi^2}{a^2} . \tag{4.2.10}$$

It is clear that the two results agree fairly well.

4.2.2 THE EDGES OF THE PLATE ARE SIMPLY SUPPORTED

Suppose now that the plate described in section 4.2.1 is simply supported along its edges. As in chapter 3 we seek u in the form

$$u = UV, \tag{4.2.11}$$

such that

$$\left.\begin{aligned}\nabla^2 U &= -2\\ \nabla^2 V &= -2\end{aligned}\right\}, \tag{4.2.12}$$

subject to the boundary conditions

$$\begin{aligned}&\text{i)}\quad U\Big|_C = 0,\\ &\text{ii)}\quad Q(UV)\Big|_C = 0.\end{aligned} \tag{4.2.13}$$

This has solution

$$U = \frac{1}{2a}\left[x^3 - 3xy^2 - ax^2 - ay^2 + \frac{4a^3}{27}\right], \tag{4.2.14}$$

$$V = \frac{1}{2}\left[\frac{4a^2}{9} - x^2 - y^2\right], \tag{4.2.15}$$

so that without loss in generality we may consider the lines of constant deflection as

$$u = (x^3 - 3xy^2 - ax^2 - ay^2 + \frac{4a^3}{27})(\frac{4a^2}{9} - x^2 - y^2)$$

$$= \text{const.} \qquad (4.2.16)$$

Proceeding as for the clamped plate we again seek $w(u)$ in the following power series involving unknown parameters c_j, viz.

$$w(u) = \sum_{j=0}^{m} c_j u^j. \qquad (4.2.17)$$

However since the edges of the plate are simply supported the boundary conditions to be imposed on w are

$$\text{i)} \quad w\Big|_{u=0} = 0, \qquad (4.2.18)$$

$$\text{ii)} \quad \frac{d^2w}{du^2}\Big|_{u=0} = 0.$$

Consequently upon substituting for w in (4.2.16) we obtain $c_2=c_0=0$, so that we seek the solutions for w as

$$w = c_1 u + \sum_{j=3}^{m} c_j u^j. \qquad (4.2.19)$$

On substitution of w into equation (2.3.9) and applying Galerkin's technique, as explained in the previous section, we obtain values of the critical load for various values of m. These values are listed in Table 6 along with the corresponding value of $\frac{N_{cr}}{D}$ obtained by Woinowsky-Krieger [65].

TABLE 6.

k \ m	1	3	4	[65]
	4.185	4.037	4.034	4.00

Here

$$\frac{N_{cr}}{D} = \frac{k\pi^2}{a^2}. \qquad (4.2.20)$$

CHAPTER V

THE TRANSVERSE VIBRATION OF THIN ELASTIC PLATES

5.1 DERIVATION OF THE EQUATION FOR THE LINES OF CONSTANT DEFLECTION

In this chapter the exact equation for the lines of constant deflection of a thin elastic plate under free vibration is obtained. As an illustration of the procedure the problem of the transverse vibration of an equilateral triangular plate with either clamped or simply supported edges is discussed.

If we substitute the expressions for the vector quantities $\underset{\sim}{n}$, $\underset{\sim}{n}_o$, $\underset{\sim}{r}$, and $\underset{\sim}{r}_o$ into (2.4.6) and make use of (2.4.7), then after simplification equation (2.4.6) is replaced by

$$A_1 \frac{d^3W}{du^3} + A_2 \frac{d^2W}{du^2} + A_3 \frac{dW}{du} - A_6 = 0, \qquad (5.1.1)$$

where A_1, A_2, and A_3 are as given in section 3.1, and

$$A_6 = \rho h\omega^2 \left[\lambda_1 \iint_{\Omega_u} xWd\Omega + \lambda_2 \iint_{\Omega_u} yWd\Omega. \right] \qquad (5.1.2)$$

Therefore as in section 3.2 we require the following relationships to hold between the coefficients of equations (5.1.1) and (2.4.7), viz.

$$\frac{A_1}{\oint_{C_u} Rds} = \frac{A_2}{\oint_{C_u} Rds} = \frac{A_3}{\oint_{C_u} Gds} = \frac{A_6}{\iint_{\Omega_u} \rho h\omega^2 Wd\Omega} . \qquad (5.1.3)$$

From the first and the last relationships in (5.1.3) we obtain

$$\frac{\lambda_1 \oint\limits_{C_u} xRds + \lambda_2 \oint\limits_{C_u} yRds}{\oint\limits_{C_u} Rds} = \frac{\lambda_1 \iint\limits_{\Omega_u} xWd\Omega + \lambda_2 \iint\limits_{\Omega_u} yWd\Omega}{\iint\limits_{\Omega_u} Wd\Omega}, \qquad (5.1.4)$$

so that denoting

$$\kappa_1(c) = \frac{\oint\limits_{C_u} xRds}{\oint\limits_{C_u} Rds}, \quad \kappa_2(c) = \frac{\oint\limits_{C_u} yRds}{\oint\limits_{C_u} Rds}, \qquad (5.1.5)$$

we finally arrive at

$$\lambda_1 \iint\limits_{\Omega_u} (x-\kappa_1(c))Wd\Omega + \lambda_2 \iint\limits_{\Omega_u} (y-\kappa_2(c))Wd\Omega = 0. \qquad (5.1.6)$$

However since the double integrals appearing in (5.1.6) are independent of the fixed point (x_0, y_0), and since the coefficients of λ_1 and λ_2 depend only on (x_0, y_0) we have

$$\iint\limits_{\Omega_u} (x-\kappa_1(c))Wd\Omega = 0, \qquad (5.1.7)$$

and

$$\iint\limits_{\Omega_u} (y-\kappa_2(c))Wd\Omega = 0, \qquad (5.1.8)$$

which we may express as

$$-\int_{u^*}^{u} W \oint_{C_{u_0}} \frac{(x-\kappa_1(c))}{\sqrt{t}}\, ds\, du_0 = 0, \tag{5.1.9}$$

and

$$-\int_{u^*}^{u} W \oint_{C_{u_0}} \frac{(y-\kappa_2(c))}{\sqrt{t}}\, ds\, du_0 = 0, \tag{5.1.10}$$

where $0 \leqslant u \leqslant u^*$. Thus in order that (5.1.9) and (5.1.10) be satisfied for all admissible values of u in $[0,u^*]$ we require

$$\oint_{C_u} \frac{(x-\kappa_1(u))}{\sqrt{t}}\, ds = 0, \tag{5.1.11}$$

and

$$\oint_{C_u} \frac{(y-\kappa_2(u))}{\sqrt{t}}\, ds = 0, \tag{5.1.12}$$

which are precisely the same equations for $u(x,y)$ as obtained in section 3.2 for the bending of a uniformly loaded, thin, elastic plate. Consequently the lines of constant deflection for the fundamental mode of transverse vibration of a thin, elastic plate must coincide with the lines of constant deflection for the bending of the same plate under uniform load, and with identical boundary conditions.

The implications of this result together with the corresponding results of section 4.1 lead to a coupling of the bending, buckling, and vibration problems. Indeed a knowledge of the solution of any one of these problems gives us an insight into the form of the deflected surface, including the point of maximum deflection, for the remaining two problems.

It should be noted that in the particular case of a simply supported polygonal plate these results may be obtained as a consequence of the membrane analogy.

5.2 VIBRATION OF EQUILATERAL TRIANGULAR PLATES

As an illustration of the above method, let us consider the problem of the free vibration of an equilateral triangular plate with either clamped or simply supported edges. Amongst the published results for the clamped equilateral triangular plate mention should be made of

i) Cox and Klein [7,8], using skew coordinates [7], and the method of collocation [8];

ii) Ota, Hamada and Taurumoto [48], using an energy method;

iii) Yoshiki and Kawai [67], using energy principles as well as the Galerkin, and Rayleigh-Ritz techniques;

iv) Laura and Falstich [27] using conformal mapping.

Whereas amongst the published results for the simply supported equilateral triangular plate mention should be made of

i) Cox and Klein [8], employing a collocation method;

ii) Conway and Farnham [5], using a point matching method;

iii) Conway [3], by analogy;

iv) Schaefer and Havers [54].

However despite the attention these problems have received no exact solutions have yet been obtained.

The equilateral triangular plate was chosen since it is frequently necessary in the analysis of structures to determine the fundamental frequency of vibration of a triangular plate. Furthermore it gives us an excellent illustration of the methods available for determining the true form of lines of constant deflection.

5.2.1 THE EDGES OF THE PLATE ARE CLAMPED.

Consider the plate as described in figure 6, where the edges of the plate are clamped, and where the equation of the boundary is given by

$$x^3 - 3xy^2 - ax^2 - ay^2 + \frac{4a^3}{27} = 0. \qquad (5.2.1)$$

We now determine $u(x,y)$ in accordance with the analysis given in section 3.2. Thus $u(x,y)$ must satisfy

$$\nabla^2 u = -2, \qquad (5.2.2)$$

subject to

$$u\Big|_C = 0, \qquad (5.2.3)$$

so that without loss in generality we may consider as the lines of constant deflection

$$u(x,y) = x^3 - 3xy^2 - ax^2 - ay^2 + \frac{4a^3}{27} = \text{const.} \qquad (5.2.4)$$

If we now seek $W(u)$ in the following power series form involving arbitrary parameters c_j, viz.

$$W(u) = \sum_{j=0}^{m} c_j u^j, \qquad (5.2.5)$$

then applying the boundary conditions

$$\text{i)} \quad W\Big|_{u=0} = 0, \qquad (5.2.6)$$
$$\text{ii)} \quad \frac{dW}{du}\Big|_{u=0} = 0,$$

gives $c_0 = c_1 = 0$. So that we seek the solution for W as

$$W = \sum_{j=2}^{m} c_j u^j. \qquad (5.2.8)$$

Upon substituting this form for W in equation (2.4.11) and applying Galerkin's technique as explained in section 4.2.1 we obtain an eigenvalue problem for the fundamental frequency of vibration $\lambda = \sqrt{\rho h/D}\,\omega a^2$. The values of λ so obtained are given in Table 7. for various values of m, along with the corresponding results of Cox Klein [7,8]

TABLE 7.

m	2	3	5	[7,8]
$\lambda=\sqrt{\rho h/D}\,\omega a^2$	70.09	70.34	70.35	70.34

5.2.2 THE EDGES OF THE PLATE ARE SIMPLY SUPPORTED.

Suppose now that the plate described in 5.2.1 is simply supported along its edges. As in section 3.2 we seek u in the form

$$u = UV, \tag{5.2.9}$$

such that

$$\left.\begin{aligned}\nabla^2 U &= -2\\ \nabla^2 V &= -2\end{aligned}\right\}, \tag{5.2.10}$$

subject to the boundary conditions

$$\begin{aligned}&\text{i)}\quad U\Big|_C = 0,\\ &\text{ii)}\quad Q(UV)\Big|_C = 0.\end{aligned} \tag{5.2.11}$$

This has solution

$$U = (x^3 - 3xy^2 - ax^2 - ay^2 + \frac{4a^3}{27})/2a, \tag{5.2.12}$$

and

$$V = (\frac{4a^2}{9} - x^2 - y^2)/2, \tag{5.2.13}$$

so that without loss of generality we may consider the lines of constant deflection as

$$u = (x^3 - 3xy^2 - ax^2 - ay^2 + \frac{4a^3}{27})(\frac{4a^2}{9} - x^2 - y^2) = \text{const.} \tag{5.2.14}$$

Here $\qquad 0 \leqslant u \leqslant \frac{16a^5}{243}$.

Proceeding as for the clamped plate we again seek $W(u)$ in the following power series form involving unknown parameters c_j, viz.

$$W(u) = \sum_{j=0}^{m} c_j u^j. \tag{5.2.15}$$

However since the edges of the plate are simply supported the boundary conditions to be imposed are

$$\text{i)}\quad W\Big|_{u=0} = 0,$$

$$\text{ii)}\quad \frac{d^2W}{du^2}\Big|_{u=0} = 0, \tag{5.2.16}$$

which gives $c_2 = c_0 = 0$. Thus W takes the form

$$W = c_1 u + \sum_{j=3}^{m} c_j u^j.$$

On substituting for W in equation (2.4.11) and applying Galerkin's technique we again obtain values of the fundamental frequency $\lambda = \sqrt{\rho h/D}\,\omega a^2$ for various values of m. These values are shown in Table 8. along with the corresponding results obtained by Conway and Farnham [5].

TABLE 8.

m	1	3	5	[5]
$\lambda=\sqrt{\rho h/D}\,\omega a^2$	40.93	39.96	39.93	39.48

5.3 DERIVATION OF THE EQUATION FOR THE LINES OF CONSTANT DEFLECTION FOR THE TRANSVERSE VIBRATION OF THIN ELASTIC PLATES WITH INPLANE FORCES

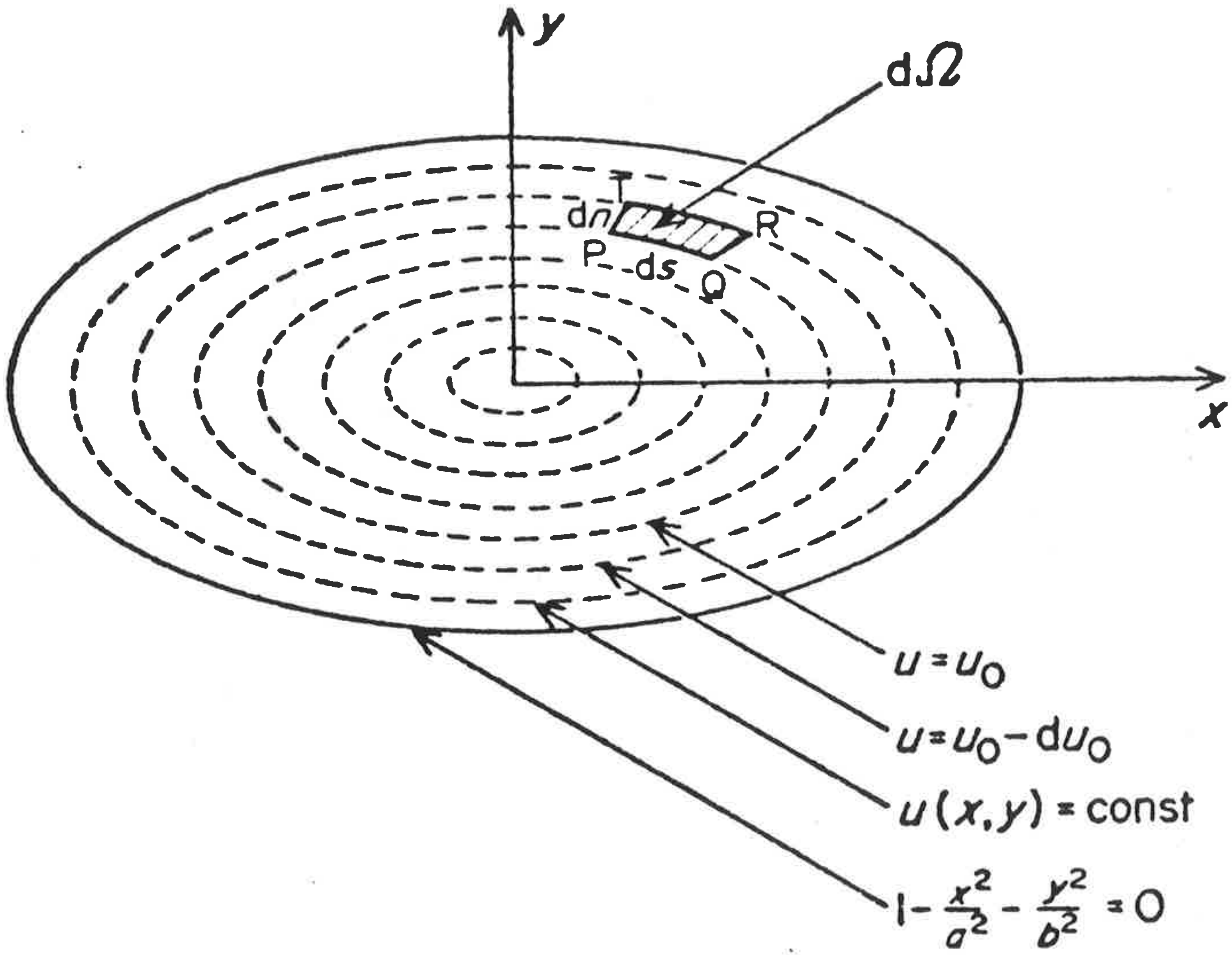
Let us first consider the transverse vibration of thin elastic plates with hydrostatic edge loading, i.e.,

$N_x = N_y = -N$, $N_{xy} = 0$. Since deflection contours for the free transverse vibration of a thin elastic plate, and for the buckling of the same plate under compressive hydrostatic edge loading coincide, we conclude they must also coincide with the lines of constant deflection for the transverse vibration of the same plate subject to hydrostatic edge loading and with the same boundary conditions.

Thus the lines of constant deflection for the transverse vibration of a hydrostatically compressed thin elastic plate are independent of the applied edge load. This is a generalization of Lurie's [34] result:"the vibration mode shapes for a simply supported polygonal plate are independent of the intensity N of the applied edge load".

5.4 TRANSVERSE VIBRATION OF ELLIPTIC PLATES WITH HYDROSTATIC EDGE LOADING

As an illustration, consider the transverse vibration of a thin elliptic plate subject to hydrostatic edge loading i.e., $N_x = N_y = N$, $N_{xy} = 0$, where the edges of the plate are either clamped or simply supported. Since there are no theoretical or experimental results with which comparisons may be made, we will consider the limiting case when the ellipse degenerates into a circle and compare our results with those of Wah [60].



y
x
dΩ
T
dn
R
P
ds
Q
$u = u_0$
$u = u_0 - du_0$
$u(x,y) = \text{const}$
$1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

Figure 13.

5.4.1 THE BOUNDARY OF THE PLATE IS CLAMPED

Let the semimajor and semiminor axes of the ellipse be taken as a, and b respectively, then with the co-ordinates as shown in figure 13. the equation of the boundary is given by

$$1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0. \tag{5.4.1}$$

Since the plate is clamped, the deflection contours may be taken as for the uniformly loaded, clamped elliptic plate, which are a family of similar and similarly situated ellipses. Therefore without loss of generality we may take

$$u(x,y) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} . \tag{5.4.2}$$

Calculating the values of R,F,G, and t to be used in equation (2.4.11) gives

$$R = -\frac{8D}{p^3} \tag{5.4.3}$$

$$F = 4Dp[\frac{(1-u)}{a^2b^2} + \frac{3x^2}{a^6} + \frac{3y^2}{b^6}]$$

$$G = 2D(1-\mu)(1/a^2 - 1/b^2)p^5(x^2/a^4 - y^2/b^4)$$

$$\cdot (1-u)/a^2b^2,$$

$$t = \frac{4}{p^2} ,$$

where

$$p^2 = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^{-1} . \tag{5.4.4}$$

And so upon substituting into (2.4.11) we obtain

$$-8D\frac{d^3W}{du^3}\oint_{C_u}\frac{1}{p^3}\,ds + 4D\frac{d^2W}{du^2}\oint_{C_u} p\left[\frac{1-u}{a^2b^2} + 3\left(\frac{x^2}{a^6}+\frac{y^2}{b^6}\right)\right]ds$$

$$+2D\frac{(1-\mu)}{a^2b^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\frac{dW}{du}\oint_{C_u} p^5\left(\frac{x^2}{a^4}-\frac{y^2}{b^4}\right)(1-u)\,ds$$

$$-\rho h\omega^2\iint_{\Omega_u} W\,d\Omega + 2N\frac{dW}{du}\oint_{C_u}\frac{1}{p}\,ds = 0, \qquad (5.4.5)$$

where the contour integrations are taken around the closed contour

$$u = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} = \text{const.} \qquad (5.4.6)$$

and the double integration extends over the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1-u. \qquad (5.4.7)$$

The values of these integrals are found to be

$$\oint_{C_u}\frac{1}{p^3}\,ds = \frac{\pi(1-u)^2}{4a^3b^3}(3a^4+2a^2b^2+3b^4)$$

$$\oint_{C_u} p\,ds = 2\pi ab$$

$$\oint_{C_u}\frac{ds}{p} = \frac{\pi(1-u)(a^2+b^2)}{ab} \qquad (5.4.8)$$

$$\oint_{C_u} p\left(\frac{x^2}{a^6}+\frac{y^2}{b^6}\right)ds = \pi ab(1-u)\left(\frac{1}{a^4}+\frac{1}{b^4}\right)$$

$$\oint_{C_u} p^5\left(\frac{x^2}{a^4} - \frac{y^2}{b^4}\right)ds = 0.$$

With the help of (5.4.8) equation (5.4.7) becomes

$$(1-u)^2 \frac{d^3W}{du^3} - 2(1-u)\frac{d^2W}{du^2} - \frac{Na^2b^2(a^2+b^2)(1-u)}{D(3a^4+2a^2b^2+3b^4)}\frac{dW}{du}$$

$$+ \frac{\rho h\omega^2 a^3b^3}{2\pi D(3a^4+2a^2b^2+3b^4)}\iint_{\Omega_u} W d\Omega = 0, \qquad (5.4.9)$$

which after differentiation with respect to u reduces to

$$(1-u)^2\frac{d^4W}{du^4} - 4\frac{d^3W}{du^3} + \frac{d^2W}{du^2}\left[2 - \frac{Na^2b^2(a^2+b^2)}{D(3a^4+2a^2b^2+3b^4)}\right.$$

$$\left.(1-u)\right] + \frac{dW}{du}\frac{N}{D}\frac{a^2b^2(a^2+b^2)}{(3a^4+2a^2b^2+3b^4)} - \frac{\rho h\omega^2 a^4b^4 W}{2D(3a^4+2a^2b^2+3b^4)}$$

$$= 0, \qquad (5.4.10)$$

which is equivalent to

$$\left((1-u)\frac{d^2}{du^2} - \frac{d}{du} + \alpha^2\right)\left((1-u)\frac{d^2}{du^2} - \frac{d}{du} - \beta^2\right)W$$

$$= 0, \qquad (5.4.11)$$

where

$$\alpha^2\beta^2 = \frac{\rho h\omega^2 a^4b^4}{2D(3a^4+2b^2a^2+3b^4)}, \qquad (5.4.12)$$

and

$$\beta^2-\alpha^2 = \frac{Na^2b^2(a^2+b^2)}{D(3a^4+2a^2b^2+3b^4)}. \qquad (5.4.13)$$

The general solution to this equation is given by

$$W = W_1 + W_2, \tag{5.4.14}$$

where W_1 and W_2 satisfy, respectively, the differential equations

$$LW_1 + \alpha^2 W_1 = 0, \tag{5.4.15}$$

and

$$LW_2 - \beta^2 W_2 = 0, \tag{5.4.16}$$

where L denotes the differential operator

$$L \equiv (1-u)\frac{d^2}{du^2} - \frac{d}{du}$$

$$= \frac{d}{du}(1-u)\frac{d}{du} . \tag{5.4.17}$$

To solve the last two equations a new independent variable f, is introduced and defined by

$$f^2 = 1-u, \tag{5.4.18}$$

with respect to which equations (5.4.15) and (5.4.16) become

$$f\frac{d^2W_1}{df^2} + \frac{dW_1}{df} + 4\alpha^2 fW_1 = 0, \tag{5.4.19}$$

$$f\frac{d^2W_2}{df^2} + \frac{dW_2}{df} - 4\beta^2 fW_2 = 0. \tag{5.4.20}$$

Equations (5.4.19) and (5.4.20) are respectively a Bessels equation and a modified Bessels equation. Their general solutions are

$$W_1 = B_1J_0(2\alpha f) + B_2Y_0(2\alpha f), \tag{5.4.21}$$

$$W_2 = B_3I_0(2\beta f) + B_4K_0(2\beta f), \tag{5.4.22}$$

where J_0 and Y_0 are regular Bessel functions of the first and second kinds respectively, I_0, and K_0 being modified Bessel functions of the first and second kinds respectively. Thus

$$W = B_1 J_0(2\alpha f) + B_2 I_0(2\beta f) + B_3 I_0(2\alpha f) + B_4 K_0(2\beta f). \tag{5.4.23}$$

If α and β can be obtained the natural frequency of vibration can easily be calculated.

The Boundary conditions at the edge $u=0$, and the condition at the centre $u=1$, must now be imposed. Since the plate is clamped we have

$$W\Big|_{u=0} = \frac{dW}{du}\Big|_{u=0} = 0, \tag{5.4.24}$$

which are equivalent to

$$W\Big|_{f=1} = \frac{dW}{df}\Big|_{f=1} = 0. \tag{5.4.25}$$

In order to avoid infinite deflection at the centre $f=0$, it is necessary while dealing with the full plate to omit the second and the fourth term in (5.4.23), giving

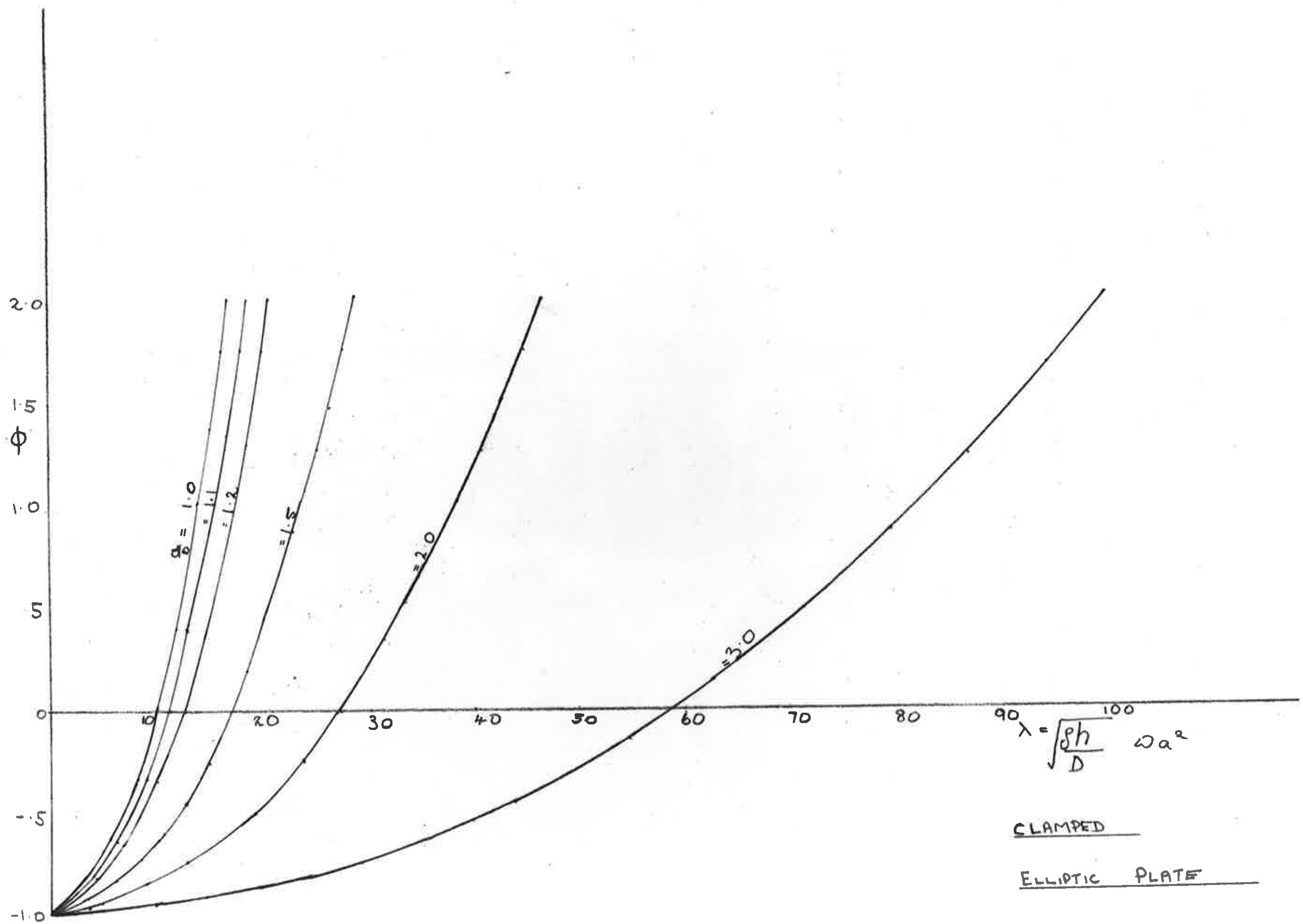
$$W = B_1 J_0(2\alpha f) + B_3 I_0(2\beta f). \tag{5.4.26}$$

Substitution of the conditions for the clamped edges into (5.4.26) yields

$$B_1 J_0(2\alpha) + B_3 I_0(2\beta) = 0, \tag{5.4.27}$$

and

$$2\alpha B_1 J_0'(2\alpha) + 2\beta B_3 I_0'(2\beta) = 0. \tag{5.4.28}$$



2·0
1·5
φ
1·0
5
0
-·5
-1·0
10
20
30
40
50
60
70
80
90
100
a/b = 1·0
= 1·1
= 1·2
= 1·5
= 2·0
= 3·0
λ = √(ρh/D) ωa²
CLAMPED
ELLIPTIC PLATE

where primes are used to indicate differentiation with respect to the arguments, which in this case are $2\alpha f$ and $2\beta f$ respectively. Therefore aside from the trivial solution $B_1=B_3=0$, solutions with non vanishing constants are obtained if and only if

$$\begin{vmatrix} J_0(2\alpha) & I_0(2\beta \\ 2\alpha J_0'(2\alpha) & 2\beta I_0'(2\beta) \end{vmatrix} = 0, \tag{5.4.29}$$

or using the derivative formulae for Bessel functions if

$$2\alpha \frac{J_1(2\alpha)}{J_0(2\alpha)} + 2\beta\frac{I_1(2\beta)}{I_0(2\beta)} = 0. \tag{5.4.30}$$

It is convenient to introduce the nondimensional parameter

$$\frac{N}{N^*} = \phi, \tag{5.4.31}$$

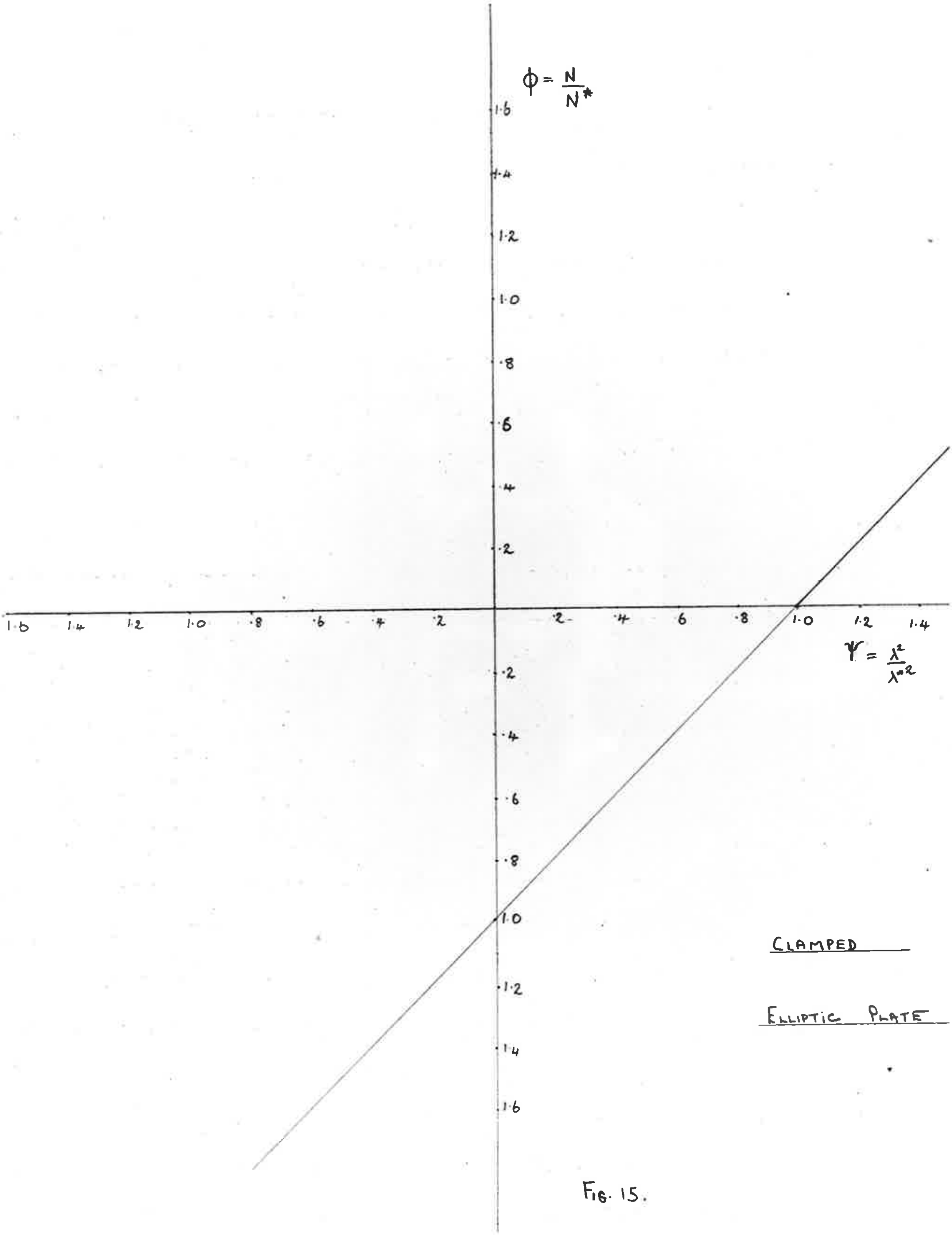
where

$$\frac{N^*}{D} = 3.67\,\frac{(3a^4+2a^2b^2+3b^4)}{a^2b^2(a^2+b^2)}, \tag{5.4.32}$$

then from (5.4.13) we obtain

$$\beta^2 - \alpha^2 = 3.67\phi. \tag{5.4.33}$$

The natural frequency of the plate may now be determined from equations (5.4.33) and (5.4.30) for various values of the parameter ϕ. Negative values of ϕ represent compression, and in particular the value of N corresponding to $\phi = -1$ represents the critical buckling load of a hydrostatically compressed elliptic plate, as determined by Mazumdar [36]. The numerical values for



$\phi = \frac{N}{N^*}$
$\Psi = \frac{\lambda^2}{\lambda^{*2}}$
CLAMPED
ELLIPTIC PLATE

FIG. 15.

the fundamental frequency

$$\lambda = \sqrt{\rho h/D}\ \omega a^2 = \alpha\beta\sqrt{2(3\delta^4+2\delta^2+3)}, \qquad (5.4.34)$$

where

$$\delta = a/b, \qquad (5.4.35)$$

are shown in figure 14. for various values of ϕ and δ, while figure 15. shows the relationship between ϕ and the nondimensionalized frequency parameter $\psi = \lambda^2/\lambda^{*2}$, where

$$\lambda^* = 10.22\sqrt{2(3\delta^4+2\delta^2+3)}, \qquad (5.4.36)$$

is the fundamental frequency of an elliptic plate, corresponding to ϕ=0.

TABLE 9.

ϕ \ a/b	1	1.1	1.2	1.5	2	3
2	8.55	9.47	10.51	14.40	33.86	49.16
1.5	7.81	8.68	9.60	13.15	30.93	44.86
1.0	6.99	7.74	8.60	11.77	27.68	40.15
.5	6.05	6.70	7.44	10.19	23.98	34.95
.25	5.52	6.12	6.79	9.30	21.86	37.10
0	4.94	5.47	6.08	8.32	19.56	28.38
-.25	5.27	4.73	5.25	7.19	16.91	24.53
-.5	3.46	3.83	4.26	5.83	13.70	19.88
-1.0	0	0	0	0	0	0

These results agree with those of Wah [60], in the particular case of a circular plate, a/b = 1. More importantly, figure 15. reveals that $\phi - \psi$ diagram is a straight line passing through the points (1,0) and (0,1) and is independent of the aspect ratio a/b. This is particularly important since it now enables us to bypass the analysis, provided the values of the critical buckling load, and the fundamental eigenvalue are known. It should be pointed out that negative values of $\psi = \lambda^2/\lambda^{*2}$ correspond to the case of plates on elastic foundation.

5.6.2 THE BOUNDARY OF THE PLATE IS SIMPLY SUPPORTED

Suppose now that the plate described in 5.6.1 is simply supported along the boundary. The deflection contours, will be as for the uniformly loaded, simply supported, elliptic plate, and as such are discussed in Section 3.5. However, since the deflection contours $u(\xi,\eta) = \text{const.}$ are complex functions of elliptical coordinates ξ, and η, and since it has been shown in section 3.5 that for moderate aspect ratio's $a/b \leqslant 2$ these deflection contours coincide almost exactly with a family of similar and similarly situated ellipses, we proceed taking

$$u(x,y) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \,. \qquad (5.4.37)$$

Proceeding as for the clamped plate we obtain

$$W = B_1 J_0(2\alpha f) + B_2 I_0(2\beta f), \qquad (5.4.38)$$

where the boundary conditions to be satisfied are

$$\text{i)} \quad W\Big|_{u=0} = 0,$$

$$(5.4.39)$$

$$\text{ii)} \quad M_n\Big|_{u=0} = 0,$$

which are equivalent to

$$\text{i)}\quad W\Big|_{f=1} = 0,$$

$$\text{ii)}\quad \frac{dW}{df^2} + \frac{\mu}{f}\frac{dW}{df}\Big|_{f=1} = 0. \qquad (5.4.40)$$

Substitution of the boundary conditions (5.4.40) i) and ii) into (5.4.38) now gives

$$B_1 J_0(2\alpha) + B_3 I_0(2\beta) = 0, \qquad (5.4.41)$$

and

$$B_1[4\alpha^2 J_0''(2\alpha) + 2\mu\alpha J_0'(2\alpha)] + B_3[4\beta^2 I_0''(2\beta) + 2\mu\beta I_0'(2\beta)] = 0. \qquad (5.4.42)$$

After detailed algebraic computation the resulting determinant can be reduced to the frequency equation

$$2\alpha \frac{J_1(2\alpha)}{J_0(2\alpha)} + 2\beta \frac{I_1(2\beta)}{I_0(2\beta)} = 4 \frac{(\beta^2+\alpha^2)}{1-\mu} . \qquad (5.4.43)$$

If we again introduce the nondimensional parameters

$$\phi = \frac{N}{N^*} , \qquad (5.4.44)$$

and

$$\psi = \frac{\lambda^2}{\lambda^{*2}} , \qquad (5.4.45)$$

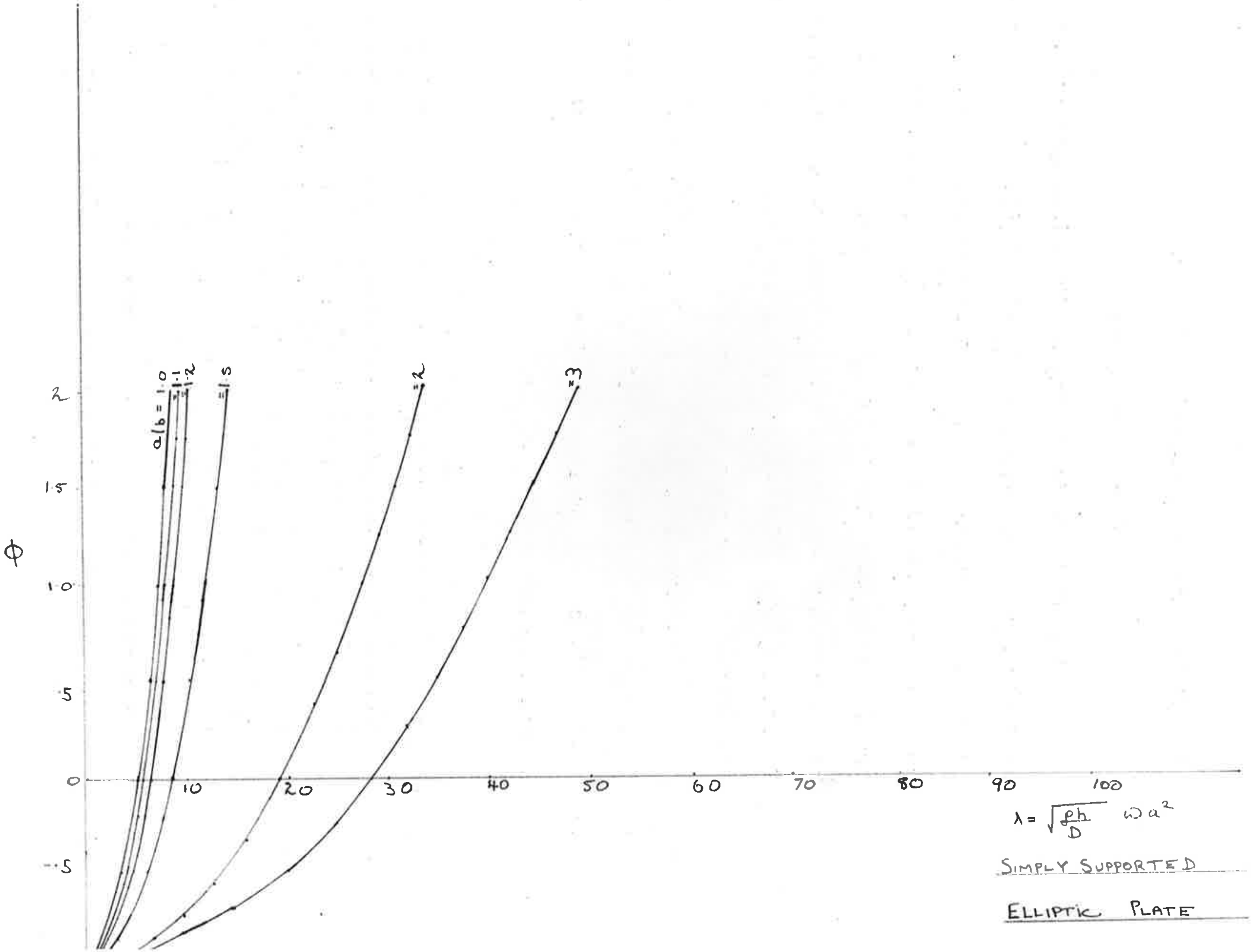
where

$$N^* = \frac{1.05D(3a^4+2a^2b^2+3b^4)}{a^2b^2(a^2+b^2)} , \qquad (5.4.46)$$

and

$$\lambda^* = 4.94\sqrt{2(3\delta^4+2\delta^2+3)}, \qquad (5.4.47)$$

and where



a/b = 1.0
=1.1
=1.2
=1.5
=2
=3
Φ
2
1.5
1.0
.5
-.5
0
10
20
30
40
50
60
70
80
90
100
λ = √(ρh/D) ω a²
SIMPLY SUPPORTED
ELLIPTIC PLATE

$$\delta = \frac{a}{b} \,, \tag{5.4.48}$$

then from equation (5.4.13) we obtain

$$\beta^2 - \alpha^2 = 1.05\phi. \tag{5.4.49}$$

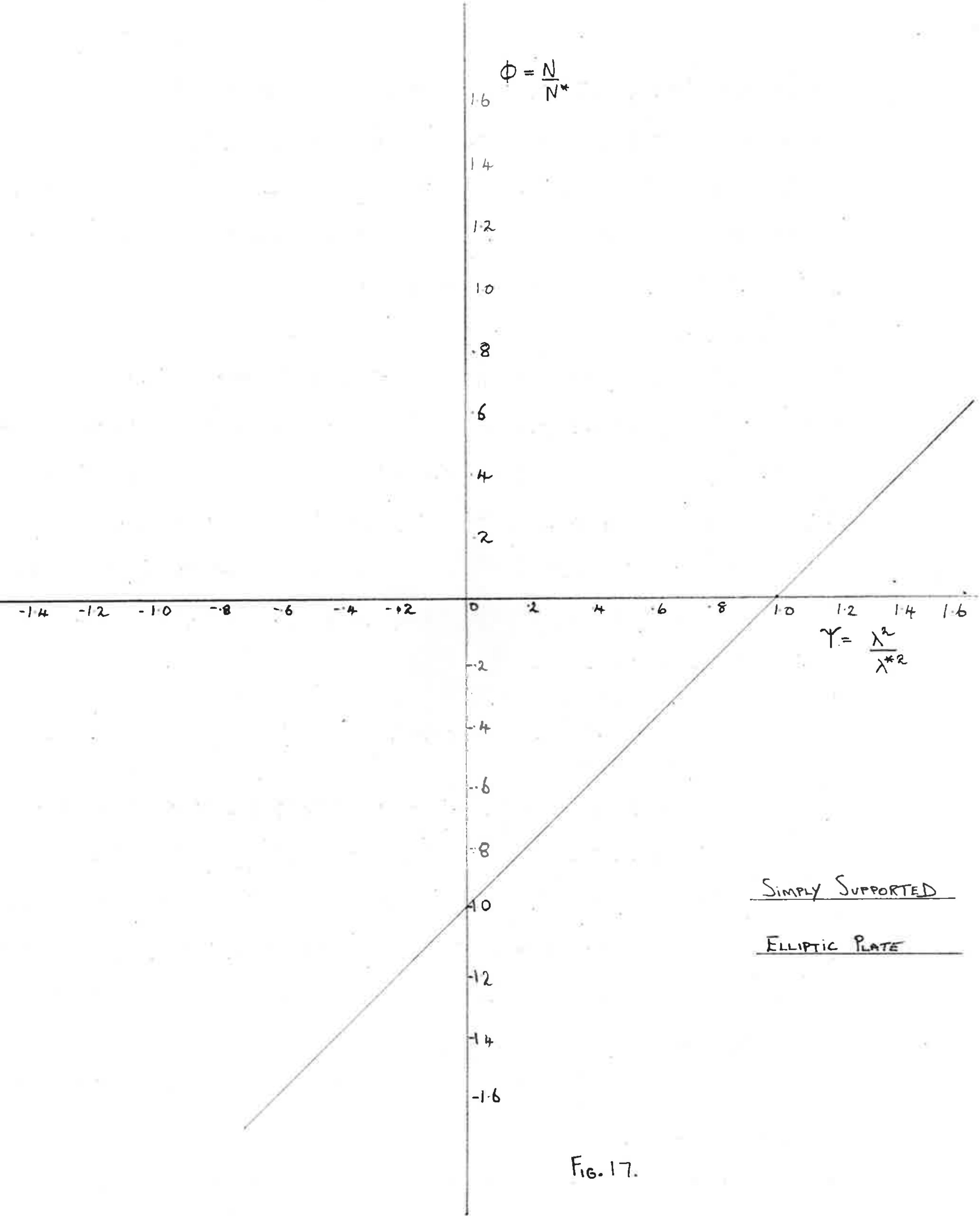
The natural frequency of the plate may now be determined from equations (5.4.49) and (5.4.43) for various values of the parameter ϕ and aspect ratio $\delta = a/b$. As previously negative values of ϕ represent compression, and in particular the value of N corresponding to $\phi = -1$ represents the critical buckling load, as determined by Mazumdar [36]. The value λ corresponding to $\psi = -1$ represents the fundamental frequency of free vibration.

The numerical values of $\lambda = \sqrt{\rho h/D}\,\omega a^2$, are shown in figure 16. for various values of ϕ and δ, while figure 17. shows the relationship between the nondimensionalized parameters ϕ and ψ.

TABLE 10.

ϕ \ a/b	1.0	1.1	1.2	1.5	2	3
2	17.37	19.24	21.36	29.25	47.20	99.78
1.5	15.92	17.61	19.57	26.81	43.23	91.45
1	14.30	15.84	17.59	24.08	38.83	82.15
.5	12.44	13.78	15.30	26.95	33.78	71.46
.25	11.39	12.61	14.01	19.18	30.93	65.43
0	10.21	11.31	12.56	17.20	27.74	58.68
-.25	8.91	9.86	10.96	15.00	24.20	51.18
-.5	7.28	8.06	8.95	12.26	19.77	41.82
1	0	0	0	0	0	0

The solution obtained is exact for the case of a



Φ = N/N*
1·6
1·4
1·2
1·0
·8
·6
·4
·2
-·2
-·4
-·6
-·8
-1·0
-1·2
-1·4
-1·6
-1·4
-1·2
-1·0
-·8
-·6
-·4
-·2
0
·2
·4
·6
·8
1·0
1·2
1·4
1·6
Ψ = λ²/λ*²
SIMPLY SUPPORTED
ELLIPTIC PLATE

Fig. 17.

circular plate, a/b=1, and agrees with that previously obtained by Wah [60]. For aspect ratio $a/b \neq 1$ no other analytical, or numerical solutions can be found with which a comparison can be made. As for the circular plate the ψ–ϕ relationship is again linear.

5.5 REMARKS

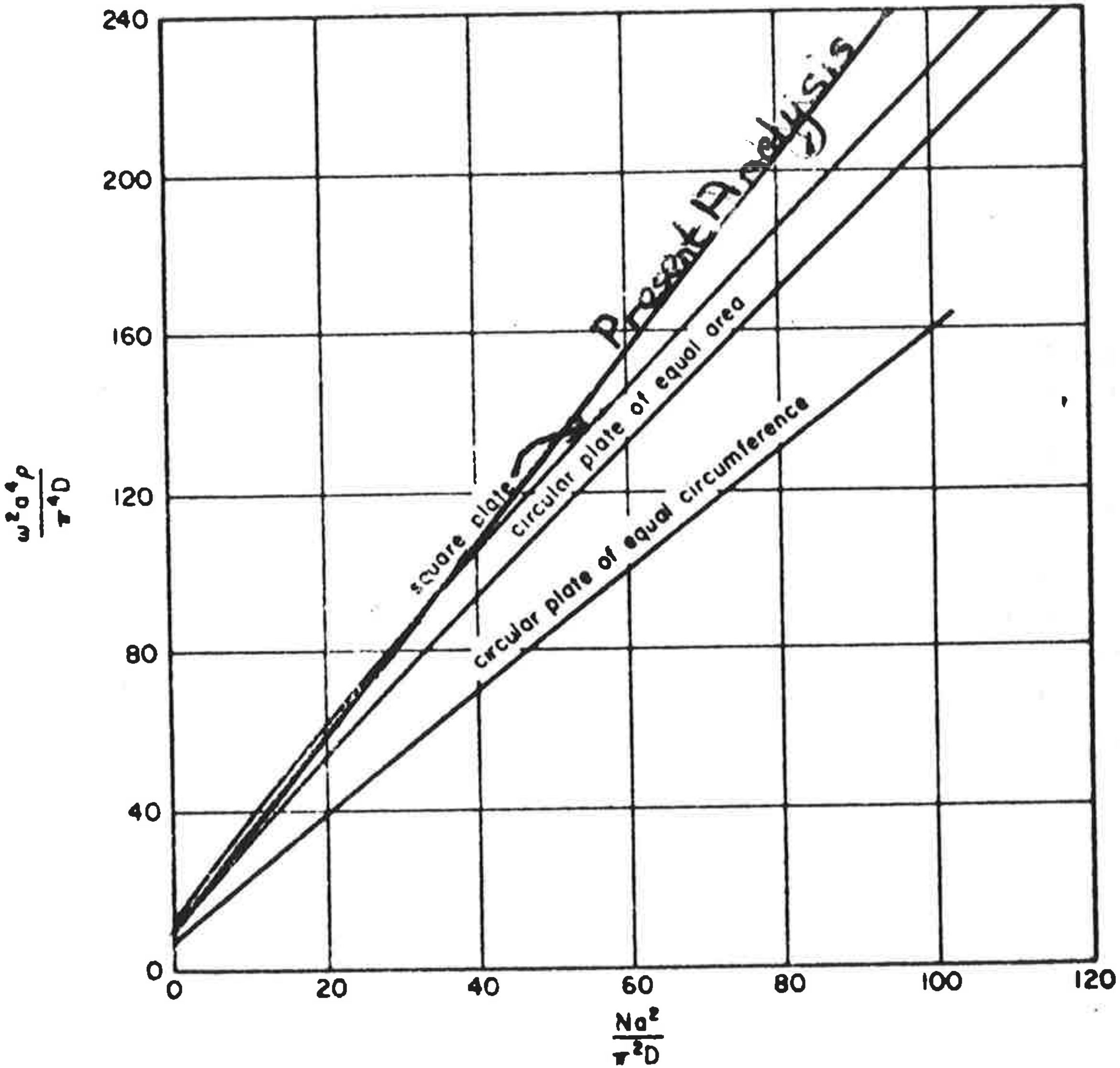
The classical results for the vibration of a simply supported polygonal plate subject to hydrostatic edge loading were first proposed by Lurie [34], who stated: "For any thin plate of polygonal shape and uniform thickness which is simply supported all along its edges and subjected to a uniform thrust N per unit length, the frequency ω follows the relationship

$$\left(\frac{\omega}{\omega^*}\right)^2 = 1 - \frac{N}{N_{cr}} \; . \qquad (5.5.1)$$

However, since when the boundary is curved Navier boundary conditions are no longer applicable" Lurie concluded that although the relationship is exact for a simply supported polygonal plate with an arbitrary number of sides, it is not valid in the limit when the number of sides becomes infinite.

It is curious that the exact linear law breaks down in the limit.

The results shown in figures 15, and 17, indicates that equation (5.5.1) is true for both a simply supported



240
200
160
120
80
40
0
0
20
40
60
80
100
120
$\frac{\omega^2 a^4 \rho}{\pi^4 D}$
$\frac{Na^2}{\pi^2 D}$
Present Analysis
square plate
circular plate of equal area
circular plate of equal circumference

Fig. 18.

elliptic plate, and a clamped elliptic plate regardless of the magnitude of the aspect ratio $\delta = a/b$. Indeed this relationship is also given in [28] for a clamped rectangular plate (see figure 18.). Consequently we conjecture that equation (5.5.1) holds for all plates subject to any combination of clamping and simple supports.

CHAPTER VI

LARGE AMPLITUDE ANALYSIS

6.1 DERIVATION OF THE EQUATION FOR THE LINES OF CONSTANT DEFLECTION FOR LARGE AMPLITUDE DEFORMATION.

Until the last decade the linear theory for the bending of thin elastic plates was sufficient for most practical purposes. However experimental evidence has indicated that the linear theory fails to mirror the physical facts if deflections as little as one fourth the thickness of the plate occur. Thus with the increased use of very thin metal plates, particularly in the structures used in aircraft, it became important from a practical point of view to obtain solutions based upon a theory which would permit deflections much greater than the thickness of the plate. One such theory, based upon the method of constant deflection lines, was discussed in section 2.5, where the differential equations for the deflected surface $w(u)$ were obtained. However this method is incomplete in the sense that the precise form of the deflection contours is unknown. Consequently in this chapter attention is focused upon obtaining the correct form for the lines of constant deflection.

The majority of research into the analysis of the large amplitude deflection of plates has been for uniformly loaded plates (i.e. $q(x,y) = \text{const}$), where equations

(2.5.8) and (2.5.9) now become identical to equations (2.3.3) and (2.3.4) for the bending of a uniformly loaded plate subject to hydrostatic edge loading $N_x = N_y = +\alpha^2 D$, $N_{xy} = 0$. However the deflection contours for the bending of a uniformly loaded plate, and for the buckling of the same plate subject to hydrostatic edge loading are identical. Thus we conclude that the lines of constant deflection for the large amplitude deformation of a uniformly loaded, thin, elastic plate must coincide with the lines of constant deflection for the same plate problem under small deflection theory.

As an illustration of this procedure let us consider the technically important problem of the large amplitude deflection of a uniformly loaded, clamped, thin, elastic plate.

6.2 THE LARGE DEFLECTION OF ELLIPTIC PLATES.

Whereas the large amplitude deformation of a uniformly loaded, clamped, circular plate was solved by Way [63], using an iteration and successive interpolation technique, only three treatments of the large amplitude deflections of elliptic plates can be found in the literature. The first, due to Perry [49], employs expansion into Mathieu functions which unfortunately are available in tabular form for only a very limited range in argument. The remaining two papers are due to Weil and Newmark [64], and Nash and Cooley [43] employing Rayleigh-Ritz, and perturbation techniques respectively. Both involve considerable numerical calculation and only yield results for discreet values of the aspect ratio a/b.

We will tackle this problem using the modified form of Berger's technique presented in equations (2.5.8) and (2.5.9), comparing our results with those of Weil and Newmark, and Nash and Cooley.

Consider the plate as described in section 5.4.1 subject to a load q uniformly distributed over its middle surface; and where the edges of the plate are rigidly clamped.

In accordance with section 6.1, above, the lines of constant deflection may be taken as for the corresponding small amplitude deflection problem. These are a family of similar and similarly situated ellipses. Therefore without loss of generality we may take

$$u(x,y) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} . \tag{6.2.1}$$

With $u(x,y)$ of this form, after substituting the values of the line integrals $\oint Rds$, $\oint Fds$, $\oint Gds$, and $\oint \sqrt{t}ds$, as calculated in section 5.4.1, into (2.5.8) we obtain the following ordinary differential equation in w, viz.

$$(1-u)^2 \frac{d^3w}{du^3} - 2(1-u)\frac{d^2w}{du^2} + q_1(1-u) - \gamma^2(1-u)\frac{dw}{du} = 0, \tag{6.2.2}$$

where

$$q_1 = \frac{qa^4b^4}{2D(3a^4+2a^2b^2+3b^4)}, \tag{6.2.3}$$

and

$$\gamma^2 = \frac{\alpha^2[1/a^2 + 1/b^2]a^4b^4}{[3a^4+3b^4+2a^2b^2]} . \tag{6.2.4}$$

This equation can be expressed as

$$(1-u)\frac{d}{du}\left[(1-u)\frac{d^2}{du^2} - \frac{d}{du} - \gamma^2\right]w + q_1(1-u) = 0, \tag{6.2.5}$$

which in terms of the new variable f defined by

$$f^2 = 1-u, \tag{6.2.6}$$

becomes

$$f\frac{d}{df}\left[\frac{1}{f}\frac{d}{df}\left(f\frac{d}{df}\right) - 4\gamma^2\right]w - 8q_1f^2 = 0. \tag{6.2.7}$$

This has solution

$$w = B_1J_0(ikf) + \frac{q_1(1-f^2)}{\gamma^2} + B_2Y_0(ikf) + B_3, \tag{6.2.8}$$

where

$$k = 2\gamma. \tag{6.2.9}$$

The boundary conditions at the edge $f=1$ must now be imposed. Since the plate is rigidly clamped we have

$$\text{i)}\ w\Big|_{f=1} = 0 \qquad \text{ii)}\ \sqrt{t}\,\frac{dw}{df}\Big|_{f=0} = 0$$

$$\text{iii)}\ \frac{dw}{df}\Big|_{f=1} = 0 \qquad \text{iv)}\ u\Big|_{f=1} = 0 \qquad \text{v)}\ v\Big|_{f=1} = 0, \tag{6.2.10}$$

where u and v are the x and y components of the horizontal displacements respectively. Consequently if we now apply the boundary conditions (6.2.10) i), ii), and iii) to equation (6.2.8) we obtain

$$w = \frac{q_1}{\gamma^2}\left[2\,\frac{[J_0(ik)-J_0(ikf)]}{ikJ_1(ik)} + 1-f^2\right]. \qquad (6.2.11)$$

Since the boundary is rigidly clamped, making use of the boundary conditions (6.2.10) iv), and v) the value of α^2 may be determined as in [61,62], giving

$$\frac{\alpha^2h^2}{12} = \frac{1}{\pi ab}\iint\limits_{\Omega_u} \tfrac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \tfrac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 d\Omega, \qquad (6.2.12)$$

where the integration is taken over the total area of the ellipse. So that after substituting for w in equation (6.2.12), and after the necessary integration has been performed we obtain

$$\frac{\alpha^2h^2}{12} = q_1{}^2\,\frac{[1/a^2 + 1/b^2]}{\gamma^4}\left[\frac{J_1(ik)^2 - J_0(ik)J_2(ik)}{J_1(ik)^2} - \frac{4J_2(ik)}{ikJ_1(ik)} + \tfrac{1}{2}\right], \qquad (6.2.13)$$

which may be written in the form

$$(16q_1)^2\,\frac{2a^4b^4[1/a^2 + 1/b^2]^2}{[3a^4+2a^2b^2+3b^4]} = \frac{k^6/3}{\frac{3}{4} + \frac{4}{k^2} + \frac{J_0(ik)}{ikJ_1(ik)} + \frac{J_0{}^2(ik)}{2J_1{}^2(ik)}}, \qquad (6.2.14)$$

and from which we obtain the value of k to be used in equation (6.2.11). It is particularly interesting to note that in the case of a clamped circular plate, $a/b = 1$, equations (6.2.11) and (6.2.14) coincide exactly with the results obtained by Berger [2].

Since the right hand side of equation (6.2.14) is independent of the ratio $\delta = a/b$, we are able to obtain k, and thus $w(u)$ directly from Berger's solution, $a/b=1$. In particular we obtain

$$w_{max} = \beta \sqrt{\frac{(3a^4+2a^2b^2+3b^4)}{2a^4b^4(1/a^2 + 1/b^2)^2}} , \qquad (6.2.15)$$

where β is the maximum value of the deflection for a circular plate acted upon by a load

$$\bar{q} = \frac{8b^4 \cdot q}{(3a^4+2b^2a^2+3b^4)} \sqrt{\frac{2a^4b^4(1/a^2 + 1/b^2)^2}{(3a^4+2a^2b^2+3b^4)}} . \qquad (6.2.16)$$

As α tends to zero this gives

$$\lim_{\alpha\to 0} w_{max} = \frac{qa^4b^4}{8(3a^4+2a^2b^2+3b^4)D}, \qquad (6.2.17)$$

which is the exact value of w_{max}. Indeed if we use the well known expansions of $J_0(ik)$ and $J_1(ik)$ for small k, we obtain

$$\lim_{\alpha\to 0} \left[\frac{wD}{q_1}\right] = \frac{1}{4}(1-f^2)^2$$

$$= \frac{1}{4}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2, \qquad (6.2.18)$$

which is the exact solution for the small amplitude deflection of a clamped, uniformly loaded, elliptic plate.

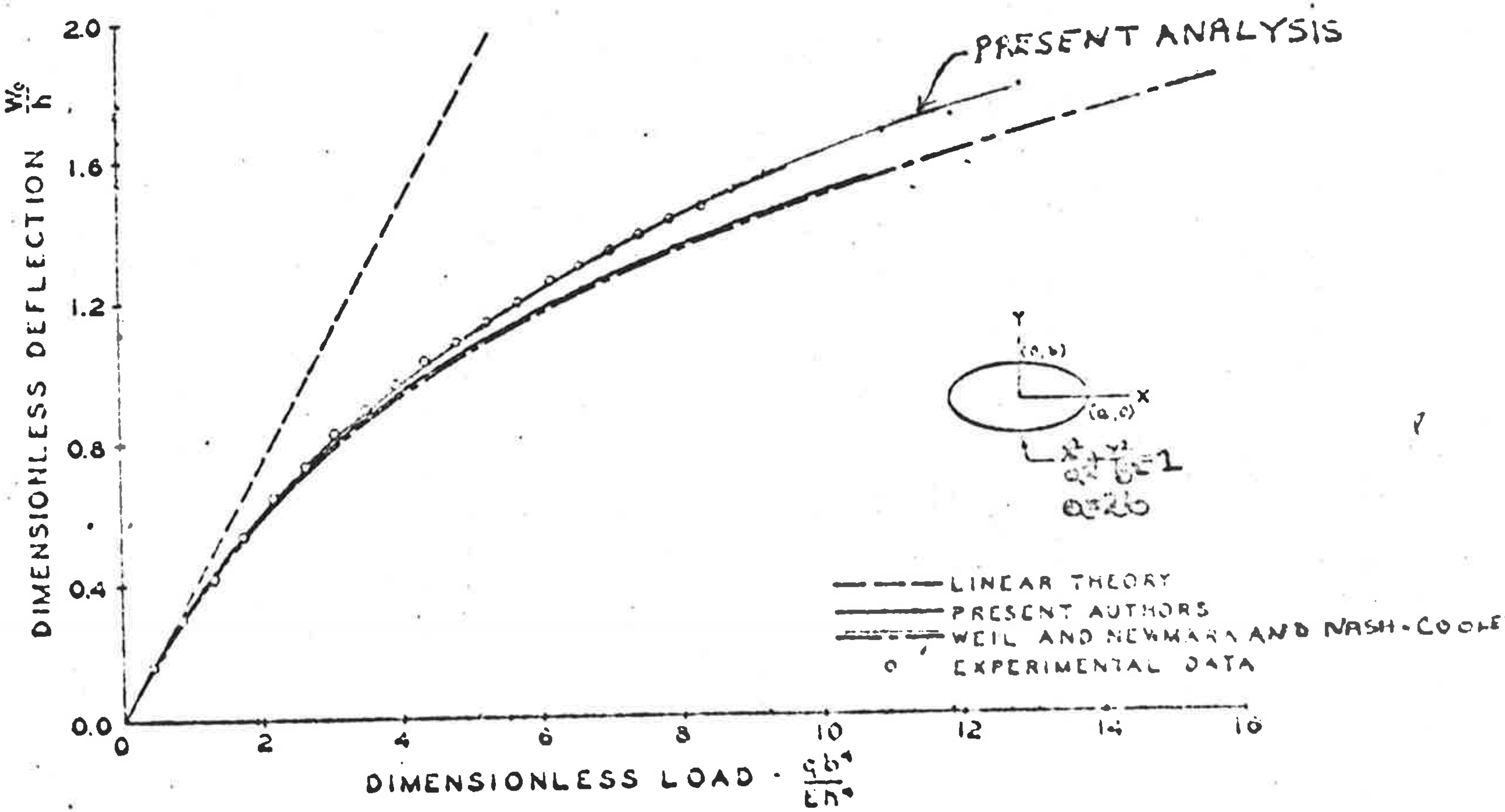

PRESENT ANALYSIS
DIMENSIONLESS DEFLECTION $\frac{W_0}{h}$
DIMENSIONLESS LOAD $\frac{qb^4}{Eh^4}$
2.0
1.6
1.2
0.8
0.4
0.0
0
2
4
6
8
10
12
14
16
Y
X
(a,0)
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
a=2b
LINEAR THEORY
PRESENT AUTHORS
WEIL AND NEWMARK AND NASH-COOLEY
EXPERIMENTAL DATA

Fig. 19

A comparison of these results with those of previous investigations is shown in figure 19. for aspect ratio $a/b = 2$.

A close examination of figure 19. reveals that the present analysis is in remarkable agreement with the experimental results, and is clearly superior to that presented in [64], and [43].

6.3 DERIVATION OF THE EQUATION FOR THE LINES OF CONSTANT DEFLECTION FOR THE LARGE AMPLITUDE VIBRATION OF THIN, ELASTIC PLATES.

In this section the exact equation for the lines of constant deflection for the large amplitude vibration of thin, elastic plates is obtained, selecting several important cases as boundary conditions.

The differential equations for the large amplitude vibration of thin, elastic plates have been obtained in section 2.6. These equations are identical to equations (2.4.2) and (2.4.3) for the transverse vibration of plates with inplane forces $N_x = N_y = \alpha^2 D$, $N_{xy} = 0$. Consequently their corresponding families of deflection contours must also be identical. Therefore we conclude that the lines of constant deflection for the large amplitude vibration of a thin, elastic plate must coincide with the lines of constant deflection for the corresponding small amplitude, vibration problem.

6.4 THE LARGE AMPLITUDE VIBRATION OF ELLIPTIC PLATES.

As an illustration of this procedure let us consider the large amplitude vibration of a clamped, thin, elastic, elliptic plate. This problem is one of great difficulty both for large, and small amplitude vibrations. However although several approximate solutions have been obtained for the small amplitude vibration problem, no published results can be found for the large amplitude vibration problem, other than for a circular plate [62].

Consider the plate as described in section 5.4.1, figure 13., where the edges of the plate are rigidly clamped. In accordance with section 6.3 above the deflection contours may be taken to be the same as for the vibration of a hydrostatically compressed, clamped, thin, elliptic plate. This problem has been discussed in section 5.4.1, where the deflection contours were found to be

$$u(x,y) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \,. \tag{6.4.1}$$

With $u(x,y)$ of this form after substituting the values of the line integrals $\oint Rds$, $\oint Fds$, $\oint Gds$, $\oint \frac{1}{\sqrt{t}}ds$, and $\oint \sqrt{t}\, ds$, as calculated in section 5.4.1, into (2.6.3) we obtain the following integro-differential equation

$$(1-u)^2 \frac{d^3W}{du^3} - 2(1-u)\frac{d^2W}{du^2} - \alpha^2\lambda_1{}^2(1-u)\frac{dW}{du}$$

$$+ \lambda_2^2 \int_1^u \frac{W(u_o)}{\theta} \frac{d\theta^2}{d\tau^2} d_{u_o} = 0, \qquad (6.4.2)$$

where

$$\lambda_1^2 = \frac{[1/a^2 + 1/b^2]a^4b^4}{[3a^4+2a^2b^2+3b^4]} \qquad (6.4.3)$$

and

$$\lambda_2^2 = \frac{\rho h a^4 b^4}{2D(3a^4+2a^2b^2+3b^4)}, \qquad (6.4.4)$$

and where the boundary conditions to be imposed are as given in (6.2.10) i), ii), iii), iv), and v).

Since the boundary of the plate is rigidly clamped the value of α^2 may be determined as in [61,62], giving

$$\alpha^2 = \frac{6}{\pi abh^2} \iint_{\Omega_u} \left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right] d\Omega, \qquad (6.4.5)$$

where the double integration is taken over the total area of the plate.

Following Wah [61,62] we now take $W(u)$ as the solution to the corresponding small amplitude vibration problem [37], viz.

$$W(u) = c(J_o(kf) - \frac{J_o(k)}{I_o(k)} I_o(kf)), \qquad (6.4.6)$$

$$= c\, W_o(u),$$

where c is an arbitrary parameter

$$f^2 = 1-u, \tag{6.4.7}$$

and $k = 3.1961$. Substituting for $W(u)$ in (6.4.5) we obtain

$$\alpha^2 = 6\,\frac{[1/a^2 + 1/b^2]}{h^2}\,c^2\theta(t)^2 \int_0^1 f\left(\frac{dW_0}{df}\right)^2 df, \tag{6.4.8}$$

so that if we substitute for α^2, and $W(u)$ in equation (6.4.2) and differentiate it with respect to u we obtain

$$\lambda_2^2\,\omega^2 c\;\theta(t)\;W_0(f) + c\lambda_2^2\,\ddot{\theta}(t)\;W_0(f) - \frac{\alpha^2\lambda_1^2}{4f}c\,\frac{d}{df}\left(f\,\frac{dW_0}{df}\right)\theta(t)$$

$$= 0. \tag{6.4.9}$$

Consequently if we now apply Galerkin's technique, then upon multiplying equations (6.4.9) by $fW_0(f)$ and integrating from 0 to 1 we obtain

$$\ddot{\theta} + \omega^2\theta + \frac{12\beta^2\chi[1/a^2 + 1/b^2]\lambda_1{}^2\theta^3}{8\lambda_2^2} = 0, \tag{6.4.10}$$

where we have defined

$$\phi^2 = \int_0^1 fW_0\,df, \tag{6.4.11}$$

$$\psi = \int_0^1 f\left(\frac{dW_0}{df}\right) df, \tag{6.4.12}$$

and

$$\chi = \frac{\psi^2}{\phi^2}, \tag{6.4.13}$$

where $\psi = .706$, $\phi^2 = .1016$, and where $\beta = c/h$ is the nondimensionalized value of the amplitude.

Introducing the nondimensionalized time

$$\xi = t\sqrt{\left(\frac{\lambda_1}{\lambda_2}\right)\frac{(1/a^2 + 1/b^2)}{8}}$$

$$= \frac{t(1/a^2 + 1/b^2)}{2}\sqrt{\frac{D}{\rho h}}, \qquad (6.4.14)$$

equation (6.4.10) becomes

$$\theta_{\xi\xi} + \gamma^4\theta + 12\beta^2\chi\theta^3 = 0, \qquad (6.4.15)$$

in which

$$\gamma^4 = \frac{8\omega^2\lambda_2^2}{[1/a^2 + 1/b^2]\lambda_1^2}$$

$$= \frac{4\omega^2\rho h}{D[1/a^2 + 1/b^2]^2} \qquad (6.4.16)$$

The solution of this equation depends upon the choice of initial conditions. If, for example these conditions are given by

$$\text{i)}\quad \theta\Big|_{\xi=0} = 1$$

$$\text{ii)}\quad \frac{d\theta}{d\xi}\Big|_{\xi=0} = 0, \qquad (6.4.17)$$

the solution is simply

$$\theta = \mathrm{cn}(\bar{\omega}\xi|m), \qquad (6.4.18)$$

in which cn is the elliptic cosine

$$\bar{\omega} = (12\beta^2\chi + \gamma^4)^{\frac{1}{2}}, \qquad (6.4.19)$$

and

$$m = \frac{6\beta^2\chi}{(12\beta^2\chi + \gamma^4)} . \qquad (6.4.20)$$

If on the other hand, the initial conditions are

$$\text{i)} \quad \theta\Big|_{\xi=0} = 0 \qquad (6.4.21)$$

$$\text{ii)} \quad \frac{d\theta}{d\xi}\Big|_{\xi=0} = 1,$$

then the solution is

$$\theta = \frac{1}{\omega^*} \operatorname{sn}(\omega^*\xi|m^*), \qquad (6.4.22)$$

where sn is the elliptic sine,

$$\omega^* = (6\beta^2\psi + \gamma^4)^{\frac{1}{2}}, \qquad (6.4.23)$$

and

$$m^* = \frac{6\beta^2\chi}{(6\beta^2\chi + \gamma^4)} . \qquad (6.4.24)$$

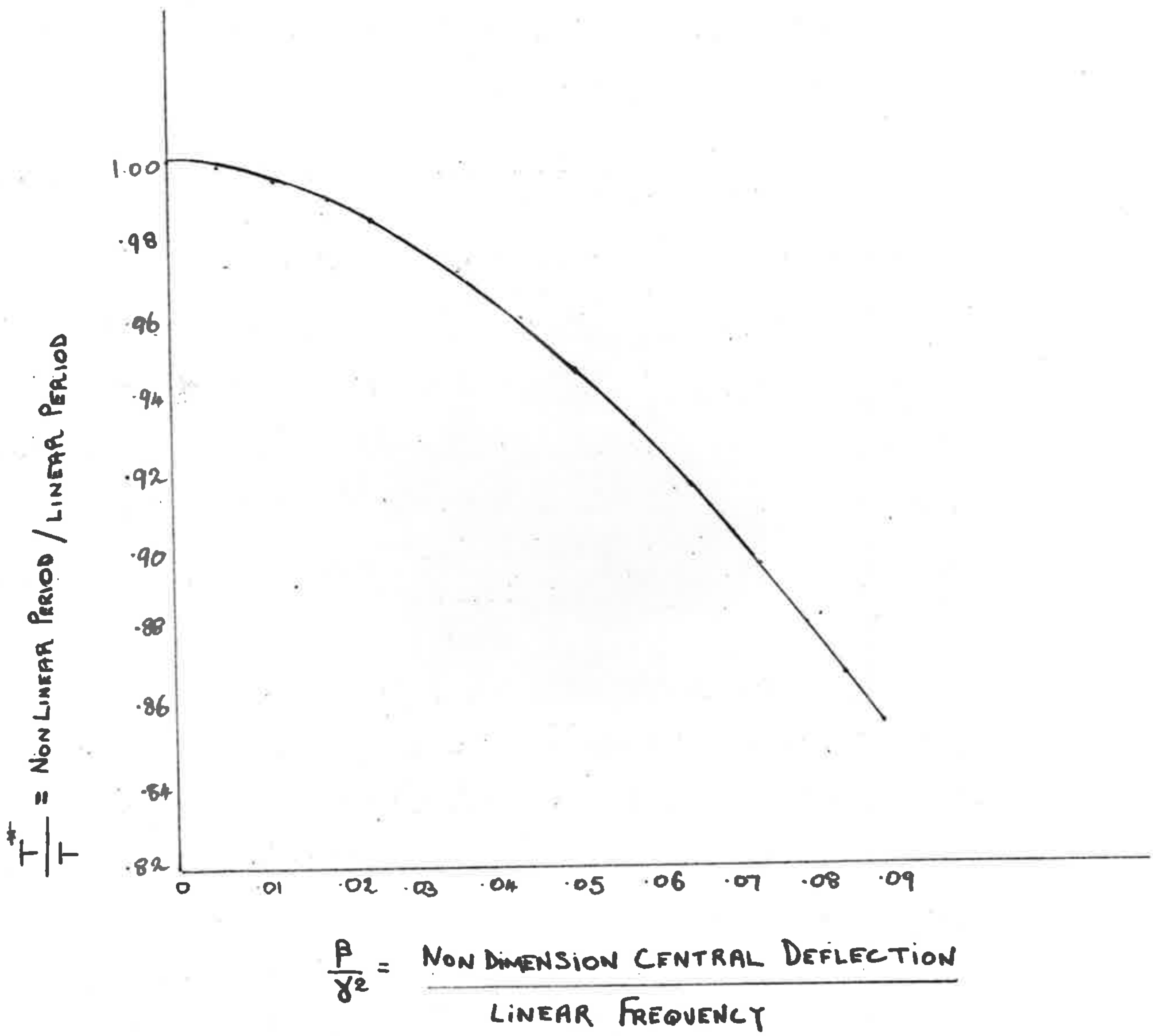
It appears that the two frequencies are different for the two sets of initial conditions. However, since m^* is a negative parameter equation (6.4.22) may be transformed to

$$\theta = \frac{1}{\bar{\omega}} \operatorname{sd} (\bar{\omega}\xi|m), \qquad (6.4.25)$$

where both cn and sd have the same real period 4K. Thus the period of vibration is given by

$$T^* = \frac{4K}{\bar{\omega}}$$

$$= \frac{4K}{(12\beta^2\chi + \gamma^4)^{\frac{1}{2}}}, \qquad (6.4.26)$$


1.00
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·88
·86
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·07
·08
·09
$\frac{T^*}{T}$ = NON LINEAR PERIOD / LINEAR PERIOD
$\frac{\beta}{\gamma^2}$ = NON DIMENSION CENTRAL DEFLECTION / LINEAR FREQUENCY

FIG. 20.

in which K is the complete elliptic integral of the first kind. Since the linear period is given by

$$T = \frac{2\pi}{\gamma^2}, \tag{6.4.27}$$

we obtain the following expression for the ratio of the non linear and the linear period, viz.

$$\frac{T^*}{T} = \frac{2K}{\pi[12\beta^2/\gamma^4+1]^{\frac{1}{2}}}, \tag{6.4.28}$$

from which it follows that as $\beta \to 0$, $\frac{T^*}{T} \to 1$.

The ratio $\frac{T^*}{T}$ is shown in figure 20. for various values of $\frac{\beta}{\gamma^2}$. Alternatively if we now adopt the method proposed by Denman [9] for the solution of (6.4.10) we obtain a more convenient form for the ratio $\frac{T^*}{T}$, viz.

$$\frac{T^*}{T} = \frac{1}{\left(1 + \frac{9\beta^2\chi}{\gamma^4}\right)^{\frac{1}{2}}} \tag{6.4.29}$$

This relationship cannot be separately shown in figure 20. since the difference between equations (6.4.29) and (6.4.28) is always less than 1%, and cannot be adequately represented. So when computing $\frac{T^*}{T}$ it is advantageous to consider equation (6.4.29).

Clearly when $a/b = 1$ the above analysis coincides exactly with that given by Wah [62] for the large amplitude vibration of a rigidly clamped circular plate.

PART II

CHAPTER VII

THE METHOD OF CONSTANT DEFLECTION LINES FOR SHALLOW SHELL - ANALYSIS

7.1 FUNDAMENTAL CONCEPTS AND ASSUMPTIONS

In the theory of elasticity the term shell is applied to bodies bounded by two curved surfaces, the distance "h" between the surfaces being small in comparison with the other dimensions. The locus of points which lie at equal distances from these two surfaces defines the middle surface of the shell. By specifying the form of this surface, and the shell thickness "h" at every point, the geometry of the shell is completely defined.

Although many variable thickness shells have been constructed, the majority of shells are of constant thickness and with middle surfaces which can be defined by continuous mathematical functions. If the shell has no boundary besides the above two surfaces (the inner surface or intrados, and the outer surface or extrados), it is called complete, e.g.; spherical boilers, balloons, eggs, etc. The shell is incomplete if its edges are plane curves with cuts perpendicular to the middle surface, e.g.; cylindrical shell roof, dome etc.

There are two different classes of shells, thick shells and thin shells. A shell will be called thin if its thickness h, is small in comparison with the principal radii of curvature. A suggested upper limit of the thickness is

$$\text{Max}\left(\frac{h}{R}\right) \leqslant \frac{1}{20} \,. \tag{7.1.1}$$

In most practical applications the thickness of the shell lies in the range

$$\frac{1}{1000} \leqslant \frac{h}{R} \leqslant \frac{1}{50} \,, \tag{7.1.2}$$

and so the shell may be classified as thin. When the ratio h/R becomes relatively large the shell may be classified as thick, and the problem of analysis changes from a two, to a three dimensional problem with a subsequent increase in the length and complexity of the problem. For example,the exact analysis of archdams is a thick shell problem.

The most general classification of thin shells is by their Gaussian curvature k, which is defined mathematically as

$$k = \frac{1}{R_x R_y} \tag{7.1.3}$$

where R_x, and R_y are the two principal radii of curvature. Shells of positive Gaussian curvature, sometimes called synclastic shells, are formed by two families of curves both with the same direction, e.g. ; spherical domes, elliptic paraboloids. Shells of zero Gaussian curvature are formed by one family of curves, e.g. ;cylinders, cones. Anticlastic shells, or shells of negative Gaussian curvature are formed by two families of curves in the opposite directions, e.g; hyperbolic paraboloids.

Much of the bending theory of shells has been developed from plate theory and, in fact, shells are sometimes known as curved plates. The Kirchoff method as used for shells was first presented by Aron [1] in 1874, and more accurately by Love [30,31] in 1888. Love's theory does however contain several inconsistencies, some small terms being retained while others, possibly more important, are neglected. These deficiences have been considered in detail by Galerkin [11,12], Lurie [32,33] and Goldenviezer [13,14].

Love's work on shell bending has been well presented by Novozhilov [46], where the same assumptions as made for plate theory, with the necessary extensions for the curvature effects, are made for shells. These are

i) the deflections under load are small enough so that changes in the geometry of the shell will not alter the static equilibrium of the system;

ii) the material is homogeneous and isotropic, and exhibits a linear elastic behaviour;

iii) points on lines normal to the middle surface before deformation, remain on lines normal to the middle surface after deformation;

iv) deformations due to shears perpendicular to the middle surface are neglected.

In recent times particular attention has been given to the theory of shallow shells, and a simplified theory has been developed by both Reissner [51,52] and Vlasov [58].

In the following analysis we adopt the thin shell assumptions listed above, as well as the assumptions inherent in the simplified shallow shell theory. We will also adopt the conventional idea of a shell being shallow if the rise H is not more than 1/5th the dimension of the smallest side of its base, or accordingly

$$\frac{H}{R} \leqslant \frac{1}{6}. \tag{7.1.4}$$

It has been shown by Aron [1], and Mushtari [41,42] that if the shell is shallow, we may neglect the terms depending upon the tangential displacements u and v in the formulae for the change of curvature and twist. This yields

$$\begin{aligned} \chi_1 &= + \frac{1}{A_1}\frac{\partial}{\partial \xi_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial \xi}\right) - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \xi_2}\frac{1}{A_2}\frac{\partial w}{\partial \xi_2}, \\ \chi_2 &= + \frac{1}{A_2}\frac{\partial}{\partial \xi_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial \xi_2}\right) - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \xi_1}\frac{1}{A_1}\frac{\partial w}{\partial \xi_1}, \\ \gamma_{12} &= \frac{-1}{A_1 A_2}\left(\frac{\partial^2 w}{\partial \xi_1 \partial \xi_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial \xi_2}\frac{\partial w}{\partial \xi_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial \xi_1}\frac{\partial w}{\partial \xi_1}\right), \end{aligned} \tag{7.1.5}$$

where A_1, and A_2 are defined by

$$ds^2 = A_1{}^2\, d\xi_1{}^2 + A_2{}^2 d\xi_2{}^2, \tag{7.1.6}$$

ξ_1 and ξ_2 being a given set of orthogonal curvilinear coordinates. With χ_1, χ_2, and γ_{12} as given above we obtain

$$M_{12} = M_{21} = D(1-\mu)\gamma_{12},$$
$$M_1 = -D(\chi_1 + \mu\chi_2), \qquad (7.1.7)$$
$$M_2 = -D(\chi_2 + \mu\chi_1),$$

$$Q_1 = \frac{-D}{A_1}\left(\frac{\partial \Delta w}{\partial \xi_1} + \underline{\frac{(1-\mu)}{R_1 R_2}\frac{\partial w}{\partial \xi_1}}\right),$$

and

$$Q_2 = \frac{-D}{A_2}\left(\frac{\partial \Delta w}{\partial \xi_2} + \underline{\frac{(1-\mu)}{R_1 R_2}\frac{\partial w}{\partial \xi_2}}\right), \qquad (7.1.8)$$

the underlined terms generally being neglected.

So far the results obtained coincide exactly with those for plate theory. However it is necessary, in order to satisfy all the equilibrium requirements, to introduce a stress function ϕ, in terms of which the membrane forces N_1, N_2 and S are determined by

$$N_1 = \frac{1}{A_2}\frac{\partial}{\partial \xi_2}\left(\frac{1}{A_2}\frac{\partial \phi}{\partial \xi_2}\right) - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \xi_1}\frac{1}{A_1}\frac{\partial \phi}{\partial \xi_1},$$
$$(7.1.9)$$
$$N_2 = \frac{1}{A_1}\frac{\partial}{\partial \xi_1}\left(\frac{1}{A_1}\frac{\partial \phi}{\partial \xi_1}\right) - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \xi_2}\frac{1}{A_2}\frac{\partial \phi}{\partial \xi_2},$$

and

$$S = -\frac{1}{A_1 A_2}\left(\frac{\partial^2 \phi}{\partial \xi_1 \partial \xi_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial \xi_2}\frac{\partial \phi}{\partial \xi_2} - \frac{1}{A_2}\frac{\partial A_2}{\partial \xi_1}\frac{\partial \phi}{\partial \xi_1}\right).$$
$$(7.1.10)$$

The conditions for the equilibrium of the shell, and for the continuity of deformation now give rise to the following two, coupled, fourth order, partial differential equations for w and ϕ, viz.

$$\nabla^4 w + \frac{K_1 N_1}{D} + \frac{K_2 N_2}{D} = \frac{q}{D} , \tag{7.1.11}$$

and

$$\nabla^4 \phi = Eh\nabla_K^2 w, \tag{7.1.12}$$

where

$$K_1 N_1 + K_2 N_2 = \nabla_K^2 \phi , \tag{7.1.13}$$

and

$$\nabla_K^2 \equiv \frac{1}{A_1 A_2} \left(\frac{\partial}{\partial \xi_1}\left(\frac{A_2 K_2}{A_1} \frac{\partial}{\partial \xi_1}\right) + \frac{\partial}{\partial \xi_2} \left(\frac{A_1 K_1}{A_2} \frac{\partial}{\partial \xi_2}\right)\right) . \tag{7.1.14}$$

Consequently the general problem of the equilibrium of a shallow shell, outlined by any surface, reduces to a system of two, consistent, symmetrically constructed, differential equations. From these equations we obtain a stress function $\phi(\xi_1,\xi_2)$ which determines the internal (membrane) forces N_1, N_2, and S, and a displacement function $w(\xi_1,\xi_2)$ which determines the bending deformations $\varkappa_1$, $\varkappa_2$, and γ_{12} and consequently all of the moments and transverse forces.

Equations (7.1.11) and (7.1.12) differ from flat plate theory in that the auxiliary terms in these equations represent the mutual effect of two systems of forces characteristic of two stressed states of the plate.

One system is characterized by normal and shearing forces, and the other by moments and transverse shears. The curvatures $K_1 = K_1(\xi_1,\xi_2)$, and $K_2 = K_2(\xi_1,\xi_2)$ contained in the expression for the auxiliary operator ∇^2_K, play the role of two characteristics of a reciprocally elastic medium. In addition they have a very important effect on the internal forces, so that a shallow shell, like a slightly curved plate, differs fundamentally from a flat plate.

7.2 DERIVATION OF THE EQUATIONS OF EQUILIBRIUM FOR THE BENDING OF THIN, ELASTIC, ISOTROPIC, SHALLOW SHELLS

Consider a thin, elastic, isotropic, shallow shell of thickness h subject to a continuously distributed normal load. Let the equation of the middle surface of the shell, referred to a system of orthogonal coordinates xyz, be given by

$$z = \frac{x^2}{2R_x} + \frac{xy}{R_{xy}} + \frac{y^2}{2R_y} , \qquad (7.2.1)$$

where the shell will be called shallow if $r = \sqrt{x^2+y^2}$ is small compared to the least of R_x, R_y, and R_{xy} (the radii of curvature) everywhere in the region, and if R_x, R_y, and R_{xy} may be taken to be constants.

When the shell is acted upon by a transverse load $q(x,y)$ then the intersections between the deflected surface and the parallels z = const. yield contours, which after projection onto the $z=0$ surface are the level curves called the Lines of Equal Deflection. We will

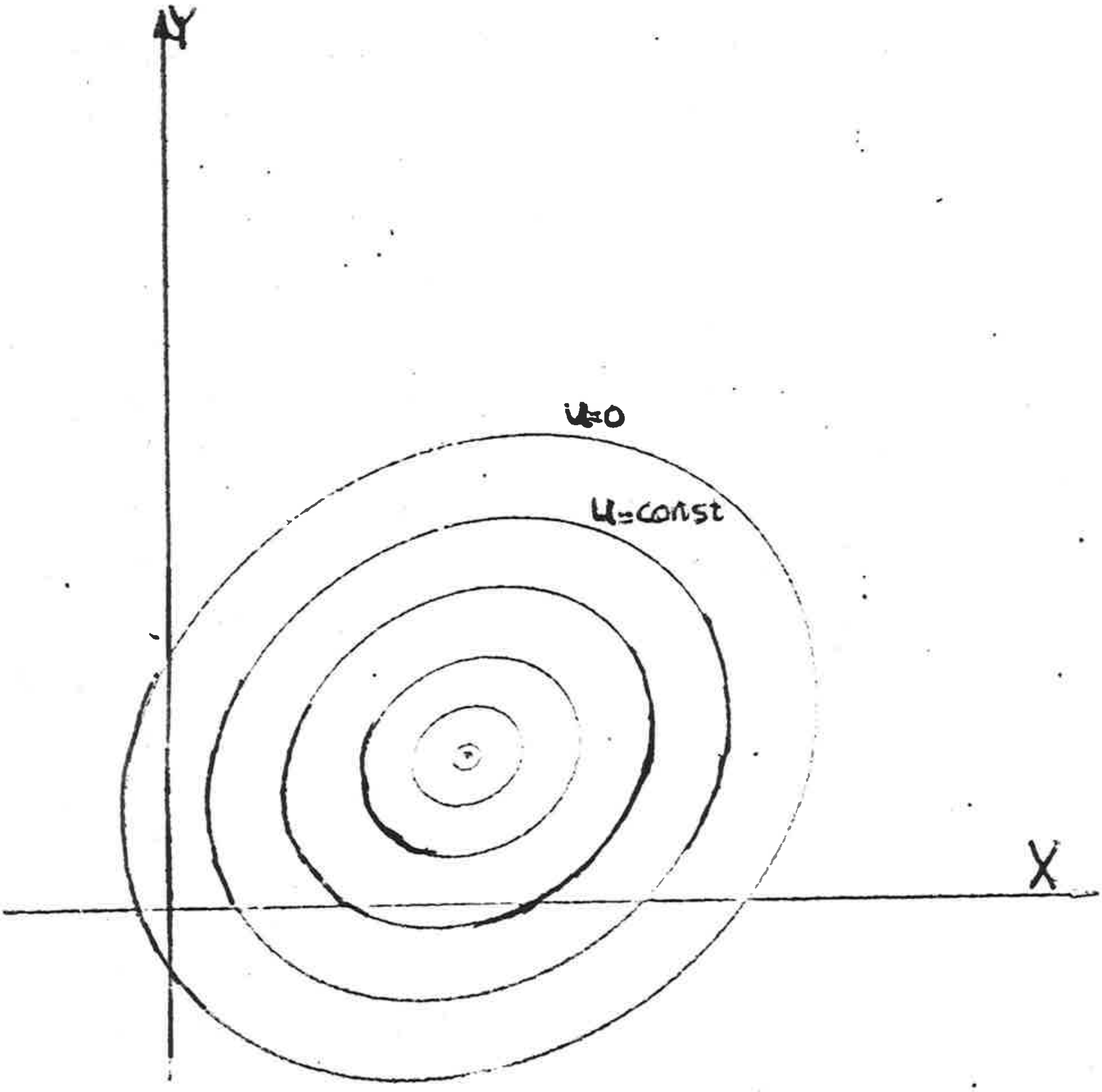
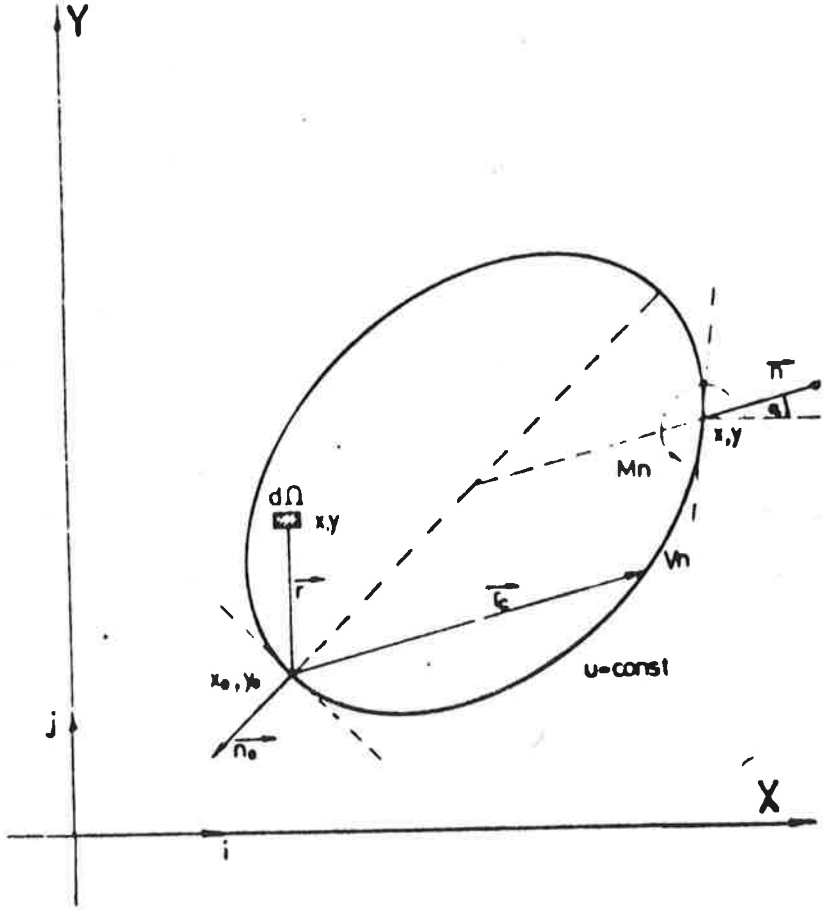


Fig 21

denote the family of such curves by $u(x,y) = \text{const.}$ If the boundary of the shell is subjected to any combination of clamping and simple support, then clearly the boundary, assumed to be a simple closed curve C, will belong to the family of lines of equal deflection and, without loss in generality, we may consider that $u=0$ on the boundary (Fig.21.). It is clear that the lines of equal deflection form a system of non intersecting closed curves starting from the closed boundary C as one of the lines. We denote the family of curves $u = \text{const.}$ by C_u, $0 \leq u \leq u^*$, so that $C_0 = C$ is the boundary of the shell, and C_{u^*} coincides with the point(s) at which the maximum $u=u^*$ is attained. Here we have assumed the value of u increases as we go toward the interior of the region.

Consider the equilibrium of an element of the shell bounded by any contour line of constant deflection. In figure 22. the interior portion of the shell bounded by any line $u = \text{const.}$ is shown, where (x_0,y_0) indicates a fixed point on the contour, $\underset{\sim}{n}$ and $\underset{\sim}{n}_0$ denote the unit vectors normal to the line $u = \text{const.}$ at any arbitrary point (x,y) and at the fixed point (x_0,y_0) respectively, $\underset{\sim}{r}$ and $\underset{\sim}{r}_0$ denote the position vectors from the fixed point (x_0,y_0) to any arbitrary point inside the contour and on the contour respectively. We thus have the relations


Y
X
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x,y
Mn
Vn
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x₀ , y₀
j
i

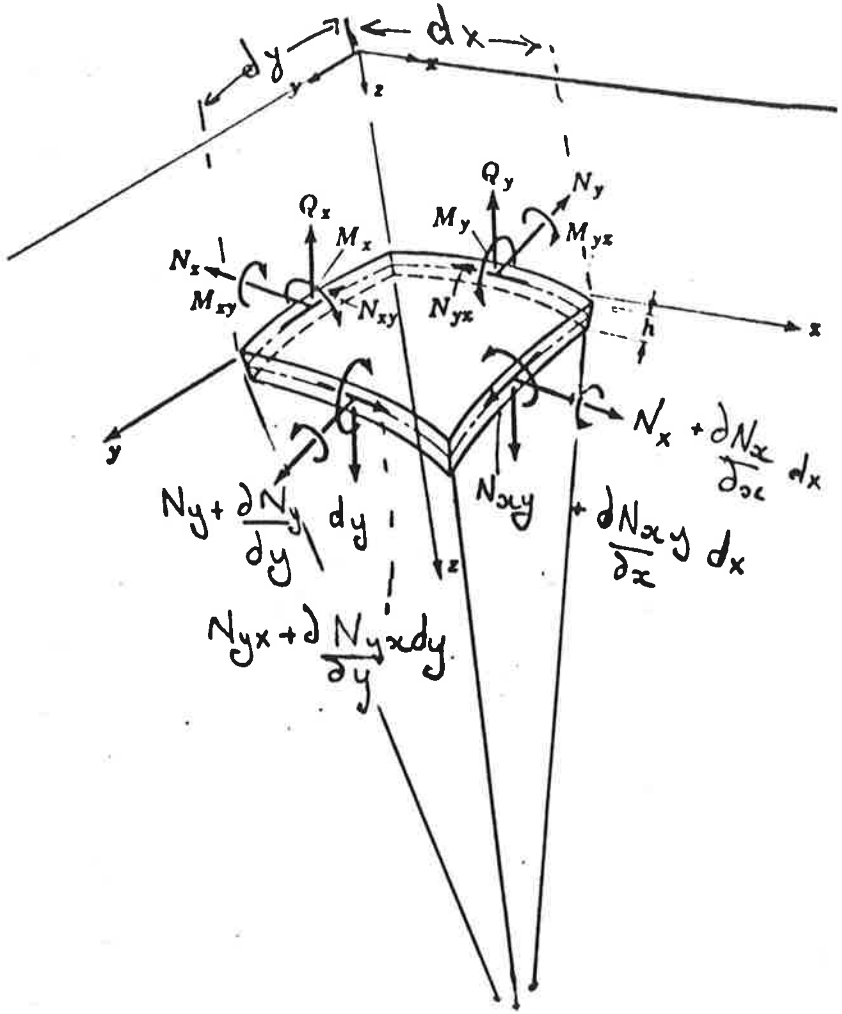
dy
dx
Q_x
Q_y
N_y
M_x
M_y
M_yx
N_x
M_xy
N_xy
N_yx
h
x
y
z
N_x + ∂N_x/∂x dx
N_y + ∂N_y/∂y dy
N_xy + ∂N_xy/∂x dx
N_yx + ∂N_yx/∂y dy

Fig. 22.

$$\underset{\sim}{r} = (x - x_o)\underset{\sim}{i} + (y - y_o)\underset{\sim}{j}$$

$$\underset{\sim}{r}_o = (x-x_o)\underset{\sim}{i} + (y-y_o)\underset{\sim}{j}\Big|_{u(x,y)=\text{const.}}$$

(7.2.2)

$$\underset{\sim}{n} = \frac{u_x\underset{\sim}{i} + u_y\underset{\sim}{j}}{\sqrt{u_x^2+u_y^2}}\Bigg|_{u(x,y)=\text{const.}}$$

$$\underset{\sim}{n}_o = \frac{u_x\underset{\sim}{i} + u_y\underset{\sim}{j}}{\sqrt{u_x^2+u_y^2}}\Bigg|_{(x_o,y_o)} .$$

The conditions for the equilibrium of an element of the shell require that the sum of the moments about the tangent line to the curve $u(x,y)=\text{const.}$ at any point (x_o,y_o) of all the forces acting on the element, and the sum of all the forces normal to the plane $z=0$ to vanish. Therefore we obtain

$$\Sigma M = \underset{\sim}{n}_o \bullet \oint_{C_u} M_n\underset{\sim}{r}ds + \underset{\sim}{n}_o \bullet \oint_{C_u} V_n\, \underset{\sim}{r}_o\, ds$$

$$- \underset{\sim}{n}_o \bullet \iint_{\Omega_u} \left[q - \frac{N_x}{R_x} - \frac{N_y}{R_y} - \frac{2N_{xy}}{R_{xy}}\right]\underset{\sim}{r}\, d\Omega$$

$$= 0, \qquad (7.2.3)$$

and

$$\Sigma Z = \oint_{C_u} V_n ds - \iint_{\Omega_u} \left[q - \frac{N_x}{R_x} - \frac{N_y}{R_y} - \frac{2N_{xy}}{R_{xy}}\right] d\Omega$$

$$= 0. \qquad (7.2.4)$$

Here, as shown in [57] the expression

$$\left[\frac{N_x}{R_x} + \frac{N_y}{R_y} + \frac{2N_{xy}}{R_{xy}}\right] d\Omega$$

represents the net downward contribution of the membrane forces N_x, N_y, and N_{xy} acting upon a small element of area $d\Omega$.

Using the simplification

$$ds^2 = dx^2 + dy^2, \tag{7.2.5}$$

which gives $A_1 = A_2 = 1$, and $A_{12} = 0$, the expressions for the moments and transverse forces become

$$\begin{aligned}
M_x &= -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\,\frac{\partial^2 w}{\partial y^2}\right)\\
M_y &= -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\,\frac{\partial^2 w}{\partial x^2}\right)\\
M_{xy} &= -M_{yx} = D(1-\mu)\,\frac{\partial^2 w}{\partial x \partial y}\\
Q_x &= -D\,\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\\
Q_y &= -D\,\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right),
\end{aligned} \tag{7.2.6}$$

where

w is the deflection of the shell

$D = Eh^3 | 12(1-\mu^2)$ is the flexural rigidity

E is Young's modulus

μ is Poisson's ratio

h is the shell thickness.

And so making use of the relations

$$\frac{\partial w}{\partial x} = \frac{dw}{du}\frac{\partial u}{\partial x} = \frac{dw}{du}\,u_x$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{dw}{du}\frac{\partial^2 u}{\partial x \partial y} + \frac{d^2 w}{du^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = \frac{dw}{du}\,u_{xy} + \frac{d^2 w}{du^2}\,u_x u_y,$$

etc.,

equations (7.2.3) and (7.2.4) finally reduce to

$$\underset{\sim}{n}_o \frac{d^2 w}{du^2}\oint_{C_u} P\,\underset{\sim}{r}\,ds + \underset{\sim}{n}_o \frac{dw}{du}\oint_{C_u} Q\underset{\sim}{n}ds + \underset{\sim}{n}_o \frac{dw}{du}\oint_{C_u} R\underset{\sim}{r}_o ds$$

$$+ \underset{\sim}{n}_o \frac{d^2 w}{du^2}\oint_{C_u} F\underset{\sim}{r}_o\,ds + \underset{\sim}{n}_o \frac{dw}{du}\oint_{C_u} G\underset{\sim}{r}_o ds - \underset{\sim}{n}_o \iint_{\Omega_u}\Big[q - \frac{N_x}{R_x} - \frac{N_y}{R_y}$$

$$- \frac{2N_{xy}}{R_{xy}}\Big]\underset{\sim}{r}d\Omega = 0, \qquad (7.2.7)$$

and

$$\frac{d^3 w}{du^3}\oint_{C_u} Rds + \frac{d^2 w}{du^2}\oint_{C_u} Fds + \frac{dw}{du}\oint_{C_u} Gds - \iint_{\Omega_u}\Big[q - \frac{N_x}{R_x} - \frac{N_x}{R_y}$$

$$- \frac{2N_{xy}}{R_{xy}}\Big]d\Omega = 0, \qquad (7.2.8)$$

where we have taken into account that w and its derivatives with respect to u are constant along the contour line u = const. Here the terms **P,Q,R,F,** and G are the same expressions involving u and its partial derivatives as given in section 2.1.

Assuming no body forces, the membrane forces N_x, N_y, and N_{xy} are determined by

$$N_x = \frac{\partial^2\phi}{\partial y^2}, \tag{7.2.9}$$

$$N_y = \frac{\partial^2\phi}{\partial x^2}, \tag{7.2.10}$$

$$N_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}, \tag{7.2.11}$$

where $\phi(x,y)$ is the stress function. Equations (7.2.7) and (7.2.8) thus reduce to

$$\underset{\sim}{n}_o \frac{d^2w}{du^2} \oint_{C_u} P \underset{\sim}{r} ds + \underset{\sim}{n}_o \frac{dw}{du} \oint_{C_u} Q \underset{\sim}{n} d_s \; \underset{\sim}{n}_o \frac{d^3w}{du^3} \oint_{C_u} R \underset{\sim}{r}_o ds$$

$$+ \underset{\sim}{n}_o \frac{d^2w}{du^2} \oint_{C_u} F \underset{\sim}{r}_o ds + \underset{\sim}{n}_o \frac{dw}{du} \oint_{C_u} G \underset{\sim}{r}_o ds - \underset{\sim}{n}_o \iint_{\Omega_u} [q-K_x \frac{d^2\phi}{dy^2}$$

$$- K_y \frac{\partial^2\phi}{\partial x^2} + 2K_{xy} \frac{\partial^2\phi}{\partial x\partial y}]\underset{\sim}{r} d\Omega = 0, \tag{7.2.12}$$

and

$$\frac{d^3w}{du^3} \oint_{C_u} R ds + \frac{d^2w}{du^2} \oint_{C_u} F ds + \frac{dw}{du} \oint_{C_u} G ds - \iint_{\Omega_u} [q-K_x \frac{d^2\phi}{dy^2}$$

$$- K_x \frac{\partial^2\phi}{\partial x^2} + 2K_{xy} \frac{\partial^2\phi}{\partial x\partial y}] d\Omega = 0. \tag{7.2.13}$$

It is to be noted that the moment equation (7.2.12), and the force equation (7.2.13) must be identical when $u(x,y) =$ const. is the correct form of the lines of constant deflection.

With N_x, N_y, and N_{xy} as determined by equations (7.2.9), (7.2.10), and (7.2.12) the condition for the continuity of deformation reduces to

$$\nabla^4\phi = Eh[K_x \frac{\partial^2 w}{\partial y^2} + K_y \frac{\partial^2 w}{\partial x^2} - 2K_{xy} \frac{\partial^2 w}{\partial x\partial y}], \tag{7.2.14}$$

where K_x, K_y, and K_{xy} denote the curvatures at a point. Consequently our problem reduces to solving equations (7.2.13) and (7.2.14) for ϕ and w, giving the exact expression for the lines of constant deflection. In the next section we will discuss how to obtain the true equation for the lines of equal deflection.

7.3 DERIVATION OF THE LINES OF CONSTANT DEFLECTION FOR SHALLOW SHELL ANALYSIS

The middle surface of the shell may be completely described by specifying the values of u and s, where u(x,y) = const. is the equation of the lines of constant deflection and s is the path length along an orbitrary closed contour C_u. Thus u and s form a system of orthogonal, curvilinear coordinates. The metric of the middle surface of the shell now takes the form given by equation (7.1.6), where A_u and A_s are the metric tensors associated with u and s respectively, and have the values

$$A_u = \frac{1}{\sqrt{t}} \quad , \tag{7.3.1}$$

and

$$A_s = 1 \quad . \tag{7.3.2}$$

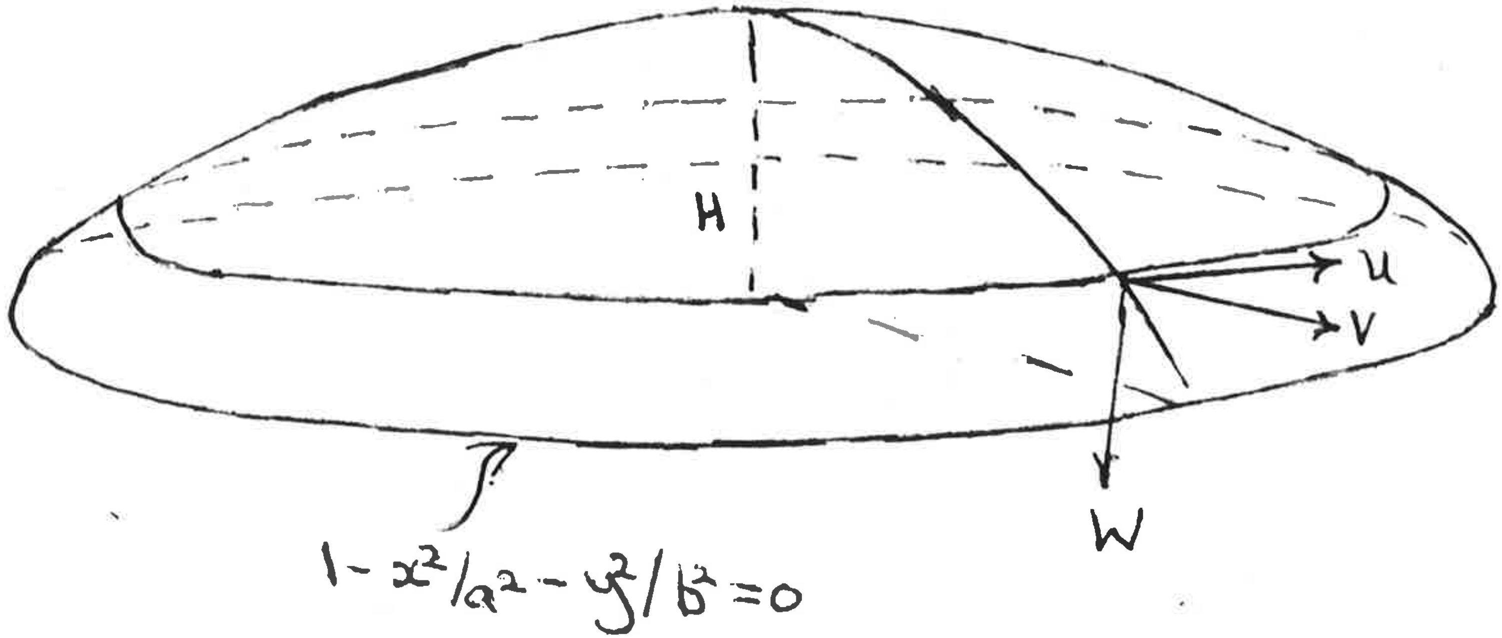

Fig. 23.

However, the intrinsic geometry of the middle surface of the shell is assumed to be Euclidean. Thus since we are considering shallow shells for which the rise of the shell is a small quantity as compared to the sides of the base of the shell, our first approach is to consider the deflection contours to be as for the corresponding flat plate problem, i.e.

$$K_x = K_y = K_{xy} = 0 \ . \tag{7.3.3}$$

Clearly the smaller the rise of the shell the smaller the error in our assumption.

7.4 BENDING OF SHALLOW SHELLS ON ELLIPTICAL BASE

As an illustration, let us consider the bending of a shallow dome of non zero Gaussian curvature upon an elliptic base. The geometry of the shell being as described in figure 23., where the origin of coordinates is taken at the centre, and where the edges of the shell are rigidly clamped. This problem of immense technical importance since shells of this type are frequently encountered in the aerospace and chemical industries, and may be used to approximate the roofs of sports arenas. The latter is a problem of growing interest since a large number of Football stadiums throughout the world are either elliptical or can be closely approximated by an ellipse (e.g. Wembley stadium in London, and the old Reichsportsfeld in Berlin). Consequently with the growing tendency to enclose the stadium, in order to protect the playing surface and the public, the problem of the bending of a shallow dome on an elliptic base comes to the fore. Yet despite the practical importance of this problem no solution can be found other than for a spherical dome [57,16].

If the shell is acted upon by a uniformly distributed load, then in accordance with section 7.3 the lines of constant deflection may be taken as for the corresponding flat plate problem, which will be a family of similar and similarly situated ellipses. Consequently we consider

$$u(x,y) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}. \qquad (7.4.1)$$

Substituting the value of the line integrals $\oint Rds$, $\oint Fds$, and $\oint Gds$ as calculated in section 5.4.1 into equation (7.2.13) we obtain

$$(1-u)^2 \frac{d^3w}{du^3} - 2(1-u)\frac{d^2w}{du^2} - \delta\iint_{\Omega_u}\left(K_x \frac{\partial^2\phi}{\partial y^2} + K_y \frac{\partial^2\phi}{\partial x^2}\right)d\Omega +$$

$$q_1(1-u) = 0, \qquad (7.4.2)$$

where

$$q_1 = \frac{qa^4b^4}{2D(3a^4+2a^2b^2+3b^4)}, \qquad (7.4.3)$$

and

$$\delta = \frac{a^3b^3}{2\pi D(3a^4+2a^2b^2+3b^4)}. \qquad (7.4.4)$$

Further since equation (7.2.14) must hold for all points in the interior of the shell, we integrate it over the region Ω_u, and obtain

$$\iint_{\Omega_u} [\nabla^4\phi - Eh(K_x \frac{\partial^2 w}{\partial y^2} + K_y \frac{\partial^2 w}{\partial x^2})] d\Omega = 0, \qquad (7.4.5)$$

which after applying Green's theorem reduces to

$$\oint_{C_u} \underset{\sim}{\nabla}(\nabla^2\phi).\underset{\sim}{n} ds + Eh \oint_{C_u} \left(K_x \frac{\partial w}{\partial y} \frac{u_y}{\sqrt{t}} + K_y \frac{\partial w}{\partial x} \frac{u_x}{\sqrt{t}}\right) ds = 0. \qquad (7.4.6)$$

Assuming that ϕ is a function of u, and making use of the relationships

$$\frac{\partial\phi}{\partial x} = \frac{d\phi}{du} u_x, \; \frac{\partial^2\phi}{\partial x \partial y} = \frac{d\phi}{du} u_{xy} + \frac{d^2\phi}{du^2} u_x u_y, \text{ etc.} \qquad (7.4.7)$$

and similar expressions for w given in (2.2.8), equation (7.4.6) finally reduces to

$$\frac{d^3\phi}{du^3} \oint_{C_u} R ds + \frac{d^2\phi}{du^2} \oint_{C_u} F ds + \frac{d\phi}{du} \oint_{C_u} G ds + EhD \frac{dw}{du} \cdot$$

$$\oint_{C_u} \frac{K_x u_y{}^2 + K_y u_x{}^2}{\sqrt{t}} ds = 0. \qquad (7.4.8)$$

After evaluating the contour integrals in (7.4.8) we obtain

$$(1-u)^2 \frac{d^3\phi}{du^3} - 2(1-u) \frac{d^2\phi}{du^2} - Eh\gamma(1-u) \frac{dw}{du} = 0, \qquad (7.4.9)$$

where

$$\gamma = \frac{(K_x/b^2 + K_y/a^2) a^4 b^4}{(3a^4 + 2a^2 b^2 + 3b^4)} \qquad (7.4.10)$$

The first integral of (7.4.9) leads to

$$\frac{d}{du}\left((1-u)\frac{d\phi}{du}\right) = Eh\gamma w + c, \tag{7.4.11}$$

where c is an arbitrary constant. Similarly, for equation (7.4.2), differentiating it with respect to u, after using Green's theorem for the double integral appearing therein, and using equation (7.4.11) we get an ordinary differential equation in w, viz.

$$(1-u)^2\frac{d^4w}{du^4} - 4(1-u)\frac{d^3w}{du^3} + \frac{2d^2w}{du^2} + \frac{Eh\gamma^2 w}{D} + \frac{c\gamma}{D} - q_1 = 0, \tag{7.4.12}$$

a particular solution of which is

$$w = \frac{q_1 D}{Eh\gamma^2} - \frac{c}{Eh\gamma}. \tag{7.4.13}$$

The general solution to the homogeneous equation corresponding to equation (7.4.12), viz.

$$(1-u)^2\frac{d^4w}{du^4} - 4(1-u)\frac{d^3w}{du^3} + 2\frac{d^2w}{du^2} + \frac{Eh\gamma^2 w}{D} = 0, \tag{7.4.14}$$

is given by

$$w = w_1 + w_2, \tag{7.4.15}$$

where w_1 and w_2 satisfy

$$Lw_1 + i\lambda w_1 = 0, \tag{7.4.16}$$

and

$$Lw_2 - i\lambda w_2 = 0. \tag{7.4.17}$$

Here L denotes the differential operator

$$\begin{aligned} L &\equiv (1-u)\frac{d^2}{du^2} - \frac{d}{du} \\ &\equiv \frac{d}{du}[(1-u)\frac{d}{du}], \end{aligned} \tag{7.4.18}$$

and

$$\lambda^2 = \frac{Eh\gamma^2}{D} . \tag{7.4.19}$$

Introducing a new variable f given by

$$f^2 = 1-u, \tag{7.4.20}$$

we have

$$w_1 = A_1 J_0 (kf) + A_2 Y_0(kf), \tag{7.4.21}$$

and

$$w_2 = A_3 I_0(kf) + A_4 K_0(kf), \tag{7.4.22}$$

where

$$k^2 = i4\lambda. \tag{7.4.23}$$

Consequently if we write

$$I_0(\sqrt{i}x) = \mathrm{Ber}(x) + i\,\mathrm{Bei}(x), \tag{7.4.24}$$

and

$$K_0(\sqrt{i}x) = \mathrm{Ker}(x) + i\,\mathrm{Kei}(x), \tag{7.4.25}$$

we have

$$w_1+w_2 = B_1\,\mathrm{Ber}(\omega f) + B_2\,\mathrm{Bei}(\omega f) + B_3\,\mathrm{Ker}(\omega f) + B_4\,\mathrm{Kei}(\omega f), \tag{7.4.26}$$

where

$$\omega^2 = 4\lambda, \tag{7.4.27}$$

and B_1, B_2, B_3, and B_4 are four arbitrary constants.

Thus w has the form

$$w = B_1\,\mathrm{Ber}(\omega f) + B_2\,\mathrm{Bei}(\omega f) + B_3\,\mathrm{Ker}(\omega f) + B_4\,\mathrm{Kei}(\omega f) + \frac{q_1 D}{Eh\gamma^2} - \frac{c}{Eh\gamma} . \tag{7.4.28}$$

The requirement of finite deflection at the centre, $f=0$, now gives

$$B_3 = B_4 = 0, \qquad (7.4.29)$$

and so

$$w = B_1 \,\mathrm{Ber}(\omega f) + B_2 \,\mathrm{Bei}(\omega f) + \frac{q_1 D}{E\gamma^2 h} - \frac{c}{Eh\gamma} \,. \qquad (7.4.30)$$

The stress function ϕ may now be determined from equation (7.4.11) which in terms of the new variable f transforms to

$$\frac{1}{f}\frac{d}{df}\left(f\,\frac{d\phi}{df}\right) = 4Eh\gamma w + 4c, \qquad (7.4.31)$$

and from which we obtain

$$\phi = \frac{4Eh\gamma}{\omega^2}\left[B_1 \,\mathrm{Bei}(\omega f) - B_2 \,\mathrm{Ber}(\omega f)\right] + \frac{q_1 D f^2}{\gamma} \,. \qquad (7.4.32)$$

The three unknown constants B_1, B_2, and c can now be determined by applying the clamping conditions

i) $\left. w \right|_{f=1} = 0,$

ii) $\left. \dfrac{dw}{df} \right|_{f=1} = 0, \qquad (7.4.33)$

iii) the vanishing of the circumferential strain[1], i.e. $\varepsilon_\theta = 0$ or equivalently[2],

$$\left. \frac{d^2\phi}{df^2} - \frac{\mu}{f}\frac{d\phi}{df} \right|_{f=1} = 0.$$

Consequently B_1, B_2 and c must satisfy

---

[1] This boundary condition was proposed by Reissner [52] and Gradowczyk [16].

[2] See appendix 4

$$B_1 \,\mathrm{Ber}(\omega) + B_2 \,\mathrm{Bei}(\omega) + \frac{q_1 D}{Eh\gamma^2} - \frac{c}{Eh\gamma} = 0, \qquad (7.4.34)$$

$$B_1 \omega \mathrm{Ber}'(\omega) + B_2 \omega \,\mathrm{Bei}'(\omega) = 0, \qquad (7.4.35)$$

and

$$Eh\gamma\, B_1[\omega^2 \,\mathrm{Bei}''(\omega) - \mu\omega \,\mathrm{Bei}'(\omega)] - Eh\gamma B_2[\omega^2 \mathrm{Ber}''(\omega) - \mu\omega \,\mathrm{Ber}'(\omega)] + \frac{(1-\mu) q_1 D\omega^2}{2\gamma} = 0. \qquad (7.4.36)$$

Fortunately equations (7.4.35) and (7.4.36) can be solved separately for B_1 and B_2 giving

$$B_1 = \frac{(1-\mu) q_1 \omega^2 \mathrm{Bei}'(\omega)}{2\gamma Eh[\omega^2 \mathrm{Ber}''(\omega)\mathrm{Ber}'(\omega) + \omega^2 \mathrm{Bei}''(\omega)\mathrm{Bei}'(\omega) - \mu\omega \mathrm{Bei}'(\omega)^2 - \mu\omega \mathrm{Ber}'(\omega)^2]} \qquad (7.4.37)$$

and

$$B_2 = \frac{(1-\mu) q_1 \omega^2 \mathrm{Ber}'(\omega)}{2\gamma Eh[\omega^2 \mathrm{Ber}''(\omega)\mathrm{Ber}'(\omega) + \omega^2 \mathrm{Bei}''(\omega)\mathrm{Bei}'(\omega) - \mu\omega \mathrm{Bei}'(\omega)^2 - \mu\omega \mathrm{Ber}'(\omega)^2]} \qquad (7.4.38)$$

so that the third constant c may be obtained from the remaining equation (7.4.34).

It is interesting to note that if we put $a=b$, and $K_x = K_y = K$ the solution coincides exactly with that obtained by Gradowczyk[1] [16] for a clamped spherical dome with curvature K. Further in the limiting case as K_x and K_y tend to zero we obtain

[1]This problem was also examined by Reissner [51], however the form of the obtained solution was such that the coefficients were difficult to evaluate analytically.

$$\lim_{\substack{K_x \to \\ K_y \to 0}} w = q_1(1-f^2)^2/4$$

$$= q_1\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2/4 \qquad (7.4.39)$$

which is the exact expression for the bending of a clamped, uniformly loaded, elliptic plate.

CHAPTER VIII.

THE TRANSVERSE VIBRATION, AND STABILITY OF SHALLOW SHELLS

8.1 TRANSVERSE VIBRATION OF SHALLOW SHELLS.

It is well known that the general problem for the vibration of a shallow shell, outlined by any surface reduces to a system of two consistent, symmetrically constructed, partial differential equations for two scalar functions. Several techniques have been used to solve them. However, despite the simplified nature of shallow shell theory relatively few exact solutions are known. The purpose of the present investigation is to develop a simple, and yet sufficiently accurate method for determining the fundamental frequency of vibration of a shallow shell.

In the work which follows, first the pertinent equations are presented for a shallow shell of arbitrary shape. The method of obtaining the equation of the lines of equal deflection is then presented. As an illustration of the method, a technically interesting example of the vibration of a shallow shell with an elliptic base is examined.

Consider an elastic, isotropic shallow shell of thickness h. Let the equation of the middle surface of the shell referred to a system of orthogonal coordinates xyz be given by

$$z = \frac{x^2}{2R_x} + \frac{xy}{R_{xy}} + \frac{y^2}{2R_y}, \qquad (8.1.1)$$

where the shell will be called shallow if $r = \sqrt{x^2+y^2}$ is small compared to the least of R_x, R_y, and R_{xy} (the radii of curvature) everywhere in the region, and if R_x, R_y, and R_{xy} may be taken to be constants.

If the shell vibrates in a normal mode, then at any instant τ the intersections between the deflected surface and the parallels z = const. yield contours which after projection onto the $z=0$ surface are the level curves called the Lines of Equal Deflection. We will denote the family of such curves by $u(x,y)$ = const.

Consider at any instant τ an element of the shell bounded by any contour line of constant deflection. Since we are primarily interested in the free transverse vibrations of a shallow shell we may neglect the effects of the longitudinal and latitudinal inertia terms. Further if $w(x,y,\tau)$ and $\phi(x,y,\tau)$ denote the transverse displacement and the stress function respectively then it is possible to write

$$w(x,y,\tau) = W(x,y) \cos (\omega\tau+\varepsilon), \qquad (8.1.2)$$

and

$$\phi(x,y,\tau) = \Phi(x,y) \cos(\omega\tau+\varepsilon), \qquad (8.1.3)$$

where $\cos(\omega\tau+\varepsilon)$ is the normal coordinate, ω is the circular frequency, W and Φ are normal functions determining the form of the deflected surface of the vibrating shell, and the stress function respectively. Since it is the free vibration of shallow shells which are of interest here, the application of D'Alembert's principle and the summing of the forces in the direction normal to the surface, and the moments about the normal to the surface yields the following dynamical equations,

$$\oint_{C_u} \left(Q_n - \frac{\partial}{\partial s} M_{nt}\right) ds + \iint_{\Omega_u} \left[\rho h \frac{\partial^2 w}{\partial \tau^2} + \frac{N_x}{R_x} + \frac{N_y}{R_y} + \frac{2N_{xy}}{R_{xy}}\right] d\Omega = 0, \tag{8.1.4}$$

and

$$\underset{\sim}{n}_o \oint_{C_u} M_n \underset{\sim}{n} ds + \underset{\sim}{n}_o \oint_{C_u} \left(Q_n - \frac{\partial M_{nt}}{\partial s}\right) \underset{\sim}{r}_o ds + \underset{\sim}{n}_o \iint_{\Omega_n} \left[\rho h \frac{\partial^2 w}{\partial \tau^2} + \frac{N_x}{R_x} + \frac{N_y}{R_y} + \frac{2N_{xy}}{R_{xy}}\right] \underset{\sim}{r} d\Omega = 0. \tag{8.1.5}$$

Here as shown in [57] the expression

$$\left[\frac{N_x}{R_x} + \frac{N_y}{R_y} + \frac{2N_{xy}}{R_{xy}}\right] d\Omega$$

represents the net downward contribution of the membrane forces N_x, N_y and N_{xy} acting upon a small element of area $d\Omega$, and the term $\rho h \frac{\partial^2 w}{\partial \tau^2}$ represents the inertia force due to the vertical acceleration of the element $d\Omega$, ρ being the mass per unit area of the shell. The double integration being taken over the region bounded by the contour line u = const. and the contour integration taken around the closed curve C_u.

As in the previous chapter the expressions for the moments and the shearing forces are

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right),$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right),$$

$$M_{xy} = -M_{yx} = D(1-\mu) \frac{\partial^2 w}{\partial x \partial y},$$

$$Q_x = -D \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right),$$

$$Q_y = -D \frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2}\right), \qquad (8.1.6)$$

where

w is the deflection of the shell

$D = Eh^3/12(1-\mu^2)$ is the flexural rigidity

E is Young's modulus

μ is Poisson's ratio

h is the thickness of the shell.

And so making use of the relations

$$\frac{\partial W}{\partial x} = \frac{dW}{du}\frac{\partial u}{\partial x} = \frac{dW}{du} u_x, \quad \frac{\partial^2 W}{\partial x \partial y} = \frac{dW}{du}\frac{\partial^2 u}{\partial x \partial y} + \frac{d^2 W}{du^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = \frac{d^2 W}{du^2} u_x u_y + \frac{dW}{du} u_{xy} \qquad (8.1.7)$$

equations (8.1.4) and (8.1.5) reduce to

$$\underset{\sim}{n}_o \frac{d^2 W}{du^2} \oint_{C_u} P\underset{\sim}{n} ds + \underset{\sim}{n}_o \frac{dW}{du} \oint_{C_u} Q\underset{\sim}{n} ds + \underset{\sim}{n}_o \frac{d^3 W}{du^3} \oint_{C_u} R\underset{\sim}{r}_o ds$$
$$+ \underset{\sim}{n}_o \frac{d^2 W}{du^2} \oint_{C_u} F\underset{\sim}{r}_o ds + \underset{\sim}{n}_o \frac{dW}{du} \oint_{C_u} G\underset{\sim}{r}_o ds - \underset{\sim}{n}_o \rho h \omega^2$$

$$\iint_{\Omega_u} W\underset{\sim}{r} d\Omega + \underset{\sim}{r}_0 \iint_{\Omega_u} \left[\frac{1}{R_x} \frac{\partial^2\Phi}{\partial y^2} + \frac{1}{R_y} \frac{\partial^2\Phi}{\partial x^2} - \frac{2\partial^2\Phi}{\partial x \partial y}\right] \underset{\sim}{r} d\Omega$$

$$= 0, \tag{8.1.8}$$

and

$$\frac{d^3W}{du^3} \oint_{C_u} R ds + \frac{d^2W}{du^2} \oint_{C_u} F ds + \frac{dW}{du} \oint_{C_u} G ds - \rho h \omega^2 \iint_{\Omega_u} W d\Omega +$$

$$\iint_{\Omega_u} \left[\frac{1}{R_x} \frac{\partial^2\Phi}{\partial y^2} + \frac{1}{R_y} \frac{\partial^2\Phi}{\partial x^2} - \frac{2}{R_{xy}} \frac{\partial^2\Phi}{\partial x \partial y}\right] d\Omega \tag{8.1.9}$$

$$= 0,$$

where the factor $\cos(\omega\tau+\varepsilon)$ has been cancelled. Here P, Q, R etc. are the same expressions involving u and its partial derivatives as given in section 2.1, while the membrane forces N_x, N_y and N_{xy} are determined by

$$N_x = \frac{\partial^2\phi}{\partial y^2},$$

$$N_y = \frac{\partial^2\phi}{\partial x^2}, \tag{8.1.10}$$

$$N_{xy} = -\frac{\partial^2\phi}{\partial x \partial y},$$

where $\phi(x,y,\tau)$ is the stress function. Consequently the condition for the continuity of deformation reduces to

$$\nabla^4\Phi = Eh\left[\frac{1}{R_x} \frac{\partial^2W}{\partial y^2} + \frac{1}{R_y} \frac{\partial^2W}{\partial x^2} - \frac{2}{R_{xy}} \frac{\partial^2W}{\partial x \partial y}\right]. \tag{8.1.11}$$

Thus our problem now reduces to solving equations (8.1.8) and (8.1.9) for Φ and W giving the exact expression for the lines of constant deflection.

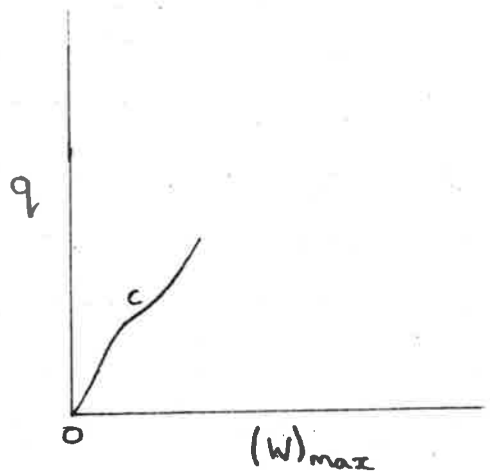

Fig. 24 (a)

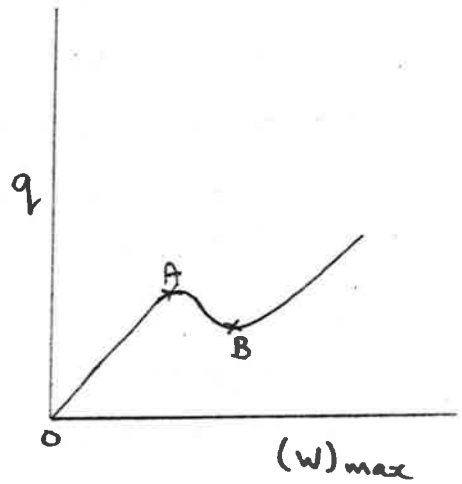

Fig. 24 (b)

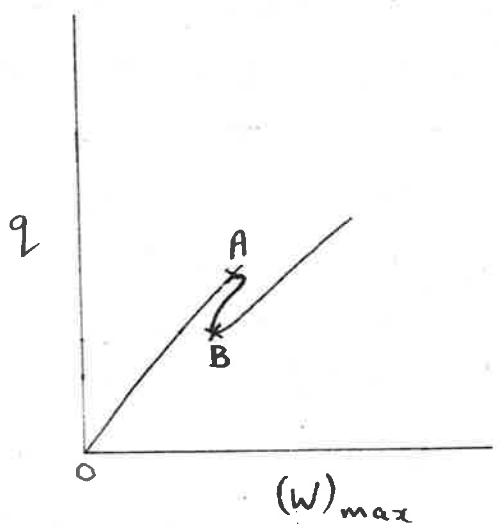

Fig. 24(c)

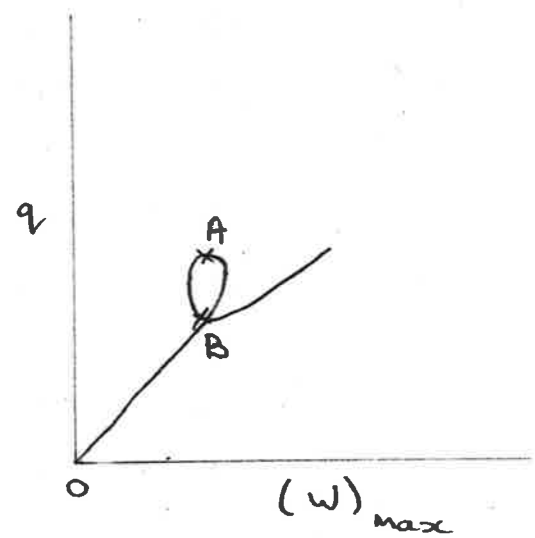

Fig. 24 (d)

Since the intrinsic geometry of the middle surface of the shell is considered to be Euclidean, and since we are considering shallow shells for which the maximum rise of the shell is a small quantity as compared with the sides of the base of the shell, our first approach is to assume the deflection contours to be as for the corresponding flat plate problem ($K_x = K_y = K_{xy} = 0$). Clearly the smaller the rise of the shell the smaller the error in our assumption.

8.2 STABILITY OF SHALLOW SHELLS

Let us now turn to problems pertaining to the stability of shallow shells, where we will assume that the shell is subjected to the action of a load directed normal to the middle surface of the shell.

The distinction between a shallow shell and a flat plate, is only that for the shell it is necessary to consider secondary stresses in its middle surface. And so in the stability analysis it is important to study large deflections from the propositions of the nonlinear theory.

Let us consider several "load-maximum deflection" curves, which are characteristic of shallow shells of different curvature. In the case of a very shallow shell the load q can be seen to increase monotonically with maximum deflection $(w)_{max}$ (see Fig.24a.), the point C being a point of inflection. On the first portion, OC, of the curve the shell rigidity drops, while on the second portion it increases. Figure 24b. corresponds to the case when the initial rise of the shell is comparable with the thickness. A limit point A is obtained where the load is "dead", and loss of stability becomes possible, expressed by clicking of the shell to a new stable equilibrium state. In figure 24c. the curvature is further increased, the resulting branch AB of unstable states lying near the initial branch OA. If we increase the curvature still further (Fig.24d.) we obtain cases where the deflection decreases and the load-maximum deflection curve is loop shaped. This corresponds to a change of wave formation.

In the following analysis we will assume that the initial form of the shell is ideal, that is there are no initial imperfections of form. The consequence of these initial imperfections, and other perturbations are generally manifested quite differently than for a flat plate, and lead to a lowering of the critical load. Recently attention has been brought to bear upon the problem of imperfections in the medium, and a summary of these works is given by Budiansky and Hutchinson [21].

The classical Vlasov-Reissner theory for the stability of a shallow shell involves two coupled, fourth order, nonlinear, partial differential equations in ϕ and w. This problem is sufficiently difficult that the works of different authors frequently appear contradictory. To overcome this shortcoming we formulate the problem using

Berg r's analysis, so that the conditions for the equilib-rium of an element of the shell bounded by any line of constant deflection now give rise to the following two differential equations, which we term the moment and the force equation, viz.

$$\underset{\sim}{n}_0 \frac{d^2w}{du^2} \oint_{C_u} P\underset{\sim}{n}ds + \underset{\sim}{n}_0 \frac{dw}{du} \oint_{C_u} Q\underset{\sim}{n}ds + \underset{\sim}{n}_0 \frac{d^3w}{du^3} \oint_{C_u} R\underset{\sim}{r}_0 ds$$

$$+ \underset{\sim}{n}_0 \frac{d^2w}{du^2} \oint_{C_u} F\underset{\sim}{r}_0 ds + \underset{\sim}{n}_0 \frac{dw}{du} \oint_{C_u} G\underset{\sim}{r}_0 ds - \underset{\sim}{n}_0 \iint_{\Omega_u} [$$

$$q+2\alpha^2KD + \alpha^2D\nabla^2w]\underset{\sim}{r}d\Omega = 0, \qquad (8.2.1)$$

and

$$\frac{d^3w}{du^3} \oint_{C_u} Rds + \frac{d^2w}{du^2} \oint_{C_u} Fds + \frac{dw}{du} \oint_{C_u} Gds - \iint_{\Omega_u} [$$

$$q + 2\alpha^2KD + \alpha^2D\nabla^2w]d\Omega = 0, \qquad (8.2.2)$$

where

$$e = \frac{\alpha^2h^2}{12}$$

$$= \text{const.} \quad . \qquad (8.2.3)$$

Here e is the first invariant of the strain tensor, and α^2 is a normalized constant of integration.

Our equations are now effectively decoupled, so that it is possible to solve separately for the deflected surface $w(u)$, and for the resultant stresses. As in the previous chapters, when the exact equation for the lines of constant deflection is known, the moment equation (8.2.1), and the force equation (8.2.2) must be identical.

One of the most interesting and most important problems in the stability of thin elastic shells, occurs when the load is uniformly distributed over the middle surface of the shell, i.e.

$$q(x,y) = q_o = \text{const.} \tag{8.2.4}$$

Equations (8.2.2) and (8.2.3) are now identical to the corresponding equations in section 2.3 for the bending of a flat plate subject to hydrostatic edge loading

$$N_x = N_y = +\alpha^2 D, \; N_{xy} = 0, \tag{8.2.5}$$

and lateral loading

$$\begin{aligned} \bar{q} &= q_o + 2\alpha^2 KD \\ &= \text{const.} \end{aligned} \tag{8.2.6}$$

However the deflection contours for the buckling of a hydrostatically compressed plate, and for the bending of the same plate under uniform load are identical. Thus we conclude that the lines of constant deflection for the stability of a uniformly loaded shallow shell may be determined from the corresponding flat plate problem under small deflection theory.

8.3 TRANSVERSE VIBRATION OF SHALLOW SHELLS ON ELLIPTICAL BASE

As an illustration let us consider the axi symmetrical vibration of a shallow dome of non zero Gaussian curvature upon an elliptic base. The geometry of the shell being as described in figure 23, where the origin of co-ordinates is taken at the centre and where the edges of the shell are rigidly clamped. As explained in section 7.4, such shells are frequently encountered in constructional practice as structural forms for the roofs of sports arenas. Yet despite the practical importance of this problem no solution can be found other than for a spherical dome [52].

If the shell vibrates in a normal mode, then in accordance with section 8.1 the deflection contours may be taken as for the corresponding flat plate problem, which will be a family of similar and similarly situated ellipses. Consequently we consider

$$u(x,y) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} . \qquad (8.3.1)$$

And so upon substituting into (8.1.9) the values of the line integrals $\oint Rds$, $\oint Fds$, $\oint Gds$, as calculated in section 5.4.1, and making use of the simplification

$$\iint_{\Omega_u} W d\Omega = -\int_1^u W(u_o)du_o \oint_{C_{u_o}} \frac{ds}{\sqrt{t}}$$

$$= -\pi ab \int_1^u W(u_o)du_o , \qquad (8.3.2)$$

equation (8.1.9) finally reduces to

$$(1-u)^2 \frac{d^3W}{du^3} - 2(1-u)\frac{d^2W}{du^2} - \delta \iint_{\Omega_u} \left[K_x \frac{\partial^2\Phi}{\partial y^2} + K_y \frac{\partial^2\Phi}{\partial x^2}\right] d\Omega - \eta^4 \int_1^u W du_o$$

$$= 0, \qquad (8.3.3)$$

where

$$\delta = \frac{a^3b^3}{2\pi D(3a^4+2a^2b^2+3b^4)}, \tag{8.3.4}$$

and

$$\eta^4 = \frac{\rho h\omega^2 a^4b^4}{2D(3a^4+2a^2b^2+3b^4)}. \tag{8.3.5}$$

Here the contour integration is taken around the closed contour

$$u = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} = \text{const.}, \tag{8.3.6}$$

and the double integration extends over the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - u. \tag{8.3.7}$$

Further, since equation (8.1.11) must hold for all points in the interior of the shell, we integrate it over the region Ω_u, and obtain

$$\iint_{\Omega_u} \left[\nabla^4\Phi - Eh\left(K_x \frac{\partial W}{\partial y^2} + K_y \frac{\partial^2 W}{\partial x^2}\right)\right] d\Omega$$

$$= 0, \tag{8.3.8}$$

which after applying Green's theorem reduces to

$$\oint_{C_u} \underset{\sim}{\nabla}(\nabla^2\Phi).\underset{\sim}{n}ds + Eh \oint_{C_u} \left(K_x \frac{\partial^2 W}{\partial y^2} \frac{u_y}{\sqrt{t}} + K_y \frac{\partial^2 W}{\partial x^2} \frac{u_x}{\sqrt{t}}\right)$$

$$ds = 0. \tag{8.3.9}$$

Assuming that Φ is a function of u and making use of the relationships

$$\frac{\partial \Phi}{\partial x} = \frac{d\Phi}{du}\frac{du}{\partial x} = \frac{d\Phi}{du} u_x, \tag{8.3.10}$$

$$\frac{\partial^2\Phi}{\partial x\partial y} = \frac{d\Phi}{du}\frac{\partial^2 u}{\partial x\partial y} + \frac{d^2\Phi}{du^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = \frac{d\Phi}{du}u_{xy} + \frac{d^2\Phi}{du^2}u_x u_y$$

etc.

and similar expressions for W given in (2.2.8), equation (8.3.9) reduces to

$$\frac{d^3\Phi}{du^3}\oint_{C_u} R\,ds + \frac{d^2\Phi}{du^2}\oint_{C_u} F\,ds + \frac{d\Phi}{du}\oint_{C_u} G\,ds + EhD\frac{dW}{du}\oint_{C_u}\left(K_x\frac{u^2_y}{\sqrt{t}} + K_y\frac{u_x^2}{\sqrt{t}}\right)ds = 0. \qquad (8.3.11)$$

After evaluating the contour integrals in (8.3.11) we obtain

$$(1-u)^2\frac{d^3\Phi}{du^3} - 2(1-u)\frac{d^2\Phi}{du^2} - Eh\gamma(1-u)\frac{dW}{du} = 0, \qquad (8.3.12)$$

where

$$\gamma = \frac{(K_x/b^2 + K_y/a^2)a^4b^4}{(3a^4 + 2a^2b^2 + 3b^4)}. \qquad (8.3.13)$$

The first integral of (8.2.12) leads to

$$\frac{d}{du}\left((1-u)\frac{d\Phi}{du}\right) = Eh\gamma W + c, \qquad (8.3.14)$$

where c is an arbitrary constant. Similarly, for equation (8.3.3) differentiating it with respect to u and, after using Green's theorem for the double integral appearing therein, and using equation (8.3.14) we get an ordinary differential equation in W, viz.

$$(1-u)^2\frac{d^4W}{du^4} - 4(1-u)\frac{d^3W}{du^3} + 2\frac{d^2W}{du^2} - \frac{\lambda^4}{16}W + \frac{\gamma c}{D} = 0, \qquad (8.3.15)$$

where

$$\lambda^4 = 16(\eta^4 - \frac{Eh\gamma^2}{D}). \tag{8.3.16}$$

Introducing a new variable f given by

$$f^2 = 1-u, \tag{8.3.17}$$

equation (8.3.15) has solution

$$W = B_0 J_0(\lambda f) + B_1 Y_0(\lambda f) + B_2 I_0(\lambda f)$$

$$+ B_3 K_0(\lambda f) + B_4, \tag{8.3.18}$$

where

$$B_4 = \frac{16\gamma c}{\lambda^4 D}. \tag{8.3.19}$$

The stress function Φ may now be determined from equation (8.3.14), which in terms of the new variable f transforms to

$$\frac{1}{f}\frac{d}{df}\left(f\frac{d\Phi}{df}\right) = 4Eh\gamma W + 4c, \tag{8.3.20}$$

and from which we obtain

$$\Phi = \frac{Eh\gamma}{4\lambda^2}[B_0 J_0(\lambda f) + B_1 Y_0(\lambda f) - B_2 I_0(\lambda f)$$

$$- B_3 K_0(\lambda f)] + B_5 \log f + D\frac{\eta^4 f^2}{\gamma} B_4, \tag{8.3.21}$$

where the boundary conditions to be imposed on W and Φ are

i) $W\Big|_{f=1} = 0$

ii) $\frac{dW}{df}\Big|_{f=1} = 0$ (8.3.22)

iii) W is regular at $f=0$

iv) the vanishing of the circumferential strain, i.e. $\varepsilon_s = 0$, or alternatively

$$\left.\frac{d^2\Phi}{df^2} - \frac{\mu}{f}\frac{d\Phi}{df}\right|_{f=1} = 0.$$

Consequently we obtain

$$B_1 = B_3 = B_5 = 0, \tag{8.3.23}$$

$$B_0 J_0(\lambda) + B_2 I_0(\lambda) + B_4 = 0, \tag{8.3.24}$$

$$B_0 \lambda_2 J_0'(\lambda) + B_2 \lambda_2 I_0(\lambda) = 0, \tag{8.3.25}$$

and

$$\frac{Eh\gamma}{4}\left[B_0\ J_0(\lambda) + B_2\ I_0(\lambda) - (1+\mu)\frac{[B_2 I_0'(\lambda) - B_0\ J_0'(\lambda)]}{\lambda}\right] + \frac{2(1-\mu)DB_4\eta^4}{\gamma} = 0. \tag{8.3.26}$$

If we denote $J_0' = -J_1$, and $I_0' = I_1$ the requirement of zero determinant gives

$$J_0(\lambda)I_1(\lambda) + J_1(\lambda)I_0(\lambda) - \frac{4M^4 J_1(\lambda)I_1(\lambda)}{\lambda(\lambda^4 - M^4)} = 0, \tag{8.3.27}$$

where the parameter

$$M^4 = \frac{16(1+\mu)Eh\gamma^2}{(1-\mu)\ D} = 48(1+\mu)^2\left(\frac{2\gamma}{h}\right)^2, \tag{8.3.28}$$

and as we have seen ω is given in terms of η^4 and M^4 by

$$16\eta^4 = \frac{16\rho h\omega^2 a^4 b^4}{2D(3a^4 + 2a^2b^2 + 3b^4)} = \frac{1-\mu}{1+\mu}M^4 + \lambda^4. \tag{8.3.29}$$

The frequency equation (8.3.27) gives λ as a function of M. Approximate values of λ for the lowest frequency may be found in Table 11.

TABLE 11

M^4	λ
0	3.196
10	3.235
20	3.273
50	3.380
100	3.537
215.56	3.382
300	4.000
400	4.180
500	4.328
600	4.460
700	4.575
800	4.680
900	4.770
1000	4.855
1200	4.990
1400	5.112
1600	5.212
1800	5.285
2000	5.360
2500	5.482
3000	5.570
4000	5.665
5000	5.723
10000	5.828
15900	5.865
21150	5.875
∞	5.910

As in [52] we obtain

$$\omega = \left(\frac{Eh^2(3a^4+2a^2b^2+3b^4)}{8\rho a^4 b^4}\right)^{\frac{1}{2}} \left[\frac{\lambda^4}{12(1-\mu^2)} + \frac{M^4(1-\mu)}{12(1+\mu)(1-\mu^2)}\right]^{\frac{1}{2}}, \tag{8.3.30}$$

so that if we denote by ω_0 the value of ω corresponding to $M=0$ and $\mu=0$, that is the value of the frequency for a flat plate with vanishing Poisson's ratio, we then have

$$\omega_0 = \left(\frac{Eh^2(3a^4+2a^2b^2+3b^4)}{8\rho a^4 b^4}\right)^{\frac{1}{2}} \left(\frac{\lambda_0{}^4}{12}\right)^{\frac{1}{2}}$$

$$= 2.948 \left(\frac{Eh^2(3a^4+2a^2b^2+3b^4)}{8\rho a^4 b^4}\right)^{\frac{1}{2}}. \qquad (8.3.31)$$

Thus ω may now be expressed in terms of ω_0, giving

$$\frac{\omega}{\omega_0} = \left[\left(\frac{\lambda}{\lambda_0}\right)^4 \frac{1}{(1-\mu^2)} + \left(\frac{M}{\lambda_0}\right)^4 \frac{1}{(1+\mu)^2}\right]^{\frac{1}{2}}, \qquad (8.3.32)$$

where

$$\lambda \begin{matrix} M=0 \\ \mu=0 \end{matrix} = \lambda_0. \qquad (8.3.33)$$

Consequently when M is large enough, the second term under the square root dominates the first, and equation (8.3.32) may be replaced by the approximation

$$\omega = \left(\frac{Eh^2(3a^4+2a^2b^2+3b^4)}{96\rho a^4 b^4}\right)^{\frac{1}{2}} \frac{M^2}{(1+\mu)}$$

$$= \left(\frac{2Ea^4b^4}{\rho(3a^4+2a^2b^2+3b^4)}\right)^{\frac{1}{2}} \left(\frac{K_x}{b^2} + \frac{K_y}{a^2}\right). \qquad (8.3.34)$$

This is particularly interesting since ω is now independent of the shell thickness h, however it must be recalled that in order for shallow shell theory to be applicable we must have

$$\frac{h}{a} < \frac{1}{10}. \qquad (8.3.35)$$

Comparing these results with those of Reissner [52] we see that as the aspect ratio $a/b \rightarrow 1$ and $K_x \rightarrow K_y$ the

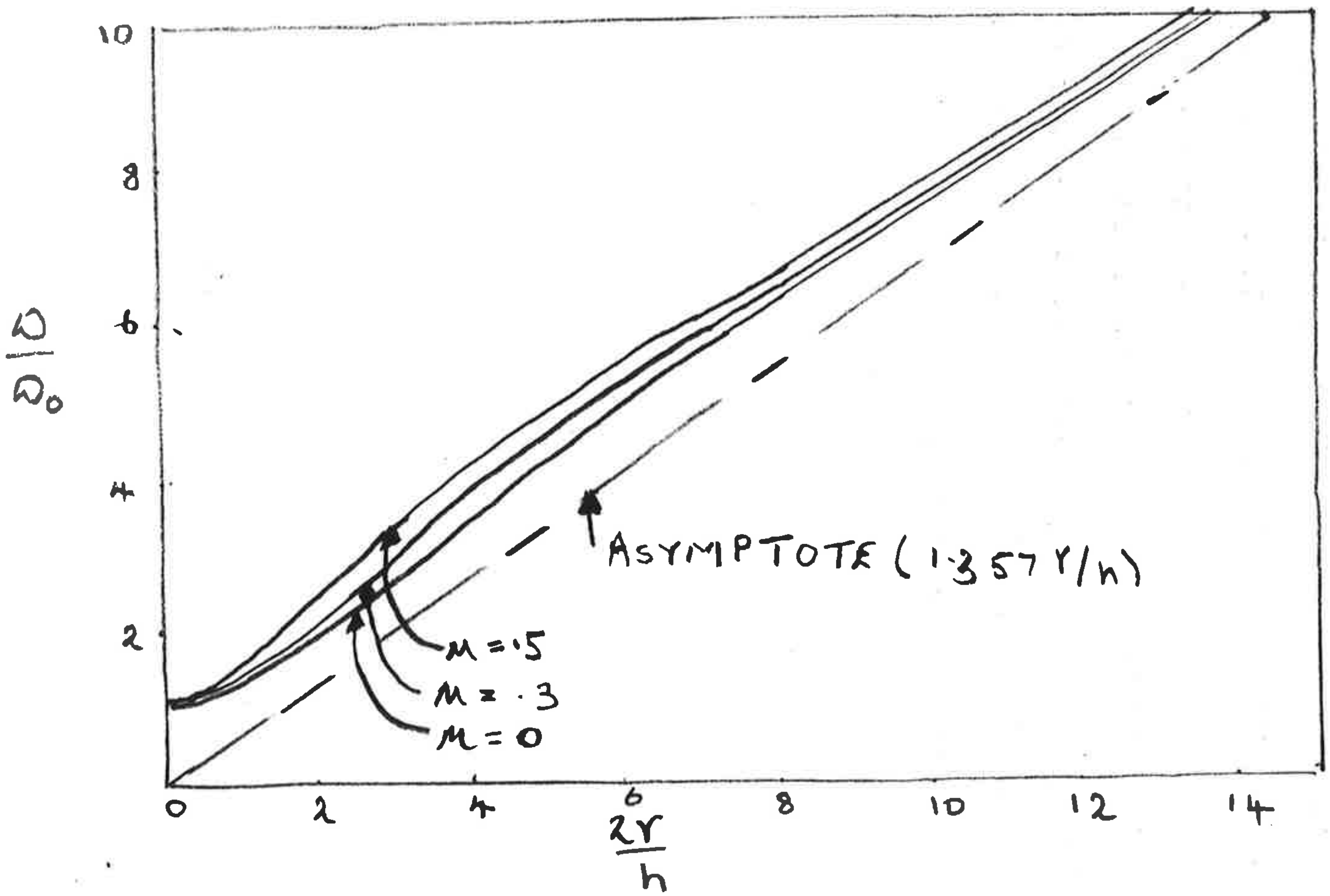

Fig. 25.

two solutions coincide. Further in the limiting case as K_x and K_y tend to zero the value of the fundamental frequency coincides exactly with that obtained in [37] for the free vibration of a clamped elliptic plate.

Table 12 gives the values of $\omega a^2/(Eh\lambda_0^4/12\rho)^{\frac{1}{2}}$ for various values of Poisson's ratio, μ; aspect ratio, a/b, and $\frac{2\gamma}{h}$, which is a measure of the shallowness of the shell. The ratio ω/ω_0 is shown in figure 25. for various values of μ, and $\frac{2\gamma}{h}$.

TABLE 12

$\frac{2\gamma}{h}$	$\frac{\omega a^2}{(Eh\lambda_0^4/12\rho)^{\frac{1}{2}}}$ $\mu=0$						
	a/b=1.0	1.1	1.2	1.5	2.0	3.0	10
0	1.00	1.11	1.23	1.68	2.72	5.74	61.44
.5	1.08	1.20	1.33	1.818	2.93	6.20	66.36
1.0	1.31	1.45	1.61	2.21	3.56	7.53	80.49
1.5	1.61	1.78	1.98	2.71	4.37	9.25	98.92
2.0	1.94	2.15	2.39	3.27	5.27	11.14	111.92
2.5	2.31	2.56	2.84	3.89	6.27	13.27	141.94
3.0	2.67	2.96	3.28	4.50	7.25	15.34	164.00
3.5	3.06	3.39	3.77	5.15	8.31	17.58	188.02
5.0	4.18	4.62	5.14	7.04	11.35	24.01	256.84
10	7.50	8.31	9.22	12.63	20.37	43.08	460.83
20	14.00	15.50	17.22	23.58	38.02	80.42	860.21

$\frac{2\gamma}{h}$	$\frac{\omega a^2}{(Eh\lambda_0^4/12\rho)^{\frac{1}{2}}}$ $\mu=.3$						
	a/b=1.0	1.1	1.2	1.5	2.0	3.0	10
0	1.05	1.16	1.29	1.77	2.85	6.02	64.41
.5	1.15	1.27	1.41	1.93	3.12	6.60	70.60
1.0	1.40	1.55	1.72	2.36	3.80	8.04	186.62
1.5	1.75	1.94	2.15	2.95	4.75	10.05	107.53
2.0	2.16	2.39	2.66	3.64	5.87	12.41	132.72
2.5	2.57	2.85	3.16	4.33	6.98	14.76	157.91
3.0	2.99	3.31	3.68	5.04	8.12	17.18	183.91
3.5	3.40	3.77	4.18	5.73	9.23	19.53	208.91
5.0	4.51	4.99	5.55	7.59	12.25	25.91	277.11
10	7.62	8.44	9.37	12.83	20.69	43.73	468.20
20	14.1	15.61	17.34	23.74	38.29	81.00	866.36

TABLE 12 (CTD.)

$\frac{2\gamma}{h}$	$\frac{\omega a^2}{(Eh\lambda_\theta^4/12\rho)^{\frac{1}{2}}}$ $\mu=.5$						
	a/b=1	1.1	1.2	1.5	2.0	3.0	10
0	1.15	1.29	1.42	1.94	3.14	6.63	70.95
.5	1.28	1.42	1.57	2.16	3.48	7.35	78.65
1.0	1.59	1.76	1.96	2.68	4.32	9.13	97.70
1.5	2.00	2.21	2.46	3.37	5.43	11.49	122.89
2.0	2.42	2.68	2.98	4.08	6.57	13.90	148.69
2.5	2.87	3.18	3.53	4.83	7.80	16.50	176.34
3.0	3.32	3.68	4.08	5.59	9.02	19.07	203.99
3.5	3.73	4.13	4.61	6.28	10.13	21.45	222.19
5.0	4.12	5.42	6.01	8.23	13.28	28.09	300.46
10	7.82	8.66	9.62	13.17	21.24	44.92	49.49
20	14.2	15.73	17.46	23.91	38.56	81.57	872.50

CHAPTER IX

CONCLUSION

From the analysis presented in the previous chapters it is apparent that one can obtain theoretically the expression for the constant deflection contours associated with the various boundary value problems of plates or shells of arbitrary shape. Once this expression is obtained our problem reduces to the solution of an ordinary differential equation, which does not pose any problem either analytically or numerically. Indeed the numerical solution to such equations has been the subject of much research, with the result that there are now numerous efficient and accurate numerical techniques available. Thus the method of constant deflection lines provides a severe challenge to most of the existing approximate methods used in plate and shell analysis. Indeed the problems considered supply ample evidence of the effectiveness of the method.

Several possible extensions of the work done in this thesis have been suggested. It is apparent that the method may be readily extended to include the bending of a shallow shell on an elastic foundation, and the transverse vibration, and buckling of a homogeneous membrane. It may also be possible to extend the method to include the large amplitude vibration of a shallow shell. However, in this case the success of the method will depend strongly upon whether it is possible to solve separately for the stresses, and the deflection as is the case when considering the large amplitude deflection of a thin, elastic plate using Berger's analysis.

It would also be interesting to see whether the method is applicable to the analysis of thick plates, and to the analysis of plates composed of viscoelastic or, plastic material.

APPENDIX 1.

By neglecting the effect of the term $\oint_{C_u} \frac{\partial M_{nt}}{\partial s} ds$

the term

$$L[w] = \frac{d^3w}{du^3} \oint_{C_u} R ds + \frac{d^2w}{du^2} \oint_{C_u} F ds + \frac{dw}{du} \oint_{C_u} G ds$$

$$= \oint_{C_u} \left(Q_n - \frac{\partial M_{nt}}{\partial s} \right) ds \qquad (1\text{-i})$$

becomes

$$L[w] = \oint_{C_u} Q_n ds$$

$$= \oint_{C_u} (Q_x \cos \alpha + Q_y \sin \alpha) ds, \qquad (1\text{-ii})$$

so that applying Green's theorem we obtain

$$L[w] = -\iint_{\Omega_u} \left(\frac{\partial}{\partial x} Q_x + \frac{\partial}{\partial y} Q_y \right) d\Omega. \qquad (1\text{-iii})$$

Making use of the well known expressions for Q_x and Q_y we finally obtain

$$L[w] = \iint_{\Omega_u} D\nabla^4 w \, d\Omega, \qquad (1\text{-iv})$$

so that equation (2.2.6) reduces to

$$\iint_{\Omega_u} (D\nabla^4 w - q) d\Omega = 0. \qquad (1\text{-v})$$

APPENDIX 2.

If we multiply equation (3.1.8) by $i = \sqrt{-1}$, and add to equation (3.1.7) we obtain

$$\iint_{\Omega_u} q(z-\xi)\,d\Omega = 0, \qquad (2\text{-}i)$$

where

$$z = x+iy, \quad \text{and} \quad \xi = \kappa_1+i\kappa_2. \qquad (2\text{-}ii)$$

And so when the load is uniform upon applying Green's theorem we obtain the requirement that

$$\oint_{C_u} (z-\xi)\overline{(z-\xi)}\,dz = 0, \qquad (2\text{-}iii)$$

which making the coordinate transformation

$$\left.\begin{aligned} X &= x-\kappa_1 \\ Y &= y-\kappa_2, \end{aligned}\right\} \qquad (2\text{-}iv)$$

transforms to

$$\oint_{C_u} r^2\,dZ = 0, \qquad (2\text{-}v)$$

where (r,θ) are polar coordinates defined by

$$\left.\begin{aligned} X &= r\cos\theta \\ Y &= r\sin\theta \end{aligned}\right\}, \qquad (2\text{-}vi)$$

and

$$Z = z-\xi. \qquad (2\text{-}vii)$$

Consequently any function $u(r,\theta)$ which satisfies equation

(2-v) as well as the required boundary conditions on u, must be an admissible form for the lines of constant deflection.

For example, if the plate is clamped then expressing the function $u(r,\theta)$, obtained in section 3.2, in power series form we obtain

$$u(r;\theta) = \sum_{n=0}^{m} r^n(a_n \cos n\theta + b_n \sin n\theta) + cr^2$$
$$= \sum_{n=0}^{m} \frac{a_n}{2} (Z^n + \overline{Z}^n) + \frac{b_n}{2i} (Z^n - \overline{Z}^n) + cZ\overline{Z}, \qquad (2\text{-viii})$$

where m is a positive integer, and where the terms involving $\log r$ and r^{-n} have been omitted since we require u to be finite at $r=0$. Thus as an alternative form for the curves $u(r,\theta)$ = const. we could consider

$$Z\overline{Z} = F(Z,u), \qquad (2\text{-ix})$$

where the function $F(Z,u)$ is some as yet unknown function of the two variables Z and u. This function can be obtained by solving equation (2-viii) for $Z\overline{Z}$ in terms of Z and u. Since equation (2-viii) may be considered as a polynomial in $Z\overline{Z}$ of degree m then applying the fundamental theorem of algebra, we must obtain m roots. Let us denote these roots by $F_1(Z,u)$, $F_2(Z,u)$,....,$F_m(Z,u)$ respectively. Then for $Z \neq 0$ these roots must be bounded. However $Z = 0$ may be a singular point. But since equation (2-viii) may also be interpreted as a polynomial in $\overline{Z}$ of degree m, with bounded coefficients, then for all bounded values of Z the roots of this equation will have a bounded modulus. And hence

$$Z\bar{Z}\Big|_{Z=0} = 0. \qquad (2\text{-x})$$

Consequently the roots $F_1(Z,u)$, $F_2(Z,u)$,....$F_m(Z,u)$ are bounded for all bounded values of Z. And so on the curve C_u, $F_1(Z,u)$, $F_2(Z,u)$,...,$F_m(Z,u)$ will be bounded functions of Z for all values of Z such that the point of Z for all values of Z such that the point $(Rl(Z), Img(Z))$ lies within the contour. Let us denote these functions by $g_{1c}(Z)$, $g_{2c}(Z)$,...,$g_{mc}(Z)$ respectively, i.e.

$$\begin{aligned} F_1(Z,u)\Big|_{C_u} &= g_{1c}(Z) \\ \vdots \quad & \quad \vdots \\ F_m(Z,u)\Big|_{C_u} &= g_{mc}(Z) \end{aligned} \qquad (2\text{-xi})$$

From equation (2-viii) we see that u is infinitely differentiable with respect to both X and Y and that its derivatives are continuous. Thus along each closed contour C_u, r^2 and its derivatives with respect to the path length s will be continuous. However since

$$ds = \frac{u_Y + iu_X}{\sqrt{t}}\, dZ \qquad (2\text{-xii})$$

this implies that r^2 and its derivatives with respect to Z must also be continuous. And so on the curve C_u $r^2 = Z\bar{Z}$ must be a single function of Z, say $g_c(Z)$, i.e.

$$r^2\Big|_{C_u} = Z\bar{Z}\Big|_{C_u} = g_c(Z). \qquad \text{(2-xiii)}$$

However, since $g(\bar{Z})$ is bounded throughout the region contained by the curve C_u it is analytic in that region, and so

$$\oint_{C_u} r^2 dZ = \oint_{C_u} g_c(Z)dZ$$

$$= 0. \qquad \text{(2-xiv)}$$

Furthermore in a process identical to that discussed above we can show that if u has the form

$$u = UV, \qquad \text{(2-xv)}$$

where U and V separately satisfy

$$\nabla^2 U = -2, \qquad \text{(2-xvi)}$$

and

$$\nabla^2 V = -2, \qquad \text{(2-xvii)}$$

then on the curve C_u, r^2 can again be represented by an analytic function of Z, and so

$$\oint_{C_u} r^2 dZ = 0. \qquad \text{(2-xix)}$$

Consequently the form of u as given by equation (3.2.19) is an admissible form for the lines of constant deflection for a uniformly loaded, simply supported plate.

APPENDIX 3

Using Green's theorem the line integral $\oint_{C_u} (z-\xi)\sqrt{t}\,ds$ can be expressed as

$$\oint_{C_u} (z-\xi)\sqrt{t}\,ds = -\iint_{\Omega_u} [(z-\xi)\nabla^2 u + u_x + iu_y]\,d\Omega, \qquad (3\text{-i})$$

so that

$$\frac{d}{du}\oint_{C_u} (z-\xi)\sqrt{t}\,ds = \oint_{C_u} \frac{(z-\xi)\nabla^2 u}{\sqrt{t}}\,ds + \oint_{C_u} \frac{(u_x+iu_y)}{\sqrt{t}}\,ds. \qquad (3\text{-ii})$$

However since $ds = \dfrac{u_y+iu_x}{\sqrt{t}}\,dz$ the second integral on the right hand side of equation (3-ii) becomes

$$\oint_{C_u} \frac{u_x+iu_y}{\sqrt{t}}\,ds = \oint_{C_u} \frac{(u_x+iu_y)(u_y+iu_x)}{t}\,dz$$

$$= i\oint_{C_u} dz$$

$$= 0, \qquad (3\text{-iii})$$

so that equation (4.1.12) reduces to

$$\frac{d}{du}\left(\frac{dw}{du}\oint_{C_u} (z-\xi)\sqrt{t}\,ds\right) = 0. \qquad (3\text{-iv})$$

APPENDIX 4

Making use of the transformation

$$\varepsilon_s = \varepsilon_x \sin^2\alpha + \varepsilon_y \cos^2\alpha + 2\,\varepsilon_{xy} \cos\alpha \sin\alpha \tag{4-i}$$

and the relationships

$$\begin{aligned} \varepsilon_x &= (N_x - \mu\, N_y)/Eh \\ \varepsilon_y &= (N_y - \mu\, N_x)/Eh \\ \varepsilon_{xy} &= (1+\mu) N_{xy}/Eh, \end{aligned} \tag{4-ii}$$

equation (7.4.33)iii) reduces to

$$\varepsilon_s = B_1 \frac{d^2\phi}{du^2} + B_2 \frac{d^2\phi}{du} = 0, \tag{4-iii}$$

where

$$B_1 = \frac{t}{Eh}, \tag{4-iv}$$

and

$$B_2 = -\frac{1}{Eht}[+\mu\, u_{xx}\, u_y^2 + \mu\, u_{yy}\, u_x^2 - u_{xx}\, u_x^2 - u_{yy}\, u_y^2 - 2(1+\mu) u_{xy}\, u_x u_y]. \tag{4-v}$$

Proceeding as in [35] equation (4-iii) ultimately simplifies to

$$(1-u) \frac{d^2\phi}{du^2} - (1-\mu) \left.\frac{d\phi}{du}\right|_{u=0} = 0, \tag{4-vi}$$

which in terms of the new variable f transforms to

$$\frac{d^2\phi}{df^2} - \frac{\mu}{f} \left.\frac{d\phi}{df}\right|_{f=1} = 0. \tag{4-vii}$$

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