THE APPLICATION OF CORRELATION TECHNIQUES TO CHECKING AND ADJUSTING MATHEMATICAL MODELS

by

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CONTENTS

SUMMARY

A mathematical approach to the problem of checking and adjusting a mathematical model of a physical device is presented. A measure of the adequacy of such a model is proposed and a detailed study is made of the case where both the system and model are linear transformations, while the inputs are realisations of stationary random processes with rational spectra. The difference between the weighting function of the system and that of the model is then the error in the weighting function of the model. If the records were error free and of unlimited length this difference could be found as the solution of a Wiener-Hopf integral equation. In the case of an input whose spectrum is sufficiently flat and of unit power density, the required solution may be found in terms of the cross-correlation function of the input and the difference between the outputs of the system and model. The particular inputs considered may all be derived by linearly transforming a realisation of a process having a flat spectrum; this problem may therefore be reduced to the determination of a cross-correlation function.

In practice the records are of limited length; the problem may then be reduced to one of estimating the above cross-correlation from a finite length of data. A formula is obtained for the variation of the expected adequacy of the corrected model with the length of data available for model

correction. The weighting function of the model should, in general, only be adjusted for an interval of its argument which is much smaller than the length of data available for model correction. A method for determining this interval is described and a simple formula, which will be a useful guide in most cases, is derived.

The effect of errors in the records is also investigated. It is shown that, providing certain statistical information concerning the errors is available, compensation for these errors may be made. The effect of recording errors on the expected adequacy of the corrected model then depends on the length of data available according to expressions derived in the text.

If the system has several inputs which are not cross-correlated, the problem may be reduced to the case of a single input with errors in the recordings. If the inputs are correlated the cross-correlation technique may be used iteratively to improve the adequacy of the corrected model, but the model weighting functions will not necessarily converge to the system weighting functions.

Experimental evidence is produced to support the theoretical work and extensions of the work to other types of system and input are discussed in the final chapter.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief the thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

R.G. KEATS

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CHAPTER 1.

INTRODUCTION

The greatly increased effort, during the last fifteen years, directed towards the design and development of physical devices of considerable complexity, for example guided weapons, has provoked a correspondingly increased interest in the application of mathematics to the study of such devices. The term "systems analysis" has been widely used to describe these studies and there is now a vast literature in this field, [1].

An important aspect of systems analysis is the description of a physical device by means of a mathematical model. Such a model can be used to obtain information about the behaviour of the physical device it represents. In some cases this information is only qualitative; but a prime requirement of each model, to be considered in this thesis, is that it be adequate, in the sense that it can be used to obtain useful, quantitative information concerning the performance of the physical device it represents. Before such a model may be used with confidence it must be checked and, if necessary, adjusted to ensure that it is adequate. The purpose of the work described in this thesis is to study the application of correlation techniques to the checking and adjusting of a mathematical model.

Each physical device to be considered may be characterized by mathematical transformations relating the input and output of each of its components. These transformations will be called the "system transformations" or simply the "system"* . If a device has only one or two components and the system transformations are elementary the system is called a "simple system". An example of a simple system is the relation between the input and output voltages of the electrical circuit illustrated in fig.1. If, on the other hand, the system transformations are numerous and complicated the system is called a complex system. Examples of complex systems are those which characterize guided weapons, chemical and other industrial plants, business undertakings, the economy of a country, modern aircraft, automatic computers, etc.

Block diagrams as shown in fig.2 are often of great assistance in the study of complex systems. In the case of a guided weapon [2] the blocks of the diagram may represent:-

input	–	information concerning the present position of the target;
comparison	–	of the weapon and target position to determine the error in the weapon's present course; any such error will result in a demand for control;

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* The term "system" is currently used both for the device and the transformations which characterize the device. In this thesis its use will be restricted to the latter.

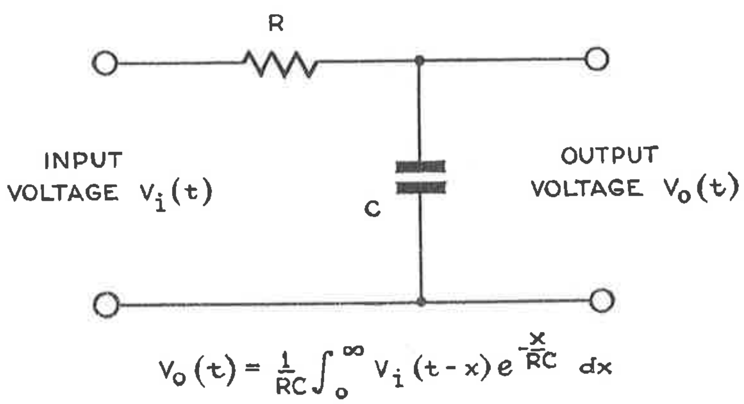

FIGURE 1. EXAMPLE OF A SIMPLE SYSTEM

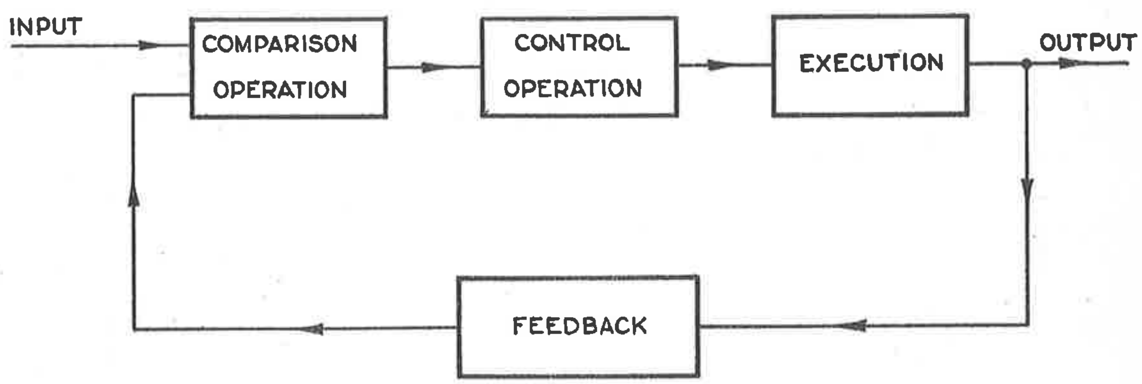

FIGURE 2. BLOCK DIAGRAM OF SYSTEM

control - of the actuators which move the control surfaces;

execution - movement of the control surfaces;

feedback - information concerning the present position of the weapon;

output - movement of the weapon.

In general, each block may be subdivided and in the case of a complex system the number of individual blocks will be large, perhaps as many as one hundred or more. The transformations characterizing individual blocks will be called components of the system; each component will have one or more inputs and outputs which are functions of time.

Recently, there have appeared a number of papers [2-5] describing the evaluation of complex systems using mathematical models; an earlier paper, Siefert [6], described the role of computing machines in the analysis of such systems. Work of this nature was rarely undertaken before 1950. Before that time mathematical models of complex systems were, with one notable exception mentioned below, greatly simplified models of the system they purported to represent. All but the highly relevant was discarded in the interests of writing a model which could be manipulated using the mathematical tools available. Non-limear equations were linearised, friction was assumed either viscous or non existent, backlash was ignored and small angle approximations were freely used. There were two major difficulties associated with the use of more complicated models:

(i) the enormous computing effort required to obtain useful information from the model was seldom available;

(ii) even should one or two numerical solutions of a set of complicated equations and inequalities be found, these solutions would be virtually useless, unless they could be given some, well understood, physical significance.

The advent of the large automatic computer has, in most cases, disposed of the first difficulty; while the work reported by Lawrence and others [2-5], shows that the second is not always valid.

There is, however, one field, viz. accounting, in which these two difficulties were faced and overcome before the advent of the large automatic computer. The extremely detailed system of book-keeping has been built up because no simple mathematical model of a business would suffice. To provide against fraud, and possibly for other valid reasons, complicated models of business undertakings are essential. Fast automatic computation considerably increases the usefulness and ease of manipulation of such models, so it is not surprising that one of the first and probably the most successful application of automatic computers has been to business accounting. On the other hand, in those activities where a mathematical model is prized mainly for its elegance and simplicity, there is sometimes considerable reluctance to take full advantage of the automatic computer in order to study the details missing from a simple model.

1.1 The development and use of a mathematical model

Before proceeding it will be useful to distinguish four types of component and their corresponding inputs and outputs.

(i) A component of a physical device, e.g., an actuator, an electrical network or a bulk store. The corresponding inputs and outputs may be shaft rotations, voltages or goods.

(ii) A component of a system; a mathematical transformation relating functions of time which are the input and output of the system component.

(iii) A component of a mathematical model; a mathematical transformation relating functions of time which are the input and output of the model component.

(iv) A component of a computer; a computing unit; e.g. a D.C. amplifier set up as an integrator, or a storage unit of a digital computer, with their electrical or other inputs and outputs.

This thesis is not concerned with physical components as such, but in their characterization as a system. It will be assumed that there is a one to one correspondence between the inputs and outputs of a physical component and those of its characterization as a system. If the voltages or shaft rotations, etc., which are the inputs and outputs of a physical component are recorded, then, assuming no recording errors, the functions of time so represented will be the inputs and outputs of the corresponding system component.

The problem discussed by Lawrence and others [2-5] is essentially one of describing, or evaluating, an existing physical device. Most problems in physics are of this type as are many problems in other fields [1], such as engineering, economics, accountancy or psychology. Four stages in this evaluation process have been recognised and described elsewhere, [1], [3]; only those points which are important to this thesis are mentioned below.

1.1.1 The formulation of a mathematical model

In the formulation of a mathematical model some system components may be ignored while others are combined and modified. In so doing it is possible to destroy the physical picture, in the sense that very few, if any, of the inputs and outputs of a model component represent inputs and outputs of a system component. However it is essential for the checking procedure described below that, in a number of cases, the inputs and outputs of a model component do represent inputs and outputs of a system component, and that recordings of the corresponding physical inputs and outputs will be available.

It will be assumed that the model is suitable for programming for an automatic computer.

1.1.2 Programming the model for a computer

Analogue computers were used for the work described by Lawrence and others [2], and Biggs and Cawthorne [5], but as shown by Keats [3] they are not necessary for such work; modern digital machines may in some cases be more suitable. The essential requirement from the computer is that records of certain physical inputs may be applied as inputs to the model as programmed, and records of the corresponding model outputs obtained. The analogue machine and the digital differential analyser meet this requirement by using computing components to represent components of the physical system. However, this is only one way and not necessarily the best way of meeting the requirement. Fast digital machines are quite capable of providing the required inputs and outputs if they are programmed to do so.

1.1.3 Checking the model

A procedure for checking the model is described by Biggs [4] and is represented essentially in fig.3. The input and output of a physical component are recorded while the device is in operation; the recorded physical input is then applied to the corresponding model component and its output is also recorded. The two output records, one from the physical component and the other from the model component, are then compared. The comparator has often been an engineer and the comparison process has been carried out by superimposing one output record on the other as illustrated in fig.4. After studying such a figure and observing the differences between the two records, a skilled engineer with a good knowledge and understanding of the system can often suggest methods for improving the model or, alternatively, decide that this component of the model is adequate for its

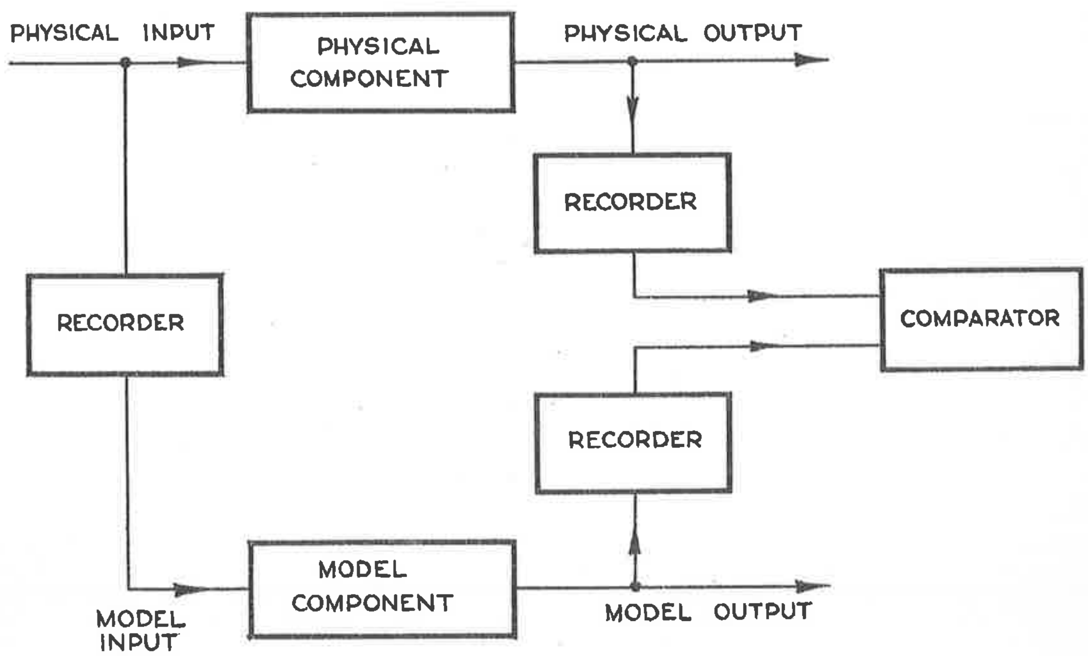

FIGURE 3. ILLUSTRATING A PROCEDURE FOR CHECKING THE MODEL

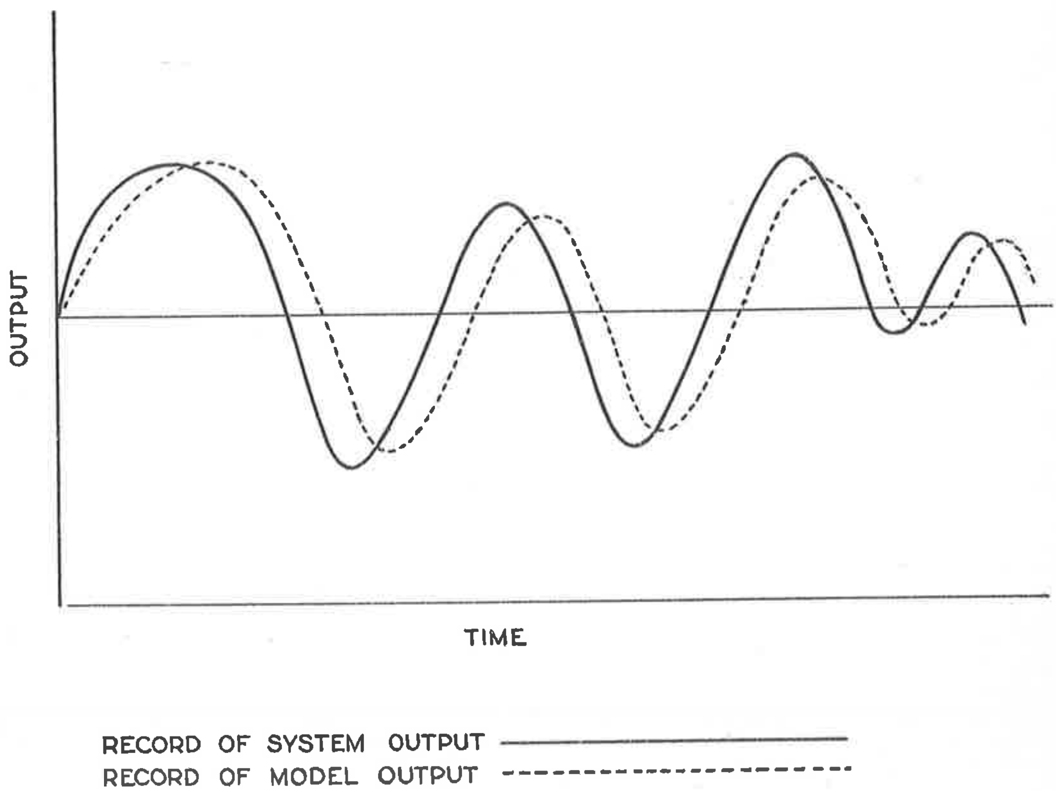

FIGURE 4. COMPARISON OF SYSTEM AND MODEL OUTPUT

purpose. Differences between the records are usually interpreted in terms of gain or phase shift errors in the model, although time dependent and non-linear effects may also be detected.

In some cases abnormal inputs to the physical component, such as step functions, may be used to assist the checking process; very often, however, it is not possible to interfere with the normal operation of the device. The work of this thesis applies to those cases where no abnormal inputs may be used.

Although the recordings of the model output may be assumed free from error, there will often be significant errors in the recordings of the input and output of a physical component. Records of information telemetred from a guided weapon, for example, are seldom free from errors which may be significant.

1.1.4 Applications of the model

Many of the uses of a checked mathematical model of a complex system are discussed in [2], [3] and [5] and also by Apostel [7]. In particular, such a model is often the only practical means for studying the performance of a physical device under certain conditions which may be of great interest and importance; experiment under these conditions being virtually impossible. For example, a guided weapon sortie against enemy aircraft is not an experiment which may be conducted in peace time; similarly experiments which interfere with the economy of a nation will seldom be permitted.

1.2 Antecedent and related work

Simulation is now well established as a powerful aid to the study of complex systems, [8], [9]. The comprehensive bibliography given by Morgenthaler [8] contains many examples of the application of simulation to a wide range of activities. Recent examples include: computer studies of the economy of India by Holland, [10]; biological applications discussed by Dallos and Jones [11], De Land [12], and Watt and Young [13]; simulation of business firms such as that described by Bonini [14]. For many years simulation has played an important part in the design and development of missiles and this application has extended naturally to studies of satellites and space vehicles, [15], [16].

The opportunity for model checking and adjustment in the manner described in section 1.3 does not arise in all applications of simulation; for example the physical device may not yet exist, [15]. However the mathematical model to be used in an evaluation exercise such as that described by Biggs and Cawthorne [5] must be checked empirically, and if necessary adjusted, to ensure that it adequately describes the system which it represents.

The mathematical approach to model checking and adjustment, described in this thesis, derives from the work of Wiener [17] on the prediction and filtering of stationary time series. Many simplifications and extensions of Wiener's theory have appeared. Levinson [17, appx.C], Bode and Shannon [18] and Darlington [19] have presented mathematically simpler versions of the theory; while Zadeh and Ragazzini [20], Davis [21] and Shinbrot [22] have considered its extension to non stationary problems. Most of this early work is now available in text books [23 - 26]. Bendat [27] gives a comprehensive bibliography of work carried out before 1959, while developments in prediction theory from 1957 to 1960 are summarised by Zadeh [28].

Kalman [29], in 1960, presented a new approach to the prediction and filtering problem applicable to both stationary and time variant problems. He observed that the prediction and filtering problem had only been effectively solved in the case of Gauss-Markov processes; i.e. processes which may be generated by exciting a linear dynamic system with Gaussian "white noise". An exposition of this work and work carried out with Bucy [30], has been given by Kalman [31].

A problem which is closely related to the prediction and filtering problem, is that of approximating a linear system using records of the input and output from the system [24,P.342]*, [32 - 34]. In 1950 Lee [24,P.342], showed that the weighting function of a linear system with constant parameters may be obtained by cross correlating its input and output when it is excited by "white noise". Modifications of this technique using specially selected inputs have been suggested; Anderson and Buland [35] describe the use of

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*References to books will include the number of the first relevant page, if appropriate.

specially selected samples of "discrete-interval binary noise" in a problem in adaptive control. Turin [36], Levin [37] and Lindenlaub and Cooper [34] have considered the estimation of the weighting function of a linear system in the presence of noise; while Grinten and Krijger [38] have recently described an analogue method for obtaining the step response of a linear filter knowing the auto-correlation function of its input and the cross correlation of its input and output.

Adaptive control or self optimization has received considerable attention during the last decade; work in this field prior to 1961 is reviewed by Jacobs [39]. Mathematical foundations for studies in adaptive control and related fields, based on the theory of dynamic programming, have been laid down by Bellman and Kalaba [40]. One aspect of adaptive control is the frequent, or continuous, estimate of a system which is slowly changing in a manner difficult or impossible to predict. On the basis of such an estimate the system may be restored to an acceptable form [35]. The work of Kerr and Surber [41] on the precision of an estimate of the impulse response of a linear system based on short, normal operating records arises from this aspect of adaptive control.

In the same way as Wiener's approach [17] to prediction and filtering has suggested methods for estimating a linear system using input and output records, Mayne [42] has shown that Kalman's approach [31] may also form the basis for estimating the parameters of a linear system, providing full knowledge of the state of the system at any instant is available.

The work discussed above is based mainly on methods for studying linear systems. Interest in non-linear methods has increased greatly during the past decade; such methods are essential when a problem is non-linear and can not be satisfactorily linearized. Bram and Saaty [43] give comprehensive bibliographies of work involving non-linear methods and their application prior to 1963; much further work has been reported since then [44]. However although some types of non-linear systems will be considered in this thesis, the methods used are essentially linear; non-linear methods may be required for further work on model checking and adjustment.

1.3 Scope of the present work

The process, described in section 1.1.3, relies heavily on the intelligent use of engineering judgment. The ultimate aim of the present work is to develop, and study, a mathematical approach to this problem of checking and adjusting a model. Such an approach would not only reduce considerably the reliance on engineering judgment, but would also provide a basis for answering important subsidiary questions such as: "How does the adequacy of the adjusted model depend on the length of the records available for checking and adjusting the model?"

A useful first step in developing a mathematical approach is to describe the intuitive approach of the engineer in a way which can be interpreted mathematically.

The engineer examines the two output records, one from the physical component and the other from the model component, looking for agreement, or lack of agreement, between them. Good agreement indicates an adequate model. If the agreement between the two records is, in his opinion, insufficient to indicate an adequate model, he will then examine the recorded input to the physical component, looking for some explanation for the discrepancy between the two output records. In other words he is looking for correlation between the input and the difference between the two output records. This correlation may also be investigated mathematically.

This thesis is primarily concerned with presenting the problem and studying the case of a stationary linear system; however the extension to non-linear systems will also be discussed. A mathematical investigation of the cross correlation mentioned above is carried out in the case where no abnormal inputs, such as the test signals used by Buland and Anderson [35], may be applied to the physical device.

Techniques other than cross correlation might also be used in the present problem. Lindenlaub and Cooper [34] have discussed three techniques; viz. cross correlation, matched filters mentioned by Turin [36], and sampled input output data discussed by Levin [37]. They have shown that these three techniques are equivalent to each other and also to an "ideal identifier", when they are applied to the related problem of system identification in the presence of noise. However, as mentioned by Kerr and Surber [41], correlation techniques are more suitable in cases such as the present one where comparatively long lengths of input and output records are available.

In Chapters 2 and 3 of this thesis the problem is stated more precisely and some preliminary definitions and theorems are given; most of these are well known or adaptations by the author of well known results. Part (i) of theorem 4, for example, is only a particular case of the work of Lee [24, P.342]. Apart from Keats [45], however, no previous statement or discussion of parts (ii) and (iii) of theorem 4 has appeared.

The case of a stationary linear system is introduced in chapter 4; alternative approaches to this case are considered and reasons are given for preferring one of these. Not unexpectedly it is shown that the problem may then be reduced to the solution of a Wiener-Hopf type integral equation. Some well known methods for attacking the Wiener-Hopf equation are mentioned and an example is given. The advantages and consequences of a "whitened" input are also discussed in preparation for the theoretical work in the following chapters. The iterative process and its convergence, which are studied in section 4.1.1, have not been mentioned by other authors.

The main contribution of this thesis is contained in chapters 5,6 and 7. Each record, available for model correction, is considered to be portion of a realisation of a stationary normal random process. Theoretically each realisation of this process may be "whitened" sufficiently for the problem to be reduced to the estimation of a cross correlation function. The effects of finite length records, and errors in these records, on the adequacy of the corrected model is discussed in detail, and mathematical expressions for these effects are obtained; some examples are given. Many of the techniques used have been applied to similar problems, but their application to the present problem has not appeared elsewhere.

The details and results of experiments carried out to confirm the theoretical work are reported in chapter 8, while in chapter 9 extensions of the work are briefly considered.

1.4 Notation

Although the subscripted notation used, especially in the case of correlation functions, becomes a little unwieldy in chapters 6 and 7, it seems, to the author, to be more meaningful than a, possibly less cumbersome, use of figures. Thus the interpretation of $\rho_{IS,OM}(\tau)$ as the cross correlation function of the input to the system and the output of the model requires less effort than the interpretation of an alternative such as $\rho_{3,6}(\tau)$. Nevertheless some compromise has been made in chapter 6. To alleviate this problem subscripts will be omitted where they are not relevant to the work of a particular chapter and a complete list of notation appears after the appendix.

CHAPTER 2.

MATHEMATICAL STATEMENT OF THE PROBLEM

In this thesis the expression "checking the model" will mean a procedure such as that outlined in 1.1.3; i.e. a comparison of two outputs, one representing system output and the other model output, to an input which, apart from recording errors, is the same for both system and model. A record of this input will always be available for consideration when making the comparison.

An input or output to a system, or model, will be regarded as a realisation $X(t,w_i)$ of a measurable random process $X(t,w)$ [46,P.60], a function of two variables, such that t belongs to the set of real numbers and represents time, while w is a point in some probability space W on which is defined a probability measure P, [46,P.605]. For fixed $w = w_i$, $X(t,w_i)$ is then a real valued function of time which is Lebesgue measurable for almost all choices of w_i. Accordingly it will be assumed that input records taken from the physical device are, apart from recording errors, values of such a realization for some interval of t. For fixed $t = t_j$, $X(t_j,w)$ is a random variable (measurable function) with expectation $\int_W X(t_j,w)dP$ and higher moments $\int_W X^n(t_j,w)dP$ assuming these exist. Similarly errors introduced in a recording will be regarded as a realization of some measurable random process $N(t,w)$.

2.1 Adjusting the model

This term will be restricted to include only minor alterations to the model. For example, an alteration to a component of a model which changed the mean square output by more than 50 per cent would not, in general, be called an adjustment. Such an alteration would usually be too drastic to be made without further study and experiment on the physical device; the resulting change to the model is then in the nature of a new formulation. However 50 per cent is rather arbitrary and some tolerance could be allowed.

2.2 Causal transformations [47,P.85]

A model or system component is a causal transformation. A transformation T, whose domain is $\mathcal{C}$, is said to be "causal" if, given any function $f(t) \in \mathcal{C}$ such that

$$f(t) = 0, \qquad t \leqslant t_1,$$

then

$$T[f(t)] = 0, \qquad t \leqslant t_1.$$

Such transformations may be mathematical representations of physical phenomena, typically:

(i) stationary, linear, phenomena causing gain and phase shift between an input and its corresponding output;

(ii) phenomena, which vary with time, (some parameters of T are then time dependent);

(iii) various non-linear phenomena, e.g. physical limits on the amount of rotation of a shaft, Coulomb friction, voltage rectification.

2.3 An adequate model

This term, which is used by Lawrence and others [2], is not, of course, meaningful, unless the purpose, for which the model is adequate, is stated. In this thesis the following measure of the adequacy of a model, or component of a model, will be used.

Consider a system component T_S, which is represented in the model by T_M. For each $w_l \in W$, a probability space having probability measure P, let $I_S(t,w_l)$, a realisation of the random process $I_S(t,w)$, be an input to T_S and let $O_S(t,w_l)$ be the corresponding output. Let the output of T_M to the input $I_S(t,w_l)$ be $O_M(t,w_l)$. The adequacy of the model at $t = t_j$ will then be measured by

$$A(t_j) = 1 - \left\{ \frac{\int_W [O_S(t_j,w) - O_M(t_j,w)]^2 dP}{\int_W O_S^2(t_j,w)dP} \right\}^{\frac{1}{2}} . \qquad (2.1)$$

An adequate model will be one which satisfies certain restrictions on $A(t)$; e.g. $A(t) > 0.9$ for all t; the actual restriction depending on the purpose of the model.

A more useful quantity for many purposes is $1 - A(t)$, which will be denoted by $\bar{A}(t)$ and called the inadequacy of the model.

$$\bar{A}(t_j) = \left\{ \frac{\int_W [O_S(t_j,w) - O_M(t_j,w)]^2 dP}{\int_W O_S^2(t_j,w)dP} \right\}^{\frac{1}{2}} . \qquad (2.2)$$

The only term in $A(t_j)$ or $\bar{A}(t_j)$ which is affected by an adjustment to the model is

$$\int_W [O_S(t_j,w) - O_M(t_j,w)]^2 dP. \qquad (2.3)$$

2.4 Statement of the problem

Given: (i) a model component T_M;

(ii) a set $\mathcal{J}$ of allowed causal transformations one member of which will normally be T_M;

(iii) for some i, $(1 \leq i \leq n)$, n typically < 12, and for some interval of t,

(a) records of $I_S(t,w_i)$ and the corresponding $O_S(t,w_i)$,

(b) a record of the output of the model, $O_M(t,w_i)$, to an input which apart from recording error $= I_S(t,w_i)$;

(iv) some information regarding the recordings and their errors;

(v) a criterion of adequacy for the model component based on $A(t)$, (2.1).

Choose from the set $\mathcal{J}$ a model component such that $A(t)$ satisfies the given criterion of adequacy.

CHAPTER 3.

STATIONARY LINEAR SYSTEM AND MODEL
PRELIMINARY DEFINITIONS AND THEOREMS

This thesis is mainly concerned with system and model components which are stationary, linear, causal transformations. The following definitions and theorems will be required; most of them are well known or adaptations of well known results.

Definition 1 - Stationary* transformation [47,P.83]

Let $\mathcal{G}$ be a linear space of real valued functions $f(\cdot)$ such that if $f(t) \in \mathcal{G}$ then $f(t + c) \in \mathcal{G}$, all real c.

Define the shift operator Δ_c such that

$$\Delta_c f(t) = f(t + c), \quad f(\cdot) \in \mathcal{G}.$$

Let T be any transformation defined on $\mathcal{G}$ and satisfying $T[f] \in \mathcal{G}$.

Then T is a stationary transformation if

$$T[\Delta_c[f]] = \Delta_c[T[f]] \quad \text{for all} \quad f(\cdot) \in \mathcal{G}.$$

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* The terms "time invariant", and "constant parameter" are also used to describe such transformations.

Definition 2 - A class of stationary, linear transformations L

The symbol L will be used for causal transformations defined on a linear space of real valued functions $f(\cdot)$ such that

$$L[f(t)] = \int_{-\infty}^{\infty} f(t - u)\, W(u)\, du;$$

where $W(t)$ is real and, in general,

$$W(t) = \sum_{j=1}^{n} P_j(t)\, e^{s_j t}, \qquad t \geq 0, \qquad (3.1)$$

$$W(t) = 0 \qquad , \qquad t < 0;$$

n is a positive integer ;

$P_j(t)$ is a polynomial in t;

s_j is a complex number whose real part is negative.

$W(t)$ may also include generalized functions of the form

$$\delta^{(n)}(t - T), \qquad T \geq 0;$$

i.e. the Dirac delta function and its derivatives defined by

$$\int_{-\infty}^{\infty} f(t - u)\, \delta(u) du = f(t),$$

$$\int_{-\infty}^{\infty} f(t - u)\, \delta^{(n)}(u - T) du = f^{(n)}(t - T).$$

In general the domain of L will be the class of measurable functions $f(t)$, such that $f(t)e^{-b(\tau - t)}$ is Lebesgue integrable $(-\infty, \tau)$ for all τ and all positive real numbers b. However this domain may be further restricted if generalized functions are included in $W(t)$. Any such operator L is stationary, linear and causal.

Definition 3 - Wide sense stationary random process [25,P.15]

The idea of a random process may be extended to include $\underset{\sim}{\xi}(t,w)$; i.e. a vector with components

$\xi_1(t,w)\ldots\xi_n(t,w)$. Such a process is stationary in the wide sense if:

(i) $\int_W \xi_i(t,w)dP = \mu_i$ is finite and independent of t;

and (ii) $\int_W \xi_i(t,w),\xi_j(t+\tau,w)dP = B_{ij}(\tau)$ exists for all i and j, and is independent of t.

$\rho_{ij}(\tau)$ will be used instead of $B_{ij}(\tau)$ if and only if, the assumption is being made that $\mu_i = 0$, all i. In the case $i = j$, $\rho_{ij}(\tau)$ is called an "autocorrelation function", otherwise it is called a "cross correlation function". If $\mu_i = 0$, then

$$\rho_{ii}(0) = \int_W \xi_i^2(t,w)dP = \sigma^2_{\xi_i}$$

which is the variance of $\xi_i(t,w)$ for every t.

If $\int_{-\infty}^{\infty} |\rho_{ii}(\tau)| d\tau$ exists, the spectrum, or spectral density function of $\xi_i(t,w)$ may be defined as

$$S_{ii}(\omega) = \int_{-\infty}^{\infty} \rho_{ii}(\tau)\, e^{-i\omega\tau} d\tau, \tag{3.2}$$

and similarly a cross spectral density function

$$S_{ij}(\omega) = \int_{-\infty}^{\infty} \rho_{ij}(\tau)\, e^{-i\omega\tau} d\tau \tag{3.3}$$

may be defined if $\int_{-\infty}^{\infty} |\rho_{ij}(\tau)| d\tau$ exists.

<u>Definition 4</u> - <u>Gaussian process</u> [46,P.71]

Let $\xi(t,w)$ be a random process and consider a finite collection of random variables $\xi_i(t_j,w)$. If for all such collections the joint distribution of the random variables is normal then the process will be called Gaussian.

Ergodic process [25,P.16]

Let $\xi(t,w)$ be a Gaussian, wide sense stationary process of zero mean. Under quite general conditions such a process is ergodic, which implies that the following equations, involving mean square limits, are true.

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \xi_i(t,w)dt = \mu_i = 0, \text{ all } i. \qquad (3.4)$$

and $$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \xi_i(t,w)\xi_j(t+\tau,w)dt = \rho_{ij}(\tau), \text{ all } i \text{ and } j. \qquad (3.5)$$

Mean square limits imply limits in probability, and, accordingly it is often assumed [26,P.90], that if $\xi(t,w_k)$ is any realisation of the above ergodic process $\xi(t,w)$,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \xi_i(t,w_k)dt = 0, \qquad (3.6)$$

and $$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \xi_i(t,w_k)\xi_j(t+\tau,w_k)dt = \rho_{ij}(\tau); \qquad (3.7)$$

where the Lebesgue integrals on the L.H.S. may be approximated by Riemann sums [46,P.63].

In this thesis the assumption that $\xi(t,w)$ is an ergodic process with zero mean will imply that equations (6) and (7) hold for each realisation $\xi(t,w_k)$ considered.

Definition 3 - Stationary Gauss Markov process

A real random process, $M(t,w)$, which is Gaussian, wide sense stationary with zero mean, and whose autocorrelation function is of the form $\rho(\tau) = \sigma^2 e^{-\alpha|\tau|}$ is a stationary Gauss Markov process [25,P.154]. The parameter α is real and positive, while σ^2 is the variance of the process $M(t,w)$. The obvious abbreviation S.G.M. will be used in this thesis.

3.1 Processes derived from stationary Gauss Markov processes

Definition 6 - A type X_1 of random process

In this thesis a random process will be called a type X_1 process if it may be written in the form

$$X_1(t,w) = \int_0^\infty M(t-u,w)W(u)du + kM(t,w) , \qquad (3.8)$$

where:

(i) $M(t,w)$ is an S.G.M.;

(ii) $W(t)$ has the exponential form (1) and therefore has a Fourier transform $\phi(\omega)$;

(iii) k is real and may be zero;

(iv) the frequency response function $k + \phi(\omega)$, corresponding to the weighting function $k\delta(t) + W(t)$, has no zeros on the real axis. This restriction, which apparently has little practical significance [25,P.162] is necessary in the sequel.

Random processes such as $X_1(t,w)$ are known to exist providing

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_M(t-u)[k\delta(t) + W(t)][k\delta(u) + W(u)]dt\, du$$

exists [25,P.53]. This condition is always satisfied in the present case since $\rho_M(t-u)$ is of the form $\sigma^2 e^{-\alpha|t-u|}$.

As will be shown in theorem 1, processes of type X_1 have a rational spectrum which has no poles on the real axis. Equations (4) and (5), with $i = j$, are therefore true for such processes [25,P.21].

The assumption will be made that each realisation of a type X_1 process, considered in this thesis, may be written in the form

$$X_1(t,w_k) = L_1[M(t,w_k)],$$

where L_1 is a transformation of type L whose weighting function satisfies (ii),(iii) and (iv) above. Such transformations will be called type L_1. The realisations $X_1(t,w_k)$ and $M(t,w_k)$ will be assumed Lebesgue measurable and when multiplied by $e^{-b(\tau-t)}$, $b > 0$, the product will be assumed Lebesgue integrable, $(-\infty,\tau)$, for all τ. The quite general conditions, under which similar assumptions hold for almost all w, are discussed by Doob (46, Chapter II).

Theorem 1

A type X_1 process has a rational spectrum with no poles or zeros on the real axis.

This spectrum may be written in the form $\psi(\omega)\,\overline{\psi(\omega)}$, where $\psi(\omega) = \frac{P_1(\omega)}{P_2(\omega)}$; the polynomials $P_1(\omega)$ and $P_2(\omega)$ have the following properties:

(i) the degree of $P_1(\omega)$ is less than that of $P_2(\omega)$;

(ii) the zeros of $P_1(\omega)$ and $P_2(\omega)$ all lie in the upper half plane and are either on the imaginary axis or occur in pairs of the form $a + ib$ and $-a + ib$, $a > 0$, $b > 0$;

(iii) the coefficient of $(\omega)^n$ in each polynomial is of the form i^n multiplied by a real positive constant.

<u>Proof</u> These results are well known, [46,P.543],[25,P.161],etc. Let the frequency response function corresponding to L_1 be $k + \phi(\omega)$ where $\phi(\omega)$ is the Fourier transform of some $W(t)$ having the exponential form (1).

<u>Lemma</u>

$\phi(-is)$, the Laplace transform of $W(t)$, may be written in the form $\frac{P_3(s)}{P_4(s)}$, where the polynomials $P_3(s)$ and $P_4(s)$ have real coefficients and the degree of $P_3(s)$ is less than that of $P_4(s)$.

<u>Proof of Lemma</u> Since $W(t)$ is the sum of terms of the form $ct^m e^{s_j t}$, $Re(s_j) < 0$, then its Laplace transform is the sum of terms of the form

$$\int_0^\infty ct^m e^{s_j t} e^{-st} dt.$$

This integral exists for $Re(s) \geq 0$, since $Re(s_j) < 0$, and is equal to

$$\frac{\partial^m}{\partial s_j{}^m} \int_0^\infty ce^{(s_j - s)t} dt$$

$$= \frac{c(m!)}{(s-s_j)^{m+1}} \ . \qquad (3.8)$$

Hence the Laplace transform of $W(t)$ is rational and may be written in the form $\frac{P_5(s)}{P_6(s)}$, where the degree of $P_5(s)$ is less than that of $P_6(s)$. Whence

$$\phi(-is) = \frac{P_5(s)}{P_6(s)} \frac{\overline{P_6(s)}}{\overline{P_6(s)}} \ . \qquad (3.9)$$

Further $\phi(-is) = \int_0^\infty W(t)e^{-st}dt$ is real for all real positive s, since $W(t)$ is real. Therefore (9) is the ratio of two polynomials which are real for all real positive s. Any such polynomial $P(s)$ has real coefficients; otherwise the polynomial $P(s) - \overline{P(s)}$ would be zero for all real positive s and at least one of its coefficients would not be zero. Hence the expression (9) is in the required form $\frac{P_3(s)}{P_4(s)}$ and the lemma is proved.

It follows from this lemma that:

(a) $k + \phi(-is)$, and therefore $k + \phi(\omega)$, is rational, and when written as the ratio of two polynomials in s, or ω, the degree of the numerator is not greater than that of the denominator;

(b) the poles and zeros of $k + \phi(-is)$ are real or they appear in conjugate pairs of the same order;

(c) the poles of $k + \phi(-is)$, which from (8) occur at $s = s_j$, all lie in the left half plane, since $Re(s_j) < 0$. The poles and zeros of $k + \phi(\omega)$ are therefore imaginary, or they appear in pairs, of the same order, having the form $a + ib$ and $-a + ib$. The poles of $k + \phi(\omega)$ all lie in the upper half plane.

A well-known result [26,P.127] gives the spectrum of $X_1(t,w)$ as

$$\frac{2\sigma^{-2}\alpha}{(\omega^2+\alpha^2)}\,(k+\phi(\omega))(k+\overline{\phi(\omega)}) \qquad (3.10)$$

This expression is rational since $k + \phi(\omega)$ is rational; it has no poles or zeros on the real axis since, $k + \phi(\omega)$

has no such poles or zeros (definition 6).

Since each pole and zero of the rational expression (10) is accompanied by its conjugate, it may be factorized to the form $\psi(\omega)\,\overline{\psi(\omega)}$, where $\psi(\omega)$ has the form $\frac{P_1(\omega)}{P_2(\omega)}$ and the zeros and poles of $\psi(\omega)$ lie in the upper half plane.

The remaining parts of the theorem may now be proved.

(i) Since the degree of the numerator of (10) is at least two less than that of the denominator then the degree of $P_1(\omega)$ is less than that of $P_2(\omega)$.

(ii) The zeros of $P_1(\omega)$ and $P_2(\omega)$, all of which lie in the upper half plane, arise either, from the zeros and poles of $\frac{1}{\omega^2+\alpha^2}$ or $(k+\phi(\omega))(k+\overline{\phi(\omega)})$. The first term gives rise to a zero of $P_2(\omega)$ on the imaginary axis at $i\alpha$; while, from (c) above, the second term gives rise to zeros which are either on the imaginary axis, or occur in pairs of the form $a + ib$ and $-a + ib$, $a > 0$, $b > 0$.

(iii) Each of the polynomials $P_1(\omega)$ and $P_2(\omega)$ may, then, be factorized into terms of the form

$$(\omega-z)(\omega+\bar{z}), \quad \text{or} \quad (\omega-ib_1), \tag{3.11}$$

where z has the form $a + ib$, $a > 0$, $b > 0$, and b_1 is also positive.

These factors may be rewritten as

$$(i\omega-iz)(i\omega-\overline{iz}), \quad \text{or} \quad (i\omega+b_1); \tag{3.12}$$

but in so doing $\psi(\omega)$ may be multiplied by i, −i or −1 and to compensate $\overline{\psi(\omega)}$ must be multiplied by −i, i or −1.

The quadratic factor in (12)

$$= (i\omega)^2 + 2b(i\omega) + a^2 + b^2 \qquad (3.13)$$

Each of the polynomials $P_1(\omega)$ and $P_2(\omega)$ may therefore be written as a polynomial in $i\omega$ which, from (12) and (13), has real coefficients of the same sign. These coefficients may all be made positive by multiplying $\psi(\omega)$ and $\overline{\psi(\omega)}$ by −1, if necessary, and the result (iii) follows

Theorem 2

If a realisation $X_1(t,w_i)$ of the random process $X_1(t,w)$ may be written in the form $X_1(t,w_i) = L_{11}[M_1(t,w_i)]$, where $M_1(t,w)$ is an S.G.M. having autocorrelation function $\sigma_1{}^2 e^{-\alpha|\tau|}$; then for every positive real σ_2 and β there exists, under the assumptions following definition 6, an S.G.M., $M_2(t,w)$, having autocorrelation function $\sigma_2{}^2 e^{-\beta|\tau|}$, such that $X_1(t,w_i) = L_{12}[M_2(t,w_i)]$. The transformations L_{11} and L_{12} are of type L_1.

Proof The random process

$$M_2(t,w) = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}} \frac{\sigma_2}{\sigma_1}\left[\int_0^\infty M_1(t-u,w)(\alpha-\beta)e^{-\beta u}du + M_1(t,w)\right]$$

exists and is of type X_1, since the frequency response function

$$\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}} \frac{\sigma_2}{\sigma_1} \left(\frac{i\omega + \alpha}{i\omega + \beta}\right) \qquad (3.14)$$

corresponding to the weighting function

$$\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}} \frac{\sigma_2}{\sigma_1} \{\delta(t) + (\alpha-\beta)e^{-\beta t}\} \qquad (3.15)$$

has no zeros on the real axis.

Let L_{13} be the type L_1 transformation having the weighting function (3.15).

The process $M_2(t,w)$ has spectrum, [26,P.127],

$$\frac{2\sigma_1{}^2\alpha}{\omega^2+\alpha^2}\left(\frac{\beta}{\alpha}\right)\frac{\sigma_2{}^2}{\sigma_1{}^2}\left(\frac{\omega^2+\alpha^2}{\omega^2+\beta^2}\right)$$

$$= \frac{2\sigma_2{}^2\beta}{(\omega^2+\beta^2)},$$

and its autocorrelation function is

$$\sigma_2{}^2 e^{-\beta|\tau|}.$$

$M_2(t,w)$ is therefore an S.G.M. and using the assumption following definition 6,

$$M_2(t,w_t) = L_{13}\, M_1(t,w_t).$$

Define the transformation L_{14} having weighting function

$$\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}\frac{\sigma_1}{\sigma_2}\,\{\delta(t) + (\beta-\alpha)e^{-\alpha t}\}.$$

Then $L_{14}[L_{13}[M_1(t,w_t)]] =$

$$= L_{14}\left[\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}}\left(\frac{\sigma_2}{\sigma_1}\right)\{M_1(t,w_t) + (\alpha-\beta)\int_0^\infty M_1(t-x,w_t)e^{-\beta x}dx\}\right]$$

$$= M_1(t,w_t) + (\alpha-\beta)\int_0^\infty M_1(t-x,w_t)e^{-\beta x}dx\ +$$

$$+ (\beta-\alpha)\int_0^\infty M_1(t-x,w_t)e^{-\alpha x}dx\ +$$

$$+ (\beta-\alpha)(\alpha-\beta)\int_0^\infty e^{-\alpha y}\left[\int_0^\infty M_1(t-y-x,w_t)e^{-\beta x}dx\right]dy.^{*} \qquad (3.16)$$

The repeated integral in (16) becomes, after the substitution $\tau = x + y$,

$$(\beta-\alpha)(\alpha-\beta)\int_0^\infty e^{-\alpha y}\int_y^\infty M_1(t-\tau,w_t)e^{-\beta(\tau-y)}d\tau\, dy. \qquad (3.17)$$

---

* The convention to be used in this thesis for repeated integrals is shown by these square brackets, which will be omitted hereafter.

Using the assumption following definition 6, that $M(t,w_K)e^{-b(\tau-t)}$, $b > 0$, is Lebesgue integrable, it may be shown that

$$\int_0^\infty e^{-\alpha y}\int_0^\infty |M_1(t-y-x,w_l)|\, e^{-\beta x}dx\ dy$$

is finite. The order of integration in (17) may therefore be reversed to give

$$(\beta-\alpha)(\alpha-\beta)\int_0^\infty M_1(t-\tau,w_l)e^{-\beta\tau}\int_0^\tau e^{-\alpha y}e^{\beta y}dy\ d\tau$$

$$= (\alpha-\beta)\int_0^\infty M_1(t-\tau,w_l)(e^{-\alpha\tau}-e^{-\beta\tau})d\tau.$$

The expression (16) is therefore equal to $M_1(t,w_l)$, whence

$$X_1(t,w_l) = L_{11}[L_{14}[M_2(t,w_l)]].$$

Also

$$L_{14}[M_2(t,w_l)] = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}\left(\frac{\sigma_1}{\sigma_2}\right)\{M_2(t,w_l) + (\beta-\alpha)\int_0^\infty M_2(t-u,w_l)e^{-\alpha u}du\},$$

so that if L_{11} has weighting function $k\delta(t) + W(t)$, then $L_{11}[L_{14}[M_2(t,w_l)]]$ equals, apart from the constant term $\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}\frac{\sigma_1}{\sigma_2}$,

$$k\ M_2(t,w_l) + k(\beta-\alpha)\int_0^\infty M_2(t-x,w_l)e^{-\alpha x}dx$$

$$+ \int_0^\infty M_2(t-x,w_l)W(x)dx + +$$

$$(\beta-\alpha)\int_0^\infty W(y)\int_0^\infty M_2(t-y-x,w_l)e^{-\alpha x}dx\ dy. \qquad (3.18)$$

As with the double integral in (16) the order of integration in the double integral in (18) may be reversed to give

$$(\beta-\alpha)\int_0^\infty M_2(t-\tau,w_l)\int_0^\tau W(y)e^{-\alpha(\tau-y)}dy\ d\tau.$$

Hence the expression $L_{11}[L_{14}[M_2(t,w_l)]]$ is equal to $L_{12}[M_2(t,w_l)]$ where L_{12} has the weighting function

$$\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}\frac{\sigma_1}{\sigma_2}\,[k\delta(t) + k(\beta-\alpha)e^{-\alpha t} + W(t) + (\beta-\alpha)\int_0^t W(y)e^{-\alpha(t-y)}dy].$$

The convolution $\int_0^t W(y)e^{-\alpha(t-y)}dy$ is zero for $t < 0$ and for $t \geqslant 0$ is the sum of terms of the form

$$\int_0^t cy^m e^{s_j y} e^{-\alpha t} e^{\alpha y} dy, \qquad \mathrm{Re}(s_j) < 0,$$

it is therefore of the form (1).

The frequency response function corresponding to L_{12} is

$$\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}} \frac{\sigma_1}{\sigma_2} \left(\frac{i\omega + \beta}{i\omega + \alpha}\right) \left(k + \phi(\omega)\right)$$

which has no zeros on the real axis, since $k + \phi(\omega)$ has no such zeros. Therefore L_{12} is of type L_1 and the theorem is proved.

Because of this theorem any realisation of the type X_1 process $X_1(t,w)$, may be written

$$X_1(t,w_l) = L_1[M_\alpha(t,w_l)],$$

where $M_\alpha(t,w)$ is an S.G.M. having autocorrelation function $\frac{\alpha}{2}e^{-\alpha|\tau|}$, and α may be chosen arbitrarily large. The S.G.M. then approaches "white noise", [25,PP.94 and 162], of unit spectral density. Its spectrum is

$$S_\alpha(\omega) = \frac{\alpha^2}{\alpha^2+\omega^2},$$

so that, given any positive ϵ and ω_0, then

$0 < |1 - S_\alpha(\omega)| < \epsilon$ for all ω such that $|\omega| < \omega_0$, providing

$$\alpha^2 > \frac{\omega_0^2(1-\epsilon)}{\epsilon}.$$

It is often useful to consider type X_1 processes to have been derived from S.G.M. processes which approximate "white noise" or, colloquially, "each realisation of $X_1(t,w)$ to have arisen from passing white noise through a filter of type L_1".

Theorem 3

Let $X(t,w)$ be a type X_1 process. Then there exists an S.G.M., $M(t,w)$, satisfying an equation of the form

$$M(t,w) = \sum_{m=0}^{n} k_m X_1^{(m)}(t,w) + \int_0^\infty X_1(t-u,w)W(u)du,$$

where the weighting function

$$\sum_{m=0}^{n} k_m \delta^{(m)}(t) + W(t)$$

corresponds to a transformation of type L.

Proof According to Theorem 1 $X_1(t,w)$ has a rational spectrum of the form $\psi(\omega)\overline{\psi(\omega)}$ where $\psi(\omega) = \frac{P_1(\omega)}{P_2(\omega)}$ and the polynomials have the following properties:

(i) their zeros are all in the upper half plane;

(ii) they may be written as polynomials in $i\omega$ with real coefficients.

Consider the expression

$$\frac{\sigma\sqrt{2\alpha}}{(i\omega+\alpha)} \frac{P_2(\omega)}{P_1(\omega)}.$$

This may be written in the form

$$P_3(i\omega) + \frac{P_4(i\omega)}{P_5(i\omega)}, \tag{3.19}$$

where:

(a) the polynomials P_3, P_4 and P_5 have real coefficients;

(b) if the degree of $P_j = p_j$ $(j = 1 - 5)$, then

$$p_5 - p_4 = 1,$$

$$p_3 = p_2 - p_1 - 1.$$

The expression (19) is the frequency response function of the required type L transformation. The individual terms in $P_3(i\omega)$ correspond to $\delta(t)$ and its derivatives, while $\frac{P_4(i\omega)}{P_5(i\omega)}$ may be transformed by contour integration to give an expression W(t) of the form (1).

Since $\int_{-\infty}^{\infty} \psi(\omega)\, \overline{\psi(\omega)}\, (\omega^2)^{p_3} d\omega$ exists, then the auto-correlation function of $X_1(t,w)$ may be differentiated $2p_3$ times, and therefore [25,P.23] the process $X_1^{(m)}(t,w)$ exists for $m \leqslant p_3$.

Hence the process

$$\sum_{m=0}^{p_3} k_m X_1^{(m)}(t,w) + \int_0^{\infty} X_1(t-u,w)W(u)du$$

is also defined; its spectrum is

$$\psi(\omega)\, \overline{\psi(\omega)}\, \left|\frac{\sigma\sqrt{2\alpha}}{i\omega + \alpha}\right|^2 \left|\frac{P_2(\omega)}{P_1(\omega)}\right|^2$$

$$= \frac{\sigma^2\, 2\alpha}{\omega^2 + \alpha^2}$$

as required by the theorem.

On the basis of this theorem it will be assumed that each realisation of a type X_1 process considered in this thesis may be transformed by a type L transformation to give a realisation of an S.G.M. The parameters σ^2 and α in the correlation function of this S.G.M. may be chosen arbitrarily. In particular they may be chosen so that $\sigma^2 = \frac{\alpha}{2}$ and α is large, in which case the S.G.M. approximates white noise.

It is well known [24,P.342], that an estimate of the weighting function of a linear transformation (filter), may often be found by applying to it an input which is a realisation of a process which approximates white noise, and then cross correlating the input and output. This procedure is now investigated more closely in the case of an S.G.M. having autocorrelation function $\frac{\alpha}{2} e^{-\alpha|\tau|}$. This investigation applies, with slight modification, to other types of stationary random process which approximate white noise [45].

Theorem 4

For each positive real number α let $M_\alpha(t,w)$ be an S.G.M. having autocorrelation function $\rho_\alpha(\tau) = \frac{\alpha}{2} e^{-\alpha|\tau|}$.

Let T be a linear causal transformation with weighting function $W(t)$ of the form (1), but restricted so that its frequency response function, $\phi(\omega)$, satisfies

$$\int_{-\infty}^{\infty} |\phi(\omega)|\,d\omega \text{ is finite.}$$

Let $$O_\alpha(t,w) = \int_0^{\infty} M_\alpha(t-u,w)W(u)\,du$$

and $$\rho_{\alpha,o}(\tau) = E[M_\alpha(t-\tau,w)O_\alpha(t,w)].$$

Then:

$$\text{(i)} \quad \lim_{\alpha\to\infty} \rho_{\alpha,o}(\tau) = W(\tau), \qquad \tau \geq 0;$$

$$\text{(ii)} \quad \lim_{\alpha\to\infty} \rho_{\alpha,o}(\tau) + \rho_{\alpha,o}(-\tau) = W(\tau), \qquad \tau \geq 0;$$

$$\text{(iii)} \quad \int_0^{\infty} \rho_{\alpha,o}(\tau) + \rho_{\alpha,o}(-\tau)\,d\tau = \int_0^{\infty} W(\tau)\,d\tau, \text{ all } \alpha.$$

Proof (i) Applying the Schwarz inequality,

$$E|M_\alpha(t-\tau,w)\, M_\alpha(t-x,w)|$$
$$\leq \{E[M_\alpha^2(t-\tau,w)]E[M_\alpha^2(t-x,w)]\}^{\frac{1}{2}}$$
$$= \frac{\alpha}{2} \text{ and is therefore finite,}$$

whence $\int_0^\infty |W(x)|E[|M_\alpha(t-\tau,w)M_\alpha(t-x,w)|]dx$

is finite.

Therefore

$$\rho_{\alpha,0}(\tau) = E[M_\alpha(t-\tau,w)\int_0^\infty M_\alpha(t-x,w)W(x)dx]$$

$$= \int_0^\infty W(x)E[M_\alpha(t-\tau,w)M_\alpha(t-x,w)]dx$$

$$= \int_0^\infty \rho_\alpha(\tau-x)W(x)dx. \qquad (3.20)$$

The Fourier transform of the convolution (20)

is $S_\alpha(\omega)\ \phi(\omega)$,

where $S_\alpha(\omega)$ is the spectrum of $M_\alpha(t,w)$, viz.

$$\frac{\alpha^2}{\alpha^2 + \omega^2}.$$

Therefore

$$\lim_{\alpha\to\infty} \rho_{\alpha,0}(\tau) = \frac{1}{2\pi} \lim_{\alpha\to\infty} \int_{-\infty}^\infty S_\alpha(\omega)\phi(\omega)e^{i\omega\tau}d\omega. \qquad (3.21)$$

But for every ω

$$\lim_{\alpha\to\infty} S_\alpha(\omega) = 1$$

$$\text{and} \quad |S_\alpha(\omega)| \leq 1;$$

so that by dominated convergence, since

$$\int_{-\infty}^\infty |\phi(\omega)|d\omega \quad \text{is finite,}$$

the limiting and integration operations in (21) may be interchanged to give

$$\lim_{\alpha\to\infty} \rho_{\alpha,0}(\tau) = \frac{1}{2\pi}\int_{-\infty}^\infty \phi(\omega)e^{i\omega\tau}d\omega$$

$$= W(\tau), \qquad \tau \geq 0. \qquad (3.22)$$

(ii) It follows from (20) that

$$\rho_{\alpha,0}(\tau) + \rho_{\alpha,0}(-\tau) = \int_0^\infty [\rho_\alpha(\tau-x) + \rho_\alpha(-\tau-x)]W(x)dx$$

which, since $\rho_\alpha(\tau)$ is an even function, becomes

$$\int_0^\infty \rho_\alpha(\tau-x)W(x)dx + \int_0^\infty \rho_\alpha(\tau+x)W(x)dx$$

$$= \int_0^\infty \rho_\alpha(\tau-x)W(x)dx + \int_{-\infty}^0 \rho_\alpha(\tau-x)W(-x)dx.$$

The Fourier transform of this expression is

$$S_\alpha(\omega)\ \phi(\omega) + S_\alpha(\omega)\ \phi(-\omega)$$

$$= 2\ S_\alpha(\omega)\ R_\phi(\omega), \qquad (3.23)$$

where $2\ R_\phi(\omega)$ = twice the real part of $\phi(\omega)$

and is the Fourier transform of the even function

which equals $W(t)$, $t \geqslant 0$.

Hence

$$\lim_{\alpha\to\infty} \rho_{\alpha,0}(\tau) + \rho_{\alpha,0}(-\tau) = \frac{1}{2\pi} \lim_{\alpha\to\infty} \int_{-\infty}^\infty 2S_\alpha(\omega)R_\phi(\omega)e^{i\omega\tau}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty 2R_\phi(\omega)e^{i\omega\tau}d\omega$$

$$= W(\tau), \qquad \tau \geqslant 0.$$

(iii) $$\int_0^\infty W(x)dx = \phi(0).$$

$$\int_0^\infty \{\rho_{\alpha,0}(x) + \rho_{\alpha,0}(-x)\}dx = \int_{-\infty}^\infty \rho_{\alpha,0}(x)dx$$

$$= S_\alpha(0)\ \phi(0)$$

$$= \phi(0).$$

The theorem is proved.

Since the process $M_\alpha(t,w)$ is ergodic the cross correlation function $\rho_{\alpha,0}(\tau)$ is equal to the expression

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} M_\alpha(t-\tau,w_K)O_\alpha(t)dt \,. \qquad (3.23)$$

Now suppose an estimate is required of the weighting function $W_1(t)$ of a linear system which is known to be of type L_1. A, realisation, $M_\alpha(t,w_t)$ of an S.G.M. whose correlation function is $\frac{\alpha}{2}\,e^{-\alpha|\tau|}$ may be applied as an input to the system and the cross correlation function $\rho_{\alpha,o}(\tau)$ estimated as a time average. Then, relying on theorem 4, it would be expected that, for large α and $\tau \geq 0$, $\rho_{\alpha,o}(\tau)$ would be a good approximation to $W_1(\tau)$.

As an illustration consider

$$W_1(\tau) = \tau e^{-a\tau}, \qquad \tau \geq 0, \quad a \text{ real and positive,}$$
$$= 0, \qquad \tau < 0 \,.$$

$$\rho_{\alpha,o}(\tau) = \int_0^\infty \frac{\alpha}{2}\,e^{-\alpha|\tau-x|}xe^{-ax}dx, \qquad \tau \geq 0,$$

$$= \frac{\alpha}{2}\,e^{-\alpha\tau}\int_0^\tau x\,e^{-(a-\alpha)x}dx + \frac{\alpha}{2}\,e^{\alpha\tau}\int_\tau^\infty x\,e^{-(a+\alpha)x}dx, \tau \geq 0,$$

$$= \frac{\alpha^2}{\alpha^2-a^2}\,\tau\,e^{-a\tau} - \frac{2a\alpha^2}{(\alpha^2-a^2)^2}\,e^{-a\tau} + \frac{\alpha}{2(\alpha-a)^2}\,e^{-\alpha\tau},\ \tau \geq 0, \qquad (3.24)$$

which for $\tau \geq 0$ approaches $\tau e^{-a\tau}$ as $\alpha \to \infty$.

The difference between the expression (24) and $W_1(\tau)$,

$$= \frac{a^2}{\alpha^2-a^2}\,\tau e^{-a\tau} - \frac{2a\alpha^2}{(\alpha^2-a^2)^2}\,e^{-a\tau} + \frac{\alpha}{2(\alpha-a)^2}\,e^{-\alpha\tau}.$$

The usefulness of the expression (24) as an approximation to $W_1(\tau)$ may be judged by considering

$$\int_0^\infty |\rho_{\alpha,o}(\tau) - W_1(\tau)|^2 d\tau$$

$$= \frac{\alpha^5 - 2a\alpha^4 - 2a^2\alpha^3 + 8a^3\alpha^2 - 7a^4\alpha + 2a^5}{8(\alpha^2 - a^2)^4}$$

$$\simeq \frac{1}{8\alpha^3} - \frac{a}{4\alpha^4}\ \ldots, \qquad \text{for } \alpha >> a.$$

The ratio of this expression to

$$\int_0^\infty |W_1(\tau)|^2 d\tau = \frac{1}{4a^3}$$

is approximately

$$\tfrac{1}{2}\left(\frac{a}{\alpha}\right)^3. \qquad (3.27)$$

However theorem 4 part (i) does not apply to all systems of type L_1. For example if $W_1(\tau) = e^{-a\tau}$, $\tau \geqslant 0$, $a > 0$,

$= 0 \quad , \tau < 0;$

then the corresponding frequency response function is $\frac{1}{i\omega+a}$ which does not satisfy the requirement

$$\int_{-\infty}^{\infty} |\phi(\omega)| d\omega \text{ is finite.}$$

In this case

$$\rho_{\alpha,0}(\tau) = \int_0^\infty \frac{\alpha}{2} e^{-\alpha|\tau-x|} e^{-ax} dx$$

$$= \frac{\alpha^2}{\alpha^2-a^2} e^{-a\tau} - \frac{\alpha}{2(\alpha-a)} e^{-\alpha\tau}, \quad \tau \geqslant 0; \qquad (3.28)$$

and therefore

$$\lim_{\alpha\to\infty} \rho_{\alpha,0}(\tau) = e^{-a\tau} = W_1(\tau), \qquad \tau > 0,$$
$$= \tfrac{1}{2} \qquad \tau = 0.$$

An error in $W_1(\tau)$ at $\tau = 0$ only is of no practical importance, but for finite α an estimate of $W_1(\tau)$ may have large errors, not only at $\tau = 0$, but also for values of τ near zero. Thus if $\alpha = 10a$

$$\rho_{\alpha,0}(\tau) = \frac{100}{99} e^{-a\tau} - \frac{5}{11} e^{-10a\tau}, \ \tau \geqslant 0,$$

which differs from $W_1(\tau)$ by

$$\frac{1}{99} e^{-a\tau} - \frac{5}{11} e^{-10a\tau}.$$

The usefulness of the expression (28) may also be judged by computing

$$\int_0^\infty |\rho_{\alpha,0}(\tau) - W_1(\tau)|^2 d\tau$$

$$= \int_0^\infty [\frac{a^2}{\alpha^2 - a^2} e^{-a\tau} - \frac{\alpha}{2(\alpha - a)} e^{-\alpha\tau}]^2 d\tau$$

$$= \frac{\alpha^3 + 2a\alpha^2 - 7\alpha a^2 + 4a^3}{8(\alpha^2 - a^2)^2}$$

$$\simeq \frac{1}{8\alpha} + \frac{a}{4\alpha^2} \cdots$$

The ratio of this expression to

$$\int_0^\infty (W_1 \ \tau)^2 d\tau$$

contains a first order term in $\left(\frac{a}{\alpha}\right)$, whereas in the first case considered, viz. $W_1(\tau) = \tau e^{-a\tau}$, the lowest order term was $\frac{1}{2}\left(\frac{a}{\alpha}\right)^3$.

As another example consider the case

$$W_1(\tau) = \delta(\tau),$$

so that

$$\rho_{\alpha,0}(\tau) = \rho_\alpha(\tau) = \frac{\alpha}{2} e^{-\alpha|\tau|}.$$

If in this case the estimate

$$W_1(\tau) = \rho_{\alpha,0}(\tau), \qquad \tau \geq 0,$$

is used to compute the output of the system to a given input the answer will in general be about half the correct answer. This follows immediately from the fact that $\int_0^\infty \frac{\alpha}{2} e^{-\alpha t} dt = \frac{1}{2}$ and not 1 as would be necessary for a good approximation to $\delta(t)$.

In the last case a more useful approximation to $W_1(\tau)$ would be the expression

$$\rho_{\alpha,0}(\tau) + \rho_{\alpha,0}(-\tau), \qquad \tau \geq 0,$$

which is considered in parts (ii) and (iii) of theorem 4. The proof of these parts is still valid if the restriction

$$\int_{-\infty}^{\infty} |\phi(\omega)|\, d\omega \quad \text{is finite,}$$

is relaxed to $\int_{-\infty}^{\infty} |R_\phi(\omega)|\, d\omega$ is finite.

This latter restriction is satisfied by all $W(t)$ of form (1), since, for such $W(t)$, $\phi(\omega)$ is rational and therefore $R_\phi(\omega)$ is rational and real. Hence $R_\phi(\omega)$ may be written in the form

$$R_\phi(\omega) = \frac{P_6(\omega)}{P_7(\omega)} ,$$

where $P_6(\omega)$ and $P_7(\omega)$ are polynomials in ω with real coefficients and the degree of $P_6(\omega)$ is less than that of $P_7(\omega)$.

Thus $$R_\phi(\omega) = \frac{P_6(\omega)}{P_7(\omega)} \; \frac{P_7(-\omega)}{P_7(-\omega)} . \qquad (3.29)$$

But $R_\phi(\omega) = \int_0^{\infty} W(t) \cos \omega t \, dt$ and is therefore even, so that (29), whose denominator is clearly even, must be the ratio of two even polynomials in ω. Since an even polynomial in ω contains only even powers of ω the degree of the numerator of (29) must be at least two less than that of the denominator. Therefore, since $\phi(\omega)$ has no poles on the real axis,

$$\int_{-\infty}^{\infty} |R_\phi(\omega)|\, d\omega \quad \text{is finite.}$$

Consider again the example

$$W_1(\tau) = e^{-a\tau} \quad , \qquad \tau \geq 0, \quad a > 0,$$
$$= 0 \quad , \qquad \tau < 0.$$

The criterion

$$\int_0^\infty |\rho_{\alpha,o}(\tau) + \rho_{\alpha,o}(-\tau) - W_1(\tau)|^2 d\tau$$

for the usefulness of the estimate

$$\rho_{\alpha,o}(\tau) + \rho_{\alpha,o}(-\tau), \qquad \tau \geq 0, \qquad (3.30)$$

of this weighting function, is equal to

$$\left(\frac{a}{\alpha + a}\right)^3.$$

In the case of the original estimate, viz.

$$\rho_{\alpha,o}(\tau), \qquad \tau \geq 0,$$

this criterion contained a first order term in $\left(\frac{a}{\alpha}\right)$; it therefore seems that (30) is the better estimate.

The above discussion is summarised in the following theorem.

<u>Theorem 5</u>

Let a linear system of type L_1 have a weighting function $k\delta(t) + W(t)$ and frequency response function $k + \phi(\omega)$. Let $\rho_{\alpha,o}(\tau)$ be the cross correlation function of the input and output of this system when the input is a realisation of an S.G.M. having autocorrelation function $\frac{\alpha}{2} e^{-\alpha|\tau|}$.

Let $f(t)$ be a bounded and piecewise continuous function of t.

Then:

(i) $\rho_{\alpha,0}(\tau) + \rho_{\alpha,0}(-\tau) = k\alpha e^{-\alpha\tau} + W_\alpha(\tau), \quad \tau \geqslant 0,$

where $\lim\limits_{\alpha\to\infty} W_\alpha(\tau) = W(\tau), \quad \tau \geqslant 0;$

(ii) $\int_0^\infty \rho_{\alpha,0}(\tau) + \rho_{\alpha,0}(-\tau)d\tau = \int_{-\infty}^\infty k\delta(\tau) + W(\tau)d\tau;$

(iii) $\lim\limits_{\alpha\to\infty} \int_0^\infty f(t-y)[\rho_{\alpha,0}(y) + \rho_{\alpha,0}(-y)]dy =$

$$= \int_{-\infty}^\infty f(t-y)[k\delta(y) + W(y)]dy.$$

<u>Proof</u> Parts (i) and (ii) follow immediately from theorem 4 and the previous discussion.

(iii) $\int_0^\infty f(t-y)[\rho_{\alpha,0}(y) + \rho_{\alpha,0}(-y)]dy$

$= \int_0^\infty f(t-y)[k\alpha e^{-\alpha y} + W_\alpha(y)]dy.$

Consider $\int_0^\infty f(t-y)k\alpha e^{-\alpha y}dy$

$$= \int_0^\infty f(t - \tfrac{z}{\alpha})ke^{-z}dz.$$

Since $f(t)$ is bounded write

$$|f(t)| < \kappa, \quad \text{all } t,$$

Then $|f(t - \frac{z}{\alpha})k\, e^{-z}| < \kappa|k|e^{-z}$

and

$$\int_0^\infty \kappa|k|e^{-z}dz \quad \text{exists.}$$

Therefore

$$\lim_{\alpha\to\infty} \int_0^\infty f(t - \tfrac{z}{\alpha})ke^{-z}dz$$

$$= \int_0^\infty \lim_{\alpha\to\infty} f(t - \tfrac{z}{\alpha})ke^{-z}dz$$

$$= k\, f(t)$$

$$= \int_{-\infty}^\infty f(t-y)k\delta(y)dy.$$

Similarly

$$\lim_{\alpha\to\infty}\int_0^\infty f(t-y)W_\alpha(y)dy$$

$$= \int_0^\infty \lim_{\alpha\to\infty} f(t-y)W_\alpha(y)dy$$

$$= \int_0^\infty f(t-y)W(y)dy.$$

The interchange of limiting and integration is justified since, as shown in the appendix, for $y \geqslant 0$

$W_\alpha(y)$ is of the form

$$f_2(\alpha)e^{-\alpha y} + \sum_{j=1}^{n} Q_j(y,\alpha)e^{s_j y}, \qquad Re(s_j) < 0,$$

where $Q_j(y,\alpha)$ is a polynomial in y whose coefficients depend on α. It is also shown that the coefficients of $Q_j(y,\alpha)$ and $f_2(\alpha)$ are bounded for $\alpha > \alpha_0 > |s_j|$, all j.

The restrictions placed on $f(t)$ in this theorem are sufficiently wide to include most, if not all, practical applications.

CHAPTER 4.

STATIONARY LINEAR SYSTEM AND MODEL
ERROR FREE RECORDS OF UNLIMITED LENGTH

In this chapter the study is restricted to system components which are linear and of type L (definition 2). The following assumptions are also made:

(a) there are no restrictions on the length of the records;

(b) there are no recording errors;

(c) the inputs to the system and model are realizations of a random process of type X_1;

(d) the original model component T_M is of type L.

This situation may be represented in two ways as shown in figs. 5A and 5B. In both figures T_S is the system component and T_M the model component, $I_S(t,w_\iota)$ is the system input and $O_S(t,w_\iota)$ is the system output, while $O_M(t,w_\iota)$ is the model output to the input $I_S(t,w_\iota)$. In A the correction to the model, T_{CA}, is shown in parallel with the original model, while in B the correction T_{CB}, is in series with the original.

Under the assumptions of this section correcting transformations T_{CA}, or T_{CB}, can be found such that the output of the corrected model will be the same as the system output to this class of input. However T_{CA} and T_{CB} are

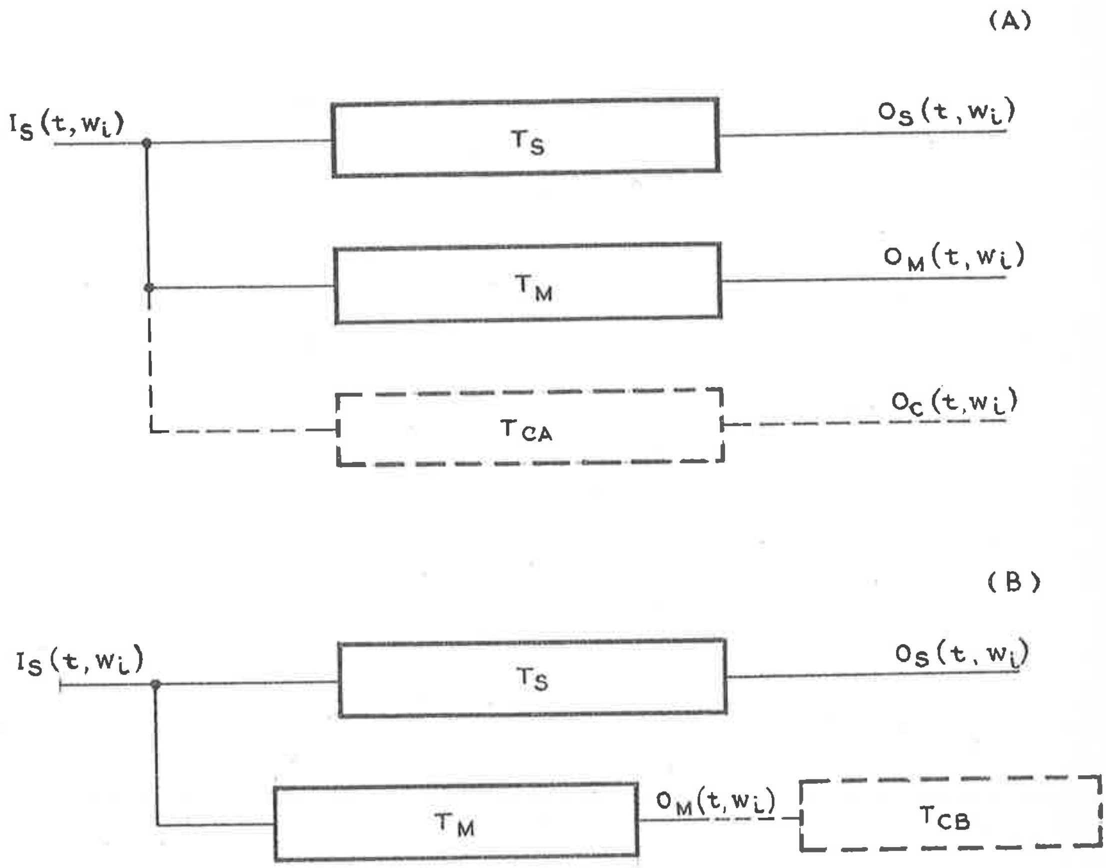

FIGURE 5. TWO APPROACHES TO MODEL CORRECTION
A. CORRECTING TRANSFORMATION T_{CA} IN PARALLEL WITH THE MODEL
B. CORRECTING TRANSFORMATION T_{CB} IN SERIES WITH THE MODEL

in general quite different from each other. In particular a T_{CA} of type L can always be found which completely corrects the model. On the other hand no T_{CB} of type L, which completely corrects the model, may exist. For example if T_S has weighting function $\delta(t)$ and T_M has weighting function $\delta(t-T)$, $T \geq 0$, then T_{CB} must have weighting function $\delta(t+T)$; hence T_{CB} is not a causal transformation and therefore not of type L. For this reason representation A is preferred and will be used in this thesis. However if T_M were restricted to the class of minimum phase transformations [48,P.282], e.g. its frequency response function were rational with all its poles and zeros above the real axis, then the method described below for representation A may be applied, with minor alterations, to B.

For representation A the following theorem applies.

<u>Theorem 6</u>

Let $I_S(t,w_\iota)$ be a realisation of the process $I_S(t,w)$ of type X_1 and let T_S and T_M be transformations of type L, with corresponding weighting functions $W_S(t)$ and $W_M(t)$, such that $T_S[I_S(t,w_\iota)]$ and $T_M[I_S(t,w_\iota)]$ exist. Then there exists a transformation T_C of type L with corresponding weighting function $W_C(t)$ such that

$$(T_M + T_C)I_S(t,w_\iota) = T_S[I_S(t,w_\iota)], \tag{4.1}$$

and $W_C(t)$ is the unique L type solution of

$$\int_{-\infty}^{\infty} \rho_I(\tau-x)W(x)dx = \rho_{I,OS}(\tau) - \rho_{I,OM}(\tau), \qquad \tau \geq 0. \tag{4.2}$$

Proof The existence of $W_C(t)$ is trivial since

$$W_C(t) = W_S(t) - W_M(t) \tag{4.3}$$

satisfies the requirement.

Now let equation (3) hold and then

$$\int_{-\infty}^{\infty} I_S(t-x,w)W_C(x)dx = \int_{-\infty}^{\infty} I_S(t-x,w)W_S(x)dx - \int_{-\infty}^{\infty} I_S(t-x,w)W_M(x)dx. \tag{4.4}$$

Multiplying each side of equation (4) by $I_S(t-\tau,w)$ and taking expectations yields

$$\int_{-\infty}^{\infty} \rho_I(\tau-x)W_C(x)dx = \int_{-\infty}^{\infty} \rho_I(\tau-x)W_S(x)dx - \int_{-\infty}^{\infty} \rho_I(\tau-x)W_M(x)dx$$

$$= \rho_{I,OS}(\tau) - \rho_{I,OM}(\tau). \tag{4.5}$$

The inversion of the expectation and integration operations is justified for weighting functions of form (3.1) since

$$\int_{-\infty}^{\infty} E[\,|I_S(t-\tau,w)I_S(t-x,w)W(x)|\,]dx$$

is finite. On the other hand for weighting functions containing generalised functions, assuming $I_S(t,w)$ may be differentiated n times,

$$E[I_S(t-\tau,w)\int_{-\infty}^{\infty} I_S(t-x,w)\delta^{(n)}(x-T)dx]$$

$$= E[I_S(t-\tau,w)I_S^{(n)}(t-T,w)]$$

$$= \rho_I^{(n)}(\tau-T) \qquad [49,P.165]$$

$$= \int_{-\infty}^{\infty} \rho_I(\tau-x)\delta^{(n)}(x-T)dx.$$

Assume now that equation (2) has two L type solutions, viz. $W_C(\tau)$ and $W_D(\tau)$. It will be shown that $W_C(\tau) - W_D(\tau) = 0$. Similar theorems are proved by Titchmarsh [50, chapter XI].

If $W_C(\tau)$ and $W_D(\tau)$ both satisfy (2) then

$$\int_{-\infty}^{\infty} \rho_I(\tau - x)\{W_C(x) - W_D(x)\}dx = 0, \qquad \tau \geqslant 0, \tag{4.6}$$

where $W_C(x) - W_D(x)$ is of the form

$$\sum_{j=1}^{n} P_j(x)e^{s_j x} + \sum_{j=0}^{m} \sum_{l=1}^{l_j} k_{jl}\delta^{(j)}(x - T_{jl}), \quad Re(s_j) < 0,\ T_{jl} \geqslant 0,\ x \geqslant 0, \tag{4.7}$$

and is zero for $x < 0$.

Let

$$\Sigma(x) = \sum_{j=1}^{n} P_j(x)e^{s_j x}, \qquad x \geqslant 0,$$
$$= 0, \qquad x < 0,$$

have Fourier transform $\phi(\omega)$, which is then a rational function of ω with all its poles above the real axis. Also assume, for the moment, that all the T_{jl} are zero. Then, writing k_j in place of $\sum_{l=1}^{l_j} k_{jl}$, equation (6) may be written

$$\int_{-\infty}^{\infty} \rho_I(\tau - x)\Sigma(x)dx + \sum_{j=0}^{m} k_j \frac{d^j}{d\tau^j} \rho_I(\tau) = F(\tau), \tag{4.9}$$

where

$$F(\tau) = 0 \qquad , \quad \tau \geqslant 0. \tag{4.10}$$

If $W_D(x)$ represents an admissible transformation of type L for this input then the Fourier transform of the left hand side of (9) exists, so that

$$\psi_F(\omega) = S_I(\omega)[\phi(\omega) + \sum_{j=0}^{m} k_j(i\omega)^j]$$
$$= S_I(\omega)[\phi(\omega) + P(\omega)]; \tag{4.11}$$

where:

$\psi_F(\omega)$ is the transform of $F(\tau)$;

$P(\omega)$ is a polynomial in ω;

$S_I(\omega)$ is the spectrum of $I_S(t,w)$.

The equation (11) may be written in the form

$$\psi_F(\omega) = \frac{P_{2n}(\omega)}{P_{2m}(\omega)} \quad \frac{P_q(\omega)}{P_r(\omega)}, \tag{4.12}$$

where $\frac{P_{2n}(\omega)}{P_{2m}(\omega)} = S_I(\omega)$ is a rational spectrum, since $I_S(t,w)$ is of type X_1. It may be assumed that the polynomials $P_{2n}(\omega)$ and $P_{2m}(\omega)$ have no common factors, they are both of even degree $2n$ and $2m$ respectively, and have half their zeros above and half below the real axis (theorem 1). It may also be assumed that $P_q(\omega)$ and $P_r(\omega)$ have no common factors.

Now $\psi_F(\omega)$ is the transform of a function which is zero for $t > 0$, therefore it has no poles above the real axis. Hence the r zeros of $P_r(\omega)$, which are all above the real axis, since $\Sigma(x) = 0$, $t < 0$, and the m zeros of $P_{2m}(\omega)$ above the real axis must also occur in $P_{2n}(\omega)$ and $P_q(\omega)$ respectively.

Therefore $$q \geqslant m \quad \text{and} \quad n \geqslant r. \tag{4.13}$$

Now consider $$\frac{P_{2n}(\omega)}{P_{2m}(\omega)} \quad \left|\frac{P_q(\omega)}{P_r(\omega)}\right|^2 \tag{4.14}$$

which, if (7) represents an admissible transformation, is the spectrum of the output of a system having input $I_S(t,w_l)$ and weighting function $W_C(x) - W_D(x)$, [26,P.126]. The inequalities (13) show that the highest power of ω in the numerator of (14) is at least equal to that in the denominator, and therefore the only permitted value for (14) is zero for all real ω.

Since $S_I(\omega) \neq 0$, then $\left|\frac{P_a(\omega)}{P_r(\omega)}\right| \equiv 0$,

$$\text{i.e.}\quad \phi(\omega) + P(\omega) \equiv 0; \qquad (4.15)$$

whence both the rational function $\phi(\omega)$ whose numerator is of lower degree than the denominator and the polynomial $P(\omega)$ must be zero for all real ω.

Thus, in the case where the T_{jl} are all zero, the expression (7) is zero for all x and therefore

$$W_C(x) = W_D(x).$$

In case some of the T_{jl} are not zero, let T_s be the smallest of these, they are all positive. Equation (6) now becomes, after inserting the expression (7) for $W_C(x) - W_D(x)$ and rearranging the terms,

$$\int_{-\infty}^{\infty} \rho_I(\tau-x)\Sigma(x)dx + \sum_{\substack{j=0\\ T_{jl}=0}}^{m} \sum_{l=1}^{l_j} k_{jl} \frac{d^j}{d\tau^j} \rho_I(\tau)$$

$$= F(\tau) - \sum_{\substack{j=0\\ T_{jl}\neq 0}}^{m} \sum_{l=1}^{l_j} k_{jl} \frac{d^j}{d\tau^j} \rho_I(\tau - T_{jl}). \qquad (4.16)$$

Consider the open interval, $0 < \tau < T_s$, in which $F(\tau)$ is zero. The left hand side of (16) has a Fourier transform which is a rational function of ω, (11). It may therefore be evaluated, for $\tau > 0$, by contour integration of this transform multiplied by $e^{i\omega\tau}$, around the upper semicircle; i.e. by evaluating residues at poles with positive imaginary parts. In the interval $0 < \tau < T_s$ then, the left hand side of (16) will be a sum of terms each of which contains an exponential of the form $e^{-z\tau}$, but no exponential of the form $e^{z\tau}$, where $Re(z) > 0$.

On the other hand a similar evaluation of the right hand side for $0 < \tau < T_s$ will, because of the presence of $e^{-i\omega T_{j\ell}}$ in each transform, involve contour integration around the lower semicircle, and hence give rise to terms containing $e^{z\tau}$ but none containing $e^{-z\tau}$, $Re(z) > 0$. Clearly equation (6) can not be satisfied, in the interval $0 < \tau < T_s$, by any function of the form (7) unless all the $T_{j\ell}$ are zero, which is the case first dismissed.

<u>4.1 Model correction not involving generalised functions</u>

Under the assumptions of this chapter there is, then, no difficulty in reducing the problem of correcting the model to one of finding an L type solution of the Wiener-Hopf like integral equation (2). In this section it is also assumed that the model correction $W_C(x)$ contains no generalised functions. The correlation functions $\rho_I(\tau)$, $\rho_{I,OS}(\tau)$ and $\rho_{I,OM}(\tau)$ may, in principle, be calculated from the data since the records are supposed free from error and of unlimited length.

The Wiener-Hopf equation has been extensively studied, (see, e.g., [23]). Wiener gives the solution [17] in the form

$$\phi_C(\omega) = \frac{1}{\psi(\omega)} \frac{1}{2\pi} \int_0^\infty e^{-i\omega t} dt \int_{-\infty}^{\infty} \frac{S_{I,OS}(u) - S_{I,OM}(u)}{\psi(u)} e^{iut} du; \quad (4.17)$$

where:

$\phi_C(\omega)$ is the frequency response function corresponding to $W_C(t)$;

$S_{I,OS}(\omega)$ is the Fourier Transform of $\rho_{I,OS}(\tau)$, i.e. the cross spectral density of $I_S(t,w)$ and $O_S(t,w)$;

$S_{I,OM}(\omega)$ is the cross spectral density of $I_S(t,w)$ and $O_M(t,w)$;

$S_I(\omega) = \psi(\omega)\,\overline{\psi(\omega)}$.

That the right hand side of (17) is the frequency response function corresponding to $W_C(x)$ is readily confirmed by direct substitution, since

$$S_{I,OS}(u) - S_{I,OM}(u) = \psi(u)\,\overline{\psi(u)}\,\phi_C(u),$$

i.e. $$\phi_C(u) = \frac{S_{I,OS}(u) - S_{I,OM}(u)}{S_I(u)} \qquad (4.18)$$

and therefore the right hand side of (17) is

$$\frac{1}{\overline{\psi(\omega)}}\,\frac{1}{2\pi}\int_0^{\infty} e^{-i\omega t}dt \int_{-\infty}^{\infty}\psi(u)\,\phi_C(u)\,e^{iut}du. \qquad (4.19)$$

Moreover the rational expression $\psi(u)\,\phi_C(u)$ has no poles below the real axis and therefore

$$\int_{-\infty}^{\infty}\psi(u)\,\phi_C(u)\,e^{iut}du \quad \text{is zero for} \quad t < 0,$$

whence (19) becomes

$$\frac{1}{\overline{\psi(\omega)}}\,\frac{\psi(\omega)}{1}\,\phi_C(\omega) = \phi_C(\omega). \qquad (4.20)$$

Both the solution (17) and the less general form (18) are of restricted practical application, since they involve Fourier transformations of the data. However it is useful to represent (17) as two L type transformations in series, as in fig. 6, where the frequency function of each transformation is shown.

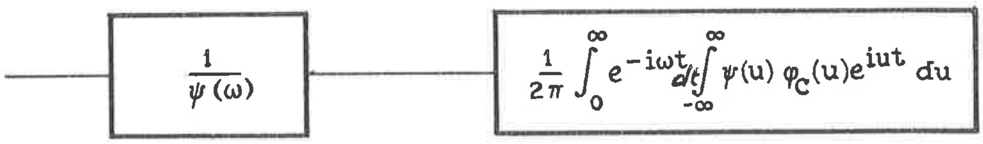

FIGURE 6. SOLUTION TO WIENER-HOPF EQUATION SHOWN AS TWO COMPONENTS IN SERIES

As shown in theorem 3 there exists an L type transformation such that, for almost all w_ι,

$$L[I_S(t,w_\iota)] = M_\alpha(t,w_\iota),$$

where $M_\alpha(t,w)$ is an S.G.M. having autocorrelation function $\frac{\alpha}{2}\, e^{-\alpha|\tau|}$ and α may be chosen arbitrarily large. The frequency response function corresponding to this L is

$$\frac{1}{\psi(\omega)} \cdot \frac{\alpha}{(\alpha+i\omega)} \tag{4.21}$$

which, for large α, approaches the frequency response function of the first component in fig. 6.

Similarly, since $I_S(t,w)$ is a type X_1 random process, its realisations may be written in the form,

$$L_1[M_\alpha(t,w_\iota)] = I_S(t,w_\iota),$$

where L_1 is a transformation of the type described in definition 6. In this case the frequency response function corresponding to L_1 is

$$\psi(\omega)\left(\frac{\alpha+i\omega}{\alpha}\right),$$

and hence that corresponding to the product transformation $T_C\,.L_1$ is

$$\phi_C(\omega)\,\psi(\omega)\left(\frac{\alpha+i\omega}{\alpha}\right). \tag{4.22}$$

For large α the expression (22) approaches

$$\phi_C(\omega)\,\psi(\omega),$$

the frequency response function of the second component in fig. 6.

The first component of fig. 6 may therefore be considered to transform the input to "white noise". Theorem 4 shows that the weighting function corresponding to the second component may then be found by cross correlating its input and output. Since $S_I(\omega)$ and therefore $\psi(\omega)$ is known the weighting function of the first component is also known. The convolution of these two weighting functions is the required weighting function $W_C(t)$.

The possibility of applying these ideas to the present problem is illustrated in fig. 7 which is an elaboration of fig. 5A introducing these two transformations L and L_1. If the parameter α in $M_\alpha(t,w)$ is large enough, then, as is clear from this figure and theorem 4 the weighting function corresponding to T_C may be estimated by cross correlating $M_\alpha(t,w_t)$ with

$$L[O_S(t,w_t) - O_M(t,w_t)].$$

A number of components in fig. 7 are included only to explain the ideas more fully, the figure may clearly be simplified to fig. 8.

The procedure just discussed is equivalent to modifying the input to the components T_M and T_S so that the autocorrelation function, $\rho(\tau)$, of this modified input is small, except for values of τ near $|\tau| = 0$, and

$$\int_{-\infty}^{\infty}\rho(\tau)d\tau = 1.$$

The corresponding spectrum is then virtually constant and approximately equal to 1, $0 \leq |\omega| < \omega_0$, where ω_0 is large and positive. This procedure is known as "whitening" the input; its usefulness has been recognised in similar activities, e.g. the measurement of power spectra [51,P.28]. It

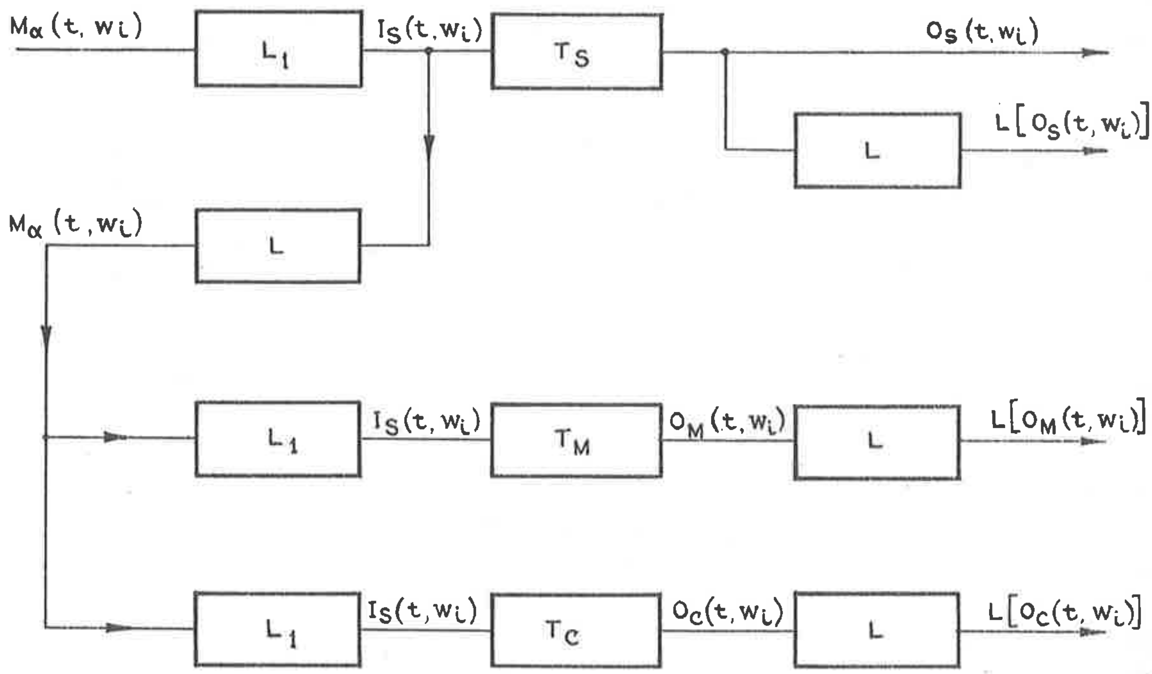

FIGURE 7. AN ELABORATION OF FIGURE 5A ILLUSTRATING A METHOD OF APPLYING THEOREM 4 TO THE CALCULATION OF THE WEIGHTING FUNCTION CORRESPONDING TO T_C .

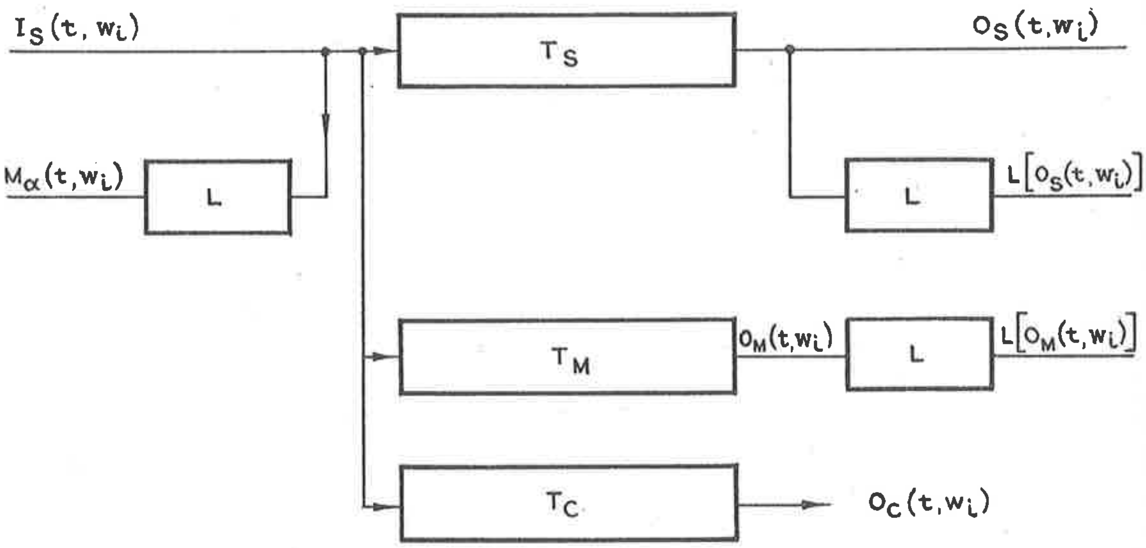

FIGURE 8. A SIMPLIFICATION OF FIGURE 7

has also been intuitively recognised in the earlier work on model checking [2 - 5]. As already indicated the computation problem is often easier if "whitened" inputs are used. Cross correlation techniques are then more easily applied and, as discussed at the end of this section, other computation techniques are also assisted by a "whitened" input.

A further advantage of a "whitened" input may be mentioned here although it is not relevant to this chapter. Errors introduced in recording often approximate a realisation of a random process whose spectrum is significant at high frequencies. "Whitening" of the input, if carried out before recording, will help maintain a satisfactory signal to noise ratio at these higher frequencies [51, P.28].

Two possible methods for achieving the effect of a "whitened" input are illustrated in figs. 9A and 9B. In each figure physical components are shown, vice the system components of fig. 8. The symbols P and P_1 denote the physical components which are characterized as systems by L and L_1. The first method, fig. 9A, may be impractical in many cases since it interferes with the operation of the physical system.

A third method, viz. to use for model checking, only those inputs which are judged by inspection to be rich in high frequency content, has also been used, [2].

A further possibility is to use specially selected inputs; e.g. step functions, or discrete interval binary noise repeated periodically in the manner suggested by Anderson and others [35]. However the use of such inputs

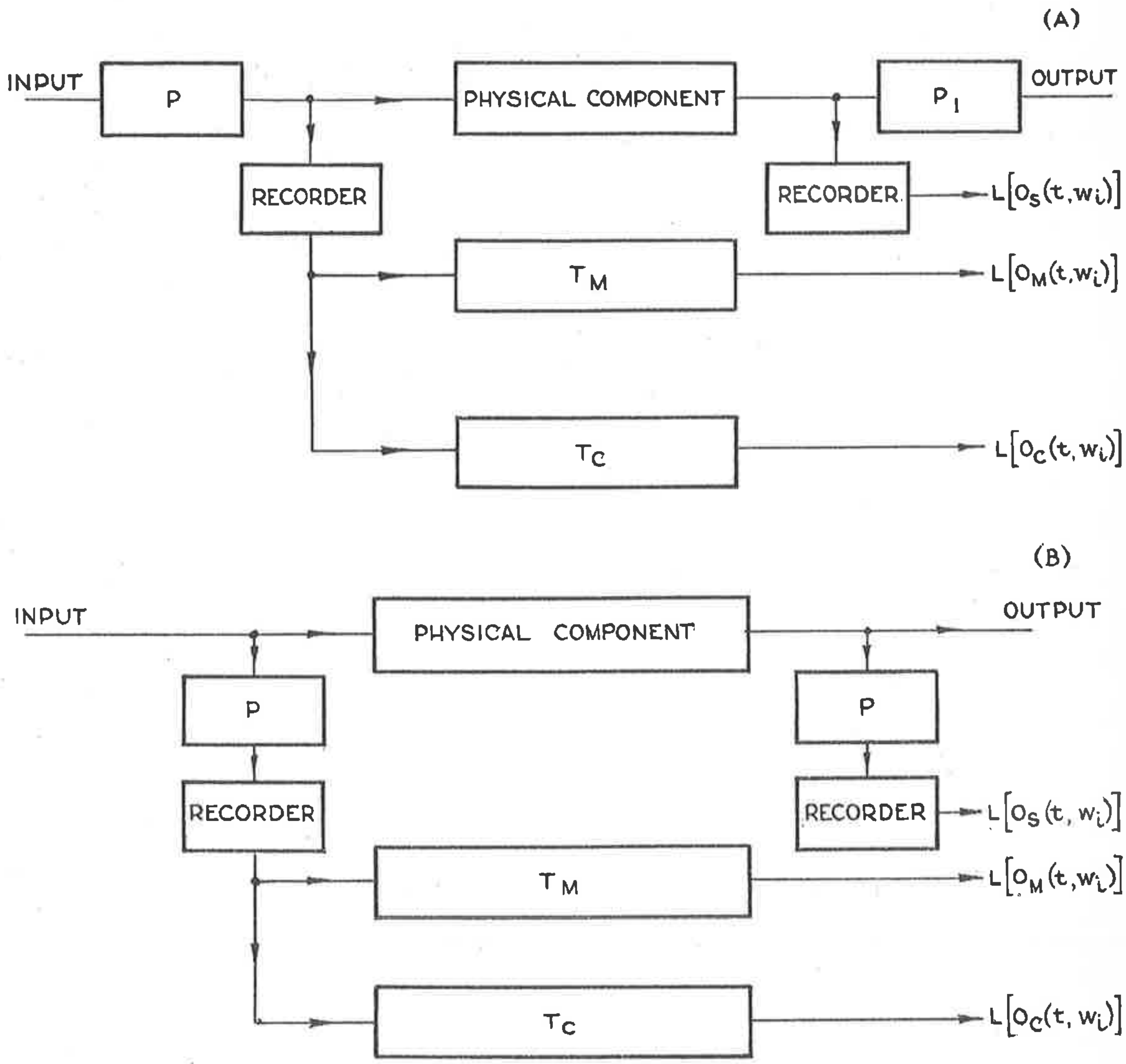

FIGURE 9. TWO POSSIBLE METHODS OF OBTAINING A "WHITENED" INPUT.

involves a degree of interference with the normal operation of the physical system which may not be permitted or possible.

Several other methods for tackling the Wiener-Hopf equation are known. Laning and Battin [23, P.283] describe in considerable detail a method, used several years earlier by the author*, for solving this equation when the kernel and the right hand side are expressed as sums of exponentials. They also describe a complete set of exponentials (some of which they tabulate), orthogonal over $(0,\infty)$, which may be used to approximate the kernel and the right hand side of this equation. Laguerre functions have also been proposed for this purpose, [17],[26].

An effective, if elementary, method for the present purpose, assuming no generalised functions are involved, is that described in reference [52, P.448]. Assume a solution of equation (2) of the form

$$W_C(\tau) = a_1 g_1(\tau) + a_2 g_2(\tau) + + + a_n g_n(\tau), \qquad (4.23)$$

where the $g_\iota(\tau)$ are functions which are orthogonal $(0,\infty)$; e.g. the exponentials of reference [23]. Then determine values for the a_ι by a least squares method. Thus the expression

$$\int_0^T [\int_0^\infty \rho_I(\tau-x)\{\sum_{\iota=1}^n a_\iota g_\iota(x)\}dx - \{\rho_{I,OS}(\tau) - \rho_{I,OM}(\tau)\}]^2 d\tau \quad (4.24)$$

may be minimised with respect to the a_ι. The limit of integration T may be taken as ∞ in the present case where the length of data available is unlimited; in practice T

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* Classified paper published by Royal Aeronautical Establishment, Farnborough, U.K.

will be finite. Similarly the number of terms taken in the approximation $\sum_{l=1}^{n} a_l g_l(x)$, i.e. the value of n, will depend on the particular application.

The minimisation of (24) leads to equations for the a_l in the form

$$\sum_{l=1}^{n} C_{kl} a_l = D_k, \tag{4.25}$$

where

$$C_{kl} = C_{lk} = \int_0^T \int_0^\infty \rho_I(\tau - x_1) g_k(x_1) dx_1 \int_0^\infty \rho_I(\tau - x_2) g_l(x_2) dx_2 \, d\tau,$$

$$D_k = \int_0^T \{\rho_{I,OS}(\tau) - \rho_{I,OM}(\tau)\} \int_0^\infty \rho_I(\tau - x) g_k(x) dx \, d\tau.$$

The following, somewhat trivial, example illustrates the effectiveness of this process.

Let the system component have weighting function

$$\frac{0.9}{2.2} e^{-\frac{t}{2.2}}, \qquad t \geq 0,$$

while that of the model component is

$$\tfrac{1}{2} e^{-\frac{1}{2}t}, \qquad t \geq 0,$$

and suppose

$$\rho_I(\tau) = e^{-|\tau|}.$$

Direct calculation shows that

$$\rho_{I,OM}(\tau) = \tfrac{1}{3} e^{\tau}, \qquad \tau \leq 0,$$

$$= \tfrac{4}{3} e^{-\frac{1}{2}\tau} - e^{-\tau}, \qquad \tau \geq 0,$$

$$\rho_{I,OS}(\tau) = \frac{0.9}{3.2} e^{\tau}, \qquad \tau \leq 0$$

$$= \frac{3.3}{3.2} e^{-\frac{\tau}{2.2}} - \tfrac{3}{4} e^{-\tau}, \qquad \tau \geq 0;$$

so that

$$\rho_{I,OS}(\tau) - \rho_{I,OM}(\tau) = \frac{3.3}{3.2}\, e^{-\frac{\tau}{2.2}} - \frac{4}{3}\, e^{-\frac{1}{2}\tau} + \frac{1}{4}\, e^{-\tau}, \qquad \tau \geq 0.$$

The orthogonal exponentials [23] are formed from the sequence

$$e^{-ct},\ e^{-2ct},\ e^{-3ct} \ldots .$$

The parameter c must be determined; accordingly the assumption has been made that the correction, $W_C(t)$, required to the model weighting function, decays at approximately the same rate as the model weighting function itself. This assumption suggests that the value $c = \frac{1}{2}$ should be used in this example; however the more conservative value $c = \frac{1}{4}$ has been used. With this value of c the orthogonal exponentials are,

$$g_1(t) = \frac{\sqrt{2}}{2}\, e^{-\frac{1}{4}t},$$

$$g_2(t) = 3e^{-\frac{1}{2}t} - 2e^{-\frac{1}{4}t},$$

$$g_3(t) = \frac{\sqrt{6}}{2}\left(10e^{-\frac{3}{4}t} - 12e^{-\frac{1}{2}t} + 3e^{-\frac{1}{4}t}\right),$$

etc. Only the first two have been used, i.e. the approximation

$$W_C(t) \simeq a_1 g_1(t) + a_2 g_2(t)$$

has been assumed.

After performing the required calculations the values

$$a_1 = -0.0588,$$

$$a_2 = -0.0380$$

were found and the weighting function for the corrected model is, then,

$$0.386e^{-\frac{1}{2}t} + 0.035e^{-\frac{1}{4}t}, \qquad t \geq 0.$$

A comparison of the weighting functions for the original model, the system component and the corrected model is shown in the following table.

Table 1.

Weighting Function

t	Original Model,	System Component,	Corrected Model
0	0·500	0·409	0·421
$\frac{1}{2}$	0·389	0·326	0·331
1	0·303	0·259	0·261
2	0·184	0·165	0·153
3	0·111	0·105	0·103
4	0·067	0·066	0·065
6	0·025	0·026	0·027
8	0·009	0·011	0·012
10	0·0034	0·0044	0·0053

However a more useful criterion for comparing the adequacy of the two models is $A(t)$, (2.1), which may be written, by virtue of the ergodic hypothesis,

$$1 - \left\{ \frac{\int_0^\infty\int_0^\infty \rho_I(\tau - x) W_C(\tau) W_C(x) d\tau \; dx}{\int_0^\infty\int_0^\infty \rho_I(\tau - x) W_S(x) W_S(\tau) d\tau \; dx} \right\}^{\frac{1}{2}} . \qquad (4.26)$$

This expression increases from approximately 0·85 for the original model to 0·98 for the corrected model.

The following similar approach to the problem also involves approximating $W_C(t)$ by an expression of the form (23). The a_t may be chosen to have values which minimise, in a least squares manner, the quantity

$$\int_0^\infty I_S(t-x)\{\sum_{l=1}^{n} a_l g_l(x)\}dx - [O_S(t) - O_M(t)]. \qquad (4.27)$$

This leads to a different, though similar, set of equations for the a_l, viz.

$$\sum_{l=1}^{n} C'_{kl} a_l = D'_k ; \qquad (4.28)$$

where

$$C'_{kl} = C'_{lk} = \int_0^\infty\int_0^\infty \rho_I(\tau-x) g_l(x) g_k(\tau) dx\, d\tau,$$

$$D'_k = \int_0^\infty [\rho_{I,OS}(\tau) - \rho_{I,OM}(\tau)] g_k(\tau) d\tau.$$

In general these values for the a_l will differ from those obtained from (25) with $T = \infty$. However if $\rho_I(\tau)$ is of the form $\frac{\alpha}{2} e^{-\alpha|\tau|}$, then calculations similar to those in the appendix show:

$$\lim_{\alpha\to\infty} C_{kl} = \lim_{\alpha\to\infty} C'_{kl} = 1, \qquad i = k,$$
$$= 0, \qquad i \neq k;$$

$$\lim_{\alpha\to\infty} D_k = \lim_{\alpha\to\infty} D'_k ;$$

i.e. the values of a_l from (28) approach those from (25) as α approaches ∞.

Also, if the input is "whitened", then C_{kl} and C'_{kl} will be small for $i \neq k$ compared with their value for $i = k$, and the computation of the a_l from (25) or (28) will often be assisted; the off-diagonal terms in the matrix (C_{kl}) or (C'_{kl}) being small compared with the diagonal terms.

4.1.1 Correction by iterated cross correlation

As mentioned above the results of theorem 4 may be applied to the problem considered in this section. Providing $I_S(t,w)$ approximates white noise, a good estimate of the correcting weighting function $W_C(t)$ may be found by cross correlating the input $I_S(t,w_l)$ with the difference between the two outputs $O_S(t,w_l)$ and $O_M(t,w_l)$. The correction to be made to the model weighting function is then

$$\rho_{I,OC}(t) + \rho_{I,OC}(-t), \qquad t \geqslant 0,$$

where

$$\rho_{I,OC}(t) = \rho_{I,OS}(t) - \rho_{I,OM}(t).$$

After this correction has been made the model weighting function becomes

$$W_M(t) + \rho_{I,OC}(t) + \rho_{I,OC}(-t), \qquad t \geqslant 0,$$

so that there remains an error

$$W_S(t) - W_M(t) - [\rho_{I,OC}(t) + \rho_{I,OC}(-t)], \qquad t \geqslant 0,$$

$$= W_C(t) - [\rho_{I,OC}(t) + \rho_{I,OC}(-t)], \qquad t \geqslant 0.$$

In principle this procedure may be repeated; it is the purpose of the next theorem to determine conditions which ensure that such an iterative procedure converges. The particular question answered is: "how closely must $I_S(t,w)$ approximate white noise in order that the iteration will converge?"

<u>Theorem 7</u>

Let $I_S(t,w)$ be a random process of type X_1 having autocorrelation function $\rho_I(\tau)$ and spectrum $S_I(\omega)$ such that

$$S_I(\omega) < 2, \quad \text{all } \omega.$$

Let the weighting functions, $W_S(t)$ and $W_M(t)$, have the exponential form (3.1) and correspond to the system and model

components whose inputs are realisations of $I_S(t,w)$.
Let $W_{M(1)}(t)$, $W_{M(2)}(t)$,..$W_{M(n)}(t)$,... be a sequence of model weighting functions such that

$$W_{M(1)}(t) = W_M(t)$$

$$W_{M(n+1)}(t) = W_{M(n)}(t) + \rho_{I,OC(n)}(t) + \rho_{I,OC(n)}(-t), \qquad t \geq 0,$$

$$= 0 \qquad , \qquad t < 0;$$

where

$$O_{C(n)}(t,w_i) = \int_0^\infty I_S(t-x,w_i)W_{C(n)}(x)dx$$

and

$$W_{C(n)}(t) = W_S(t) - W_{M(n)}(t).$$

Then

$$\lim_{n\to\infty} W_{C(n)}(t) = 0 \qquad , \quad \text{all } t.$$

<u>Proof</u> Let the frequency response function corresponding to each $W_{C(n)}(t)$ have real part $R_n(\omega)$.
Then, as in theorem 4,

$$\rho_{I,OC(1)}(t) + \rho_{I,OC(1)}(-t) = \int_0^\infty [\rho_I(t-x) + \rho_I(t+x)]W_{C(1)}(x)dx$$

has Fourier transform

$$S_I(\omega)\ 2R_1(\omega),$$

which is rational. Therefore for $t > 0$, the expression

$$\rho_{I,OC(1)}(t) + \rho_{I,OC(1)}(-t)$$

has the form (3.1); hence

$$W_{M(2)}(t) = W_{M(1)}(t) + \int_0^\infty [\rho_I(t-x) + \rho_I(t+x)]W_{C(1)}(x)dx, \quad t \geq 0,$$

also has this form.
Similarly each of the $W_{M(n)}(t)$ is of the form (3.1).

Since

$$W_{M(2)}(t) = W_{M(1)}(t) + \int_0^\infty [\rho_I(t-x) + \rho_I(t+x)] W_{C(1)}(x)dx, \quad t \geq 0,$$

$$= 0 \qquad , \quad t < 0,$$

then

$$W_{C(2)}(t) = W_S(t) - W_{M(2)}(t)$$

$$= W_{C(1)}(t) - \int_0^\infty [\rho_I(t-x) + \rho_I(t+x)] W_{C(1)}(x)dx, \quad t \geq 0,$$

$$= 0 \qquad , \quad t < 0.$$

So, as in theorem 4,

$$R_2(\omega) = R_1(\omega) - S_I(\omega)R_1(\omega)$$

$$= R_1(\omega)[1 - S_I(\omega)].$$

Similarly

$$R_{n+1}(\omega) = R_n(\omega)[1 - S_I(\omega)]$$

$$= R_1(\omega)[1 - S_I(\omega)]^n.$$

Now, for a type X_1 process, $S_I(\omega) > 0$ and therefore for this process

$$0 < S_I(\omega) < 2,$$

$$\text{i.e. } |1 - S_I(\omega)| < 1.$$

Hence

$$\lim_{n\to\infty} R_n(\omega) = 0, \quad \text{all } \omega.$$

Also $$W_{C(n)}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2R_n(\omega)e^{i\omega t}d\omega \qquad t \geq 0,$$

and therefore

$$\lim_{n\to\infty} W_{C(n)}(t) = \frac{1}{2\pi}\lim_{n\to\infty}\int_{-\infty}^{\infty} 2R_n(\omega)e^{i\omega t}d\omega, \qquad t \geq 0,$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{n\to\infty} 2R_n(\omega)e^{i\omega t}d\omega, \qquad t \geq 0,$$

$$= 0, \qquad t \geq 0.$$

Since, for all n,

$$W_{C(n)}(t) = 0, \qquad t < 0,$$

the theorem is proved.

In agreement with the assumptions of this chapter, the strict application of theorem 7 demands the impractical requirement that the length of data available for model checking is unlimited. Nevertheless this theorem gives some hope that quite large departures from a flat spectrum may be tolerated if the records are not too short and several iterations are possible.

4.2 Model correction when generalised functions are involved

If the correction $W_C(t)$ contains generalised functions then it will usually be necessary to modify the procedures discussed in section 4.1. Thus referring to figs. 7 and 8 and the related discussion it is possible, under these circumstances, that the expression

$$L[O_S(t,w_\iota) - O_M(t,w_\iota)]$$

may not exist.

A suitable modification, which will overcome this difficulty, is illustrated in fig. 10, where the transformations L, L_1, T_S, T_M and T_C have the same meaning as in fig. 7. With this modification the methods discussed in 4.1 may be used to estimate the weighting function corresponding to the product transformation $L_1 \cdot T_C$. The correction to the model is then the product of the two transformations L

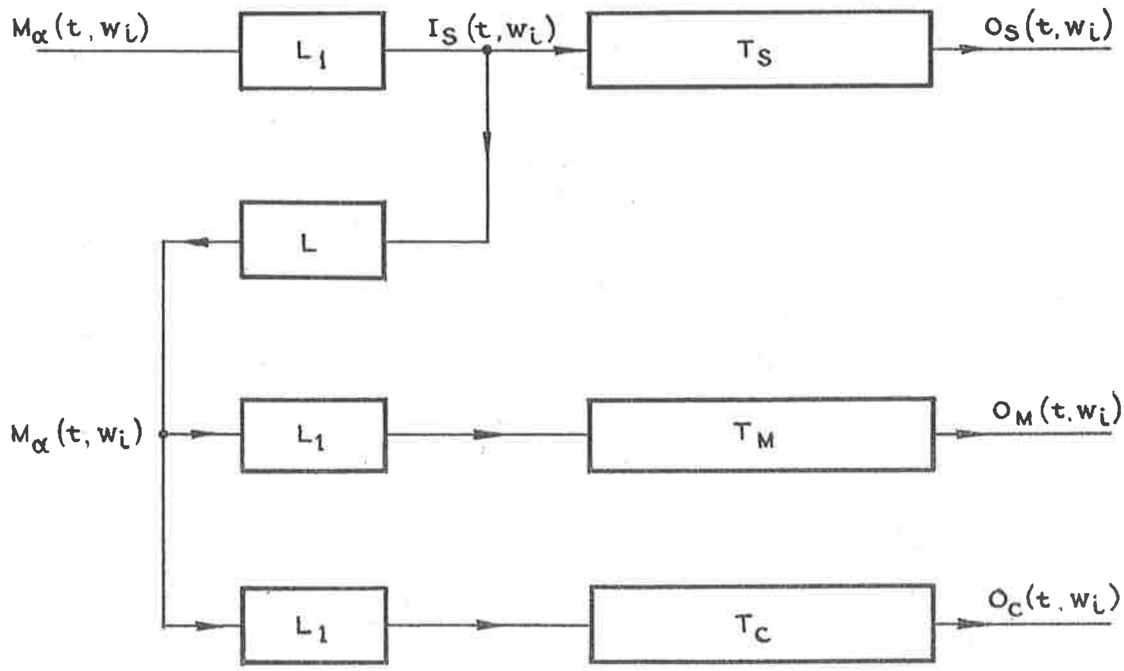

FIGURE 10. ILLUSTRATING MODIFIED APPROACH TO MODEL CORRECTION WHEN THE CORRECTING TRANSFORMATION MAY CONTAIN GENERALISED FUNCTIONS

and $L_1 \cdot T_C$. The weighting functions corresponding to both these transformations are known and the required correction may therefore be made. If it is necessary to estimate $W_C(t)$ explicitly it may be found as the convolution of the weighting functions corresponding to L and $L_1 \cdot T_C$.

CHAPTER 5.

STATIONARY LINEAR SYSTEM AND MODEL
ERROR FREE RECORDS OF FINITE LENGTH

In chapter 4 the length of the records available for model checking was assumed to be unlimited. This assumption is now relaxed and the more practical case of finite length records is considered; all the other assumptions of chapter 4 are retained. The case of one set of records of length T_n is first considered.

The following three difficulties arise from this restriction on the length of the records.

(i) Since the output of the model, $O_M(t,w_l)$, to an input, $I_S(t,w_l)$, applied at time t_0 and removed at time $T_n + t_0$, is

$$\int_0^{t-t_0} I_S(t-x,w_l)W_M(x)dx, \qquad t_0 < t \leq T_n + t_0,$$

then no useful estimate of the weighting function

$$W_C(t) = W_S(t) - W_M(t)$$

can be expected for values of its argument greater than some T_m which is less than T_n. If accordingly, no correction is made to the model weighting function $W_M(t)$ for $t > T_m$, the corrected model will, due to this fact alone, have an error in its output at time t of

$$\int_{T_m}^{\infty} I_S(t-x,w_t)W_C(x)dx. \tag{5.1}$$

Consequently even though

$$W_M(t) = W_S(t), \qquad 0 \leqslant t \leqslant T_m$$

the adequacy of the corrected model, as measured by A(t), (2.1) will not be better than

$$1 - \left\{ \frac{\int_{T_m}^{\infty}\int_{T_m}^{\infty} \rho_I(\tau-x)W_C(x)W_C(\tau)dxd\tau}{\sigma_{OS}^2} \right\}^{\frac{1}{2}}, \tag{5.2}$$

where $\sigma_{OS}^2 = \int_0^{\infty}\int_0^{\infty} \rho_I(\tau-x)W_S(x)W_S(\tau)dxd\tau.$

The expression (2) can not be calculated exactly without knowing $W_C(t)$, $T_m \leqslant t < \infty$; but in practice a useful estimate of the value of T_m, which ensures (2) has an acceptable value, may often be found from known characteristics of the physical component, its inputs and outputs. For example the design of the physical component will usually have entailed some consideration of the spectrum of $I_S(t,w)$ and therefore some information on which to base an estimate of $\rho_I(\tau)$ will be available. A conservative estimate of $W_C(t)$ for this purpose would be $W_M(t)$, the known weighting function of the model, and the expression

$$\sigma_{OM}^2 = \int_0^{\infty}\int_0^{\infty} \rho_I(\tau-x)W_M(x)W_M(\tau)dxd\tau$$

may be used as an estimate of σ_{OS}^2. It will be assumed, then, that a useful estimate of T_m, based on the expression

$$\frac{\int_{T_m}^{\infty}\int_{T_m}^{\infty}\rho_I(\tau-x)W_M(x)W_M(\tau)dxd\tau}{\sigma^2_{OM}}, \tag{5.3}$$

may be made.

(ii) Some length T_m' at the beginning of the record of $O_M(t,w_l)$ must be discarded, since irrelevant transients will be set up when the record of $I_S(t,w_l)$ is first applied to the model; some settling time [41] must be allowed for these to die out. Zero time will be taken as the time of commencement of an acceptable output from the model after discarding this length T_m'.

The output of the model at $t = 0$ will then differ from the output which would have resulted had the complete input prior to $t = 0$ been known, by

$$\int_{T_m'}^{\infty} I(-x,w_l)W_M(x)dx. \tag{5.4}$$

The square of this quantity averaged over all w_l, i.e. all realisations of $I_S(t,w)$, and divided by σ^2_{OM} is

$$\frac{\int_{T_m'}^{\infty}\int_{T_m'}^{\infty}\rho_I(\tau-x)W_M(x)W_M(\tau)dxd\tau}{\sigma^2_{OM}}; \tag{5.5}$$

which may therefore be used as a measure of the significance of this effect. The value of this expression will usually decrease rapidly as t increases from zero since the lower limits of integration become $T_m' + t$.

The expressions (3) and (5) are identical apart from T_m and T_m', the lower limits of integration; further, the total length of record available for model checking will usually be much greater than T_m. In general therefore it will be of no practical importance and of some theoretical convenience to take $T_m = T_m'$.

(iii) Estimates of autocorrelation and cross-correlation functions based on records of restricted length will not be exact. Such estimates however must be used if the techniques discussed in previous chapters are to be applied to model correction; they will lead to an estimate, $W_E(t)$, of $W_C(t)$, the usefulness of which will depend on the length of the available records.

The next section is concerned with the case of a whitened S.G.M. input. The relation between the adequacy of the corrected model and the length of record available for model correction is investigated. The results obtained may easily be extended to cover the case of any type X_1 input and this is done in section 5.2.

5.1 Gauss Markov input. Single sample

Let the system input be a realisation of an S.G.M., $M_\alpha(t,w)$, whose correlation function $\rho_{M\alpha}(\tau) = \frac{\alpha}{2}e^{-\alpha|\tau|}$, and let the correcting weighting function $W_C(t)$ contain no generalised functions. Because of the residual error, $W_C(t) - W_E(t)$, the adequacy, $A(t)$, of the corrected model is in this case

$$1 - \left[\frac{\int_0^\infty \int_0^\infty \rho_{M\alpha}(\tau - x)[W_C(x) - W_E(x)][W_C(\tau) - W_E(\tau)]dxd\tau}{\sigma^2_{OS}} \right]^{\frac{1}{2}}. \quad (5.6)$$

Setting $W_E(t) = 0, \quad t > T_m$, and using theorem 4 to estimate $W_C(t), \ 0 \leqslant t \leqslant T_m$, gives

$$W_E(t) = \rho_{M\alpha,OC}(t) + \rho_{M\alpha,OC}(-t), \qquad 0 \leqslant t \leqslant T_m,$$

where

$$O_C(t,w_l) = O_S(t,w_l) - O_M(t,w_l).$$

From the data sample values $R_{M\alpha,OC}(\tau,w_l)$, $R_{M\alpha,OC}(-\tau,w_l)$, of these cross-correlation functions $\rho_{M\alpha,OC}(\tau)$ and $\rho_{M\alpha,OC}(-\tau)$ may be calculated for some values of τ. For example

$$R_{M\alpha,OC}(\tau,w_l) = \frac{1}{T_n'} \int_0^{T_n'} M_\alpha(t-\tau,w_l)O_C(t,w_l)dt \qquad (5.7)$$

where T_n' is the effective length of the record after discarding an amount $T_m' = T_m$;

i.e. $T_n' = T_n - T_m$.

For $0 \leqslant \tau \leqslant T_m$ the expression (7) will be used as an estimate of $\rho_{M\alpha,OC}(\tau)$ and accordingly the function $R'_{M\alpha,OC}(\tau,w_l)$ is defined such that

$$R'_{M\alpha,OC}(\tau,w_l) = R_{M\alpha,OC}(\tau,w_l), \qquad 0 \leqslant \tau \leqslant T_m,$$

$$\qquad\qquad (5.8)$$

$$= 0 \qquad \tau > T_m.$$

However, as shown in theorem A.1. in the appendix, $\rho_{M\alpha,OC}(-\tau)$ decays like $e^{-\alpha\tau}$ for $\tau > 0$ and hence the definition of $R'_{M\alpha,OC}(\tau,w_l)$ is completed by

$$R'_{M\alpha,OC}(-\tau,w_l) = R_{M\alpha,OC}(-\tau,w_l), \quad 0 \leqslant \tau \leqslant T(\alpha), \tag{5.9}$$

$$= 0 \qquad \tau > T(\alpha),$$

where $T(\alpha)$ approaches zero as α tends to ∞. Actually an effective length $T'_n + T(\alpha)$ of data would be required in order to compute $R'_{M\alpha,OC}(\tau)$, $-T(\alpha) \leqslant \tau \leqslant T_m$, using (7). However, since $T'_n >>> T(\alpha)$, the difference between the expression (7) and

$$\frac{1}{T'_n - T(\alpha)} \int_0^{T'_n - T(\alpha)} M_\alpha(t-\tau,w_l)O_C(t,w_l)dt$$

has no practical or theoretical significance here.

The expression

$$\int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)[W_C(x) - W_E(x)][W_C(\tau) - W_E(\tau)]dxd\tau \tag{5.10}$$

is the only part of (6) which is affected by $W_E(\tau)$. Apart from the constant σ^2_{OS} this expression is equal to the square of the inadequacy of the corrected model.

Since

$$W_E(\tau) = R'_{M\alpha,OC}(\tau,w_l) + R'_{M\alpha,OC}(-\tau,w_l), \quad \tau \geqslant 0, \tag{5.11}$$

the expression (10) depends on the particular realisation, w_l, which was used for checking the model. The expectation of (10) is then, apart from the constant σ^2_{OS}, the mean square inadequacy of the corrected model, where the average is taken over all the realisations which may have been available for model checking. The value of this expectation will depend on T_h'; a small value will correspond to an adequate model. It is useful therefore to study the variation of this expectation with T_h'.

Substituting the expression (11) for $W_E(\tau)$ in (10) and taking the expectation yields

$$E\Big[\int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)\{W_C(x)-[R'_{M\alpha,OC}(x,w)+R'_{M\alpha,OC}(-x,w)]\}\{W_C(\tau) - -[R'_{M\alpha,OC}(\tau,w)+R'_{M\alpha,OC}(-\tau,w)]\}dxd\tau\Big]$$

$$= \int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)E[\{W_C(x)-[R'_{M\alpha,OC}(x,w)+R'_{M\alpha,OC}(-x,w)]\}\{W_C(\tau) - -[R'_{M\alpha,OC}(\tau,w)+R'_{M\alpha,OC}(-\tau,w)]\}]dxd\tau. \quad (5.12)$$

Calculation of the required expectations is trivial apart from terms of the form

$$E[R'_{M\alpha,OC}(x,w)R'_{M\alpha,OC}(\tau,w)], \quad (5.13)$$

which for, $0 \leqslant x \leqslant T_m$, and, $0 \leqslant \tau \leqslant T_m$,

$$= E[R_{M\alpha,OC}(x,w)R_{M\alpha,OC}(\tau,w)],$$

$$= E\Big[\frac{1}{(T_h')^2}\int_0^{T_h'}\int_0^{T_h'} M_\alpha(t_1-x,w)O_C(t_1,w)M_\alpha(t_2-\tau,w)O_C(t_2,w)dt_1dt_2\Big]. \quad (5.14)$$

Since the random variables $M_\alpha(t_1-x,w)$, $O_C(t_1,w)$, $M_\alpha(t_2-\tau,w)$ and $O_C(t_2,w)$ have a joint normal distribution, a well known result, Bendat [27,P.288], enables (14) to be written

$$\frac{1}{(T_h')^2}\int_0^{T_h'}\int_0^{T_h'} \rho_{M\alpha,OC}(x)\rho_{M\alpha,OC}(\tau)+\rho_{M\alpha}(t_2-t_1-\tau+x)\rho_{OC}(t_2-t_1) + \rho_{M\alpha,OC}(t_2-t_1+x)\rho_{M\alpha,OC}(t_1-t_2+\tau)dt_1dt_2. \quad (5.15)$$

The important case is that of a "whitened" input and accordingly the limit of (12) as α tends to ∞ will be considered here. The following results are justified by the work in the appendix.

$$\lim_{\alpha\to\infty} \int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)E[W_C(x)W_C(\tau)]\,dx\,d\tau$$

$$= \int_0^\infty [W_C(x)]^2\,dx. \tag{5.16}$$

$$\lim_{\alpha\to\infty} \int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)E[-W_C(x)R'_{M\alpha,OC}(\tau,w)]\,dx\,d\tau$$

$$= \lim_{\alpha\to\infty} - \int_0^{T_m}\int_0^\infty \rho_{M\alpha}(\tau-x)W_C(x)\rho_{M\alpha,OC}(\tau)\,dx\,d\tau$$

$$= - \int_0^{T_m} [W_C(x)]^2\,dx. \tag{5.17}$$

$$\lim_{\alpha\to\infty} - \int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)E[W_C(x)R'_{M\alpha,OC}(-\tau,w)]\,dx\,d\tau$$

$$= \lim_{\alpha\to\infty} - \int_0^{T(\alpha)}\int_0^\infty \rho_{M\alpha}(\tau-x)W_C(x)\rho_{M\alpha,OC}(-\tau)\,dx\,d\tau$$

$$= 0. \tag{5.18}$$

$$\lim_{\alpha\to\infty} - \int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)E[R'_{M\alpha,OC}(+x,w)W_C(\tau)]\,d\tau\,dx$$

$$= - \int_0^{T_m} [W_C(x)]^2\,dx. \tag{5.19}$$

$$\lim_{\alpha\to\infty} - \int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)E[R'_{M\alpha,OC}(-x,w)W_C(\tau)]\,d\tau\,dx$$

$$= 0. \tag{5.20}$$

$$\lim_{\alpha\to\infty} \int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)E[R'_{M\alpha,OC}(x,w)R'_{M\alpha,OC}(\tau,w)]\,dx\,d\tau$$

is made up of three terms corresponding to (15):

(i) the limit as $\alpha \to \infty$

$$\int_0^{T_m}\int_0^{T_m} \rho_{M\alpha}(\tau-x)\{\frac{1}{(T_h')^2}\int_0^{T_h'}\int_0^{T_h'} \rho_{M\alpha,OC}(x)\rho_{M\alpha,OC}(\tau)dt_1 dt_2\}dxd\tau$$

$$= \lim_{\alpha\to\infty}\int_0^{T_m}\int_0^{T_m} \rho_{M\alpha}(\tau-x)\rho_{M\alpha,OC}(x)\rho_{M\alpha,OC}(\tau)dxd\tau$$

$$= \int_0^{T_m}[W_C(x)]^2 dx; \qquad (5.21)$$

(ii) the limit as $\alpha \to \infty$

$$\int_0^{T_m}\int_0^{T_m} \rho_{M\alpha}(\tau-x)\{\frac{1}{(T_h')^2}\int_0^{T_h'}\int_0^{T_h'} \rho_{M\alpha}(t_2-t_1-\tau+x)\rho_{OC}(t_2-t_1)dt_1 dt_2\}dxd\tau$$

$$= \frac{T_m}{T_h'}\sigma^2_{OC}; \qquad (5.22)$$

(iii) the limit as $\alpha \to \infty$

$$\int_0^{T_m}\int_0^{T_m} \rho_{M\alpha}(\tau-x)\frac{1}{(T_h')^2}\int_0^{T_h'}\int_0^{T_h'} \rho_{M\alpha,OC}(t_2-t_1+x)\rho_{M\alpha,OC}(t_1-t_2+\tau)dt_1 dt_2 d\tau dx$$

$$= \frac{2}{(T_h')^2}\int_0^{T_m}\int_0^{\tau}(T_h'-t_3)W_C(\tau-t_3)W_C(\tau+t_3)dt_3 d\tau. \qquad (5.23)$$

$$\lim_{\alpha\to\infty}\int_0^{\infty}\int_0^{\infty} \rho_{M\alpha}(\tau-x)E[R'_{M\alpha,OC}(x,w)R'_{M\alpha,OC}(-\tau,w)]dxd\tau$$

$$= 0. \qquad (5.24)$$

$$\lim_{\alpha\to\infty}\int_0^{\infty}\int_0^{\infty} \rho_{M\alpha}(\tau-x)E[R'_{M\alpha,OC}(-x,w)R'_{M\alpha,OC}(\tau,w)]dxd\tau$$

$$= 0. \qquad (5.25)$$

$$\lim_{\alpha\to\infty}\int_0^{\infty}\int_0^{\infty} \rho_{M\alpha}(\tau-x)E[R'_{M\alpha,OC}(-x,w)R'_{M\alpha,OC}(-\tau,w)]dxd\tau$$

$$= 0. \qquad (5.26)$$

The limit as α tends to ∞ of the expression (12) then becomes, after dividing by σ^2_{OS},

$$\frac{T_m}{T_n'}\frac{\sigma^2_{OC}}{\sigma^2_{OS}} + \frac{2}{\sigma^2_{OS}(T_n')^2}\int_0^{T_m}\int_0^{\tau}(T_n'-t_3)W_C(\tau-t_3)W_C(\tau+t_3)dt_3\,d\tau$$

$$+ \frac{1}{\sigma^2_{OS}}\int_{T_m}^{\infty}W_C{}^2(x)dx. \qquad (5.27)$$

In order to study this expression further some assumption must be made concerning the unknown correction $W_C(t)$. The following is a simple assumption which seems to lead to useful results. Assume that the error, $W_C(t)$, in the model decays with the same time constant as the weighting function of the model. Estimate, from the model, a time constant $\frac{1}{a}$ which approximately describes the decay of $W_M(t)$ and assume, for this purpose,

$$W_C(t) = ke^{-at}, \qquad k > 0, \quad a > 0. \quad (5.28)$$

Then

$$\sigma^2_{OC} = \lim_{\alpha\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\alpha^2}{(\alpha^2+\omega^2)}\;\frac{k^2}{(\omega^2+a^2)}\,d\omega$$

$$= \frac{k^2}{2a};$$

whence, writing $aT_m = p$, and $aT_n' = q$, and substituting in (27) yields

$$\frac{\sigma^2_{OC}}{\sigma^2_{OS}}\frac{p}{q} + \frac{2a^2}{\sigma^2_{OS}q^2}\int_0^{T_m}\int_0^{\tau}(T_n'-t_3)\,k^2e^{-a\tau}e^{at_3}e^{-a\tau}e^{-at_3}dt_3\,d\tau + \frac{k^2}{\sigma^2_{OS}}\,\frac{e^{-2p}}{2a}$$

$$= \frac{\sigma^2_{OC}}{\sigma^2_{OS}}\left[\frac{p}{q} + \frac{4a^3}{q^2}\int_0^{T_m}e^{-2a\tau}(T_n'\tau - \frac{\tau^2}{2})d\tau + e^{-2p}\right]$$

$$= \frac{\sigma^2_{OC}}{\sigma^2_{OS}}\left\{\frac{p}{q} + \frac{4a^3}{q^2}\Big[T_n'\Big(\frac{\tau e^{-2a\tau}}{-2a} + \frac{e^{-2a\tau}}{-4a^2}\Big) + \tfrac{1}{2}\Big(\frac{\tau^2e^{-2a\tau}}{2a} + \frac{2e^{-2a\tau}}{8a^3}\Big)\Big]_0^{T_m} + e^{-2p}\right\}$$

$$= \frac{\sigma^2_{0C}}{\sigma^2_{0S}} \frac{p}{q}[1 + \frac{1}{p} - \frac{1}{2pq} - e^{-2p}(2 + \frac{1}{p} - \frac{p}{q} - \frac{1}{q} - \frac{1}{2pq} - \frac{q}{p})]; \tag{5.29}$$

which for $q >> p > 1$ is approximately

$$\frac{\sigma^2_{0C}}{\sigma^2_{0S}} \frac{p}{q} [1 + \frac{1}{p} - e^{-2p}(- \frac{q}{p})]$$

$$= \frac{\sigma^2_{0C}}{\sigma^2_{0S}} [\frac{p+1}{q} + e^{-2p}]. \tag{5.30}$$

For fixed q, i.e. fixed length of record, differentiation of (29) with respect to p shows that (29) has a minimum value if

$$1 + e^{-2p}[4p - \frac{2p^2}{q} - 2q] = 0,$$

or for $q >> p$

$$p \simeq \tfrac{1}{2} \ln(2q). \tag{5.31}$$

As an example, if the length of record available, T_n', were $\frac{100}{a}$ then T_m should be chosen about $\frac{2.7}{a}$ and the expected value of the square of the inadequacy of the corrected model, as measured by the expression (29), is about $\frac{1}{25} \frac{\sigma^2_{0C}}{\sigma^2_{0S}}$.

The value of (29) is, in many practical cases, not very sensitive to that of p. Thus for $q = 100$ and $p = 2$ the value of (29) is $0.047 \frac{\sigma^2_{0C}}{\sigma^2_{0S}}$, whereas for $q = 100$ and $p = 3$ it is $0.042 \frac{\sigma^2_{0C}}{\sigma^2_{0S}}$.

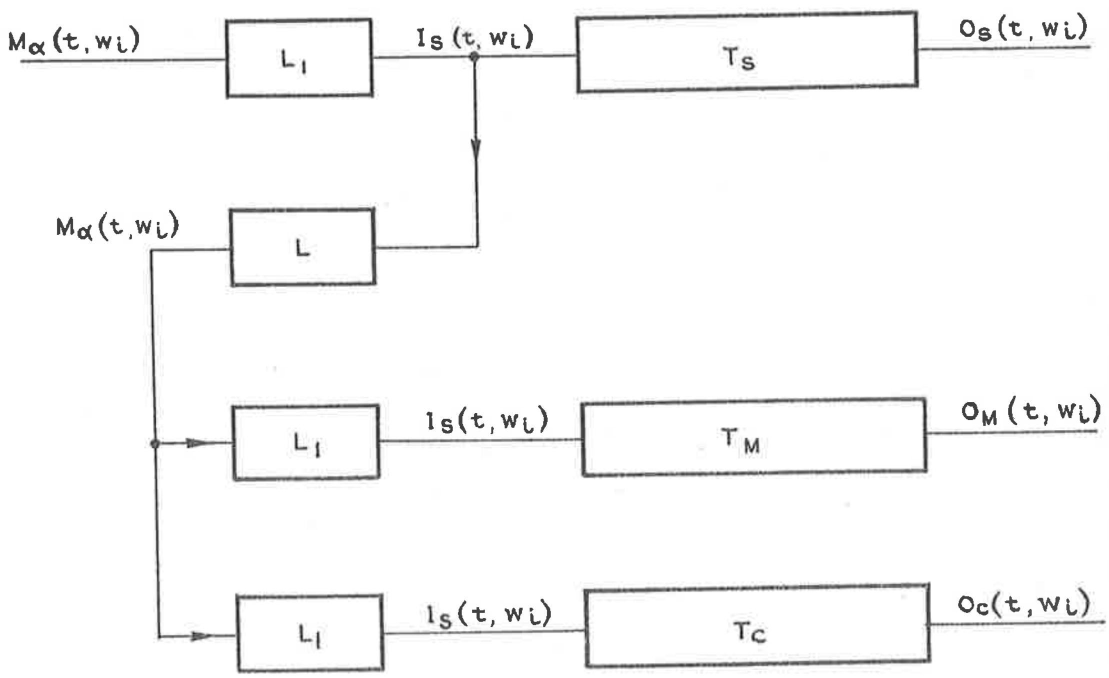

FIGURE 11. ILLUSTRATING MODEL CORRECTION WHEN SYSTEM INPUT IS A REALISATION OF A TYPE X_1 PROCESS

5.2 Input derived from Gauss Markov process. Single sample.

The case of a system input $I_S(t,w_\iota)$ which is a realisation of a type X_1 process is considered in this section. Such an input may be generated by operating on a realisation, $M_\alpha(t,w_\iota)$, of an S.G.M., $M_\alpha(t,w)$, having autocorrelation function $\frac{\alpha}{2}\, e^{-\alpha|\tau|}$. The situation is illustrated in fig. 11; the transformations L and L_1 being chosen so that

$$I_S(t,w_\iota) = L_1[M_\alpha(t,w_\iota)]$$

and $$M_\alpha(t,w_\iota) = L\,[I_S(t,w_\iota)].$$

The existence of these transformations was discussed in chapter 3.

If the parameter α is chosen large enough then, as illustrated in fig. 11, it may be expected that the expression

$$\rho_{M\alpha,OC}(\tau) + \rho_{M\alpha,OC}(-\tau) \qquad , \tau \geqslant 0, \qquad (5.32)$$

will be a good approximation to the convolution

$$W_1(\tau) * W_C(\tau); \qquad (5.33)$$

where $W_1(\tau)$ is the weighting function corresponding to the transformation L_1. As explained previously the frequency response function corresponding to L_1 is

$$\frac{\alpha + i\omega}{\alpha}\,\psi(\omega); \qquad (5.34)$$

where $\psi(\omega)\,\overline{\psi(\omega)}$ is the spectrum of $I_S(t,w)$ and $\psi(\omega)$ is free of poles and zeros in the lower half plane. It follows from (34) that, in general,

$$\lim_{\alpha\to\infty} W_1(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(\omega)e^{i\omega\tau}d\omega. \qquad (5.35)$$

The problem of estimating the convolution (33) from a finite length T_n of the realisations $M_\alpha(t,w_l)$, $O_S(t,w_l)$ and $O_M(t,w_l)$ is identical to the one discussed in the previous section; providing $W_C(t)$ is replaced by the convolution

$$W_1(t) * W_C(t).$$

With this change the expression (27) applies to the present case. The value of the parameter "a" used in the assumption (28) must now be based on the convolution of $W_C(\tau)$ with the expression (35); this value will, in general, be smaller than one based on $W_C(\tau)$ only. The results of the analysis following the assumption (28) show that the smaller the value of "a" the greater must be the length of record, T_n, to give the same mean square inadequacy (27) in the corrected model. It may be concluded that, in general, the "whiter" the input, $I_S(t,w)$, the smaller is the length T_n required for the same improvement in the model.

The foregoing approach to the problem of this section leads to an estimate of the convolution (33) from which $W_C(t)$ may be estimated if this is required specifically. On the other hand the relation

$$T_C = L[L_1[T_C]]$$

shows that the correction to the model may be incorporated as the transformation corresponding to (33) in series with the known transformation L. If it is necessary to estimate

$W_C(t)$ directly the method discussed in section 4.1 and illustrated by fig. 7 may be used. In this case $W_C(t)$ is estimated by correlating $M_\alpha(t,w_\iota)$ with

$$O_L(t,w_\iota) = L[O_S(t,w_\iota) - O_M(t,w_\iota)] \qquad (5.36)$$

providing the latter exists.

The mean square inadequacy of a model corrected in this way is, as before, the expected value of

$$\frac{1}{\sigma^2_{OS}} \int_0^\infty\int_0^\infty \rho_I(\tau-x)[W_C(x) - W_E(x)][W_C(\tau) - W_E(\tau)]dxd\tau, \qquad (5.37)$$

where

$$\begin{aligned} W_E(t) &= R'_{M\alpha,OL}(t) + R'_{M\alpha,OL}(-t) && ,t \geqslant 0, \\ &= 0 && t < 0; \\ R'_{M\alpha,OL}(\tau) &= \frac{1}{T'_n}\int_0^{T'_n} M_\alpha(t-\tau,w_\iota)O_L(t,w_\iota)dt, && 0 \leqslant \tau \leqslant T_m, \\ &= 0 && , \quad \tau > T_m; \\ R'_{M\alpha,OL}(-\tau) &= \frac{1}{T'_n}\int_0^{T'_n} M_\alpha(t+\tau,w_\iota)O_L(t,w_\iota)dt, && 0 \leqslant \tau \leqslant T(\alpha), \\ &= 0. && \tau > T(\alpha). \end{aligned}$$

The limit of this expectation as α tends to ∞ may be derived in a similar way to the expressions (16) to (27). It is again the sum of three terms, viz.:

$$\int_{T_m}^\infty\int_{T_m}^\infty \rho_I(\tau-x)W_C(x)W_C(\tau)dxd\tau; \qquad (5.36)$$

$$\frac{2}{(T'_n)^2}\int_0^{T_m} \rho_I(y)(T_m-y)(T'_n-y)\rho_{OL}(y)dy; \qquad (5.37)$$

$$\frac{2}{(T'_n)^2}\int_0^{T_m}\int_0^{T_m} \rho_I(\tau-x)\int_0^\tau (T'_n-y)W_C(x+y)W_C(\tau-y)dydxd\tau. \qquad (5.38)$$

The contribution (36) is due to the truncation of $W_E(\tau)$ at $\tau = T_m$. As shown in the appendix, the expression (37) approaches $\frac{T_m}{T_{h}'}\sigma^2_{OC}$ as T_{h}' and T_m approach infinity in such a way that $\frac{T_m}{T_{h}'}$ remains finite; i.e., for large T_{h}' (37) may be approximated by $\frac{T_m}{T_{h}'}\sigma^2_{OC}$. There does not seem to be any useful simplification of (38).

5.3 Several input and output samples

If there are several input and output samples available for model checking they may be treated in the following way which is very similar to the previous treatment for one sample.

Let the inputs to the linear system T_S whose weighting function is of the form (3.1), i.e.

$$W_S(t) = \sum_{j=1}^{m} P_j(t)e^{s_j t} \quad , \mathrm{Re}(s_j) < 0, \quad t \geq 0, \qquad (5.39)$$

$$= 0 \qquad\qquad t < 0,$$

be realisations of an S.G.M., $M_\alpha(t,w)$, whose correlation function is $\frac{\alpha}{2}e^{-\alpha|t|}$. Let there be a model, T_M, of this system, whose weighting function $W_M(t)$ is of a form similar to (39). Suppose there are available l sets of records, the i^{th} set consisting of a finite length $T_{h\iota}$ from the realisation $M_\alpha(t,w_\iota)$ and the corresponding outputs, $O_S(t,w_\iota)$ and $O_M(t,w_\iota)$, from the system and model. Consider the case of large α where the error in the weighting function of the model is estimated as

$$W_E(\tau) = \left(\sum_{\iota=1}^{l} T_{h\iota}'\right)^{-1} \sum_{\iota=1}^{l} T_{h\iota}'[R'_{M\alpha,OC}(\tau,w_\iota) + R'_{M\alpha,OC}(-\tau,w_\iota)], \qquad (5.40)$$

where as before:

$$R'_{M\alpha,OC}(\tau,w_i) = \frac{1}{T'_{ni}} \int_0^{T'_{ni}} M_\alpha(t-\tau,w_i)O_C(t,w_i)dt, \quad 0 \leq \tau \leq T_m,$$

$$= 0 \qquad , \qquad \tau > T_m;$$

$$R'_{M\alpha,OC}(-\tau,w_i) = \frac{1}{T'_{ni}} \int_0^{T'_{ni}} M_\alpha(t+\tau,w_i)O_C(t,w_i)dt, \quad 0 \leq \tau \leq T(\alpha),$$

$$= 0 \qquad , \qquad \tau > T(\alpha);$$

$$\lim_{\alpha\to\infty} T(\alpha) = 0.$$

T'_{ni} is the length remaining of the i^{th} sample after discarding sufficient to ensure that the effect of transients set up in the output, $O_M(t,w_i)$, is negligible. It will be assumed that $T'_{ni} > T_m$, all i.

The expectation

$$E[\int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)[W_C(x)-W_E(x)][W_C(\tau)-W_E(\tau)]dxd\tau \qquad (5.41)$$

must be calculated. This calculation is similar to those previously carried out except there are now terms of the form

$$E[\int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)T'_{ni}T'_{nj}R'_{M\alpha,OC}(\tau,w_i)R'_{M\alpha,OC}(x,w_j)dxd\tau] \qquad (5.42)$$

$$= \int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)T'_{ni}T'_{nj}E[R'_{M\alpha,OC}(\tau,w_i)R'_{M\alpha,OC}(x,w_j)dxd\tau].$$

The symbols w_i and w_j are here to be interpreted as points in two similar probability spaces W_i and W_j. The expectation is to be taken with respect to the product space. For, $0 < x < T_m$, and, $0 < \tau < T_m$, the required expectation is given by

$$E[R'_{M\alpha,OC}(\tau,w_i)R'_{M\alpha,OC}(x,w_j)]$$

$$=E[\frac{1}{T'_{ni}T'_{nj}}\int_0^{T'_{ni}} M_\alpha(t_1-\tau,w_i)O_C(t_1,w_i)dt_1\int_0^{T'_{nj}} M_\alpha(t_2-x,w_j)O_C(t_2,w_j)dt_2]$$

$$= \frac{1}{T\acute{n}_i T\acute{n}_j}\int_0^{T\acute{n}_i}\int_0^{T\acute{n}_j} E[M_\alpha(t_1-\tau,w_i)O_C(t_1,w_i)M_\alpha(t_2-x,w_j)O_C(t_2,w_j)]dt_1dt_2$$

$$= \frac{1}{T\acute{n}_i T\acute{n}_j}\int_0^{T\acute{n}_i}\int_0^{T\acute{n}_j} \rho_{M\alpha,OC}(\tau)\rho_{M\alpha,OC}(x)dt_1dt_2$$

$$= \rho_{M\alpha,OC}(\tau)\rho_{M\alpha,OC}(x); \tag{5.43}$$

since the random variables defined on W_i and W_j are independent for $i \neq j$.

Hence the expression (42) equals, for $i \neq j$,

$$\int_0^{T_m}\int_0^{T_m}\rho_{M\alpha}(\tau-x)T\acute{n}_i T\acute{n}_j\rho_{M\alpha,OC}(\tau)\rho_{M\alpha,OC}(x)d\tau dx. \tag{5.44}$$

The limit as α tends to ∞ of the expectation (41) now becomes the sum of the following terms.

$$\lim_{\alpha\to\infty}\int_0^{\infty}\int_0^{\infty}\rho_{M\alpha}(\tau-x)W_C(x)W_C(\tau)dxd\tau$$

$$= \int_0^{\infty}W_C^2(x)dx\ . \tag{5.45}$$

$$-\lim_{\alpha\to\infty} 2\int_0^{\infty}\int_0^{T_m}\rho_{M\alpha}(\tau-x)W_C(x)\{\rho_{M\alpha,OC}(\tau)+\rho_{M\alpha,OC}(-\tau)\}dxd\tau$$

$$= -2\int_0^{T_m}W_C^2(x)dx. \tag{5.46}$$

$$\lim_{\alpha\to\infty}\Big(\sum_{i=1}^{l}T\acute{n}_i\Big)^{-2}\int_0^{T_m}\int_0^{T_m}\rho_{M\alpha}(\tau-x)\Big\{\sum_{i=1}^{l}\sum_{j=1}^{l}T\acute{n}_i T\acute{n}_j\rho_{M\alpha,OC}(\tau)\rho_{M\alpha,OC}(x)\Big\}d\tau dx$$

$$= \int_0^{T_m}W_C^2(x)dx. \tag{5.47}$$

$$\Big(\sum_{i=1}^{l}T\acute{n}_i\Big)^{-2}T_m\Big(\sum_{i=1}^{l}T\acute{n}_i\Big)\sigma^2_{OC} = \frac{T_m}{\Sigma T\acute{n}_i}\sigma^2_{OC}. \tag{5.48}$$

$$2\Big(\sum_{i=1}^{l}T\acute{n}_i\Big)^{-2}\int_0^{T_m}\int_0^{\tau}\Big(\sum_{i=1}^{l}T\acute{n}_i-ly\Big)W_C(\tau+y)W_C(\tau-y)dyd\tau. \tag{5.49}$$

The three terms (45), (46) and (47) together contribute

$$\int_{T_m}^{\infty} W_C^2(x)dx;$$

i.e. the amount due to the truncation of $W_C(x)$ at $x = T_m$. The fourth and fifth terms arise only from the expectations

$$E[R'_{M\alpha,OC}(\tau,w_i)R'_{M\alpha,OC}(x,w_j)] \quad \text{for} \quad i = j$$

and are therefore the sum of l similar expressions, one for each sample.

As in the case of one sample this expression may be studied further by making the assumption $W_C(t) = ke^{-at}$ and writing

$$aT_m = p \quad \text{and}$$

$$a \sum_{i=1}^{l} T'_{mi} = q.$$

The expression (41) when divided by σ^2_{OS} then becomes

$$\frac{\sigma^2_{OC}}{\sigma^2_{OS}} \frac{p}{q}[1 + \frac{1}{p} - \frac{l}{2pq} - e^{-2a}(2 + \frac{1}{a} - \frac{lp}{q} - \frac{l}{q} - \frac{l}{2pq} - \frac{q}{p})]. \qquad (5.50)$$

In most applications the inequalities $q >> p > 1$ and $q >> l$ will hold and in such cases (50) is approximately

$$\frac{\sigma^2_{OC}}{\sigma^2_{OS}} (\frac{p + 1}{q} + e^{-2p}). \qquad (5.51)$$

For fixed q the derivative of (50) with respect to p is zero if

$$1 + e^{-2p}(4p - \frac{2lp^2}{q} - 2q) = 0$$

$$\text{i.e.} \quad p \simeq \tfrac{1}{2} \ln 2q. \qquad (5.52)$$

Comparing these results with the results for one sample, (30) and (31), it appears that in general there is little to choose between l samples of total effective length

T_n' and one sample of length T_n'. In the former case, however, more of the total length of record must be discarded because of transients set up in the output of the model.

The above discussion assumes that $T_{n_i}' \geq T_m$ for all samples. Only rarely, if at all, would samples of such small length that $T_{n_i}' < T_m$ be used; the analysis could be extended to cover such cases if such an extension were useful.

5.4 Correction by iteration

The discussion in the last section related to the case where all the available records were used to estimate the correction, $W_C(t)$, to the original model. An alternative procedure is possible, viz. each set of records is used to adjust the model before proceeding to use the next set. If there are 1 sets of records there will then be 1 successive corrections made to the model. The convergence of such an iterative procedure is now discussed in the case of a system input, $M_\alpha(t,w)$, which is an S.G.M. having autocorrelation function $\frac{\alpha}{2}\, e^{-\alpha|\tau|}$ and α is arbitrarily large. It is assumed that each set of records is obtained independently and the parameter T_m is the same for each iteration; the method described at the beginning of this chapter could be used for determining a value for T_m.

Let

$$\int_{T_m}^{\infty} W_C^2(x)dx = K,$$

and let an effective length T_{n_i}' of the input $M_\alpha(t,w_i)$ be used in making the i^{th} adjustment to the model.

Then according to the expression (27), the mean square error in the output of the model after the first adjustment, when averaged over all possibilities for w_1 is,

$$\leqslant \frac{T_m}{T'_{n1}} \sigma^2_{OC} + \frac{2}{(T'_{n1})^2} \int_0^{T_m} \int_0^{\tau} |(T'_{n1}-t_3) W_C(\tau - t_3) W_C(\tau + t_3)| \, dt_3 \, d\tau + K$$

$$\leqslant \frac{T_m}{T'_{n1}} \sigma^2_{OC} + \frac{1}{T'_{n1}} \int_0^{T_m} \int_0^{\tau} W^2_C(\tau - t_3) + W^2_C(\tau + t_3) \, dt_3 \, d\tau + K$$

$$= \frac{T_m}{T'_{n1}} \sigma^2_{OC} + \frac{1}{T'_{n1}} \int_0^{T_m} \int_0^{2\tau} W^2_C(y) \, dy \, d\tau + K$$

$$\leqslant 2 \frac{T_m}{T'_{n1}} \sigma^2_{OC} + K.$$

Denote by σ^2_{OC1} the mean square error in the output of the model after this first adjustment and consider the situation after the second adjustment.

The mean square error in the output of the model after this second adjustment when averaged over all possibilities for w_2 is

$$\leqslant 2 \frac{T_m}{T'_{n2}} \sigma^2_{OC1} + K.$$

This expression when averaged over all possibilities for w_1 is

$$\leqslant 2 \frac{T_m}{T'_{n2}} [2 \frac{T_m}{T'_{n1}} \sigma^2_{OC} + K] + K. \qquad (5.53)$$

Thus the mean square error in the output of the model averaged over all possible independent choices of the first 2 sets of records is given by (53).

Proceeding by induction, after the l^{th} correction to the model the mean square error in its output averaged over all possible independent choices of the first l sets of records is

$$\leqslant j^l \sigma^2_{OC} + K\left(\frac{1-j^l}{1-j}\right); \qquad (5.54)$$

where j is the greatest of the $\frac{2T_m}{T_n'{}_i}$. This expression (54) converges to $\frac{K}{1 - j}$ as $l \to \infty$ providing $0 \leqslant j < 1$.

This iterative method should not then be used unless T_m may be chosen such that:

$\left(\frac{K}{\sigma^2_{OS}}\right)^{\frac{1}{2}}$ < the required value of the inadequacy $\overline{A}$;

$\frac{2T_m}{T_n'{}_i} < 1$ all i.

However if, for example, T_m may be chosen so that

$\left(\frac{K}{\sigma^2_{OS}}\right)^{\frac{1}{2}} < \frac{1}{2}\overline{A}$,

$T_m < \frac{1}{4}T_n'{}_i$, all i,

then the procedure should be satisfactory.

In some cases it may be useful to divide one set of records of long length into two or more sets of shorter length and use an iterative procedure. Consider for example the problem discussed at the end of section 5.1 and take $aT_n' = q = 100$ and $aT_m = p = 3$. The value of the mean square inadequacy after correction using this one set of records was found to be approximately $0.04\,\sigma^2_{OC}$. The parameter K in this case is

$$\int_{T_m}^{\infty} W_C^2(x)dx$$

$$= \int_{\frac{3}{a}}^{\infty} k^2 e^{-2ax}dx$$

$$= 0{\cdot}0025\,\sigma^2_{OC}.$$

The length T_n' may be divided into l equal shorter lengths, but at each division an amount T_m (say) must be discarded to allow for transients in the model and to ensure the records are independent. Hence the length of the shortened records will be

$$\frac{T_n' - (l - 1)T_m}{l}$$

and the parameter j will be

$$\frac{2lT_m}{T_n' - (l - 1)T_m}$$

$$= \frac{2lp}{q - (l - 1)p}$$

$$= \frac{6l}{103 - 3l} .$$

The following table gives the value of the expression (54) calculated for some values of l.

Table 2.

Value of l.	$j^l\sigma^2_{0C} + K\left(\frac{1-j^l}{1-j}\right)$
1	$0.063\sigma^2_{0C}$
2	$0.019\sigma^2_{0C}$
3	$0.009\sigma^2_{0C}$
4	$0.008\sigma^2_{0C}$
5	$0.008\sigma^2_{0C}$
6	$0.010\sigma^2_{0C}$
7	$0.014\sigma^2_{0C}$
8	$0.025\sigma^2_{0C}$
10	$0.15\ \sigma^2_{0C}$

Clearly there is a distinct advantage in dividing up this long record of effective length T_n' into four or five shorter lengths and correcting the model by iteration.

5.5 Corrections involving generalised functions

The work of this chapter may be extended to the case of transformations T_S and T_M whose weighting functions include the generalised functions $\delta(t)$ and its derivatives. It would be meaningless in such a case to consider an input which is a realisation of an S.G.M. whose autocorrelation function is $\frac{\alpha}{2} e^{-\alpha|\tau|}$ and then allow the parameter α to approach ∞; the mean square outputs $\sigma^2_{OS}, \sigma^2_{OM}$ and, probably, σ^2_{OC} would also approach ∞.

However the general case of an input $I_S(t, w_\iota)$ which is a realisation of a type X_1 process and for which the mean squares of the corresponding outputs are finite, presents no difficulty. The weighting functions corresponding to the transformations $L_1 \cdot T_S$, $L_1 \cdot T_M$, and $L_1 \cdot T_C$ · shown on fig. 11 will all be of the exponential form (3.1). Accordingly the first method described in section 5.2 may be used to estimate the effect of finite sample length on the adequacy of the corrected model.

CHAPTER 6.

STATIONARY LINEAR SYSTEM AND MODEL RECORDS WITH ERRORS

In this chapter the effect of errors in the records of $I_S(t,w_\iota)$ and $O_S(t,w_\iota)$ are considered, both in the case where these records are of unlimited length and also in the case where the length of the records is finite. The other assumptions of chapter 4 are retained.

The situation is represented in fig.12 where:

(a) $I_S(t,w_\iota)$, the system input, is a realisation of a random process of type X_1;

(b) T_S and T_M are the system and model components;

(c) $N_I(t,w_\iota)$ and $N_O(t,w_\iota)$ are realisations of random processes of type X_1 which appear as errors, or noise, in the recordings of $I_S(t,w_\iota)$ and $O_S(t,w_\iota)$;

(d) b_I and b_O are constant bias errors introduced in the recording.

6.1 Records of unlimited length

Suppose, first, that the records are of unlimited length, the weighting functions corresponding to T_S and T_M are of the exponential form (3.1) and the bias errors b_I and b_O are both zero.

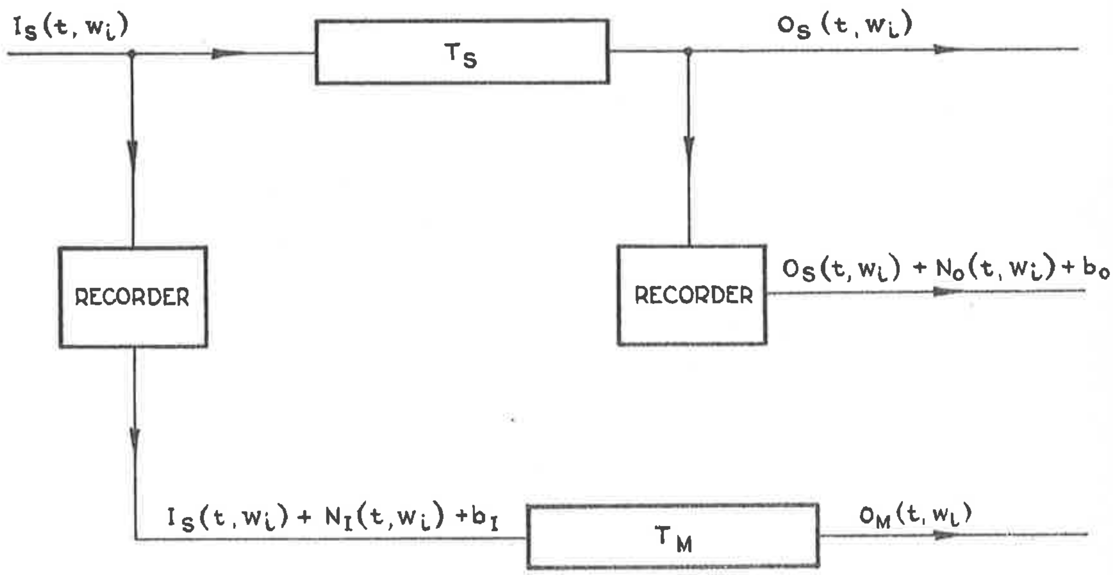

FIGURE 12. BLOCK DIAGRAM ILLUSTRATING RECORDING ERRORS

According to theorem 6 the error in the weighting function of the model is the solution of the Wiener-Hopf equation

$$\int_0^\infty \rho_I(\tau - x)W(x)dx = \rho_{I,OC}(\tau), \qquad \tau \geq 0;$$

where $\rho_{I,OC}(\tau)$ is the cross-correlation function of $I_S(t,w_t)$ and the difference, $O_C(t,w_t)$, between $O_S(t,w_t)$ and $O_M(t,w_t)$ in the absence of recording errors. However, owing to the errors $N_I(t,w_t)$ and $N_O(t,w_t)$ in the records, neither $\rho_I(\tau)$ nor $\rho_{I,OC}(\tau)$ may be calculated exactly, unless certain information concerning these errors is available. If these errors are ignored and correlation is carried out as before, the quantities

$$\rho_I(\tau) + \rho_{NI,I}(\tau) + \rho_{I,NI}(\tau) + \rho_{NI}(\tau) \tag{6.1}$$

and

$$\rho_{I,OC}(\tau) + \rho_{NI,OC}(\tau) - \int_0^\infty \rho_{I,NI}(\tau - x)W_M(x)dx - \int_0^\infty \rho_{NI}(\tau - x)W_M(x)dx$$
$$+ \rho_{NI,NO}(\tau) + \rho_{I,NO}(\tau), \tag{6.2}$$

will be obtained instead of $\rho_I(\tau)$ and $\rho_{I,OC}(\tau)$. Clearly some knowledge, or assumption, concerning the correlation functions involving $N_I(t,w_t)$ and $N_O(t,w_t)$, is required.

The following assumptions, which are not unrealistic, will be made in this section.

(a) It would be surprising if the process $N_O(t,w)$ were correlated with either $I_S(t,w)$ or $N_I(t,w)$. Accordingly it will be assumed that

$$\rho_{I,NO}(\tau) \equiv \rho_{NI,NO}(\tau) \equiv 0. \qquad (6.3)$$

Since these are the only terms involving $N_0(t,w_\iota)$ in (1) and (2) it follows that, under this assumption, $N_0(t,w_\iota)$ may be ignored.

(b) For many recording devices it will be safe to assume that the error in the record is not correlated with the signal being recorded or with records of other quantities. Accordingly it will be assumed that

$$\rho_{NI,I}(\tau) = \rho_{I,NI}(\tau) = \rho_{NI,OC}(\tau) = 0. \qquad (6.4)$$

(c) It will often be satisfactory to assume the spectrum of $N_I(t,w_\iota)$ to be sufficiently flat that, with negligible error,

$$\int_0^\infty [\rho_{NI}(\tau-x)+\rho_{NI}(-\tau-x)]W_M(x)dx = k^2 W_M(\tau), \quad \tau \geqslant 0 \; ; \qquad (6.5)$$

where k^2 is the height of the spectrum of $N_I(t,w)$ at low frequency.

Consider now the case where $I_S(t,w_\iota) = M_\alpha(t,w_\iota)$, i.e. a realisation of an S.G.M. whose autocorrelation function is $\frac{\alpha}{2} e^{-\alpha|\tau|}$. For large α the correction $W_C(\tau)$ may, as before, be estimated by

$$\rho_{M\alpha,OC}(\tau) + \rho_{M\alpha,OC}(-\tau), \qquad (\tau \geqslant 0).$$

If the expression (2) is used as an approximation to this quantity, there will, under the above assumptions, be an error of $-k^2 W_M(\tau)$ due to the presence of the noise $N_I(t,w_\iota)$. The residual mean square error in the output of the corrected model will then be

$$\int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)k^2W_M(\tau)k^2W_M(x)d\tau dx \tag{6.6}$$

$= k^4$ times the mean square output of the original model or, approximately,

$$k^4\sigma^2_{OS}. \tag{6.7}$$

Since k^2 is the ratio of the power density of $N_I(t,w)$ to that of $M_\alpha(t,w)$ at low frequency it will, in most cases, be much less than one. The error (6) may then be so small that it can be accepted; alternatively a knowledge of k^2 would allow a simple correction to be made to the estimate of $W_C(\tau)$.

A similar result is obtained if the input $I_S(t,w_i)$ is a realisation of a type X_1 process which is "whitened" to an S.G.M. having autocorrelation function $\frac{\alpha}{2}e^{-\alpha|\tau|}$, before recording. As illustrated in fig. 13, providing the realisations $O_M(t,w_i)$ and $L[O_S(t,w_i)] + N_O(t,w_i)$ exist, an estimate of $W_C(x)$ may be obtained by cross correlating the recorded "whitened" input, $M_\alpha(t,w_i) + N_I(t,w_i)$, with the difference between the recorded, "whitened" system output, $L[O_S(t,w_i)] + N_O(t,w_i)$, and the output of the model, $O_M(t,w_i)$, to the recorded, "whitened" input. The result of this cross correlation will be, under the above assumptions,

$$\rho_{M\alpha,OL}(\tau) - \int_0^\infty \rho_{NI}(\tau-x)W_M(x)dx;$$

where $O_L(t,w_i) = L[O_S(t,w_i)] - T_M[M_\alpha(t,w_i)]$. The error in the estimate of $W_C(\tau)$, due to the recording errors, is then the same as that obtained for $I_S(t,w_i) = M_\alpha(t,w_i)$.

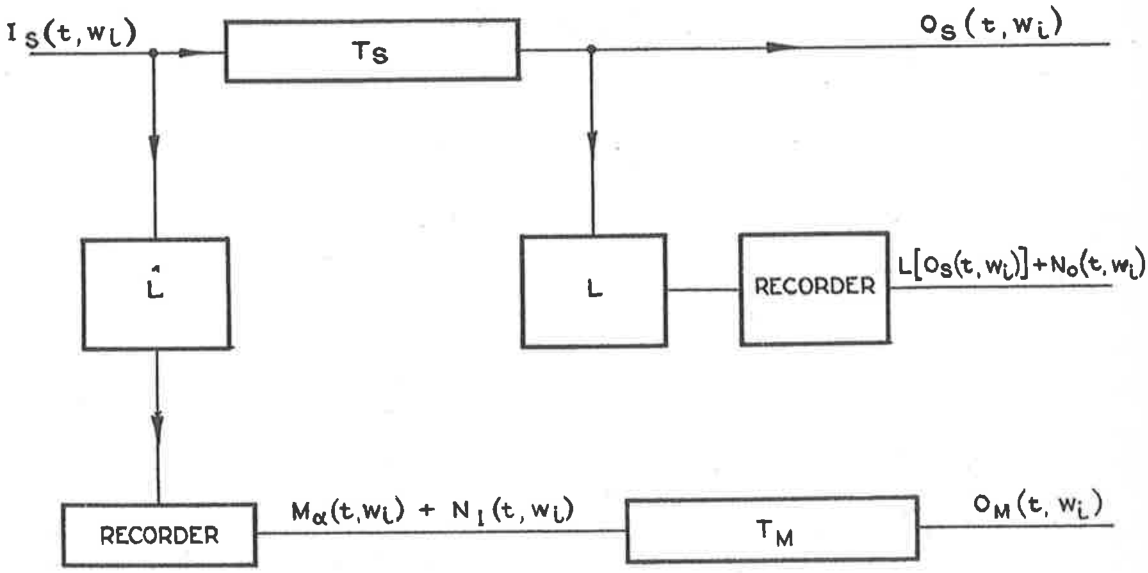

FIGURE 13. INPUT AND OUTPUT "WHITENED" BEFORE RECORDING

If the weighting functions corresponding to T_S and T_M in fig. 13 contain generalised functions then the quantities $L[O_S(t,w_\iota)] + N_O(t,w_\iota)$ and $O_M(t,w_\iota)$ may not exist. To overcome this difficulty a suitable transformation L_2 may be placed in series with T_M so that the weighting function of the product transformation $L_2 \cdot T_M$ has the exponential form (3.1). The output of the system $O_S(t,w_\iota)$ should then be transformed by the product transformation $L_2 \cdot L$ instead of L as shown on fig. 13. A procedure similar to that described above will then lead to an estimate of the weighting function corresponding to the product transformation $L_2 \cdot T_M$.

If the "whitening" is carried out after recording then the transformation L acts on the recording errors $N_I(t,w_\iota)$ and $N_O(t,w_\iota)$ as well as on the system input and output. The previous results would still apply if the recording errors could be replaced by $L[N_I(t,w_\iota)]$ and $L[N_O(t,w_\iota)]$. However these two expressions may not exist as realisations of a type X_1 process, in which case corresponding practical difficulties will also arise; e.g., if $I_S(t,w)$ has spectrum $S_I(\omega) = [\frac{1}{(\omega^2 + \beta^2)}]^3$ and $N_I(t,w)$ has spectrum $(\frac{1}{\omega^2 + \gamma^2})$, the frequency response function corresponding to L will then be $\frac{\alpha(\beta + i\omega)^3}{\alpha + i\omega}$ so that $L[N_I(t,w_\iota)]$ is not a realisation of a type X_1 process.

This difficulty may be overcome by modifying the transformation L. One possible modification, which is illustrated in fig. 14, would be to modify L to L_3 such that

$$L_3[I_S(t,w) + N_I(t,w)] = M_\alpha(t,w); \qquad (6.8)$$

this will be satisfactory providing $L_3[N_0(t,w)]$ exists. Under the assumptions (3), (4) and (5), of this section, the existence of an L_3 satisfying (8) may be shown in the following way. The spectrum of the process $I_S(t,w) + N_I(t,w)$ is the Fourier transform of its correlation function

$$\rho_{IS}(\tau) + \rho_{I,NI}(\tau) + \rho_{NI,I}(\tau) + \rho_{NI}(\tau)$$

which under the assumption, (4), of this section

$$= \rho_{IS}(\tau) + \rho_{NI}(\tau).$$

The spectrum of this sum is, therefore, the sum of the spectra of $I_S(t,w)$ and $N_I(t,w)$ which may be written (theorem 1) as the ratio of two polynomials in ω^2 with real coefficients and no zeros on the real axis. This spectrum may, therefore, be factorised to the form $\psi(\omega)\overline{\psi(\omega)}$, where $\psi(\omega)$ has all its zeros and poles above the real axis; these poles and zeros are either imaginary or occur in pairs of the form $a + ib$, and $-a + ib$, $b > 0$. Hence, as shown in theorem 1, $\psi(\omega)$ may be written as the ratio of two polynomials in $(i\omega)$ with real coefficients.

Let L_3 be the transformation whose frequency response function is $\dfrac{1}{\psi(\omega)} \dfrac{\alpha}{\alpha + i\omega}$ then $L_3[I_S(t,w_t) + N_I(t,w_t)]$ is a realisation of a process which exists and has spectrum

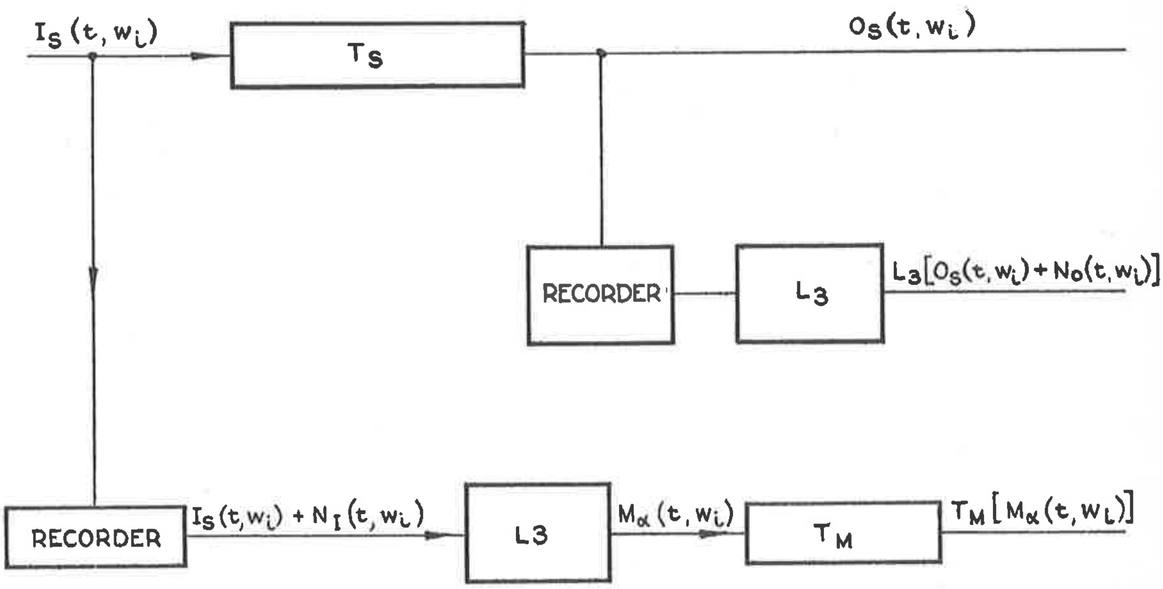

FIGURE 14. INPUT AND OUTPUT "WHITENED" AFTER RECORDING

$$\frac{\psi(\omega)\overline{\psi(\omega)}}{1} \times \frac{1}{\psi(\omega)\overline{\psi(\omega)}} \times \frac{\alpha^2}{\alpha^2 + \omega^2}$$

$$= \frac{\alpha^2}{\alpha^2 + \omega^2} .$$

Hence this transformation L_3 satisfies the equation (8).

As illustrated in fig. 14, the output of the system, after recording and transforming by L_3, is

$$L_3[O_S(t,w_l) + N_O(t,w_l)].$$

Assuming this expression exists it is equal to

$$L_3(T_S[I_S(t,w_l)+N_I(t,w_l)])+L_3[N_O(t,w_l)]-L_3(T_S[N_I(t,w_l)])$$
$$= T_S[M_\alpha(t,w_l)]+L_3[N_O(t,w_l)]-L_3(T_S[N_I(t,w_l)]). \quad (6.9)$$

The cross correlation of the input to the model, $M_\alpha(t,w_l)$, with the difference between this expression (9) and the output of the model may be written

$$\rho_{M\alpha,OL}(\tau) + \rho_{M\alpha,N2}(\tau) - \rho_{M\alpha,N3}(\tau); \quad (6.10)$$

where, for brevity, the notation,

$$O_L(t,w_l) = (T_S - T_M)[M_\alpha(t,w_l)],$$
$$N_2(t,w_l) = L_3[N_O(t,w_l)],$$
$$N_3(t,w_l) = L_3(T_S[N_I(t,w_l)]),$$

has been introduced.

Under the previous assumptions (3), (4), (5), that various cross correlation terms are zero, the expression (10) becomes

$$\rho_{M\alpha,OL}(\tau) - \rho_{N,N3}(\tau)$$
$$= \rho_{M\alpha,OL}(\tau) - \rho_{N,N4}(\tau) - \rho_{N,N5}(\tau); \quad (6.11)$$

where,

$$N\ (t,w_l) = L_8[N_I(t,w_l)],$$
$$N_4(t,w_l) = L_8(T_M[N_I(t,w_l)]),$$
$$N_5(t,w_l) = L_8(T_C[N_I(t,w_l)]).$$

In order to proceed as above the correlation function $\rho_{NI}(\tau)$ must be known and therefore the expression $\rho_{N,N4}(\tau)$ may be calculated and allowed for in (11). The third term in (11) involves the unknown transformation T_C, but as shown below an iteration process such as that described in theorem 7 will converge.

After allowing for $\rho_{N,N4}(\tau)$, the correction made to the model weighting function is

$$\rho_{M\alpha,OL}(\tau) + \rho_{M\alpha,OL}(-\tau) - \rho_{N,N5}(\tau) - \rho_{N,N5}(-\tau), \qquad \tau \geq 0,$$

so that the error remaining is

$$W_C(\tau) - \rho_{M\alpha,OL}(\tau) - \rho_{M\alpha,OL}(-\tau) + \rho_{N,N5}(\tau) + \rho_{N,N5}(-\tau), \qquad \tau \geq 0.$$

The real part of the Fourier transform of this quantity is

$$R(\omega)[1 - S_\alpha(\omega) + S_N(\omega)]$$

where $R(\omega)$ is the real part of the frequency response function corresponding to $W_C(\tau)$. An argument similar to that used in the proof of theorem 7 now shows that this iterative procedure will converge providing

$$|1 - S_\alpha(\omega) + S_N(\omega)| < 1. \qquad (6.12)$$

Since $S_\alpha(\omega) - S_N(\omega)$

$$= S_\alpha(\omega) - \frac{S_{NI}(\omega)}{S_{NI}(\omega) + S_I(\omega)} S_\alpha(\omega)$$

$$= S_\alpha(\omega)\left(\frac{S_I(\omega)}{S_{NI}(\omega) + S_I(\omega)}\right)$$

the condition (12) is theoretically always satisfied.

6.1.1 Records with bias errors

A common type of recording error is a constant error or bias. As shown in fig. 12 the records of input and output are, in this case, of the form $I_S(t,w_l) + b_I$ and $O_S(t,w_l) + b_O$ where b_I and b_O are constants. In the case $I_S(t,w_l) = M_\alpha(t,w_l)$ an estimate of $W_C(\tau)$ based on the correlation of $I_S(t,w_l) + b_I$ with $O_S(t,w_l) + b_O - T_M[I_S(t,w_l) + b_I]$ will be

$$\rho_{M\alpha,OC}(\tau) + \rho_{M\alpha,OC}(-\tau) + 2b_I b_O - 2b_I^2\int_0^\infty W_M(x)dx.$$

Since the length of each record is unlimited then each bias may be determined and an allowance made. If this is not done then $W_C(\tau)$ as estimated will, due to biases, be in error by a constant. Proceeding mechanically this will produce a mean square error in the corrected model of

$$b^2\int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)d\tau dx \tag{6.12}$$

where b is this constant error. This expression is not finite because

$$\int_0^\infty \rho_{M\alpha}(\tau-x)dx = 1 - \tfrac{1}{2}e^{-\alpha\tau}, \qquad \tau \geqslant 0.$$

However in practice the estimate of $W_C(\tau)$ would be truncated so that this estimate is zero for $\tau > T_m$ (say) and the expression (12) then becomes

$$b^2\int_0^{T_m}\int_0^{T_m} \rho_{M\alpha}(\tau-x)dxd\tau, \tag{6.13}$$

which is finite and approaches b^2T_m as α approaches ∞.

6.2 Finite length records with errors

In this section the work of chapter 5, dealing with finite length records, is extended to take account of errors in the records. The case in which the following assumptions hold, is considered in detail.

(a) The system component T_S and the model component T_M both have weighting functions of the exponential form (3.1).

(b) The system input is an S.G.M. $M_\alpha(t,w)$ whose autocorrelation function is $\frac{\alpha}{2} e^{-\alpha|\tau|}$.

(c) One set consisting of finite lengths, T_n, of recordings of the realisation $M_\alpha(t,w_l)$, the corresponding system output, $O_S(t,w_l)$, and model output, $O_M(t,w_l)$, is available for model checking. The length of the record of $O_M(t,w_l)$ is reduced to T_n' when sufficient to allow for transients has been discarded.

(d) The errors in the recordings are portions of realisations of type X_1 random processes $N_I(t,w)$ and $N_O(t,w)$ together with small constant biases. These biases vary with each realisation in such a way that for every choice of j_1, j_2 and j_3 the normal random variables $M_\alpha(t_{j_1},w)$, $N_I(t_{j_2},w)$, $N_O(t_{j_3},w)$, $b_I(w)$, $b_O(w)$ are independent and have zero mean. This assumption, which is similar to that made in section (6.1) is reasonable in many practical cases.

(e) The correlation functions $\rho_{NI}(\tau)$, $\rho_{NO}(\tau)$, and the variances σ^2_{bI}, σ^2_{bO}, are known.

The records available for model checking are, then, finite lengths of the realisations

$$M_\alpha(t,w_\iota) + N_I(t,w_\iota) + b_I(w_\iota), \tag{6.14}$$

$$T_S[M_\alpha(t,w_\iota)] + N_O(t,w_\iota) + b_O(w_\iota), \tag{6.15}$$

and $T_M[M_\alpha(t,w_\iota)] + T_M[N_I(t,w_\iota)] + T_M[b_I(w_\iota)]$, (6.16)

for one realisation, w_ι, of the random processes $M_\alpha(t,w)$, $N_I(t,w)$, $b_I(w)$, $N_O(t,w)$, $b_O(w)$. For the present purpose $b_I(w)$ and $b_O(w)$ may be considered to be random processes whose realisations are constants, i.e. independent of t. It will be convenient to consider the vector process having realisation

$$\xi(t,w_\iota) = \begin{bmatrix} M_\alpha(t,w_\iota) \\ N_I(t,w_\iota) \\ b_I(t,w_\iota) \\ O_C(t,w_\iota) \\ N_O(t,w_\iota) \\ b_O(t,w_\iota) \\ N_M(t,w_\iota) \\ b_M(t,w_\iota) \end{bmatrix} ; \tag{6.17}$$

where

$$O_C(t,w_\iota) = (T_S - T_M)[M_\alpha(t,w_\iota)], \tag{6.18}$$

$$N_M(t,w_\iota) = - T_M[N_I(t,w_\iota)], \tag{6.19}$$

$$b_M(t,w_\iota) = - T_M[b_I(t,w_\iota)]. \tag{6.20}$$

The following table distinguishes correlation functions which, under the assumption (d) above, are identically zero, from those which are not; a tick means $\rho_{hj}(\tau) \neq 0$, a dash means $\rho_{hj}(\tau) \equiv 0$.

Table 3

j \ h	1	2	3	4	5	6	7	8
1	√	-	-	√	-	-	-	-
2	-	√	-	-	-	-	√	-
3	-	-	√	-	-	-	-	√
4	√	-	-	√	-	-	-	-
5	-	-	-	-	√	-	-	-
6	-	-	-	-	-	√	-	-
7	-	√	-	-	-	-	√	-
8	-	-	√	-	-	-	-	√

The quantities $\rho_{33}(\tau)$, and $\rho_{66}(\tau)$ are to be interpreted as σ^2_{bI} and σ^2_{bO} for all values of τ while $\rho_{38}(\tau)$ is also independent of τ and equal to

$$-\sigma^2_{bI}\int_0^\infty W_M(y)dy.$$

The definition of $R(\tau,w_\iota)$ and $R'(\tau,w_\iota)$ are extended so that for $h = 1,2,3,$ and $j = 4 - 8$:

$$R'_{hj}(\tau,w_\iota) = \frac{1}{T_n{}'}\int_0^{T_n{}'} \xi_h(t-\tau,w_\iota)\xi_j(t,w_\iota)dt, \quad 0 \leq \tau \leq T_m, \tag{6.21}$$

$$= 0 \qquad\qquad , \quad \tau > T_m;$$

$$R'_{hj}(-\tau,w_\iota) = \frac{1}{T_n{}'}\int_0^{T_n{}'} \xi_h(t+\tau,w_\iota)\xi_j(t,w_\iota)dt, \quad 0 < \tau \leq T(\alpha), \tag{6.22}$$

$$= 0 \qquad\qquad , \quad \tau > T(\alpha);$$

where $\xi_h(t,w_\iota)$ and $\xi_j(t,w_\iota)$ are components of $\underset{\sim}{\xi}(t,w_\iota)$, and $\lim_{\alpha\to\infty} T(\alpha) = 0$.

The expectation of $\sum_{h=1}^{3}\sum_{j=4}^{8} R'_{hj}(\tau,w_\iota) + R'_{hj}(-\tau,w_\iota)$ is then equal to

$$\begin{aligned}
&\sum_{h=1}^{3}\sum_{j=4}^{8} \rho_{hj}(\tau) + \rho_{hj}(-\tau), && 0 \leqslant \tau \leqslant T(\alpha),\\
&\sum_{h=1}^{3}\sum_{j=4}^{8} \rho_{hj}(\tau) \qquad , && T(\alpha) < \tau \leqslant T_m,\\
&0 \qquad , && \tau > T_m.
\end{aligned}$$

For large α the error, $W_C(\tau)$, in the weighting function $W_M(\tau)$ of the model may be estimated by cross correlating the expression (14) with the difference between (15) and (16), and then compensating, where possible, for the errors in the records. The quantities required are

$$\begin{aligned}
&\rho_{M\alpha,OC}(\tau) + \rho_{M\alpha,OC}(-\tau), && 0 \leqslant \tau \leqslant T(\alpha),\\
&\rho_{M\alpha,OC}(\tau) \qquad , && T(\alpha) < \tau \leqslant T_m ;
\end{aligned}$$

they may be estimated from the data and the given correlation functions as,

$$\sum_{h=1}^{3}\sum_{j=4}^{8} R'_{hj}(\tau,w_\iota) + R'_{hj}(-\tau,w_\iota) - \sum_{h=1}^{3}{}'\sum_{j=4}^{8}{}'[\rho_{hj}(\tau) + \rho_{hj}(-\tau)], \qquad 0 \leqslant \tau \leqslant T(\alpha),$$

$$\sum_{h=1}^{3}\sum_{j=4}^{8} R'_{hj}(\tau,w_\iota) - \sum_{h=1}^{3}{}'\sum_{j=4}^{8}{}' \rho_{hj}(\tau), \qquad T(\alpha) < \tau \leqslant T_m. \tag{6.23}$$

The dashes on the summations, $\Sigma'\Sigma'$, indicate that ρ_{14} is to be omitted from the summation. The quantities (23) may be computed since all the required $\rho_{hj}(\tau)$ except

ρ_{14} are known or assumed to be zero.

In order to determine the mean square inadequacy of the corrected model the expectation of

$$\int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)[W_C(\tau) - W_E(\tau)][W_C(x) - W_E(x)]d\tau dx \qquad (6.24)$$

must be investigated, where $W_E(\tau)$ is given by (23) for $0 \leqslant \tau \leqslant T_m$, and $W_E(\tau) = 0$ for $\tau < 0$, and $\tau > T_m$.

In the intervals, $0 \leqslant \tau \leqslant T_m$, $0 \leqslant x \leqslant T_m$, the expectation

$$E[R'_{hj}(\tau)R'_{kl}(\tau)]$$

may be written, as in Chapter 4,

$$E\,\frac{1}{(T'_n)^2}\int_0^{T'_n}\int_0^{T'_n}\xi_h(t_1-\tau)\xi_j(t_1)\xi_k(t_2-x)\xi_l(t_2)dt_1dt_2$$

$$= \frac{1}{(T'_n)^2}\int_0^{T'_n}\int_0^{T'_n}\rho_{hj}(\tau)\rho_{kl}(x)dt_1dt_2 \;+ +$$

$$+ \frac{1}{(T'_n)^2}\int_0^{T'_n}\int_0^{T'_n}\rho_{hk}(t_2-t_1+\tau-x)\rho_{jl}(t_2-t_1)dt_1dt_2 \;+ +$$

$$+ \frac{1}{(T'_n)^2}\int_0^{T'_n}\int_0^{T'_n}\rho_{hl}(t_2-t_1+\tau)\rho_{jk}(t_2-t_1-x)dt_1dt_2 \qquad (6.25)$$

$$= \rho_{hj}(\tau)\rho_{kl}(x) + \text{other terms.} \qquad (6.26)$$

This result is true for all h,k (1-3), and j,l (4-8), providing the terms involving b_I, b_O and b_M are interpreted as above.

If the other terms in (26) were not present, i.e. if

$$E[R'_{hj}(\tau)R'_{kl}(x)] = E[R'_{hj}]E[R'_{kl}(\tau)],$$

then the expectation of (24) would equal

$$\int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)\{W_C(\tau) - E[W_E(\tau)]\}\{W_C(x) - E[W_E(x)]\}d\tau dx$$

which, as α tends to ∞, approaches

$$\int_{T_m}^{\infty} W_C^2(x)dx; \tag{6.27}$$

i.e. the mean square error in the output of the corrected model due to the truncation of the estimate of $W_C(\tau)$ at $\tau = T_m$.

In this limiting case where α approaches ∞, the contribution to the mean square error arising from the other terms in (26) may be written as the limit $\alpha \to \infty$

$$\int_0^{T_m}\int_0^{T_m} \rho_{M\alpha}(\tau-x) \sum_{h=1}^{3} \sum_{j=4}^{8} \sum_{k=1}^{3} \sum_{l=4}^{8} \frac{1}{(T_h')^2}\int_0^{T_h'}\int_0^{T_h'} \rho_{hk}(t_2-t_1+\tau-x)\rho_{jl}(t_2-t_1)+$$
$$+ + \rho_{hl}(t_2-t_1+\tau)\rho_{jk}(t_2-t_1-x)dt_1 dt_2 dx d\tau. \tag{6.28}$$

The parameter $T(\alpha)$ appearing in the estimate (23) of $W_C(\tau)$ has, in this case, been given the value zero and accordingly the terms in (23) with negative arguments have been omitted. These terms may be of interest in other cases depending on the value of α.

The mean square error in the output of the corrected model now consists of the contribution (27) due to the truncation at T_m plus the non-zero terms in (28). These latter terms will be considered individually.

If $h \neq k$ and $j \neq l$ there are 6 non zero terms, equal in pairs and corresponding to the subscripts shown in table 4.

Table 4

h	j	k	l
1	7	2	4
2	4	1	7
1	8	3	4
3	4	1	8
2	8	3	7
3	7	2	8

Corresponding to the subscripts 1,7,2,4 and 2,4,1,7 there are two terms; viz:

$$\lim_{\alpha\to\infty}\int_0^{T_m}\int_0^{T_m}\rho_{M\alpha}(\tau-x)\frac{1}{(T_h')^2}\int_0^{T_h'}\int_0^{T_h'}\rho_{14}(t_2-t_1+\tau)\rho_{72}(t_2-t_1-x)dt_1dt_2d\tau dx,$$

and $$\lim_{\alpha\to\infty}\int_0^{T_m}\int_0^{T_m}\rho_{M\alpha}(\tau-x)\frac{1}{(T_h')^2}\int_0^{T_h'}\int_0^{T_h'}\rho_{27}(t_2-t_1+\tau)\rho_{41}(t_2-t_1-x)dt_1dt_2d\tau dx,$$

Since $\rho_{hj}(\tau) = \rho_{jh}(-\tau)$ the second of these terms may be written

$$\lim_{\alpha\to\infty}\int_0^{T_m}\int_0^{T_m}\rho_{M\alpha}(\tau-x)\frac{1}{(T_h')^2}\int_0^{T_h'}\int_0^{T_h'}\rho_{72}(t_1-t_2-\tau)\rho_{14}(t_1-t_2+x)dt_1dt_2d\tau dx.$$

Further the variables of integration t_1 and t_2 may be interchanged, as also may x and τ since $\rho_{M\alpha}(\tau-x)$ is an even function. Hence these two terms are equal and their sum is

$$2\lim_{\alpha\to\infty}\int_0^{T_m}\int_0^{T_m}\rho_{M\alpha}(\tau-x)\frac{1}{(T_h')^2}\int_0^{T_h'}\int_0^{T_h'}\rho_{14}(t_2-t_1+\tau)\rho_{27}(t_1-t_2+x)dt_1dt_2dxd\tau$$

which, as in the appendix, is equal to

$$2\lim_{\alpha\to\infty}\int_0^{T_m}\int_0^{T_m}\rho_{M\alpha}(\tau-x)\frac{1}{(T_h')^2}\int_0^{T_h'}(T_h'-t_3)[\rho_{14}(t_3+\tau)\rho_{27}(x-t_3)]dt_3dxd\tau +$$

$$+\ 2\lim_{\alpha\to\infty}\int_0^{T_m}\int_0^{T_m}\rho_{M\alpha}(\tau-x)\frac{1}{(T_h')^2}\int_0^{T_h'}(T_h'-t_3)[\rho_{14}(\tau-t_3)\rho_{27}(x+t_3)]dt_3dxd\tau \quad (6.29)$$

$$= \frac{2}{(T_h')^2}\int_0^{T_m}\int_0^{T_h'}(T_h'-t_3)[W_C(t_3+\tau)\rho_{27}(\tau-t_3)+W_C(\tau-t_3)\rho_{27}(\tau+t_3)]dt_3\,d\tau.$$

The term corresponding to the subscripts 1,4,3,8 is, similarly, twice the limit $\alpha \to \infty$

$$\int_0^{T_m}\int_0^{T_m}\rho_{M\alpha}(\tau-x)\frac{1}{(T_h')^2}\int_0^{T_h'}\int_0^{T_h'}\rho_{14}(t_2-t_1+\tau)\rho_{38}(t_1-t_2+x)dt_1\,dt_2\,dx\,d\tau,$$

which, since ρ_{38} is constant and equal to $-\sigma^2_{bI}\int_0^{\infty}W_M(y)dy$, becomes

$$-2\sigma^2_{bI}\left(\int_0^{\infty}W_M(y)dy\right)\frac{1}{(T_h')^2}\int_0^{T_m}\int_0^{T_h'}(T_h'-t_3)[W_C(t_3+\tau)+W_C(\tau-t_3)]dt_3\,d\tau. \qquad (6.30)$$

The remaining term corresponding to the subscripts 2,7,3,8, is

$$-2\sigma^2_{bI}\left(\int_0^{\infty}W_M(y)dy\right)\frac{1}{(T_h')^2}\int_0^{T_m}\int_0^{T_h'}(T_h'-t_3)[\rho_{27}(t_3+\tau)+\rho_{27}(\tau-t_3)]dt_3\,d\tau. \qquad (6.31)$$

If $h \neq k$ and $j = 1$ there are no non zero terms in (28); if $h = k$ and $j \neq 1$ there are again no non zero terms in (28); it therefore remains to consider the terms for which $h = k$ and $j = 1$.

In the case $h = k = 1$ there is for each $j, j = 4-8$, the term

$$\lim_{\alpha\to\infty}\int_0^{T_m}\int_0^{T_m}\rho_{M\alpha}(\tau-x)\frac{1}{(T_h')^2}\int_0^{T_h'}\int_0^{T_h'}\rho_{M\alpha}(t_2-t_1+\tau-x)\rho_{jj}(t_2-t_1)dt_1\,dt_2\,dx\,d\tau,$$

and, according to equation (5.22) and the appendix, these will sum to

$$\frac{T_m}{T_h'}\left(\sigma^2_{OC}+\sigma^2_{NO}+\sigma^2_{bO}+\sigma^2_{NM}+\sigma^2_{bM}\right). \qquad (6.32)$$

There is also the additional term (5.23) when $h = k = 1$ and $j = l = 4$, viz.

$$\frac{2}{(T_n')^2}\int_0^{T_m}\int_0^{\tau}(T_n'-t_3)W_C(\tau-t_3)W_C(\tau+t_3)dt_3\,d\tau. \qquad (6.33)$$

In the case $h = k = 2$ there is for each $j, j = 4\text{-}8$, the term

$$\lim_{\alpha\to\infty}\int_0^{T_m}\int_0^{T_m}\rho_{M\alpha}(\tau-x)\frac{1}{(T_n')^2}\int_0^{T_n'}\int_0^{T_n'}\rho_{NI}(t_2-t_1+\tau-x)\rho_{JJ}(t_1-t_2)dt_1\,dt_2\,dx\,d\tau$$

and the sum of these five terms is

$$\sum_{J=4}^{8}\frac{1}{(T_n')^2}\int_0^{T_m}\int_0^{T_n'}\int_0^{T_n'}\rho_{NI}(t_2-t_1)\rho_{JJ}(t_2-t_1)dt_1\,dt_2\,d\tau$$

$$=\sum_{J=4}^{8}\frac{2T_m}{(T_n')^2}\int_0^{T_n'}(T_n'-t_3)\rho_{NI}(t_3)\rho_{JJ}(t_3)dt_3. \qquad (6.34)$$

There is also an additional term when $j = l = 7$, viz., the limit as $\alpha\to\infty$

$$\frac{1}{(T_n')^2}\int_0^{T_m}\int_0^{T_m}\rho_{M\alpha}(\tau-x)\int_0^{T_n'}\int_0^{T_n'}\rho_{27}(t_2-t_1+x)\rho_{27}(t_1-t_2+\tau)dt_1\,dt_2\,dx\,d\tau$$

$$=\frac{1}{(T_n')^2}\int_0^{T_m}\int_0^{T_n'}\int_0^{T_n'}\rho_{27}(t_2-t_1+\tau)\rho_{27}(t_1-t_2+\tau)dt_1\,dt_2\,d\tau$$

$$=\frac{2}{(T_n')^2}\int_0^{T_m}\int_0^{T_n'}(T_n'-t_3)\rho_{27}(t_3+\tau)\rho_{27}(\tau-t_3)dt_3\,d\tau. \qquad (6.35)$$

In the case $h = k = 3$ there are the terms

$$\sum_{J=4}^{8}\sigma_{bI}^2\lim_{\alpha\to\infty}\int_0^{T_m}\int_0^{T_m}\rho_{M\alpha}(\tau-x)\frac{1}{(T_n')^2}\int_0^{T_n'}\int_0^{T_n'}\rho_{JJ}(t_2-t_1)dt_1\,dt_2\,dx\,d\tau$$

$$=\frac{\sigma_{bI}^2T_m}{(T_n')^2}\sum_{J=4}^{8}\int_0^{T_n'}\int_0^{T_n'}\rho_{JJ}(t_2-t_1)dt_1\,dt_2$$

$$=\frac{2\sigma_{bI}^2T_m}{(T_n')^2}\sum_{J=4}^{8}\int_0^{T_n'}(T_n'-t_3)\rho_{JJ}(t_3)dt_3. \qquad (6.36)$$

For the two cases $j = 6$ and $j = 8$ this expression (36) simplifies to $T_m\sigma^2_{bI}\sigma^2_{bO}$ and $T_m\sigma^4_{bI}[\int_0^\infty W_M(y)dy]^2$ respectively. Again there is an additional term when $j = 8$, viz.

$$\lim_{n\to\infty}\int_0^{T_m}\int_0^{T_m}\rho_{M\alpha}(\tau-x)\frac{1}{(T_n')^2}\int_0^{T_n'}\int_0^{T_n'}\rho_{38}(t_2-t_1+\tau)\rho_{38}(t_1-t_2+x)dt_1dt_2dxd\tau$$

$$= T_m\sigma^4_{bI}[\int_0^\infty W_M(y)dy]^2 \, . \qquad (6.37)$$

Many of the expressions (29) - (36) are still quite complicated. Further simplification is possible in cases where suitable assumptions are appropriate. For example it may be appropriate to assume that the process $N_I(t,w)$ is "white" of power density k_1^2 over a frequency range wide enough that the following approximations are useful

$$\rho_{27}(\tau) = -k_1^2W_M(t), \qquad t > 0, \qquad (6.38)$$

$$\int_0^{T_n'}\rho_{NI}(t_2-t_1)\rho_{JJ}(t_2-t_1)dt_1 = k_1^2\sigma_J^2, \qquad 0 < t_2 < T_n'. \qquad (6.39)$$

Under these assumptions the expressions (29) - (36) become

$$\frac{-2k_1^2}{(T_n')^2}\int_0^{T_m}\int_0^{\tau}(T_n'-t_3)[W_C(t_3+\tau)W_M(\tau-t_3)+W_C(\tau-t_3)W_M(\tau+t_3)]dt_3d\tau. \qquad (6.29a)$$

$$-2\sigma^2_{bI}\left(\int_0^\infty W_M(y)dy\right)\frac{1}{(T_n')^2}\int_0^{T_m}\int_0^{T_n'}(T_n'-t_3)[W_C(t_3+\tau)+W_C(\tau-t_3)]dt_3d\tau. \qquad (6.30a)$$

$$+2\sigma^2_{bI}k_1^2\left(\int_0^\infty W_M(y)dy\right)\frac{1}{(T_n')^2}\int_0^{T_m}\int_0^{T_n'}(T_n'-t_3)[W_M(t_3+\tau)+W_M(\tau-t_3)]dt_3d\tau. \qquad (6.31a)$$

$$\frac{T_m}{T_n'}(\sigma^2_{OC} + \sigma^2_{NO} + \sigma^2_{bO} + \sigma^2_{NM} + \sigma^2_{bM}) \, . \qquad (6.32a)$$

$$\frac{2}{(T_n')^2}\int_0^{T_m}\int_0^{\tau}(T_n'-t_3)W_C(\tau-t_3)W_C(\tau+t_3)dt_3d\tau. \qquad (6.33a)$$

$$\frac{k_1^2 T_m}{T_h'} (\sigma_{OC}^2 + \sigma_{NO}^2 + \sigma_{bO}^2 + \sigma_{NM}^2 + \sigma_{bM}^2). \qquad (6.34a)$$

$$\frac{2k_1^4}{(T_h')^2} \int_0^{T_m} \int_0^{\tau} (T_h' - t_3) W_M(t_3 + \tau) W_M(\tau - t_3)\, dt_3\, d\tau. \qquad (6.35a)$$

$$\frac{2\sigma_{bI}^2 T_m}{(T_h')^2} \int_0^{T_h'} (T_h' - t_3) \rho_{OC}(t_3)\, dt_3. \qquad (6.36a)$$

$$\frac{2\sigma_{bI}^2 T_m}{(T_h')^2} \int_0^{T_h'} (T_h' - t_3) \rho_{NO}(t_3)\, dt_3. \qquad (6.36b)$$

$$T_m \sigma_{bI}^2 \sigma_{bO}^2. \qquad (6.36c)$$

$$\frac{2\sigma_{bI}^2 T_m}{(T_h')^2} \int_0^{T_h'} (T_h' - t_3) \rho_{NM}(t_3)\, dt_3. \qquad (6.36d)$$

$$T_m\ \sigma_{bI}^4 \left[\int_0^{\infty} W_M(y)\, dy\right]^2. \qquad (6.36e)$$

$$T_m\ \sigma_{bI}^4 \left[\int_0^{\infty} W_M(y)\, dy\right]^2. \qquad (6.37a)$$

For the purpose of estimating the effect of recording errors these expressions may be rearranged in the following five categories.

(i) Those which are present when there are no recording errors, viz.,

$$\frac{T_m}{T_h'} \sigma_{OC}^2, \qquad (6.38)$$

$$\frac{2}{(T_h')^2} \int_0^{T_m} \int_0^{\tau} (T_h' - t_3) W_C(\tau - t_3) W_C(\tau + t_3)\, dt_3\, d\tau. \qquad (6.39)$$

(ii) Those which contain the recording error N_O, viz.,

$$(1 + k_1^2) \frac{T_m}{T_h'} \sigma_{NO}^2, \qquad (6.40)$$

$$\frac{2\sigma_{bI}^2 T_m}{(T_h')^2} \int_0^{T_h'} (T_h' - t_3) \rho_{NO}(t_3)\, dt_3. \qquad (6.41)$$

(iii) Those which contain the recording error b_O, viz.,

$$(1 + k_1^2) \frac{T_m}{T_h'} \sigma_{bO}^2, \qquad (6.42)$$

$$T_m \, \sigma_{bI}^2 \, \sigma_{bO}^2. \qquad (6.43)$$

(iv) Those terms, not already contained in categories (i) - (iii), which contain N_I, viz.,

$$\frac{-2k_1^2}{(T_h')^2}\int_0^{T_m}\int_0^{\tau}(T_h'-t_s)[W_C(t_s+\tau)W_M(\tau-t_s)+W_C(\tau-t_s)W_M(\tau+t_s)]dt_s d\tau. \qquad (6.44)$$

$$2\sigma_{bI}^2 k_1^2 \left(\int_0^{\infty}W_M(y)dy\right)\frac{1}{(T_h')^2}\int_0^{T_m}\int_0^{T_h'}(T_h'-t_s)[W_M(t_s+\tau)+W_M(\tau-t_s)]dt_s d\tau. \qquad (6.45)$$

$$(1 + k_1^2) \frac{T_m}{T_h'} \sigma_{NM}^2 + k_1^2 \left(\frac{T_m}{T_h'}\right)(\sigma_{OC}^2 + \sigma_{bM}^2) \, . \qquad (6.46)$$

$$\frac{2k_1^4}{(T_h')^2}\int_0^{T_m}\int_0^{\tau}(T_h'-t_s)[W_M(t_s+\tau)W_M(\tau-t_s)]dt_s d\tau \, . \qquad (6.47)$$

$$2\sigma_{bI}^2 \, T_m \frac{1}{(T_h')^2}\int_0^{T_h'}(T_h'-t_s) \, \rho_{NM}(t_s) \, dt_s \, . \qquad (6.48)$$

(v) Those terms, not already contained in categories (i) - (iv), which contain b_I, viz.,

$$-2\sigma_{bI}^2\left[\int_0^{\infty}W_M(y)dy\right]\frac{1}{(T_h')^2}\int_0^{T_m}\int_0^{T_h'}(T_h'-t_s)[W_C(t_s+\tau)+W_C(\tau-t_s)]dt_s d\tau. \qquad (6.49)$$

$$\frac{T_m}{T_h'} \sigma_{bM}^2 \, . \qquad (6.50)$$

$$2\sigma_{bI}^2 \, T_m \frac{1}{(T_h')^2}\int_0^{T_h'}(T_h'-t_s)\rho_{OC}(t_s)dt_s \, . \qquad (6.51)$$

$$2T_m\sigma_{bI}^4\left[\int_0^{\infty}W_M(y)dy\right]^2. \qquad (6.52)$$

The terms in the first category have already been discussed in chapter 5 . The significance of the terms in the other categories may be estimated by comparing them with the known quantity $\sigma^2_{OM} = \int_0^\infty W^2_M(t)dt$. All these terms may be calculated from the given data excepting (44), (49), (51) and portion of (46); estimates of these four quantities may be made using the assumption made in chapter 4, viz. the error in the weighting function of the model decays with the same time constant as the weighting function of the model.

The terms in categories (ii) and (iii), i.e. those which are zero if there are no errors in the record of the system output, are fewer, easier to calculate and, in general, of less significance than those in categories (iv) and (v). If there were no errors in the recording of the system input, i.e. $\sigma^2_{bI} = \sigma^2_{NI} = 0$, then the remaining terms in categories (ii) - (v), viz., $\frac{T_m}{T_h}(\sigma^2_{NO} + \sigma^2_{bO})$, would in general be less than the term $\frac{T_m}{T_h}\sigma^2_{OC}$ in category (i). On the other hand the terms in categories (iv) and (v) remain, even though the errors in the recording of the system output are eliminated.

If the bias error in the input, b_I, is zero then all the terms in categories (i) - (v) approach zero as the length of record $T_h \to \infty$. However the terms (43) and (52) are independent of T_h and proportional to T_m; bias errors in the recording of the input therefore require special consideration when long lengths, T_h, are available for correcting the model. A bias in the recording of the system output needs no such consideration.

In section 5.1 an expression (5.31) leading to an optimum value of T_m for a given $T_{\acute{n}}$ was determined. The present section could be extended along those lines; but in practice it will usually be satisfactory to determine T_m from (5.31) and then compare the additional terms, categories (ii) - (v), with those already taken into account, category (i). If any of the additional terms are significant it may be necessary to revise the value of T_m.

This chapter may be extended along the lines of chapter 5 to cover the two cases:

(a) $I_S(t,w)$ is any type X_1 random process,

(b) more than one set of records are available.

CHAPTER 7.

UNCORRELATED MULTIPLE INPUTS

A physical component may have more than one input and output. In order to characterize such a component as a system, a number of transformations must be known, since the relation between the system inputs and outputs will be of the form

$$\sum_{k=1}^{n} T_{Sjk}[I_{Sk}(t,w_\iota)] = O_{Sj}(t,w_\iota); \qquad (7.1)$$

where each of the n inputs $I_{Sk}(t,w_\iota)$, as operated on by the appropriate transformation T_{Sjk}, provides, in general, some contribution to the output $O_{Sj}(t,w_\iota)$. A model of the system will then involve corresponding transformations T_{Mjk}; the task of adjusting such a model will involve adjusting these T_{Mjk}.

The following restrictions apply to the problem discussed in this chapter:

(a) the system components T_{Sjk} all have real weighting functions of the form

$$\sum_{l=1}^{n} P_{ljk}(t)\, e^{s_{ljk}t}, \qquad Re\,(s_{ljk}) < 0,$$

where the $P_{ljk}(t)$ are polynomials in t;

(b) the model components T_{Mjk} all have real known weighting functions of the form

$$\sum_{l=1}^{m} Q_{ljk}(t)\, e^{r_{ljk}t}, \qquad Re\,(r_{ljk}) < 0,$$

where the $Q_{1jk}(t)$ are polynomials in t;

(c) the system inputs are all realisations of type X_1 random processes;

(d) one set of recordings of portions of the inputs $I_{Sk}(t,w_l)$, and the outputs $O_{Sj}(t,w_l)$ and $O_{Mj}(t,w_l)$ are available;

(e) the inputs are not cross correlated, i.e.

$$\rho_{ISk,ISl}(\tau) \equiv 0, \qquad k \neq l.$$

This problem may be reduced to one of those already considered in previous chapters. Since it will not be necessary to distinguish between the system outputs, the subscript j will be omitted when used for this purpose; e.g. in T_{Sjk}, T_{Mjk}, $O_{Sj}(t,w_l)$.

7.1 Error free records of unlimited length

If there are no recording errors and the records are of unlimited length then

$$\sum_{k=1}^{n}(T_{Sk} - T_{Mk})[I_{Sk}(t,w_l)] = O_C(t,w_l), \qquad (7.2)$$

where $O_C(t,w_l) = O_S(t,w_l) - O_M(t,w_l)$. Multiplying each side of this equation by $I_{Sl}(t-\tau,w_l)$ and taking expectations yields

$$\sum_{k=1}^{n}(T_{Sk} - T_{Mk})[\rho_{ISl,ISk}(\tau)] = \rho_{ISl,OC}(\tau).$$

This becomes, since the inputs are uncorrelated,

$$(T_{Sl} - T_{Ml})[\rho_{ISl}(\tau)] = \rho_{ISl,OC}(\tau),$$

so that $$\int_0^\infty \rho_{ISl}(\tau-x)W_{Cl}(x)dx = \rho_{ISl,OC}(\tau), \qquad \tau \geq 0; \qquad (7.3)$$

where $W_{C1}(t)$ is the weighting function corresponding to $T_{C1} = T_{S1} - T_{M1}$.

Equation (3) is the Wiener-Hopf equation already discussed in chapter 4.

7.2 Records of finite length

If the records are of finite length then, as with a single input, the adequacy of the corrected model will vary with the lengths of the records available for model checking. The case considered here is one where there are no recording errors and, for each k,

$$I_{Sk}(t,w) = M_{\alpha k}(t,w),$$

i.e. an S.G.M. of zero mean and correlation function $\frac{\alpha_k}{2}e^{-\alpha_k|t|}$. The weighting function $W_{C1}(t)$ corresponding to the transformation $T_{C1} = T_{S1} - T_{M1}$ is then equal to

$$\lim_{\alpha_1\to\infty} \rho_{M\alpha 1,OC1}(t) + \rho_{M\alpha 1,OC1}(-t), \qquad t \geqslant 0;$$

where $O_{C1}(t,w_l) = T_{C1}[M_{\alpha 1}(t,w_l)]$.

The recordings available may be used to calculate, for each l, the quantity

$$\begin{aligned} R'_{M\alpha 1,OC}(\tau,w_l) &= \frac{1}{T_h'}\int_0^{T_h'} M_{\alpha 1}(t-\tau,w_l)O_C(t,w_l)dt, \quad 0 \leqslant \tau \leqslant T_{m1}, \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad, \quad \tau > T_{m1}; \end{aligned}$$

and

$$\begin{aligned} R'_{M\alpha 1,OC}(-\tau,w_l) &= \frac{1}{T_h'}\int_0^{T_h'} M_{\alpha 1}(t+\tau,w_l)O_C(t,w_l)dt, \quad 0 \leqslant \tau \leqslant T(\alpha_1), \\ &= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \tau > T(\alpha_1); \end{aligned}$$

where $\lim_{\alpha_1\to\infty} T(\alpha_1) = 0$, and T_h' is the effective length of the records after discarding sufficient to allow for transients in the output of the model; T_{ml} may be chosen differently for each T_{Cl}.

Since

$$R'_{M\alpha l,OC}(\tau,w_t) = R'_{M\alpha l,OCl}(\tau,w_t) + \sum_{k\neq l} R'_{M\alpha l,OCk}(\tau,w_t), \qquad (7.4)$$

then

$$\begin{aligned} &E[R'_{M\alpha l,OC}(\tau,w_t) + R'_{M\alpha l,OC}(-\tau,w_t)] \qquad (7.5)\\ &= \rho_{M\alpha l,OCl}(\tau) + \rho_{M\alpha l,OCl}(-\tau), \qquad 0 \leqslant \tau \leqslant T(\alpha_l), \end{aligned}$$

and $$\rho_{M\alpha l,OCl}(\tau), \qquad T(\alpha_l) < \tau \leqslant T_{ml}.$$

Hence as α_l tends to ∞ the expression (5) approaches

$$W_{Cl}(\tau), \qquad 0 \leqslant \tau \leqslant T_{ml}.$$

The adequacy of a model corrected in this way will now be investigated. Since the inputs are not correlated with each other

$$\sigma^2_{OC} = \sum_{k=1}^{n} \sigma^2_{OCk}. \qquad (7.6)$$

Equation (2) may be rewritten in the form

$$O_C(t,w_t) = O_{Cl}(t,w_t) + \sum_{k\neq l} O_{Ck}(t,w_t);$$

whence $O_C(t,w_t)$ may be interpreted as the difference between recordings of $O_{Sl}(t,w_t)$ and $O_{Ml}(t,w_t)$ where the record of $O_{Sl}(t,w_t)$ has a recording error of $\sum_{k\neq l} O_{Ck}(t,w_t)$ uncorrelated with the input $M_{\alpha l}(t,w_t)$.

The present problem is therefore identical to that considered in section 6.2, restricted so that

$$N_I(t,w_\iota) = b_I(t,w_\iota) = b_O(t,w_\iota) = N_M(t,w_\iota) = b_M(t,w_\iota) = 0$$

and

$$N_O(t,w_\iota) = \sum_{k\neq 1} O_{Ck}(t,w_\iota).$$

Hence it may be deduced from the results in section 6.2 that the mean square inadequacy of a model adjusted as above will, as α_1 approaches infinity, contain the following terms.

$$\int_{T_{m1}}^{\infty} W_{C1}^2(x)dx; \tag{7.7}$$

$$\frac{T_{m1}}{T_n'} \sum_{k=1}^{n} \sigma^2_{OCk} = \frac{T_{m1}}{T_n'} \sigma^2_{OC}; \tag{7.8}$$

$$\frac{2}{(T_n')^2} \int_0^{T_{m1}} \int_0^{\tau} (T_n' - t_3) W_{C1}(\tau - t_3) W_{C1}(\tau + t_3) dt_3\, d\tau. \tag{7.9}$$

If the calculation is repeated for each input then the mean square inadequacy of the components of the model contributing to the particular output difference, $O_C(t,w_\iota)$, being considered will be $\sum_{1=1}^{n}$

$$\int_{T_{m1}}^{\infty} W_{C1}{}^2(x)dx + \frac{T_{m1}}{T_n'} \sigma^2_{OC} + \frac{2}{(T_n')^2} \int_0^{T_{m1}} \int_0^{\tau} (T_n' - t_3) W_{C1}(\tau - t_3) W_{C1}(\tau + t_3) dt_3\, d\tau. \tag{7.10}$$

As in section 5.1 it may be useful to approximate $W_{C1}(x)$ by $k_1 e^{-a_1 x}$ and endeavour to determine a criterion for each T_{m1} assuming T_n' fixed. Writing

$$a_1 T_{m1} = p_1,$$

$$a_1 T_n' = q_1,$$

and substituting in (10) gives, as in section 5.1,

$$\sum_{l=1}^{n}\{\sigma^2_{OCl}e^{-2p_l}+\sigma^2_{OC}\frac{p_l}{q_l}+\sigma^2_{OCl}[\frac{1}{q_l}-\frac{1}{2q_l^2}-e^{-2p_l}(2\frac{p_l}{q_l}+\frac{1}{q_l}-\frac{p_l^2}{q_l^2}-\frac{p_l}{q_l^2}-\frac{1}{2q_l^2})]\} \qquad (7.11)$$

Differentiating (11) with respect to p_l gives

$$\frac{1}{q_l}\sigma^2_{OC}+\sigma^2_{OCl}\,e^{-2p_l}\left(\frac{4p_l}{q_l}-\frac{2p_l^2}{q_l^2}-2\right),$$

which is equal to zero if

$$e^{2p_l}=\frac{\sigma^2_{OCl}}{\sigma^2_{OC}}\left(2+\frac{2p_l^2}{q_l^2}-\frac{4p_l}{q_l}\right)q_l, \qquad (7.12)$$

or

$$p_l \doteq \tfrac{1}{2}\ln\left(\frac{2\sigma^2_{OCl}}{\sigma^2_{OC}}q_l\right). \qquad (7.13)$$

If one is prepared to make a further assumption concerning the ratio $\frac{\sigma^2_{OCl}}{\sigma^2_{OC}}$ then expression (12) or (13) may be used to determine a useful value of T_{ml} for a given T_l'. Those model components which contribute most to the inadequacy will then, reasonably enough, receive the most attention. On the other hand any for which $\frac{2\sigma^2_{OCl}}{\sigma^2_{OC}}q_l \leq 1$ should not be adjusted.

This discussion may be extended as in section 5.2 to cover the case of type X_1 inputs.

In this section all except one of the model outputs have been treated as a recording error. In many cases the genuine recording errors will not be important in comparison with this term; if necessary they may be dealt with as in section 6.2.

CHAPTER 8.

SOME EXPERIMENTAL CONFIRMATION

Experimental work to confirm some of the preceding theoretical results has been carried out using analogue computing equipment at the Weapons Research Establishment, Salisbury, South Australia. In addition to standard computing equipment for integration, addition and multiplication, the multi-channel digital to analogue converter IDAC proved very useful in this work. IDAC will convert up to 20 channels of digital data stored on magnetic tape into voltages which appear at each of the 20 outputs of the machine. These voltages may be changed every $\frac{1}{50}$ second. Thus, given any bounded function of time $f(t)$, a voltage output $V_0(t)$ may be obtained from IDAC which, apart from a scale factor, is a step function approximation to $f(t)$;

$$\text{e.g.} \quad V_0(t) = f(\tfrac{n}{50}), \qquad \tfrac{n}{50} \leqslant t < \tfrac{n+1}{50};$$

where n is a non-negative integer. This converter is fully described by Dunne [53]. In the experiments to be described here, IDAC was used in the following way to generate synthetic inputs to the system and model.

Random numbers from a population normally distributed with zero mean and generated using a method described by Taussky and Todd [54], were stored on the magnetic tape for

IDAC. Each IDAC channel contained the same numbers, except that they were displaced so that the voltages $V_0(t)$ and $V_0(t-\tau)$ were available simultaneously at the output of IDAC for a number of values of τ, $0 \leqslant \tau \leqslant 3$. As explained later this facility was extremely useful for computing points on the experimental cross-correlation function $R'_{IS,OC}(\tau)$.

The output, $V_0(t)$, of random numbers from IDAC was assumed to be portion of a realisation $X(t,w_\iota)$ of a stationary normal process having zero mean and autocorrelation function

$$= \sigma_a^2 \left(1 - \frac{|\tau|}{T}\right), \qquad |\tau| \leqslant T,$$

$$= 0 \qquad , \qquad |\tau| > T; \quad (8.1)$$

where

T equals the constant step length, i.e. $\frac{1}{50}$ second, and σ_a^2 is the variance of the random numbers. This autocorrelation function may be derived by a method similar to that described by Solodovnikov [26,P.105], for example. The corresponding spectrum,

$$2\int_0^T \sigma_a^2 \left(1 - \frac{\tau}{T}\right) \cos \omega\tau d\tau$$

$$= \sigma_a^2 T \frac{\sin^2\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)^2},$$

is almost flat for $|\omega T| \leqslant \frac{1}{2}$; in particular for $|\omega| \leqslant 24$ it lies between $0\cdot98\sigma_a^2 T$ and $\sigma_a^2 T$.

If $X(t,w_\iota)$ is operated on by a transformation whose weighting function is $\alpha e^{-\alpha t}$ the result will be a realisation,

$I_S(t, w_l)$, of a stationary normal process having spectrum

$$\frac{\sigma_a^2 T \sin^2(\frac{\omega T}{2})}{(\frac{\omega T}{2})^2} \frac{\alpha^2}{(\alpha^2 + \omega^2)} . \qquad (8.2)$$

Apart from the scale factor $\sigma_a^2 T$ this expression differs from the spectrum of an S.G.M. having autocorrelation function $\frac{\alpha}{2} e^{-\alpha|\tau|}$ by < 2% for $|\omega| \leqslant 24$.

For the experiments described in this chapter the quantity

$$I_S(t) = \int_0^\infty V_0(t-x) 10 e^{-10x} dx$$

was used as system input. This quantity was assumed to be portion of a realisation of a stationary normal process of zero mean which differs from a Markov process in spectrum and other characteristics by amounts which are unimportant for the present experiments.

Two experiments will be described. The first, which confirms that the quantity $R'_{M\alpha,OC}(t) + R'_{M\alpha,OC}(-t)$ may in some cases give a better estimate of the weighting function of a system component than does $R'_{M\alpha,OC}(t)$ alone, was also used to check the experimental method. The second and main experiment confirms some of the results of chapter 5 using an electromechanical servomechanism as the physical component.

8.1 Experiment 1: comparison of cross correlation approximations to an exponential weighting function

8.1.1 Objects of the experiment

(i) To show that $R'_{IS,OC}(t) + R'_{IS,OC}(-t)$, $(t \geqslant 0)$, provides a better estimate of the weighting function $W_C(t)$ than does $R'_{IS,OC}(t)$ in the case:

(a) $I_S(t)$, the system input, is generated as previously described in this chapter;

(b) $W_C(t) = e^{-t}$, $t \geq 0$.

(ii) To check the usefulness of the equation

$$p = \tfrac{1}{2} \ln 2q, \tag{5.31}$$

and the expression

$$\frac{\sigma^2_{OC}}{\sigma^2_{OS}}\left[\frac{p+1}{q} + e^{-2p},\right] \tag{5.30}$$

derived in chapter 5, which relate the quantities T_m, T'_n and the mean square inadequacy of the corrected model.

(iii) To check the experimental technique.

8.1.2 Method

The computing units were interconnected as shown in fig. 15. Each of the outputs $V_0(t)$ and $V_0(t-\tau)$ from IDAC was applied to a filter, having frequency response function $\frac{10}{10+i\omega}$, to produce the system input $I_S(t)$, and $I_S(t-\tau)$. $I_S(t)$ was then applied to the system whose weighting function was e^{-t} and frequency response function $\frac{1}{1+i\omega}$ to produce the system output $O_S(t)$. In this experiment there is no model, i.e. $O_M(t) = 0$, and therefore

$$O_C(t) = O_S(t).$$

The two quantities $O_S(t)$ and $I_S(t-\tau)$ were then multiplied together and the result applied to the integrator, as shown,

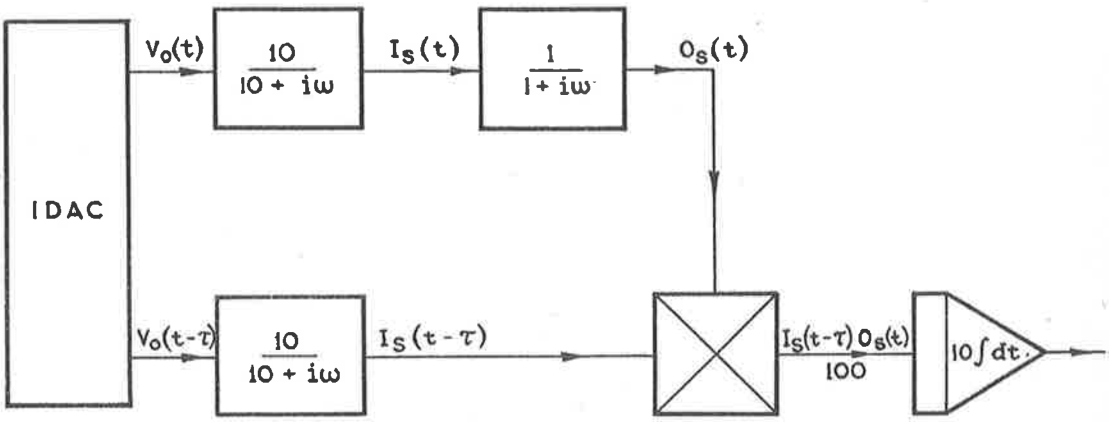

FIGURE 15. BLOCK DIAGRAM OF COMPUTER INTERCONNECTIONS FOR EXPERIMENT 1

to give

$$\frac{1}{10}\int I_S(t-\tau)\ O_S(t)dt,$$

$$= \frac{1}{10}\int I_S(t-\tau)\ O_C(t)dt.$$

A continuous record of this quantity was displayed on a digital voltmeter which was read at 3 seconds, 103 seconds and 203 seconds after the commencement of each run; the three seconds were allowed for transients to die out. Pen recordings of the quantities $V_o(t)$, $I_S(t)$, $O_S(t)$ and

$$\frac{1}{10}\int I_S(t-\tau)\ O_C(t)dt$$

were taken throughout the experiment. A portion of such a recording of the last three quantities is shown as fig. 16.

8.1.3 Results

The main data obtained from the experiment are shown in table 5, where for each of eleven values of τ the output $\frac{1}{10}\int_0^{T_n} I_S(t-\tau)O_C(t)dt$ is shown for T_n = 3 secs, 103 secs., and 203 seconds. The quantities in this table have units of volt² seconds. In order to estimate the system weighting function their value at 3 seconds must be deducted from those at 103 and 203 seconds and the result divided by a scale factor. This scale factor is the product of the following quantities:

(a) $\sigma_a^2 T = \frac{244}{50} = 4{\cdot}88$ volt² secs.;

(b) the gain of the computer $= \frac{1}{10}$;

(c) the time of integration = 100 or 200 seconds.

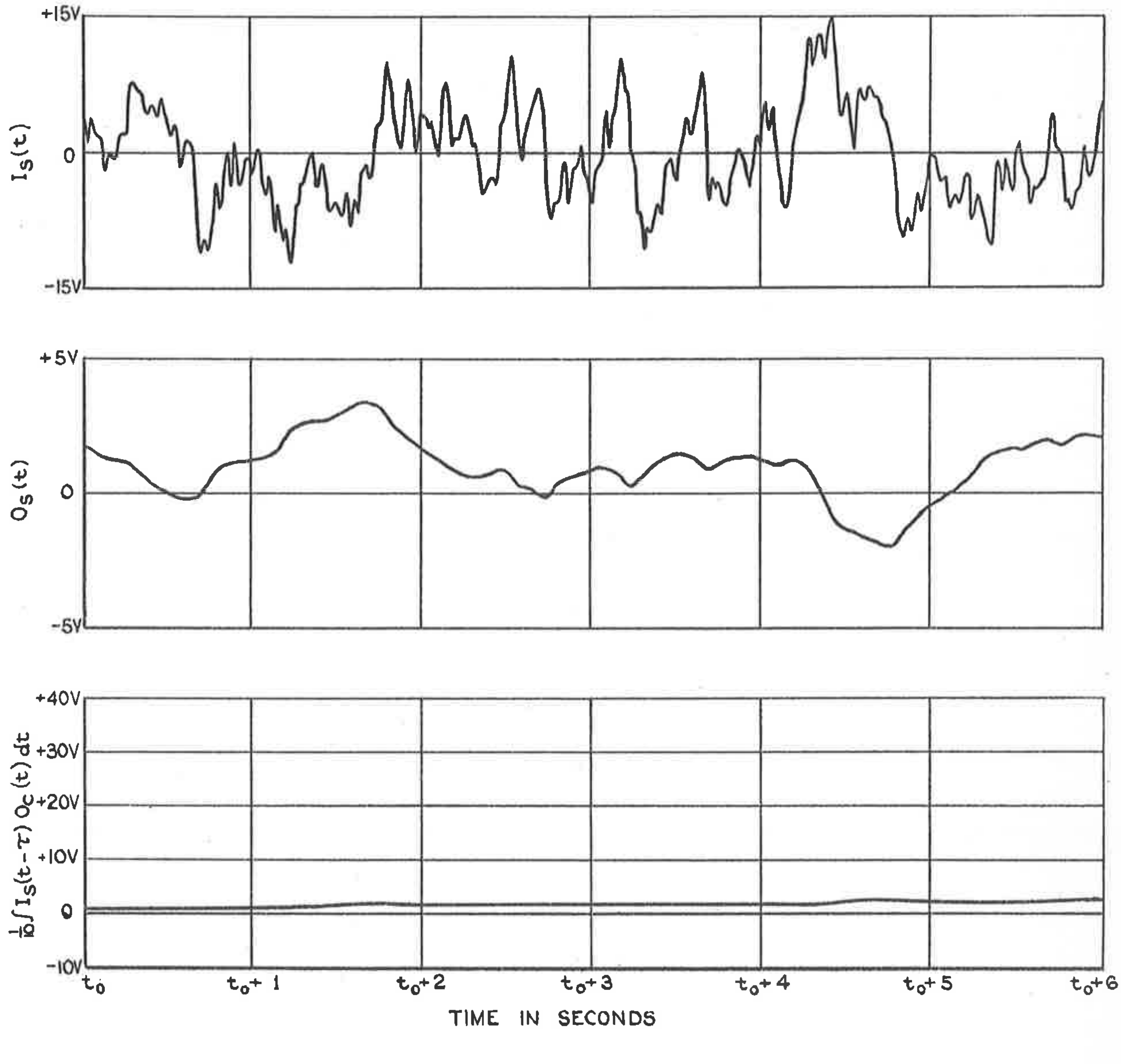

FIGURE 16. PORTION OF PEN RECORDS TAKEN DURING EXPERIMENT 1

Table 5

τ in Secs.	$\frac{1}{10}\int_0^{T_n} I_S(t-\tau)O_C(t)dt$ at values of T_n =		
	3 Secs	103 Secs	203 Secs
-0·25	0	0·20	-0·30
-0·10	0·4	10·0	18·6
0·00	0·4	24·9	45·8
0·10	0·6	37·5	69·5
0·25	0·7	37·3	68·3
0·50	0·6	31·9	55·9
0·75	0·4	20·7	44·0
1·00	0·6	20·0	41·1
1·50	0·2	11·6	21·6
2·00	0·2	5·5	11·4
3·00	0·1	5·3	7·7

The scale factor is therefore, numerically, 48·8 for 100 seconds integration time, and 97·6 for 200 seconds integration time.

When adjusted in this way the data provide the quantities $R'_{IS,OC}(\tau)$ for some τ in the range, $0 \leqslant \tau \leqslant 3$, and $R'_{IS,OC}(-\tau)$ for $\tau = 0$, 0·10 and 0·25. Tables 6A and 6B show the quantities $R'_{IS,OC}(\tau) + R'_{IS,OC}(-\tau)$ and $R'_{IS,OC}(\tau)$, for the values of τ covered by the experiment and the two integration times of 100 seconds and 200 seconds. These results are also shown on the graphs, figs. 17 and 18, one for each integration time. Each graph also shows curves for $e^{-\tau}$, the system weighting function, and calculated values for $\rho_{M\alpha,OC}(\tau)$ and $\rho_{M\alpha,OC}(\tau) + \rho_{M\alpha,OC}(-\tau)$; where $M_\alpha(t,w_t)$ is a realisation of the stationary Gauss Markov process having correlation function $\frac{\alpha}{2}\, e^{-\alpha|t|}$, $\alpha = 10$; and

$$O_C(t,w_t) = \int_0^\infty M_\alpha(t-x,w_t)\, e^{-x} dx.$$

8.1.4 Discussion

(i) Clearly the first object has been achieved since for values of τ where $R'_{IS,OC}(-\tau)$ is significant, i.e. $\tau = 0$ and $\tau = 0{\cdot}1$, the values of $R'_{IS,OC}(\tau) + R'_{IS,OC}(-\tau)$ are much closer to those of $e^{-\tau}$ than is the value of $R'_{IS,OC}(\tau)$ alone.

(ii) The equation $p = \frac{1}{2} \ln 2q$ (5.31) gives values of $p = 2{\cdot}65$ for $q = 100$ and $p = 3$ for $q = 200$. Since the time constant of the

Table 6A

τ in Secs.	Integration time 100 Secs. $R'_{IS,OC}(\tau) + R'_{IS,OC}(-\tau)$	$R'_{IS,OC}(\tau)$
0	1·00	0·50
0·10	0·95	0·76
0·25	0·76	0·75
0·50	0·64	0·64
0·75	0·42	0·42
1·00	0·40	0·40
1·50	0·23	0·23
2·00	0·11	0·11
3·00	0·10	0·10

Table 6B

τ in Secs.	Integration time 200 Secs. $R'_{IS,OC}(\tau) + R'_{IS,OC}(-\tau)$	$R'_{IS,OC}(\tau)$
0	0·92	0·46
0·10	0·88	0·70
0·25	0·68	0·69
0·50	0·56	0·56
0·75	0·46	0·46
1·00	0·41	0·41
1·50	0·21	0·21
2·00	0·11	0·11
3·00	0·08	0·08

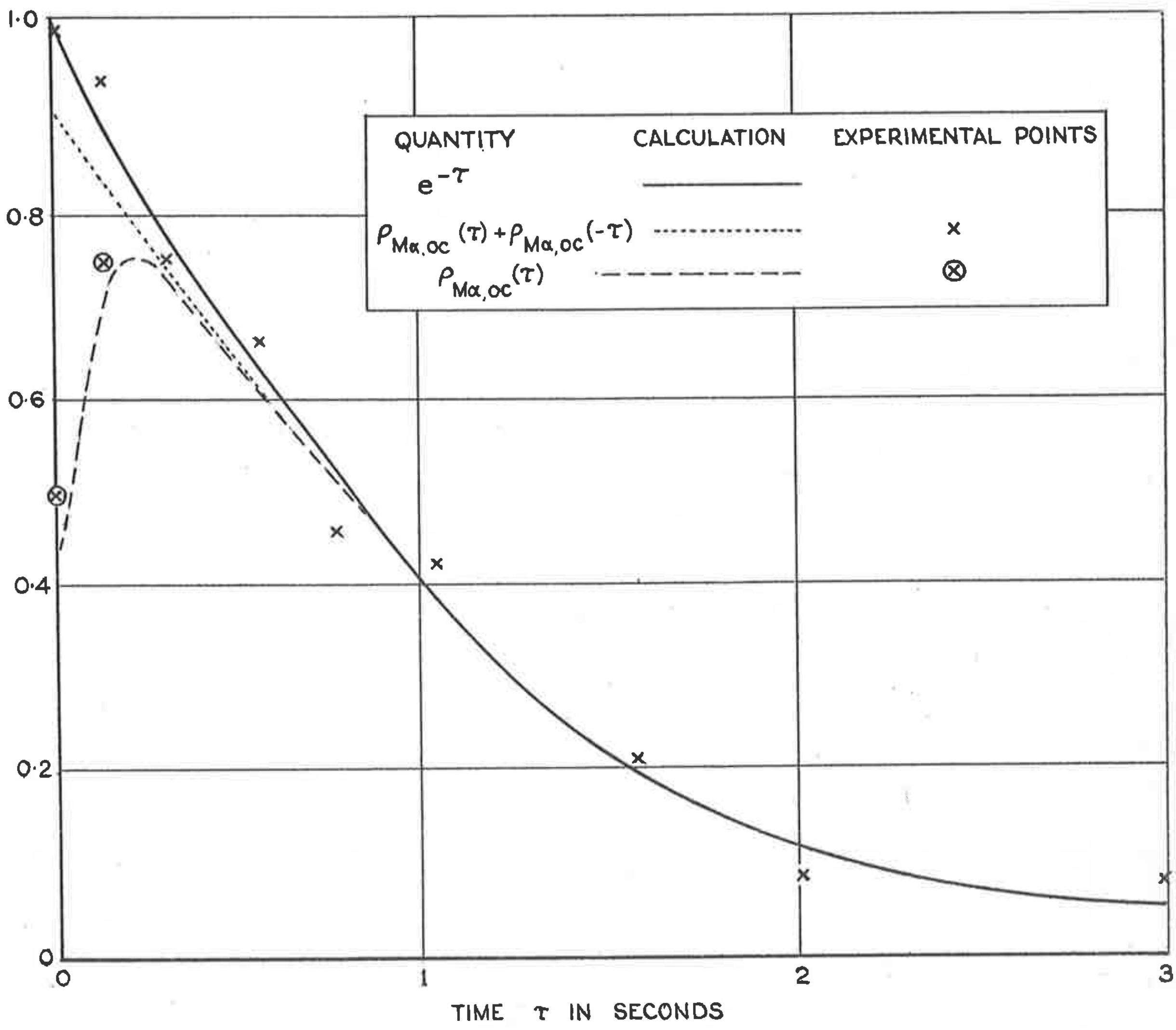

FIGURE 17. GRAPH OF CALCULATED CORRELATION FUNCTIONS AND EXPERIMENTAL POINTS OBTAINED IN EXPERIMENT 1 - INTEGRATION TIME 100 SEC

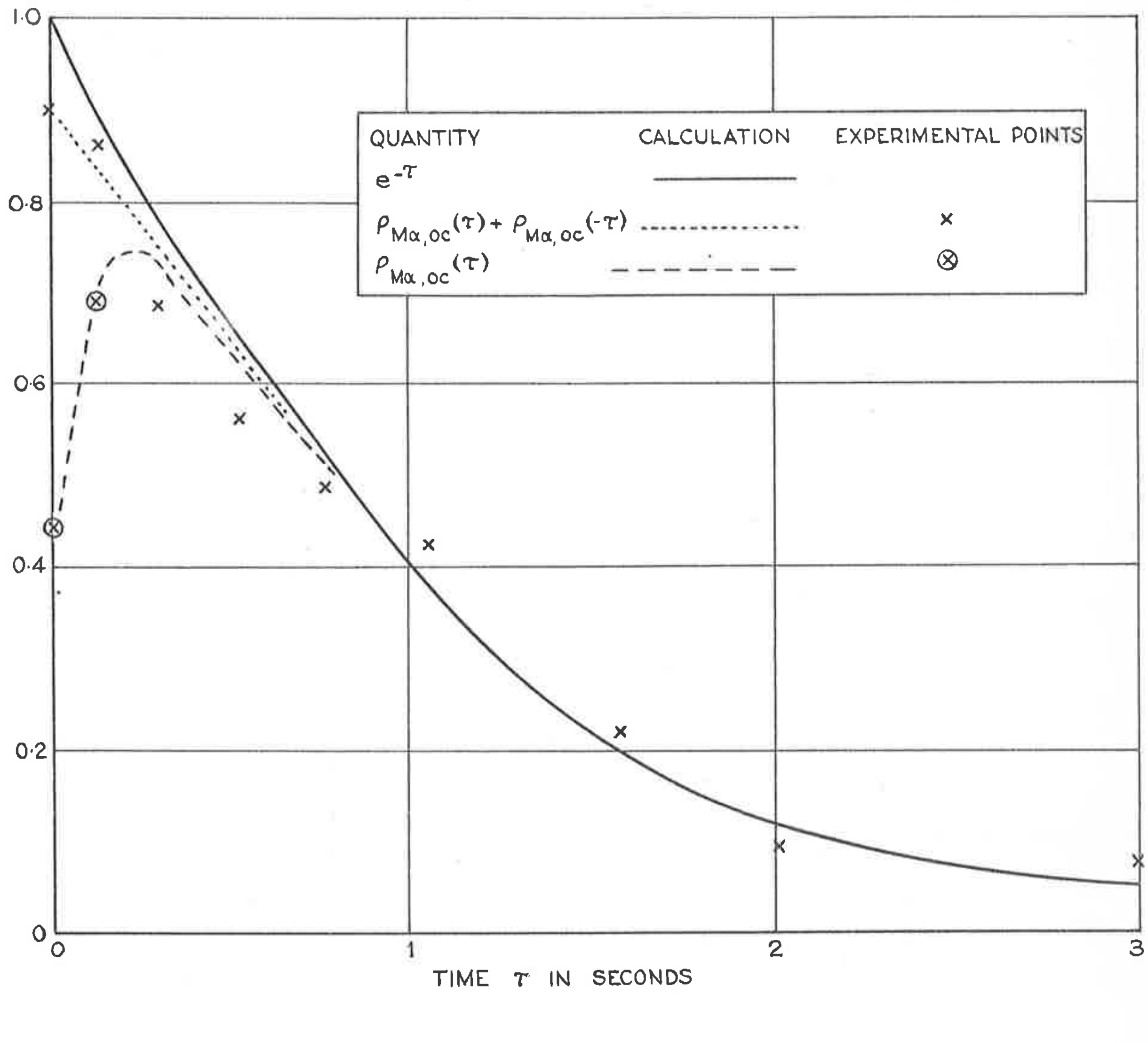

FIGURE 18. GRAPH OF CALCULATED CORRELATION FUNCTIONS AND EXPERIMENTAL POINTS OBTAINED IN EXPERIMENT 1 - INTEGRATION TIME 200 SEC

system weighting function is 1 second then a value of

T_m = 2·65 seconds corresponds to T_{h} = 100 seconds,

and

T_m = 3 seconds corresponds to T_{h} = 200 seconds.

The corresponding value for the mean square inadequacy of the corrected model as given by

$$\frac{\sigma^2_{OC}}{\sigma^2_{OS}}\left(\frac{p+1}{q} + e^{-2p}\right) \tag{5.30}$$

$$= 0{\cdot}042 \quad \text{for} \quad q = 100,$$

and $$0{\cdot}023 \quad \text{for} \quad q = 200.$$

An upper limit to the inadequacy,

$$\bar{A} = \left[\frac{\int_0^\infty\int_0^\infty \rho_I(\tau-x)\{e^{-x}-W_E(x)\}\{e^{-\tau}-W_E(\tau)\}\,dx\,d\tau}{\sigma^2_{OS}}\right]^{\frac{1}{2}},$$

of a model based on the data from this experiment may be found in the following way.

$$\sigma^2_{OS} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_I(\omega)\,\frac{1}{(1+\omega^2)}\,d\omega$$

$$= \frac{\sigma^2_a T}{2\pi}\int_{-\infty}^{\infty}\frac{\sin^2(\frac{\omega T}{2})}{(\frac{\omega T}{2})^2}\,\frac{\alpha^2}{\alpha^2+\omega^2}\,\frac{1}{1+\omega^2}\,d\omega$$

$$> \frac{\sigma^2_a T}{2\pi}\,({\cdot}98)\int_{-24}^{24}\frac{100}{(100+\omega^2)}\,\frac{1}{(1+\omega^2)}\,d\omega$$

$$= \frac{\sigma^2_a T}{2\pi}\,({\cdot}98)\int_{-24}^{24}\frac{100}{99(1+\omega^2)} - \frac{100}{99(100+\omega^2)}\,d\omega$$

$$= \frac{\sigma_a^2 T}{2\pi}\,\frac{98}{99}\,2(\text{arc tan } 24 - \frac{1}{10}\text{ arc tan } 2\cdot 4)$$

$$= 0\cdot 44\ \sigma_a^2 T\,. \qquad (8.3)$$

$$\int_0^\infty\!\!\int_0^\infty \rho_I(\tau - x)\{e^{-x} - W_E(x)\}\{e^{-\tau} - W_E(\tau)\}\,dx\,d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_I(\omega)\ |\Phi(\omega)|^2\ d\omega,$$

where $\Phi(\omega) = \int_0^\infty (e^{-x} - W_E(x))e^{-i\omega x}dx.$

The above expression is therefore

$$< \frac{\sigma_a^2 T}{2\pi}\int_{-\infty}^{\infty}|\Phi(\omega)|^2 d\omega$$

$$= \sigma_a^2 T \int_0^\infty |e^{-x} - W_E(x)|^2 dx. \qquad (8.4)$$

In the case $q = 100$, i.e. $T_m = 2\cdot 65$ seconds and $T_h = 100$ seconds, and for the values of τ in table 6A the difference $|e^{-\tau} - W_E(\tau)| \leqslant 0\cdot 05$. Hence in this case the expression (4) will be less than

$$\sigma_a^2 T[25 \times 10^{-4} \times 2\cdot 65 + \int_{2\cdot 65}^{\infty} e^{-2x}dx]$$

$$= 0\cdot 009\ \sigma_a^2 T. \qquad (8.5)$$

The inadequacy of the corrected model, i.e. one using a weighting function based on the data of this experiment should therefore be less than

$$\left(\frac{0\cdot 009}{0\cdot 44}\right)^{\frac{1}{2}} = (0\cdot 02)^{\frac{1}{2}} = 0\cdot 14; \qquad (8.6)$$

so that in this case the adequacy actually obtained is better than that predicted by the expression (5.30).

As mentioned below the accuracy of the computations for $T_h' = 200$ seconds was affected by drift in the multiplication unit. For this reason the difference $|e^{-\tau}-W_E(\tau)|$ is for some values of τ, notably $\tau = \frac{1}{4}$, much greater than the value obtained for $T_h' = 100$ seconds. A calculation as above would have to be based on

$$|e^{-\tau}-W_E(\tau)| \leqslant 0\cdot10,$$

which leads in this case to an inadequacy of $0\cdot26$. A fairer estimate of the $\int_0^3 (e^{-x}-W_E(x))^2 dx$ would be one based on Simpson's rule for integration, e.g.,

$$\frac{1}{12}(64\times10^{-4}+4\times10^{-2}+2\times21\times10^{-4}+4\times10^{-4}+16\times10^{-4}) +$$
$$+ \frac{1}{6}(16\times10^{-4}+4\times10^{-4}+6\times10^{-4})+9\times10^{-4}$$
$$= 0\cdot0057.$$

This estimate plus $\int_3^\infty e^{-2x}dx$, combined with $\sigma^2_{OS} = 0\cdot44\,\sigma^2_a T$ leads to an inadequacy in this case of

$$\bar{A} = 0\cdot13. \qquad (8.7)$$

The equation (5.31) and the expression (5.30) appear then to give a useful prediction of the adequacy of the corrected model in the present case.

(iii) The computing equipment interconnected as in fig. 15 was extremely useful for this work. The use of a multiplier and integrator, together with the recording of the input on several IDAC channels, proved a most efficient method for computing points on the required cross correlation functions. Many methods for computing such functions have been described [26,P.116], [24,P.276], some similar to the one actually used. One alternative, viz., reading the pen records of $I_S(t-\tau)$ and $O_S(t)$, and feeding these to a digital computer to compute the cross correlation functions, was at first contemplated. Many hours of work would have been required to produce the results which were available immediately from the experimental arrangement, fig. 15.

The only difficulty encountered was some drift in the multiplying unit which introduced some inaccuracy in the readings at 203 seconds. Errors of 3 or 4 volts are possible in these readings. As a result of this experience the time of integration was restricted to 103 seconds in the second experiment. It was later found that this drift could be greatly reduced by using a servo-multiplier rather than the electronic multiplier used in this experiment.

8.2 Experiment 2: adjusting the mathematical model of a servomechanism

8.2.1 Objects of the experiment

(i) To demonstrate the use of cross correlation for adjusting a model of an electromechanical device.

(ii) To demonstrate the adequacy of a model adjusted as in (i) above.

8.2.2 Method

The physical component used in this experiment was a servomechanism which, it was known, could be approximately characterized by a transformation having the frequency response function $\frac{4}{-\omega^2+3{\cdot}4i\omega+4}$. This component was interconnected with the computing units as shown in fig. 19.

The system input $I_S(t)$ was generated as in experiment 1 and applied to both the physical component and the model as set up on the computer. The model chosen had a frequency response function of $\frac{1{\cdot}96}{-\omega^2+2{\cdot}52i\omega+1{\cdot}96}$ and a corresponding weighting function of $3{\cdot}16\ e^{-1{\cdot}26t}\sin 0{\cdot}62t$; this choice was based mainly on a desire to ensure that some significant adjustment would be necessary. The difference, $O_C(t)$, between the physical output and the model output was then multiplied by $I_S(t-\tau)$, which was also generated as in experiment 1. The result of this multiplication was integrated to give $\frac{1}{10}\int I_S(t-\tau)O_C(t)dt$. A continuous record of this quantity was displayed on a digital voltmeter which was read at three seconds and 103 seconds after the commencement of each run. Pen recordings of the quantities $V_0(t)$, $I_S(t)$, $O_S(t)$, $O_M(t)$, $O_C(t)$ and $\frac{1}{10}\int I_S(t-\tau)O_C(t)dt$ were also taken throughout the experiment.

8.2.3 Results

The main data obtained from the experiment are shown in Table 7 where, for each of 16 values of τ, the output

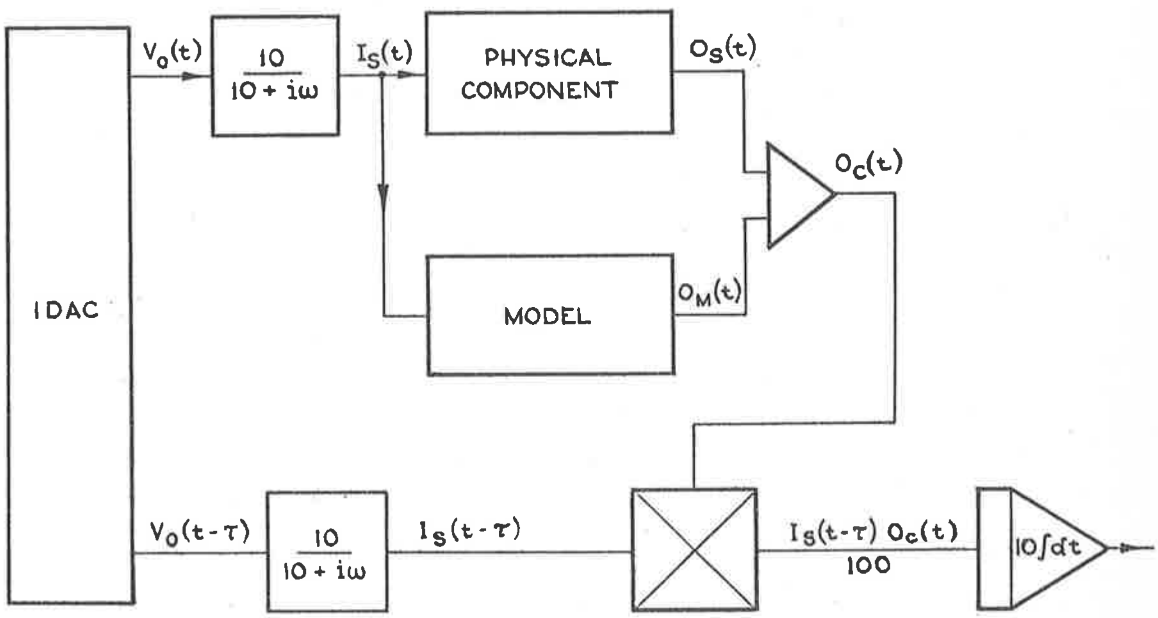

FIGURE 19. BLOCK DIAGRAM OF INTERCONNECTIONS FOR EXPERIMENT 2

$\frac{1}{10}\int_0^{T_n} I_S(t-\tau)O_C(t)dt$ is shown for T_n = 3 seconds and T_n = 103 seconds.

Table 7

τ in secs	$\frac{1}{10}\int_0^{T_n} I_S(t-\tau)O_C(t)dt$ at values of T_n = 3 seconds	103 seconds
$-\frac{1}{4}$	0·0	-0·9
$-\frac{1}{10}$	0·0	-0·4
0	0·2	1·6
$\frac{1}{10}$	0·0	5·4
$\frac{1}{4}$	0·2	13·9
$\frac{1}{2}$	0·1	15·3
$\frac{3}{4}$	0·3	13·3
1	0·1	6·0
$1\frac{1}{4}$	0·0	2·2
$1\frac{1}{2}$	0·1	-1·3
$1\frac{3}{4}$	-0·1	-4·3
2	0·0	-5·3
$2\frac{1}{4}$	0·0	-6·8
$2\frac{1}{2}$	0·0	-6·7
$2\frac{3}{4}$	0·0	-5·7
3	0·0	-4·5

In order to adjust the model, the figures in this table at 3 seconds must be deducted from those at 103 seconds and the result divided by the scale factor 48·8 as for experiment 1. The result of so doing is shown in Table 8 together with values of the weighting function after adjustment i.e. $3{\cdot}16\, e^{-1{\cdot}26\tau} \sin 0{\cdot}62\tau + R'_{IS,OC}(\tau) + R'_{IS,OC}(-\tau)$.

Table 8

τ in secs	$R'_{IS,OC}(\tau)+R'_{IS,OC}(-\tau)$	Model weighting function after adjustment
0	0·06	0·06
$\frac{1}{10}$	0·10	0·27
$\frac{1}{4}$	0·26	0·60
$\frac{1}{2}$	0·31	0·81
$\frac{3}{4}$	0·27	0·80
1	0·12	0·62
$1\frac{1}{4}$	0·04	0·48
$1\frac{1}{2}$	-0·03	0·34
$1\frac{3}{4}$	-0·09	0·21
2	-0·11	0·13
$2\frac{1}{4}$	-0·14	+0·03
$2\frac{1}{2}$	-0·14	-0·01
$2\frac{3}{4}$	-0·12	-0·02
3	-0·09	-0·02

These results are also shown on the graph fig. 20 together with curves of the weighting function of the original model and an estimate of the system weighting function obtained independently. This independent estimate was supplied by

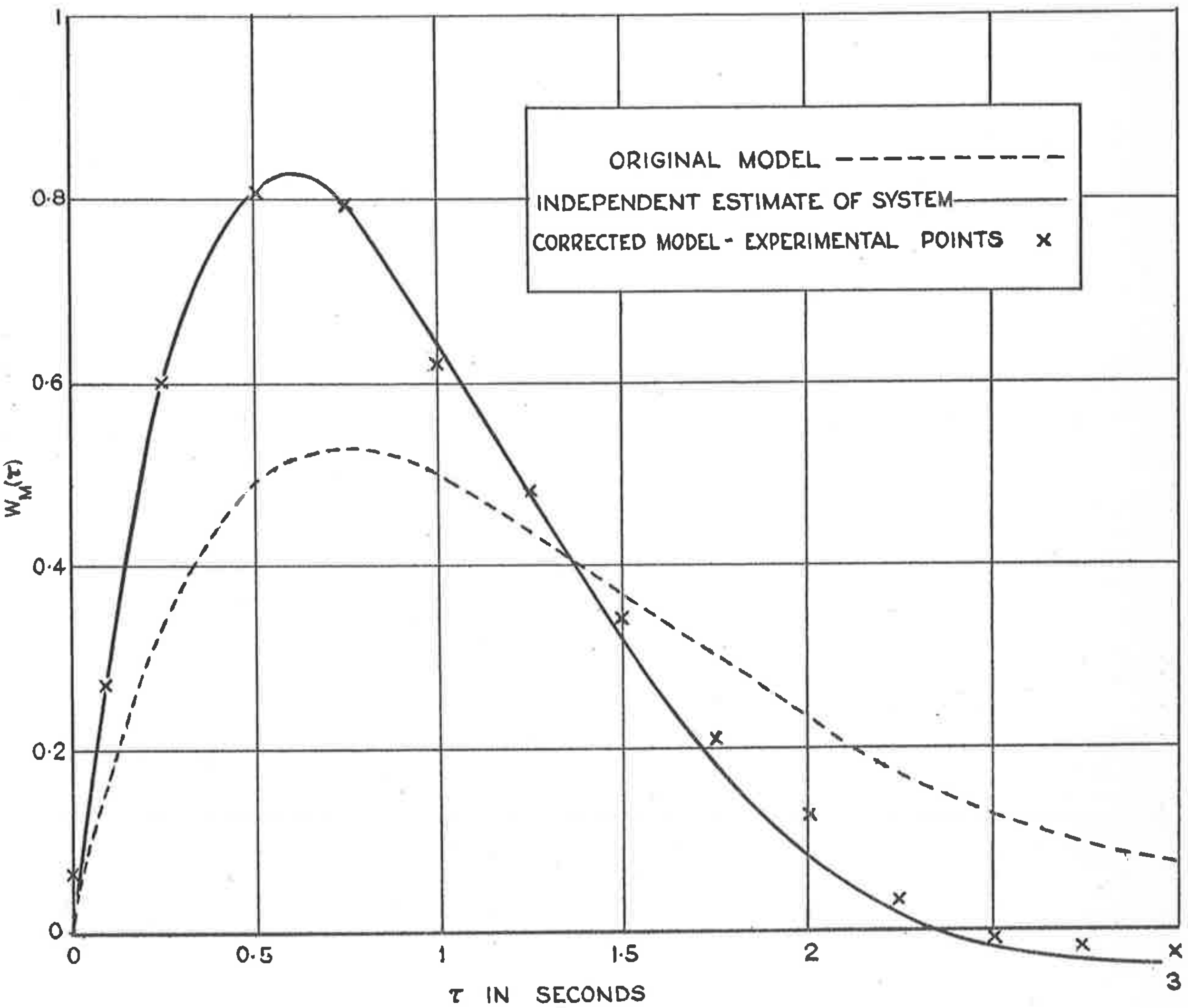

FIGURE 20. ILLUSTRATING RESULT OF EXPERIMENT 2

staff of the Weapons Research Establishment; it was obtained using a transfer function analyser.

8.2.4 Discussion

It is clear from fig. 20 that cross correlation has been extremely effective in providing data required to adjust the model. The inadequacy of the model before adjustment may be estimated in the following way. For this purpose the system is assumed to have the frequency response function $\frac{4}{-\omega^2+3\cdot 4i\omega+4}$ and the input to have spectrum $\sigma_a^2 T \frac{100}{100+\omega^2}$.

$$\sigma_{OS}^2 = \frac{1}{2\pi}\sigma_a^2 T \int_{-\infty}^{\infty} \frac{100}{100+\omega^2} \frac{16}{|-\omega^2+3\cdot 4i\omega+4|^2}\, d\omega$$

$$= \frac{1}{2\pi}\sigma_a^2 T \int_{-\infty}^{\infty} \frac{1600}{|(10+i\omega)(-\omega^2+3\cdot 4i\omega+4)|^2} d\omega. \qquad (8.8)$$

Integrals such as this have been computed by Solodovnikov* [26]. His table for

$$I_n = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{G_n(\omega)}{H_n(\omega)H_n(-\omega)}\, d\omega, \qquad (n = 3),$$

where $G_n(\omega) = b_0\omega^4 + b_1\omega^2 + b_2$,

$H_n(\omega) = a_0\omega^3 + a_1\omega^2 + a_2\omega + a_3$,

and the equation $H_n(\omega) = 0$ has all its roots in the upper half plane, gives

$$I_3 = \frac{-a_2b_0+a_0b_1 - \frac{a_0a_1b_2}{a_3}}{2a_0(a_0a_3-a_1a_2)}.$$

---

* Appendix IV of reference [26], which contains these tables, states they are tables of

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{G_n(i\omega)}{H_n(i\omega)H_n(-i\omega)}\, d\omega$$

and defines $G_n(\omega)$ differently. This statement is not correct. The correct statement may be found on P.165 of the reference.

Substituting the values $b_0 = b_1 = 0$, $b_2 = 1600i$, $a_0 = -i$, $a_1 = -13{\cdot}4$, $a_2 = 38i$, and $a_3 = 40$ in this expression for I_3 yields

$$\sigma^2_{OS} = \sigma^2_a T\ i\ \frac{-13{\cdot}4i \times 40}{2(-i)(-40i+509i)} = \sigma^2_a T\ \frac{268}{469}$$

$$= 0{\cdot}57\ \sigma^2_a T = 2{\cdot}8, \quad \text{since} \quad \sigma^2_a = 244 \text{ and } T = \frac{1}{50}. \quad (8.9)$$

Similarly

$$\sigma^2_{OC} = \frac{1}{2\pi}\,\sigma^2_a T \int_{-\infty}^{\infty} \frac{100}{(100+\omega^2)} \left| \frac{4}{-\omega^2+3{\cdot}4i\omega+4} - \frac{1{\cdot}96}{-\omega^2+2{\cdot}52i\omega+1{\cdot}96} \right|^2 d\omega,$$

$$= 100\sigma^2_a T\ \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4{\cdot}16\omega^4 + 11{\cdot}7\omega^2}{|i\omega^5+15{\cdot}92\omega^4-73{\cdot}73i\omega^3-162\omega^2+175{\cdot}2i\omega+78{\cdot}4|^2}\,d\omega.$$

Solodovnikov's formula for this case, where $n = 5$ and $b_0 = b_1 = b_4 = 0$, is

$$I_5 = \frac{M_5}{2a_0\Delta_5},$$

where M_5 reduces to

$$a_0 b_2(a_0 a_5 - a_1 a_4) + a_0 b_3(-a_0 a_3 + a_1 a_2)$$

and Δ_5 is equal to

$$a_0^2 a_5^2 - 2a_0 a_1 a_4 a_5 - a_0 a_2 a_3 a_5 + a_0 a_3^2 a_4 + a_1^2 a_4^2 + a_1 a_2^2 a_5 - a_1 a_2 a_3 a_4 .$$

On substituting the values of the a_i and b_i in the expression

$$\sigma^2_{OC} = 100\ \sigma^2_a T\ i\ I_5$$

it is found that

$$\sigma^2_{OC} = 0{\cdot}074\ \sigma^2_a T\ . \quad (8.10)$$

Hence the inadequacy of the model before adjustment was approximately equal to

$$\left(\frac{0{\cdot}074}{0{\cdot}57}\right)^{\frac{1}{2}} = 0{\cdot}36\ . \quad (8.11)$$

To determine the adequacy of the model after adjustment, the new model must be determined completely by fitting some curve to the experimental points shown in fig. 20 and table 8. The following procedure was adopted in the present case. The form of the model weighting function, i.e. $ke^{-at}\sin bt$, was not changed, but new values for the parameters k, a and b were determined in the following way. From fig. 20 the first zero of the weighting function occurs at $t \simeq 2{\cdot}45$ seconds so that $2{\cdot}45b = \pi$ i.e. $b = 1{\cdot}28$. The maximum value of the corrected model will clearly occur at about $t = 0{\cdot}625$ seconds at which value

$$-ak\,e^{-at}\sin bt + k\,b\,e^{-at}\cos bt = 0;$$

$$\text{i.e.}\quad a = b\cot bt$$

$$= 1{\cdot}25.$$

The value of k was computed as 2·5 by taking a least squares fit of the data to a curve of the form

$$k\,e^{-1{\cdot}25t}\sin 1{\cdot}28t.$$

With this new model set up on the computer an estimate of σ^2_{OC} was computed as 0·013. The inadequacy of the model after correction in this way was therefore approximately

$$\left(\frac{0{\cdot}013}{2{\cdot}8}\right)^{\frac{1}{2}} = 0{\cdot}07. \qquad (8.12)$$

The equation $p = \frac{1}{2}\ln 2q$ (5.31) and the expression $\frac{\sigma^2_{OC}}{\sigma^2_{OS}}\left(\frac{p+1}{q} + e^{-2p}\right)$ (5.30) were used in planning the experiment; for this purpose the error in the model was assumed to die out with a time constant $\frac{1}{a} = 1$ second. Equation (5.31) then gives a value of $T_m = 2{\cdot}65$ seconds and so the cross correlation was not carried beyond $\tau = 3$ seconds. The expression (5.30) applied to the present case predicts an inadequacy for the corrected model of

$$\left[\frac{0{\cdot}074}{0{\cdot}57}\left(\frac{3{\cdot}65}{100} + {\cdot}005\right)\right]^{\frac{1}{2}}$$

$$= \left(\frac{0{\cdot}31}{57}\right)^{\frac{1}{2}}$$

$$= 0{\cdot}08. \qquad (8.13)$$

The remarkable agreement between this figure and the result actually obtained is, to some extent, fortuitous since the assumptions on which (5.30) and (5.31) are based, are not satisfied in the present case. However the equation (5.30) and the expression (5.31) appear to be a useful guide in planning work of this nature.

CHAPTER 9.

DISCUSSION AND SOME EXTENSIONS OF THE WORK

The work described in chapters 4 to 8 of this thesis deals with some particular cases of the general problem detailed in the statement 2.4. For these cases the problem was restricted in the following way.

(a) The inputs to the system were realisations of a type X_1 process as defined in section 3.1. When more than one input was considered, as in chapter 7, cross-correlation functions were assumed to be zero.

(b) The system and model components were both required to be stationary and to have weighting functions of the form (3.1), viz.,

$$W(t) = \sum_{j=1}^{n} P_j(t)\, e^{s_j t}, \quad \mathrm{Re}\,(s_j) < 0, \quad t \geq 0,$$
$$= 0, \qquad\qquad t < 0.$$

In some cases the generalised functions $\delta(t)$ and its derivatives were also allowed.

(c) The errors in the recordings of the inputs and outputs were assumed to be either constant biases or realisations of type X_1 processes. It was also assumed that autocorrelation functions and other statistical information concerning the input and the errors in the recordings, are known.

Under these restrictions, methods of adjusting the model have been considered and, in particular, it has been demonstrated that the cross-correlation technique described in section 3.1 is extremely effective for this purpose. This technique has been studied, in considerable detail, for the practical case of finite length samples; particularly the problem of determining the effect of this finite length on the adequacy of the corrected model. The relations developed in chapter 5, between the length of record, T_n, the extent, T_m, of the adjustment to the model and the inadequacy of the corrected model, should be very useful in planning experiments to provide data for adjusting a model. It should be noted that, in this application, an accurate knowledge of the spectrum of the input or other processes involved is not required. As shown in the experiments described in chapter 8, fairly crude information concerning these quantities can lead to useful results.

If several samples are available for model checking, then they may be used, as described in section 5.3, to provide one correction to the model. In that case there is little to choose, as far as adequacy of the corrected model is concerned, between several samples of effective length T_n' and one sample of the same effective length. However, as discussed in section 5.4, it is sometimes more efficient to use an iterative method, even to the extent that a single long length of record is divided into several shorter lengths.

Recording errors were discussed in chapter 6. It was shown that the cross-correlation technique, with some small modifications, could also be very effective in the presence of errors. However a very large number of terms now appeared in the estimate of the adequacy of the corrected model. Under certain assumptions, which it is thought would be satisfied by many good recording systems, most of these terms are zero; these assumptions are summarised in table 3. The non zero terms admit of some simplification and their importance will vary with the particular application. The expressions developed in chapter 6 should help in determining the importance of these terms in comparison with those present in the absence of "noise".

Uncorrelated multiple inputs were considered in chapter 7, where it was shown that this problem could be reduced to one already considered in previous chapters. Each model transformation may be adjusted separately; all other inputs, except the one relevant to the transformation being considered, may be treated as recording errors.

In this chapter some additional aspects of the general problem, stated in 2.4, will be discussed in an incomplete and less detailed manner. This discussion could be the starting point for future work.

9.1 Correlated stationary multiple inputs

If condition (e) of chapter 7 is relaxed so that the inputs to the system may be cross-correlated, then it is no longer necessary that the problem (2.4) has a unique solution.

For example, let the system have two inputs both of which are realisations of the type X_1 processes, $I_{S1}(t,w)$ and $I_{S2}(t,w)$, such that for every w_l

$$I_{S1}(t,w_l) = T_{12}[I_{S2}(t,w_l)]; \qquad (9.1)$$

where T_{12} is a linear causal transformation of type L whose weighting function contains no generalised functions. Then any of the systems corresponding to real k in fig. 21 have the same inputs and output. Any of these systems which characterizes a particular physical device must, of course, correspond to a particular value of k; this value can not however be determined from the data available.

If, for example:

(a) $I_{S2}(t,w)$ was an S.G.M. having correlation function $\frac{\alpha}{2} e^{-\alpha|t|}$, α large and positive;

(b) there was no original model;

(c) the first input cross-correlated with $O_S(t,w_l)$ was $I_{S2}(t,w_l)$;

then, neglecting the effects of finite α and finite length of record, the cross-correlation technique would yield the model corresponding to $k = 0$. Similarly, if there had been an original model such that

$$T_{M1}[I_{S1}(t,w_l)] + T_{M2}[I_{S2}(t,w_l)] = O_M(t,w_l),$$

then under these conditions the model would, after the first adjustment of T_{M2}, have no error in its output. If the model is required solely to predict the output of the system given inputs satisfying (1) then any model found by the above procedure will be satisfactory; its adequacy (2.1) would

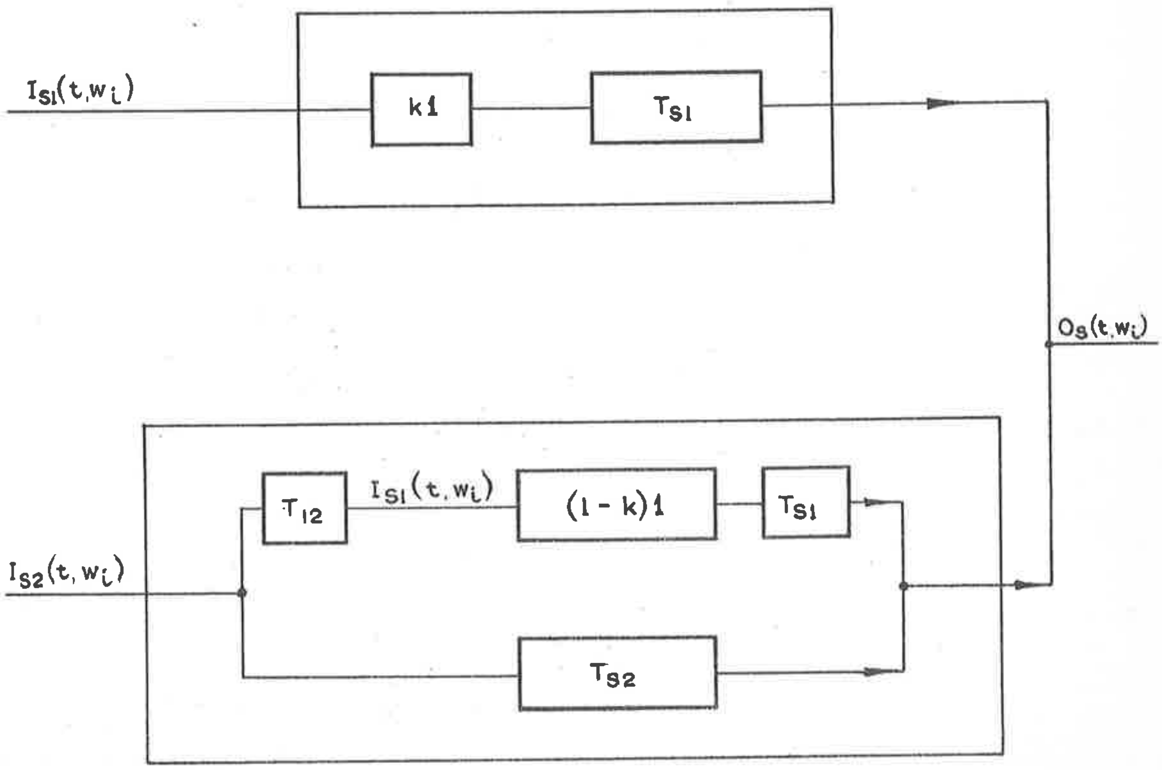

FIGURE 21. ILLUSTRATING THE CASE WHERE THE MODEL CORRECTION IS NOT UNIQUE

have the maximum value, 1. Nevertheless care would be needed in using such a model since its adequacy may deteriorate markedly if the inputs were changed so that (1) was no longer satisfied.

If the records are error free and of unlimited length then a procedure such as that in section 7.1 yields, for each l, the equation

$$\sum_{k=1}^{n}(T_{Sk} - T_{Mk})[\rho_{ISl,ISk}(\tau)] = \rho_{ISl,OC}(\tau),$$

or
$$\sum_{k=1}^{n}\int_{0}^{\infty}\rho_{ISl,ISk}(\tau-x)\, W_{Ak}(x)dx = \rho_{ISl,OC}(\tau). \qquad (9.2)$$

Taking the Fourier transform of each side of (9.2) gives

$$\sum_{k=1}^{n} S_{l,k}(\omega)\ \phi_k(\omega) = S_{l,OC}(\omega); \qquad (9.3)$$

where the notation has been simplified in an obvious manner.

This set of equations, one for each l, which is known to have one solution of the required form, has no other solutions, providing the Hermitian matrix, whose elements are the spectral density functions $S_{l,k}(\omega)$, is positive definite [17,P.106]. However the solution of (2) by taking Fourier transforms, solving the set of equations (3) and transforming back is hardly a practical proposition. The problem is only slightly reduced in complexity if the inputs are modified to stationary Gauss Markov processes with the parameter α so large that

$$\int_{0}^{\infty}\rho_{ISl,ISk}(\tau-x)W_k(x)dx = W_l(x) \quad \text{for} \quad l = k,$$

since no such simplification is possible for $l \neq k$.

On the other hand the correlation technique may still be useful in the case of correlated inputs; since, as shown below, the adequacy of the model is not reduced if we add to the model weighting function, $W_{M1}(\tau)$, the solution of the Wiener-Hopf integral equation,

$$\int_0^\infty \rho_{IS1}(\tau-x)W(x)dx = \rho_{IS1,OC}(\tau), \qquad \tau \geqslant 0. \qquad (9.4)$$

Let $W^*(x)$ be the solution of (4) and $O_M^*(t,w_l)$ be the output of the model when $W_{M1}(x)$ has been replaced by

$$W_{M1}(x) + W^*(x).$$

Then

$$O_C^*(t,w_l) = O_S(t,w_l) - O_M^*(t,w_l)$$

$$= O_S(t,w_l) - O_M(t,w_l) - \int_0^\infty I_{S1}(t-x,w_l)W^*(x)dx;$$

i.e. $$O_C^*(t,w_l) = O_C(t,w_l) - \int_0^\infty I_{S1}(t-x,w_l)W^*(x)dx;$$

and therefore

$$\sigma^2_{OC^*} = \sigma^2_{OC} - 2\int_0^\infty \rho_{IS1,OC}(x)W^*(x)dx + \int_0^\infty\int_0^\infty \rho_{IS1}(\tau-x)W^*(x)W^*(\tau)dxd\tau.$$

But, from equation (4) after multiplying by $W^*(\tau)$ and integrating with respect to τ,

$$\int_0^\infty\int_0^\infty \rho_{IS1}(\tau-x)W^*(x)W^*(\tau)dxd\tau = \int_0^\infty \rho_{IS1,OC}(\tau)W^*(\tau)d\tau,$$

so that

$$\sigma^2_{OC^*} = \sigma^2_{OC} - \int_0^\infty\int_0^\infty \rho_{IS1}(\tau-x)W^*(x)W^*(\tau)dxd\tau,$$

$$= \sigma^2_{OC} - \sigma^2_{IS1^*};$$

where $$I_{S1^*}(t,w_l) = \int_0^\infty I_{S1}(t-x,w_l)W^*(x)dx.$$

Therefore

$$\sigma^2_{OC^*} \leqslant \sigma^2_{OC} \quad \text{and hence}$$

$$1 - \left(\frac{\sigma^2_{OC}{}^*}{\sigma^2_{OS}}\right)^{\frac{1}{2}} \geq 1 - \left(\frac{\sigma^2_{OC}}{\sigma^2_{OS}}\right)^{\frac{1}{2}}.$$

If $\rho_{IS1,OC}(\tau) = 0, \quad \tau \geq 0,$ then there will be no improvement in the adequacy of the model, since

$$\sigma^2_{IS1}{}^* = \int_0^\infty \rho_{IS1,OC}(\tau) W^*(\tau) d\tau.$$

Conversely if there is no improvement in the adequacy of the model then $\rho_{IS1,OC}(\tau) = 0, \quad \tau \geq 0.$ This may be shown by considering the equation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{IS1}(\omega)|\phi^*(\omega)|^2 d\omega = \sigma^2_{IS1}{}^*,$$

where $\phi^*(\omega)$ is the Fourier transform of $W^*(\tau)$. If there is no improvement in adequacy then $\sigma^2_{IS1}{}^* = 0$. It follows, since $S_{IS1}(\omega) > 0$ for all ω, $I_{S1}(\omega)$ being a type X_1 process, that $\phi^*(\omega) \equiv 0$. Therefore $W^*(\tau) \equiv 0$ and hence,(4), $\rho_{IS1,OC}(\tau) = 0, \qquad \tau \geq 0.$

Further, since

$$O_C(t, w_\iota) = \sum_{1=1}^{n} \int_0^\infty I_{S1}(t-x, w_\iota) W_{A1}(x) dx,$$

then multiplying both sides by $O_C(t, w_\iota)$ and taking expectations yields

$$\sigma^2_{OC} = \sum_{1=1}^{n} \int_0^\infty \rho_{IS1,OC}(x) W_{A1}(x) dx;$$

so that, if $\sigma^2_{OC} \neq 0$, then all of the $\rho_{IS1,OC}(\tau)$ can not be zero for $\tau \geq 0$.

It therefore seems, that with whitened inputs, some improvement in the adequacy of the model will be achieved by cross correlating each input in turn with the current $O_C(t)$, each model weighting function being adjusted before proceeding to the next. However, even though the adequacy of the model be improved considerably by this process, the individual model weighting functions may, at the conclusion of the process, bear less resemblance to the corresponding system weighting functions than they did at the beginning.

A possible method of achieving the effect of whitened inputs is shown in fig. 22. Only the input $I_{S1}(t,w_l)$ is considered and it is assumed the corresponding type X_1 process is such that $I_{S1}(t,w_l) = L_1 M_{\alpha 1}(t,w_l)$ and $M_{\alpha 1}(t,w_l) = L[I_{S1}(t,w_l)]$. If $M_{\alpha 1}(t,w_l)$ is cross-correlated with $O_C(t,w_l) = O_S(t,w_l) - O_M(t,w_l)$ then, for large α, an expression will be obtained which approximates the solution, $W^*(x)$, of the equation

$$\int_0^{\infty} \rho_{M\alpha 1}(\tau - x) W(x) dx = \rho_{M\alpha 1, OC}(\tau), \qquad \tau \geq 0.$$

According to the previous discussion, if the product transformation $L_1 \cdot T_{M1}$, in fig. 22, is replaced by $L_1 \cdot T_{M1} + T^*_{M1}$, where T^*_{M1} has weighting function $W^*(x)$, then the mean square error in the output of the model will, in general, be reduced. This adjustment is equivalent to replacing T_{M1} by $T_{M1} + L \cdot T^*_{M1}$ and so improving the adequacy of the model.

9.2 Non-linear and other systems not of type L

In all cases so far considered in this thesis the system components have been assumed to be of type L as defined in chapter 3. Although this is a satisfactory

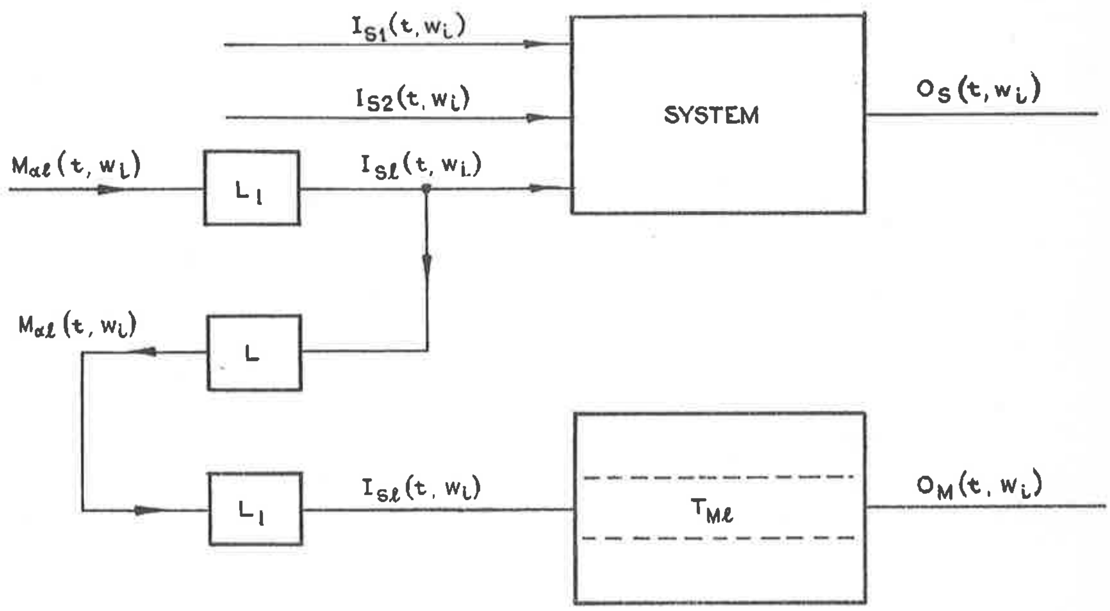

FIGURE 22. ILLUSTRATING A METHOD OF "WHITENING" AN INPUT IN THE CASE OF MULTIPLE INPUTS

assumption in a large number of practical situations, there are also many in which it is unsatisfactory. If no restriction is placed on the type of transformation which may be incorporated in the model in order to obtain better agreement between the sample system output and model output, then, in most cases, good agreement will ultimately be obtained. However the transformations of the adjusted model may then be unjustified physically and quite unlike those of the system. It is therefore considered that no adjustment to a model should involve the introduction of non linear, or other transformations not of type L, which were not present in the original model. If as a result of model checking it is suspected that some type of transformation exists in the system other than those represented in the original model, then a new formulation is required.

Accordingly this preliminary discussion will be restricted so that:

(i) the model checking and adjustment consists of adjusting the value of a few parameters in model components which are not of type L, combined with adjusting type L components;

(ii) if the inputs to the system and model are specified for all t then there is a unique system output and a unique model output;

(iii) both the system and model are stable, i.e., if $I_1(t) = I_2(t), \qquad t \geq t_0,$ and T is the system or model transformation having $I_1(t)$ and $I_2(t)$ as possible inputs, then

$$\lim_{t\to\infty} T[I_1(t)] - T(I_2(t)] = 0:$$

further, the difference between these two outputs is negligible for $t > t_0 + t_1$, where $t_1 << T_n$, the length of record available for model checking.

A simple example, of the type of system being considered, is the quite common situation of the output of an L type transformation being limited in amplitude, so that it does not exceed some constant value, Λ, in absolute value, i.e.

$$|O_S(t,w_\iota)| \leq \Lambda \quad \text{for all } t \text{ and } i.$$

There is now a considerable literature dealing with time variant [27], non-linear [43] and other transformations not of type L. Different methods are available for tackling different types of non linearity or other departure from type L. However the process of model checking, restricted as above, is often simpler than general problems involving non-linearities. The remainder of this chapter is devoted to two or three examples.

9.2.1 L type model of system not of type L

Very often it is known that, although the system is not of type L, nevertheless, it should be possible to represent it adequately by an L type model. In some such cases the system output may be assumed to be of the form

$$L[I_S(t,w_\iota)] + J(t,w_\iota),$$

where both $I_S(t,w_\iota)$ and $J(t,w_\iota)$ are processes of type X_1 which are, in general, cross correlated. If the input is an S.G.M. with correlation function $\frac{\alpha}{2} e^{-\alpha|t|}$ and this input is cross-correlated with

$$L[I_S(t,w_\iota)] + J(t,w_\iota) - O_M(t,w_\iota),$$

then, for α large and a long length of record, a good approximation to

$$W_S(\tau) - W_M(\tau) + \rho_{IJ}(\tau), \qquad \tau \geqslant 0, \qquad (9.5)$$

will be obtained. Here $W_S(\tau)$ is the weighting function corresponding to the system transformation L.

If the model transformation is adjusted by adding a transformation having (5) as weighting function, the error in the output of the adjusted model becomes

$$O_C^*(t,w_\iota) = J(t,w_\iota) - \int_0^\infty I_S(t-x,w_\iota)\rho_{I,J}(x)dx.$$

Multiplying each side of this expression by $I_S(t-\tau,w_\iota)$ and taking expectations shows that

$$\lim_{\alpha\to\infty} \rho_{IS,OC^*}(\tau) = 0, \qquad \tau \geqslant 0.$$

Since, in such circumstances, if $W(x)$ is any weighting function corresponding to an L type transformation,

$$E[(O_C^*(t,w_\iota) - \int_0^\infty I_S(t-x,w_\iota)W(x)dx)]^2$$

$$= \sigma^2_{OC^*} - 2\int_0^\infty \rho_{IS,OC^*}(x)W(x)dx + \int_0^\infty\int_0^\infty \rho_{IS}(\tau-x)W(\tau)W(x)d\tau dx$$

$$= \sigma^2_{OC^*} + \sigma^2_{IS^*} ,$$

where $$I_{S^*}(t-x,w_\iota) = \int_0^\infty I_S(t-x,w_\iota)W(x)dx,$$

then there is no point in adjusting the model further.

Therefore, in this case a process very similar to that described in previous chapters will lead to the L type model of highest adequacy. This adequacy will however be less than 1 if the system is not of type L.

Two further comments are necessary.

(i) In general it will be wrong to whiten the input to the system. If the system is non linear it may be impossible to predict its output to the original input knowing only its output to a whitened input. However as mentioned in section 4.1 it is usually undesirable, for other reasons, to interfere with the operation of the system in this way. Of course there is no objection to whitening the model input.

(ii) Theorem 4 part (i) has been used in deriving the expression (5). If part (ii) of this theorem were used the corresponding expression would be

$$W_S(\tau) - W_M(\tau) + \rho_{IJ}(\tau) + \rho_{IJ}(-\tau), \quad \tau \geq 0. \quad (9.6)$$

The discussion following (5) would not then be valid since $\rho_{IJ}(-\tau)$ is not necessarily zero. In many cases of this type, therefore, only the result of the correlation for non negative τ should be used in adjusting the model.

9.2.2 L type system with limited output

It is often necessary that the amplitude of the output of a physical device be limited. If the system output is of the form

$$\begin{aligned} O_S(t,w_\iota) &= L[I_S(t,w_\iota)], & |L[I_S(t,w_\iota)]| &< \Lambda, \\ O_S(t,w_\iota) &= \Lambda \quad , & L[I_S(t,w_\iota)] &\geq \Lambda, \\ O_S(t,w_\iota) &= -\Lambda \quad , & L[I_S(t,w_\iota)] &\leq -\Lambda, \end{aligned}$$

where Λ is a positive constant, the procedure given in previous chapters may be used to determine an adequate model of L providing that portions of the record where $|L[I_S(t,w_\iota)]| \geqslant \Lambda$ are not used when correlating $I_S(t,w_\iota)$ with $O_C(t,w_\iota)$. Normally the period of time for which the output is limited is only a small portion of the total record. The correlation may be carried out taking $O_C(t,w_\iota) = 0$ for the period t_1 in which the output is limited; the total effective length of record for this purpose is then reduced from T_n' to $T_n' - t_1$. Since Λ is not known accurately, in general, some conservative value less than Λ should be used in deciding the intervals during which $O_C(t,w_\iota)$ is to be taken as zero.

Having determined the linear part of the model by the above method, the correct value of Λ may be estimated by varying this parameter in the model. Since an approximate value of Λ for the system is known there will usually be little difficulty in determining a value of Λ which provides an adequate model.

9.2.3 Stationary systems defined by a few parameters

Suppose it is known that $O_S(t,w_\iota)$ may be expressed as a function of the independent variables $I_S(t,w_\iota)$, a finite number of the derivatives of $I_S(t,w_\iota)$, together with a few parameters $\lambda_j, (j = 1-n)$, whose values are known approximately,

i.e. $O_S(t,w_\iota) = F[I_S(t,w_\iota), \ldots, \lambda_1, \ldots \lambda_n]$.

Let the model be of the form

$$O_M(t,w_l) = F[I_S(t,w_l),\ldots, \lambda_{10},\ldots \lambda_{n0}],$$

where $\lambda_{10},\ldots\lambda_{n0}$ are the approximate values of the λ_j, used in the model.

Then $O_C(t,w_l) = O_S(t,w_l) - O_M(t,w_l)$

and, assuming F is a differentiable function, we may write $O_C(t,w_l) = \sum_{j=1}^{n} \left(\frac{\partial F}{\partial \lambda_j}\right) \Delta\lambda_j$ plus terms of higher order in the $\Delta\lambda_j$. The expressions $\frac{\partial F}{\partial \lambda_j}$ are to be evaluated at the values, λ_{j0}, of the λ_j; they will be of the form, $f_j(t,w_l)$, and they may be found knowing the input $I_S(t,w_l)$. Hence if the higher order terms in

$$\Delta \lambda_j = \lambda_j - \lambda_{j0}$$

are neglected

$$O_C(t,w_l) \simeq \sum_{j=1}^{n} f_j(t,w_l)\, \Delta \lambda_j .$$

If the effective length of the records is T_n' then

$$\frac{1}{T_n'} \int_0^{T_n'} [O_C(t,w_l) - \sum_{j=1}^{n} f_j(t,w_l)\Delta\lambda_j]^2\, dt$$

may be minimised with respect to the $\Delta\lambda_j$ to give n equations to be solved for the $\Delta\lambda_j$.

Assuming these equations have been solved to give useful values for the $\Delta\lambda_j$ and the model adjusted accordingly, a further improvement in the adequacy of the model may, in general, be obtained by introducing a type L transformation in parallel with the adjusted model. The advisability and necessity for doing so will depend on the particular case.

A simple example of this procedure is the case,

$$O_S(t,w_l) = \lambda I_S^2(t,w_l),$$

$$O_M(t,w_l) = \lambda_0 I_S^2(t,w_l),$$

$$f(t,w_l) = I_S^2(t,w_l),$$

so that the expression to be minimised is

$$\frac{1}{T_n'}\int_0^{T_n'}[O_C(t,w_l) - \Delta\lambda I_S^2(t,w_l)]^2 dt;$$

this clearly has a minimum value for $\Delta\lambda = \lambda - \lambda_0$. In practice this method would yield $\Delta\lambda$ as the ratio

$$\frac{\int_0^{T_n'} I_S^2(t,w_l)O_C(t,w_l)dt}{\int_0^{T_n'} I_S^4(t,w_l)dt}$$

the two integrals being computed from the records available for model checking.

CHAPTER 10

CONCLUSIONS

A mathematical approach to the problem of checking and adjusting a mathematical model of a physical system has been described. This formulation of the problem has provided answers to some outstanding questions which could not be answered using the subjective approach described in earlier papers [2 - 5].

The important case of a stationary linear system whose inputs are realisations of a particular type of stationary random process has been studied in considerable detail. It has been shown, both theoretically and experimentally, that in this case, the process of checking and adjusting the model may be represented as one of estimating the cross-correlation between the input, "whitened" if necessary to give a suitable spectrum, and the difference between the system output and the model output. This representation leads to a relation between the length of record available for model checking and the improvement to be expected in the adequacy of the model as defined in chapter 2. This improvement increases with the total effective length of record available according to the expression (5.27) given in chapter 5. However if the length of record available for model checking is considerable, or if there are several samples available, an iterative method, such as that described in section 5.4, may be more efficient than a single correction based on the total effective length of record available. As shown in chapter 8 the relation (5.27) is useful in planning experiments to check a mathematical model; it should be very useful in cases where, because of cost or for some other reason, there is an advantage in not producing more data for model checking than is necessary.

The effect of recording errors on the adequacy of the corrected model is studied in chapter 6. In order to compensate for such errors, statistical information concerning the variance, correlation and cross-correlation of these errors is required. For most types of recording error, the adequacy of the corrected model increases with the length of record available for model correction. However the effect of a bias error present in the recording of the system input is, in part, independent of the length of the records available.

The cross-correlation technique may also be applied to the case of multiple inputs. If these inputs are not themselves correlated then the transformations between each input and the output may be adjusted individually. That portion of the output which does not arise from the input under consideration may, as shown in chapter 7, be treated as an error in the output. In this way a relation between the improvement in adequacy of the model and length of record available was found for the case of uncorrelated multiple inputs.

Other cases have been discussed in less detail in chapter 9. The cross-correlation technique may be applied, sometimes with modifications, to some of these cases. However there is still a large number of cases, e.g. the case of non stationary inputs, which has not been studied. These could be the subject of future work.

APPENDIX

ROUTINE CALCULATIONS USED MAINLY IN CHAPTER 5

A number of results, e.g. (5.16) to (5.26), have been quoted, without proof, in the main text. The proofs have been omitted since in many cases they are long and tedious. In this appendix a few of these results are proved, the others may be proved in a similar manner.

Theorem A1 Let a linear system have weighting function

$$W_C(t) = \sum_{j=1}^{n} P_j(t)e^{s_j t}, \qquad \text{Re } (s_j) < 0,\ t \geqslant 0,$$
$$= 0, \qquad t < 0;$$

where $P_j(t)$ is, for each j, a polynomial in t.
Let the input to this system, $M_\alpha(t,w_\iota)$, be a realisation of an S.G.M. having correlation function $\frac{\alpha}{2}e^{-\alpha|t|}$. Let $O_C(t,w_\iota)$ be the corresponding output from the system.

Then for $\alpha > |s_j|$, all j,

$$\rho_{M\alpha,OC}(\tau) = f_1(\alpha)e^{\alpha\tau}, \qquad \tau \leqslant 0,$$
$$= f_2(\alpha)e^{-\alpha\tau} + \sum_{j=1}^{n} Q_j(\tau,\alpha)e^{s_j\tau}, \quad \tau \geqslant 0;$$

where:

(i) $\lim_{\alpha\to\infty} f_1(\alpha)$ and $\lim_{\alpha\to\infty} f_2(\alpha)$ both exist;

(ii) the $Q_j(\tau,\alpha)$ are polynomials in τ whose coefficients are functions of α and have a finite limit as α tends to infinity; these coefficients are therefore bounded for $\alpha > \alpha_0 > |s_j|$, all j.

PROOF $\int_0^\infty M_\alpha(t-x,w_t)W_C(x)dx = O_C(t,w_t)$,

whence, multiplying each side by $M_\alpha(t-\tau,w_t)$ and taking expectations,

$$\int_0^\infty \rho_{M\alpha}(\tau-x)W_C(x)dx = \rho_{M\alpha,OC}(\tau). \qquad (A_1).$$

Now consider $\int_0^\infty \frac{\alpha}{2}\, e^{-\alpha|\tau-x|}x^m e^{s_j x}dx$

$$= \int_0^\tau \frac{\alpha}{2}\, e^{-\alpha(\tau-x)}x^m e^{s_j x}dx + \int_\tau^\infty \frac{\alpha}{2}\, e^{-\alpha(x-\tau)}x^m e^{s_j x}dx, \qquad \tau \geq 0,$$

$$= \frac{\partial^m}{\partial s_j{}^m}\left[\int_0^\tau \frac{\alpha}{2}\, e^{-\alpha(\tau-x)}e^{s_j x}dx + \int_\tau^\infty \frac{\alpha}{2}\, e^{-\alpha(x-\tau)}e^{s_j x}dx\right], \qquad \tau \geq 0,$$

$$= \frac{\partial^m}{\partial s_j{}^m}\left[\frac{\alpha}{2}\frac{2\alpha}{(\alpha^2-s_j{}^2)}\, e^{s_j\tau} - \frac{\alpha}{2(\alpha+s_j)}\, e^{-\alpha\tau}\right], \qquad \tau \geq 0,$$

$$= \frac{\partial^m}{\partial s_j{}^m}\left[\left(1 - \frac{s_j{}^2}{\alpha^2}\right)^{-1}e^{s_j\tau} - \tfrac{1}{2}\left(1 + \frac{s_j}{\alpha}\right)^{-1}e^{-\alpha\tau}\right], \qquad \tau \geq 0,$$

$$= e^{s_j\tau}\left[\sum_{k=0}^{m}\binom{m}{k}\tau^k\, \frac{(m-k)!}{2\alpha^{m-k}}\left\{\left(1 - \frac{s_j}{\alpha}\right)^{k-m-1} + (-1)^{m-k}\left(1 + \frac{s_j}{\alpha}\right)^{k-m-1}\right\}\right]$$

$$+ e^{-\alpha\tau}\left[\frac{m!}{2\alpha^m}(-1)^{m+1}\left(1 + \frac{s_j}{\alpha}\right)^{-m-1}\right], \qquad \tau \geq 0.$$

(A_2).

For $\tau \leq 0$, $\int_0^\infty \frac{\alpha}{2}\, e^{-\alpha|\tau-x|}x^m e^{s_j x}dx$

$$= \int_0^\infty \frac{\alpha}{2}\, e^{-\alpha(x-\tau)}x^m e^{s_j x}dx$$

$$= \frac{\partial^m}{\partial s_j{}^m}\int_0^\infty \frac{\alpha}{2}\, e^{-\alpha x}e^{s_j x}e^{\alpha\tau}dx$$

$$= \frac{\partial^m}{\partial s_j{}^m}\left[e^{\alpha\tau}\frac{\alpha}{2(\alpha-s_j)}\right] = e^{\alpha\tau}\frac{m!}{2\alpha^m}\left(1 - \frac{s_j}{\alpha}\right)^{-1-m}, \qquad \tau \leq 0.$$

(A_3).

For $\tau \geq 0$, $f_2(\alpha)$ is then a linear combination of terms of the form

$$\frac{m!}{2\alpha^m}(-1)^{m+1}\left(1+\frac{s_j}{\alpha}\right)^{-m-1}, \qquad (A_4)$$

the limit of which, as α approaches ∞, is $-\frac{1}{2}$ for $m = 0$ and zero otherwise.

The coefficient of τ^k in the polynomial $Q_j(\tau,\alpha)$ will similarly be a linear combination of terms of the form

$$\binom{m}{k}\frac{(m-k)!}{2\alpha^{m-k}}\left\{\left(1-\frac{s_j}{\alpha}\right)^{k-m-1}+(-1)^{m-k}\left(1+\frac{s_j}{\alpha}\right)^{k-m-1}\right\}, \qquad (A_5)$$

the limit of which is 1 for $m = k$ and zero otherwise. For $\tau \leqslant 0$, $f_1(\alpha)$ is a linear combination of terms of the form

$$\frac{m!}{2\alpha^m}\left(1-\frac{s_j}{\alpha}\right)^{-1-m}, \qquad (A_6)$$

the limit of which is $\frac{1}{2}$ for $m = 0$ and zero otherwise.

Since $f_1(\alpha)$, $f_2(\alpha)$ and the coefficients of the polynomials $Q_j(\tau,\alpha)$ are continuous for all α such that $\alpha > \alpha_0 > |s_j|$, all j, and in each case $\lim_{\alpha\to\infty}$ exists, then they are bounded for all $\alpha > \alpha_0$. Hence $\rho_{M\alpha,OC}(\tau)$ is bounded for $\alpha > \alpha_0$ and all τ.

Several corollaries follow from the above results.

<u>Corollary 1</u>

For all $\tau \neq 0$ $\lim_{\alpha\to\infty} \rho_{M\alpha,OC}(\tau) = W_C(\tau)$.

This result follows from a consideration of the limits of expressions (A_4), (A_5) and (A_6).

<u>Corollary 2</u>

For all $\alpha > \alpha_0$ the following integrals exist

$\int_a^b |\rho_{M\alpha,OC}(x)|\,dx$, a and b any real numbers,

$$\int_{-\infty}^{\infty} |\rho_{M\alpha,OC}(x)|dx.$$

Further $\lim_{\alpha\to\infty}$ of each of these integrals exists and

$$\lim_{\alpha\to\infty} \int_a^b \rho_{M\alpha,OC}(x)dx = \int_a^b \lim_{\alpha\to\infty} \rho_{M\alpha,OC}(x)dx;$$

$$\lim_{\alpha\to\infty} \int_a^\infty \rho_{M\alpha,OC}(x)dx = \int_a^\infty \lim_{\alpha\to\infty} \rho_{M\alpha,OC}(x)dx.$$

These results follow from the form of the integrand which is dominated by an absolutely integrable sum of expressions similar to $W_C(t)$ for $t \geqslant 0$ and to $W_C(-t)$ for $t \leqslant 0$.

Corollary 3

$$\operatorname{Lim}_{\alpha\to\infty} \int_0^{T_m} \rho_{M\alpha}(\tau-x)\rho_{M\alpha,OC}(x)dx = W_C(\tau), \qquad 0 < \tau < T_m.$$

This result may be obtained by performing the integration.

Corollary 4

$$\operatorname{Lim}_{\alpha\to\infty} \int_0^{T_m} \rho_{M\alpha}(\tau-x)\rho_{M\alpha,OC}(t_3+x)dx = W_C(\tau+t_3),\ 0 < \tau < T_m.$$

Theorem A1 and its corollaries may be used to establish the results quoted in the main text. Two examples will be given.

Example 1

$$\operatorname{Lim}_{\alpha\to\infty} \int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)E[W_C(x)W_C(\tau)]dxd\tau$$

$$= \lim_{\alpha\to\infty} \int_0^\infty\int_0^\infty \rho_{M\alpha}(\tau-x)W_C(x)W_C(\tau)dxd\tau$$

$$= \lim_{\alpha\to\infty} \int_0^\infty \rho_{M\alpha,OC}(\tau)W_C(\tau)d\tau \qquad \text{(equation } A_1\text{)}$$

$$= \int_0^\infty \lim_{\alpha\to\infty} \rho_{M\alpha,OC}(\tau)W_C(\tau)d\tau$$

$$= \int_0^\infty W_C^2(\tau)d\tau, \qquad \text{(Corollary 1)}. \tag{5.16}$$

The integration and limiting operations may be interchanged since $\rho_{M\alpha,OC}(\tau)$ is bounded, $\alpha > \alpha_0$, and

$\int_0^\infty |W_C(\tau)|d\tau$ exists.

<u>Example 2</u>

Consider

$$\int_0^{T_m}\int_0^{T_m} \frac{\rho_{M\alpha}(\tau-x)}{(T_n')^2} \int_0^{T_n'}\int_0^{T_n'} \rho_{M\alpha,OC}(t_2-t_1+x)\rho_{M\alpha,OC}(t_1-t_2+\tau)dt_2\,dt_1\,dx\,d\tau.$$

Substituting $t_3 = t_2 - t_1$ and inverting the order of integration reduces this expression to

$$\int_0^{T_m}\int_0^{T_n'} \frac{2(T_n'-t_3)}{(T_n')^2}\int_0^{T_m} \rho_{M\alpha}(\tau-x)\rho_{M\alpha,OC}(t_3+x)\rho_{M\alpha,OC}(-t_3+\tau)dx\,dt_3\,d\tau$$

$$= \int_0^{T_m}\int_0^{T_n'} \frac{2(T_n'-t_3)}{(T_n')^2} \rho_{M\alpha,OC}(\tau-t_3)\int_0^{T_m} \rho_{M\alpha}(\tau-x)\rho_{M\alpha,OC}(t_3+x)dx\,dt_3\,d\tau.$$

The inner integral is bounded for all $\alpha > \alpha_0$ and has a finite limit as $\alpha \to \infty$, so the limit $\alpha \to \infty$ of this expression is equal to

$$\int_0^{T_m}\int_0^{T_n'} \frac{2(T_n'-t_3)}{(T_n')^2} \lim_{\alpha\to\infty} \rho_{M\alpha,OC}(\tau-t_3)\int_0^{T_m} \rho_{M\alpha}(\tau-x)\rho_{M\alpha,OC}(t_3+x)dx\,dt_3\,d\tau$$

$$= \int_0^{T_m}\int_0^{T_n'} \frac{2(T_n'-t_3)}{(T_n')^2} W_C(\tau-t_3)W_C(\tau+t_3)dt_3\,d\tau, \qquad \text{(Corollaries 1 and 4)}$$

$$= \int_0^{T_m}\int_0^{\tau} \frac{2}{(T_n')^2} (T_n'-t_3)W_C(\tau-t_3)W_C(\tau+t_3)dt_3\,d\tau \tag{5.23}$$

since $W_C(\tau-t_3) = 0, \qquad t_3 > \tau.$

The final expression to be examined is

$$\frac{2}{(T_n')^2}\int_0^{T_m}\rho_I(y)(T_m-y)(T_n'-y)\rho_{OL}(y)dy \qquad (5.37)$$

$$= 2\int_0^{T_m}\rho_I(y)\left(\frac{T_m}{T_n'}-\frac{y}{T_n'}\right)\left(1-\frac{y}{T_n'}\right)\rho_{OL}(y)dy.$$

Allowing T_m and T_n' to tend to infinity in such a way that $\left(\frac{T_m}{T_n'}\right)$ is a finite constant, this expression becomes

$$2\,\frac{T_m}{T_n'}\int_0^{\infty}\rho_I(y)\;\rho_{OL}(y)dy$$

$$= \frac{T_m}{T_n'}\int_{-\infty}^{\infty}\rho_I(y)\;\rho_{OL}(y)dy$$

$$= \frac{T_m}{T_n'}\,\frac{1}{2\pi}\int_{-\infty}^{\infty}S_I(\omega)\;S_{OL}(\omega)d\omega.$$

$S_I(\omega)$ is the spectrum of $I(t,w)$ and it may therefore be factorised to the form $\psi(\omega)\,\overline{\psi(\omega)}$. $S_{OL}(\omega)$ is the spectrum of $O_L(t,w)$ and is therefore equal to $S_{OC}(\omega)$ multiplied by $(\psi(\omega)\,\overline{\psi(\omega)})^{-1}$ so that the expression (5.37) approaches

$$\frac{T_m}{T_n'}\,\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{OC}(\omega)d\omega$$

$$= \left(\frac{T_m}{T_n'}\right)\sigma^2_{OC}.$$

NOTATION

A	Adequacy of the model.
$\bar{A}$	Inadequacy of the model $\bar{A} = 1 - A$.
B	Correlation function for random process.
C	As subscript indicates correction.
E	Expectation of random variable. As subscript indicates estimate. Also used to denote an error.
G	With subscript; functions used in tabulated integrals of reference [26].
H	With subscript; functions used in tabulated integrals of reference [26].
I	Random process, realisations of which constitute inputs to the system and model; with subscripts. As subscript, indicates input.
L	A stationary, linear, causal transformation. As subscript indicates the function has been transformed by such a transformation.
L_1	A type of stationary linear causal transformation defined in chapter 3.
M	Stationary Gauss Markov process. As subscript usually denotes model; but in combination M_α, or M_β denotes stationary Gauss Markov process. Also an expression used in the tabulated integrals of reference [26].
N	Random process, realisations of which appear as errors in recordings.

O	Random process, realisations of which constitute outputs of the system and model. As subscript indicates output.
O_C	Difference between the system output and the model output in the absence of recording errors.
P	Probability measure.
$P(\cdot)$	A polynomial expression; lower case subscripts if necessary.
$Q(\cdot)$	A polynomial expression; lower case subscripts if necessary.
R	An estimate of a correlation function. Real part of a complex function; often with subscript.
R'	Estimate of a correlation function; but zero outside stated values of its argument.
S	Spectrum of random process, subscripts indicate the processes concerned. As subscript indicates system.
T	Transformation; subscripted in upper case where necessary. A constant representing time; lower case subscripts if necessary.
T_m	Time after which no correction is made to the model weighting function.
T_n	Length of record.
T_n'	Effective length of record.
T_m'	$T_n - T_n'$.
$T(\alpha)$	Time which approaches zero as α approaches infinity.

V_o	Voltage.
W	Probability space of points w.
$W(\cdot)$	Weighting function, subscripted as necessary.
X	A random process.
X_1	A type of random process defined in chapter 3.
$\mathcal{C}$	Domain of transformation; a linear space of functions $f(t)$.
$\mathcal{T}$	A set of transformations.
a	A constant; $\frac{1}{a}$ used as time constant.
b	A constant, usually a bias error.
f	A function, usually of time; often with subscript.
g	With subscript, a member of a class of orthogonal functions.
k^2	Constant spectral density.
p	Degree of polynomial. Non dimensional quantity aT_m.
q	Non dimensional quantity aT_n'.
s	Complex number, with subscript. Argument of Laplace transform.
w	A point in a probability space W.
z	A complex number.
Δ	Shift operator. Expression used in tabulated integrals of reference [26].
Λ	A constant.
Σ	Function which is sum of several functions.

α	Parameter, usually of stationary Gauss Markov process, often as subscript.
β	Parameter of stationary Gauss Markov process.
δ	The Dirac delta function.
μ	Mean of a stationary random process.
ρ	Correlation function whose mean is zero; subscripted to indicate the process or processes concerned.
σ^2	Variance of stationary random process; subscripted where necessary.
τ	Time, usually as argument of correlation function.
ϕ	Fourier transform; often frequency response function.
ξ	As a vector indicates random process; subscripted to indicate its components.
ψ	Fourier transform. Also factor of spectrum having all its poles and zeros in the upper half plane.
ω	Argument of Fourier transform.

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