THE DEVELOPMENT OF THE BOSON CALCULUS FOR THE ORTHOGONAL AND SYMPLECTIC GROUPS

by

M. A. Lohe

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Department of Mathematical Physics,
The University of Adelaide,
South Australia

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CONTENTS

Page No.

ABSTRACT

The boson calculus has been used extensively in the study of the unitary groups U(n) as a means of constructing explicitly all irreducible unitary representations. However, the boson calculus in this form cannot be applied directly to the subgroups Sp(n) and O(n); our aim is to develop the boson calculus in a form which is immediately applicable to all the classical groups.

The U(n) representation space is constructed from tensors, which in the restriction to Sp(n) and O(n) must be traceless if the representations are to be irreducible. Instead of the usual boson operators we introduce "modified boson operators" a_i^α which satisfy in particular the traceless condition $\rho_{pq}\ a_p^\alpha\ a_q^\beta = 0$ (summation), where ρ is the metric for Sp(n) or O(n). With these operators, which behave as vectors under Sp(n) or O(n), we are able to construct manifestly traceless tensors (multivectors) of arbitrary symmetry. Furthermore, all objects to be studied may be defined in terms of these operators. In general, we find that the modified boson calculus which we develop has, for Sp(n) and O(n), a domain of application which not only includes that of ordinary boson operators, but is considerably larger.

We are now able to construct simply the irreducible spaces which carry all representations of Sp(n) and O(n). We calculate maximal and semi-maximal basis states, and all states in symmetric tensor representations of O(n) and Sp(n), and also general states for arbitrary tensor representations of O(3), O(4). The O(3) states appear as monomials and the O(4) states as Jacobi polynomials in modified boson operators.

In the application to Sp(n) we encounter the problem of state-labelling. We restrict our attention to Sp(4), although keeping in mind the general problem, and we seek a solution using the parameters appearing in the branching theorem for Sp(4) restricted to Sp(2). We utilize modified boson operators in the construction of the non-orthogonal Weyl states, and carry out a suitable Gram-Schmidt orthogonalization to obtain explicitly the orthogonal basis states. Although in principle the extra labelling operator which is required may be found from these states, its form will necessarily be complicated. It is found that a satisfactory solution to the state-labelling problem, exhibiting the structure and simplicity which is apparent for U(n) and O(n), does not exist.

We carry out a further development of the boson calculus for O(n) to enable the explicit construction of all spinor (double-valued) representations in spaces of traceless tensors. For the lower order groups this is done in such a way as to obtain the representation space of the covering group, by finding operators which satisfy the traceless condition, but are different from modified boson operators. Some of these operators satisfy simple triple commutation relations which are of interest for both group theory and field theory.

In order to enable the construction to be made of all spinor representations of O(n) in general in a space of traceless tensors, or equivalently, harmonic homogeneous polynomials, we establish firstly the relation between the methods of the boson calculus and of Zhelobenko [1]. This latter method uses polynomials over a homogeneous space defined by a certain triangular subgroup, and we show the two methods can be directly related, so that one construction can be mapped into the other. Zhelobenko's formalism includes the spinor representations in a natural way, and we show how to tranfer to the boson calculus so as to retain this construction; this is achieved ultimately by finding realizations of the Lie algebra of O(n) which are new. Results are written out explicitly for O(3).

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[1] D. P. Zhelobenko, Russ. Math. Surv. XVII, 1 (1962).

STATEMENT

This thesis contains no material which has been accepted for the award of any other degree, and to the best of my knowledge and belief, contains no material previously published or written by another person except where due reference is made in the text.

Max Adolph Lohe

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CHAPTER 1

INTRODUCTION

§1. Historical Account

The theory of group representations is well established as being of central importance in several branches of physics, particularly elementary particle physics and nuclear and atomic physics. This was already clear in the initial investigations of the quantum theory of angular momentum, which it was found [1] could be based on the algebraic results derived from the commutation relations for angular momentum $\underline{J} \times \underline{J} = i\,\underline{J}$; these relations are also the defining relations of the Lie algebra of the groups O(3) and SU(2), and therefore these groups are of fundamental importance.

The reasons why the representations of groups in general are significant, have been clearly explained by Dyson [2] as follows:

"(1) States of a system which obeys the laws of quantum mechanics are described as vectors in a vector-space V over the field of complex numbers.

(2) An atomic system usually has some degree of symmetry, described by a group G of operations under which the equations describing the system are invariant.

(3) Each symmetry operation g in G defines a linear transformation L(g) of the vector space V into itself, and the transformations L(g) constitute a representation of G.

(4) The character of a state, so far as the physical properties connected with G are concerned, is completely specified by the irreducible representation of G within the (irreducible) subspace V_i to which the state belongs.

(5) States of the system occur in multiplets, the states within a multiplet having the same energy and transforming into each other under the operations of G. The number of states in a multiplet is equal to the dimension of the corresponding irreducible representation of G. "

Let us consider as an example of such a system the 3-dimensional harmonic oscillator. The Hamiltonian energy for a single particle is

$$H = \tfrac{1}{2} \sum_j (p_j^2 + x_j^2) ,$$

where we have chosen units such that the mass, frequency, and $\hbar$ are unity. $\underline{x} = (x_1, x_2, x_3)$ and $\underline{p} = (p_1, p_2, p_3)$ denote the position and linear momentum of the particle and satisfy $[x_i, p_j] = i\,\delta_{ij}$, with other commutators being zero. The equations of motion of the particle are $p_j = \dot{x}_j$, $\dot{p}_j = -x_j$ and are invariant under O(3). The states of the oscillator are grouped in multiplets with dimensions 1,3,5,.., the states in the multiplet $2\ell+1$ having angular momentum equal to ℓ. However, the true invariance group of the system is U(3), for if we let

$$a_j = \frac{1}{\sqrt{2}} (x_j - i\, p_j) \qquad (1)$$

then the equations of motion are $\dot{a}_j = i\, a_j$, which are invariant under all complex unitary transformations of the 3- component vector $\underline{a}$. More generally, the invariance group of the n-dimensional harmonic oscillator is known ([3], [4]) to be U(n); using this knowledge the structure of the state space can be revealed, in particular all particle states can be classified [5].

If we define also

$$\bar{a}_j = \frac{1}{\sqrt{2}} (x_j + i\, p_j) , \qquad (1')$$

the adjoint of a_j, then the essential Hamiltonian may be written as $H = \sum_i a_i \bar{a}_i$, and the eigenvectors provide a basis for an irreducible representation of U(3). As Dirac has shown ([6] §34), the eigenvectors may be written as polynomials P(a) in the operators a_i, acting on the state $|0\rangle$ of lowest energy satisfying $\bar{a}_i |0\rangle = 0$. Operator wave functions of this form were used by Fock [7], and so the space is known as a Fock space.

From the commutation relations of $\underline{x}$ and $\underline{p}$ we find that a_i, $\bar{a}_i$ satisfy $[\bar{a}_i, a_j] = \delta_{ij}$, $[a_i, a_j] = 0$, showing that the operators, in terms of which the problem has been formulated, are simply boson operators, and eigenvectors of H are homogeneous polynomials in a_i acting on $|0\rangle$. In this way boson operators provide an irreducible representation space of U(3), whilst appearing initially in the context of the simple harmonic oscillator.

The formalism of second quantisation of a system of n identical particles is closely connected to the boson realization of the harmonic oscillator, as Dirac has described ([6], §60). The states in a system of n identical particles are symmetric and are labelled $|n_1, n_2, .. n_i, ..\rangle$, where n_i is the number of particles in the i^{th} state, and $n = \sum_i n_i$. These particles are known as bosons because the corresponding statistics were first studied by Bose [8] (and also Einstein [9]). We define the operator $\bar{a}_i$ by

$$\bar{a}_i \, |n_1, n_2, .. n_i, ..\rangle = \sqrt{n_i} \, |n_1, n_2 .., n_i - 1, ..\rangle$$

and also the adjoint a_i, for which

$$a_i \, |n_1, n_2, .. n_i, ..\rangle = \sqrt{n_i + 1} \, |n_1, n_2, .., n_i + 1, ..\rangle \, .$$

The following commutation relations of these operators, which annihilate and create particles, may be verified:

$[\bar{a}_i, a_j] = \delta_{ij}$, $[a_i, a_j] = 0 = [\bar{a}_i, \bar{a}_j]$, so that boson operators have again appeared. The operator $N_i = a_i \bar{a}_i$ is known as the number operator for the state i because it has eigenvalues n_i, and the states $|n_1, n_2 \, .. \, n_i, \, ..\rangle$ may then be regarded as basis vectors in the representation for which all N_i are diagonal.

Let us also consider a system composed of several 1-dimensional harmonic oscillators, each of which is described using a boson operator and its adjoint. We obtain a general state vector by allowing the boson operators for each oscillator to act on the vacuum $|0\rangle$. The state $|0\rangle$, the "standard ket" for the Fock representation of the assembly of oscillators, is the product of the vacuum states for each oscillator. The state corresponding to n bosons is $a_1^{n_1} a_2^{n_2} \, .. \, a_j^{n_j} |0\rangle$ with $n = n_1 + .. + n_j$. This state may be regarded also as the symmetrization of an arbitrary state in an assembly of bosons, which means that the dynamical system consisting of an assembly of bosons is equivalent to the dynamical system consisting of a set of oscillators. In this way the boson realization of the simple harmonic oscillator is of fundamental importance for systems of identical particles, and consequently for the theory of radiation. The normalized state in the system with occupation numbers n_1, n_2 is

$$\frac{a_1^{n_1} a_2^{n_2} |0\rangle}{\sqrt{n_1! \, n_2!}}$$

where $n = n_1 + n_2$, an expression which appears also in the representation theory of SU(2).

We have now seen how boson operators acting in a Fock space appear naturally in the discussion of physical systems. These systems may also be invariant under group transformations, and the boson states then provide a representation space on which the group acts. This method of constructing representation spaces, which is known as the boson calculus, has appeared in this way as a powerful tool for the study of group representations in an explicit form.

As Biedenharn has emphasized ([10] ,[11]) the representation theory of Lie groups in general has received several classic treatments, but which are not as explicit and constructive as the physicist requires. The boson calculus uses the classic methods of Weyl ([12], [13]) in the form of the explicit realization of boson operators. In this formalism the functions of the representation space, and the operators which act on this space such as the group generators, are all expressed in terms of boson operators. It is then possible to carry out direct and explicit calculations, permitting the computation of matrix elements of generators and tensor operators in general, and enabling a thorough investigation of the properties of the group under consideration.

The connection between boson operators appearing in a physical system and group representations was first revealed by Jordan [14] in a discussion of the relationship between the linear group and an n-particle system. Using the method of second quantisation of the system, boson operators a_i were introduced and Jordan then constructed

$$E_{k\ell} = a_k \bar{a}_\ell \qquad (2)$$

which were shown to satisfy $[E_{k\ell}, E_{jm}] = \delta_{j\ell} E_{km} - \delta_{km} E_{j\ell}$, the commutation relations of the Lie algebra of GL(n). This analysis was carried out for fermions also, so that here we have the beginnings of the boson and fermion calculus.

We shall regard the boson calculus in a wider sense as the construction of representations in spaces of homogeneous polynomials of complex variables. A purely operator construction can then be carried out simultaneously ~~by regarding each variable~~ z_i as a boson operator a_i, with adjoint $\bar{a}_i = \frac{\partial}{\partial z_i}$. The relation of these isomorphic methods has been clarified by Bargmann [15], as will be discussed later. In this wider view the boson calculus may be considered to have originated earlier than the work of Jordan. As explained in Chapter 2, the boson calculus is a realization of a space of tensors on which the group acts, and such a realization was described previously by Weyl [13]. Elements of GL(n) act on a vector $x = (x_1, x_2, \ldots x_n)$ from which quadratic and higher order forms can be constructed, and Weyl pointed out ([13] p.124) that these forms themselves constitute a representation space of tensors on which the group acts. By introducing additional arbitrary vectors y, z, etc., arbitrary tensors can be constructed. Weyl described [12] how to reduce this space using the symmetric group, and this investigation was published already in 1925 [16], including O(n) and Sp(n) as well as GL(n). Already, then, the tensors constituting the "substratum" of the representation space have appeared as monomials $x_1^{f_1} x_2^{f_2} \ldots x_n^{f_n}$, and these are later to be known as Weyl states. For SU(2) these normalized monomials take the form $x(m) = \frac{\xi^i \eta^k}{\sqrt{i!\,k!}}$ $(i + k = 2j,\ i - k = 2m)$ where ξ, η are vector components.

As we have mentioned, Jordan subsequently introduced the operator formalism but it was not until Schwinger [17] carried out a thorough analysis with abstract boson operators that the full capabilities of the boson calculus were realized. Schwinger introduced boson operators a_1, a_2 (and their adjoints) and all other objects to studied were defined in terms of the a_i. The physical reasons motivating this approach are those which we have mentioned in general, that angular momentum in quantum mechanics can be regarded as a superposition of an assembly of elementary "spins" with angular momentum $j = \frac{1}{2}$. Such an assembly may be regarded as a Bose-Einstein system which may be discussed using second quantisation, whereby boson operators are introduced. The basis states appear as

$$\frac{a_1^{\ell+m}\, a_2^{\ell-m}\, |0>}{\sqrt{(\ell + m)!(\ell - m)!}}$$

and the generators are expressed as in the realization (2). Using this formalism Schwinger was able to present a very complete account of angular momentum and SU(2), deriving matrix elements and relevant properties for both finite and infinitesimal group elements, together with an account of the addition of two, three and four angular momenta and of the theory of tensor operators. The methods used by Schwinger are of such power because all objects to be studied can be introduced in the explicit realization of boson operators, which themselves are easily manipulated.

Bargmann [15] has reviewed the theory of angular momentum in a formulation which is isomorphic to Schwinger's operator method. Irreducible representations of SU(2) are obtained by considering homogeneous polynomials in two complex variables, defined in a Hilbert space, and the standard methods of analysis are available at each step. In the viewpoint which we adopt Bargmann's method is not distinguished from that of the boson calculus, so that a boson operator a_i and the variable z_i are treated interchangeably according to what is most convenient.

The work of Jordan, and in particular Schwinger, has led to extensive research on the development and application of the boson calculus (e.g. [18],[19]). Dirac [20] carried out independent work using a method suggested to him by Fock's quantum theory of the harmonic oscillator; representations of the rotation and Lorentz groups were constructed in the space of coefficients of a power series in the variables ξ_i, i = 1, .. 4. These variables, called "expansors", are easily transformed to the components x_μ of a four-vector, which can be regarded as boson variables, so that Dirac independently encountered the boson calculus.

Considerable development of the boson calculus, in the generalisation to higher order unitary groups, has been carried out by Moshinsky ([21],[22],[23]). The motivation here has again been the study of the harmonic oscillator, and boson operators have appeared as in (1). Results are obtained concerning the form of the polynomial bases, in a completely group theoretical context ([24], [25]), in particular the state of highest weight becomes of prime importance. For an irreducible representation this state is unique ([26] p.37) and so its explicit knowledge for arbitrary representations of U(n) is an important advance. With the expressions for lowering operators known [27], it is now possible to calculate arbitrary Gelfand basis states.

A complete and general account of the boson calculus was given by Baird and Biedenharn [10], in which the full power of the integral approach against the infinitesimal methods was demonstrated. Here the boson calculus is revealed as a realization of the tensor spaces employed by Weyl, and the expansion of the Gelfand states in terms of the Weyl tensors is carried out explicitly for U(2) and SU(3). Full use is made of the Young tableaux which define the symmetry of the basis tensors, even to the extent of associating with the Young pattern a measure, or normalization of the state, by means of an explicit algorithm involving hook lengths. With such techniques an explicit derivation of matrix elements of the U(n) generators becomes possible, to give the results which had been stated previously by Gelfand and Zetlin [28].

Since the time of this work, one development of the boson calculus has been concerned with the explicit form of the basis states ([29],[30],[31]) and the relevant combinatorial structure ([32],[33]). Another development has occurred as part of a general program concerning the investigation of tensor operators in the unitary groups. In what is termed the Racah-Wigner angular momentum calculus ([34], [11]), it is required to investigate the following: reduction of direct products and Wigner coefficients of the group; irreducible tensor operators and Wigner-Eckart theorem; Racah coefficients of the group. The role of the boson calculus in this program has been explained by Louck [34], and has been used ([35],[36],[37]) for the explicit calculation of Wigner coefficients, in a method which is a generalization of Wigner's original calculation [38].

Other modern work involving the boson calculus has concerned the application to nuclear structure ([35] Vol. II, p.340) and to the n-dimensional harmonic oscillator [5], and also to problems of state-labelling [39]. Generally, where explicit group theoretical results are required the boson calculus formalism provides the most powerful method of investigation.

§2. The Need for a Further Development of the Boson Calculus

It is noticeable that the boson calculus has been developed so as to apply primarily to the unitary groups. O(n) and Sp(n) are subgroups of U(n) and therefore many properties of U(n) apply also to O(n) and Sp(n). It is of considerable interest to have available an explicit boson calculus for O(n) and Sp(n) which incorporates particularly those properties which result from the extra structure which O(n) and Sp(n) possess through leaving invariant a certain quadratic form. The interest in O(n) and Sp(n), for physics, stems from the fact that these groups are as rich in structure as U(n), and therefore frequently appear as symmetry groups of physical systems. This is particularly true for O(n), whereas Sp(n) has been somewhat neglected. The importance for physics of O(3), which is locally isomorphic to SU(2) and Sp(2), is well known. O(4) appears as the symmetry group of the bound state problem for the hydrogen atom, under a Coulomb potential ([40],[41]). SO(5), which is locally isomorphic to Sp(4), has been studied by Hecht [42] with a view to applications in nuclear spectroscopy and elsewhere, but our methods will simplify much of his work. Another study [43] of SO(5) arises from the description of nuclei states with the 5-dimensional harmonic oscillator. The symplectic group Sp(2j+4) has appeared [44] in the classification of shell model states, and the further development of this work [18] has revealed the presence of another Sp(4) group. In his study of complex spectra, Racah [45] showed how both the general orthogonal and symplectic groups could be used in the theory of seniority, leading also to further developments on symplectic symmetry [46].

From the mathematical point of view a boson calculus for O(n) and Sp(n) is of interest as an explicit construction of all unitary irreducible representations. Together with SU(n), O(n) and Sp(n) constitute the classical groups, and with the exceptional groups make up all the semi-simple compact Lie groups. Because of these properties, the knowledge of the group structure is certain to be of consequence for mathematical interest alone. For example, basis states would be special functions which could be studied using group theoretical properties, as has been done by Miller [47], Talman [48], and Vilenkin [49]. In general it is expected that a suitable boson calculus would enable an investigation of O(n) and Sp(n) in the way that has been done for U(n) with consequent applications in both mathematical and physical areas of interest.

As we have mentioned, O(n) and Sp(n) are subgroups of U(n), so that the boson calculus for U(n) can apply to O(n) and Sp(n) by considering a suitable irreducible subspace of the U(n) representation space. As will be explained, we need to carry out a restriction from the space of tensors of certain symmetry to the subspace of traceless tensors, or in the equivalent viewpoint, we need to consider the harmonic subspace of homogeneous polynomials. This restriction is well known for O(3), where the basis states become the spherical harmonic functions, or equivalently, symmetric traceless tensors ([50], p.397). Symmetric representations of O(n) have been studied [51] where the basis states are regarded as polynomials in boson operators, acting on a vacuum state, and similar calculations have been done for general representations of Sp(4) [52]. States of highest weight have been found for O(n) [53], together with lowering operators ([54],[55]), so that it is possible to calculate arbitrary basis states. It would seem from this that the boson calculus applies to O(n) and Sp(n) in the same way as it does to U(n), simply by considering O(n) and Sp(n) as subgroups of U(n).

However, there are some clear and definite deficiencies in this approach which are apparent already for O(3). Firstly, the basis states for O(3) are very complicated by comparison with the monomial basis states of SU(2), although SU(2) is of comparable complexity to O(3); secondly, the basis states are not manifestly harmonic i.e.

the representation space is not irreducible in an obvious way, and is not manifestly invariant under the operators on the space. Connected with these deficiencies is the fact that, although for U(n) Weyl states can be written down from the Young tableau by inspection as monomials, the same cannot be done for O(n) and Sp(n) i.e. the traceless tensors of the space are not determined in an obvious way from each Young tableau, and there are no monomials which could be regarded as the traceless Weyl tensors.

It is clear that the boson calculus for U(n) is not immediately applicable to O(n) and Sp(n) and requires a modification in order to overcome the deficiencies described. The starting point of the development required will be to find operators, "modified bosons" a_i, which behave as ordinary bosons in many ways, e.g. they commute and are vectors under O(n) and Sp(n), but which satisfy in addition $a_1^2 + .. a_n^2 = 0$ (for O(n)). Tensors constructed as polynomials in a_i acting on $|0>$ will then be manifestly traceless, since modified boson operators will operate only within such a space, unlike ordinary bosons. It is necessary to generalize a_i to several sets a_i^σ which can be used for arbitrary representations of Sp(n) as well as O(n).

With the basic mechanism of the boson calculus set up the development takes place as for U(n), so that Weyl tensors may be constructed and orthogonal Gelfand states calculated. Although in principle these states may be calculated using ordinary bosons, the simplicity which modified bosons permit allows the construction of some general states, which had been previously prevented because of the greater complexity if ordinary bosons are used. It will be shown that there are two applications where modified boson operators are indispensible, i.e. ordinary bosons do not permit the discovery of these results at all.

The first is in establishing the relation of the boson calculus formalism to an apparently completely different and powerful method used by Zhelobenko [56], and secondly in the construction of all spinor representations of O(n) in harmonic spaces. In general it will be clear that, for O(n) and Sp(n), the domain of application of modified bosons not only includes that of ordinary bosons, but is considerably larger.

§3. Summary of Thesis

The development of the boson calculus for O(n) and Sp(n) will be described as follows. In Chapter II we give an account of the boson calculus and its development for U(n), partly to set the notation, and partly to indicate how the development for O(n) and Sp(n) should be carried out. Mainly however this material serves as a prerequisite for the study of O(n) and Sp(n) because these groups are subgroups of U(n), and many properties of U(n) apply almost immediately to O(n) and Sp(n). We introduce here the direct product space in which are defined the multivectors known as Weyl tensors, and we give the equivalent description, due to Bargmann, in the space of homogeneous polynomials. The labelling of states, the Gelfand basis states, and the method of their calculation are also described.

In Chapter 3 we carry out the reduction into the subspaces which are irreducible under O(n) and Sp(n), involving the appearance of traceless tensors and harmonic homogeneous polynomials. The considerations for both groups can be combined by using a general metric ρ_{ij}; it is for this reason that the development of the boson calculus for O(n) is also a development for Sp(n), and vice versa. We introduce in Chapter 3 the modified boson operators which form the basis of our investigations, both as abstract operators in a Fock space, and as differential operators acting on functions of complex variables. We examine their properties, particularly with regard to the realization of the group generators they provide, and their properties as vectors. Aside from group theoretical concepts, modified bosons are useful in the construction of traceless tensors; given a multivector we can write down immediately its traceless part, not just for symmetric multivectors as has been done before [57], but quite generally.

In Chapter 4 we apply modified bosons to the task of calculating basis states for O(n). We show how to obtain Weyl states quite generally, and we write down the maximal and semi-maximal states of the Gelfand basis. This leads also to a derivation of the branching theorems, which are used as a means of labelling the states. We calculate general basis states for O(2), O(3) and O(4) and symmetric basis states for O(n). In some cases, these are different, simpler formulations of known results, but the expression for O(4) basis states as Jacobi polynomials would appear to be new.

In Chapter 5 we use modified bosons in the calculation of Sp(n) basis states. We are able to calculate maximal states and symmetric states of Sp(n), but for the general states we encounter the problem of state-labelling. Sp(n) does not possess a suitable subgroup chain, as do U(n) and O(n), which could be used to label the states completely. For this problem we restrict our attention to Sp(4) keeping in mind the general case. We show how the basis states can be labelled using the branching theorems, and we calculate the general Gelfand state by carrying out a suitable orthogonalization of the non-orthogonal Weyl states. Our solution to the problem is therefore on a global scale, utilizing state vectors, and it becomes clear that suitable labelling operators constructed from the group generators are very complicated. From this point of view no suitable solution to the problem exists. Our methods apply also to the well known $SU(3) \supset O(3)$ labelling problem.

In Chapter 6 we examine in detail the triple commutation relations encountered when first considering modified bosons. These relations are of interest both for the group theory involved, and also as a means of defining a new field theory. We show that the commutator algebra of these relations is isomorphic to the Lie algebra of $O(n,2)$, and we find the corresponding representations in a Fock space. There are also solutions of these triple commutation relations as double bosons, and these are all classified.

From Chapter 7 to Chapter 9 we turn our attention to the construction of the spinor representations of $O(n)$, a construction not achieved in Chapter 3. We examine firstly the role of the covering groups as a means of obtaining the spinor representations. The triple commutation relations of Chapter 6 appear again in the study of certain operators which satisfy the traceless conditions, but are different from modified bosons. These operators provide a global mapping from $O(n)$ to its covering group, for $n = 2, .. 6$, and in particular for $n = 5$ involve the modified bosons for $Sp(4)$.

In Chapter 8 we demonstrate the relation between the boson calculus, for $U(n)$ and also $O(n)$ and $Sp(n)$, and a different method developed by Zhelobenko. This latter method can be regarded as being more mathematical in origin compared to the rather more physical motivation of the boson calculus. Zhelobenko has developed his method to construct all finite dimensional representations of the complex classical groups, including the spinor representations for $O(n)$. The method relies on the existence for these groups of a Gauss decomposition, and all representations are induced by the subgroup of diagonal matrices. Zhelobenko's method identifies clearly the homogeneous spaces in which the representations are constructed, and by establishing the relation with the boson calculus we reveal the true mathematical origin of the boson calculus, in terms of the homogeneous spaces involved.

In Chapter 9 we show how to transfer from the formalism of Zhelobenko to the boson calculus in a way that is general enough to retain the construction of all representations of U(n), O(n) and Sp(n). As a result we are able to construct all spinor representations of O(n) in harmonic spaces, which for O(3) amounts to using as a basis the same set of functions, the spherical harmonics, as is used for the tensor representations. The two types of representations, single- and double-valued, are distinguished by the different form of the generators; here modified bosons play an important part.

In conclusion, then, we have completed the principle task, which was under investigation, the development of a boson calculus for O(n) and Sp(n) which would enable the explicit construction of all unitary representations in manifestly irreducible form. It remains now to develop these methods further as has been done for U(n), and to apply the techniques to problems in both mathematics and physics.

CHAPTER 2

DEVELOPMENT OF THE BOSON CALCULUS FOR U(n)

§1. Space of Tensors

The boson calculus has been developed fully from its initial stages by Biedenharn [10,32,35] and Moshinsky [21,24,25] and co-workers. The description of the carrier space in terms of tensors realized with boson operators is as follows.

The fundamental (defining) representation of U(n) has as its carrier space an n-dimensional vector space A. The Wigner-Stone theorem [35, Vol. II, p2] shows that we may construct all unitary irreducible representations of a compact matrix group by taking repeated direct products of the fundamental representation of the group.

Hence we form $B = A^{(1)} \times A^{(2)} \times .. \ A^{(\lambda)}$, the direct product of λ spaces like A. B is then the carrier space of tensors of rank λ. B is reducible because the transformation induced by the operations of U(n) commute with transformations permuting the λ vector spaces among themselves. To see this let $T_{(i)} = T_{i_1 i_2 .. i_\lambda}$ be a tensor of rank λ; then the transformation of T under U(n) is $T'_{(i)} = T_{(p)} \, g_{(p)(i)}$ i.e.

$$T'_{i_1..i_\lambda} = T_{p_1..p_\lambda} \, g_{p_1 i_1} .. \, g_{p_\lambda i_\lambda} \quad , \quad g \in U(n)$$

(summation over p).

Let $S \in S_\lambda$, the symmetric group of order λ. Then

$$(ST)'_{(i)} = T'_{S(i)} = T_{(p)} \, g_{(p)S(i)}$$

$$= T_{S(p)} \, g_{S(p)} \, S_{(i)}$$

$$= (ST)_{(p)} \, g_{(p)(i)} \ ,$$

~~where we have used the fact that the tensor transformation~~ is bisymmetric. We have shown that B can be reduced into subspaces each of which is spanned by tensors of the same symmetry. A given symmetry i.e. an element of S_λ, corresponds to a Young symmetry pattern defined by a partition $[m]$ of λ. Hence each pattern uniquely denotes an irreducible subspace of B. Each Young tableau (the Young pattern filled lexically with the integers denoting the λ vector spaces) defines an operator, the Young symmetrizer which projects the direct product space B into the invariant subspace defined by the Young tableau.

The basis vectors of the irreducible representation of U(n) are determined in a one to one correspondence with the lexical Young tableau in which the indices of the Young symmetrizer tableau have been assigned numerical values (1 to n).

The boson calculus appears when we realize each abstract space $A^{(\sigma)}$ with boson operators, so that the elements of $A^{(\sigma)}$ are bosons a_i^σ i=1, .. n which behave as vectors under U(n):

$$a' = ag \quad \text{i.e.} \quad a_i^{\sigma\prime} = a_p^\sigma g_{pi}$$

$$\text{where } g \in U(n) \; .$$

These boson operators are defined by the commutation relations

$$[a_i^\sigma, a_j^\tau] = 0 = [\bar{a}_i^\sigma, \bar{a}_j^\tau] \; ,$$

$$[\bar{a}_i^\sigma, a_j^\tau] = \delta_{ij}\, \delta^{\sigma\tau} \qquad (1)$$

where $\bar{a}_i^\sigma$ is the destruction operator adjoint to the creation operator a_i^σ . A tensor in B is now constructed from boson operators acting on the unique vacuum $|0\rangle$. The effect of the Young symmetrizer acting on these tensors is to introduce the following antisymmetric combinations (symmetric combinations appear automatically):

$$a_{i_1..i_k} = \Sigma\ \varepsilon(i_1..i_k)\ a^1_{i_1}\ ..\ a^k_{i_k} \quad , \tag{2}$$

which is the determinant of the k × k matrix $M_{ij} = a^i_j$. The state which corresponds to a diagram is called the Weyl state and can now be written down explicitly; for example the Weyl state with diagram

1	3
2	

is $a_3\ a_{12}\ |0\rangle$.

The abstract generators E_{ij} of U(n) satisfy

$$[E_{ij}, E_{k\ell}] = \delta_{jk}\ E_{i\ell} - \delta_{i\ell}\ E_{kj} \quad . \tag{3}$$

These generators can be divided into weight generators, and lowering and raising generators. The classification in terms of the generators E_α corresponding to root α is

$$H_i = E_{ii}$$

$$E_{ei-ej} = E_{ij} \quad , \quad i \neq j = 1, .. n \ . \tag{4}$$

The E_{ij} can be realized explicitly with boson operators as

$$E_{ij} = a^p_i\ \bar{a}^p_j \ , \tag{5}$$

which is checked using (1). The hermiticity property $E^*_{ij} = E_{ji}$ ensures that the representations are unitary. Bosons behave as vectors under these generators (as previously noted):

$$[E_{ij}, a^\sigma_k] = \delta_{jk}\ a^\sigma_i \ . \tag{6}$$

The Weyl basis is very useful, partly because of its simplicity being always a monomial, and because it can be written down immediately by inspection from the Young tableau. However the Weyl basis has the disadvantage that in general it is not an orthogonal basis.

A labelling scheme which refers to orthogonal basis states was originated by Gelfand and Zetlin [28]. In this scheme a basis state $|(m)\rangle$ for U(n) is written

$$|(m)\rangle = \left| \begin{matrix} m_{1n} & & m_{2n} & & \cdots & & m_{nn} \\ & m_{1n-1} & & \cdots & & m_{n-1\,n-1} & \\ & & & m_{11} & & & \end{matrix} \right\rangle \qquad (7)$$

where the m_{ij} are non-negative integers satisfying

$$m_{i+1,j+1} \leqslant m_{ij} \leqslant m_{ij+1} \; . \qquad (8)$$

The meaning of the m_{ij} is that the numbers $(m_{1i}, m_{2i} \cdots m_{ii})$ form the highest weight vector of the subgroup U(i) contained in the decomposition $U(i + 1) \supset U(1) \times U(i)$. The inequalities (7) are then simply a statement of the Weyl branching theorem for U(i + 1) restricted to U(i). These Gelfand basis states are necessarily orthogonal because of the group theoretic meaning of the m_{ij}.

Since the Gelfand labelling enumerates the states correctly, the Gelfand states can be put into a 1 - 1 correspondence with the Weyl states. The natural correspondence is the following: with the Gelfand state $|(m)\rangle$ we associate the Weyl state determined by the Young tableau containing in the k^{th} row m_{kk} k's followed by

$(m_{k,k+1} - m_{kk})$ k+1's " "

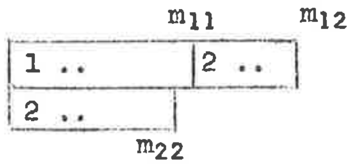
.. .. " "

$(m_{kn} - m_{k\,n-1})$ n's , for $k = 1, \cdots \lambda$.

Generally we put $\lambda = n$ in order to obtain all representations of U(n). Taking for example U(2), the Gelfand state is

$$|(m)\rangle = \left| \begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix} \right\rangle$$

and is associated with the Weyl state (denoted with round brackets)

$$\begin{vmatrix} m_{12} & m_{22} \\ m_{11} & \end{pmatrix} =$$ 

(9)

$$= a_1^{m_{11}-m_{22}}\, a_2^{m_{12}-m_{11}}\, a_{12}^{m_{22}}\, |0\rangle \ .$$

In this case the Gelfand and Weyl states are the same, but in general the Gelfand state will be a linear combination of the Weyl states. These expansions have been studied extensively for the unitary groups [10,32,33].

§2. Homogeneous Polynomials

The boson calculus can also be described in the following way, in which the carrier space is constructed from homogeneous polynomials.

We can define a representation T of U(n) in a function space R, consisting of functions f defined on n-dimensional complex variables z, by

$$Tg\ f(z) = f(zg) \qquad (10)$$

where $g \in U(n)$.

This reducible representation will be irreducible if we restrict f to lie in the subspace R^ℓ of polynomials homogeneous of degree ℓ in z i.e. such that $f(\lambda z) = \lambda^\ell f(z)$ or equivalently

$$N\ f(z) = z_p \frac{\partial}{\partial z_p} f(z) = \ell\ f(z) \ .$$

The space R^ℓ is not large enough to carry all representations of U(n). We enlarge the space by introducing more variables so that a function f is defined on a set of variables z^σ, where $\sigma = 1, \ldots \lambda$. f(z) is now a polynomial homogeneous of degree ℓ_σ in z_i^σ , for $\sigma = 1, \ldots \lambda$, and belongs to the irreducible space

$$R^{(\ell_\lambda)} = R^{(\ell_1,\ell_2,..\ell_\lambda)} .$$

Tg is defined in $R^{(\ell_\lambda)}$ by (10), but where z now stands collectively for λn variables z_i^σ.

Given $f(z) \in R^{(\ell_\lambda)}$ we can obtain a new polynomial in $R^{(\ell_\lambda)}$ which is of degree $\ell_\sigma + 1$ in z^σ, namely $z_i^\sigma f(z)$. This process is simply that of applying a creation operator a_i^σ to an arbitrary state, hence we make the association

$$a_i^\sigma = z_i^\sigma \quad . \tag{11}$$

Similarly to each $f(z) \in R^{(\ell_\lambda)}$ there corresponds a polynomial of of degree $\ell_\sigma - 1$ in z^σ, which is $\frac{\partial}{\partial z_i^\sigma} f(z)$. The operator $\frac{\partial}{\partial z_i^\sigma}$ has the effect of a destruction operator (removing a particle) and so we write

$$\bar{a}_i^\sigma = \frac{\partial}{\partial z_i^\sigma} \quad . \tag{11'}$$

The operators z_i^σ and $\frac{\partial}{\partial z_j^\tau}$ obey the commutation relations (1) of boson operators. In this way we introduce bosons and the method (10) of constructing representations is an equivalent view of the boson calculus. The basis states, homogeneous polynomials, can be equally well regarded as tensors or multivectors constructed from the vectors z_i^σ. The generators become differential operators

$$E_{ij} = z_i^p \frac{\partial}{\partial z_j^p} \quad . \tag{12}$$

This can be seen either by substituting for (5) with (11), or by calculating E_{ij} from (10). In this calculation we put $g = 1 + te_{ij}$ where e_{ij} is an $n \times n$ matrix with elements $(e_{ij})_{k\ell} = \delta_{ik}\delta_{j\ell}$. Then

$$E_{ij} = \frac{d}{dt} f(zg)\bigg|_{t=0} = z_i^p \frac{\partial}{\partial z_j^p} \quad .$$

We have outlined two apparently different ways of obtaining the U(n) representations, both involving the boson calculus. Bargmann [15] has discussed the equivalence of the abstract operator approach carried out by Schwinger [17] for SU(2), and that using Hilbert spaces of homogeneous polynomials. He points out the characteristic differences, that in the first case the boson operators a_i^σ, $\bar{a}_j^\tau$ and their commutation relations are postulated. Other objects to be studied, such as the basis states (tensors) are defined in terms of these operators with the emphasis being on the construction of representations using the infinitesimal generators. In the second case the function space R is postulated and studied with the methods of analysis, with the representations defined directly on R. The space R with its differential operators can be regarded as a realization of the more abstractly defined system constructed from bosons.

§3. Scalar Product

By postulating the operators a_i^σ and their adjoints we have defined a scalar product. An arbitrary state has the form $f(a)\ |0\rangle$ where f is a function in the boson operators and $|0\rangle$ is the vacuum state. The scalar product (f,f') is then defined as

$$(f,f') = \langle 0|\ f(\bar{a})\ f'(a)\ |0\rangle \ . \tag{13}$$

Within R we need to define the scalar product so that $a_i = z_i$ is the adjoint of $\bar{a}_i = \frac{\partial}{\partial z_i}$. Bargmann [58,59] has studied this space and the scalar product in detail and has defined

$$(f,f') = \int_{\mathbb{C}_n} \bar{f}(z)\ f'(z)\ d\mu_n(z) \tag{14}$$

where

$$d\mu_n(z) = \pi^{-n} \exp(-\bar{z}\cdot z) \prod_k dx_k\, dy_k$$

$$(z_k = x_k + iy_k) .$$

We then have

$$(f, z_i f') = \left(\frac{\partial}{\partial z_i} f, f'\right) \qquad \text{as required.}$$

Alternatively, given a homogeneous polynomial $f(z) = \sum_h \alpha_h z^{[h]}$ we may put

$$f^*\left(\frac{\partial}{\partial z}\right) = \sum_h \bar{\alpha}_h \left(\frac{\partial}{\partial z}\right)^{[h]}$$

where $h = (h_1, .. h_n)$ and $z^{[h]} = z_1^{h_1} .. z_n^{h_n}$. Then we define the complex number

$$(f, f') = \left[f^* \left(\frac{\partial}{\partial z}\right) f'(z)\right]_{z=0} \qquad (15)$$

which is easily checked to satisfy all the requirements of a scalar product.

§4. Basis States

Having constructed a suitable representation space we now wish to calculate the Gelfand basis states, which is done as follows. Firstly we obtain the state |max.> which is of highest weight ie., the state which is an eigenvalue of the weight generators $H_i = E_{ii}$ in which the eigenvalues, or the weight, take the highest possible values. This will be the case [26] if E_α |max.> = 0 where E_α are all the raising generators of U(n). In an irreducible space |max.> is unique, by Cartan's theorem [26] (see also Zhelobenko [56] p12 Corr. 2). The state |max.> is also cyclic, i.e. by applying the generators E_{ij} of U(n) we obtain all other states in the space ([56] p12 Corr. 2). By applying suitable combinations of the E_{ij} we reach states which are the orthogonal Gelfand states. These combinations, called lowering and raising operators, have been calculated by Nagel and Moshinsky [27].

It is not necessary that our space be irreducible initially. Since U(n) is compact and therefore completely reducible the representation space can be written as the direct sum of irreducible spaces. Equivalently the space will contain several states of highest weight, one for each irreducible subspace. We select the most convenient of these states and obtain the basis vectors spanning the irreducible subspace by application of the E_{ij}. This is the case when we form the space $R^{\ell_1} \times R^{\ell_2} \times .. R^{\ell_\lambda}$ from which we choose the irreducible subspace $R^{(\ell_\lambda)} = R^{(\ell_1,..\ell_\lambda)}$. The E_{ij} cannot lead to states outside the space because the E_{ij} commute with the group invariants, the eigenvalues of which determine the representation and remain unchanged.

For U(n) the state of highest weight is

$$|max.\rangle = M^{-\frac{1}{2}} a_1^{m_1-m_2} a_{12}^{m_2-m_3} .. a_{12..n}^{m_n} |0\rangle \qquad (16)$$

where $m_1..m_n$ are the representation labels and M the normalization, which can be calculated from (13) as has been done by Biedenharn and Ciftan [32]. To show that (16) is of highest weight we need to show that

$$E_{ij} |max.\rangle = 0 \qquad \text{for } i < j . \qquad (17)$$

To check this it is sufficient to show only that

$$E_{i,i+1} |max.\rangle = 0 \qquad (18)$$

because E_{ij} can be written as the repeated commutator of $E_{i,i+1}$, $E_{i+1,i+2}$ .. $E_{j-1,j}$. This is so because $E_{i\ i+1}$ corresponds to the simple root $e_i - e_{i+1}$ from which all other roots can be obtained by addition. Now (16) belongs to an irreducible space (it is a Weyl state), and since there is only one solution of (17) in an irreducible space, by Cartan's theorem we find that (16) is the required state of highest weight. By applying the diagonal generators H_i to (16) we find that the numbers m_i, which must satisfy $m_1 \geqslant m_2 \geqslant .. \geqslant m_n \geqslant 0$, are not only the degrees of the polynomial $|max.\rangle$ but are also the actual representation labels.

An alternative method of finding (16) is to use the fact that the representation space $R^{(\ell_\lambda)}$ carries representations of not just U(n), but $U(n) \times U(n)$ where the second U(n) group is generated by

$$E^{\alpha\beta} = a_p^\alpha \bar{a}_p^\beta \quad , \quad \text{and} \quad [E_{ij}, E^{\alpha\beta}] = 0 \ . \qquad (19)$$

The only representations of $U(n) \times U(n)$ which the space carries are those in which the labels of each U(n) group are the same $(m_1,..m_n)$. This is because the invariants formed from E_{ij} and those from $E^{\alpha\beta}$ are the same when the substitutions (19) and (5) are made (Louck [5]). This direct product has been denoted $U(n) * U(n)$ [60], and the two groups referred to as complimentary [61].

The method of Moshinsky [24] is to note that an arbitrary basis state of U(n) generated by E_{ij} can be chosen to be of highest weight in U(n) generated by $E^{\alpha\beta}$ i.e., the polynomials f of $R^{(\ell_\lambda)}$ can be characterized as the solutions of the equations

$$\begin{aligned} E^{\alpha\alpha} f &= m_\alpha f \qquad \alpha = 1,\ldots n \ , \\ E^{\alpha\beta} f &= 0 \qquad \alpha < \beta \ . \end{aligned} \qquad (20)$$

Moshinsky has solved these partial differential equations [21] and by requiring that f be of highest weight in both U(n) groups has obtained (16). This method has been followed also in the treatment of O(n) [62] and Sp(n) [63]. However as a means of obtaining the state of highest weight this approach is unnecessarily complicated by comparison with that first outlined. We will use the simpler method to find states of highest weight for O(n) and Sp(n).

Moshinsky [24] has distinguished between his approach (carried out first by Wigner [38] for O(3)) using homogeneous polynomials, and that of Weyl using tensors of definite symmetries. However when we construct tensors with bosons, with the associations (11) the two methods become isomorphic. We will regard the approaches as equivalent and use the methods of each as convenient. This situation will appear also for O(n) and Sp(n) e.g. for O(3) the basis states will be regarded both as symmetric traceless tensors and as harmonic homogeneous polynomials (spherical harmonics).

With the state $|\mathrm{max.}\rangle$ known we can readily calculate basis states. This has been done for U(2) and SU(3) by Baird and Biedenharn [10] :

for U(2)

$$|(m)\rangle = \left|\begin{matrix} m_{12} & m_{22} \\ m_{11} & \end{matrix}\right\rangle = M^{-\frac{1}{2}} \, a_{12}{}^{m_{22}} \, a_1{}^{m_{11}-m_{22}} \, a_2{}^{m_{12}-m_{11}} \, |0\rangle \tag{21}$$

where

$$M = \frac{(m_{11} - m_{22})! \, (m_{12} - m_{11})!}{(m_{12} + 1)! \, m_{22}! \, (m_{12} - m_{22} + 1)!}$$

and for SU(3)

$$\left|\begin{matrix} m_{13} & m_{23} & 0 \\ m_{12} & m_{22} & \\ m_{11} & & \end{matrix}\right\rangle = N \; a_{12}{}^{m_{22}} \, a_{13}{}^{m_{23}-m_{22}} \, a_1{}^{m_{11}-m_{23}} \times a_2{}^{m_{12}-m_{11}} \, a_3{}^{m_{13}-m_{12}} \times {}_2F_1\left(m_{22} - m_{23},\, m_{11} - m_{12},\, m_{11} - m_{23} + 1;\, \frac{a_1\, a_{23}}{a_2\, a_{13}}\right)$$

where N is known. (22)

From the normalized basis states it is possible to calculate the matrix elements of the group generators, and this has been done for U(2) and SU(3). It is possible [10] also to calculate matrix elements for arbitrary n, and then the original results of Gelfand and Zetlin [28] are obtained.

The form of the Gelfand basis states for other unitary groups has been investigated ([29], [33], [32]). The expressions obtained are of interest in special function theory because of the many properties of special functions which can be derived from group theory ([49], [48], [47]).

CHAPTER 3

MODIFIED BOSON OPERATORS

§1. ρ-Orthogonal Groups

The orthogonal group O(n) in n dimensions is defined as the set of $n \times n$ matrices g such that $gg^t = I$. If g acts on a space of vectors z (denoting a complex or real row n-vector) then we may equally well define an orthogonal matrix as one leaving invariant the quadratic form zz^t. More generally we define the set G(n) of ρ-orthogonal matrices as those leaving invariant the general form $z\rho z^t$ where ρ is an $n \times n$ nonsingular matrix. These ρ-orthogonal matrices g must then satisfy $g\rho g^t = \rho$. If ρ is symmetric we obtain the orthogonal group O(n), and if ρ is antisymmetric we have the symplectic group Sp(n). In the latter case the requirement that $z\rho z^t$ be a nondegenerate bilinear form restricts n to even values. We can accommodate both choices for ρ by choosing the symmetry condition $\rho_{ij} = \eta\rho_{ji}$ where $\eta = \pm 1$. In addition we assume $\rho\rho^t = I$. We will be concerned with the compact subgroups of U(n) but our approach will still be useful in obtaining finite-dimensional (non-unitary) representations for the non-compact groups.

We wish to develop a boson calculus for O(n) and Sp(n). To do this we approach the problem as for the unitary group U(n) i.e. we take repeated direct products of the n-dimensional carrier space A of the defining representation to form the reducible space $B = A^{(1)} \times A^{(2)} \times .. A^{(\lambda)}$. The Wigner-Stone theorem ([35] Vol. II, p2) then shows that all irreducible unitary representations of G(n) can be extracted from B. B is decomposed by application of the Young symmetrizer which projects B into the subspace defined by the corresponding Young tableau. This process is that of decomposing an arbitrary tensor into symmetry types as described in Chpt II, §1. For U(n) this decomposition is sufficient i.e., the various subspaces are irreducible. The boson calculus is introduced by realizing each space $A^{(\sigma)}$ with boson operators.

§2. Traceless Tensors

For G(n) B must be decomposed further because of the appearance of the metric ρ. From each tensor $T_{i_1..i_r}$ of rank r we may form another tensor of rank r-2 by contraction with ρ:

$$T_{i_1..i_r} \to \rho_{pq}\, T_{i_1..p..q..i_r} .$$

It was observed by Weyl [12] that the operation of contraction (taking the trace) of tensors belonging to B commutes with the ρ-orthogonal transformations. If T' is the transform of T under G(n) then

$$T'_{i_1..i_r} = T_{p_1..p_r}\, g_{p_1 i_1} .. g_{p_r i_r} .$$

Contracting, for example over the first two indices, we have

$$\begin{aligned}\rho_{pq}\, T'_{pqi_3..\, i_r} &= T_{p_1..p_r}\, g_{p_1 p}\, g_{p_2 q}\, g_{p_3 i_3} .. g_{p_r i_r}\, \rho_{pq} \\ &= T_{p_1..p_r}\, (g\rho g^t)_{p_1 p_2} .. g_{p_r i_r} \\ &= \rho_{p_1 p_2}\, T_{p_1 p_2 .. p_r}\, g_{p_3 i_3} .. g_{p_r i_r} \\ &= [\rho_{pq}\, T_{pqi_3..i_r}]' .\end{aligned}$$

We see now that the subspace of tensors of zero trace is invariant. In order to obtain an irreducible representation space we must start from the subspace of traceless tensors and apply the

Young symmetrizer to obtain traceless tensors of a given symmetry type. A tensor $T_{i_1..i_r}$ can be decomposed uniquely into a traceless tensor T^o plus a tensor of the form ([50] p392):

$$F_{i_1..i_r} = \rho_{i_1 i_2} G^{(12)}_{i_3..i_r} + .. + \rho_{i_\alpha i_\beta} G^{(\alpha\beta)}_{i_1..i_{\alpha-1} i_{\alpha+1}..i_{\beta-1} i_{\beta+1}..i_r}$$

$$\left(\frac{r(r-1)}{2} \text{ terms}\right) .$$

Hence the traceless part of a tensor can be written

$$T^o_{i_1..i_r} = T_{i_1..i_r} - F_{i_1..i_r} . \qquad (1)$$

Our problem is to project from T to T^o.

The requirement that a tensor be both traceless and of a given symmetry is a strong condition and it has been shown ([12,50]) that traceless tensors of certain symmetries are identically zero, i.e., some Young diagrams are not admissible. For O(n) we have that traceless tensors corresponding to Young diagrams in which the sum of the lengths of the first two columns exceeds n must be identically zero. For Sp(n) traceless tensors corresponding to Young diagrams in which there are more than $\nu = \frac{n}{2}$ rows are identically zero.

If we realize B with boson operators as before we could attempt to reduce B by projecting out the traceless part of products of boson operators. This is the method previously used for obtaining an irreducible representation space, but one which is unnecessarily complicated. The complications arise because functions in the representation space appear in the expanded form (1).

A simpler method is to realize B with operators chosen so that B is immediately traceless. In this method we realize each space $A^{(\sigma)}$ with a set of n operators a_i^σ which behave as vectors under ρ-orthogonal transformations and which commute with each other. An arbitrary tensor $T_{i_1..i_\lambda}$ in B then is the sum of terms such as $a^1_{i_1}..a^\lambda_{i_\lambda}$. For T to be traceless i.e., $\rho_{pq}\ T_{i_1..p..q..i_\lambda} = 0$ we require that the operators a^σ_i should satisfy

$$\rho_{pq}\ a^\sigma_p\ a^\tau_q = 0 \qquad \text{for arbitrary } \sigma,\ \tau. \tag{2}$$

Clearly these operators cannot be ordinary boson operators. The complexity which was present in (1) is now absorbed into the operators a^σ_i which then, as will be seen, do not have all the simple properties of bosons.

Having realized B as a space of traceless tensors we apply the Young symmetrizer to project into irreducible subspaces, as before. In order to see how the operators a^σ_i may be defined and to reveal their properties, we consider the problem with the following equivalent approach.

§3. Harmonic Spaces

Beginning as for U(n) before we define a representation T of $g \in G(n)$ in the space R^ℓ by

$$Tg\ f(z) = f(zg) \qquad \text{where} \quad f(z) \in R^\ell . \tag{3}$$

We consider initially only one set of variables z_i so that the tensors of the representation space are all symmetric. Hence R^ℓ is the space of homogeneous polynomials of degree ℓ in z. Now although R^ℓ is irreducible under unitary transformations it becomes reducible under transformations of G(n). The subspace $(z,z)R^{\ell-2}$ is invariant because $(z,z) = \rho_{pq}\ z_p\ z_q$ remains invariant under G(n). Let H^ℓ denote the orthogonal complement of $(z,z)R^{\ell-2}$ in R^ℓ. Then if $h^\ell \in H^\ell$ we have $(h^\ell,\ (z,z)\ f^{\ell-2}) = 0$ for all $f^{\ell-2} \in R^{\ell-2}$, where (,) denotes the

the scalar product in R^{ℓ} (Chpt II §3). Therefore $(\nabla^2 h^{\ell}, f^{\ell-2}) = 0$ where

$$\nabla^2 = \rho_{pq} \frac{\partial^2}{\partial z_p \, \partial z_q} ,$$

since the adjoint of z_i in R^{ℓ} is $\frac{\partial}{\partial z_i}$. $f^{\ell-2}$ is arbitrary therefore we have $\nabla^2 h^{\ell} = 0$, i.e., H^{ℓ} is characterized as the subspace of R^{ℓ} consisting of harmonic polynomials. We can write R^{ℓ} as a direct sum ([64], p129):

$$R^{\ell} = H^{\ell} \oplus (z,z)\, R^{\ell-2} . \tag{4}$$

This result has previously been noted by Vilenkin ([49], p444).

As $(z,z)R^{\ell-2}$ is still reducible it is clear that H^{ℓ} is the irreducible space we are seeking. We can see directly that the space H^{ℓ} is invariant under $G(n)$ because the operator ∇^2 is invariant under $G(n)$.

More generally we need to consider polynomials of several sets of variables. We form the space $R^{\ell_1} \times R^{\ell_2} \, .. \times R^{\ell_\lambda}$ which then consists of polynomials homogeneous of degree ℓ_σ in z_i^{σ}, for $\sigma = 1,.. \lambda$. The representation T is defined as before by (3) but now z stands collectively for all variables z_i^{σ}. Now the subspace is

$$\bigcup_{\alpha,\beta=1}^{\lambda} (z^{\alpha},z^{\beta})R^{\ell_1} \times .. R^{\ell_\alpha -1} \times .. R^{\ell_\beta -1} \times ... R^{\ell_\lambda}$$

invariant under $G(n)$ because $(z^{\alpha},z^{\beta}) = \rho_{pq}\, z_p^{\alpha}\, z_q^{\beta}$ is invariant under $G(n)$. If

$$h^{(\ell_\lambda)} \in H^{(\ell_\lambda)} = H^{(\ell_1,\ell_2,..\ell_\lambda)} ,$$

defined as the orthogonal complement of

$$\bigcup_{\alpha,\beta=1}^{\lambda} (z^{\alpha},z^{\beta})R^{\ell_1} \times .. R^{\ell_\lambda} ,$$

then $(h^{(\ell_\lambda)}, (z^{\alpha},z^{\beta})f) = 0$ for all $\alpha,\beta = 1,..\lambda$ and for all $f \in R^{\ell_1} \times .. R^{\ell_\lambda}$. As before we find that $h^{(\ell_\lambda)}$ satisfies $\nabla^2_{\alpha\beta}\, h^{(\ell_\lambda)} = 0$

for $\alpha,\beta = 1,..\lambda$ where $\nabla^2_{\alpha\beta} = \rho_{pq} \frac{\partial^2}{\partial z^\alpha_p \, \partial z^\beta_q}$. We may write the decomposition of the space as

$$R^{\ell_1} \times .. R^{\ell_\lambda} = H^{(\ell_\lambda)} \oplus \bigcup_{\alpha,\beta=1}^{\lambda} (z^\alpha, z^\beta) R^{\ell_1} \times .. R^{\ell_\alpha - 1} \times .. R^{\ell_\beta - 1} \times .. R^{\ell_\lambda} . \tag{5}$$

The irreducible space $H^{(\ell_\lambda)}$ with which we will carry irreducible representations of G(n) may be characterized as the space of harmonic homogeneous polynomials or equivalently as traceless tensors of definite symmetry type. This fact has been known and used in most treatments of O(n) and Sp(n) ([24,48]). Working as for U(n) we write down the state of highest weight in the harmonic space. This state is cyclic so that one can generate all the basis states by application of the group generators. For O(3) this leads to the familiar spherical harmonic functions ([21], [47] Chpt 2). These functions can also be regarded as traceless tensors but because they are constructed with ordinary bosons they appear in a form unnecessarily complicated.

The state of highest weight for O(n) has been calculated by Wong [53]. All states for symmetric representations (using only symmetric tensors) have been calculated ([51], [65]). Holman [52] has calculated basis states for Sp(4), but the attempt to project out the traceless part of tensors is incomplete. Nevertheless the results are correct because the state of highest weight is chosen correctly. Again, these calculations are unnecessarily complicated and the structure of the results is clarified by the use of operators satisfying (2).

§4. Modified Bosons

We need to find operators a^σ_i and their adjoints $\bar{a}^\sigma_i$ which behave as vectors under G(n) and which satisfy $\rho_{pq} \, a^\sigma_p \, a^\tau_q = 0$. By forming polynomials from these a_i acting on the vacuum $|0\rangle$ we will obtain traceless tensors or equivalently harmonic polynomials. If $h^{(\ell_\lambda)} \in H^{(\ell_\lambda)}$ we require that $a_i^\sigma h^{(\ell_\lambda)} \in H^{(\ell_1,..\ell_\sigma+1,..\ell_\lambda)}$ and $\bar{a}_i^\sigma h^{(\ell_\lambda)} \in H^{(\ell_1,..\ell_\sigma-1,..\ell_\lambda)}$.

Consider firstly only symmetric representations i.e., $\ell_2,..\ell_\lambda = 0$. For U(n) we have $a_i = z_i$, $\bar{a}_i = \frac{\partial}{\partial z_i}$. For the ρ-orthogonal groups G(n) we find that the annihilation operator is unchanged since

$$\frac{\partial}{\partial z_i} h^\ell \in H^{\ell-1} \qquad \text{if} \quad h^\ell \in H^\ell .$$

Hence we put $\bar{a}_i = \frac{\partial}{\partial z_i}$. However $z_i h^\ell \notin H^{\ell+1}$ because $\nabla^2 z_i h^\ell \neq 0$. Now from (4) we see that

$$z_i h^\ell = g_i^{\ell+1} + (z,z) f_i^{\ell-1}$$

where

$$g_i^{\ell+1} \in H^{\ell+1} \quad \text{and} \quad f_i^{\ell-1} \in R^{\ell-1} .$$

Clearly $g_i^{\ell+1}$ is the polynomial we require by applying a_i to h^ℓ i.e., $g_i^{\ell+1} = a_i h^\ell$. a_i must then have the form $a_i = z_i - (z,z)A_i$ for some operator A_i. The requirement $a_i h^\ell \in H^{\ell+1}$ enables us to find that

$$A_i = 2[\nabla^2,(z,z)]\ \rho_{ip} \frac{\partial}{\partial z_p}$$

(noting that $[\nabla^2, [\nabla^2,(z,z)]] = 0$ in H^ℓ).

Hence

$$a_i = (1 - (z,z)[\nabla^2,(z,z)]^{-1} \nabla^2) z_i$$

$$= z_i - (z,z)(n + 2N)^{-1} \rho_{ip} \frac{\partial}{\partial z_p} \qquad (6)$$

together with $\bar{a}_i = \frac{\partial}{\partial z_i}$ (6'). These operators, which have been encountered by Vilenkin ([49] p442), are modified from the ordinary boson operators and so we call them "modified boson operators". We will see they have all the properties we require of them. The following commutation relations are readily checked:

$$[\bar{a}_i, \bar{a}_j] = 0 \qquad (7)$$

and hence

$$[a_i, a_j] = 0$$

and

$$[\bar{a}_i, a_j] = \delta_{ij} - \rho_{ip}\, a_p \left(\frac{n}{2} + N\right)^{-1} \rho_{jq}\, \bar{a}_q \,.$$

We could define modified bosons by these relations or by their explicit realizations (6). We will use either definition as convenient. The difference in approach is that discussed earlier (Chap. II, §2) between the abstract system constructed with bosons and the explicit realization in a Hilbert space of polynomials. Now if we define the unique vacuum by $\bar{a}_i\,|0\rangle = 0$ for all $i=1,..n$ then it follows from (7) that $\bar{a}_i\, a_j\,|0\rangle = \delta_{ij}\,|0\rangle$ and then also $\bar{a}_i\,\rho_{pq}\, a_p\, a_q\,|0\rangle = 0$. From the uniqueness of the vacuum we have $\rho_{pq}\, a_p\, a_q\,|0\rangle = K\,|0\rangle$ for some constant K. An arbitrary state $|\ell\rangle$ can be written as the sum of products of ℓ creation operators and therefore we have

$$\rho_{pq}\, a_p\, a_q\,|\ell\rangle = K\,|\ell\rangle \qquad \text{i.e.,}$$

$$\rho_{pq}\, a_p\, a_q = K\,.$$

Hence

$$\rho_{pq}\, \bar{a}_p\, \bar{a}_q = \bar{K}$$

and then

$$\rho_{pq}\, \bar{a}_p\, \bar{a}_q\,|0\rangle = \bar{K}\,|0\rangle = 0$$

so that we must have $K = 0$ i.e.,

$$\rho_{pq}\, a_p\, a_q = 0 \,.$$

This is the required traceless condition (2). The adjoint relation $\rho_{pq}\, \bar{a}_p\, \bar{a}_q = 0$ is the harmonic condition on our space as can be seen by substituting (6′). For symmetric representations (one row in the Young pattern) we use the operators defined by (7) to obtain states of irreducible representations of O(n). For Sp(n) (when ρ is anti-symmetric) bosons are sufficient because $\rho_{pq}\, a_p\, a_q = 0$ is satisfied without modification.

In order to obtain other representations we require more variables with which to construct tensors of the various symmetries and consequently more operators a_i^σ, $\sigma = 1,..\lambda$. We require that a_i^σ, $\bar{a}_i^\sigma$ satisfy $\nabla^2_{\alpha\beta}\, a_i^\sigma\, h^{(\ell_\lambda)} = 0$ and $\nabla^2_{\alpha\beta}\, \bar{a}_i^\sigma\, h^{(\ell_\lambda)} = 0$ for $\alpha,\beta = 1,..\lambda$. The annihilation operator remains unchanged so that

$$\bar{a}_i^\sigma = \frac{\partial}{\partial z_i^\sigma} \,.$$

The creation operator must have the form, using (5)

$$a_i^\sigma = z_i^\sigma - (z^\alpha, z^\beta)\, A_{\alpha\beta}(\sigma)$$

(α,β summed)

where $A_{\alpha\beta}(\sigma)$ is an operator to be determined. We require

$$\nabla^2_{\gamma\varepsilon}[\, z_i^\sigma - (z^\alpha,z^\beta)\, A_{\alpha\beta}(\sigma)]\; h^{(\ell_\lambda)} = 0$$

i.e.,

$$[\,\nabla^2_{\gamma\varepsilon},\, z_i^\sigma] = [\,\nabla^2_{\gamma\varepsilon},\, (z^\alpha,z^\beta)]\; A_{\alpha\beta}(\sigma)$$

provided

$$[\,\nabla^2_{\gamma\varepsilon}, A_{\alpha\beta}(\sigma)] = 0 \,. \qquad (8)$$

Let

$$\Delta(\lambda)_{(\gamma\varepsilon)(\alpha\beta)} = [\nabla^2_{\gamma\varepsilon}, (z^\alpha, z^\beta)] \quad . \tag{9}$$

Then Δ is a $\lambda^2 \times \lambda^2$ matrix of operators satisfying the symmetries

$$\Delta_{(\gamma\varepsilon)(\alpha\beta)} = \eta\, \Delta_{(\varepsilon\gamma)(\alpha\beta)} = \eta\, \Delta_{(\gamma\varepsilon)(\beta\alpha)} = \Delta_{(\varepsilon\gamma)(\beta\alpha)} \; .$$

We have now

$$A_{\alpha\beta}(\sigma) = \Delta^{-1}(\lambda)_{(\alpha\beta)(\gamma\varepsilon)} [\nabla^2_{\gamma\varepsilon}, z^\sigma_i]$$

$$= 2\, \Delta^{-1}(\lambda)_{(\alpha\beta)(\sigma\gamma)}\, \rho_{ip} \frac{\partial}{\partial z^\gamma_p} \quad ,$$

where $\Delta^{-1}(\lambda)$ is defined by

$$\Delta^{-1}{}_{(\mu\nu)(\alpha\beta)}\, \Delta_{(\alpha\beta)(\sigma\tau)} = \delta_{(\mu\nu)(\sigma\tau)} \tag{10}$$

$$= \tfrac{1}{2}(\delta_{\mu\sigma}\delta_{\nu\tau} + \eta\, \delta_{\mu\tau}\delta_{\nu\sigma}) \; .$$

Now

$$\Delta_{(\mu\nu)(\sigma\tau)} = 2[n\, \delta_{(\mu\nu)(\sigma\tau)} + P^{\alpha\mu}\, \delta_{(\alpha\nu)(\sigma\tau)} + P^{\alpha\nu}\, \delta_{(\mu\alpha)(\sigma\tau)}] \tag{11}$$

where

$$P^{\mu\nu} = z^\mu_p \frac{\partial}{\partial z^\nu_p}$$

and satisfies

$$[\nabla^2_{\sigma\tau}, P^{\mu\nu}] = 0$$

within $H^{(\ell_\lambda)}$. Hence the condition (8) is satisfied.

We now have for our operators, which depend on the number of rows λ,

$$a^\sigma_i(\lambda) = [1 - (z^\alpha, z^\beta)\Delta^{-1}_{(\alpha\beta)(\gamma\varepsilon)}\nabla^2_{\gamma\varepsilon}]\; z^\sigma_i$$

$$= z^\sigma_i - 2(z^\alpha, z^\beta)\Delta^{-1}_{(\alpha\beta)(\sigma\gamma)}\, \rho_{ip} \frac{\partial}{\partial z^\gamma_p} \quad , \tag{12}$$

$$\bar{a}^\sigma_i = \frac{\partial}{\partial z^\sigma_i} \; .$$

These operators satisfy

$$[\bar{a}^\sigma_i, \bar{a}^\tau_j] = 0$$

and therefore

$$[a_i^\sigma , a_j^\tau] = 0 \ . \qquad (13)$$

We also have

$$[\bar{a}_i^\sigma , a_j^\tau] = \delta_{ij}\delta^{\sigma\tau} - 4 \rho_{ip} a_p^\alpha \Delta^{-1}_{(\alpha\sigma)(\tau\beta)}\rho_{jq} \bar{a}_q^\beta$$

where Δ^{-1} is expressed implicitly in terms of the a's according to (10) and (11) (noting that $P^{\mu\nu} = a_p^\mu \bar{a}_p^\nu$). To verify these relations we need to calculate

$$[\frac{\partial}{\partial z_i^\sigma} , \Delta^{-1}_{(\alpha\beta)(\tau\gamma)}]$$

which is done by using (10):

$$[\frac{\partial}{\partial z_i^\sigma} , \Delta^{-1}]\Delta = - \Delta^{-1} [\frac{\partial}{\partial z_i^\sigma} , \Delta] \ .$$

The relations (13) are the defining relations for our operators when considering λ rows. We can show that these modified boson operators satisfy the required traceless conditions in the following way. We define the unique vacuum state by $\bar{a}_i^\sigma \ |0\rangle = 0$ for all i, σ. We have immediately $\bar{a}_i^\sigma a_j^\tau \ |0\rangle = \delta_{ij}\delta^{\sigma\tau} \ |0\rangle$ and then also $\bar{a}_i^\mu \rho_{pq} a_p^\sigma a_q^\tau \ |0\rangle = 0$. For this we need to know

$$\Delta^{-1}_{(\mu\nu)(\sigma\tau)} \ |0\rangle = \frac{1}{2n} \delta_{(\mu\nu)(\sigma\tau)} \ |0\rangle$$

calculated from $\Delta^{-1} \Delta \ |0\rangle = I \ |0\rangle$. In the same way as before, for one row, we obtain the result that our operators obey the traceless condition $\rho_{pq} a_p^\sigma a_q^\tau = 0$.

§5. Determination of Δ^{-1}

We have defined Δ^{-1} formally by (9) and (10) which are the means by which we can manipulate Δ^{-1}. However the appearance of negative powers of operators requires some explanation and to do this we need to know Δ^{-1} explicitly. This $\lambda^2 \times \lambda^2$ matrix of operators is complicated even for $\lambda = 2$. We have in this case (putting $\rho = I$):

$$[\bar{a}_i, a_j] = \delta_{ij} - a_i\left(\frac{n}{2} + N\right)^{-1}\bar{a}_j$$
$$- [b_i - a_i\left(\frac{n}{2} + N\right)^{-1}Q]\ \Omega^{-1}[\bar{b}_j - P\left(\frac{n}{2} + N\right)^{-1}\bar{a}_j] \ , \qquad (14)$$

$$[\bar{b}_i, a_j] = - [a_i - b_i\left(\frac{n}{2} + M\right)^{-1}P]\ \Omega^{-1}[\bar{b}_j - P\left(\frac{n}{2} + N\right)^{-1}\bar{a}_j]$$

where

$$a_i = a_i^{(1)} \ , \ b_i = a_i^{(2)}$$

and

$$\Omega = n + N + M - Q\left(\frac{n}{2} + M\right)^{-1}P - P\left(\frac{n}{2} + N\right)^{-1}Q$$

and

$$N = P^{11}, \quad M = P^{22}, \quad P = P^{12}, \quad Q = P^{21} \ .$$

The expressions for $[\bar{b}_i, b_j]$ and $[\bar{a}_i, b_j]$ are readily deduced from (14).

We will find that fortunately it will not be necessary to know the explicit form of Δ^{-1} in calculations of basis states. Our method of calculating these states is to apply suitable lowering operators, which are functions of the generators, to the state of highest weight $|max.\rangle$. Modified bosons are vectors under the group generators, as will be shown, and so properties of Δ^{-1} are not needed in the process of applying lowering operators. Furthermore modified bosons appear in the state of highest weight in such a way that the contribution of terms involving Δ^{-1} cancel out (to be shown). However properties of Δ^{-1} may need to be known in calculating normalizations when the commutation relations are involved.

Let us outline a recursive method for the calculation of Δ^{-1}. The method is to form the space $H^{(\ell_\lambda)}$ by a means other than decomposing $R^{\ell_1} \times .. R^{\ell_\lambda}$ as was done earlier. Suppose we have already formed $H^{(\ell_{\lambda-1})}$ with modified bosons $a_i^\sigma(\lambda - 1)$ i.e., we have formed a space which is irreducible when $\lambda - 1$ rows are considered in the Young pattern. We now form $H^{(\ell_\lambda)} = H^{(\ell_1,\ell_2..\ell_\lambda)}$ by decomposing

$H^{(\ell_{\lambda-1})} \times H^{\ell_\lambda}$ as

$$H^{(\ell_\lambda)} \oplus \bigcup_{\alpha=1}^{\lambda-1} (a^\alpha(\lambda-1),\ a^\lambda(1))\ H^{(\ell_1..\ell_\alpha-1,..\ell_{\lambda-1})} \times H^{\ell_\lambda-1}$$

Hence $a_i^\sigma(\lambda)$ has the form

$$a_i^\sigma(\lambda) = a_i^\sigma(\lambda-1) - (a^\alpha(\lambda-1), a^\lambda(\lambda-1))A_\alpha(i,\sigma)$$

for $\sigma = 1, .. \lambda$.

Here α is summed from 1 to $\lambda - 1$, and we have defined $a_i^\lambda(\lambda-1) = a_i^\lambda(1)$, and A_α is to be determined. We require that $a_i^\sigma(\lambda)$ satisfy $\nabla^2_{\lambda\alpha}\ a_i^\sigma(\lambda)\ h^{(\ell_\lambda)} = 0$ for $h^{(\ell_\lambda)} \in H^{(\ell_\lambda)}$, for $\alpha = 1,..\lambda - 1$ (the condition $\nabla^2_{\alpha\beta}\ a_i^\sigma(\lambda)\ h^{(\ell_\lambda)} = 0$ holds already for $\alpha,\beta < \lambda$ and for $\alpha = \beta = \lambda$ by construction). Hence we require

$$[\nabla^2_{\lambda\beta},\ a_i^\sigma(\lambda-1)] = [\nabla^2_{\lambda\beta},\ (a^\alpha(\lambda-1),\ a^\lambda(\lambda-1))]\,A_\alpha$$

provided

$$[\nabla^2_{\lambda\beta},\ A_\alpha] = 0\ .$$

Let

$$\Omega_{(\beta\alpha)} = [\nabla^2_{\lambda\beta},\ (a^\alpha(\lambda-1),\ a^\lambda(\lambda-1))]\ ,$$

a $\lambda-1 \times \lambda-1$ matrix of operators. Then

$$A_\alpha = \Omega^{-1}_{(\alpha\beta)}\ [\nabla^2_{\lambda\beta},\ a_i^\sigma(\lambda-1)]$$

where

$$\Omega^{-1}_{(\gamma\beta)}\ \Omega_{(\beta\alpha)} = \delta_{\gamma\alpha}\ .$$

Hence

$$a_i^\sigma(\lambda) = a_i^\sigma(\lambda-1) - (a^\alpha(\lambda-1),\ a^\lambda(\lambda-1))\Omega^{-1}_{(\alpha\beta)}\ [\nabla^2_{\lambda\beta},\ a_i^\sigma(\lambda-1)]$$

$$= [1 - (a^\alpha(\lambda-1),\ a^\lambda(\lambda-1))\Omega^{-1}_{(\alpha\beta)}\ \nabla^2_{\lambda\beta}]\ a_i^\sigma(\lambda-1)\ .$$

This expression involves $\Delta^{-1}(\lambda-1)$ where it appears in $a_i^\sigma(\lambda-1)$. By comparison with (12), which involves $\Delta^{-1}(\lambda)$, we can obtain a recursive expression for $\Delta^{-1}(\lambda)$, in terms of $\Delta^{-1}(\lambda-1)$.

Let us carry out this process explicitly for $\lambda = 2$ (putting $\rho = I$). We have two sets of modified bosons $a_i(1) = a_i^1(1)$, $b_i(1) = a_i^2(1)$ for one row each. We form $H^{(\ell_1,\ell_2)}$ from H^{ℓ_1} and H^{ℓ_2}:

$$H^{\ell_1} \times H^{\ell_2} = H^{(\ell_1,\ell_2)} \oplus (a(1),b(1))\, H^{\ell_1-1} \times H^{\ell_2-1} .$$

Now $a_i^\sigma(2) = a_i^\sigma(1) - (a(1),b(1))\, A(i,\sigma)$ must satisfy

$$\nabla_{12}^2\, a_i^\sigma(2)\, h^{(\ell_1,\ell_2)} = 0 .$$

Let

$$\Omega = \Omega_{(11)} = [\nabla_{12}^2,\ (a(1),b(1))]$$

$$= n + N + M - Q\left(\frac{n}{2} + M\right)^{-1}P - P\left(\frac{n}{2} + N\right)^{-1}Q$$

where

$$N = P^{11},\quad M = P^{22},\quad P = P^{12},\quad Q = P^{21}$$

and

$$P^{\mu\nu} = a_p^\mu(1)\,\frac{\partial}{\partial z_p^\nu} = z_p^\mu\,\frac{\partial}{\partial z_p^\nu}$$

in $H^{(\ell_1,\ell_2)}$. Putting $\sigma = 1$, we have

$$[\nabla_{12}^2,\ a_i(1)] = \frac{\partial}{\partial y_i} - P\left(\frac{n}{2} + N\right)^{-1}\frac{\partial}{\partial x_i}$$

where

$$x_i = z_i^{(1)}\ ,\ y_i = z_i^{(2)} .$$

The requirement $[\nabla_{12}^2,\ A_\alpha] = 0$ is now easily seen to be satisfied.

We have now

$$a_i(2) = a_i(1) - (a(1),\ b(1))\Omega^{-1}\left(\frac{\partial}{\partial y_i} - P\left(\frac{n}{2} + N\right)^{-1}\frac{\partial}{\partial x_i}\right)$$

with

$$(a(1),b(1)) = (x,y) - (x,x)(n + 2N)^{-1}Q - (y,y)(n + 2M)^{-1}P .$$

Also

$$a_i(2) = x_i - 2(z^\alpha,z^\beta)\Delta^{-1}(2)_{(\alpha\beta)(1\gamma)}\frac{\partial}{\partial z_i^\gamma} .$$

By comparison we find that the independent components of $\Delta^{-1}(2)$ are (remembering that $\Delta(1)^{-1}_{(11)(11)} = \frac{1}{4}\left(\frac{n}{2} + N\right)^{-1}$):

$$\Delta(2)^{-1}{}_{(11)(11)} = \frac{1}{4}\left(\frac{n}{2} + N\right)^{-1}\left[1 + Q\,\Omega^{-1}\,P\left(\frac{n}{2} + N\right)^{-1}\right]$$

$$\Delta(2)^{-1}{}_{(11)(12)} = -\frac{1}{4}\left(\frac{n}{2} + N\right)^{-1} Q\,\Omega^{-1}$$

$$\Delta(2)^{-1}{}_{(12)(11)} = -\frac{1}{4}\,\Omega^{-1}\,P\left(\frac{n}{2} + N\right)^{-1}$$

$$\Delta(2)^{-1}{}_{(12)(12)} = \frac{1}{4}\,\Omega^{-1}$$

$$\Delta(2)^{-1}{}_{(22)(11)} = \frac{1}{4}\left(\frac{n}{2} + M\right)^{-1} P\,\Omega^{-1}\,P\left(\frac{n}{2} + N\right)^{-1}$$

$$\Delta(2)^{-1}{}_{(22)(12)} = -\frac{1}{4}\left(\frac{n}{2} + M\right)^{-1} P\,\Omega^{-1}\ .$$

With the explicit form of Δ^{-1} known we can see how to interpret the inverse operators which appear in the formal expressions. For $\lambda = 1$ there is no problem because the operator N is replaced by a non-negative number so that $\left(\frac{n}{2} + N\right)^{-1}$ is always well defined. For $\lambda = 2$ we need to understand the meaning of Ω^{-1}. We have

$$\Omega^{-1}\,h^{(\ell_1,\ell_2)} = \left(n + \ell_1 + \ell_2 - \frac{QP}{\frac{n}{2} + \ell_2 - 1} - \frac{PQ}{\frac{n}{2} + \ell_1 - 1}\right)^{-1} h^{(\ell_1,\ell_2)}$$

$$= \left[n + \ell_1 + \ell_2 + \frac{\ell_1 - \ell_2}{\frac{n}{2} + \ell_2 - 1} - \frac{(n + \ell_1 + \ell_2 - 2)}{(\frac{n}{2} + \ell_2 - 1)(\frac{n}{2} + \ell_1 - 1)}\,PQ\right]^{-1} h^{(\ell_1,\ell_2)} .$$

We expand this inverse as an infinite series to obtain the form

$$\Omega^{-1}\,h^{(\ell_1,\ell_2)} = \sum_m C_m\,(PQ)^m\,h^{(\ell_1,\ell_2)} \qquad (15)$$

for some coefficient C_m. Now $(PQ)^m$ is a differential operator acting on $h^{(\ell_1,\ell_2)}$ which is of finite degree. As a consequence for m large enough the differential operators give no contribution, so that an infinite series of differential operators with convergence problems does not in fact appear in (15), and Ω^{-1} is well defined. For arbitrary λ we need to understand the meaning of $\Omega^{-1}{}_{(\alpha\beta)}$ and this can be done in the same way as for Ω^{-1}.

Generally Δ^{-1} has an explicit form which is too complicated to be useful and in most cases we will be able to derive the necessary properties from the definitions (9) and (10).

Modified bosons enable us to write down traceless tensors which appear in a simple form, but the complexity of equation (1) has been absorbed into the structure of Δ^{-1} where it is more easily handled.

§6. Scalar Product

We have chosen the scalar product in $H^{(\ell_\lambda)}$ in such a way that the adjoint of a_i^σ is $\bar{a}_i^{\sigma}$. The scalar product is defined as

$$(h,h') = \langle 0|\; h(\bar{a})\; h'(a)\; |0\rangle \qquad (16)$$

where $h,\, h' \in H^{(\ell_\lambda)}$ are functions of modified bosons. When substituting for a_i^σ with the realizations (12) we find that the scalar product in $H^{(\ell_\lambda)}$ is simply that for $R^{(\ell_\lambda)}$ (see Chapt. II, §3) restricted to $H^{(\ell_\lambda)}$. Considered as a scalar product in $R^{(\ell_\lambda)}$ we have

$$\begin{aligned}(h,\, a_i^\sigma h') &= \left(h,\, \left[z_i - 2(z^\alpha,z^\beta)\Delta^{-1}{}_{(\alpha\beta)(\sigma\gamma)}\rho_{ip}\frac{\partial}{\partial z_p^\gamma}\right] h'\right) \\ &= \left(\left[\frac{\partial}{\partial z_i^\sigma} - 2\,\rho_{ip}\, z_p^\gamma\, \Delta^{-1}{}_{(\alpha\beta)(\sigma\gamma)}\nabla^2_{\alpha\beta}\right] h,h'\right) \\ &= \left(\frac{\partial}{\partial z_i^\sigma} h,h'\right) \qquad \text{because } \nabla^2_{\alpha\beta}\, h = 0 \\ &= (\bar{a}_i^{\sigma}\, h,h')\ .\end{aligned}$$

Here we have used the fact that Δ^{-1} is hermitean in $R^{(\ell_\lambda)}$.

Hence we may evaluate (h,h') in two ways. If $h,\, h' \in H^{(\ell_\lambda)}$ then h,h' may be written as functions of a_i^σ (modified bosons) and (h,h') is determined from (16) using the commutation relations (13). Alternatively we may regard h,h' as functions not of a_i^σ but z_i^σ and evaluate (h,h') as a scalar product in $R^{(\ell_\lambda)}$ using the commutation relations of ordinary bosons. This entails writing $h(a)$ as a different, generally more complicated function $f(z)$, a problem which we consider next.

§7. Construction of Traceless Tensors

Using modified bosons we can write down thetraceless part of a tensor constructed from elementary vectors z. Let $f(z_i^\sigma)$ denote such a tensor, then the traceless part of $f(z_i^\sigma)$ is $f(a_i^\sigma)$ where a_i^σ are modified bosons, This may be seen if we write down $f(a_i^\sigma)$ using the realization (12). In expanding

$$f(a_i^\sigma) = f(z_i^\sigma - 2(z^\alpha, z^\beta)\Delta^{-1}{}_{(\alpha\beta)(\sigma\gamma)}\rho_{ip} \frac{\partial}{\partial z_p^\gamma})$$

we find the leading term to be $f(z_i^\sigma)$. Clearly we have expressed a tensor $f(z_i^\sigma)$ as a sum of a traceless tensor $f(a_i^\sigma)$ plus a remainder, as in the form (1).

If $f(z) \in H^{(\ell_\lambda)}$ so that $f(z)$ is already a traceless tensor then $f(a)$ is the same tensor but expressed with a_i^σ which satisfy the traceless conditions. These extra conditions enable us to express $f(z)$ in a simpler form. For example the spherical harmonics are [65]

$$Y_{\ell m}(z) \propto \sum_k \frac{(-z_1 - iz_2)^{k+m}(z_1 - iz_2)^k \, z_3{}^{\ell-m-2k}}{2^{2k+m}\,(k+m)!\,k!(\ell-m-2k)!} \tag{17}$$

(z real).

Now $Y_{\ell m}(z) = Y_{\ell m}(a)$ where the a's are modified bosons satisfying $a_1^2 + a_2^2 + a_3^2 = 0$ (here $\rho_{ij} = \delta_{ij}$). Hence (17) is equal to

$$\sum_k \frac{(-a_1 - ia_2)^{k+m}\,(a_1 - ia_2)^k \, a_3{}^{\ell-m-2k}}{2^{2k+m}\,(k+m)!k!(\ell-m-2k)!} \,|0\rangle$$

$$\propto (-a_1 - ia_2)^m \, a_3{}^{\ell-m} \,|0\rangle ,$$

which is a much simpler expression, and is manifestly a harmonic polynomial (traceless tensor).

In the reverse process we may obtain from f(a) the corresponding traceless tensor h(z) as a function of z by substituting for a_i^σ and expanding the result as above, so that

$$f(a_i^\sigma)\;|0\rangle = f(z_i^\sigma - 2(z^\alpha,z^\beta)\Delta^{-1}{}_{(\alpha\beta)(\sigma\gamma)}\rho_{ip}\frac{\partial}{\partial z_p^\gamma}) = h(z_i^\sigma)\;.$$

Of course $h(z_i^\sigma) = h(a_i^\sigma)$ and by using the traceless conditions $h(a_i^\sigma) = f(a_i^\sigma)$.

Given a homogeneous polynomial $f(z_i^\sigma)$ we can project into the harmonic subspace to obtain a harmonic polynomial $f(a_i^\sigma)$. It is useful to know explicitly such a harmonic projector operator, i.e., to know an operator H such that

$$f(a_i^\sigma) = H\;f(z_i^\sigma)\;.$$

In the case of symmetric tensors, such an operator has been found by Vilenkin [49]. For a metric ρ this has the form

$$H = \sum_k \frac{(-)^k(n + 2N - 2k - 4)!!}{2^k\;k!\;(n + 2N - 4)!!}\;(z,z)^k\;\nabla^{2k}\;. \tag{18}$$

For tensors of other symmetries it is much more difficult to write down H. Such an H could be regarded as originating from the expression

$$a_i^\sigma = L\;z_i^\sigma$$

where

$$L = 1 - (z^\alpha,z^\beta)\Delta^{-1}{}_{(\alpha\beta)(\gamma\epsilon)}\nabla^2_{\gamma\epsilon}\;.$$

H is the result of moving the operators L to the left of a polynomial in a_i^σ, and will in general be of a complicated form.

However we can write down H for the case λ = 2, when ρ is antisymmetric:

$$H = \sum_m \frac{(-)^m(n + N + M - m - 2)!}{m!(n + N + M - 2)!}\;(x,y)^m\nabla_{12}{}^{2m} \tag{19}$$

where

$$N = P^{11},\quad M = P^{22},\; x = z^{(1)},\quad y = z^{(2)}\;.$$

Using H we can write down a tensor in terms of bosons rather than modified bosons, which is useful for the evaluation of the scalar product, as described in §6. This will be done in finding normalizations of Sp(4) basis states.

§8. Properties of Modified Bosons

Modified bosons a_i^σ do not appear explicitly in the basis states (except for $\sigma = 1$) because the application of the Young symmetrizer means that only the antisymmetric combinations $a_{i_1..i_k}$ (Chapt. II, Eq. (2)) appear. The traceless condition imposes the following relations on these tensors:

$$\rho_{pq}\ a_{i_1..i_k p}\ a_{j_1..j_\ell q} = 0\ . \tag{20}$$

This follows by expanding according to Eq. (II.2), when the factor $\rho_{pq}\ a_p^\alpha\ a_q^\beta$ (for some α,β) appears in each term which is then zero. In the same way we can also show

$$\rho_{pq}\ a_{pqi_1..i_k} = 0\ . \tag{21}$$

This relation is non-trivial only when ρ is antisymmetric.

Modified bosons possess several properties which enable them to be used equally well as bosons in some aspects, besides satisfying in addition the traceless condition. We find that the group generators can be realized in terms of modified bosons. Since G(n) is a subgroup of U(n) (or GL(n) as applicable) the generators G_{ij} may be written as a linear combination of the generators E_{ij} of U(n):

$$\begin{aligned} G_{ij} &= \rho_{ip}\ E_{pj} - \eta\ \rho_{jp}\ E_{pi} \\ &= \rho_{ip}\ E_{pj} - \rho_{pj}\ E_{pi} \\ &= -\ \eta\ G_{ji}\ . \end{aligned} \tag{22}$$

The commutation relations are

$$[G_{ij},G_{k\ell}] = \rho_{kj}\, G_{i\ell} + \rho_{\ell i}\, G_{jk} + \rho_{\ell j}\, G_{ki} + \rho_{ki}\, G_{\ell j} \,. \qquad (23)$$

The realization of the G_{ij} is

$$G_{ij} = \rho_{ip}\, a_p^{\alpha}\, \bar{a}_j^{\alpha} - \rho_{pj}\, a_p^{\alpha}\, \bar{a}_i^{\alpha} \,. \qquad (24)$$

This expression can be obtained from the representation Tg defined by (3). The hermiticity property $G_{ij} = -\rho_{ip}\,\rho_{jq}\, G_{pq}^{*}$ ensures that the representations are unitary. We can verify that G_{ij} satisfies (23) by using the commutation relations (13). However this is more easily seen with the explicit substitution of (12) in (24) when we find that only the boson parts of the a_i^{σ} contribute:

$$\begin{aligned} G_{ij} &= \left(z_p^{\alpha} - 2(z^{\beta},z^{\gamma})\Delta^{-1}{}_{(\beta\gamma)(\alpha\varepsilon)}\rho_{pq}\frac{\partial}{\partial z_q^{\varepsilon}}\right)\left(\rho_{ip}\frac{\partial}{\partial z_j^{\alpha}} - \rho_{pj}\frac{\partial}{\partial z_i^{\alpha}}\right) \\ &= \rho_{ip}\, z_p^{\alpha}\frac{\partial}{\partial z_j^{\alpha}} - \rho_{pj}\, z_p^{\alpha}\frac{\partial}{\partial z_i^{\alpha}} \end{aligned}$$

because

$$\Delta^{-1}{}_{(\beta\gamma)(\alpha\varepsilon)}\left[\rho_{pq}\,\rho_{ip}\frac{\partial^2}{\partial z_q^{\varepsilon}\,\partial z_j^{\alpha}} - \rho_{pq}\,\rho_{pj}\frac{\partial^2}{\partial z_q^{\varepsilon}\,\partial z_i^{\alpha}}\right] = 0 \,.$$

Hence in the expression (24) the a's may be regarded as either ordinary or modified bosons.

We may also show, as we required in initial considerations, that the a_i^{σ} are vectors under G_{ij}:

$$[G_{ij},a_k^{\sigma}] = \delta_{jk}\,\rho_{ip}\, a_p^{\sigma} - \delta_{ik}\,\rho_{pj}\, a_p^{\sigma} \,.$$

Let us check this using (13):

$$\begin{aligned} [G_{ij},a_k^{\sigma}] &= \rho_{ip}\, a_p^{\alpha}\,[\bar{a}_j^{\alpha},a_k^{\sigma}] - \rho_{pj}\, a_p^{\alpha}\,[\bar{a}_i^{\alpha},a_k^{\sigma}] \\ &= \delta_{jk}\,\rho_{ip}\, a_p^{\sigma} - \delta_{ik}\,\rho_{pj}\, a_p^{\sigma} \\ &\quad - 4[\rho_{ip}\, a_p^{\alpha}\,\rho_{jq}\, a_q^{\beta} - \rho_{pj}\, a_p^{\alpha}\,\rho_{iq}\, a_q^{\beta}]\,\Delta^{-1}{}_{(\beta\alpha)(\sigma\gamma)}\rho_{kr}\,\bar{a}_r^{\gamma} \end{aligned}$$

and the last term is zero.

We see that both modified and ordinary bosons behave in the same way under G(n). In this way the complexity of Δ^{-1} does not affect the properties of a_i^σ. This fact is of importance in calculating basis states.

The representation space carries representations not only of G(n) but also of U(λ). We find that the polarization operators

$$P^{\mu\nu} = a_p^\mu \bar{a}_p^\nu \tag{25}$$

satisfy the U(λ) commutation relations:

$$[P^{\mu\nu}, P^{\sigma\tau}] = \delta^{\nu\sigma} P^{\mu\tau} - \delta^{\mu\tau} P^{\sigma\nu}$$

as is verified by substituting for (12):

$$\begin{aligned} a_p^\mu \bar{a}_p^\nu &= z_p^\mu \frac{\partial}{\partial z_p^\nu} - 2(z^\alpha, z^\beta)\Delta^{-1}{}_{(\alpha\beta)(\mu\gamma)} \nabla^2_{\gamma\nu} \\ &= z_p^\mu \frac{\partial}{\partial z_p^\nu} \end{aligned}$$

because $\nabla^2_{\gamma\nu} = 0$ in $H^{(\ell_\lambda)}$.

Also we find $[P^{\mu\nu}, G_{ij}] = 0$ so that in fact we can obtain representations of G(n) × U(λ). This result is in analogy with the case for ordinary bosons where the space carries representations of U(n) × U(n) imbedded in $U(n^2)$.

The two groups G(n) and U(λ) are complementary [61] i.e., the only representations of each which appear are those with the same representation labels $(m_1, .. m_\lambda)$. This fact is proved in Chpt. IV but we can see why it is so for a simple case O(3) × U(1). The invariant for O(3) is $\rho_{rs}\, \rho_{pq}\, G_{pr}\, G_{qs} = 2(z,z)\nabla^2 - 2N(N+1)$ and the invariant for U(1) is N, where $N = z_p \frac{\partial}{\partial z_p}$. Since $\nabla^2 = 0$ we see that each invariant operator specifies a label which must be the same for the two groups.

The question arises as to whether we can find a space which carries irreducible representations of $G(n) \times G(n)$ where the two groups are complementary. This could be done by restricting the space to the subspace of homogeneous polynomials $h^{(\ell_\lambda)}$ satisfying further

$$\rho_{pq} \frac{\partial^2}{\partial z_i^p \, \partial z_j^q} h^{(\ell_\lambda)} = 0$$

$$\left(\text{in addition to } \rho_{pq} \frac{\partial^2}{\partial z_p^\alpha \, \partial z_q^\beta} h^{(\ell_\lambda)} = 0\right) .$$

In this way we can choose spaces to carry any one of the following: $O(n) \times O(n)$, $Sp(n) \times Sp(n)$, $Sp(n) \times O(n)$. This would lead to another modification of the a_i^σ so that they satisfy the traceless condition with respect to the upper indices, i.e., $\rho_{pq} \, a_i^p \, a_j^q = 0$. These operators are of further complexity which is unnecessary, as our interest is primarily in $G(n)$ and not $G(n) \times G(n)$ and so this modification need not be considered.

§9. Choice of ρ

For $O(n)$ it is usual to put $\rho_{ij} = \delta_{ij}$. In this form it is possible to understand a geometrical picture of rotations, for then the invariant (z,z) has the form $r^2 = z_1^2 + .. + z_n^2$ where the z_i are Cartesian coordinates which are real. Hence in this choice of ρ the compact $O(n)$ is also real. There is also the advantage that the chain of subgroups $O(n) \supset O(n-1) \, .. \supset O(2)$ are easily selected.

The disadvantage of putting $\rho = I$ is that the generators $J_{ij} = - i \, G_{ij}$ are not in immediate Cartan standard form. Wong [54] has listed the linear combinations of the J_{ij} which are in the Cartan form. These linear combinations also appear in the structure of the basis states which then involve linear combinations of the vectors a_i.

For example when n=3 the vector components which appear are the spherical components $\frac{1}{\sqrt{2}}(a_1 \pm i\ a_2)$, a_3.

This awkwardness can be avoided by choosing $\rho = \sigma$ where $\sigma_{ij} = \delta_{i,n+1-j}$. In this case the generators $K_{ij} = G_{ij}$ are in Cartan form, and the classification in terms of the generators E_α corresponding to the root α is as follows:

$$H_i = K_{n+1-i,i} \qquad i = 1,..\nu$$

$$E_{ep+eq} = K_{n+1-q,n+1-p} \;,\; E_{-ep-eq} = K_{pq}$$

$$E_{ep-eq} = K_{n+1-p,q} \tag{26}$$

$$\text{where } p,q = 1,..\nu \text{ and } \nu = \left[\frac{n}{2}\right].$$

In addition for $O(2\nu+1)$ we have $E_{ep} = K_{2\nu+2-p,\nu+1}$ and $E_{-ep} = K_{\nu+1,p}$, for $p = 1,..\nu$.

With this metric the spherical and Cartesian vector components are the same, with a consequent simplification in the appearance of basis states. For n=3 the vector components in spherical coordinates are

$$z_1 = \frac{r}{\sqrt{2}} e^{i\phi} \sin\theta$$

$$z_2 = r\cos\theta \tag{27}$$

$$z_3 = \frac{r}{\sqrt{2}} e^{-i\phi} \sin\theta$$

which for this choice of metric are complex. On a global scale the choice $\rho = \sigma$ allows a Gauss decomposition [56] (for the complex group). We will use both metrics $\rho = \sigma$ or $\rho = I$ as convenient.

In the case of Sp(n) we will put $\rho = \varepsilon$

where

$$\varepsilon = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}.$$

The generators $S_{ij} = G_{ij}$ are then in Cartan form classified as follows:

$$H_i = S_{2i,2i-1} \qquad i = 1,..\nu$$

$$E_{2e_p} = \tfrac{1}{2} S_{2p2p} \;, \quad E_{-2e_p} = -\tfrac{1}{2} S_{2p-1,2p-1} \tag{28}$$

$$E_{e_p+e_q} = S_{2p2q} \;, \quad E_{-e_p-e_q} = - S_{2p-1,2q-1}$$

$$E_{e_p-e_q} = S_{2_p,2q-1} \qquad p,q = 1,..\nu$$

and $\nu = \left(\frac{n}{2}\right)$ is the rank.

Another useful choice is $\rho = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix}$ where σ is a $\nu \times \nu$ matrix as above. In this case we can carry out a Gauss decomposition [56].

The condition that $g \in G(n)$ satisfies $g\rho g^t = \rho$ is a weaker condition when ρ is antisymmetric than for ρ symmetric. As a consequence modified bosons for Sp(n) have a simpler form than for O(n), for a given number of rows. Hence for one row bosons are sufficient for Sp(n) since $\epsilon_{pq}\, a_p\, a_q = 0$ is satisfied trivially. For two rows, as would be required for Sp(4), we use modified boson operators $a_i = a_i^{(1)}$, $b_i = a_i^{(2)}$ defined by

$$[\bar{a}_i, b_j] = \epsilon_{ip}\, b_p\, (n + N)^{-1}\, \epsilon_{jq}\, \bar{a}_q$$

$$[\bar{a}_i, a_j] = \delta_{ij} - \epsilon_{ip}\, b_p\, (n + N)^{-1}\, \epsilon_{jq}\, \bar{b}_q \tag{29}$$

$$[\bar{b}_i, b_j] = \delta_{ij} - \epsilon_{ip}\, a_p\, (n + N)^{-1}\, \epsilon_{jq}\, \bar{a}_q$$

$$[\bar{b}_i, a_j] = \epsilon_{ip}\, a_p\, (n + N)^{-1}\, \epsilon_{jq}\, \bar{b}_q$$

where $N = a_p\, \bar{a}_p + b_p\, \bar{b}_p$ (total number operator). These operators satisfy $\epsilon_{pq}\, a_{pq} = 0 = a_{12} + a_{34}$ (from (21)).

§10. Further Properties of Modified Bosons

The expression for symmetric modified bosons i.e., when $\lambda = 1$, may be written in the following way:

$$a_i\, h^\ell = \left(z_i - \frac{(z,z)}{n+2\ell-2}\,\rho_{ip}\,\frac{\partial}{\partial z_p}\right) h^\ell$$

$$= -\frac{r^{n+2\ell}}{n+2\ell-2}\,\rho_{ip}\,\frac{\partial}{\partial z_p}\, r^{-n-2\ell+2}\, h^\ell \qquad (30)$$

where $h^\ell \in H^\ell$, $(z,z) = r^2$, and we may take ρ to be symmetric. It follows then that

$$(a_i)^m\, h^\ell \propto r^{n+2\ell+2m-2}\left(\rho_{ip}\frac{\partial}{\partial z_p}\right)^m \frac{1}{r^{n+2\ell-2}}\, h^\ell \,. \qquad (31)$$

Hence for n=3, $\rho_{ij} = \delta_{ij}$ we have

$$(a_i)^m\, |0\rangle \propto r^{2m+1}\left(\frac{\partial}{\partial z_i}\right)^m \frac{1}{r}\,.$$

This expression appears in the theory of multipoles [66], in connection with a linear distribution of charge. The potential of a point charge of m^{th} order is

$$\phi_m = \frac{p^{(m)}}{4\pi\varepsilon}\,\frac{(-)^m}{m!}\,\frac{\partial^m}{\partial z^m}\,\frac{1}{r}$$

where $p^{(m)}$ is a multipole of m^{th} order, and the total potential is $\phi = \sum_m \phi_m$.

From (30) we see that a convenient expression for a_i can be found by renormalizing in the following way:

We put $\quad a_i \to a_i\left(\frac{n}{2} + N - 1\right)$

with $\bar{a}_i$ unchanged, or alternatively, in order that the scalar product be unchanged,

$$a_i \to a_i\left(\frac{n}{2} + N - 1\right)^{\frac{1}{2}}$$
$$\bar{a}_i \to \left(\frac{n}{2} + N - 1\right)^{\frac{1}{2}} \bar{a}_i \,. \qquad (32)$$

These renormalized modified bosons satisfy the following triple commutation relations:

$$[a_i, a_j] = 0 = [\bar{a}_i, \bar{a}_j]$$
$$[a_i, [\bar{a}_j, a_k]] = -\delta_{jk}\, a_i - \delta_{ij}\, a_k + \delta_{ik}\, a_j \,. \qquad (33)$$

These relations can serve as an alternative definition for symmetric modified bosons, but cannot be generalized to include multiple sets of operators. The operators defined by (33) are discussed more fully in Chapter 6.

In our construction of representations we have used essentially boson operators. However there exists also a fermion calculus [67] in which basis states are constructed from fermion operators. The disadvantage of this approach is that to construct a sufficient number of basis states it is necessary that a large number of fermions be introduced. It is possible to extend this fermion calculus to O(n) and Sp(n) by constructing, with the methods used above, modified fermions satisfying the traceless condition. This can also be done for parafermions and parabosons.

CHAPTER 4

BASIS STATES FOR O(n)

§1. State Labelling and Branching Theorem

We have seen how in modified bosons we have a simple way to construct irreducible carrier spaces for representations of O(n). In this chapter we calculate basis functions in both the non-orthogonal Weyl basis and the orthogonal Gelfand basis.

The first problem in this calculation is the method of labelling the basis states. For O(n) this problem is easily solved because the chain of subgroups

$$O(2) \subset O(3) \,..\, \subset O(n-1) \subset O(n) \qquad (1)$$

provide sufficient labels. From the infinitesimal viewpoint this is to say that the commuting hermitean invariants of the groups in the chain (1) are sufficient in number to provide state labels from their eigenvalues. This was shown by Gelfand and Tsetlin [68] with the independent invariants being listed. The problem has also been examined by Racah [26] and Louck [34,69] and suitable invariants investigated by Bracken and Green [70,71] .

An immediate solution to the problem of state labelling is given by the branching theorem which was stated by Gelfand and Tsetlin [68] . The representations of O(n) may be labelled by numbers $m_1, .. m_\nu$ which are simultaneously integers or semi-integers (i.e. half odd integers) and satisfy

$$m_1 \geqslant m_2 \geqslant .. \geqslant m_\nu \geqslant 0 \qquad \text{for } O(2\nu+1) \qquad (2)$$

and

$$m_1 \geqslant m_2 \geqslant .. \geqslant |m_\nu| \qquad \text{for } O(2\nu)$$

where $\nu = \left[\frac{n}{2}\right]$. The branching theorem may be stated as follows: On restricting the representation of $O(2\nu+1)$ with labels $(m_1, \ldots m_\nu)$ to $O(2\nu)$, the representations $(q_1, \ldots q_\nu)$ of $O(2\nu)$ appear just once each such that

$$m_1 \geq q_1 \geq m_2 \geq q_2 \geq \ldots \geq m_\nu \geq q_\nu \geq - m_\nu . \tag{3}$$

On restricting the representation of $O(2\nu)$ with labels $(m_1, \ldots m_\nu)$ to $O(2\nu-1)$, the representations $(p_1, \ldots p_{\nu-1})$ of $O(2\nu-1)$ appear just once each such that

$$m_1 \geq p_1 \geq m_2 \geq p_2 \geq \ldots \geq p_{\nu-1} \geq |m_\nu| . \tag{3'}$$

The numbers q_i, p_i are simultaneously all integers or semi-integers according as the m_i are integers or semi-integers.

We can now label basis states in the irreducible representation space R of $O(n)$. Suppose it is known how to do this for $O(n-1)$. Under $O(n-1)$ the space R breaks up into a direct sum of subspaces irreducible under $O(n-1)$; within each of these subspaces it is possible to select a basis with a known labelling. By taking all of these bases together we obtain basis states in R. By induction then we obtain a labelling using the subgroup chain (1) and involving the branching theorems (3). The Gelfand pattern for $O(n)$ then has the form

$$\left|\begin{matrix} m_{2\nu+1,1} & m_{2\nu+1,2} & \cdots & m_{2\nu+1,\nu} \\ m_{2\nu,1} & m_{2\nu,2} & \cdots & m_{2\nu,\nu} \\ m_{2\nu-1,1} & m_{2\nu-1,2} & m_{2\nu-1,\nu-1} & \\ \vdots & & & \\ m_{21} & & & \end{matrix}\right\rangle$$

for $O(2\nu+1)$ and the same for $O(2\nu)$, except without the labels $m_{2\nu+1,i}$. The restrictions on the m_{ij} are simply those given by the branching theorem (3). In this chapter the m_{ij} will be integers only.

These basis states were introduced by Gelfand and Tsetlin [68] and have been explained by Pang and Hecht [55], and also described by Zhelobenko [56].

§2. Weyl States

The group theoretic meaning of the m_{ij} ensures that the basis is orthogonal, but the system of numbers used in the labelling can also be used to ennumerate the non-orthogonal Weyl states. The advantage of the Weyl states is that, using modified bosons, they are monomials and may be written down from the Young tableau of the representation by inspection.

Consider for example n=3. The Gelfand state takes the form $\left|\begin{matrix}\ell \\ m\end{matrix}\right\rangle$ where $-\ell \leqslant m \leqslant \ell$. Here ℓ is a non-negative integer labelling the representation of O(3) and m is the label of the O(2) subgroup. The general Young tableau with one row associated with this Gelfand state is

$\longleftarrow \ell \longrightarrow$

1..	2..

$\longleftrightarrow$ m

which is filled with the modified bosons a_1 and a_2. The Weyl states then take the form (denoted with round brackets):

$$\left|\begin{matrix}\ell \\ m\end{matrix}\right) = a_1^m \, a_2^{\ell-m} \, |0\rangle \ .$$

This expression covers also the case when m is negative since the traceless condition reads (for $\rho = \sigma$) $2a_1a_3 + a_2^2 = 0$ so that

$$a_1^m \, a_2^{\ell-m} \, |0\rangle \propto a_3^{-m} \, a_2^{\ell+m} \, |0\rangle \ .$$

For O(4) the Gelfand state is

$$|(m)\rangle = \left|\begin{matrix}m_1 & m_2 \\ \ell & \\ m & \end{matrix}\right\rangle \qquad \text{with } m_2 \leqslant \ell \leqslant m_1 \ .$$

The Young tableau for two rows is

m	ℓ	m_1
1 ..	2 ..	3 ..
2 ..		
m_2		

corresponding to

$$\left|\begin{matrix}m_1 & m_2 \\ \ell & \\ m & \end{matrix}\right) = a_1^{m-m_2} \, a_2^{\ell-m} \, a_3^{m_1-\ell} \, a_{12}^{\;m_2} \, |0\rangle \ .$$

This holds for $m \geqslant m_2 \geqslant 0$, and for $m_2 \geqslant |m| \geqslant 0$:

$$\begin{array}{l} \quad\quad m \quad\quad\quad \ell \quad\quad\quad m_1 \\ \begin{array}{|l|l|l|}\hline 1\ .. & 2\ .. \quad\quad & 3\ .. \\ \hline 2\ .. & 3\ .. & \\ \hline \end{array} \\ \quad\quad\quad\quad m_2 \end{array} = a_2^{\ell-m_2}\ a_3^{m_1-\ell}\ a_{23}^{m_2-m}\ a_{12}^{m}\ |0>$$

which holds for all negative values of m by using the following equations derived from III (20) (for $\rho = \sigma$):

$$a_{12}a_{13} = 0 = a_{12}a_{24} = a_{13}a_{24}$$

$$a_{12}a_{34} = -\ a_{23}^{\ 2}$$

$$a_4a_{23} = -\ a_2a_{34}\ .$$

Similar Weyl states can be written down for $m_2 \leqslant 0$.

§3. State of Highest Weight

Our method of calculating the Gelfand basis state is to apply certain lowering operators to the state of highest weight as explained in II §4. Suitable lowering operators for O(n) have been calculated by Pang and Hecht [55] and Wong [54]. These Gelfand states will be linear combinations of the Weyl states.

Since our space is irreducible there is only one state of highest weight |max.>, and this is specified by the labels $(m_1,\ldots m_\nu)$ of the representation. The explicit expressions for |max.> are (putting $\rho = I$):

$$|max.> = M^{-\frac{1}{2}}\ A_1^{\ m_1-m_2}\ A_{12}^{\ m_2-m_3}\ ..\ A_{12..\nu}^{\ m_\nu}\ |0> \qquad (4)$$

where M is the normalization, and where we have defined

$$A_i^\sigma = a_{2i-1}^\sigma - i\ a_{2i}^\sigma \qquad i = 1,..\nu\ ,$$

and the antisymmetric combinations $A_{12..k}$ are defined as in II (2). We see that for this expression to be meaningful all the m_i $i=1,..\nu$ must be non negative integers satisfying (2). As indicated earlier, we obtain with this construction only the tensor or integer representations and the m_i cannot be half integers. There is the possibility for $O(2\nu)$ that m_ν can take negative values and in this case we have

$$|max.> = M^{-\frac{1}{2}} A_1^{m_1-m_2} \,..\, A_{12..\nu-1}^{m_{\nu-1}+m_\nu} \tilde{A}_{12..\nu}^{-m_\nu} |0> \qquad (4')$$

where

$$\tilde{A}_{i_1..i_\nu} = \Sigma\varepsilon_{(i_1..i_\nu)} A_1^{i_1} \,..\, A_{\nu-1}^{i_{\nu-1}} \bar{A}_\nu^{i_\nu}$$

and

$$\bar{A}_i^\sigma = a_{2i-1}^\sigma + i\, a_{2i}^\sigma \,.$$

The weight of the state (4), (4′) is given by

$$J_{2\alpha,2\alpha-1} |max.> = m_\alpha |max.> \qquad \alpha = 1,..\nu \qquad (5)$$

where

$$J_{ij} = - i\, G_{ij} \,.$$

We have

$$[J_{2\alpha,2\alpha-1}, A_i^\sigma] = \delta_{i\alpha} A_i^\sigma$$

and

$$[J_{2\alpha,2\alpha-1}, \bar{A}_\nu^\sigma] = - \delta_{\nu\alpha} \bar{A}_\nu^\sigma$$

so that

$$[J_{2\alpha,2\alpha-1}, A_{12..q}] = A_{12..q} \qquad q \geq \alpha \,,$$
$$= 0 \qquad q < \alpha$$

and

$$[J_{2\alpha,2\alpha-1}, \tilde{A}_{12..\nu}] = - \tilde{A}_{12..\nu} \qquad \alpha = \nu \,,$$
$$= \tilde{A}_{12..\nu} \qquad \alpha < \nu$$

Eq. (5) then follows.

The states (4), (4′) are of highest weight because they are annihilated by all the raising generators:

$$D^{p}_{p+1} \, |\text{max.}\rangle = 0 \qquad p = 1,..\nu-1$$

$$A^{\nu-1}_{\nu} \, |\text{max.}\rangle = 0 \qquad \text{for } O(2\nu) \tag{6}$$

$$E^{\nu}_{2\nu+1} \, |\text{max.}\rangle = 0 \qquad \text{for } O(2\nu+1)$$

where

$$D^{p}_{p+1}, \; A^{\nu-1}_{\nu}, \; E^{\nu}_{2\nu+1}$$

are raising generators of $O(2\nu)$ and $O(2\nu+1)$ corresponding to the simple roots, as defined by Wong [54]. We have

$$[D^{p}_{p+1}, A^{\sigma}_{i}] = \delta_{i-1\,p} \, A^{\sigma}_{i-1}$$

$$(= 0 \text{ for } i = 1)$$

and

$$[D^{p}_{p+1}, \bar{A}^{\sigma}_{\nu}] = 0$$

so that

$$[D^{p}_{p+1}, A_{12..q}] = 0 \qquad q = 1,..\nu$$

and

$$[D^{p}_{p+1}, \tilde{A}_{12..\nu}] = 0 \; .$$

Also

$$[A^{\nu-1}_{\nu}, A^{\sigma}_{i}] = 0$$

so that

$$[A^{\nu-1}_{\nu}, A_{12..q}] = 0$$

and

$$[A^{\nu-1}_{\nu}, \bar{A}^{\sigma}_{\nu}] = - A^{\sigma}_{\nu-1}$$

so that

$$[A^{\nu-1}_{\nu}, \tilde{A}_{12..\nu}] = 0 \; .$$

Also

$$[E^{\nu}_{2\nu+1}, A^{\sigma}_{i}] = 0 \; .$$

Eq. (6) now follows.

We have shown that (4), (4′) are solutions of (6) with weights given by (5), and since our space is irreducible they are the only solutions by Cartan's theorem, and are therefore the required states of highest weight.

It is useful to know the form of $|max.\rangle$ when $\rho = \sigma$, because of its simple appearance:

$$|max.\rangle = M^{-\frac{1}{2}} a_1^{m_1-m_2} a_{12}^{m_2-m_3} \cdots a_{12..\nu}^{m_\nu} |0\rangle . \qquad (7)$$

For $O(2\nu)$ there is also the case $m_\nu \leqslant 0$:

$$|max.\rangle = M^{-\frac{1}{2}} a_1^{m_1-m_2} \cdots a_{12..\nu-1}^{m_{\nu-1}+m_\nu} a_{12..\nu-1\,\nu+1}^{-m_\nu} |0\rangle \qquad (7')$$

It is possible to see now more clearly why the two cases arise for $O(2\nu)$. It would seem at first that several states of highest weight could be formed, namely

$$|max.\rangle = a_1^{m_1-m_2} \cdots a_{12..\nu-2}^{m_{\nu-2}-m_{\nu-1}} a_{12..\nu-1}^{m_{\nu-1}+m_\nu-2p}$$

$$\times\, a_{12..\nu}^{p}\, a_{12..\nu-1\,\nu+1}^{p-m_\nu} |0\rangle$$

where p is not uniquely determined. This state is annihilated by all raising generators, and belongs to the irreducible space labelled with the numbers $(m_1,..m_\nu)$ as is checked by application of H_i (III (25)). However it follows easily from III (20) that $a_{12..\nu}\, a_{12..\nu-1\,\nu+1} = 0$, so that either $p = 0$ or $p = m_\nu$, leading to the two possibilities (7) and (7′).

We have shown that for $n = 2\nu$ the space reduces into the sum of two irreducible spaces determined by the two states (7), (7′). These two spaces are irreducible only under the proper orthogonal group $SO(n)$, and the representations are known as mirror conjugate representations [56]. Under the full orthogonal group the two mirror conjugate representations combine to give a single irreducible representation ([12] Chapt. V). For example if

$$g = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & 1 & & & \\ & & & 1 & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix}$$

then $g \in O(n)$ with $\det g = -1$, and g permutes the coordinate axes with the numbers ν and $\nu + 1$. Hence under g (7) and (7′) are interchanged.

(7) and (7′) are Weyl states which correspond to Young diagrams of no more than ν rows. However it is possible to have other diagrams for which the sum of the lengths of the first two columns is not greater than n. These other diagrams are known as associate diagrams ([12,50]) and under SO(n) give representations which are equivalent to those determined by (7) and (7′). Under the full orthogonal group however, representations corresponding to associate diagrams are non-equivalent because of the different transformation properties of basis tensors under reflections. This will be examined more fully for O(3).

An important property of the states of highest weight ((4), (4′) or (7), (7′)) is that the modified bosons appearing there may be regarded as ordinary bosons. Taking (7), (7′) for example, and using the explicit realization for a_i^σ :

$$a_i^\sigma = z_i^\sigma - (z^\alpha, z^\beta)\Delta^{-1}{}_{(\alpha\beta)(\sigma\gamma)} \frac{\partial}{\partial z_{n+1-i}^\gamma}$$

we have

$$a_i^\sigma \; a_j^\tau \; |0\rangle = z_i^\sigma \; z_j^\tau$$

because here $i,j \leq \nu$ so that $n + 1 - i > \nu$. It follows then by induction that $a_i^\sigma \; |\ell\rangle = z_i^\sigma \; |\ell\rangle$ where $|\ell\rangle$ is a state composed of modified bosons $a_j{}^\tau$ such that $j \leq \nu$. Alternatively we could show in a more abstract way that

$$[\bar{a}_i^\sigma , a_j^\tau] \; |\ell\rangle = \delta_{ij} \; \delta^{\sigma\tau} \; |\ell\rangle \; .$$

This result explains why previous authors [24,53] have been able to construct states of highest weight adequately with ordinary bosons. Such states f(z) belong to the harmonic space so that f(z) = f(a), where f(a) is |max.> expressed with modified bosons. This result can be seen as deriving from the fact that the state of highest weight is determined completely by the condition that it be annihilated by the raising generators, and the generators can be regarded as composed of bosons or equally well modified bosons (shown in III, §8).

Since the complicated Δ^{-1} expression does not appear in |max.> our task of normalizing this state is greatly simplified. The normalization $M^{-\frac{1}{2}}$ is calculated from M = <max.| max.> by using the boson commutation relations. This problem has been solved already in the context of U(n). From the Young tableau associated with (7), (7´) we can write down immediately the normalization in terms of hook lengths using known algorithms [32].

In the form (7), (7) |max.> is the same as the state of highest weight for U(ν) (see II (16)). This is expected in view of the fact that our space carries representations of O(n) × U(ν). We can now show easily that the groups O(n), U(ν) are complementary when the generators G_{ij}, $P^{\mu\nu}$ have the form III (24) and III (25). We need to show that the representation space of O(n) × U(ν) carries representations in which the labels of each group are the same $(m_1,..m_\nu)$. Now the representation space of O(n) × U(ν) is the space H of harmonic homogeneous polynomials in the variables z_i^σ, i = 1,..n, σ = 1,..ν . This space is larger than $H^{(m_\nu)} = H^{(m_1,..m_\nu)}$ which carries irreducible representations of O(n), because we have not yet applied the Young symmetrizer. H is reducible under either U(ν), or O(n); for example, all states in $H^{(m_\nu)} \subset H$ are states of highest weight for U(ν). However H is irreducible under U(ν) × O(n). This is because the space

$\mathcal{R} = R^{m_1} \times R^{m_2} .. \times R^{m_\nu}$ is irreducible under $U(\nu) \times U(n)$ ([5]) and the and the restriction to the subspace H of harmonic polynomials ensures that H is irreducible under $U(\nu) \times O(n)$. Hence H contains only one state which is of highest weight in both $U(\nu)$ and $O(n)$; this state is (7) as is verified by applying all raising generators of $U(\nu) \times O(n)$. Now finally by applying the weight generators P^{ii}, $G_{n+1-i,i}$ $i = 1,..\nu$, we find that both $U(\nu)$ and $O(n)$ have the same representation labels $(m_1,..m_\nu)$.

In this proof we have found it unnecessary to carry out explicit calculations with the invariants, but have introduced the representation labels through the state of highest weight. Complementary groups have been utilized by other authors [62,63] to calculate states of highest weight for $O(n)$, and $Sp(n)$. However because the restriction to the harmonic subspace has not been carried out, the groups complementary to $O(n)$ and $Sp(n)$ are non-compact and thus without the simpler properties of $U(\nu)$. The calculations ([54,63]) to obtain the state of highest weight are unnecessarily complicated by comparison with our method, when we have taken advantage of the fact that the harmonic subspace is irreducible.

§4. SEMI-MAXIMAL STATE

The semi-maximal state $|s.m.\rangle$ is a Gelfand basis state which is maximal in the immediate subgroup $O(n-1)$. The Gelfand pattern is

$$|s.m.\rangle = \left|\begin{matrix} m_1 & .. & m_{\nu-1} & m_\nu \\ \ell_1 & .. & \ell_{\nu-1} & \ell_\nu \\ \ell_1 & .. & \ell_{\nu-1} & \\ \vdots & & & \\ \ell_1 & & & \end{matrix}\right\rangle \quad \text{for } O(2\nu+1) \qquad (8)$$

with $m_1 \geq \ell_1 \geq .. \ell_{\nu-1} \geq m_\nu \geq \ell_\nu \geq - m_\nu$ (all integers) and

$$|s.m.\rangle = \left|\begin{matrix} m_1 & .. & m_{\nu-1} & m_\nu \\ \ell_1 & .. & \ell_{\nu-1} & \\ \vdots & & & \\ \ell_1 & & & \end{matrix}\right\rangle \quad \text{for } O(2\nu) \qquad (8')$$

with $m_1 \geq \ell_1 \geq .. \ \ell_{\nu-1} \geq |m_\nu|$ (all integers).

In the choice of metric $\rho = I$, the $O(n-1)$ subgroup is chosen to be that which leaves invariant the vector component a^τ_n. For $\rho = \sigma$ we choose the $O(2\nu)$ subgroup of $O(2\nu+1)$ as that which leaves invariant $a^\tau_{\nu+1}$, and the $O(2\nu-1)$ subgroup of $O(2\nu)$ as that which leaves invariant $a^\tau_\nu + a^\tau_{\nu+1}$.

Let us consider the reduction $O(2\nu+1)$ to $O(2\nu)$. The explicit form of $|s.m.\rangle$ is determined by the requirement that it be annihilated by the raising generators of $O(2\nu)$, and that it be an eigenvector of the diagonal generators H_i with eigenvalue ℓ_i, $i = 1,..\nu$. Hence in forming the expression for $|s.m.\rangle$ we may use polynomials not only in $a_1, a_{12}, .. \ a_{12..\nu}$, as was used for $|max.\rangle$, but also in $a_{\nu+1}$, $a_{1\nu+1}, .. \ a_{12..\nu-1\ \nu+1}$ since $a^\tau_{\nu+1}$ is invariant under $O(2\nu)$.

Therefore we have that $|s.m.\rangle$ is a linear combination of polynomials of the form

$$a_1^{r_1} \ a_{12}^{r_2} \ .. \ a_{12..\nu}^{r_\nu} \ a_{\nu+1}^{s_1} \ a_{1\nu+1}^{s_2} \ .. \ a_{12..\nu-1\ \nu+1}^{s_\nu} .$$

Because each term must be a polynomial homogeneous of degree m_j in a^σ_j we have

$$m_1 - m_2 = r_1 + s_1$$
$$m_2 - m_3 = r_2 + s_2$$
$$m_\nu = r_\nu + s_\nu .$$

Each term is already an eigenvector of H_i and in order that $|s.m.\rangle$ be an eigenvector with eigenvalue ℓ_1 we have (noting that

$$[H_i, a^\tau_j] = \delta_{ij} \ a^\tau_j \quad \begin{matrix} i = 1,..\nu \\ j \leq \nu + 1 \end{matrix}):$$

$$r_1 + r_2 + .. \ r_\nu + s_2 + .. \ s_\nu = \ell_1$$

$$r_2 + .. r_\nu + s_3 + .. s_\nu = \ell_2$$
$$\vdots$$
$$r_\nu = \ell_\nu \quad .$$

We have 2ν independent equations for 2ν unknowns, and the solution is

$$s_i = m_i - \ell_i$$
$$r_i = \ell_i - m_{i+1} \quad i=1,..\nu$$
$$(\text{with } m_{\nu+1} = 0) \ .$$

Hence

$$|s.m.> = M^{-\frac{1}{2}} a_1^{\ell_1-m_2} a_{\nu+1}^{m_1-\ell_1} a_{12}^{\ell_2-m_3} a_{1\nu+1}^{m_2-\ell_2}$$

$$\times .. a_{12..\nu}^{\ell_\nu} a_{12..\nu-1\ \nu+1}^{m_\nu-\ell_\nu} |0> \ . \qquad (9)$$

From III (20) it is easily deduced that

$$2a_{1..\nu-1\nu}\, a_{1..\nu-1\ \nu+2} + a^2_{1..\nu-1\ \nu+1} = 0$$

and using this we may rewrite (9) as

$$|s.m.> = M^{-\frac{1}{2}} a_1^{\ell_1-m_2} .. a_{12..\nu-1\ \nu+2}^{-\ell_\nu} a_{12..\nu-1\ \nu+1}^{m_\nu+\ell_\nu} |0> \ . \qquad (9')$$

We require that all exponents be non-negative integers, and this leads to the restriction of ℓ_1 to those values (8) given above. We have in fact proved the branching theorem (2) for $O(2\nu+1)$ restricted to $O(2\nu)$. The representation space for $O(n)$ is reducible under $O(n-1)$ and we have found the explicit reduction by identifying the states $|s.m.>$ of highest weight in $O(n-1)$.

In the reduction $O(2\nu)$ to $O(2\nu-1)$ $|s.m.>$ is formed from polynomials in a_1, a_{12}, .. $a_{12..\nu}$ as before, with also $a_\nu + a_{\nu+1}$, $a_{1\nu} + a_{1\nu+1}$, .. $a_{12..\nu-2\ \nu} + a_{1..\nu-2\ \nu+1}$. The factor $a_{12..\nu-1\nu} + a_{12..\nu-1\ \nu+1}$ need not be included because $a_{12..\nu}\, a_{12..\nu-1\ \nu+1} = 0$, so that only $a_{12..\nu}$ contributes (for $m_\nu > 0$). We find in the same way that

$$|s.m.\rangle = M^{-\frac{1}{2}}\, a_1^{\ell-m_2} (a_\nu + a_{\nu+1})^{m_1-\ell_1}\, a_{12}^{\ell_2-m_3} (a_{1\nu} + a_{1\nu+1})^{m_2-\ell_2}$$

$$\times\, a_{12..\nu-1}^{\ell_\nu - m_\nu} (a_{12..\nu-2\nu} + a_{1..\nu-2\ \nu+1})^{m_{\nu-1}-\ell\nu}\, a_{1..\nu}^{m_\nu} |0\rangle \tag{10}$$

which holds for $m_\nu \geqslant 0$. Again we have proved the branching theorem (8′).

The normalization $M = \langle s.m.|s.m.\rangle$ cannot be calculated as easily as for the state of highest weight. Methods of calculation include the following: we could use the commutation relations for the a_i^τ, which would be difficult in general because the properties of Δ^{-1} would be required; we could expand the modified bosons with the help of the projection operator H described in III §7, and then use the commutation relations of ordinary bosons. A third method would be to use the normalized lowering operators of Wong [54] to reach $|s.m.\rangle$ from $|max.\rangle$; we will find this method the most practical for $n \geqslant 4$.

It should be noted that, unlike the situation for $|max.\rangle$ modified bosons are essential in this construction of $|s.m.\rangle$. The corresponding expression with ordinary bosons would be much more complicated and could not in general be written down immediately.

§5. Basis States

The general state for O(2) is also the state of highest weight and is labelled by one number m. We have (putting $\rho = \sigma$) and using (7), (7′):

$$|m\rangle = \frac{a_1^{\,m}}{\sqrt{m!}}|0\rangle \qquad m \geqslant 0$$

$$= \frac{a_2^{\,-m}}{\sqrt{-\,m!}}|0\rangle \qquad m \leqslant 0\ ,\quad m \text{ integral}\ . \tag{11}$$

The traceless condition reads $a_1 a_2 = 0$. We can change to polar coordinates by putting

$$z_1 = \frac{r}{\sqrt{2}} e^{i\theta} \quad , \quad z_2 = \frac{r}{\sqrt{2}} e^{-i\theta}$$

so that $r^2 = 2z_1z_2$, and then $|m\rangle \propto r^m e^{im\theta}$. The exponential function in connection with O(2) has been studied by Vilenkin([49] Chpt. II). In the restriction to SO(2) we have all the single-valued representations, but for O(2) there is also the representation in which the Young tableau has the form  , to which corresponds the state vector $a_{12}\ |0\rangle$ of scalar angular momentum.

The general state for O(3) is also the semi-maximal state, i.e., is of highest weight in O(2). Hence using (9)

$$\left|\begin{matrix}\ell \\ m\end{matrix}\right\rangle = M^{-\frac{1}{2}} a_1{}^m a_2{}^{\ell-m}\ |0\rangle \qquad (12)$$

where $-\ \ell \leqslant m \leqslant \ell$. Negative values for m are included by using $2a_1a_3 + a_2{}^2 = 0$, so that the minimum state is

$$\left|\begin{matrix}\ell \\ -\ell\end{matrix}\right\rangle = M^{-\frac{1}{2}} a_3{}^{\ell}\ |0\rangle \ .$$

The generators are

$$\begin{aligned} J_+ &= K_{32} \\ J_- &= K_{21} \\ J \ &= K_{31} \ . \end{aligned} \qquad (13)$$

The value of M can be found by using the commutation relations of the a_i, or alternatively using

$$J_- \left|\begin{matrix}\ell \\ m\end{matrix}\right\rangle = \left[(\ell - m + 1)(\ell + m)\right]^{\frac{1}{2}} \left|\begin{matrix}\ell \\ m-1\end{matrix}\right\rangle \ .$$

We find

$$M = 2^{\ell-m} \frac{(\ell - m)!\ (\ell + m)!\ \ell!}{(2\ell)!} \ .$$

The general state (12) is necessarily an operator form of the solid spherical harmonic functions. This can be seen directly by putting

$$a_i = z_i - r^2(3 + 2N)^{-1} \sigma_{ip} \frac{d}{dz_p}$$

where $r^2 = 2z_1z_3 + z_2{}^2$. The expansion of $a_2{}^{\ell-m}$ is carried out most easily with the projection operator H (III (18)):

$$\left|\begin{matrix}\ell\\m\end{matrix}\right\rangle \propto H\, z_1{}^m\, z_2{}^{\ell-m}$$

$$\propto \sum_k \frac{(-)^k(2\ell - 2k - 1)!!\; z_1{}^m\, r^{2k}\, z_2{}^{\ell-m-2k}}{2^k\, k!\, (\ell - m - 2k)!}$$

$$\propto r^{\ell-m}\, z_1{}^m\, C^{m+\frac{1}{2}}_{\ell-m}\left(\frac{z_2}{r}\right) \quad , \qquad (14)$$

where $C^{m+\frac{1}{2}}_{\ell-m}$ is a Gegenbauer polynomial. Another expression for $\left|\begin{matrix}\ell\\m\end{matrix}\right\rangle$ can be written by putting

$$a_2\, h^\ell = -\frac{1}{2\ell+1}\, r^{2\ell+3}\, \frac{d}{dz_2}\, \frac{1}{r^{2\ell+1}}\, h^\ell$$

where $h^\ell \in H^\ell$ is of degree ℓ (see III, §10). We have then

$$\left|\begin{matrix}\ell\\m\end{matrix}\right\rangle \propto r^{2\ell+1}\, z_1{}^m \left(\frac{d}{dz_2}\right)^{\ell-m} \frac{1}{r^{2m+1}} \;.$$

We can transform to the more familiar spherical polar coordinates by putting

$$z_1 = \frac{r}{\sqrt{2}}\, e^{i\phi} \sin\theta$$

$$z_2 = r\cos\theta \qquad (15)$$

$$z_3 = \frac{r}{\sqrt{2}}\, e^{-i\phi} \sin\theta$$

and then

$$\left|\begin{matrix}\ell\\m\end{matrix}\right\rangle \propto r^\ell\, Y_{\ell m}(\theta,\phi) \;.$$

The expression (12) for the state is much simpler than that for the spherical harmonics in the usual expanded form. One could treat the state of highest weight as consisting of bosons and by applying $(J_-)^{\ell-m}$ obtain the expanded expression (14). However in this context modified bosons have all the simple properties of bosons while satisfying further the traceless condition. It is a simple matter to switch from (14) to (12) by simply regarding the z_i as modified bosons

a_i and putting $2a_1a_3 + a_2^2 = 0$, as explained in III, §7. The difference in complexity between (12) and (14) is that difference which appears between the decomposition of a tensor into a traceless part and a remainder (III, §2), and the composition of a tensor with modified bosons, which is manifestly traceless.

The basis state (12) corresponds to a Young tableau of one row, so that only symmetric traceless tensors appear. It is also permissible to have Young tableaus with two and three rows. The corresponding basis functions are

$$\left|\begin{matrix}\ell \\ m\end{matrix}\right\rangle \propto a_{12}\ a_1^{m-1}\ a_2^{\ell-m}\ |0\rangle\ . \qquad (16)$$

If we calculate this state by applying $J_-^{\ell-m}$ to the state of highest weight (which is $a_{12}\ a_1^{\ell-1}\ |0\rangle$) then we need to use the relations $a_1a_{13} + a_2a_{12} = 0$. In (16) takes integral values with $\ell \geqslant 1$. The state for $\ell = 0$ is the triple scalar product $a_{123}\ |0\rangle$. For the full orthogonal group O(3) these basis states carry new representations since the tensors (12) are polar but the tensors (16) are axial.

As was described in III, §2 traceless tensors are zero when, in the corresponding Young pattern the sum of the lengths of the first two columns is greater than three. We can see this explicitly in the following relations derived from the traceless condition:

$$a_{ij}\ a_{k\ell} = 0 = a_{123}^2 = a_i\ a_{123} = a_{ij}\ a_{123}$$

$$i,j,k,\ell = 1,2,3\ .$$

In the choice of metric $\rho = I$ (12) takes a slightly different form:

$$\left|\begin{matrix}\ell \\ m\end{matrix}\right\rangle = M^{-\frac{1}{2}}\ (a_1 - i\ a_2)^m\ a_3^{\ell-m}\ |0\rangle\ , \qquad (17)$$

and the change to spherical polars is made by putting

$$z_1 = r \sin\theta \cos\phi$$

$$z_2 = r \sin\theta \sin\phi$$

$$z_3 = r \cos\theta \ .$$

For O(4) the state of highest weight is (for $\rho = I$):

$$\begin{vmatrix} m_1 & m_2 \\ m_1 & \\ m_1 & \end{matrix}\Bigg\rangle = M\binom{m_1}{m_1}^{-\frac{1}{2}} A_1^{m_1-m_2} A_{12}^{m_2} |0\rangle \qquad m_2 \geqslant 0$$

where

$$M\binom{m_1}{m_1} = \frac{2^{m_1+m_2}(m_1 + 1)!\ m_2!\ (m_1 - m_2)!}{(m_1 - m_2 + 1)!}$$

is calculated according to the algorithm given in [32]. The semi-maximal state is (§4):

$$\Bigg|\begin{matrix} m_1 & m_2 \\ \ell & \\ \ell & \end{matrix}\Bigg\rangle = M\binom{\ell}{\ell}^{-\frac{1}{2}} a_4^{m_1-\ell} A_1^{\ell-m_2} A_{12}^{m_2} |0\rangle \ . \qquad (18)$$

In order to find the normalization we need to apply the normalized lowering operator given by Wong [54]. This operator is

$$L_- = (K_- J_{21} + i J_- J_{43})(2J_{21} + 1) + \tfrac{1}{2} J_-^2 K_-$$

where

$$J_\pm = J_{32} \pm i J_{31} \quad , \quad K_\pm = J_{42} \pm i J_{41} \ .$$

Now

$$L_- \Bigg|\begin{matrix} m_1 & m_2 \\ \ell & \\ \ell & \end{matrix}\Bigg\rangle = N^{\frac{1}{2}} \Bigg|\begin{matrix} m_1 & m_2 \\ \ell-1 & \\ \ell-1 & \end{matrix}\Bigg\rangle$$

where

$$N = 2\ell(2\ell + 1)(m_1 - \ell + 1)(m_1 + \ell + 1)(\ell + m_2)(\ell - m_2) \ .$$

Applying $(L_-)^{m_1-\ell}$ to $|\text{max.}\rangle$ we obtain (18) with

$$M\binom{\ell}{\ell} = 2^{2\ell+m_2-m_1} \frac{(m_1 + 1)!\ m_2!\ (m_1 - \ell)!\ (\ell - m_2)!\ (\ell + m_2)!}{(2\ell + 1)!\ (m_1 + m_2)!\ (m_1 - m_2 + 1)!}$$

In this calculation we must use in particular

$$(a_1 - i\ a_2)(a_{12} + a_{34}) = i\ (a_3 + i\ a_4)A_{12}. \qquad (19)$$

The general state of the Gelfand basis is now obtained by lowering m from its highest value ℓ to a general value m, by applying $J_-^{\ell-m}$ to (18). One obtains Jacobi polynomials:

$$\left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m & \end{matrix}\right\rangle = M\binom{\ell}{m}^{-\frac{1}{2}} P_{\ell-m}^{(m+m_2,m-m_2)} \left(\frac{a_3}{i\ a_4}\right)$$

$$\times\ a_4^{m_1-m} A_1^{m-m_2} A_{12}^{m_2} |0\rangle \ , \tag{20}$$

where

$$M\binom{\ell}{m} = \frac{2^{2m+m_2-m}\ (m_1+1)!\ m_2!\ (m_1-\ell)!\ (\ell-m_2)!\ (\ell+m_2)!\ (\ell+m_1+1)!}{(2\ell+1)(m_1+m_2)!\ (m_1-m_2+1)!(\ell+m)!\ (\ell-m)!} .$$

The easiest way to show that (20) is the correct expression is to apply J_+ and show

$$J_+ \left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m & \end{matrix}\right\rangle = [\ (\ell - m)(\ell + m - 1)]^{\frac{1}{2}} \left|\begin{matrix} m_1 & m_2 \\ \ell & \\ m+1 & \end{matrix}\right\rangle .$$

This is done with the help of the formula

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}(\alpha + \beta + n + 1)\ P_{n-1}^{(\alpha+1,\beta+1)}(x)$$

for positive and negative α,β. Negative values of α,β are interpreted with the formula

$$\binom{n}{\alpha} P_n^{(-\alpha,\beta)}(x) = \binom{n+\beta}{\alpha}\left(\frac{x-1}{2}\right)^{\alpha} P_{n-\alpha}^{(\alpha,\beta)}(x) .$$

It should be noted that although the argument of $P_n^{(\alpha,\beta)}$ is $\left(\frac{a_3}{i\ a_4}\right)$ this expression is symbolic and no inverse of a_4 appears explicitly. In the Gelfand pattern m may take negative values, and the negative exponents of A_1 which may appear are interpreted with the relation (19) and

$$\begin{aligned} i\ A_2(a_{12} + a_{34}) &= A_{12}\bar{A}_1 \\ A_{12}\bar{A}_{12} &= -\ (a_{12} + a_{34})^2 . \end{aligned} \tag{21}$$

Hence for example if $m \leqslant -m_2$

$$\begin{vmatrix} m_1 & m_2 \\ \ell & \\ m & \end{matrix}\Big\rangle = (-)^{m_2}\, 2^{2m}\, M\binom{\ell}{m}^{-\frac{1}{2}}\, P_{\ell+m}^{(-m-m_2,\, m_2-m)}\left(\frac{a_3}{i\ a_4}\right)$$

$$\times\ a_4^{\,m_1+m}\ \bar{A}_1^{\,-m_2-m}\ \bar{A}_{12}^{\,m_2}\ |0\rangle\ .$$

For symmetric representations we put $m_2 = 0$ and then the general state (20) is expressed in terms of Gegenbauer polynomials:

$$\begin{vmatrix} m_1 & 0 \\ \ell & \\ m & \end{matrix}\Big\rangle = M^{-\frac{1}{2}}\,(a_1 - i\ a_2)^m\ a_4^{\,m_1-m}\ C_{\ell-m}^{m+\frac{1}{2}}\left(\frac{a_3}{i\ a_4}\right)\ |0\rangle \qquad (22)$$

This could also be deduced immediately from (14). The connection between symmetric representations of O(n) and Gegenbauer polynomials has been previously noted by Vilenkin ([49], Chpt IX).

From the general state one can find the matrix elements of J_{43}:

$$J_{43}\begin{vmatrix} m_1 & m_2 \\ \ell & \\ m & \end{matrix}\Big\rangle =$$

$$i\left[\frac{(\ell+m+1)(\ell-m+1)(m_1-1)(m_1+\ell+2)(\ell-m_2+1)(\ell+m_2+1)}{(2\ell+1)(2\ell+3)(\ell+1)^2}\right]^{\frac{1}{2}}\begin{vmatrix} m_1 & m_2 \\ \ell+1 & \\ m & \end{matrix}\Big\rangle$$

$$+\ \frac{m(m_1+1)m_2}{\ell(\ell+1)}\begin{vmatrix} m_1 & m_2 \\ \ell & \\ m & \end{matrix}\Big\rangle$$

$$-\ i\left[\frac{(\ell+m)(\ell-m)(m_1-\ell+1)(m_1+\ell+1)(\ell-m_2)(\ell+m_2)}{(2\ell+1)(2\ell-1)\ell^2}\right]^{\frac{1}{2}}\begin{vmatrix} m_1 & m_2 \\ \ell-1 & \\ m & \end{matrix}\Big\rangle$$

which is the required result [72]. This calculation is carried out by means of the standard differentiation formulas and recurrence relations for Jacobi polynomials [73].

One can also carry out a similar analysis when $m_2 \leqslant 0$. The state of highest weight is

$$\begin{vmatrix} m_1 & m_2 \\ m_1 & \\ m_1 & \end{matrix}\Big\rangle = M^{-\frac{1}{2}}\ A_1^{\,m_1+m_2}\ \tilde{A}_{12}^{\,-m_2}\ |0\rangle$$

where

$$M = \frac{2^{m_1-m_2}(-m_2)!\,(m_1+1)!}{(m_1+m_2+1)}.$$

Formulas similar to (19) hold, e.g.,

$$A_1(a_{12} - a_{34}) = i\,\tilde{A}_{12}\,A_2$$

and the analysis proceeds in the same way.

Using modified bosons we can calculate simply for O(n) basis states in symmetric representations, for which $m_2 = m_3 = \ldots = m_\nu = 0$. This has been done before ([51,65]) using bosons, in which case the structure of the states is necessarily more complicated.

These basis states are labelled by (n - 1) integers $\ell_2, \ldots \ell_n$ where ℓ_i refers to the label of the O(i) subgroup, representations of which appear according to the branching theorem:

$$\ell_n \geq \ell_{n-1} \geq \ldots \ell_3 \geq |\ell_2| \quad . \tag{23}$$

The semi-maximal state $|s.m.\rangle = |\ell_n, \ell_{n-1}, \ell_{n-1}, \ldots \ell_{n-1}\rangle$ can be written down immediately:

$$|s.m.\rangle = M^{-\frac{1}{2}}\, a_n^{\ell_n-\ell_{n-1}}\,(a_1 - i\,a_2)^{\ell_{n-1}}\,|0\rangle \tag{24}$$

and by using the commutation relations we find

$$M = \frac{2^{\ell_{n-1}}\,\ell_{n-1}!\,(\ell_n-\ell_{n-1})!\,(n+\ell_n+\ell_{n-1}-3)!\,(n+2\ell_{n-1}-4)!!}{(n+2\ell_{n-1}-3)!\,(n+2\ell_n-4)!!}.$$

The modified bosons a_n appearing in (24) depend on n, but the factors $(a_1 - i\,a_2)$ are independent of n because only the boson parts contribute. Now the generators of O(n-1) act only on $(a_1 - i\,a_2)^{\ell_{n-1}}$ because a_n is invariant under O(n-1). The state which is semi-maximal in O(n-2) is known, so that

$$|\ell_n, \ell_{n-1}, \ell_{n-2} \ldots \ell_{n-2}\rangle = M^{-\frac{1}{2}}\, a_n^{\ell_n-\ell_{n-1}}\, a_{n-1}^{\ell_{n-1}-\ell_{n-2}}\,(a_1 - i\,a_2)^{\ell_{n-2}}\,|0\rangle$$

where the a_{n-1} are modified bosons in n-1 dimensions. Continuing this way we find

$$|\ell_n, \ell_{n-1}, \ldots \ell_2\rangle = M^{-\frac{1}{2}}\, a_n^{\ \ell_n-\ell_{n-1}}\, a_{n-1}^{\ \ell_{n-1}-\ell_{n-2}} \ldots a_3^{\ \ell_3-\ell_2}\, (a_1 - i\, a_2)^{\ell_2}\, |0\rangle \tag{25}$$

where a_i is the modified boson for O(i), i.e., we have varied the dimension n on which the modified bosons depend.

The expression (25) has a much simpler appearance than that previously obtained ([51,65]) and the structure imposed by the sub-group chain $O(n) \supset O(n-1) \ldots \supset O(2)$ is clearly visible. It is possible to write these symmetric states using modified bosons for a constant n dimensions only, but the result does not exhibit the same simplicity and structure. For example, for n = 5 we would have

$$\left|\begin{matrix}\ell_1 & 0\\ m_1 & 0\\ \ell & \\ \ell & \end{matrix}\right\rangle \propto (a_1 - i\, a_2)^{\ell}\, a_5^{\ \ell_1-\ell}\, C^{\ell+1}_{m_1-\ell}\left(\frac{a_4}{i\ a_5}\right) |0\rangle\ .$$

Then we apply J_-, which acts only on $(a_1 - i\, a_2)^{\ell}$. However $a_1^{\,2} + a_2^{\,2} + a_3^{\,2}$ equals not zero, but $-\,a_4^{\,2} - a_5^{\,2}$, and so (from (14)):

$$\left|\begin{matrix}\ell_1 & 0\\ m_1 & 0\\ \ell & \\ m & \end{matrix}\right\rangle \propto (a_1 - i\, a_2)^{m}\, a_5^{\ \ell_1-\ell}\, C^{\ell+1}_{m_1-\ell}\left(\frac{a_4}{i\ a_5}\right) \times C^{m+\frac{1}{2}}_{\ell-m}\left(\frac{a_3}{i\sqrt{a_4^{\,2} + a_5^{\,2}}}\right) (a_4^{\,2} + a_5^{\,2})^{\frac{\ell-m}{2}}\, |0\rangle\ .$$

Using the techniques we have described it would be possible to calculate basis functions for arbitrary representations of O(5), and the higher order groups. The basis functions would be complicated in structure but would be of interest in special function theory. Those functions obtained would be generalizations, probably new, of Jacobi polynomials in the same way that basis functions for SU(4) and higher orders are generalizations of the hypergeometric function ([29,33]).

CHAPTER 5

BASIS STATES AND LABELLING FOR Sp(n)

§1. Branching Theorems

We have constructed with modified bosons a space which carries all irreducible representations of Sp(n). It is of interest to know explicitly the form of the orthogonal basis states in this space, for the calculation of matrix elements and for the study of the special functions which might arise. Such properties of Sp(n) have received little attention in the literature partly because Sp(n) is lacking in some properties which the other classical groups possess. There is no symplectic group of odd dimension and so the subgroup chain

$$Sp(2\nu) \supset Sp(2\nu-2) \,..\, \supset Sp(2) \tag{1}$$

is not complete as for U(n) and O(n). As a result there is a problem in the labelling of basis states because the chain (1) does not provide a complete set of labels. $Sp(2\nu)$ is of order $\nu(2\nu+1)$ (the number of generators) and so we need to label uniquely the elements of a matrix of $\nu(2\nu+1)$ parameters. As Racah [26] and Biedenharn [95] have explained, it is necessary to find $\frac{r - 3\ell}{2}$ operators in addition to the group invariants and the diagonal generators H_i, where r is the order of the group, and ℓ is the rank. We need to provide, therefore, $\frac{1}{2}(\nu(2\nu+1) - 3\nu) = \nu^2 - \nu$ further labels. Of these the subgroup chain (1) provides $1 + .. + \nu - 1 = \frac{\nu}{2}(\nu - 1)$ labels by means of the subgroup invariants, so that $\frac{\nu}{2}(\nu - 1)$ operators are still required to label the basis states. For example, for Sp(2) (isomorphic to SU(2)) no more labels are required, but for Sp(4) one more operator has to be found.

The maximal subgroup of $Sp(2\nu)$ is $Sp(2\nu - 2) \times Sp(2)$ and this group can be inserted in the chain of subgroups

$$Sp(2\nu) \supset Sp(2\nu - 2) \times Sp(2) \supset Sp(2\nu - 2) .$$

However the $Sp(2)$ group provides only one more label which is still not sufficient except for $Sp(4)$. The reduction of $Sp(4)$ with respect to $Sp(2) \times Sp(2)$ has been studied by Holman [52] and basis states calculated. The methods employed here can be simplified by the use of modified bosons, and by imposing the traceless condition which reads $a_{12} + a_{34} = 0$ for $Sp(4)$ (see III (21)). This enables the basis states to be put in a simpler form.

In general we will try to solve the problem of state labelling in a manner analogous to that for $U(n)$ and $O(n)$, through the branching theorem. The representations of $Sp(2\nu)$ are specified by numbers $m_1,..m_\nu$ which are non-negative integers such that $m_1 \geqslant m_2 \geqslant .. \geqslant m_\nu \geqslant 0$. The reduction $Sp(2\nu)$ to $Sp(2\nu - 2)$ was first considered by Zhelobenko [56], who gave a statement of the branching theorem: The restriction of the representation $(m_1, .. m_\nu)$ of $Sp(2\nu)$ to $Sp(2\nu - 2)$ contains in its spectrum all representations $(q_1, .. q_{\nu-1})$ of $Sp(2\nu - 2)$ such that

$$m_1 \geqslant p_1 \geqslant m_2 \geqslant p_2 \geqslant m_3 \geqslant .. p_{\nu-1} \geqslant m_\nu \geqslant p_\nu \geqslant 0 ,$$

$$p_1 \geqslant q_1 \geqslant p_2 \geqslant q_2 \geqslant .. p_{\nu-1} \geqslant q_{\nu-1} \geqslant p_\nu \qquad (2)$$

where the indices q_i, p_j are all non-negative integers. This theorem was later proved also by Hegerfeldt [74], and results for $\nu = 2,3$ given by Whippman [74]. We have considered the reduction $Sp(2\nu)$ directly to $Sp(2\nu - 2)$ rather than through $Sp(2\nu - 2) \times Sp(2)$ the maximal subgroup, since we are seeking a labelling similar to that for $U(n)$ and $O(n)$.

Using the branching theorem Zhelobenko wrote down basis vectors $|(m)>$ as a "Gelfand" pattern which can be done in the following way ([94]):

$$(m) = \left| \begin{matrix} m_{\nu 1} & \cdot\cdot & m_{\nu\nu} \\ p_{\nu 1} & \cdot\cdot & p_{\nu\nu} \\ m_{\nu-1\ 1} & \cdot\cdot & m_{\nu-1\ \nu-1} \\ \vdots & & \\ m_{11} & & \\ p_{11} & & \end{matrix} \right\rangle \qquad (3)$$

where the m_{ij}, p_{ij} satisfy inequalities according to (2). The use of the branching theorem to write down basis states is the same as for O(n) (IV, §1), except that here the "intermediate" integers p_{ij} do not have any group theoretic significance. The p_{ij} are attached arbitrarily to representations of $Sp(2\nu - 2)$ labelled by the same numbers, which appear more than once and need to be distinguished by different p_{ij} labels. For Sp(4), for example, the representations of Sp(2) labelled ℓ appear in the representation (m_1, m_2) such that $0 \leqslant \ell \leqslant m_1$ and with multiplicity $(m_1 - \ell + 1)(m_2 + 1)$ for $\ell \geqslant m_2$, and $(m_1 - m_2 + 1)(\ell + 1)$ for $\ell \leqslant m_2$. The labels p_{ij} can be attached arbitrarily because representations of $Sp(2\nu - 2)$ with the same labels are all equivalent. The restriction of p_{ij} to the ranges given in (2) ensures that the $Sp(2\nu - 2)$ representations are counted correctly. The states (3) can be chosen to be orthogonal with respect to the labels p_{ij} as well as m_{ij} because the reduction $Sp(2\nu)$ to $Sp(2\nu - 2)$ is completely reducible, so that subspaces carrying representations of $Sp(2\nu - 2)$ labelled by the same numbers can be made orthogonal; the difference between such subspaces is their p_{ij} labels.

It follows from the considerations of Zhelobenko that the states (3) are also weight vectors, i.e., they are eigenvectors of the diagonal generators $H_1, \ldots H_\nu$. This is also the case for U(n), but not for O(n) where the Gelfand states are not weight vectors. The eigenvalues of $H_1, \ldots H_\nu$ may be found from the analysis of Zhelobenko:

$$H_j |(m)\rangle = \left[2 \sum_{i=1}^{j} p_{ji} - \sum_{i=1}^{j} m_{ji} - \sum_{i=1}^{j-1} m_{j-1,i}\right] |(m)\rangle \quad .$$

By approaching the labelling of states through the branching theorem we have cast the problem in a form very similar to that for U(n) and O(n). For Sp(n) however the labels p_{ij} do not have a group theoretic meaning because Sp(n) does not contain suitable subgroups. It is hoped that nevertheless the p_{ij} share some of the important properties possessed by the m_{ij}, in particular that they appear as eigenvalues of certain labelling operators. This seems reasonable considering that, for most properties, Sp(n) is "not quite so simple as GL(n) and not so complicated as O(n)" (Weyl [12] p.229).

In order to reveal the properties of the p_{ij} labels we will carry out explicit calculations for Sp(4) with the labelling (3). As will become apparent, this labelling is different from the subgroup labelling Sp(2) × Sp(2) which solves this particular problem, but which does not generalize suitably. Firstly however we calculate quite generally the state of highest weight, and symmetric basis states.

§2. State of Highest Weight

Since the representation space constructed from modified bosons is irreducible there is one state of highest weight. This state is specified by the representation labels $(m_1, \ldots m_\nu)$ and so can be calculated independently of any labelling problems. Our method here follows clearly that for O(n) (IV, §3). The state of highest weight is

$$|max.\rangle = M^{-\frac{1}{2}} a_1^{\,m_1-m_2} a_{13}^{\,m_2-m_3} \ldots a_{13..2\nu-1}^{\,m_\nu} |0\rangle \quad . \qquad (4)$$

To prove this it is sufficient to show that $|max.\rangle$ is annihilated by all raising generators (listed in III (28)). By applying the diagonal generators H_i we deduce that the numbers $(m_1, .. m_\nu)$ are not only the polynomial degrees, but also the representation labels. The requirement that the exponents be non-negative integers shows that we have obtained representations for which $m_1 \geq m_2 \geq .. \geq m_\nu \geq 0$, where the m_i are all integers i.e., we have obtained all representations of Sp(n).

Again as for O(n) it is easy to see that only the boson parts of the modified bosons in (4) contribute. As a result the normalization M is calculated using boson operator techniques [32].

The expression (4) has been used previously for Sp(4) by Holman [52]. Using a different metric the general state of highest weight has been calculated by Quesne [63] using the properties of complementary groups. However our method provides a more direct solution, without using the complementary group U(ν). The remarks of IV, §3 apply here also.

We note that (4) is also a Weyl state, corresponding to a Young diagram of $\nu = \frac{1}{2}n$ rows which is the maximum allowed (III, §2). We can write down other Weyl states by using the system of parameters which appear in the branching theorem, in the same way as we have done for U(n) and O(n) (IV, §2). It is a simple matter then to write down a complete set of non-orthogonal basis states. Taking Sp(4) for example the general Gelfand state is

$$\left|\begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ m & \end{matrix}\right\rangle$$

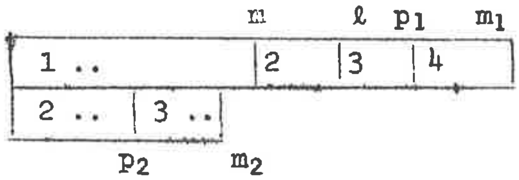
where all parameters are non-negative integers and

$$\begin{aligned} m_1 \geq p_1 \geq m_2 \geq p_2 \geq 0 \\ p_1 \geq \ell \geq p_2 \\ \ell \geq m \geq 0 \,. \end{aligned}$$

Here (m_1, m_2) are the Sp(4) labels and ℓ is the Sp(2) label. The corresponding Weyl state is

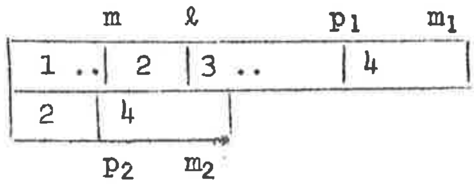
$$\left| \begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ m & \end{matrix} \right)$$

and the explicit expressions are obtained from the Young diagrams:



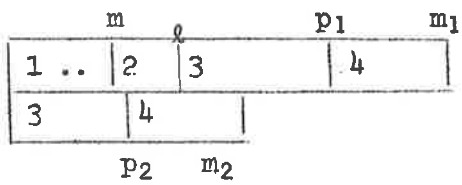
$$= a_4^{m_1-p_1} a_3^{p_1-\ell} a_2^{\ell-m} a_1^{m-m_2} a_{13}^{m_2-p_2} a_{12}^{p_2} |0>$$

$$(\ell \geqslant m_2, \quad m \geqslant m_2)$$



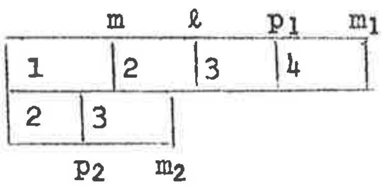
$$= a_4^{m_1-p_1} a_3^{p_1-m_2} a_{34}^{m_2-\ell} a_{24}^{\ell-m} a_{14}^{m-p_2} a_{12}^{p_2} |0>$$

$$(\ell < m_2, \quad m \geqslant p_2)$$



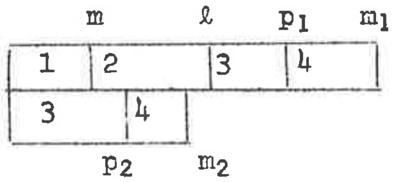
$$= a_4^{m_1-p_1} a_3^{p_1-m_2} a_{34}^{m_2-\ell} a_{24}^{\ell-p_2} a_{23}^{p_2-m} a_{13}^{m} |0>$$

$$(\ell < m_2, \quad m < p_2)$$



$$= a_4^{m_1-p_1} a_3^{p_1-\ell} a_2^{\ell-m_2} a_{23}^{m_2-m} a_{13}^{m-p_2} a_{12}^{p_2} |0>$$

$$(\ell \geqslant m_2, \quad p_2 \leqslant m < m_2)$$



$$= a_4^{m_1-p} \; a_3^{p_1-\ell} \; a_2^{\ell-m_2} \; a_{24}^{m_2-p_2} \; a_{23}^{p_2-m} \; a_{13}^{m} \; |0>$$

$$(\ell \geqslant m_2 , \;\; m < p_2) \; .$$

Weyl states can be written down in a similar way for all the symplectic groups. Recently [75] it was shown how to construct operators which are polynomials in the group generators and which can be used to obtain basis states by application to the state of highest weight. However the states obtained in this way are non-orthogonal. Clearly modified bosons have made these operators obsolete since we may write down by inspection, without use of lowering operators, the non-orthogonal but very simple Weyl states. Lowering operators are useful only when they allow the calculation of the orthogonal Gelfand states.

§3. Symmetric Basis States

It is possible to calculate basis states for O(n) in a symmetric representation (i.e., for which $m_2 = 0 = m_3 = .. m_\nu$) (see IV .23). The same is true for Sp(2ν), where the basis states may be labelled $|(\ell_\nu, p_\nu, .. \ell_1, p_1)$. Here ℓ_k refers to the label of the subgroup Sp(2k), and the numbers $p_1, .. p_\nu$ are defined from the eigenvalues of the commuting generators H_k, which have eigenvalues $2p_k - \ell_k - \ell_{k-1}$, for k = 1, .. ν. In this way the states are adequately labelled.

The state of highest weight is $|max.> = a_1^{\ell_\nu} |0>$. The Sp(2ν) state which is maximal in Sp(2ν-2) is $\psi = |\ell_\nu, p_\nu, \ell_{\nu-1}, .. \ell_{\nu-1}>$ and is constructed from the ordinary bosons $a_1, a_{2\nu-1}, a_{2\nu}$. This state is then a linear combination of the polynomials $a_1^p \, a_{2\nu-1}^q \, a_{2\nu}^r$, where $p + q + r = \ell_\nu$ since ℓ_ν is the polynomial degree. Also, because

$$H_\nu \psi = (2p_\nu - \ell_\nu - \ell_{\nu-1})\psi$$

$$H_1 \psi = \ell_{\nu-1} \psi$$

we have

$$q - r = 2p_\nu - \ell_\nu - \ell_{\nu-1} \quad , \quad p = \ell_{\nu-1} \ .$$

Therefore

$$\psi = a_1^{\ell_{\nu-1}} a_{2\nu-1}^{p_\nu - \ell_{\nu-1}} a_{2\nu}^{\ell_\nu - p_\nu} |0\rangle \ .$$

The generators of $Sp(2\nu-2)$ act only on $a_1^{\ell_{\nu-1}}$, so that continuing as above we obtain

$$|\ell_\nu, p_\nu, \ldots \ell_1, p_1\rangle$$

$$= N^{-\frac{1}{2}} a_{2\nu}^{\ell_\nu - p_\nu} a_{2\nu-1}^{p_\nu - \ell_{\nu-1}} a_{2\nu-2}^{\ell_{\nu-1} - p_{\nu-1}} \ldots a_2^{\ell_1 - p_1} a_1^{p_1} |0\rangle$$

where

$$N = (\ell_\nu - p_\nu)! \, (p_\nu - \ell_{\nu-1})! \, \ldots (\ell_1 - p_1)! \, p_1!$$

§4. Labelling Operators

In order to calculate arbitrary basis states we need to understand the method of labelling states, and for this reason we carry out explicit calculations on $Sp(4)$. In the usual approach we try to find a sufficient number of independent commuting operators whose eigenvectors then comprise the basis states labelled by the eigenvalues. We require that these operators be hermitean in order that the basis states be orthogonal.

For $Sp(4)$ we need to find 6 independent operators. Furthermore these must be chosen so as to specify the labels appearing in the Gelfand pattern

$$\left| \begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ m & \end{matrix} \right\rangle \ .$$

Our problem differs slightly from the usual state labelling problem in that we are seeking a particular labelling system. For Sp(4) the labelling of states is solved by the reduction

$$Sp(4) \supset Sp(2) \times Sp(2) \supset Sp(2)$$

but this solution is not the Gelfand type labelling.

Of the 6 operators required two are the Sp(2) subgroup labels. These are

$$H_1 = S_{21} \quad \text{with eigenvalue } 2m - \ell \ ,$$

and

$$\Lambda^2 = \tfrac{1}{2}\,\mathrm{Tr}(S^2) = H_1(H_1 + 2) - S_{11}\, S_{22}$$
$$\text{with eigenvalue } \ell(\ell + 2) \ .$$

We have used here the notation $(S^2)^i{}_j = S^i{}_p\, S^p{}_j$ (summation over p) where $S^i{}_j = \varepsilon^{ip}\, S_{pj}$ with $\varepsilon^{ij} = -\,\varepsilon_{ij}$. Therefore $\mathrm{Tr}(S^2) = S^q{}_p\, S^p{}_q$ $= \varepsilon_{qr}\, \varepsilon_{ps}\, S_{rp}\, S_{sq}$ (summation). (This notation is useful for the discussion of group invariants and characteristic identities; see [70,71].) H_1 and Λ^2 are hermitean and commute with each other. There are also the two group invariants of Sp(4). These invariants and their eigenvalues have been studied extensively in a general context ([70,71,76,77]) and may be listed as follows:

$$C_2 = \tfrac{1}{2}\,\mathrm{Tr}(S^2) = \tfrac{1}{2}\, S^p{}_q\, S^q{}_p \quad (p,q = 1, \ldots 4)$$
$$= H_1(H_1 + 4) + H_2(H_2 + 2) - S_{11}\, S_{22} - S_{33}\, S_{44}$$
$$+ 2\, S_{41}\, S_{32} - 2S_{31}S_{42}$$

with eigenvalue

$$S_2 = m_1(m_1 + 4) + m_2(m_2 + 2) \ ,$$

$$C_4 = \mathrm{Tr}(S^4) \ , \quad \text{with eigenvalue } 2S_4 + 6S_2 \qquad (5)$$

where

$$S_4 = (m_1 + 2)^4 + (m_2 + 1)^4 - 17 \ .$$

Another labelling operator is $H_2 = S_{43}$ with eigenvalue

$$2p_1 + 2p_2 - m_1 - m_2 - \ell \ .$$

As indicated earlier there remains one operator P to be found, which must be an Sp(2) invariant, hermitean, and commute with H_2. One possibility is $L = -(S^2)_{43} = S_{42}S_{31} - S_{41}S_{32}$. However, with what is a characteristic typical of state-labelling problems, this choice is not unique and there is another independent possibility:

$$\begin{aligned} M &= -(S^3)_{43} \\ &= S_{42}S_{11}S_{32} + S_{41}S_{22}S_{31} \\ &\quad - S_{41}S_{32}(H_1+1) - S_{42}S_{31}(H_1-1) . \end{aligned}$$

It is straightforward to show that other possibilities depend on the seven operators we have listed, namely

$$H_1, H_2, \Lambda^2, C_2, C_4, L, M \qquad (6)$$

We could choose for example $(S^r)_{43}$ for $r > 3$ but then we use the characteristic identity for Sp(2) ([71]), which reads $S^2 = 2S + \Lambda^2$, to express $(S^r)_{43}$ in terms of the operators above, e.g.,

$$\begin{aligned} (S^4)_{43} &= S_{4p}(S^2)^p{}_q\, S^q{}_3 \qquad (p,q = 1,2) \\ &= -4M - L(\Lambda^2 - 3) \end{aligned}$$

using

$$[\Lambda^2, S_{q3}] = 2\, S_{qr}\, S^r{}_3 - 3\, S_{3q} .$$

Similarly operators such as $S_{33}S_{44}$ and more generally $(S^r)_{33}(S^r)_{44}$ also are not independent. The invariant of the second Sp(2) group is $\Omega^2 = H_2(H_2+2) - S_{33}S_{44}$ which depends on C_2 in the following way:

$$C_2 - \Omega^2 = \Lambda^2 - 2L - 2H_2 .$$

Therefore by choosing L to be diagonal we would recover the Sp(2) × Sp(2) labelling. In this way the eigenvalues of L are easy to find. However the Sp(2) × Sp(2) labelling is not the Gelfand labelling as we can easily show. Using the notation of Holman [52] the states of $Sp(4) \supset Sp(2) \times Sp(2)$ are written

$$|J_m, \Lambda_m; \;\; J, \Lambda; \;\; M_J, M_\Lambda\rangle \tag{7}$$

where J_m, Λ_m are the $Sp(4)$ labels and J, M_J and Λ, M_Λ are the $Sp(2) \times Sp(2)$ subgroup labels. In our notation

$$m_1 = 2J_m, \; m_2 = 2\Lambda_m, \quad m = M_J + J, \quad \ell = 2J \; .$$

The branching theorems are given by the inequalities:

$$\begin{aligned} J_m - \Lambda_m \leqslant J + \Lambda \leqslant J_m + \Lambda_m \\ - J_m + \Lambda_m \leqslant J - \Lambda \leqslant J_m - \Lambda_m \\ - \Lambda \leqslant M_\Lambda \leqslant \Lambda \\ - J \leqslant M_J \leqslant J \; . \end{aligned} \tag{8}$$

If the $Sp(2) \times Sp(2)$ labelling is the same as the Gelfand labelling the state (7) will be equal to

$$\left| \begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ m & \end{matrix} \right\rangle$$

with

$$m_1 = 2J_m, \; m_2 = 2\Lambda_m, \quad \ell = 2J, \quad m = M_J + J \; .$$

Now p_1, p_2 may be defined as $p_1 = \max.(\ell)$, $p_2 = \min.(\ell)$ since $p_2 \leqslant \ell \leqslant p_1$. From (8) we see that both the following inequalities must be satisfied:

$$- 2\Lambda + 2J_m - 2\Lambda_m \leqslant 2J \leqslant 2J_m + 2\Lambda_m - 2\Lambda$$

$$2\Lambda - 2J_m + 2\Lambda_m \leqslant 2J \leqslant 2J_m - 2\Lambda_m + 2\Lambda$$

therefore

$$p_1 = \min.(2\Lambda + 2J_m - 2\Lambda_m, \; 2J_m + 2\Lambda_m - 2\Lambda)$$

$$p_2 = \max.(2\Lambda - 2J_m + 2\Lambda_m, \; - 2\Lambda + 2J_m - 2\Lambda_m) \; .$$

This mapping, which is the only one possible, is not acceptable because here p_1, p_2 are dependent, since they do not depend on M_Λ, whereas in the Gelfand basis state they are independent labels. In this way we deduce that the Gelfand labelling is not the $Sp(2) \times Sp(2)$ labelling and we cannot choose L to be the sixth labelling operator.

The required operator P will be a function of the operators (6). There is now the problem as to how the form of P may be determined. It was found that no method appeared to exist which led to a suitable choice of P.

One method tried is the following, and was used by Louck [69] for obtaining O(n) matrix elements. We try to find matrix elements of a generator in a certain basis by taking repeated commutators with a labelling operator to find the selection rules on the corresponding label. This method depends on being able to express the commutators in terms of the generator and labelling operators. For example, we have

$$[\Lambda^2,[\Lambda^2,S_{42}]] = 3S_{42} + 2S_{42}\Lambda^2 + 2\Lambda^2 S_{42}. \qquad (9)$$

By taking matrix elements of both sides we find

$$\left\langle \begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell' & \\ m' & \end{matrix} \right| S_{42} \left| \begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ m & \end{matrix} \right\rangle = 0 \qquad \text{unless } \ell' = \ell \pm 1.$$

This selection rule is a result of the fact that S_{42} is a vector component under Sp(2). Similar rules hold for the other generators. In the same way we take commutators using M and L and from these expressions hope to choose a combination of M and L which enables us to calculate a commutator in the form (9). However a suitable such combination could not be found and so this method fails. These type of calculations with repeated commutators are hindered by the fact that M is of third order and C_4, which may also be involved, is of fourth order, leading to complicated expressions. The expanded expression for C_4 is

$$C_4 = 4S_{33}\, MS_{44} + L(16H_2 - 8\Lambda^2 + 16 + 4M - 4L) + 2C_2 + 16H_2$$
$$+ M(4L + 4C_2 - 4H_2{}^2 - 4\Lambda^2 - 8H_2 - 16) + 2C_2{}^2 - 4C_2\,\Lambda^2 + 4\Lambda^4$$
$$+ 40\Lambda^2 + 20H_2{}^2$$

and the eigenvalue (5) is checked by applying C_4 to the state of highest weight.

Another method tried was to write down matrix elements in the most general possible form and impose the condition on these elements required to satisfy the commutation relations. Then by choosing expressions for P it was hoped this further information would allow the equations to be solved. Again this method was not successful.

Part of the problem here is that we are seeking a particular labelling scheme and therefore also a particular labelling operator. Without any knowledge of this operator, infinitesimal techniques are not effective. For example it is not possible to calculate lowering operators as has been done for U(n) and O(n). As mentioned earlier the operators of Mickelsson [75] lead only to non-orthogonal basis states.

§5. Sp(4) Labelling Parameters

In order to reveal the origin of the labels p_1, p_2 let us derive the branching theorem by identifying, as did Zhelobenko [56], the states of highest weight of the subgroup Sp(2).

The irreducible space which carries all representations (m_1,m_2) of Sp(4) consists of polynomials homogeneous of degree $m_1 - m_2$ in a_i, and of degree m_2 in a_{ij}. The subspace, denoted R, which is of highest weight in Sp(2) consists of polynomials which are annihilated by S_{22}, the raising generator of Sp(2). Now $[S_{22},a_i] = 0$ except for

$i = 2$. Therefore polynomials in R are constructed from a_1, a_3, a_4, a_{13}, a_{14}, a_{34} (we have $[S_{22}, a_{12}] = 0$ also, but $a_{12} = -a_{34}$), and these variables are not connected by any traceless condition.

Now let us consider simultaneously the irreducible space R´ which carries the representation $(m_1, m_2, 0)$ of U(3). This space consists of polynomials homogeneous of degree $m_1 - m_2$ in the variables α_1, α_2, α_3 and of degree m_2 in the variables α_{12}, α_{13}, α_{23} where the α's are ordinary bosons (the variable α_{123} does not appear because $m_3 = 0$).

We can establish a 1 - 1 mapping between the spaces R and R´ by corresponding the variables in the following way:

$$a_1 \leftrightarrow \alpha_1, \quad a_3 \leftrightarrow \alpha_2, \quad a_4 \leftrightarrow \alpha_3, \quad a_{13} \leftrightarrow \alpha_{12}$$

$$a_{14} \leftrightarrow \alpha_{13}, \quad a_{34} \leftrightarrow \alpha_{23} .$$

This correspondence determines a mapping between a function $f(a) \in R$ and the corresponding function $f(\alpha) \in R'$. In this mapping weight vectors are mapped into weight vectors. This is because the effect of H_1, H_2 on the a's is the same as the effect of E_{11}, $E_{22} - E_{33}$ (defined by II.5) on the α's. This is easily seen from the commutation relations:

$$\begin{aligned}
[H_1, a_1] &= a_1 & \qquad [E_{11}, \alpha_1] &= \alpha_1 \\
[H_1, a_3] &= 0 & [E_{11}, \alpha_2] &= 0 \\
[H_1, a_4] &= 0 & [E_{11}, \alpha_3] &= 0 \\
[H_2, a_1] &= 0 & [E_{22} - E_{33}, \alpha_1] &= 0 \\
[H_2, a_3] &= a_3 & [E_{22} - E_{33}, \alpha_2] &= +\alpha_2 \\
[H_2, a_4] &= -a_4 & [E_{22} - E_{33}, \alpha_3] &= -\alpha_3 .
\end{aligned}$$

The basis states $F(\alpha)$ in R´ may be denoted

$$F(\alpha) = \left| \begin{matrix} m_1 & m_2 & 0 \\ p_1 & p_2 & \\ \ell & & \end{matrix} \right\rangle$$

where p_1, p_2 are the U(2) subgroup labels, and ℓ the U(1) subgroup label. From our knowledge of U(3) we may write

$$E_{11}\ F(\alpha) = \ell\ F(\alpha)$$

$$(E_{22} - E_{33})F(\alpha) = (2p_1 + 2p_2 - m_1 - m_2 - \ell)F(\alpha).$$

It follows immediately that

$$H_1\ F(a) = \ell\ F(a)$$

$$H_2\ F(a) = (2p_1 + 2p_2 - m_1 - m_2 - \ell)F(a)\ ,$$

which shows that the representations of Sp(2) appearing in R are labelled ℓ. The number of times this representation appears is known from the branching theorem for $U(3) \supset U(2) \supset U(1)$, and so is determined by the inequalities

$$m_1 \geqslant p_1 \geqslant m_2 \geqslant p_2 \geqslant 0$$

$$p_1 \geqslant \ell \geqslant p_2 \quad .$$

In this way the numbers p_1, p_2 appear originally as U(2) labels, but once applied to Sp(4) are significant only as parameters which count correctly the multiplicity of the Sp(2) representations. Using these parameters we label the states f(a) in R as

$$\left|\begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ \ell & \end{matrix}\right\rangle .$$

However it does not follow that the corresponding state $f(\alpha)$ in R′ is labelled

$$\left|\begin{matrix} m_1 & m_2 & 0 \\ p_1 & p_2 & \\ \ell & & \end{matrix}\right\rangle ,$$

or vice versa. This is because although the states f(a) are orthogonal, the states $f(\alpha)$ will not be orthogonal because of the different commutation relations of the α's. In other words, the states F(a) corresponding to

$$F(\alpha) = \left|\begin{matrix} m_1 & m_2 & 0 \\ p_1 & p_2 & \\ \ell & & \end{matrix}\right\rangle$$

form a non-orthogonal basis in R, and the orthogonal states

$$\left| \begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ \ell & \end{matrix} \right\rangle$$

are a linear combination of the states F(a).

If we calculate matrix elements in the basis states F(a) then the assumption that these are orthogonal leads to contradictions, i.e., the matrices do not in fact represent the generators of Sp(4). A technique for writing down matrix elements of Sp(2ν) has been described by Gilmore [78]. The results for Sp(4) are those obtained with the above assumption and it is not difficult to show that the results are incorrect. If it is desired to write down matrix elements in a non-orthogonal basis it would be simpler to use the Weyl states.

§6. Calculation of Orthogonal Basis States

We have seen how the numbers p_1, p_2 have been introduced to label basis states, but no operators significance has been established for these numbers. If we choose a labelling operator P then its eigenfunctions will be linear combinations of the basis states in the representation space R of highest weight in Sp(2). (States not of highest weight in Sp(2) are easily obtained by application of the Sp(2) lowering generator). A set of non-orthogonal basis states in R is obtained from the U(3) representation space R′, namely

$$F(\alpha) = \left| \begin{matrix} m_1 & m_2 & 0 \\ p_1 & p_2 & \\ \ell & & \end{matrix} \right\rangle$$

which are hypergeometric functions (see II. 22) However there is a much simpler basis in R, the Weyl states of highest weight in Sp(2), which are:

$$\left| \begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ \ell & \end{matrix} \right) = \begin{array}{cccc} & \ell & p_1 & m_1 \\ \cline{1-3} \multicolumn{1}{|c|}{1\ \cdots} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} & \\ \hline \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} & & \\ \cline{1-2} p_2 & m_2 & & \end{array}$$

$$= a_1^{\ell-m_2} \; a_3^{p_1-\ell} \; a_4^{m_1-p_1} \; a_{14}^{m_2-p_2} \; a_{13}^{p_2} \; |0\rangle$$

$$(\ell \geq m_2)$$

and

$$\begin{array}{cccc} & \ell & p_1 & m_1 \\ \cline{1-3} \multicolumn{1}{|c|}{1\ \cdots} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} & \\ \hline \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} & & \\ \cline{1-2} p_2 & m_2 & & \end{array}$$

$$= a_3^{p_1-m_2} \; a_4^{m_1-p_1} \; a_{14}^{\ell-p_2} \; a_{34}^{m_2-\ell} \; a_{13}^{p_2} \; |0\rangle$$

$$(\ell \leq m_2) \; .$$

These Weyl states $|(m))$ are also weight vectors:

$$H_1 \; |(m)) = \ell \; |(m))$$

and

$$H_2 \; |(m)) = (2p_1 + 2p_2 - m_1 - m_2 - \ell) \; |(m)) \; . \qquad (10)$$

The Gelfand states are a linear combination of the Weyl states:

$$|(m)\rangle = \left| \begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ \ell & \end{matrix} \right\rangle = \sum_{p_1', p_2'} A_{p_1' p_2'} \left| \begin{matrix} m_1 & m_2 \\ p_1' & p_2' \\ \ell & \\ \ell & \end{matrix} \right) \; . \qquad (11)$$

We now show how to calculate exactly this linear combination, i.e., to obtain an explicit expression for the coefficients $A_{p_1' p_2'}$. We do this by firstly obtaining restrictions on the parameters of summation, and then by imposing the orthogonality requirements on the Gelfand states. We are in effect carrying out a Gram-Schmidt orthogonalization of the Weyl states, but only in the one way possible which leads to the Gelfand states. The final expression for the coefficients $A_{p_1' p_2'}$ involves the inverse of a certain matrix of scalar products, and by calculating some special cases we will see that these expressions cannot be further simplified.

In taking the linear combination (11) there is no summation over m_1, m_2 which are the group invariants, or over ℓ since we must have $H_1 \,|(m)\rangle = \ell \,|(m)\rangle$. In addition since

$$H_2 \,|(m)\rangle = (2p_1 + 2p_2 - m_1 - m_2 - \ell)\,|(m)\rangle$$

we have from (10) that the summation over p_1', p_2' is such that $p_1' + p_2' = p_1 + p_2$. (11) can now be written

$$|(m)\rangle = \sum_q A_q(\ell) \left|\begin{matrix} m_1 & m_2 \\ p_1-q & p_2+q \\ \ell & \\ \ell & \end{matrix}\right\rangle \qquad (12)$$

where $A_q(\ell)$ depends also on the parameters m_1, m_2, p_1, p_2. The summation over q is further restricted by the requirement that the inequality $p_1 \geq \ell \geq p_2$ must be satisfied. On the right hand side of (12) we see that $p_2 + q \leq \ell \leq p_1 - q$ so that the maximum value of ℓ is p_1 - min.(q). Similarly min.$(\ell) = p_2$ + min.(q). On the left hand side we have max.$(\ell) = p_1$, min.$(\ell) = p_2$ so that we must have min.(q) = 0, i.e., $q \geq 0$. (12) now becomes

$$\left|\begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ \ell & \end{matrix}\right\rangle = \sum_{q=0} A_q(\ell) \left|\begin{matrix} m_1 & m_2 \\ p_1-q & p_2+q \\ \ell & \\ \ell & \end{matrix}\right\rangle \qquad (13)$$

where the maximum value of q is determined by the inequalities

$$\begin{gathered} p_1 - q \geq m_2 \geq p_2 + q \\ p_1 - q \geq \ell \geq p_2 + q \quad . \end{gathered} \qquad (14)$$

The expression (13) enables us to write down several states explicitly, in particular the following state which in analogy with O(n) (IV, §4) we call the semi-maximal state:

$$\left|\begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ p_1 & \\ p_1 & \end{matrix}\right\rangle = |s.m.\rangle = M^{-\frac{1}{2}}\, a_1^{p_1-m_2}\, a_4^{m_1-p_1}\, a_{13}^{p_2}\, a_{14}^{m_2-p_2}\, |0\rangle \;.$$

The orthogonality of the basis states is assured here because p_1 is also the Sp(2) subgroup label. A similar expression can be written down quite generally for the semi-maximal state of $Sp(2\nu)$:

$$|s.m.\rangle = M^{-\frac{1}{2}}\, a_1{}^{p_1-m_2}\, a_{2\nu}{}^{m_1-p_1}\, a_{13}{}^{p_2-m_3}\, a_{12\nu}{}^{m_2-p_2} \cdots$$

$$\times\, a_{13..2\nu-3\,2\nu}{}^{m_\nu-p_\nu}\, a_{13..2\nu-1}{}^{p_\nu}\, |0\rangle \;.$$

At this stage we might try to deduce the form of P from the knowledge that P is diagonal on $|s.m.\rangle$. Since such an operator distinguishes between spaces labelled by p_1 and p_2 we might expect this to be the required operator. By calculating matrix elements of the operators (6) we can show that the following P is diagonal on $|s.m.\rangle$, with an eigenvalue that is known but is complicated:

$$P = 2M^2 - 3ML - 3LM + 3L^2 - 2L^2\Lambda^2 + 4LH_2\Lambda^2 + 2MH_2$$
$$- 22\Lambda^2 L - 12L + 4M\Lambda^2 + 3H_2{}^2 + 4H_2{}^2\Lambda^2 - 12H_2 - 16H_2\Lambda^2. \quad (15)$$

However this approach fails because from (13) we have that

$$\left|\begin{matrix} m_1 & m_2 \\ p_1 & m_2 \\ \ell & \\ \ell & \end{matrix}\right\rangle = M^{-\frac{1}{2}}\, a_1{}^{\ell-m_2}\, a_3{}^{p_1-\ell}\, a_4{}^{m_1-p_1}\, a_{13}{}^{m_2}\, |0\rangle$$

and P is not diagonal on this state. This means that a function of the form (13) cannot in general be an eigenfunction of (15). Clearly it is necessary to choose a more general P which is diagonal on the state $|(m)\rangle$.

In order to calculate the coefficients A_q we impose the orthogonality conditions required on the Gelfand basis states. The conditions may be written

$$\left\langle\begin{matrix} m_1 & m_2 \\ p_1-n & p_2+n \\ \ell & \\ \ell & \end{matrix}\right|\left.\begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ \ell & \end{matrix}\right\rangle = 0 \quad (16)$$

for $n = 1, \ldots p_1 - \ell$.

Orthogonality between the labels m_1, m_2, ℓ and $p_1 + p_2$ is already satisfied because (13) is an eigenvector of C_2, C_4, H_1, H_2. In the equations (16) it is not necessary to consider $n = 0$, because this would give simply the normalization. We are in fact carrying out a Gram-Schmidt orthogonalization with respect to the semi-maximal state since, beginning with $n = p_1 - \ell$ we ensure that all states

$$\left| \begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ \ell & \end{matrix} \right\rangle$$

are orthogonal to

$$|s.m.\rangle = \left| \begin{matrix} m_1 & m_2 \\ \ell & p_1+p_2-\ell \\ \ell & \\ \ell & \end{matrix} \right\rangle$$

and then, by putting $n = p_1 - \ell - 1,\ p_1 - \ell - 2,\ ..\ 1$, successively we ensure that

$$\left| \begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ \ell & \end{matrix} \right\rangle$$

is orthogonal to all other states.

The equations (16) can be greatly simplified by the knowledge that it is sufficient to require

$$\left(\begin{matrix} m_1 & m_2 \\ p_1-n & p_2+n \\ \ell & \\ \ell & \end{matrix} \middle| \begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ \ell & \end{matrix} \right\rangle = 0 \tag{17}$$

i.e., the Gelfand state is orthogonal to the Weyl states. We note that

$$\left| \begin{matrix} m_1 & m_2 \\ \ell & p_1+p_2-\ell \\ \ell & \\ \ell & \end{matrix} \right\rangle \propto \left| \begin{matrix} m_1 & m_2 \\ \ell & p_1+p_2-\ell \\ \ell & \\ \ell & \end{matrix} \right)$$

so that (16) is equivalent to (17) for $n = p_1 - \ell$. Suppose (16) is equivalent to (17) for $n = p_1 - \ell,\ p_1 - \ell - 1,\ ..\ p_1 - \ell - k$. Now

$$\left\langle \begin{matrix} m_1 & m_2 \\ \ell+k+1 & p_1+p_2-\ell-k-1 \\ \ell & \\ \ell & \end{matrix} \right| = \bar{A}_{p_1-\ell} \left(\begin{matrix} m_1 & m_2 \\ \ell & p_1+p_2-\ell \\ \ell & \\ \ell & \end{matrix} \right| + \;..$$

$$+ \bar{A}_1 \left(\begin{matrix} m_1 & m_2 \\ \ell+k & p_1+p_2-\ell-k \\ \ell & \\ \ell & \end{matrix} \right| + \bar{A}_0 \left(\begin{matrix} m_1 & m_2 \\ \ell+k+1 & p_1+p_2-\ell-k-1 \\ \ell & \\ \ell & \end{matrix} \right| .$$

(In this expression the Weyl states and corresponding coefficients for which the inequalities (14) are not satisfied will be zero.) The first k terms of this expression are already orthogonal to

$$\left| \begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ \ell & \end{matrix} \right\rangle$$

by assumption, so that (16) implies that (17) holds for $n = p_1 - \ell - k - 1$ also. By induction then (16) is equivalent to (17).

(17) can be rewritten:

$$\sum_{q=0}^{p_1-\ell} A_q(\ell) \left(\begin{matrix} m_1 & m_2 \\ p_1-n & p_2+n \\ \ell & \\ \ell & \end{matrix} \middle| \begin{matrix} m_1 & m_2 \\ p_1-q & p_2+q \\ \ell & \\ \ell & \end{matrix} \right) = 0 . \qquad (18)$$

We may assume that $A_0(\ell) = 1$ by bringing out a normalization factor $N^{-\frac{1}{2}}(\ell)$, which is determined from $\langle(m)|(m)\rangle = 1$, once A_q is known. (18) now consists of $p_1 - \ell$ simultaneous equations for the same number of unknowns, and may be written $MA = - K$ where M is a $p_1 - \ell \times p_1 - \ell$ matrix with elements

$$M_{nq} = \left(\begin{matrix} m_1 & m_2 \\ p_1-n & p_2+n \\ \ell & \\ \ell & \end{matrix} \middle| \begin{matrix} m_1 & m_2 \\ p_1-q & p_2+q \\ \ell & \\ \ell & \end{matrix} \right)$$

and

$$K_n = \left(\begin{matrix} m_1 & m_2 \\ p_1-n & p_2+n \\ \ell & \\ \ell & \end{matrix} \middle| \begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ \ell & \end{matrix} \right) .$$

Hence

$$A_q(\ell) = - [M(\ell)^{-1}]_{qp} K_p(\ell) .$$

The normalization is given by

$$N(\ell) = \sum_{q,r=0} \bar{A}_r(\ell)\, A_q(\ell) \left(\begin{array}{ll} m_1 & m_2 \\ p_1-r & p_2+r \\ \ell & \\ \ell & \end{array}\right|\left.\begin{array}{ll} m_1 & m_2 \\ p_1-q & p_2+q \\ \ell & \\ \ell & \end{array}\right)$$

$$= \sum_{q=0}^{p_1-\ell} A_q(\ell) \left(\begin{array}{ll} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ \ell & \end{array}\right|\left.\begin{array}{ll} m_1 & m_2 \\ p_1-q & p_2+q \\ \ell & \\ \ell & \end{array}\right)$$

since from (18) only the term $r=0$ contributes, for which $A_o = 1$.

Therefore

$$N(\ell) = \left(\begin{array}{ll} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ \ell & \end{array}\right|\left.\begin{array}{ll} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ \ell & \end{array}\right) - K(\ell)\, M^{-1}(\ell)\, K(\ell)\ .$$

The evaluation of A_q and N depends on being able to calculate $M_{pq}(\ell)$. This scalar product can be calculated using the commutation relations (III.28), but only with some difficulty. An easier method is to use that outlined in III, §7 taking advantage of the projection operator H. The functions in R are homogeneous polynomials in modified bosons but may be written as polynomials in ordinary bosons with the help of H. A function f in R then takes the form H f(a,b) where

$$H = \sum_{r=0}^{m_2} \frac{(-)^r(m_1 + m_2 - r + 2)!}{r!(m_1 + m_2 + 2)!} (a_{12} + a_{34})^r\, (\bar{a}_{12} + \bar{a}_{34})^r$$

and the a's are now ordinary bosons. f does not contain any operators a_2 or b_2 (since R is of highest weight in Sp(2)) and so the terms $\bar{a}_{12}$ and a_{12}, when acting to the right and left respectively give no contribution. In R then H takes the form

$$H = \sum_{r=0}^{m_2} \frac{(-)^r(m_1 + m_2 + 2 - r)!}{r!(m_1 + m_2 + 2)!}\, a_{34}^r\, \bar{a}_{34}^r\ .$$

We are now able to calculate any basis state

$$\left|\begin{array}{ll} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ m & \end{array}\right\rangle$$

where the $U(1) \subset Sp(2)$ label m is obtained by applying $(S_{11})^{\ell-m}$ to

$$\left|\begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ \ell & \end{matrix}\right\rangle .$$

In order to obtain some idea of the complexity of the coefficients A_q let us calculate the expression for

$$\left|\begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ p_1-1 & \\ p_1-1 & \end{matrix}\right\rangle$$

which is

$$N^{-\frac{1}{2}}_{(p_1-1)}\, a_1{}^{p_1-m_2-1}\, a_3\, a_4{}^{m_1-p_1}\, a_{13}{}^{p_2}\, a_{14}{}^{m_2-p_2}\left(1 + A_1(p_1 - 1)\frac{a_4 a_{13}}{a_3 a_{14}}\right)|0\rangle .$$

We need to know

$$\left(\begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ p_1 & \\ p_1 & \end{matrix}\middle|\begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ p_1 & \\ p_1 & \end{matrix}\right)$$

$$= \frac{(m_1 - p_1)!^2 p_2!^2 (p_1 + 1)!(m_1 - p_2 + 1)!(m_2 - p_2)!}{(m_1 + m_2 + 2)!(m_1 - m_2 + 1)!}$$

$$\times \sum_r \frac{(-)^r (m_1 + m_2 + 2 - r)!(p_1 - m_2 + r)!}{r!(m_1 - p_1 - r)!(p_2 - r)!(p_1 - p_2 + r + 1)!}$$

and

$$\left(\begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ p_1-1 & \\ p_1-1 & \end{matrix}\middle|\begin{matrix} m_1 & m_2 \\ p_1-1 & p_2+1 \\ p_1-1 & \\ p_1-1 & \end{matrix}\right)$$

$$= \frac{(p_1+m_1+2)(m_1-p_1+1)!(p_2+1)!p_2!(m_1-p_1)!p_1!(m_2-p_2)!(m_1-p_2)!}{(m_1+m_2+2)!(m_1-m_2+1)!}$$

$$\times \sum_r \frac{(m_1 + m_2 + 1 - r)!(-)^r (p_1 - m_2 - 1 + r)!}{r!(m_1 - p_1 - r)!(p_2 - r)!(p_1 - p_2 + r)!}$$

We find then

$$A_1(p_1 - 1) = - \frac{(p_1+m_1+2)(m_2-p_2)\,{}_3F_2\left(\begin{matrix}p_1-m_2,\ -m_1+p_1,\ -p_2;\\ -m_1-m_2-1,\ p_1-p_2+1\end{matrix}\right)}{(m_1+m_2+2)(p_1-p_2)\,{}_3F_2\left(\begin{matrix}p_1-m_2,\ -m_1+p_1-1,\ -p_2-1;\\ -m_1-m_2-2,\ p_1-p_2\end{matrix}\right)}$$

which cannot be simplified further. This result is much more complicated than had been expected by analogy with U(n) and O(n). It is possible to treat the problem of state labelling for U(3) and O(4) in the same way as we have done here for Sp(4), without using properties of U(2) and O(3) respectively. We would carry out the reductions $U(3) \supset U(1)$ and $O(4) \supset O(2)$ directly and then obtain the required basis states (II.22 and IV.18) which are much simpler in form. For example the U(3) basis function to be compared with (19) is

$$\left|\begin{matrix} m_1 & m_2 & 0 \\ p_1 & p_2 & \\ p_1-1 & & \end{matrix}\right\rangle$$

$$\propto a_1^{p_1-m_2-1}\, a_2\, a_3^{m_1-p_1}\, a_{12}^{p_2}\, a_{13}^{m_2-p_2} \left(1 + \frac{m_2 - p_2}{p_1 - m_2}\,\frac{a_1\, a_{23}}{a_2\, a_{13}}\right) |0\rangle$$

The calculation of further coefficients, such as $A_1(p_1 - 2)$, $A_2(p_1 - 2)$ shows only that these are also of similar complexity, and lacking in the structure we expected.

With the basis states known explicitly, it is possible to calculate matrix elements of the group generators. It is sufficient to calculate only the matrix elements of one generator, S_{32} for example, since all other representation matrices are then determined by commutation. It was hoped that the following selection rules would apply: $\Delta p_1, \Delta p_2 = 0, \pm 1$, i.e.,

$$\left\langle \begin{matrix} m_1 & m_2 \\ p_1' & p_2' \\ \ell' & \\ m' & \end{matrix} \right| S_{32} \left| \begin{matrix} m_1 & m_2 \\ p_1 & p_2 \\ \ell & \\ m & \end{matrix} \right\rangle = 0$$

$$\text{unless } p_1' = p_1 \text{ or } p_1 \pm 1, \quad p_2' = p_2 \text{ or } p_2 \pm 1 .$$

Calculations with the basis states for special cases show this is not the case. The matrix element is non-zero for Δp_1, $\Delta p_2 = \pm 2$, and higher values.

When the representation matrices are determined it is possible to find the matrix elements of L and M and then obtain a function of these operators which is diagonal, and which will be the labelling operator P. That such an operator exists is known because the set of representation matrices form a complete matrix algebra (Weyl [12] p91). The diagonal matrix with the required eigenvalues can then be expressed as a representation dependent function of the generator representation matrices.

§7. Conclusion

We have labelled the basis states of Sp(2ν) using the branching theorem, in the hope that these labels will possess all the properties we require of them, even though they do not all have the significance of group invariants. We have approached the problem in a way which would also be applicable to U(n) and O(n), where the results are simple with a clear structure. Calculations for Sp(4) however show that corresponding expressions for Sp(2ν) are much more complicated and lacking in structure. The intermediate labels p_{ij} do not possess all the properties of group invariant labels so that, for example, selection rules for matrix elements are more complicated. Although labelling operators exist, and are determined in principle from the orthogonal basis states, their form is complicated to the point that explicit expressions for them are of no practical use.

In seeking a solution to the state labelling problem there is the question of the kind of solution required. For the SU(3) ⊃ O(3) problem it is necessary to find one label which is not a group invariant. In the approach of Bargmann and Moshinsky [23] a labelling operator is chosen for its physical significance. Eigenfunctions are linear combinations of the known non-orthogonal basis states, and eigenvalues may then also be calculated in principle. Our situation is the same, where we have the non-orthogonal Weyl states and a labelling operator may be chosen using L and M. In our case, as for SU(3) ⊃ O(3) also, results are simpler in a non-orthogonal basis than in an orthogonal basis. Although we have not solved the problem if it is required that a labelling operator and its eigenvalues be known explicitly, we have specified the explicit form of a complete set of orthogonal basis states. A labelling operator is then determined implicitly.

Our approach is also possible for the well known SU(3) ⊃ O(3) state labelling problem. The number q which appears in the branching theorem without group significance can be used to list the non-orthogonal basis states [23]. By an orthogonalization process we obtain the required states from which a labelling operator can be implicitly defined. The reduction SU(3) ⊃ O(3) corresponds to the reduction of the representation space as described in III, §3. Modified bosons can be used within the harmonic subspaces on which O(3) acts.

In the application of our approach to the SU(3) ⊃ O(3) problem we seek to label the states as

$$\left| \begin{matrix} m_1 & & m_2 \\ & \ell & \\ & m & \end{matrix} , q \right\rangle ,$$

where m_1, m_2 are the SU(3) labels, ℓ, m the O(3), O(2) labels, and q is the extra quantum number appearing in the branching theorem. These orthogonal states will be a linear combination of the non-orthogonal states listed by Bargmann and Moshinsky [23], and this combination is found by carrying out a Gram-Schmidt orthogonalization in the single way which produces the required states, as we have done for Sp(4). It should be noted that in this approach we seek a particular labelling scheme, in which the extra quantum number q has the simplest possible properties, i.e., q takes integer values between known limits. Other labelling schemes are possible where the extra label has less simple properties, but where, on the other hand, the labelling operators may be less complicated. The problems in finding such operators have been explained by Racah [96], and are similar to those we have encountered for Sp(4).

CHAPTER 6

TRIPLE COMMUTATION RELATIONS

§1. Definition and Properties of Operators

In the study of the commutation relations satisfied by modified bosons (Chpt. III), triple commutation relations of the following form were encountered:

$$[a_i, a_j] = 0$$
$$[a_i, \tfrac{1}{2}[\bar{a}_j, a_k]] = \delta_{ik}\, a_j - \delta_{jk}\, a_i - \delta_{ij}\, a_k \; . \qquad (1)$$

In this chapter we investigate these relations more fully and determine realizations other than modified boson operators. Techniques for analysing triple commutation relations are known from parastatistics [79]. In fact the parafermi relations,

$$[a_i, [a_j, a_k]] = 0$$
$$[a_i, \tfrac{1}{2}[\bar{a}_j, a_k]] = \delta_{ij}\, a_k \qquad (2)$$

are similar to (1) but slightly more complicated because both relations are trilinear. For dimension n = 1 the relations (1) become

$$[a, \tfrac{1}{2}[\bar{a}, a]] = -\, a \qquad (3)$$

which is the same as for parafermions in one dimension except for the sign, and is identical to a relation studied by Kademova and Kraev [80,81]. These authors generalized (3) to the following:

$$[a_i, [a_j, a_k]] = 0$$
$$[a_i, \tfrac{1}{2}[\bar{a}_j, a_k]] = -\,\delta_{ij}\, a_k \qquad (4)$$

which is a simple modification of the parafermi relations. The operators satisfying (4) were introduced ([80,81,82]) as the definition of a new parastatistics in field theory, but it has been shown [83] that a self consistent theory is not possible with such a generalization.

The trilinear relations (1) offer a different generalization of the relations (4). It is therefore of interest to study these operators so as to reveal properties likely to be important in both field theory and group theory.

In addition to (1) we have the conjugate relations

$$[\bar{a}_i, \bar{a}_j] = 0$$

and

$$[\bar{a}_i, \tfrac{1}{2}[a_j, \bar{a}_k]] = \delta_{ik}\,\bar{a}_j - \delta_{jk}\,\bar{a}_i - \delta_{ij}\,\bar{a}_k \, . \tag{5}$$

The Jacobi identity requires that

$$[a_i, [\bar{a}_j, a_k]] = [a_k, [\bar{a}_j, a_i]]$$

which is seen to be satisfied. Let

$$N_{ij} = \tfrac{1}{2}[\bar{a}_i, a_j] \, .$$

Then from (1)

$$[N_{ij}, N_{k\ell}] = \delta_{ik}\,N_{j\ell} - \delta_{jk}\,N_{i\ell} - \delta_{j\ell}\,N_{ki} + \delta_{i\ell}\,N_{kj} \, . \tag{6}$$

By interchanging the indices $(ij) \leftrightarrow (k\ell)$ we have

$$[N_{k\ell}, N_{ij}] = \delta_{ik}\,N_{\ell j} - \delta_{i\ell}\,N_{kj} - \delta_{\ell j}\,N_{ik} + \delta_{kj}\,N_{i\ell} \, . \tag{7}$$

Comparing (6) and (7) we obtain

$$\delta_{ik}\,(N_{\ell j} + N_{j\ell}) = \delta_{\ell j}\,(N_{ik} + N_{ki}) \tag{8}$$

so that

$$N_{ij} + N_{ji} = 0 \qquad \text{for } i \neq j$$

and

$$N_{ii} = N_{jj} = M \qquad \text{for all } i, j.$$

Generally therefore $N_{ij} + N_{ji} = 2\,\delta_{ij}\,M$. The independent operators we can form from a_i, $\bar{a}_j$ are then

$$J_{ij} = \frac{i}{2}(N_{ij} - N_{ji}) \qquad \left(\frac{(n-1)n}{2} \text{ operators}\right)$$

and M, a_i, $\bar{a}_i$, a total of $\frac{(n+1)(n+2)}{2}$ operators.

It follows from (6) that

$$[J_{ij}, J_{k\ell}] = i(\delta_{ik} J_{j\ell} + \delta_{j\ell} J_{ik} - \delta_{jk} J_{i\ell} - \delta_{i\ell} J_{jk})$$

so that the operators J_{ij} are the generators of the Lie algebra of O(n). From (1) we see that

$$[J_{ij}, a_k] = i(\delta_{ik} a_j - \delta_{jk} a_i)$$

showing that a_i, $\bar{a}_i$ behave as vectors under transformations of O(n).

We have defined J_{ij} for $i,j = 1, .. n$. In addition let

$$J_{n+1,i} = \tfrac{1}{2}(a_i + \bar{a}_i) = - J_{i\ n+1} \qquad i = 1, .. n ,$$

$$J_{n+2,j} = \frac{i}{2}(\bar{a}_j - a_j) = - J_{j\ n+2} \qquad j = 1, .. n$$

and

$$J_{n+2,n+1} = M = - J_{n+1,n+2}$$

with

$$J_{n+1,n+1} = 0 = J_{n+2,n+2} .$$

The operators J_{ij} are all hermitean and they satisfy the following:

$$[J_{ij}, J_{k\ell}] = i(\rho_{ik} J_{j\ell} + \rho_{j\ell} J_{ik} - \rho_{jk} J_{i\ell} - \rho_{i\ell} J_{jk})$$

$$i,j,k,\ell = 1, .. n+2 ,$$

where

$$\rho_{ij} = \delta_{ij} \qquad i,j = 1, .. n$$

$$= - \delta_{ij} \qquad i,j = n+1, n+2$$

$$= 0 \qquad \text{elsewhere.}$$

We have shown that the algebra of the 2n operators defined by (1), is isomorphic to the Lie algebra of O(n,2). By comparison the operators defined by (4) generate the Lie algebra of O(2n,1), and both cases reduce to O(2,1) ≈ SU(1,1) for n=1.

Let us find the representations of the commutation relations (1) in a Fock space, i.e., we find a certain class of representations of $O(n,2)$. In this space there is a unique state $|0\rangle$ which obeys $\bar{a}_i |0\rangle = 0$ for $i = 1, .. n$, and which we call the vacuum state. The Fock space is then a Hilbert space which is the closure of vectors of the form $P(a)\,|0\rangle$ where P is an arbitrary polynomial. In this space we have from (5)

$$\bar{a}_i\, \bar{a}_j\, a_k\, |0\rangle = 0 \qquad \text{for all } i \;,$$

so that, because $|0\rangle$ is unique, $\bar{a}_j\, a_k\, |0\rangle = C_{jk}\, |0\rangle$. Now, by applying (6) to the state $|0\rangle$ we find

$$\delta_{ik}\, C_{j\ell} + \delta_{i\ell}\, C_{kj} = \delta_{jk}\, C_{i\ell} + \delta_{j\ell}\, C_{ki} \;.$$

Putting $j = k$, $i \neq j \neq \ell$, we obtain $C_{i\ell} = 0$, for $i \neq \ell$, and putting $k = j$, $\ell = i$ we obtain $C_{ii} = C_{jj} = p$, a constant. Generally we have

$$\bar{a}_i\, a_j\, |0\rangle = p\, \delta_{ij}\, |0\rangle \tag{9}$$

and therefore

$$p = \langle 0|\, \bar{a}_i\, a_i\, |0\rangle \geqslant 0 \;,$$

and is independent of i.

The representations of the commutation relations (1) in the Fock space are characterized by the number p, since each space for a given p is invariant under a_i and $\bar{a}_i$. The representations are infinite-dimensional since the polynomials $P(a)\,|0\rangle$ can be made of arbitrary high degree without vanishing, for $p > 0$. p can be any non-negative number since

$$\langle 0|\, \bar{a}_i^{\,q}\, a_i^{\,q}\, |0\rangle = q!\, \frac{\Gamma(p + q)}{\Gamma(p)} \;,$$

which is always non-negative, so that states in the Hilbert space are always well defined. The number p, to which we refer as the order of the statistics defined by (1), takes continuous values whereas for parafermi statistics p can be integral only.

We have found a realization of the trilinear commutation relations using modified bosons (see III.32). This realization satisfies the further property $\bar{a}_q^2 = 0 = a_q^2$ (summation). However not all operators defined by (1) possess this property. From (1) and (9)

$$\bar{a}_i \, a_q^2 \, |0\rangle = 2(p + 2 - n) \, a_i \, |0\rangle$$
$$= 0 \quad \text{if} \quad p = n-2.$$

It follows in the same way as for modified bosons (III, §4) that $a_q^2 = 0$ in the Fock space such that $p = n - 2$; on the other hand $p \neq n - 2$ implies $a_q^2 \neq 0$. We note that the realization as modified bosons satisfies $p = n - 2$. A more general realization for arbitrary p is given by

$$a_j = p \, z_j - r^2 \frac{d}{dz_j} + 2 \, z_j \, z_p \frac{d}{dz_p} ,$$
$$\bar{a}_i = \frac{d}{dz_i}$$

where $r^2 = z_q^2$. The vacuum state becomes the constant 1. For this case

$$[\bar{a}_i, a_j] = \delta_{ij}(p + 2N) + 2(z_j \frac{d}{dz_i} - z_i \frac{d}{dz_j})$$

where

$$N = z_q \frac{d}{dz_q} ,$$

and we can check (8) directly.

§2. Realizations as Double Bosons

There are other realizations of the trilinear commutation relations. Let

$$a_1 = \tfrac{1}{2} \alpha_1^2 + \tfrac{1}{2} \alpha_2^2$$
$$a_2 = \tfrac{1}{2} i \alpha_1^2 - \frac{i}{2} \alpha_2^2 \qquad (10)$$
$$a_3 = i \, \alpha_1 \, \alpha_2$$

where α_1, α_2 are ordinary bosons. It is readily checked that the equations (1) are satisfied for $i,j,k = 1,2,3$. We can let the unique boson vacuum state, defined by $\bar{\alpha}_1 |0\rangle = 0 = \bar{\alpha}_2 |0\rangle$, be also the vacuum state of the operators a_i, and in this Fock space we find that $p = 1$. Now since $n=3$ we have $p = n - 2$ so that $a_1^2 + a_2^2 + a_3^2 = 0$ as can be checked directly. Thus we have found operators which satisfy the traceless conditions but are different from modified bosons. The group theoretical origin of these operators will be examined in Chapter 7. It is of interest to find other double boson realizations of the triple commutation relations, since these operators offer an alternative to modified bosons.

In general we seek realizations in the following form:

$$a_i = \tfrac{1}{2} \gamma_i^{pq} \alpha_p \alpha_q \qquad (11)$$

$$i = 1, \ldots n$$

where γ_i is an $m \times m$ matrix which we can choose to be symmetric since the ordinary bosons α_p, α_q commute. We require that $n \leqslant \frac{m(m+1)}{2}$ in order that $a_1, \ldots a_n$ be independent.

Now

$$\bar{a}_i = \tfrac{1}{2} \bar{\gamma}_i^{pq} \bar{\alpha}_p \bar{\alpha}_q$$

where $\bar{\gamma}_i$ is the complex conjugate of γ_i and is also the hermitean conjugate. The relation $[a_i, a_j] = 0$ is satisfied identically. We have also

$$\begin{aligned} 2 N_{ij} &= [\bar{a}_i, a_j] \\ &= \tfrac{1}{2} \mathrm{Tr}(\gamma_j \bar{\gamma}_i) + (\gamma_j \bar{\gamma}_i)^{pq} \alpha_p \bar{\alpha}_q \end{aligned} \qquad (12)$$

and

$$[a_i, [\bar{a}_j, a_k]] = -(\gamma_i \bar{\gamma}_j \gamma_k)^{pq} \alpha_p \alpha_q .$$

We require

$$\tfrac{1}{2}(\gamma_i \bar{\gamma}_j \gamma_k + \gamma_k \bar{\gamma}_j \gamma_i) = \delta_{ij} \gamma_k + \delta_{jk} \gamma_i - \delta_{ik} \gamma_j . \qquad (13)$$

Therefore

$$\gamma_i \bar{\gamma}_i \gamma_i = \gamma_i \ , \tag{14}$$

and so

$$(\gamma_i \bar{\gamma}_i)^2 = \gamma_i \bar{\gamma}_i \ . \tag{15}$$

Now from (8), using (12) we deduce that

$$\gamma_i \bar{\gamma}_i = \gamma_j \bar{\gamma}_j \qquad \text{for all } i,j$$

and

$$\gamma_i \bar{\gamma}_j + \gamma_j \bar{\gamma}_i = 2\delta_{ij} \gamma_i \bar{\gamma}_i \ . \tag{16}$$

In finding representations of the matrices γ_k we will choose $\gamma_i \bar{\gamma}_i$, which is independent of i, to be diagonal. From (15) $\gamma_i \bar{\gamma}_i$ has eigenvalues 0 and 1 and therefore can be represented in block form as

$$\gamma_i \bar{\gamma}_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ .$$

If the matrices γ_k are all nonsingular then $\gamma_i \bar{\gamma}_i = 1$ and from (16)

$$\gamma_i \bar{\gamma}_j + \gamma_j \bar{\gamma}_i = 2\delta_{ij} \ . \tag{17}$$

Straightforward calculations show that, with the assumption of non-singularity, (17) is equivalent to (13).

If some of the matrices γ_k are singular then $\gamma_k \bar{\gamma}_k = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and from (14) we can write γ_k in block form:

$$\gamma_k = \begin{pmatrix} \Gamma_k & 0 \\ 0 & 0 \end{pmatrix} \ . \tag{18}$$

By using (16) we find that the matrices Γ_k must satisfy (17). We can see now that it is not necessary to consider where the γ_k are singular because from (18) this possibility has already been accounted for in (11), by taking smaller values of m.

We need to find symmetric nonsingular matrices satisfying (17). These equations appear similar to those satisfied by Clifford matrices, $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$. We can in fact show that all solutions are obtained from the Clifford algebra.

Beginning with γ_1 we may write

$$\gamma_1 = \gamma_{1R} + i\,\gamma_{1I}$$

where γ_{1R} and γ_{1I} are real. Then

$$\gamma_1\,\bar{\gamma}_1 = \gamma_{1R}^2 + \gamma_{1I}^2 - i[\gamma_{1R},\gamma_{1I}] = 1$$

so that

$$[\gamma_{1R},\gamma_{1I}] = 0 \ .$$

Therefore γ_{1R}, γ_{1I} are a pair of commuting symmetric matrices and so there exists a real orthogonal matrix such that γ_{1R}, γ_{1I} can be simultaneously diagonalized. γ_1 is therefore chosen to be diagonal, and $\gamma_1\,\bar{\gamma}_1 = 1$ requires that the eigenvalues λ satisfy $|\lambda|^2 = 1$. We can put

$$(\gamma_1)_{jk} = e^{i\theta_j}\,\delta_{jk} \ .$$

Now with

$$\gamma_2 = \gamma_{2R} + i\,\gamma_{2I} \ ,$$

the condition

$$\gamma_1\,\bar{\gamma}_2 + \gamma_2\,\bar{\gamma}_1 = 0$$

requires

$$0 = e^{i\theta_j}\,[(\gamma_{2R})_{jk} - i\,(\gamma_{2I})_{jk}] + [(\gamma_{2k})_{jk} + i(\gamma_{2I})_{jk}]\,e^{-i\theta_k} \ .$$

The real and imaginary parts reduce to the same equation, giving

$$(\gamma_{2I})_{jk} = -\frac{\cos\theta_j + \cos\theta_k}{\sin\theta_j + \sin\theta_k}\,(\gamma_{2R})_{jk}$$

$$= -\cot\left(\frac{\theta_j + \theta_k}{2}\right)\,(\gamma_{2R})_{jk} \ .$$

i.e.,

$$(\gamma_2)_{jk} = \left(1 - i\cot\frac{\theta_j + \theta_k}{2}\right)(\gamma_{2R})_{jk}$$

$$= -i\,e^{i\frac{\theta_j + \theta_k}{2}}\,(\sigma_2)_{jk} \ ,$$

where

$$(\sigma_2)_{jk} = (\gamma_{2R})_{jk}/\sin\frac{\theta_j + \theta_k}{2} \ ,$$

which is a real matrix.

Now $\gamma_2 \bar{\gamma}_2 = 1$ implies

$$\delta_{jk} = - i e^{i \frac{\theta_j + \theta_p}{2}} (\sigma_2)_{jp} \, i \, e^{-i \frac{(\theta_p + \theta_k)}{2}} (\sigma_2)_{pk}$$

$$= e^{i \frac{(\theta_j - \theta_k)}{2}} (\sigma_2{}^2)_{jk} \quad \text{i.e.} \quad \sigma_2{}^2 = 1 \, .$$

Putting

$$\Lambda_{jk} = e^{i\theta_j} \delta_{jk} \, ,$$

we have now

$$\gamma_1 = \Lambda \ (= \Lambda^{-1})$$

$$\gamma_2 = - i\Lambda^{\frac{1}{2}} \sigma_2 \Lambda^{\frac{1}{2}}$$

where σ_2 is an arbitrary real symmetric square root of unity.

By applying this result to γ_3 we find that γ_3 must have the form

$$\gamma_3 = i\Lambda^{\frac{1}{2}} \sigma_3 \Lambda^{\frac{1}{2}} \quad \text{where} \quad \sigma_3{}^2 = 1 \, .$$

Now

$$\gamma_2 \bar{\gamma}_3 + \gamma_3 \bar{\gamma}_2 = 0$$

implies

$$\Lambda^{\frac{1}{2}}\sigma_2\Lambda^{\frac{1}{2}}\Lambda^{-\frac{1}{2}}\sigma_3\Lambda^{-\frac{1}{2}} + \Lambda^{\frac{1}{2}}\sigma_3\Lambda^{\frac{1}{2}}\Lambda^{-\frac{1}{2}}\sigma_2\Lambda^{-\frac{1}{2}} = 0$$

i.e.

$$\sigma_2\sigma_3 + \sigma_3\sigma_2 = 0 \, .$$

We can see now that in general all solutions of (17) have the form

$$\gamma_1 = \Lambda$$

$$\gamma_k = - i\Lambda^{\frac{1}{2}} \sigma_k \Lambda^{\frac{1}{2}} \qquad k \geqslant 2$$

where $\{\sigma_j, \sigma_k\} = 2\delta_{jk}$ and where σ_k is real and symmetric. In this way all solutions are determined by the real, symmetric Clifford matrices. The representation properties of Clifford matrices are known [84] , and we find that there exist at most $2\nu + 1$ irreducible matrices of degree 2^ν for $\nu \geqslant 1$ satisfying the Clifford algebra. For $\nu = 1$ for example, the Clifford matrices are simply the three Pauli matrices:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

Of these only the first two are symmetric, giving the following γ matrices (where we have put $\Lambda = 1$):

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \gamma_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad , \quad \gamma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad ,$$

and we have then the realization (10).

This realization is of order p = 1, as indicated earlier. The order of the general realization (11) is found in the following way. From (12)

$$\bar{a}_i \, a_j \, |0\rangle = p \, \delta_{ij} \, |0\rangle$$

$$= \tfrac{1}{2} \, \mathrm{Tr}(\bar{\gamma}_i \, \gamma_j) \, |0\rangle + (\gamma_j \, \bar{\gamma}_i)^{pq} \, \alpha_p \, \bar{\alpha}_q \, |0\rangle$$

i.e.

$$p \, |0\rangle = \tfrac{1}{2} \, \mathrm{Tr}(\gamma_i \, \bar{\gamma}_i) \, |0\rangle + \alpha_q \, \bar{\alpha}_q \, |0\rangle \quad . \tag{19}$$

Although $\bar{a}_i \, |0\rangle = 0$ for all i it does not follow that $\bar{\alpha}_i \, |0\rangle = 0$ since the boson Fock space need not be the same as the Fock space of the operators a_i. In the choice of operators (10) for example the states $|0\rangle = \Phi$, $\alpha_1 \, \Phi$, $\alpha_2 \, \Phi$ all satisfy $\bar{a}_i \, |0\rangle = 0$, where Φ is the boson vacuum state. In this way the boson Fock space is reducible under a_i, $\bar{a}_j$. We should note however for this example that the space built up from the state $\alpha_1 \, \Phi$ includes also $\alpha_2 \, \Phi$, so that the vacuum state is not unique, and some of the above analysis does not apply.

From (19), using $\mathrm{Tr}(\gamma_i \, \bar{\gamma}_i) = m$, we obtain $p = \frac{m}{2} + N$ where N is the number of bosons in the state $|0\rangle$. In this way the realizations (11) can be of order $p = \tfrac{1}{2}, 1, \frac{3}{2}, 2 \ldots$. All of these realizations can be obtained in the boson Fock space (when N = 0). This space also has a unique vacuum and so is our most convenient choice.

The realization (11) will satisfy the traceless condition if p = n - 2. Now from Boerner [84] we find that of the $2\nu + 1$ matrices of dimension 2^{ν} satisfying the Clifford algebra $\nu + 1$ are real and symmetric. Therefore by including γ_1 we can obtain a maximum number of $n = \nu + 2$ matrices satisfying (17).

Now $p = \frac{m}{2} = 2^{\nu-1}$ so that if $p = n - 2$ is to be satisfied we require $\nu = 2^{\nu-1}$ i.e., $\nu = 1$ or $\nu = 2$. We see that there is only one other possibility for operators satisfying the traceless condition. This solution is obtained from the 4×4 Clifford matrices giving the following realization:

$$\begin{aligned} a_1 &= \alpha_1\alpha_2 + \alpha_3\alpha_4 \\ a_2 &= i\alpha_1\alpha_2 - i\alpha_3\alpha_4 \\ a_3 &= i\alpha_1\alpha_3 + i\alpha_2\alpha_4 \\ a_4 &= \alpha_1\alpha_3 - \alpha_2\alpha_4 \quad . \end{aligned} \tag{20}$$

The group theoretic origin of these operators will be examined more closely in Chpt 7.

We have shown how to obtain all solutions of the trilinear relations, as double boson operators and essentially only two of these satisfy the traceless condition. We have also recovered the realizations used by Kraev and Kademova for $n = 1$. If we take $m = 1$, then $\gamma = 1$ and $a = \frac{1}{2}\alpha^2$. For $m = 2$ we can choose $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ i.e., $a = \alpha_1\alpha_2$. Our analysis has revealed many other possibilities as well.

The trilinear relations we have found are not inconsistent as are those of Kraev and Kademova in general. If we define the number operator to be

$$N = \tfrac{1}{2}[\bar{a}_1, a_1] - \frac{p}{2} = M - \frac{p}{2}$$

then $[N, a_k] = a_k$, and N can be assumed consistently to have a spectrum of all non-negative integers. This assumption is consistent because we have found realizations for which this is true.

§3. Other Triple Commutation Relations

The relations (1) were found from modified bosons a_i by the substitution

$$a_i \to a_i(n + 2N - 2)^{\frac{1}{2}}$$
$$\bar{a}_i \to (n + 2N - 2)^{\frac{1}{2}} \bar{a}_i \ .$$

The same can be done with ordinary bosons, i.e., we put

$$a_i \to a_i(p + N)^{\frac{1}{2}}$$
$$\bar{a}_j \to (p + N)^{\frac{1}{2}} \bar{a}_j \ ,$$

where a_i are ordinary bosons, $N = a_p \bar{a}_p$ is the number operator, and p is any non-negative number. The operators obtained in this way also satisfy trilinear commutation relations:

$$[[a_i,\bar{a}_j],a_k] = -\delta_{ij}\, a_k - \delta_{jk}\, a_i \qquad (21)$$
$$[a_i,a_j] = 0 \ .$$

For n = 1 these equations reduce to (3), and so form another generalization of the statistics considered by Kademova and Kraev [80]. The algebra of the operators satisfying (21) is isomorphic to the Lie algebra of SU(n,1) ([85]). Realizations as double bosons exist, as for (1). It is likely that (21) will be more useful than (1) as a definition of a new parastatistics, since it is possible, unlike (1), to define an individual number operator, as well as the total number operator.

We mention also that it is possible to obtain trilinear commutation relations involving anti-commutators, by using fermions and modified fermions.

CHAPTER 7

SPINOR REPRESENTATIONS AND COVERING GROUPS

§1. Problems of Construction

The irreducible representations of O(n) are labelled by numbers $m_1, \ldots m_\nu$ $(\nu = [\frac{n}{2}])$ satisfying

$$m_1 \geq m_2 \geq \ldots m_\nu \geq 0 \qquad \text{for } O(2\nu + 1)$$

$$m_1 \geq m_2 \quad \ldots |m_\nu| \qquad \text{for } O(2\nu) .$$

In Chpt IV we constructed all the tensor representations, i.e. the single-valued representations for which $m_1 \ldots m_\nu$ are all integers. However there exist also the spinor representations which are double-valued and for which the labels are all semi-integers (half odd integers). These representations do not appear in the previous construction because the labels m_i are also the polynomial degrees in $a^i_{\ j}$ so that the m_i are necessarily integers. This fact has been overlooked by Wong [53], who has allowed the labels to be semi-integers without establishing a meaning for corresponding states.

The spinor representations arise because the group manifold of SO(n) is doubly connected for $n \geq 3$ (see Weyl [12], Chpt VIII, §12). Double-valued continuous functions may then be defined, in particular matrix elements of the representation can be double-valued, and so there will exist double-valued representations. Now it is important to be able to construct these representations because of their physical importance. For example, some particles have a spin angular momentum $\underline{S}$ which satisfies the commutation relations of O(3), $\underline{S} \times \underline{S} = i\,\underline{S}$. The eigen-value of $\underline{S}^2$ is then $s(s + 1)$ and s, the spin of the particle, labels the representation of $\underline{S}$. Many particles are known with half integer spin, such as electrons, protons, neutrons for which $s = \frac{1}{2}$. Clearly then the spinor representations cannot be ignored.

The easiest way to include these representations is to consider not actually the orthogonal group but its covering group, as is the case for SO(3) covered by SU(2) (see for example Bargmann [15]). The double-valued representations of O(n) become single-valued representations of the covering group, the spin group Spin (n). This method is effective only for the low order groups, where the covering group is also a classical group which is readily handled. The covering groups of SO(n) are SU(2), SU(2) × SU(2), Sp(4), SU(4), for n = 3, 4, 5, 6 respectively.

We will see how to carry out a simple mapping from the orthogonal group to its covering group, including in this way the spinor representations. As this method does not generalize to all orthogonal groups we seek also a construction independent of the covering group. Possible approaches are mentioned in this chapter, but a complete solution, using the same harmonic space as for the tensor representations, is described in Chpt 9.

§2. Unified Treatment of $O(3) \approx SU(2)$

Operators satisfying triple commutation relations of the following form were studied in Chpt VI:

$$[a_i, a_j] = 0$$

$$[a_i, [\bar{a}_j, a_k]] = \rho_{ik}\, \rho_{jp}\, a_p - \delta_{jk}\, a_i - \delta_{ij}\, a_k ,$$

where previously we had $\rho_{ij} = \delta_{ij}$, but here we choose $\rho = \sigma$. One possible realization of these operators uses modified bosons which satisfy $\sigma_{pq}\, a_p\, a_q = 0$. There are also other realizations satisfying the traceless condition, such as (for n = 3):

$$a_1 = \tfrac{1}{2}\,\alpha_1{}^2$$

$$a_2 = \frac{1}{\sqrt{2}}\,\alpha_1\alpha_2 \qquad (2)$$

$$a_3 = -\tfrac{1}{2}\,\alpha_2{}^2$$

where α_1. α_2 are ordinary bosons (these operators are the spherical components of those given in VI 10).

Let

$$K_{ij} = \tfrac{1}{2}\,\sigma_{ip}\,[\bar{a}_j, a_p] - \tfrac{1}{2}\,\sigma_{jp}\,[\bar{a}_i, a_p] \ . \qquad (3)$$

Then K_{ij} generate the Lie algebra of O(n), and the a_i transform as vectors. As shown in Chpt. VI, we can construct symmetric representations of O(n) in the Fock space of the operators defined by (1); the representation space will be irreducible provided the a_i are of order $p = n - 2$, so that $\sigma_{pq}\, a_p\, a_q = 0$. For SO(3) we can calculate basis states simply, using in particular $K_{ij}\, |0\rangle = 0$, and that the a_i behave as SO(3) vectors. In general we have

$$\left|\begin{matrix}\ell\\ m\end{matrix}\right\rangle = M^{-\frac{1}{2}}\, a_1{}^m\, a_2{}^{\ell-m}\, |0\rangle \ . \qquad (4)$$

Now if we use the realization of a_i as modified bosons we recover the expression for the spherical harmonics, as was studied in IV §5, and the generators (3) reduce to the usual form given in IV (13). As previously indicated, we obtain only the tensor representations, since the state of highest weight $|max.\rangle = a_1{}^\ell\, |0\rangle$ has meaning only for integral ℓ.

Suppose however we substitute for a_i with the realization (2). Then $|max.\rangle = \alpha_1{}^{2\ell}\, |0\rangle$ which is well-defined for half integral values of ℓ. In this way the operators (2) permit the construction of all spinor representations of SO(3). The general basis state has the form (from (4)): $\left|\begin{matrix}\ell\\ m\end{matrix}\right\rangle = N^{-\frac{1}{2}}\, \alpha_1{}^{\ell+m}\, \alpha_2{}^{\ell-m}\, |0\rangle$ and the generators are (from (3)):

$$K_{31} = \tfrac{1}{2}(\alpha_1\bar{\alpha}_1 - \alpha_2\bar{\alpha}_2) = \tfrac{1}{2}(E_{11} - E_{22})$$

$$K_{21} = \frac{1}{\sqrt{2}}\,\alpha_2\bar{\alpha}_1 \qquad = \frac{1}{\sqrt{2}}\,E_{21} \qquad (5)$$

$$K_{32} = \frac{1}{\sqrt{2}}\,\alpha_1\bar{\alpha}_2 \qquad = \frac{1}{\sqrt{2}}\,E_{12} \quad ,$$

where $E_{ij} = \alpha_i\bar{\alpha}_j$.

It is clear now that we have constructed simply the representation space for SU(2), and the spinor representations of SO(3) have appeared as single-valued tensor representations of SU(2), the covering group generated by E_{ij}. The operators α_1,α_2 are spinor components under SO(3).

It is usual to carry out the mapping from the orthogonal group to its covering group on an infinitesimal scale, by identifying corresponding generators. However (2) provides a mapping on a global scale since the representation space, on which the group as a whole acts, is mapped over by replacing a function f(a) with the corresponding function g(α) using (2). The global mapping then determines an infinitesimal mapping of the generators through the common expression (3). The operators (2) have been encountered before in this context e.g. Weyl [13] , p146, and Brinkman [86] .

We have revealed now the group theoretic origin of the operators (2) which appeared in Chpt. VI. The following operators (where α_i are ordinary bosons)

$$\begin{aligned} a_1 &= \alpha_1\alpha_3 \\ a_2 &= \alpha_1\alpha_4 \\ a_3 &= \alpha_2\alpha_3 \\ a_4 &= -\alpha_2\alpha_4 \end{aligned} \qquad (6)$$

appeared also, essentially as the operators VI (19). They satisfy (1), and also the traceless condition $a_1a_4 + a_2a_3 = 0$. If we calculate the generators according to (3) then the substitution of modified bosons gives the usual SO(4) generators K_{ij}, but the substitution of (6) gives the generators of SU(2) × SU(2). The correspondence of the generators appears as follows (where $E_{ij} = \alpha_i\bar{\alpha}_j$):

$\frac{1}{2}(K_{41} + K_{32})$	$\frac{1}{2}(E_{11} - E_{22})$
$\frac{1}{2}(K_{41} - K_{32})$	$\frac{1}{2}(E_{33} - E_{44})$
K_{43}	E_{12}
K_{21}	E_{21}
K_{42}	E_{34}
K_{31}	E_{43}

(7)

In the same way as for SO(3) we have determined a global map from SO(4) to the covering group SU(2) × SU(2). In this case however the spinor representations do not appear because we have considered only symmetric representations. We can see now that the existence of the operators (2) and (6) is due to the existence of a suitable covering group, and the fact that such operators do not generalize to higher n, as was found in Chpt. VI, is because the covering groups do not generalize suitably.

§3. Mappings to Covering Groups

It is useful to know explicitly the operators which map the space carrying only tensor representations of the orthogonal group, into the representation space of the covering group. The mappings may be found systematically in the following way.

Firstly we equate the corresponding generators as in (5) and (7); we can then write down the states of highest weight in terms of the same set of labels. By comparing these states, which must be mapped into each other, we obtain the mapping for $a_1, a_{12}, \ldots$. Taking n = 3 for example, we have

$$|\text{max.}\rangle = \alpha_1^{2\ell}\ |0\rangle = a_1^{\ell}\ |0\rangle$$

so that $a_1 = \alpha_1^2$ (ignoring constant factors). Now we require that a_i be an SO(3) vector i.e.

$$[K_{ij}, a_k] = \delta_{jk}\ \sigma_{ip}\ a_p - \delta_{ik}\ \sigma_{jp}\ a_p$$

which must still hold when the corresponding generators of the covering group are substituted. In this way we find $a_2 = \sqrt{2}\ \alpha_1\alpha_2$, $a_3 = -\alpha_2^2$, and the traceless condition is necessarily satisfied, because the representations are irreducible. For $SO(4) \approx SU(2) \times SU(2)$ we have

$$|\text{max.}\rangle = a_1^{m_1-m_2}\ a_{12}^{m_2}\ |0\rangle \quad , \quad m_2 \geqslant 0$$

$$= \alpha_1^{m_1+m_2}\ \alpha_3^{m_1-m_2}\ |0\rangle$$

so that $a_1 = \alpha_1\alpha_3$, $a_{12} = \alpha_1^2$ and again, using properties as vectors we obtain (6). We note that with the mapping $a_{12} \to \alpha_1^2$ we may allow m_2 to take semi-integer values. Expressions for $a_{ij} = a_i\ b_j - a_j\ b_i$ are readily obtained and so we may solve for all b_i. It is possible only to express the b's in terms of any one of them, b_1 say. We find

$$b_2 = \frac{\alpha_1}{\alpha_3} + \frac{\alpha_4}{\alpha_3} b_1$$

$$b_3 = \frac{\alpha_2}{\alpha_1} b_1$$

$$b_4 = -\frac{\alpha_2\alpha_4}{\alpha_1\alpha_3} b_1 - \frac{\alpha_2}{\alpha_3} \quad .$$

The variables a_{ij} which actually appear are nevertheless independent of b_1. The a's and b's found in this way satisfy the traceless conditions i.e. $a_1a_4 + a_2a_3 = 0$, $b_1b_4 + b_2b_3 = 0$, $a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1 = 0$. In fact they satisfy more than this, because we have

$a_1b_4 + a_3b_2 = 0 = a_4b_1 + a_2b_3$. This is because a_{13} and a_{24} are zero in the space for which $m_2 \geqslant 0$. When $m_2 < 0$ the expressions for the b's are

$$b_2 = \frac{\alpha_4}{\alpha_3} b_1$$

$$b_3 = \frac{\alpha_2}{\alpha_1} b_1 + \frac{\alpha_3}{\alpha_1}$$

$$b_4 = -\frac{\alpha_2\alpha_4}{\alpha_1\alpha_3} b_1 - \frac{\alpha_4}{\alpha_1} ,$$

and then $a_{12} = 0 = a_{34}$.

For SO(5) a_i, b_i $i = 1, .. 5$ are expressed in terms of modified bosons α_i, β_i for Sp(4) which satisfy $\varepsilon_{pq}\, \alpha_p\, \beta_q = 0$, i.e. $\alpha_{12} + \alpha_{34} = 0$. The correspondence between the generators of Sp(4) and SO(5) is as follows, where

$$S_{ij} = \varepsilon_{ip}\, \alpha_p\bar{\alpha}_j + \varepsilon_{jp}\, \alpha_p\bar{\alpha}_i + \varepsilon_{ip}\, \beta_p\bar{\beta}_j + \varepsilon_{jp}\, \beta_p\bar{\beta}_i \qquad i,j = 1,..4$$

are the generators of Sp(4):

S_{21}	$K_{51} + K_{42}$
S_{43}	$K_{51} - K_{42}$
$\frac{1}{\sqrt{2}} S_{42}$	K_{53}
$\frac{1}{\sqrt{2}} S_{32}$	K_{43}
$-\frac{1}{\sqrt{2}} S_{31}$	K_{31}
$\frac{1}{\sqrt{2}} S_{41}$	K_{32}
$\frac{1}{2} S_{22}$	K_{54}
$-\frac{1}{2} S_{44}$	K_{52}
$-\frac{1}{2} S_{11}$	K_{21}
$\frac{1}{2} S_{33}$	K_{41}

The operators a_i, b_i which have all the required properties are

$$a_1 = \alpha_{13},\ a_2 = -\alpha_{14},\ a_3 = \sqrt{2}\,\alpha_{12},\ a_4 = \alpha_{23},\ a_5 = \alpha_{24}\ ,$$

$$b_2 = \frac{\alpha_1{}^2}{\alpha_{13}} - \frac{\alpha_{14}}{\alpha_{13}} b_1$$

$$b_3 = \sqrt{2}\,\frac{\alpha_1\alpha_3}{\alpha_{13}} + \sqrt{2}\,\frac{\alpha_{12}}{\alpha_{13}} b_1$$

$$b_4 = -\frac{\alpha_3{}^2}{\alpha_{13}} + \frac{\alpha_{23}}{\alpha_{13}} b_1$$

$$b_5 = -\frac{\alpha_1\alpha_2}{\alpha_{13}} - \frac{\alpha_3\alpha_4}{\alpha_{13}} + \frac{\alpha_{24}}{\alpha_{13}} b_1\ .$$

For SO(6) the operators a_i, b_i, c_i $i = 1, \ldots 6$ are expressed in terms of ordinary bosons which appear for SU(4). The results for a_i are $a_1 = \alpha_{12}$, $a_2 = \alpha_{13}$, $a_3 = \alpha_{23}$, $a_4 = \alpha_{14}$, $a_5 = -\alpha_{24}$, $a_6 = \alpha_{34}$.

§4. Construction in a Non-Harmonic Space

We can exploit our complete knowledge of the representations of SU(2) to find a means of constructing the spinor representations of SO(3) in a form which can be generalized. We do this by carrying out a stereographic projection, from the representation space of SO(3) consisting of functions defined on a sphere of radius r, onto the SU(2) functions defined on the two-dimensional plane.

Instead of two complex variables z_1, z_2 we may use $z = \frac{z_2}{z_1}$ as a variable on which to define the functions of the SU(2) representation space. This realization has been described by Vilenkin ([49], Chpt III). Homogeneous polynomials $f(z_1,z_2)$ are related to the functions $\phi(z)$ defined on $z = \frac{z_2}{z_1}$ by $f(z_1,z_2) = z_1{}^{\ell}\,\phi(z)$, where ℓ is the degree of f, and is also the SU(2) label. The operators are

$$J_1 = \tfrac{1}{2}\left(-z^2\frac{d}{dz} + \ell z + \frac{d}{dz}\right)$$

$$J_2 = -\tfrac{1}{2}i\left(-z^2\frac{d}{dz} + \ell z - \frac{d}{dz}\right)$$

$$J_3 = z\frac{d}{dz} - \tfrac{1}{2}\ell\ .$$

Now we may enlarge the representation space to consist of functions $F(z,\bar{z})$ in $\bar{z}$ as well as z, giving a reducible representation. F will be a polynomial of maximum degree ℓ in z and of maximum degree ℓ' in $\bar{z}$. From z, $\bar{z}$ we carry out the stereographic projection onto the sphere of radius r by means of the formula

$$z = \frac{x_1 + ix_2}{r - x_3} \quad \text{where} \quad r^2 = x_1{}^2 + x_2{}^2 + x_3{}^2 \ .$$

The space of functions on x_1, x_2, x_3 will be reducible and we can choose an irreducible subspace in several ways. If we choose the subspace determined by $\ell = \ell'$ we recover the familiar SO(3) basis states as the polynomials $(r - x_3)^{\ell}\ F(z,\bar{z})$ which can be shown to be the solid spherical harmonics. The generators when acting on these functions take the usual form.

Another subspace is obtained by putting $\ell' = 0$. Instead of the functions $F(z)$ we may consider $(r - x_3)^{\ell}\ F(z)$ which is a homogeneous polynomial in x_1, x_2, x_3 of degree ℓ. When acting on these functions the generators take the following form:

$$\begin{aligned} J_1 &= i\left(x_3 \frac{d}{dx_2} - x_2 \frac{d}{dx_3} - \tfrac{1}{2} i\ell \ \frac{x_1 - ix_2}{r - x_3}\right) \\ J_2 &= i\left(x_1 \frac{d}{dx_3} - x_3 \frac{d}{dx_1} + \tfrac{1}{2}\ell \ \frac{x_1 - ix_2}{r - x_3}\right) \qquad (8) \\ J_3 &= i\left(x_2 \frac{d}{dx_1} - x_1 \frac{d}{dx_2} + \tfrac{1}{2} i\ell\right) \ . \end{aligned}$$

We have derived these expressions without giving details, but it is readily checked directly that the commutation relations $\underline{J} \times \underline{J} = i\ \underline{J}$ are satisfied. Now $J_+ = J_1 + i\ J_-$ takes the usual form, since the terms involving $\dfrac{x_1 - ix_2}{r - x_3}$ cancel. The state of highest weight is therefore the usual function $|\text{max.}\rangle = (x_1 + ix_2)^{\ell}$ (the expression from IV. 17 is slightly different, due to a small change in the definition of the generators). By applying J_3 to this state we find that the eigenvalue, which is also the representation label, is $m_1 = \frac{1}{2}\ell$. This means that m_1 can be a semi-integer, and we obtain all spinor and tensor representations. The minimum state is $|min.\rangle = (r - x_3)^{\ell}$ which is annihilated by $J_- = J_1 - i J_2$. The general state will be a homogeneous polynomial in x_1, x_2, x_3 but will be different from the spherical harmonic functions. This is because the generators (8) do not commute with the Laplacian, so that basis functions are not harmonic. The general state is

$$\left|\begin{matrix} m_1 \\ m \end{matrix}\right\rangle = (x_1 + ix_2)^{m+m_1} (r - x_3)^{m_1-m} \qquad (m_1 = \tfrac{1}{2}\ell) .$$

The generators (8) are in a form which can be generalized to all orthogonal groups. The way this might be done can be seen from a similar construction in Chpt IX, but we will not carry out this generalization. Although this would enable the construction of all spinor representations there would be two defects. The basis functions would not be harmonic, and the generators would depend on the representation label and so would need to be redefined for each representation. The tensor representations are constructed without these deficiencies, and as is shown in Chpt IX the same can be done for the spinor representations.

§5. Realizations with Fermions

The fundamental spinor representation is labelled $(\frac{1}{2},\frac{1}{2} .. \frac{1}{2})$ for $O(2\nu+1)$ and $(\frac{1}{2},\frac{1}{2} .. \pm \frac{1}{2})$ for $O(2\nu)$. It is known how to construct these representations using the Clifford algebra ([84] Chpt VIII). This formulation can be carried out using fermions; for $O(3)$ we require only one fermion α, and its conjugate α^* satisfying $\alpha^2 = 0$, $\{\alpha,\alpha^*\} = 1$. The generators are $J_+ = \alpha$, $J_- = \alpha^*$, $J_3 = \frac{1}{2} - \alpha^*\alpha$ and

and the basis states are $\left|\begin{matrix}\frac{1}{2}\\ \frac{1}{2}\end{matrix}\right\rangle = |0\rangle$ and $\left|\begin{matrix}\frac{1}{2}\\ -\frac{1}{2}\end{matrix}\right\rangle = \alpha^* |0\rangle$ where $|0\rangle$ is the fermion vacuum state.

We can form an arbitrary spinor representation by taking the direct product of a tensor representation with the fundamental spinor representation, and reducing the result:

$$(\ell) \times (\tfrac{1}{2}) = (\ell + \tfrac{1}{2}) + (\ell - \tfrac{1}{2}) .$$

The representation space is constructed with modified bosons and fermions, which commute. The conditions which characterize the basis functions of each irreducible subspace are found by requiring that the Casimir invariant be diagonal. It is possible then to determine operators which carry us directly into each irreducible subspace. However these operators, being a combination of fermions and modified bosons, do not satisfy simple commutation relations. As a result this approach does not lead to a simple construction of all spinor representations, although it will generalize suitably for orthogonal groups of odd dimension. The construction of the fundamental spinor representation only, with fermions, is a simple formulation and is useful in some applications.

CHAPTER 8

RELATION BETWEEN THE BOSON CALCULUS AND ZHELOBENKO'S METHOD

§1. Zhelobenko's Method

The method of the boson calculus has been extensively developed for the construction of finite-dimensional irreducible tensor representations of U(n), and we have shown in Chpt III how to extend this calculus so as to apply to O(n) and Sp(n). As there is already in existence a fully developed formalism for constructing these representations, due to Zhelobenko [56], it is of interest to establish the connection between the two methods. In addition, the formalism of Zhelobenko includes in a natural way the construction of all spinor representations of O(n), and so we hope to carry over this construction so as to apply to the boson calculus. Some discussion of the relationship between the two methods is given by Zhelobenko in the paper referred to, if the boson calculus is interpreted as the construction of irreducible tensor representations, but we will give a detailed correspondence showing exactly how to relate the polynomial bases which occur in both methods. In this way we are able to transfer easily from one formalism to the other, taking advantage of the results which might be shown more easily in one.

The formalism of Zhelobenko appears to be more appropriate in a mathematical context, since the construction employs a smaller homogeneous space, which allows more economy in the use of the basic variables, but which still allows the construction of all irreducible representations. On the other hand the boson calculus is of greater importance from the physical viewpoint, since the basis functions are tensors or multivectors which can be interpreted as physical objects.

This will be apparent in the case of O(3), where the basis functions are the spherical harmonics, but which in Zhelobenko's method appear as monomials with a structure indistinguishable from SU(2) basis functions. In general the boson calculus has the advantage also that the scalar products, and consequently unitary representations, are very easily defined.

Zhelobenko has considered the finite- dimensional representations of the complex classical groups, without defining a scalar product since the representations must be non-unitary, whereas the boson calculus is usually concerned with the construction of unitary representations of unitary groups (including the unitary orthogonal and symplectic groups). However many of the considerations involved in the construction of the finite-dimensional representations are indifferent to whether a complex group or one of the real or compact forms is being considered, and so long as questions of adjoint operators and scalar products do not arise, one may discuss representations without worrying about which particular field of numbers is chosen. Our procedure then is to apply the boson calculus to the complex groups in order to demonstrate the relationship, with the knowledge that the restriction to the compact form is easily carried out for the boson calculus.

The starting point of both methods is to consider, say for the group GL(n), polynomials in the matrix elements g_{ij}, and to define a representation by the right regular representation:

$$T_g\, f(g') = f(g'g) .$$

Instead of the full group manifold of GL(n), one may choose the homogeneous space $GL(n)/ZD(n)$ where Z is a lower triangular group with unit diagonal elements and D is a diagonal matrix, and then the polynomial $f(g')$ is replaced by an inhomogeneous polynomial $f(z)$ where z_{ij}

are the elements of an upper triangular matrix, which serve to label the right cosets. Then multiplier representations of GL(n) may be defined by right translations on this space according to

$$T_g\, f(z) = \alpha(zg)\, f(z\circ g) \qquad (1)$$

where $z\circ g$ means the right component in the Gauss decomposition of the element $zg \in GL(n)$. This is the basis of Zhelobenko's method, and it is fully discussed in [56].

In its simplified form the boson calculus starts with an n-dimensional vector (a_i), $i = 1, \ldots n$, which transforms according to the fundamental n-dimensional representation of GL(n), homogeneous polynomials f(a) in the (a_i) are constructed, and a representation is defined by

$$T_g\, f(a) = f(ag)\ . \qquad (2)$$

If we write $f(a) = a_1^{m_1}\, f\left(\frac{a_i}{a_1}\right)$, then $f\left(\frac{a_i}{a_1}\right)$ is a function over the homogeneous space GL(n)/H where H is the subgroup for which $g_{1i} = 0$, $i > 1$, and in this simple case we see that the boson calculus is also concerned with a homogeneous space defined by a lower triangular subgroup. A more detailed examination given in the subsequent sections shows that the general boson calculus can be expressed in terms of polynomials over exactly the same homogeneous space as that employed by Zhelobenko.

Zhelobenko's method relies on the existence for the group G of a Gauss decomposition, in which G can be factorized as G = ZDZ. For the classical groups, Z(Z resp.) is the subgroup of upper (lower) triangular matrices with unit diagonal elements, and D is the subgroup of diagonal matrices. The representations of D are 1-dimensional and are all known; these are used to induce representations of G of the form (1). An important theorem, first noted by Godement [87], is stated by Zhelobenko as follows:

> "Every irreducible representation of a group G is induced by some character $\alpha(\delta)$ of the subgroup D; two irreducible representations of G induced by the characters α_1 and α_2 are equivalent if and only if $\alpha_1 = \alpha_2$."

Thus we obtain all irreducible finite-dimensional representations of the complex groups and therefore also of the compact groups, and in particular for O(n) this theorem ensures that the spinor representations appear naturally in this formalism.

For the classical groups the representation (1) takes the form

$$Tg\ f(z) = \Delta_1^{m_1 - m_2}(zg) \ldots \Delta_n^{m_n}(zg)\ f(\tilde{z}) \tag{3}$$

where $\Delta_i(zg)$ is the diagonal minor of zg and $\tilde{z} = z \circ g$ has elements $\tilde{z}_{ij} = \frac{\Delta_{ij}}{\Delta_i}$, where Δ_{ij} is the minor obtained from Δ_i by substituting the column with the number j in place of the column with the number i. The restrictions on the exponents $m_i - m_{i+1}$, which show what representations are being constructed, are obtained by considering the subgroup SL(2,C). According to the theorem above we obtain all representations, and so this approach solves the problem of the classification of the finite-dimensional irreducible representations of the classical groups, a problem first solved by Cartan using different methods.

The functions of the representation space are polynomials on Z, and are characterized completely as the null space of a system of differential operators. On the other hand, for the boson calculus we will show that the basis functions may be regarded as being defined on the subgroup DZ, and are homogeneous polynomials which for O(n) and Sp(n) are also harmonic.

§2. The Unitary and Linear Group

We can describe the irreducible space of the boson calculus for U(n) and GL(n) in the following way, showing exactly which variables appear in the basis functions. In this description the representation labels appear in the form

$$r_i = m_i - m_{i+1} \qquad \text{for } i = 1, \ldots n (m_{n+1} = 0) . \qquad (4)$$

In general we require n sets of bosons $a_i^{\alpha} (i, \alpha = 1, \ldots n)$ with adjoints $\bar{a}_i^{\alpha}$ in order to obtain sufficient polynomials. It will be convenient to think of these operators as n^2 complex variables, i.e., $a_i^{\alpha} = z_i^{\alpha} \varepsilon C_{n^2}$, with adjoints $\bar{a}_i^{\alpha} = \frac{\partial}{\partial z_i^{\alpha}}$, and the vacuum state $|0>$ becomes the constant 1. The representation space consists of polynomials homogeneous of degree m_k in the n variables a_i^k, for $k = 1, \ldots n$. We can form an irreducible representation of U(n) in the subspace of these polynomials in which the bosons a_i^{α}, for fixed α, appear only in antisymmetric combinations with $a_{i_1}^{\alpha-1}$, $a_{i_2}^{\alpha-2}$, .. a_i^1. This subspace R_n appears through the application of the Young symmetrizer to an arbitrary polynomial, which may be regarded as a tensor under U(n) transformations, to produce a polynomial (tensor) of a certain symmetry. The irreducible space now consists of polynomials homogeneous of degree r_k in the variables $a_{i_1 \ldots i_k}$, for $k = 1, \ldots n$ (r_k defined by (4)).

In this space R_n we define the irreducible representation Tg by (2) where "a" now stands collectively for the variables $a_{i_1 \ldots i_k}$, and ag stands for the same variables, in which each a_i^{α} has been transformed to $(a^{\alpha} g)_i = a_p^{\alpha} g_{pi}$. In this representation the generators of U(n) have the form

$$E_{ij} = a_i^p \bar{a}_j^p . \qquad (5)$$

We wish to demonstrate the relation of these representations to those obtained by Zhelobenko [56] in a different formalism. In order to do this we will obtain another realization of the representation Tg, in a projective space P_n which is set up in the following way (see e.g. Hermann [88]). Two non-zero tensors a and a′ are said to be equivalent if there is a non-zero scalar λ such that $a = \lambda a'$. P_n is then defined to be the set of all these equivalence classes, so that a point of P_n is an equivalence class of such tensors. A function f defined on P_n must then satisfy $f(\lambda a) = f(a)$ i.e., is homogeneous of zeroth degree. These functions can be constructed by taking functions in the inhomogeneous coordinates

$$\frac{a_{i_1 i_2 .. i_k}}{a_{12\,..\,k}}$$

for $k = 1,..n$. These coordinates are not defined everywhere, but on those points for which $a_{12..k} \neq 0$. To each homogeneous polynomial defined in R_n there corresponds a single polynomial defined on P_n, since using the properties of $f \in R_n$ as a homogeneous polynomial we can write

$$f(a_i, a_{j_1 j_2}, \ldots, a_{i_1 .. i_n}) = a_1^{m_1 - m_2}\, a_{12}^{m_2 - m_3} \ldots a_{12..n}^{m_n}$$

$$\times f\left(\frac{a_i}{a_1}, \frac{a_{j_1 j_2}}{a_{12}}, \ldots, \frac{a_{i_1 .. i_n}}{a_{1\,..\,n}}\right) . \qquad (6)$$

The functions on P_n are obtained by dividing the homogeneous polynomials by the state of highest weight. In this way the representation space can be characterized not as the space R_n of polynomials homogeneous in the variables a, but as the space of polynomials on the projective space P_n. This construction has been described before for SU(2) and SL(2,C) (Vilenkin [49,89]). We have obtained the functions on P_n from the space R_n, in which the polynomial degrees m_k are also the representation labels. However there are other ways to carry out the construction described, in which the degrees m_k are not the representation labels themselves.

The coordinate functions of P_n

$$\frac{a_{i_1 \,..\, i_k}}{a_{1 \,..\, k}} , \; k = 1, \,..\, n ,$$

are not independent, but an independent set may be taken as

$$z_{ij} = \frac{a_{12 \,..\, i-1j}}{a_{12 \,..\, i-1i}} \; i,j = 1, \,..\, n .$$

This follows from the identities

$$a_{12..m}\, a_{i_1 i_2 .. i_m\, i_{m+1}} = \sum_{p=1}^{m+1} (-)^{m^2+1+p}\, a_{i_1 .. i_{p-1} i_{p+1} \,..\, i_{m+1}} \times a_{1 \,..\, m i_p} . \tag{7}$$

Using these identities for $m = 1, \,..\, n-1$ successively, we see using (6), that each

$$\frac{a_{i_1 \,..\, i_k}}{a_{1 \,..\, k}}$$

can be expressed in terms of

$$\frac{a_{12 \,..\, m-1i}}{a_{12 \,..\, m-1m}} \quad \text{for } m = 1, \,..\, k .$$

Eq. (7) is proved by considering the following $2m + 1 \times 2m + 1$ determinant. The upper left $m \times m$ block has elements $M_{kj} = a^k_j$; the upper right $m \times m + 1$ block is zero; the lower left $m + 1 \times m$ block has elements $M_{kj} = a^k_j$; and the lower right $m + 1 \times m + 1$ block has elements $L_{kj} = a^k_{i_j}$. The value of this determinant is $a_{12..m}\, a_{i_1..i_{m+1}}$ as is shown by carrying out a Laplacian expansion according to the first m and the last $m + 1$ rows (see Aitken [90]). Now replace the i^{th} row by the i^{th} row minus the $(i + m)^{th}$ row, for $i = 1, \,..\, m$. The determinant is unchanged, but the upper left m × m block is now zero, and the upper right m + 1 × m block has elements $L_{kj} = - a^k_{i_j}$. Again carry out a Laplacian expansion, and we obtain the right hand side of (7).

The function

$$f\left(\frac{a_i}{a_1}, \frac{a_{j_1 j_2}}{a_{12}}, \ldots, \frac{a_{i_1 \ldots i_n}}{a_{1 \ldots n}}\right)$$

on P_n can now be written as a function ϕ of the variables z_{ij}. Hence the correspondence (6) may now be written

$$f(a) = a_1^{m_1-m_2} a_{12}^{m_2-m_3} \ldots a_{12..n}^{m_n} \phi(z) \tag{8}$$

where $f \in R_n$, and ϕ is defined on $z = (z_{ij}) \in P_n$, where z is upper triangular. The function ϕg which corresponds to f(ag) is then given by

$$f(ag) = a_1^{m_1-m_2} \ldots a_{12..n}^{m_n} \phi g$$

$$= (ag)_1^{m_1-m_2} (ag)_{12}^{m_2-m_3} \ldots (ag)_{12..n}^{m_n} \phi\left(\frac{(ag)_{1 \ldots j}}{(ag)_{1 \ldots i}}\right)$$

$$= a_1^{m_1-m_2} a_{12}^{m_2-m_3} \ldots a_{12..n}^{m_n}$$

$$\times \left[\frac{(ag)_1}{a_1}\right]^{m_1-m_2} \left[\frac{(ag)_{12}}{a_{12}}\right]^{m_2-m_3} \ldots \left[\frac{(ag)_{12..n}}{a_{12..n}}\right]^{m_n} \phi\left[\frac{(ag)_{1..j}}{(ag)_{1..i}}\right] .$$

Therefore the irreducible representation Tg defined on the space of functions ϕ on P_n is given by

$$Tg\, \phi(z) = \left[\frac{(ag)_1}{a_1}\right]^{m_1-m_2} \ldots \left[\frac{(ag)_{12..n}}{a_{12..n}}\right]^{m_n} \phi(\tilde{z}) , \tag{9}$$

where $\tilde{z}$ is the matrix with elements

$$\tilde{z}_{ij} = \frac{(ag)_{12..i-1j}}{a_{12..i-1i}} \Big/ \frac{(ag)_{12..i-1i}}{a_{12..i-1i}} .$$

The factors $\frac{(ag)_{12..k}}{a_{12..k}}$ for k = 1, .. n are functions of z_{ij}, the explicit form of which is given by

$$\frac{(ag)_{12..k}}{a_{12..k}} = \Delta_k(zg) \qquad (10)$$

where $\Delta_k(zg)$ is the minor formed from the first k rows and columns of the matrix zg. To prove this, consider the k × k determinant D with elements

$$D_{ij} = a_{12..i-1q}\, g_{qj} = a_{12..i-1i}\, z_{iq}\, g_{qj} \qquad \text{(q summed)}$$

which has the value

$$D = a_1\, a_{12} \,..\, a_{12..k}\, \Delta_k(zg) \,. \qquad (11)$$

We will show that the (i,j) element of D may be written as $a^i_q\, g_{qj}\, a_{12..i-1}$, without changing the value of D. This is clearly true for i = 1 (with the convention that $a_{12..i-1} = 1$ for i = 1). Suppose it is true for i = 1, .. m − 1. We carry out the following row operations on the m^{th} row leaving the determinant unchanged. Firstly note that by expanding the determinant $a_{12..m-1\,q}$ down the m^{th} column, we can write

$$a_{12..m-1\,q} = a^m_q\, a_{12..m-1} + \sum_{r=1}^{m-1} a^r_q\, C_r$$

for some coefficient C_r (depending on a^α_ℓ, $\ell \neq q$). Therefore

$$a_{12\,..\,m-1\,q}\, g_{qj} = a^m_q\, g_{qj}\, a_{12\,..\,m-1} + \sum_{r=1}^{m-1} a^r_q\, g_{qj}\, C_r \,.$$

Now replace the m^{th} row of D by

$$(\text{the } m^{th} \text{ row}) - \sum_{r=1}^{m-1} \frac{C_r}{a_{12..r-1}} (\text{the } r^{th} \text{ row}).$$

The element D_{rj} (for $r \leq m-1$) is $a^r_q\, g_{qj}\, a_{12..r-1}$, so that now the the element D_{mj} is equal to $a^m_q\, g_{qj}\, a_{12..m-1}$. By induction, and by bringing out the factor $a_{12..i-1}$, for i = 2 .. k, we find that D_k is equal to $a_1\, a_{12} \,..\, a_{12..k-1}$ multiplied by the determinant with elements $a^i_q\, g_{qj}$ which is $(ag)_{12..k}$. This proves (10). If $\Delta_{ij}(zg)$ is the minor obtained from $\Delta_i(zg)$ by substituting the column with the number j in place of the column with the number i, then the same proof shows that

$$\Delta_{ij}(zg) = \frac{(ag)_{12..i-1j}}{a_{12..i-1i}} .$$

The irreducible representation Tg can now be written

$$Tg\ \phi(z) = \Delta_1(zg)^{m_1-m_2}\ \Delta_2(zg)^{m_2-m_3} \ .. \ \Delta_n(zg)^{m_n}\ \phi(\tilde{z})$$

where now we may write $\tilde{z}_{ij} = \Delta_{ij}(zg)/\Delta_i(zg)$. In this form we can see that the representation Tg is the same as that obtained by Zhelobenko by a different method. The results he has obtained can be immediately applied to our case where the functions ϕ are defined on the space P_n, with coordinates $z_{ij} = \dfrac{a_{12..j}}{a_{12..i}}$. On the other hand, in the formalism of Zhelobenko, the functions ϕ are defined on Z, the subgroup of GL(n) consisting of upper triangular matrices with elements z_{ij}. However we can exhibit P_n as a homogeneous space of GL(n), and idemtify P_n with the coset space GL(n)/H, where H = ZD(n) is the subgroup of lower triangular matrices referred to in §1. Let us determine the isotropy subgroup H of GL(n) at the point in P_n determined by the tensors, denoted a, with the values $a_{i_1..i_k} = 0$ except $a_{12..k} = 1$, for k = 1, .. n. The matrix g leaves the point in P_n fixed if there exist non-zero scalars $\lambda = \lambda(k)$ such that $ag = \lambda a$. Firstly, we show that $a^m_j = 0 \quad j > m$, and $a^m_m = 1$. Clearly this is true for m = 1. If it is true for m = 1,2 .. k - 1, then writing $a_{12..k-1j}$ as a k × k determinant, we find

$$a_{12..k-1j} = a^1_1\, a^2_2 \ .. \ a^{k-1}_{k-1}\, a^k_j = 0 \quad \text{for} \quad j > k, = 1 \quad \text{for} \quad j = k .$$

The result follows by induction. Now the condition $ag = \lambda a$ implies the result $g_{mj} = 0 \quad j > m$, with $g_{mm} \neq 0$. This is true for $m = 1$, because $(ag)_i = \lambda(1)\, a_i = a_p\, g_{pi} = g_{1i} = \lambda(1)\, \delta_{1i}$. Hence $g_{1i} = 0$ $i > 1$, and $g_{11} = \lambda(1)$ is non-zero. Suppose the result is true for $m = 1,2, \ldots k - 1$. The $ag = \lambda a$ means

$$(ag)_{12..k-1j} = \lambda(k)\, a_{12..k-1j} = 0 \quad \text{for} \quad j > k, = \lambda(k) \quad \text{for} \quad j = k\,.$$

Writing $(ag)_{12..k-1j}$ as a determinant, we see that elements above the diagonal are zero. Therefore

$$\begin{aligned}(ag)_{12..k-1j} = g_{11}g_{22} \cdots g_{k-1\,k-1}\, g_{kj} &= 0, \quad j > k \\ &= \lambda(k) \neq 0, \quad j = k\,.\end{aligned}$$

Hence $g_{kj} = 0$ for $j > k$, and $g_{kk} \neq 0$. By induction then we have shown that H is the subgroup of GL(n) consisting of lower triangular matrices and we can put $P_n = GL(n)/H$. This demonstrates the asserted identity of the two methods for GL(n).

We can find the homogeneous spaces which are used in the boson calculus by putting $\lambda(k) = 1$ in the above analysis, and we see that the representation functions are defined on GL(n)/Z instead of GL(n)/ZD(n) as for Zhelobenko's method. As Zhelobenko has noted ([56] p64) the boson calculus is "a realization on the group DZ, and after a necessary normalization also on the group Z". This normalization is carried out by dividing by the state of highest weight, as shown in (8), "causing a 'contraction' of all dominant vectors to a single point".

§3. Orthogonal and Symplectic Groups

In order to consider the groups with a metric some changes are necessary. The methods for O(n) and Sp(n) are the same, and can be combined using the metric ρ introduced in Chpt III. Zhelobenko chooses, for O(n), $\rho = \sigma$ and for Sp(n) $\rho = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix}$, since it is then possible to carry out a Gauss decomposition, for the complex groups, into subgroups of upper and lower triangular matrices, as for GL(n). We find it convenient, but not at all necessary, to use also this choice of ρ, since then the states of highest weight take the simplest possible form, and the mapping between the two methods is correspondingly simpler. Generally therefore $\rho_{ij} = \pm \delta_{i,n+1-j}$.

As before, the representations of G(n) (= O(n) or Sp(n)) are realized in a space H_n of homogeneous polynomials. In order to ensure that these representations are irreducible, the variables a_i^α (i,   = 1, .. n) on which the polynomials are defined must be constrained with the condition $\rho_{pq}\, a_p^\alpha\, a_q^\beta = 0$. This is in order that the tensors of the representation space are traceless, or from the same viewpoint, the polynomials which appear are harmonic between all variables. We will think of the a_i^α as modified bosons, i.e.,

$$a_i^\alpha = z_i^\alpha - 2(z^p, z^q)^{\Delta-1}{}_{(pq)(\alpha r)}\rho^{is} \frac{\partial}{\partial z^r{}_s}$$

with adjoints $\bar{a}_i^\alpha = \frac{\partial}{\partial z_i{}^\alpha}$, but there are other possibilities.

In this space H_n we define the irreducible representation Tg by Tg f(a) = f(ag), $g \in G(n)$, $f \in H_n$. The generators then have the form

$$K_{ij} = \rho_{ip}\, a_p^q\, \bar{a}_j^q - \rho_{pj}\, a_p^q\, \bar{a}_i^q \ .$$

For Sp(n) the state of highest weight is (for this metric)

$$|max.\rangle = a_1{}^{m_1-m_2}\, a_{12}{}^{m_2-m_3} \,..\, a_{12..\nu}{}^{m_\nu}\, |0\rangle \ .$$

The variables of the space will then be $a_{i_1..i_k}$, and the polynomials will be homogeneous of degree $r_k = m_k - m_{k+1}$ in $a_{i_1..i_k}$, for k = 1, .. ν

($m_{\nu+1} = 0$). For $O(n)$ $|max.\rangle$ is given by IV, (7), (7´). For $SO(2\nu)$ we will not consider the case $m_\nu \leqslant 0$ since the situation is entirely analogous to that for $m_\nu \geqslant 0$. Again the variables of the space will be $a_{i_1..i_k}$ for $k = 1, .. \nu$, appearing with degree r_k, except in the case of the full orthogonal group, when $a_{i_1..i_k}$ for $k = 1, .. n$ will appear.

Now we define a projective space P_n in the same way as before, i.e., a point of P_n is an equivalence class of tensors, where two tensors a, a' are defined to be equivalent if $a = \lambda a'$ for a non-zero scalar λ. Functions on P_n are constructed from the coordinates

$$z_{ij} = \frac{a_{12..i-1j}}{a_{12..i-1i}} .$$

Here we let $i,j = 1, .. n$, but if $a_{i_1..i_k}$ does not appear in H_n for some value of k, the corresponding z_{ij} will not be an independent variable, as we shall see below. The polynomials ϕ on P_n are obtained from those in H_n according to

$$f(a) = a_1^{r_1} .. a_{12..\nu}^{r_\nu} \phi(z) , \tag{12}$$

(for the case $m_\nu \geqslant 0$) i.e., by dividing by the state of highest weight. The a's here are constructed from modified bosons and will be manipulated formally, but the polynomials which actually appear are well defined as before because the variables $a_{12..k}$ appearing in $|max.\rangle$ may be regarded as ordinary bosons (as shown in IV, §3).

The z_{ij} defined as above are not all independent, and the relations between these variables are expressed in the restriction that z is ρ-orthogonal, i.e., $z\rho z^t = \rho$. Now $\rho_{pq} z_{ip} z_{jq} = \rho_{ij}$ holds identically for $i \geqslant n + 1 - j$ because z is upper triangular, with unit diagonal elements. For $i < n + 1 - j$ we need to show

$$\rho_{pq} a_{1..i-1\,p} a_{1..j-1\,q} = 0 ,$$

and this follows immediately from III. 20. These relations between the z_{ij} are the same as those between the z_{ij} which appear in the formalism described by Zhelobenko. The representations obtained in both approaches are therefore the same, with the same representation spaces, i.e., we have

$$Tg\ \phi(z) = \Delta_1^{r_1}(zg)\ ..\ \Delta_\nu^{r_\nu}(zg)\ \phi(\tilde{z})$$

where in our formalism

$$\Delta_k(zg) = \frac{(ag)_{12..k}}{a_{12..k}}\ , \quad k = 1,\ ..\ \nu.$$

We have considered here the case when the state of highest weight takes the form $|max.\rangle = a_1^{r_1}\ ..\ a_{12..\nu}^{r_\nu}\ |0\rangle$ so that (12) applies, but the same procedure still holds when $|max.\rangle$ takes a different form e.g. when the choice of metric is different, or when $m_\nu \leq 0$ for $O(2\nu)$. In these cases we divide elements of H_n by $|max.\rangle$ and look for a mapping such that the matrix $z = (z_{ij})$ is upper triangular and satisfies $z\rho z^t = \rho$.

In the same way as for $GL(n)$ we can show that the homogeneous spaces employed in both methods are the same, and the explicit mapping for z_{ij} then demonstrates exactly the relationship of the two methods. We can use the important results obtained by Zhelobenko, and apply them to our case; in particular we can transfer back to H_n in such a way as to include both spinor and tensor representations together in a natural way.

CHAPTER 9

SPINOR REPRESENTATIONS IN HARMONIC SPACES

§1. Inadequacy of Angular Momentum Generators

We have developed the boson calculus so as to apply to all tensor representations of O(n) and Sp(n), and we now wish to carry out a further development which will include all the spinor representations of O(n), for arbitrary n. This means we must try to realise the spinor representations in the same harmonic space used for the tensor representations, since then basis functions can be expressed as polynomials in modified bosons, and the operators which act within this space, such as the generators, will also be expressible with modified bosons.

Let us see now exactly why the construction in Chpt IV of the tensor representations fails for the spinor representations. Taking $n = 3$ for example, the state of highest weight is $|\text{max.}\rangle = a_1^{\ell}\,|0\rangle$ which is defined only for integral values of the label ℓ. However by writing a_1 as a differential operator, and changing to the spherical polar coordinates defined by (IV. 15), $|\text{max.}\rangle$ becomes equal to $e^{i\ell\phi}\sin^{\ell}\theta$ which is defined for semi-integer ℓ. By applying

$$J_- = e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\phi}\right)$$

to $|\text{max.}\rangle$ we obtain a set of functions $Y_{\ell m}(\theta,\phi)$ defined for both integral and semi-integral ℓ. This representation is realized in a $(2\ell+1)$-dimensional space of complex valued functions, so that it is necessary for $J_-\,Y_{\ell,-\ell}$ to be identically zero. It is readily checked that this condition, which of course holds for integral ℓ, is not satisfied in the cases when ℓ takes semi-integer values; therefore this construction of the spinor representation fails.

Pandres ([91,92]) has attempted to overcome this fact by interpreting $Q_{\ell,-\ell-1} = J_- Y_{\ell,-\ell}$ as a "representation" of the zero vector. The property motivating this interpretation is mainly that $Q_{\ell,-\ell-1}$, and in general the functions $Q_{\ell m} = J_-^{-\ell-m} Y_{\ell,-\ell}$ are orthogonal to all $Y_{\ell m}$, which must be the case for all eigenvectors of the hermitean operator J_3. Pandres has defined a "scalar product", a bilinear mapping which satisfies

$$(Q_{\ell m}, Q_{\ell',m'}) = 0 = (Y_{\ell m}, Q_{\ell',m'}) \ ,$$

$$(Y_{\ell m}, Y_{\ell' m'}) = \delta_{\ell\ell'} \, \delta_{mm'} \ .$$

However this mapping is not a true scalar product, because the requirement $(\psi,\psi) = 0 \Leftrightarrow \psi = 0$ does not hold. A close examination of Pandres' argument shows that this requirement has been used as the definition of a zero vector i.e., ψ is "essentially" zero if $(\psi,\psi) = 0$, and this will then apply to $\psi = Q_{\ell m}$. In this way $Q_{\ell,-\ell-1}$ is interpreted as being "essentially" zero, and the functions $Y_{\ell m}$ for semi-integer ℓ, m are claimed to span a suitable space. We reject this argument because the $Q_{\ell m}$ functions are nevertheless non-zero in the normal sense, so that the representation space is not finite-dimensional as we require. We note that also the questions of orthogonality are irrelevant, because in a representation space where no scalar product has been defined it is still necessary that J_- annihilates the minimum state (see for example Miller [47]).

We can see now that when the generators of O(3) take the usual form as the angular momentum operators,

$$J_{ij} = -i \left(x_i \frac{d}{dx_j} - x_j \frac{d}{dx_i}\right) \ , \qquad (1)$$

the spinor representations cannot be constructed. This is because firstly these operators conserve particle number i.e., basis functions will be homogeneous polynomials f^{ℓ}, and $Nf^{\ell} = \ell f^{\ell}$ where $N = x_p \frac{d}{dx_p}$; secondly these polynomials will also be eigenfunctions of the Casimir invariant $J^2 = r^2\nabla^2 - N(N+1)$, with eigenvalue $\ell(\ell + 1)$, and therefore the polynomials f^{ℓ} must also be harmonic. But as we have seen above, it is not possible to construct spinor representations in the harmonic space when the generators take the form (1).

We need to look for other realizations of the group generators. We will do this by taking advantage of Zhelobenko's construction, which we have described in Chpt VIII using as a starting point the formalism of the boson calculus; by transferring back to this formalism we show that the space of harmonic homogeneous polynomials is a suitable space with which to carry the spinor representations. This is achieved by finding realizations of the Lie algebra of O(n) which are new. The methods used here lead also to new realizations of the Lie algebra of U(n). It is likely that these realizations will be important in obtaining infinite dimensional representations of the non-compact groups, especially considering that our approach is, in the words of Zhelobenko "the theory of finite-dimensional representations from the infinite-dimensional point of view".

§2. Transfer from Zhelobenko's Formalism

We have shown in Chpt VIII how to construct multiplier representations in the space of functions ϕ on a set of $n \times n$ upper triangular matrices z with elements z_{ij}. The functions ϕ, homogeneous of zeroth degree, are constructed by taking rational functions in modified boson operators $a_i^{\alpha}(i, \alpha = 1 \ldots n)$, and the coordinate functions z_{ij} can be put equal to $\frac{a_{1 \ldots i-1j}}{a_{1 \ldots i-1i}}$. If f(a) is a harmonic polynomial, homogeneous of degree $r_k = m_k - m_{k+1}$ in $a_{i_1 \ldots i_k}$

(for $k = 1, \ldots \nu$, $m_{\nu+1} = 0$), then the 1-1 correspondence between $f(a)$ and $\phi(z)$ is given by

$$f(a) = a_1^{m_1-m_2} \ldots a_{12..\nu-1}^{m_{\nu-1}-m_\nu} a_{12..\nu}^{m_\nu} \phi(z) . \qquad (2)$$

The representation Tg in the space of functions ϕ is given by

$$Tg \, \phi(z) = \Delta_1^{m_1-m_2} \ldots \Delta_\nu^{m_\nu} \phi(\tilde{z}) , \qquad (3)$$

where $\Delta_k(z,g)$ is the minor formed from the first k rows and columns of the matrix zg and is equal to $\frac{(ag)_{12..k}}{a_{12..k}}$, and $\tilde{z}$ is a known function of z. The important results which Zhelobenko has obtained and which apply to Tg defined by (3) are as follows:

For $n = 2\nu+1$ there exists a polynomial $\mathfrak{S}_o(z,g)$ on P_n such that $\Delta_\nu(zg) = \mathfrak{S}_o^2(z,g)$. The fundamental spinor representation labelled by $(\frac{1}{2},\frac{1}{2} \ldots \frac{1}{2})$ can now be constructed according to $Sg \, \phi(z) = \mathfrak{S}_o(z,g) \, \phi(\tilde{z})$, where Sg is a representation in the sense that $Sg_1g_2 = \pm Sg_1Sg_2$. The multiplier for an arbitrary representation Tg is now written

$$\Delta_1^{m_1-m_2} \ldots \Delta_{\nu-1}^{m_{\nu-1}-m_\nu} \mathfrak{S}_o^{2m_\nu} \qquad (4)$$

so that m_ν may be a semi-integer, in which case $m_1, \ldots m_{\nu-1}$ are also semi-integers.

For $n = 2\nu$ there exist two polynomials on P_n, $\mathfrak{S}_-$ and $\mathfrak{S}_+$ such that

$$\Delta_{\nu-1}(zg) = \mathfrak{S}_-(z,g) \, \mathfrak{S}_+(z,g)$$

$$\Delta_\nu(zg) = \mathfrak{S}_+^2(z,g) .$$

The two fundamental spinor representations, labelled by $(\frac{1}{2},\frac{1}{2}, \ldots, \pm\frac{1}{2})$ are constructed according to $Sg \, \phi(z) = \mathfrak{S}_\pm \, \phi(\tilde{z})$ and in general the multiplier has the form

$$\Delta_1^{m_1-m_2} \ldots \Delta_{\nu-2}^{m_{\nu-2}-m_{\nu-1}} \mathfrak{S}_-^{m_{\nu-1}-m_\nu} \mathfrak{S}_+^{m_{\nu-1}+m_\nu} . \qquad (5)$$

Again m_ν, $m_{\nu-1}$ can be semi-integers simultaneously, in which case m_1, .. $m_{\nu-2}$ are all semi-integers. In the form (5) the representation includes naturally the case for which $m_\nu < 0$, even though the transfer (2) to the functions ϕ has been carried out from polynomials f for which $m_\nu \geqslant 0$ only. If we had begun with functions f for which $m_\nu < 0$, then we would reach the same space of functions $\phi(z)$ by putting

$$f(a) = a_1^{m_1-m_2} \cdot\cdot\, a_{12..\nu-1}^{m_{\nu-1}+m_\nu}\; a_{1..\nu-1\nu+1}^{-m_\nu}\; \phi(z)$$

with

$$z_{\nu j} = \frac{a_{12..\nu-1j}}{a_{12..\nu-1\nu+1}}, \; j \geqslant \nu + 2 \;\; (z_{\nu\nu+1} = \frac{a_{1..\nu-1\nu}}{a_{1..\nu-1\nu+1}} = 0$$

as before).

We wish to transfer back to the harmonic space H_n in such a way as to retain this construction of the spinor representations. We obtain a polynomial f(a) in H_n from $\phi(z)$ by multiplying ϕ with a certain polynomial which becomes the state of highest weight in H_n. This is expressed in the formula (2) which holds in the case when the degrees r_i of $f(a) \varepsilon H_n$ are connected with the representation labels m_i by $r_i = m_i - m_{i+1}$. As previously noted, the formula (2) restricts each m_i to non-negative integral values. More generally however we can also transfer back to H_n by multiplying each $\phi(z)$ with a polynomial of degrees r_i such that $r_i \geqslant m_i - m_{i+1}$. We do this in the following way. In the representation space of polynomials $\phi(z)$ on P_n we replace each m_i by $m_i - p_i = \ell_i$, for $i = 1, .. \nu$, so that the representation labels are now $\ell_i = m_i - p_i$. Now transfer back to H_n, the space of harmonic polynomials in the a's with degrees r_i where r_i is not equal to $\ell_i - \ell_{i+1} = (m_i - m_{i+1}) - (p_i - p_{i+1})$ as before, but $r_i = m_i - m_{i+1}$. This transfer is carried out by multiplying each $\phi(z)$ with the polynomial $a_1^{m_1-m_2} \ldots a_{12..\nu}^{m_\nu}$ i.e. (2) still holds, but now the m_i are no longer the representation labels. We can be sure that f(a) obtained according to (2) is actually a polynomial in the a's if $m_i - m_{i+1} \geqslant (m_i - p_i) - (m_{i+1} - p_{i+1})$ for $i = 1, \ldots \nu$ $(p_{\nu+1} = 0)$, i.e. if $p_1 \geqslant p_2 \geqslant \ldots \geqslant p_\nu \geqslant 0$. Since the $m_i - p_i$ are representation labels they satisfy

$$m_1 - p_1 \geqslant m_2 - p_2 \geqslant \ldots \geqslant m_\nu - p_\nu \geqslant 0 \qquad n = 2\nu + 1$$

and

$$m_1 - p_1 \geqslant m_2 - p_2 \geqslant \ldots \geqslant |m_\nu - p_\nu| \qquad n = 2\nu \,. \tag{6}$$

The m_i are integers, hence the p_i are either all integers or all semi-integers.

In this way we obtain the representations $(\ell_1, \ldots \ell_\nu)$ in the space of harmonic polynomials homogeneous of degrees $r_i = m_i - m_{i+1}$. By choosing p_i suitably we can obtain any of the permissible values of $(\ell_1, \ldots \ell_\nu)$. All the tensor representations for $\ell_\nu \geqslant 0$ are obtained by putting $p_i = 0$ for all i, and all the spinor representations for $\ell_\nu \geqslant 0$ by putting $p_i = \frac{1}{2}$ for all i. We could also obtain, for $n = 2\nu$, all representations for which $\ell_\nu < 0$ but these are constructed more conveniently in the space H_n for which $m_\nu < 0$.

The representation Tg in the space of functions $\phi(z)$ has the form (putting $m_i \to m_i - p_i$)

$$Tg\, \phi(z) = \{\Delta_1^{m_1-m_2} \ldots \Delta_\nu^{m_\nu}\, \phi(\tilde{z})\} \left[\frac{1}{\Delta_1}\right]^{p_1-p_2} \ldots \left[\frac{1}{\Delta_\nu}\right]^{p_\nu}$$

$$= \phi g \left[\frac{1}{\Delta_1}\right]^{p_1-p_2} \ldots \left[\frac{1}{\Delta_\nu}\right]^{p_\nu} .$$

We transfer back to H_n and using the fact that ϕg corresponds to $f(ag)$ and $\Delta_k = \dfrac{(ag)_{12..k}}{a_{12..k}}$ we find that Tg is defined in H_n by

$$Tg\, f(a) = \left[\frac{a_1}{(ag)_1}\right]^{p_1-p_2} \ldots \left[\frac{a_{12..\nu}}{(ag)_{12..\nu}}\right]^{p_\nu} f(ag) \,. \tag{7}$$

For $n = 2\nu + 1$ this may be written

$$Tg\ f(a) = \left[\frac{a_1}{(ag)_1}\right]^{p_1-p_2} \cdot\cdot \left[\frac{a_{12..\nu-1}}{(ag)_{12..\nu-1}}\right]^{p_{\nu-1}-p_\nu} \mathfrak{S}_o^{-2p_\nu} f(ag) \tag{8}$$

while for $n = 2\nu$ we have

$$Tg\ f(a) = \left[\frac{a_1}{(ag)_1}\right]^{p_1-p_2} \cdot\cdot \left[\frac{a_{12..\nu-2}}{(ag)_{12..\nu-2}}\right]^{p_{\nu-2}-p_{\nu-1}} \mathfrak{S}_-^{p_\nu-p_{\nu-1}} \cdots$$

$$\mathfrak{S}_+^{-p_{\nu-1}-p_\nu} f(ag) \tag{9}$$

where $\mathfrak{S}_o$, $\mathfrak{S}_\pm$ are polynomials in $\dfrac{a_{12\,..\,i-1j}}{a_{12\,..\,i-1i}}$.

The space H_n is invariant under Tg provided the parameters p_i are restricted to the values indicated above. From Tg we can calculate the form of the generators. However we will find it easier to use the correspondence (2) to calculate the dependence of the generators on m_i in the space of functions ϕ, and then to put $m_i \to m_i - p_i$ and transfer back to H_n. If

$$K_{ij}(a) = \sigma_{iq}\, a_q^{\ p}\, \bar{a}_j^p \ - \sigma_{jq}\, a_q^{\ p}\, \bar{a}_i^p$$

then the generators $K_{ij}(z)$ acting on $\phi(z)$ are determined by

$$K_{ij}(a)\ f(a) = K_{ij}(a)\ a_1^{m_1-m_2} \cdot\cdot\ a_{12..\nu}^{\ m_\nu}\ \phi\left(\frac{a_{1\,..\,k-1\ell}}{a_{1\,..\,k-1k}}\right)$$

$$= a_1^{m_1-m_2} \cdot\cdot\ a_{12..\nu}^{\ m_\nu}\ K_{ij}(z)\ \phi(z)\ . \tag{10}$$

The dependence of $K_{ij}(z)$ on the m_i, which is found from the action of $K_{ij}(a)$ on $a_1^{m_1-m_2} \cdot\cdot\ a_{12..\nu}^{\ m_\nu}$, is established in this way for integral m_i but will also hold in the case when m_i takes semi-integer values. The classification of the K_{ij} as raising or lowering generators, or weight generators, has been given in III. 26. The raising generators commute with $a_1^{m_1-m_2} \cdot\cdot\ a_{12..\nu}^{\ m_\nu}$ and therefore when acting on $\phi(z)$ are independent of m_i. Hence they are unchanged in H_n. The weight generators are $H_i = K_{n+1-i,i}$ and using (10) we see that $K_{n+1-i,i}(z) = m_i + D_i(z)$ where D_i is a differential operator in z, independent of $m_1, .. m_\nu$. Putting $m_i \to m_i - p_i$ and transferring to H_n we have that $K_{n+1-i,i}(a)$ is replaced by $K_{n+1-i,i}(a) - p_i$. In order to specify the changes necessary for the lowering generators it is sufficient to consider only the generators corresponding to the simple roots, since all other lowering generators are obtained from these by commutation. For $n = 2\nu + 1$ the lowering generators corresponding to the simple roots are $K_{2\nu+1-j,j}$ for $j = 1, .. \nu$. We find $K_{2\nu+1-j,j}(z) = (m_j - m_{j+1})z_{jj+1} + D_j'(z)$ for some D'. Hence in H_n $K_{2\nu+1-j,j}(a)$ is replaced by

$$K_{2\nu+1-j,j}(a) - (p_j - p_{j+1}) \frac{a_{1\,..\,j-1j+1}}{a_{1\,..\,j-1j}}, \qquad j = 1, .. \nu.$$

For $n = 2\nu$ all lowering generators can be obtained from $K_{2\nu,j}$ $j = 1, .. \nu - 1$ and $K_{\nu,\nu-1}$. We find $K_{2\nu-j,j}(z) = (m_j - m_{j+1})z_{jj+1} + D_j''(z)$ and $K_{\nu,\nu-1}(z) = (m_{\nu-1} + m_\nu)z_{\nu-1\nu+1} + D_j'''(z)$ for differential operators D'', D'''. Hence in H_n

$$K_{2\nu-j,j}(a) \to K_{2\nu-j,j}(a) - (p_j - p_{j+1}) \frac{a_{1\,..\,j-1j+1}}{a_{1\,..\,j-1j}}$$

and

$$K_{\nu,\nu-1}(a) \to K_{\nu,\nu-1}(a) - (p_{\nu-1} + p_\nu) \frac{a_{1\,..\,\nu-2\nu+1}}{a_{1\,..\,\nu-2\nu-1}}.$$

These replacements are considerably simplified in the case $p_i = 0$ (tensor representations) and $p_i = \frac{1}{2}$ (spinor representations). Although the generators involve ratios of the variables $a_{i_1..i_k}$ their range is a ~~subspace~~ of H_n ~~and no rational function of polynomials~~ appears provided the p_i satisfy (6) and $p_1 \geqslant p_2 \geqslant .. \geqslant p_\nu \geqslant 0$. The representations constructed are not unitary in general, although they are equivalent to unitary representations. They can be made unitary by redefining the scalar product in H_n which can always be done because SO(n) is compact (see Vilenkin [49] p44).

It is possible to find realizations of the type just described for the generators of U(n) also. We can construct representations labelled by $\ell_i = m_i - p_i$ $(i = 1, .. n)$ in the space of homogeneous polynomials of degree $r_k = m_k - m_{k+1}$ in the variables $a_{i \, .. \, i_k}$ for $k = 1, .. n$, where a_i^{α} are now ordinary bosons. We require $p_1 \geqslant p_2 .. \geqslant p_n \geqslant 0$ where the p_i are all integers, and also $m_1 - p_1 \geqslant m_2 - p_2 .. \geqslant m_n - p_n$. The representation Tg in this space is given by

$$Tg\, f(a) = \left[\frac{a_1}{(ag)_1}\right]^{p_1-p_2} .. \left[\frac{a_{12..n}}{(ag)_{12..n}}\right]^{p_n} f(ag) \tag{11}$$

and the generators E_{ij} satisfying

$$[E_{ij}, E_{k\ell}] = \delta_{jk} E_{i\ell} - \delta_{i\ell} E_{kj}$$

are specified by

$$E_{ij} = a_i^{\,p} \bar{a}_j^{\,p}, \quad j > i \quad \text{(raising generators)}$$

$$H_i = E_{ii} = a_i^{\,p} \bar{a}_i^{\,p} - p_i \tag{12}$$

$$E_{i+1,i} = a^p_{i+1} \bar{a}_i^{\,p} - (p_i - p_{i+1}) \frac{a_{1\, ..\, i-1\, i+1}}{a_{1\, ..\, i-1 i}}$$

(lowering generators).

This construction leads to no new representations except that now the labels ℓ_i can be negative in addition to the usual non-negative values.

§3. Results for O(3)

In order to illustrate the construction described above we will write down the results explicitly for SO(3). We begin in the space H of harmonic homogeneous polynomials of degree m_1 in the a_i where $m_1 = \ell$ is also the representation label, and the a_i are modified bosons with $2a_1a_3 + a_2^2 = 0$. We have

$$J_+ = K_{32} = a_1\bar{a}_2 - a_2\bar{a}_3$$

$$J_- = K_{21} = a_2\bar{a}_1 - a_3\bar{a}_2$$

$$J_3 = K_{31} = a_1\bar{a}_1 - a_3\bar{a}_3$$

and an arbitrary basis state is $\left|\begin{matrix}\ell\\ m\end{matrix}\right\rangle = a_1^{\,m}\, a_2^{\,\ell-m}\, |0\rangle$. We put

$z_{12} = \dfrac{a_2}{a_1} = z$, so that $\dfrac{a_3}{a_1} = -\tfrac{1}{2}z^2$ and then

$$f(a_1,a_2,a_3) = a_1^{\,m_1}\, f\left(1, \frac{a_2}{a_1}, \frac{a_3}{a_1}\right)$$

$$= a_1^{\,m_1}\, \phi(z)$$

where

$$\phi(z) = f(1,z, -\tfrac{1}{2}z^2) .$$

The representation Tg in the space of functions $\phi(z)$ is given by

$$Tg\ \phi(z) = \left[\frac{(ag)_1}{a_1}\right]^{m_1} \phi\left[\frac{(ag)_2}{a_1}\bigg/\frac{(ag)_1}{a_1}\right] .$$

Now $g = (g_{ij})$ satisfies $g\sigma g^t = \sigma$, so that if g_{11}, g_{12}, g_{21} are taken to be independent, we have

$$g_{31} = -\frac{g_{21}^{\,2}}{2g_{11}}, \qquad g_{13} = -\frac{g_{12}^{\,2}}{2g_{11}},$$

$$g_{32} = -\frac{g_{12}g_{21}^{\,2}}{2g_{11}^{\,2}} - \frac{g_{21}}{g_{11}}, \qquad g_{23} = -\frac{g_{21}g_{12}^{\,2}}{2g_{11}^{\,2}} - \frac{g_{12}}{g_{11}},$$

$$g_{22} = \frac{g_{21}g_{12}}{g_{11}} + 1, \qquad g_{33} = \frac{1}{g_{11}}\left(1 + \frac{g_{21}g_{12}}{2g_{11}}\right)^2 . \qquad (13)$$

Hence

$$\frac{(ag)_1}{a_1} = \frac{a_p}{a_1} g_{p1} = z_{1p}\, g_{p1}$$

$$= g_{11} + zg_{21} + \tfrac{1}{2}z^2 \quad \frac{g_{21}^{\ 2}}{2g_{11}}$$

$$= \left(\sqrt{g_{11}} + z\,\frac{g_{21}}{2\sqrt{g_{11}}}\right)^2$$

i.e.,

$$\bigcirc_o(z,g) = \sqrt{g_{11}} + \frac{zg_{21}}{2\sqrt{g_{11}}} \; .$$

Also

$$\frac{(ag)_2}{a_1} = \left(\sqrt{g_{11}} + \frac{zg_{21}}{2\sqrt{g_{11}}}\right)\left(\frac{g_{12}}{\sqrt{g_{11}}} + \frac{z}{\sqrt{g_{11}}} + z\,\frac{g_{12}g_{21}}{2g_{11}\sqrt{g_{11}}}\right) ,$$

therefore

$$Tg\ \phi(z) = \left(\sqrt{g_{11}} + \frac{zg_{21}}{2\sqrt{g_{11}}}\right)^{2m_1} \phi\left(\frac{g_{12} + z + z\,\dfrac{g_{12}g_{21}}{2g_{11}}}{g_{11} + \tfrac{1}{2}zg_{21}}\right) . \qquad (14)$$

The basis functions are $z^{\ell-m}$ for $-\ell \leqslant m \leqslant \ell$ and the generators are

$$J_+ = \frac{d}{dz} , \qquad J_- = m_1\, z - \tfrac{1}{2}z^2\,\frac{d}{dz} , \qquad J_3 = m_1 - z\,\frac{d}{dz} \; .$$

m_1 can now be a semi-integer. Putting $m_1 \to m_1 - p = \ell$, where p can have any of the values 0, ½, .. m_1 and transferring back to H, we find the generators have the form

$$J_+ = a_1\bar{a}_2 - a_2\bar{a}_3$$

$$J_- = a_2\bar{a}_1 - a_3\bar{a}_2 - p\,\frac{a_2}{a_1} \qquad (15)$$

$$J_3 = a_1\bar{a}_1 - a_3\bar{a}_3 - p \; .$$

Tg is defined in H by

$$Tg\ f(a) = \left|\frac{2a_1\sqrt{g_{11}}}{2a_1g_{11} + a_2g_{21}}\right|^{2p} f(ag) \; . \qquad (16)$$

We obtain all tensor representations by putting p - 0, and all spinor representations by putting p = ½.

The basis states are

$$\left|\begin{matrix}\ell\\ m\end{matrix}\right\rangle = a_1{}^{m+p}\; a_2{}^{\ell-m}\; |0\rangle$$

which are the solid spherical harmonics $r^{\ell+p}\, Y_{\ell+p,m+p}(\theta,\phi)$ (shown in IV, §5). The minimum state is

$$\left|\begin{matrix}\ell\\ -\ell\end{matrix}\right\rangle = a_2{}^{2p}\; a_3{}^{\ell-p}\; |0\rangle \qquad \text{for } p \leqslant \ell$$

$$= a_1{}^{p-\ell}\; a_2{}^{2\ell}\; |0\rangle \qquad \text{for } p \geqslant \ell \;,$$

and of course is annihilated by J_-.

These representations we have constructed are non-unitary because J_+ is not the hermitean adjoint of J_-, but they can be made unitary by redefining the scalar product in H. To do this it is sufficient to specify the scalar product between the basis states, and from the requirement that

$$J_- \left|\begin{matrix}\ell\\ m\end{matrix}\right\rangle = \left[\frac{(\ell + m)(\ell - m + 1)}{2}\right]^{\frac{1}{2}} \left|\begin{matrix}\ell\\ m-1\end{matrix}\right\rangle$$

we find

$$\left(\left|\begin{matrix}\ell'\\ m'\end{matrix}\right\rangle , \left|\begin{matrix}\ell\\ m\end{matrix}\right\rangle\right) = \delta_{\ell\ell'}\delta_{mm'} \frac{2^{\ell-m}(\ell - m)!(\ell + m)!(\ell + p)!}{(2\ell)!}$$

We see that it has been necessary only to renormalize the spherical harmonics. The basis states have the necessary orthogonality properties with the former scalar product because the labelling operators J_3, and $J^2 = J_3(J_3 + 1) + 2\, J_-\, J_+$ are hermitean. This is because $\frac{a_2}{a_1} J_+ = a_2\bar{a}_2 + 2a_3\bar{a}_3$ is hermitean.

We have noted that p must take non-negative semi-integer values in order that H be invariant under the generators. However it is possible to allow p to be negative, and we obtain then inhomogeneous basis states involving $\frac{a_2}{a_1}$, which do not belong to H but which still cut off suitably. For $p = -\ell$ the basis states are $z^{\ell-m}$ where $z = \frac{a_2}{a_1}$, and in this way we can choose p so as to transform partially or completely to the space of polynomials employed by Zhelobenko.

The operators (15) satisfy the O(3) commutation relations because of the properties of modified bosons as O(3) vectors. Ordinary bosons also possess these properties, and so if the a's in (15) are all ordinary bosons the commutation relations are still satisfied. A satisfactory representation space cannot be constructed with this realization, but if we put

$$\begin{aligned} J_+ &= z_1 \frac{d}{dz_2} - z_2 \frac{d}{dz_3} \\ J_- &= z_2 \frac{d}{dz_1} - z_3 \frac{d}{dz_2} + p\,\frac{r - z_2}{z_1} \\ J_3 &= z_1 \frac{d}{dz_1} - z_3 \frac{d}{dz_3} - p \end{aligned} \qquad (17)$$

suitable basis states can be calculated. We can transfer from this space to H simply by substituting for the z's with modified bosons (when r becomes zero) and we regain (15). The realization (17) is possible only if p is equal to the representation label m_1, and we then obtain the same realization described in Chpt. VII, except for the different metric (note also that $\frac{r - z_2}{z_1} = \frac{2z_3}{r + z_2}$). It would seem possible to generalize (17) to arbitrary n as has been done for (15), but as explained in Chpt VII such realizations are not entirely satisfactory. We can generalize (17) within O(3) as follows:

$$\begin{aligned} J_+ &= z_1 \frac{d}{dz_2} - z_2 \frac{d}{dz_3} - q\,\frac{r - z_2}{z_3} \\ J_- &= z_2 \frac{d}{dz_1} - z_3 \frac{d}{dz_2} + p\,\frac{r - z_2}{z_1} \\ J_3 &= z_1 \frac{d}{dz_1} - z_3 \frac{d}{dz_3} + q - p\,. \end{aligned} \qquad (18)$$

This realization has been encountered before in polar coordinates by Hurst [93] , with $p = \frac{1}{2}\mu = -q$. In this case however the spinor representations do not appear because $2q$ must be an integer, and therefore $\mu = p - q$ is an integer. Modified bosons can be substituted in (18) to obtain what appears to be a generalization of (15), but in fact nothing is lost by putting $q = 0$.

These results do not depend on the form we have taken for the metric σ; if we put $\sigma = I$, the identity, then we would have $z = \dfrac{a_3}{a_1 - ia_2}$ This formalism also includes naturally the representations of the full orthogonal group O(3), by enlarging the representation space to include axial tensors. The state of highest weight is then written

$$\left|\begin{matrix}\ell\\ \ell\end{matrix}\right\rangle = a_1{}^{\ell+p-1}\, a_{12}\, |0\rangle ,$$

and the generators have a similar form as (15).

Although SU(2) is the covering group of SO(3) it is not obvious how Tg defined by (14) is a representation of $g \in SU(2)$. In fact we can recover the usual expression for Tg, $g \in SU(2)$ by substituting for a_1, a_2, a_3 with VII. 2. Then $z = \dfrac{a_2}{a_1} = \sqrt{2}\,\dfrac{\alpha_2}{\alpha_1}$ where α_1, α_2 are ordinary bosons. With the matrix

$$u = \begin{pmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{pmatrix} , \quad |\alpha|^2 + |\beta|^2 = 1 ,$$

which belongs to SU(2) we identify $g \in SO(3)$ determined by $g_{11} = \alpha^2$, $g_{21} = -\sqrt{2}\,\alpha\bar{\beta}$, $g_{12} = \sqrt{2}\alpha\beta$. Then from (14) T_u takes the

$$T_u\, \phi(z) = (\alpha - \bar{\beta}\, z)^{2m_1}\, \phi\left(\frac{\beta + \bar{\alpha}\, z}{\alpha - \bar{\beta}\, z}\right) ,$$

which is the familiar expression for representations of SU(2) in the space of polynomials ϕ of one variable z.

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