An investigation into the behaviour of teletraffic networks in which several streams are offered to a common link, with particular attention to partitioning of the overflow stream.

by

KYM GRAHAM WILSON

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SUMMARY

This thesis is concerned with an investigation into the behaviour of two independent teletraffic streams which are offered to a common link in a telephone network. This behaviour is characterised by the means, variances and covariances of the traffic overflowing from the common link.

A mathematical model is presented and the solution of this model is investigated by analytic, computational and approximate methods.

The model is solved explicitly for a special case, in which random traffic is offered to the common link, by the classical approach and a direct method. The non-generating function approach is used to reduce the order of the problem for the general model.

A description of a matrix formulation of the model is given and several iterative solution methods are discussed. The most suitable method was incorporated into a computer program.

Data generated by this program was used to investigate the accuracy of some approximate formulas suggested by Olsson, Wallström and Harris as well as some simple approximations and the equivalent random method. An approximation based on simple linear regression and tables for calculating the parameters used in this method are presented. Graphical representations of the approximations are included for visual comparison of their accuracy.

SIGNED STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university. To the best of my knowledge and belief the thesis contains no material previously published or written by another person, except where due reference is given in the text.

(K.G. WILSON)

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CHAPTER 1

INTRODUCTION

1.1 Objectives

This thesis is concerned with a mathematical investigation of an overflow system in an alternate routing telephone network. The aim of the research is to study the behaviour of a network in which two or more streams of traffic are offered to a common link and to determine the proportion of the total overflow traffic belonging to each stream. The mean and variance of the total overflow traffic have been investigated previously, for example [27], but comparatively little research has been undertaken on determination of the means and variances of the individual streams. The development of a method of calculating these statistics is desirable since, in many networks, these overflow streams are subsequently offered to different links on higher choice routes.

In this chapter the concepts of teletraffic theory are presented and a review of research into this and related problems is given.

In succeeding chapters a mathematical model is developed and an iterative solution to the model is derived. Analytic solutions for a special case are given and analytic techniques for solving the general model are treated. The accuracy of several approximations developed by other researchers is investigated and a new approximated method is presented.

1.2 Telephone Networks

Each individual telephone is connected to a particular exchange. When a call is made from one subscriber to another an electrical circuit must be closed between the two telephones. If the phones are connected to the same exchange then the connection is made by switching equipment in the exchange. Otherwise, an additional connection must also be made between the two different exchanges. This connection may be made directly between the exchanges or indirectly through one or more other exchanges. The system through which such connections are made is called a telephone network. The research presented in this thesis is concerned only with the network between exchanges and does not consider calls made between subscribers in the same exchange area.

The physical network consists of telephone exchanges involving switching equipment and junctions for carrying calls between them. Each junction can carry a single call at any one time and junctions may take the form of overhead telephone lines, underground cables or communication channels in satellite networks. Mathematically, for any particular stream of calls, the system may be considered as a directed graph with nodes corresponding to the exchanges and links to groups of junctions. There are two types of node : one type can act as a source or a sink for calls (corresponding to an exchange to which subscribers are connected), the second type (a tandem exchange) is purely a switching point in the graph. Directly linked to the source of a call is the origin exchange, and to the sink, the destination exchange, the two exchanges together constitute an origin-destination (O-D) pair.

An alternate routing network is one in which there is more than one route between each O-D pair. A route may consist of a single link between the origin and the destination exchanges, known as the direct link, or a succession of links which connect the two exchanges, via one or more tandem exchanges. There is a definite order of preference for using these routes and they are referred to as the first choice route, the second choice route and so on.

In many cases a direct link is present and this is usually the first choice route. As an example, a simple one O-D pair network will be considered (Figure 1.1). There are three routes between the origin exchange "O" and the destination exchange "D". The direct link, 1, makes up the first choice route. The second choice route consists of links 2 and 3 and passes through the tandem exchange X. The third choice route has three links, 4, 5 and 3 and passes through two tandems, X and Y.

In any network there is a limit to the number of calls which may be carried simultaneously. If all junctions on the direct route were busy then a newly arriving call would be offered to the second choice route and would be carried on this route if a free junction was available on both link 2 and link 3. Otherwise, it would be offered to the third choice route and would be carried if there was a free junction on each of links 4, 5 and 3, otherwise it would be lost from the system. This hierarchy of routes is called an overflow system.

The development of alternate routing in telephone networks was necessitated by the prohibitive cost of provided enough junctions on the direct route to carry the required amount of traffic between each O-D pair. Non direct links may be part of several routes, between different O-D pairs and this sharing of junctions reduces the overall cost of the network. It is impossible to provide enough junctions to always be able to carry any call, even using alternate routing. The processes of determining the number of junctions to be provided on each route is known as dimensioning and networks are dimensioned so that they will perform to a given grade of service. The grade of service is measured by the probability that a call will be unable to be carried on any of the alternate routes and will be lost from the system (in Australia, this is often set at .002). *

The introduction of alternate routing has increased the complexity of telephone networks, leading to a field of study of the mathematical properties of such networks. While the arrival and completion of individual calls is impossible to predict, the behaviour of streams of calls can be subject to statistical

* The description of dimensioning given here is tailored to Berry's model. Current network dimensioning practice in Australia does <u>not</u> use the concept of OD grades of service.

analysis. Traffic arrives, is carried or overflows according to various probability distributions and the purpose of this thesis is to investigate the overflow distributions for specific simple network configurations.

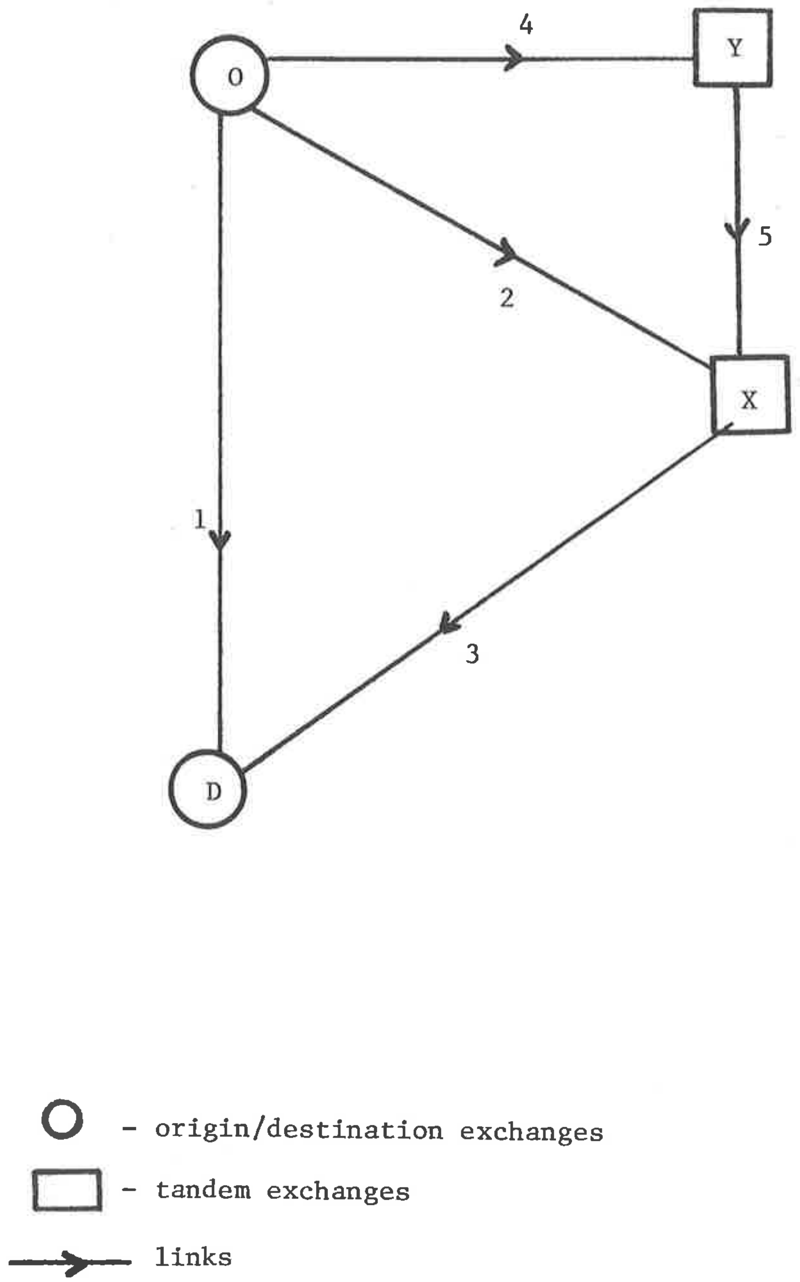


Figure 1.1 : Single O-D pair network with three alternate routes.

1.3 Telephone traffic

The traffic carried on a particular link at any instant, measured in units called erlangs, is numerically equal to the number of calls simultaneously in progress on that link.* The traffic overflowing from a particular link is equal to the traffic which would be carried if that traffic were offered to a fictitious link with an infinite number of junctions.

Traffic between each O-D pair varies with time, although it does reach a state of statistical equilibrium during parts of any day, including the time when traffic is heaviest. Most networks are designed to specifications applied to this time, known as the busy period, which ensures that the grade of service, for example, is no worse than a specified value. The assumption of statistical equilibrium is convenient for modelling the system.

Traffic is generally characterised by its mean and variance, although some recent papers consider even higher moments, for example, Freeman [9] and Schehrer [24]. It was shown by Wilkinson [27] that the mean and variance describe the traffic with sufficient accuracy for dimensioning purposes and recent dimensioning models, for example Berry [2], consider only these two parameters.

✓ A link is said to have full availability if an offered call may be carried on any unoccupied junction. For the particular case when the offered traffic is Poisson, the link to which it is offered has full availability, and lost calls are ~~cleared~~ held in later stages or route, the mean and variance of the overflow traffic can be calculated exactly. Expressions for these statistics are given in terms of the Erlang loss formula, which gives the probability that exactly n junctions on the link are busy. If the arrival rate is a and the link has d junctions, then the formula is

$$E_n(a) = \frac{\frac{a^n}{n!}}{\sum_{r=0}^{n} \frac{a^r}{r!}} \qquad 0 \le n \le d \tag{1.1}$$

* The definition of an "erlang" varies from one author to another. The best definition for this thesis relates the "erlang" to the average number of calls in progress during the busy hour, assuming statistical equilibrium during such a busy period.

Although the assumption of negative exponential holding times is commonly made, this is not a necessary assumption for (1.1) to hold, as was shown by Pollaczek [18]. $E_d(a)$ is the probability that all d junctions on the link are occupied and this is called the blocking probability* and denoted by B. Then mean of the overflow traffic is

$$M = a.E_d(a) = a\ B \tag{1.2}$$

and the variance, which was derived by Riordan [27], is

$$V = M(1 - M + \frac{a}{d+1-a+M}). \tag{1.3}$$

These formula are no longer exact when the offered traffic is non random.

* or Erlang loss formula for d trunks

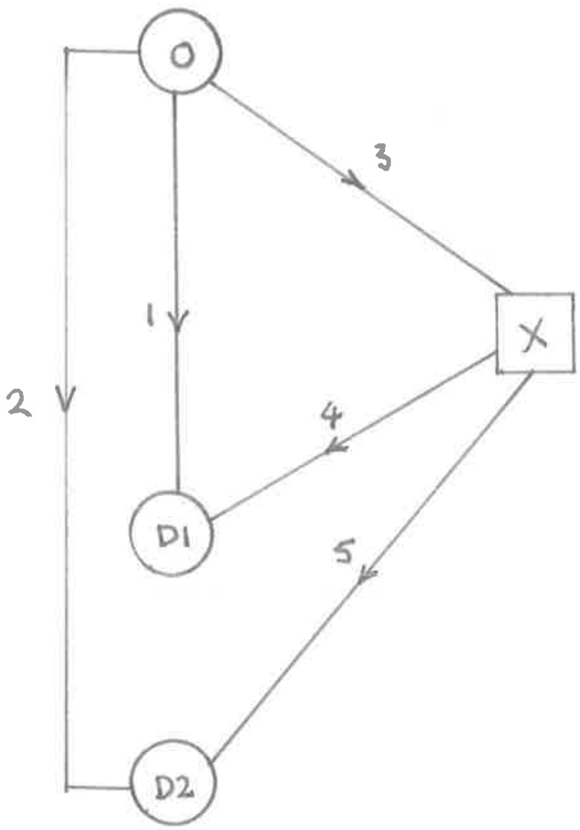


Figure 1.2 : Two O-D pair network.

1.4 The Equivalent Random Method

The overflow from a link which is offered Poisson traffic is no longer Poisson, although it is renewal traffic. The overflow from a renewal stream offered to a link is also renewal but the combination of two or more renewal streams is not itself renewal unless all component streams are Poisson. Thus the formulas described in section (1.3) are not applicable to many network situations.

An approximate method of calculating the mean and variance for the total overflow from a link offered two or more streams of non Poisson traffic was developed by Wilkinson [27]. A network of two O-D pairs is considered to illustrate this equivalent random method (Figure 1.2). The first O-D pair, O and D1, has direct route, link 1, second choice route (3,4) and arrival rate a_1. The second pair, O and D2, has direct route 2, second choice route (3,5). Third choice routes may exist but are not shown, since it is the overflow from the second choice route that is of interest. The arrivals for the two pairs are independent and have Poisson distributions, and it will be assumed that any call finding a free junction on link 3, will also find one on link 4 or 5 as required. Thus overflow from the second choice routes is caused by congestion on link 3.

The links may be considered as groups of servers. Each server corresponds to a junction in the link and may serve only one customer (call) at any one time. The two primary groups correspond to the direct links and have d_1 and d_2 servers respectively. The customers not served in these groups overflow to a common secondary group of c servers corresponding to link 3. (Figure 1.3) The terms calls and customers, junctions and servers, and so on, will be used equivalently throughout the text.

The mean and variance of the overflow from the ith primary group will be denoted by M_i, V_i and the overflow from the secondary group, corresponding to the ith stream, will have mean and variance m_i and v_i. The covariance between the two overflow streams will be denoted cov. The parameters for the combination of the two streams will be denoted by the appropriate unsubscripted symbol (e.g. M, v) and estimates of parameters by a ^ above the symbol.

The algorithm for the equivalent random method is;

(a) Calculate M_1, V_1, M_2 and V_2 using (1.2) and (1.3).

(b) Calculate the total traffic offered to the common link, $M = M_1 + M_2$, $V = V_1 + V_2$.

(c) Calculate, again from (1.2) and (1.3), the equivalent random traffic a_e, which when offered to a link with d_e junctions would give overflow traffic with mean M and variance V.

(d) Calculate the overflow from a single link with (d_e+c) junctions, offered a_e erlangs of random traffic. Denote the mean and variance thus calculated, by (1.2) and (1.3), again by $\hat{m}$ and $\hat{v}$.

The basic assumption of this method is that the overflow traffic (or equivalently the blocking probability) depends only on the total mean and variance of the offered traffic, and not on its distribution. Steps (a) and (b) give exact results and, if random traffic is offered to d junctions and the overflow from this to a further c junctions, then the overflow from the second group is identical to the overflow from a single group with d+c junctions. Hence any source of error in this method is due to the accuracy of the assumption. The approximation is widely used and the assumption gives a reasonable approximation of reality in most situations. The accuracy of this method is discussed further in Chapter 6.

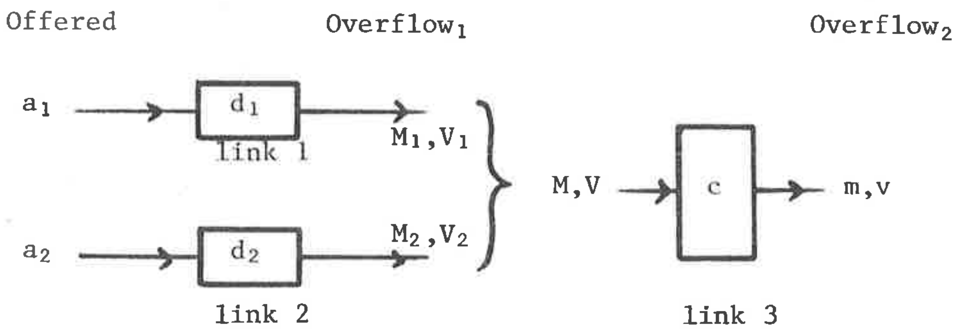
Step (c) was traditionally performed using tables, for example [5] or graphs as presented in Wilkinson's paper [27]. However, some approximate formulas have been developed by Rapp [21] and these can be used in computer programs.

$$a_e = V + \frac{3V}{M}\left(\frac{V}{M} - 1\right) \qquad (1.4)$$

$$d_e = \frac{a_e \cdot \left(M + \frac{V}{M}\right)}{M + \frac{V}{M} - 1} - M - 1 \qquad (1.5)$$

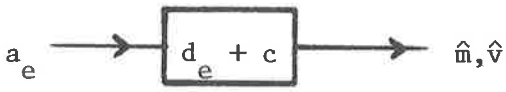
(exact if a_e is known exactly)

This method gives approximations for the total overflow mean and variance only. It is often necessary to know the means and variances of the individual overflow streams and this was the motivation for the research.



(a) The network

(b) Equivalent single link



(c) Final "equivalent" network

Figure 1.3 : The Equivalent Random Method

1.5 Review of research into this and related problems

Several papers have been published on networks in which two or more streams of traffic are offered to a single link, and some of these have attempted to find formulas for the statistics for the individual overflow streams. The network is usually considered as a system of service stages as described in the previous section. The network investigated in the thesis is illustrated in Figure 1.2, and the service stage representation is given in Figure 1.4. The service stages N_1 and N_2 corresponding to the direct links, M to the common link on the second choice route and L_1 and L_2 to the fictitious infinite links which are used to measure the overflow traffic. All network diagrams and parameters of the various authors in the literature have been translated into a common notation, consistent with Neal [16].

There are three main lines of research into this problem. The analytic approaches generally follow the technique used by Riordan [27] to obtain equations (1.2) and (1.3).

Riordan considered a system with a single primary group and an unlimited overflow group (Figure 1.5a). The state of the system is defined by the number of busy servers in each group, and the equations of state are obtained, under the assumption of statistical equilibrium. These equations are transformed, by consideration of a binomial moment generating function, into an equivalent system involving the binomial moments, which has one main equation and one boundary equation. The main equation with a constraint relaxed to allow an infinite number of servers was expressed as a differential equation in a second generating function which was solved in terms of the σ-polynomials, defined by

$$\sum_{m=0}^{\infty} \sigma_k(m)\ t^m = \frac{e^{at}}{(1-t)^k} . \qquad (1.6)$$

These σ-polynomials satisfy a number of recurrence relationships which when utilised in the boundary condition (or normalising condition) lead to the formulas for the mean and variance (1.2) and (1.3).

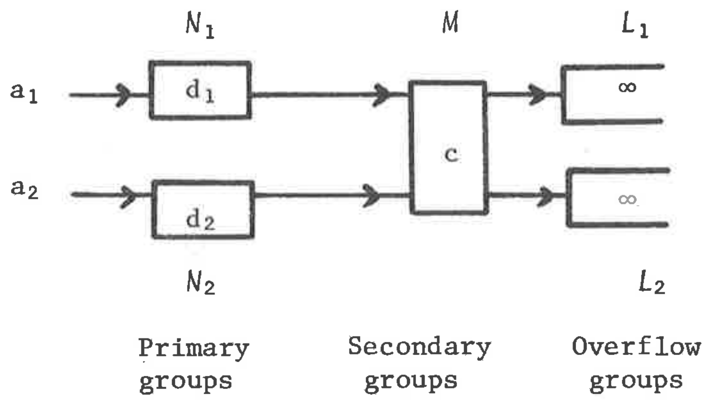
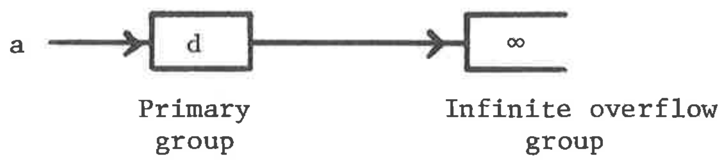


Figure 1.4 : Service group representation of two O-D pair network



(a) Riordan's model.

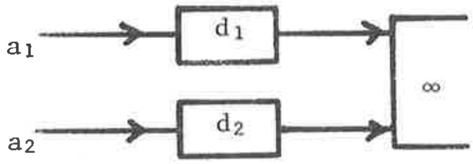
(b) Chastang's first model.

Figure 1.5 : Some one stream overflow models.

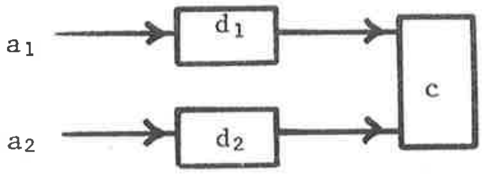
Chastang [4], investigated an extension of Riordan's results. He first considered a system in which the secondary group was finite (Figure 1.5b). The solution of this system (first investigated by Brockmeyer [3]) again involved several generating function transformations and the σ-polynomials. (Chastang refers to the traffic carried by this finite secondary group as overflow traffic, which may cause some confusion.) He then considered systems with two primary groups and one secondary group (Figure 1.6). For the finite secondary group case, he derived a set of equations in binomial moments from the state equation by using a generating function. He then sums some of the boundary conditons corresponding to one of the primary groups being full. This leads to a formula, which by the deletion of several terms, was analogous to the moment equation for the one primary group. Chastang suggested that a solution similar to the solution of the simple problem would be an approximation for the two stream model. He stated that the deleted terms were 'comparatively small' but admits that the approximations "fail however, to give a better accuracy then the approximate method of R.I. Wilkinson." He suggests they may be used to determine the decomposition of the total overflow stream into its components, but does not investigate this idea any further.

Neal [16], investigated a grading system in which two or more streams of overflow traffic were recombined (Figure 1.7a). He again uses a generating function to obtain equations involving binomial moments. He then relaxes the constraint on the main equation to allow m to go to infinity and by applying a second generating function obtains a partial differential equation, which is solved in terms of the σ-polynomials and some unknown constants. The number of unknowns corresponds exactly to the number of boundary equations and they are solved by introduction of several more generating functions. Neal does not obtain explicit results but reduces the order of the system from $(c+1)(d_1+1)(d_2+1)$ equations to $(d_1+1)(d_2+1)$ equations which for his system is a reduction from about 500 to less than 36.

A second approach to this problem has been to obtain a computational solution. This has been achieved by simulation on a computer [22], by solving the state or moment equations iteratively [12], and by replacing the primary groups by Interrupted Poisson Process (IPP) models [9].

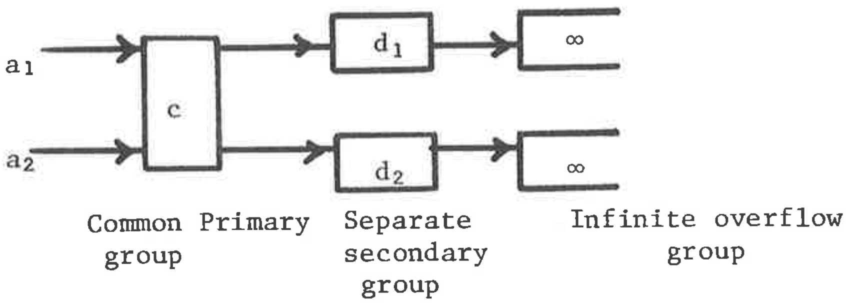


(a)

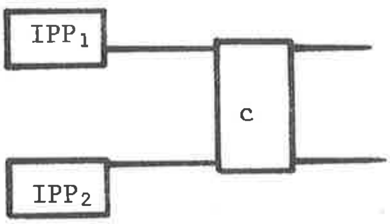


(b)

Figure 1.6 : Chastang's two stream models.



(a) Neal's model for two streams.



(b) Freeman's model for two streams.

Figure 1.7 :

Kibble published a paper [12] and again [13], in which he solved the state equations computationally, Although he does not describe either the network or the equations of state which he solves, it is believed that he is referring to the third model considered by Chastang (Figure 1.6b). The paper describes without technical details, an iterative procedure and a non iterative procedure which seems to involve as many operations. He compares his results to results from the equivalent random method, by partitioning the overflow mean in the ratio of the offered means. It is not the overflow mean but the 'blocking probabilities' which are compared. Although the term is not defined it appears to refer to the probability that a call will not be carried on either the relevant primary group or the secondary (as distinct from the definition of B in Section 1.4). The papers are useful for some numerical results to which approximate solutions can be compared.

In a recent paper Freeman [9], compares the iterative solutions of Kibble, the equivalent random method and a model involved interrupted Poisson processes. The IPP model of Kuczura [14] is used to replace a primary group (Figure 1.7b). The IPP model has two states, an 'on' state where it generates Poisson traffic and an 'off' state. This is a realistic approximation to a simple overflow system since, when there are free servers in he group an offered call is carried, and there is no overflow, and, when all servers are occupied, traffic overflows with a Poisson distribution. The IPP model has three parameters and Freeman suggests methods of choosing these. (The mean and variance of the generated traffic are the same as for the overflow from the primary group.) This system as 4(c+1) states which is a considerable reduction of the order of the problem. Freeman investigated the higher moments of the traffic distributions and claimed that the IPP model was more accurate than the Equivalent Random Method.

A third approach has been to obtain approximate formulas for the various statistics. These are usually obtained by observation and experimentation involving results obtained by a computational method. Olsson [20] and Wallström have suggested formulas for obtaining the individual overflow means, from results of simulations, and Harris [10] has derived an approximation for the overflow variances using results from the iterative solution of the author. These formulas are discussed in Chapter 6.

CHAPTER 2

A MATHEMATICAL MODEL

2.1 Definitions and Assumptions

The model investigated in this thesis is the two O-D pair model, described in section 1.4 and illustrated in fig. 1.4, within two primary groups N_1 and N_2, a shared secondary group M and two separate overflow groups L_1 and L_2. The state of the system at any instant will be defined by the number of busy servers in each group and denoted by the five dimensional vector $S = (N_1, N_2, M, L_1, L_2)$.

The following assumptions have been made in the model.

(a) Full availability conditions apply to all links.

(b) The system is in a state of equilibrium.

(c) Arrivals for the two O-D pairs occur independently, and have Poisson distributions.

(d) All holding times through the system have independent negative exponential distributions with unit mean.

(e) No more than one event, that is an arrival or departure of a call, can occur in an arbitrarily small time interval.

(f) Final links in each route are provided with sufficient circuits to carry all calls which are offered to them.

Poisson arrival rates have been assumed in the majority of models of overflow systems in which the number of subscribers connected to each exchange is large. Some papers have considered other distributions for the arrival rate; the binomial distribution, for example, is considered by Schehrer [23] and Harris and Rubas [11].

Negative exponential service times are also assumed in many models. The unit mean can be obtained by a suitable scaling of the arrival rate and holding time. This is done simply for convenience and does not affect the validity of the model in any way.

Assumption (e) is a direct consequence of assumptions (c) and (d), if in addition, the arrival rates and service times are independent.

If there is high congestion on a second link of a route then one of the following assumptions is made. The call finding a free junction on the first link of a route, but no free junction on the second link will either be lost from the system or offered to the next choice route. The former assumption corresponds to a zero holding time, the latter to a call overflowing from a link with a free junction. Normally, (in Australia, at least) the final links of second and higher choice routes are dimensioned so that most of the congestion on the route will occur on the first link, and hence assumption (f) is acceptable.

These six assumptions are commonly made in similar models, although not always specifically stated, and are accepted as being reasonable approximations to the real system.

2.2 The state equations for the model

The state of the system is defined by the 5 dimensional vector, $S = (N_1, N_2, M, L_1, L_2)$ and has a probability distribution function defined by

$$f(n_1,n_2,m,\ell_1,\ell_2) = \Pr\{N_1=n_1,N_2=n_2,M=m,L_1=\ell_1,L_2=\ell_2\} \tag{2.1}$$

Since there can never be a negative number of busy servers and the total number of primary and secondary servers is finite,

$$f(n_1,n_2,m,\ell_1,\ell_2) = 0 \text{ outside the range } 0\le n_i\le d_i,\ 0\le m\le c \text{ and } \ell_i\ge 0;\ i=1,2. \tag{2.2}$$

Under the assumption of statistical equilibrium, the probability of the system being in a particular state is independent of time and hence no parameter involving time appears in the probability function. The equations of state are uniquely determined by the definition of f and by the assumptions described in the previous section.

Consider a point in time, t, when the system is in state

$$s_0 = (n_1,n_2,m,\ell_1,\ell_2).$$

By assumption (e), in an arbitrarily small time period, Δt, only one event can occur, to first order. Hence the state of the system at time $t+\Delta t$ can be one of the following,

$$s_1 = (n_1+1,n_2,m,\ell_1,\ell_2)$$

$$s_{-1} = (n_1-1,n_2,m,\ell_1,\ell_2)$$

$$s_2 = (n_1,n_2+1,m,\ell_1,\ell_2) \text{ and so on.}$$

State s_j differs from s_0 in that the jth parameter is increased by 1, and in s_{-j} the jth parameter has decreased by 1.

The transition from state s_0 to s_1 corresponds to an arrival in the first stream. If all servers in N_1 are busy then an arrival in the first stream would cause an increase in the number of busy servers in group M, (s_0 to s_3), or if the secondary group also had no free servers an increase in the number of busy servers in group L_1 (s_0 to s_4). The probability of a first stream arrival occuring in that small time period is $a_1\Delta t + o(\Delta t)$. A similar set of transitions occur for the second stream.

If a service group has x servers busy at time t, then, since all the service times have unit mean, the probability of any particular server completing his service is Δt. Hence, the probability that exactly one of the x servers completes a service in the time interval is $x\Delta t + o(\Delta t)$, and this corresponds to a transition, s_0 to s_{-j}, for the group corresponding to the jth parameter.

If the probability of being in state s at time t is denoted by $\Pr\{s;t\}$ then the transition equation for s_0 is

$$
\begin{aligned}
\Pr\{s_0;t+\Delta t\} = \;& \Pr\{s_{-1};t\}\cdot a_1\Delta t && (N_1 \text{ arrival})\\
&+ \Pr\{s_{-2};t\}\cdot a_2\Delta t && (N_2 \text{ arrival})\\
&+ \Pr\{s_1;t\}\cdot(n_1+1)\Delta t && (N_1 \text{ departure})\\
&+ \Pr\{s_2;t\}\cdot(n_2+1)\Delta t && (N_2 \text{ departure})\\
&+ \Pr\{s_3;t\}\cdot(m+1)\Delta t && (M \text{ departure})\\
&+ \Pr\{s_4;t\}\cdot(\ell_1+1)\Delta t && (L_1 \text{ departure})\\
&+ \Pr\{s_5;t\}\cdot(\ell_2+1)\Delta t && (L_2 \text{ departure})\\
&+ \Pr\{s_0;t\}(1-(a_1+a_2+n_1+n_2+m+\ell_1+\ell_2)\Delta t) && (\text{no event})\\
&+ o(\Delta t) && (\text{more than one event}) \qquad (2.3)
\end{aligned}
$$

Equation (2.3) holds for s_0, such that $0 \le n_i \le d_i - 1$, $0 \le m \le c-1$, $\ell_i \ge 0$. (This equation, in fact, considers the probability of finishing in state s_0 after Δt, starting from the states which are one step accessible from that state at time t.)

If $\Pr\{s_0;t\}$ is subtracted from both sides of (2.3) and then both sides are divided by Δt, equation (2.3) yields

$$
\begin{aligned}
\frac{\Pr\{s_0;t+\Delta t\}-\Pr\{s_0;t\}}{\Delta t} = & -(a_1+a_2+n_1+n_2+m+\ell_1+\ell_2)\Pr(s_0;t)\\
& + a_1\Pr\{s_{-1};t\} + a_2\Pr\{s_{-2};t\}\\
& + (n_1+1)\Pr\{s_1;t\} + (n_2+1)\Pr\{s_2;t\}\\
& + (m+1)\Pr\{s_3;t\}\\
& + (\ell_1+1)\Pr\{s_4;t\} + (\ell_2+1)\Pr\{s_5;t\}\\
& + \frac{o(\Delta t)}{\Delta t} \qquad (2.4)
\end{aligned}
$$

Under the assumption of statistical equilibrium $\Pr\{s;t\} = f(s)$ and if $\Delta t \to 0$ then

$$\frac{\Pr\{s_0;t+\Delta t\}-\Pr\{s_0;t\}}{\Delta t} \to \frac{d\Pr\{s_0;t\}}{dt} = 0 \text{ and } \frac{o(\Delta t)}{\Delta t} \to 0.$$

Therefore (2.4) becomes,

$$(a_1+a_2+n_1+n_2+m+\ell_1+\ell_2)f(n_1,n_2,m,\ell_1,\ell_2)$$

$$= a_1 f(n_1-1,n_2,m,\ell_1,\ell_2) + a_2 f(n_1,n_2-1,m,\ell_1,\ell_2)$$

$$+ (n_1+1)f(n_1+1,n_2,m,\ell_1,\ell_2) + (n_2+1)f(n_1,n_2+1,m,\ell_1,\ell_2)$$

$$+ (m+1)f(n_1,n_2,m+1,\ell_1,\ell_2)$$

$$+ (\ell_1+1)f(n_1,n_2,m,\ell_1+1,\ell_2) + (\ell_2+1)f(n_1,n_2,m,\ell_1,\ell_2+1) \quad (2.5a)$$

and this holds for $0 \le n_i < d_i$, $0 \le m \leq c-1$, $\ell_i \ge 0$; $i=1,2$.

The boundary equations corresponding to states in which one or more of the service groups has no free servers can be derived in a similar way. In order to simplify these equations, the abbreviation $f_j(k)$ will be used, to indicate that the jth parameter of s_0 has been changed to k. For example,

$$f_1(n_1-1) \equiv f(n_1-1,n_2,m,\ell_1,\ell_2)$$

$$f_{4,5}(k_1,k_2) \equiv f(n_1,n_2,m,k_1,k_2) \text{ and so on}$$

and

$$f = f(n_1,n_2,m,\ell_1,\ell_2).$$

In addition, the unsubscripted terms a, n and ℓ will be used to indicate the sum of the corresponding two parameters, for example $a = a_1 + a_2$.

For $n_1 = d_1$, $n_2 < d_2$, $m < c$,

$$(a+n+m+\ell)f$$

$$= a_1 f_1(n_1-1) + a_2 f_2(n_2-1)$$

$$+ a_1 f_3(m-1) + (n_2+1)f_2(n_2+1)$$

$$+ (m+1)f_3(m+1)$$

$$+ (\ell_1+1)f_4(\ell_1+1) + (\ell_2+1)f_5(\ell_2+1). \quad (2.5b)$$

For $n_1 < d_1$, $n_2 = d_2$, $m < c$,

$$(a+n+m+\ell)f$$

$$= a_1 f_1(n_1-1) + a_2 f_2(n_2-1)$$

$$+ (n_1+1)f_1(n_1+1) + a_2 f_3(m-1)$$

$$+ (m+1)f_3(m+1)$$

$$+ (\ell_1+1)f_4(\ell_1+1) + (\ell_2+1)f_5(\ell_2+1). \quad (2.5c)$$

For $n_1 = d_1$, $n_2 = d_2$, $m < c$,

$$\begin{aligned}(a+n+m+\ell)f = {} & a_1 f_1(n_1-1) + a_2 f_2(n_2-1) \\ & + a_1 f_3(m-1) + a_2 f_3(m-1) \\ & + (m+1) f_3(m+1) \\ & + (\ell_1+1) f_4(\ell_1+1) + (\ell_2+1) f_5(\ell_2+1). \end{aligned} \tag{2.5d}$$

For $n_1 < d_1$, $n_2 < d_2$, $m = c$,

$$\begin{aligned}(a+n+m+\ell)f = {} & a_1 f_1(n_1-1) + a_2 f_2(n_2-1) \\ & + (n_1+1) f_1(n_1+1) + (n_2+1) f_2(n_2+1) \\ & + (\ell_1+1) f_4(\ell_1+1) + (\ell_2+1) f_5(\ell_2+1). \end{aligned} \tag{2.5e}$$

For $n_1 = d_1$, $n_2 < d_2$, $m = c$,

$$\begin{aligned}(a+n+m+\ell)f = {} & a_1 f_1(n_1-1) + a_2 f_2(n_2-1) \\ & + a_1 f_3(m-1) + (n_2+1) f_2(n_2+1) \\ & + a_1 f_4(\ell_1-1) \\ & + (\ell_1+1) f_4(\ell_1+1) + (\ell_2+1) f_4(\ell_2+1). \end{aligned} \tag{2.5f}$$

For $n_1 < d_1$, $n_2 = d_2$, $m = c$,

$$\begin{aligned}(a+n+m+\ell)f = {} & a_1 f_1(n_1-1) + a_2 f_2(n_2-1) \\ & + (n_1+1) f_1(n_1+1) + a_2 f_3(m-1) \\ & \qquad\qquad + a_2 f_5(\ell_2-1) \\ & + (\ell_1+1) f_4(\ell_1+1) + (\ell_2+1) f_5(\ell_2+1). \end{aligned} \tag{2.5g}$$

For $n_1 = d_1$, $n_2 = d_2$, $m = c$,

$$\begin{aligned}(a+n+m+\ell)f = {} & a_1 f_1(n_1-1) + a_2 f_2(n_2-1) \\ & + a_1 f_3(m-1) + a_2 f_3(m-1) \\ & + a_1 f_4(\ell_1-1) + a_2 f_5(\ell_2-1) \\ & + (\ell_1+1) f_4(\ell_1+1) + (\ell_2+1) f_5(\ell_2+1). \end{aligned} \tag{2.5h}$$

The state equations described in (2.5) are valid for all values of $\ell_1 \geq 0$, $\ell_2 \geq 0$ and n_1, n_2, m must also be non zero, as well as satisfying the appropriate conditions, which precede each equation.

Although some expressions may be simplified (e.g. 2.5h) by using $a_1+a_2 = a$, they have been left in the unsimplified form to illustrate the changes which occur in the equations, as each variable reaches its upper limit.

These eight equations represent an infinite system since both L_1 and L_2 are unlimited groups. The model can be simplified by considering an equivalent system of equations in which the relationships between the various states of the system are described in terms of the binomial moments of L_1 and L_2.

2.3 The system of moment equations

The binomial moments of L_1 and L_2, are defined by

$$B_{\ell_1,\ell_2}(n_1,n_2,m) = \sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty} \binom{k_1}{\ell_1}\binom{k_2}{\ell_2} f(n_1,n_2,m,k_1,k_2)$$

$$\text{for } 0 \le n_i \le d_i,\ 0 \le m \le c,\ \ell_i \ge 0$$

$$= 0 \text{ otherwise.} \tag{2.6}$$

Equations involving these moments can be derived from (2.5) using the following steps:

(1) change the dummy variables ℓ_1, ℓ_2 to k_1 and k_2.

(2) multiply the equations by $\binom{k_1}{\ell_1}\binom{k_2}{\ell_2}$.

(3) sum the equations over the ranges of k_1 and k_2, (namely $\sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty}$) and simplify.

The performance of step (3) is facilitated by use of the following lemmas.

Lemma 1:

For a sufficiently well behaved function $h(k)$ defined on the non negative integers

$$\sum_{k=\ell}^{\infty}\{kh(k)-(k+1)h(k+1)\}\binom{k}{\ell}$$

$$= \ell \sum_{k=\ell}^{\infty} h(k)\binom{k}{\ell}. \tag{2.7}$$

Proof:

The following identity holds for $k>\ell\ge 0$,

$$\binom{k}{\ell} = \binom{k+1}{\ell}\frac{k+1-\ell}{k+1}$$

$$= \binom{k+1}{\ell} - \frac{\ell}{k+1}\binom{k+1}{\ell}.$$

Hence, the L.H.S. of (2.7) $= \ell h(\ell)\binom{\ell}{\ell} + \sum_{k=\ell+1}^{\infty} kh(k)\binom{k}{\ell}$

$$- \sum_{k=\ell}^{\infty}(k+1)h(k+1)\left[\binom{k+1}{\ell} - \frac{\ell}{k+1}\binom{k+1}{\ell}\right]$$

$$= \ell h(\ell)\binom{\ell}{\ell} + \sum_{k=\ell+1}^{\infty} kh(k)\binom{k}{\ell}$$

$$- \sum_{k=\ell+1}^{\infty} kh(k)\binom{k}{\ell} + \sum_{k=\ell+1}^{\infty} \ell h(k)\binom{k}{\ell}$$

$$= \ell \sum_{k=\ell}^{\infty} h(k)\binom{k}{\ell}$$

= R.H.S. of (2.7)

Lemma 2:

$$\sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty} \binom{k_1}{\ell_1}\binom{k_2}{\ell_2} f_{4,5}(k_1,k_2-1)$$

$$= B_{\ell_1,\ell_2}(n_1,n_2,m) + B_{\ell_1,\ell_2-1}(n_1,n_2,m) \qquad (2.8)$$

And a similar relationship holds for $f_{4,5}(k_1-1,k_2)$.

Proof:

Using the identity

$$\binom{k}{\ell} = \binom{k-1}{\ell} + \binom{k-1}{\ell-1},$$

the L.H.S. of (2.8) $= \sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty} \binom{k_1}{\ell_1}\binom{k_2-1}{\ell_2} f_{4,5}(k_1,k_2-1)$

$$+ \sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty} \binom{k_1}{\ell_1}\binom{k_2-1}{\ell_2-1} f_{4,5}(k_1,k_2-1)$$

$$= \sum_{k_1=\ell_1}^{\infty} \sum_{k=\ell_2-1}^{\infty} \binom{k_1}{\ell_1}\binom{k}{\ell_2} f_{4,5}(k_1,k)$$

$$+ \sum_{k_1=\ell_1}^{\infty} \sum_{k=\ell_2-1}^{\infty} \binom{k_1}{\ell_1}\binom{k}{\ell_2-1} f_{4,5}(k_1,k)$$

$$= B_{\ell_1,\ell_2}(n_1,n_2,m) + B_{\ell_1,\ell_2-1}(n_1,n_2,m)$$

$$= \text{R.H.S. of (2.8)}.$$

The simplification of the first summation term uses the definition of $B_{\ell_1,\ell_2}(n_1,n_2,m)$ and $\binom{\ell_2-1}{\ell_2} = 0$.

Abbreviations for $B_{\ell_1,\ell_2}(n_1,n_2,m)$ will be used to simplify the new system of equations which will be obtained.

$B(,,,)$ will represent $B_{\ell_1,\ell_2}(n_1,n_2,m)$

$B(x,,) \equiv B_{\ell_1,\ell_2}(x,n_2,m)$

$B(,,y) \equiv B_{\ell_1,\ell_2}(n_1,n_2,y)$

$B_{z,}(,,,) \equiv B_{z,\ell_2}(n_1,n_2,m)$ and so on.

Application of the steps described above to (2.5a) yields, (with $\Sigma\Sigma \equiv \sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty}$),

$$(a+n+m)\Sigma\Sigma\binom{k_1}{\ell_1}\binom{k_2}{\ell_2}f + \Sigma\Sigma\binom{k_1}{\ell_1}\binom{k_2}{\ell_2}k_1 f + \Sigma\Sigma\binom{k_1}{\ell_1}\binom{k_2}{\ell_2}k_2 f$$

$$= a_1\ \Sigma\Sigma\binom{k_1}{\ell_1}\binom{k_2}{\ell_2}f_1(n_1-1) + a_2\ \Sigma\Sigma\binom{k_1}{\ell_1}\binom{k_2}{\ell_2}f_2(n_1-1)$$

$$+ (n_1+1)\ \Sigma\Sigma\binom{k_1}{\ell_1}\binom{k_2}{\ell_2}f_1(n_1+1) + (n_2+1)\ \Sigma\Sigma\binom{k_1}{\ell_1}\binom{k_2}{\ell_2}f_2(n_2+1)$$

$$+ (m+1)\ \Sigma\Sigma\binom{k_1}{\ell_1}\binom{k_2}{\ell_2}f_3(m+1)$$

$$+ \Sigma\Sigma(k_1+1)\binom{k_1}{\ell_1}\binom{k_2}{\ell_2}f_4(k_1+1) + \Sigma\Sigma(k_2+1)\binom{k_1}{\ell_1}\binom{k_2}{\ell_2}f_5(k_2+1).$$

Using the definition of $B_{\ell_1,\ell_2}(n_1,n_2,m)$ and Lemma 1, this simplifies to

$$\begin{aligned}(a+n+m+\ell)B(,,,) &= a_1B(n_1-1,,) + a_2B(,n_2-1,)\\ &+ (n_1+1)B(n_1+1,,) + (n_2+1)B(,n_2+1,)\\ &+ (m+1)B(,,m+1).\end{aligned} \tag{2.9a}$$

Equation (2.9a) is valid for $n_1 < d_1$, $n_2 < d_2$ and $m < c$, (and all five variables, $(n_1,n_2,m,\ell_1,$ and $\ell_2)$ must be non negative integers).

The boundary conditions can be represented in terms of the binomial moments and these equations can be obtained from the state equations analogously to (2.9a). Lemma 2 is used in the derivation of equations (2.9f, g and h).

For $n_1 = d_1$, $n_2 < d_2$, $m < c$,

$$\begin{aligned}(a+n+m+\ell)B(,,,) &= a_1B(n_1-1,,) + a_2B(,n_2-1,)\\ &+ a_1B(,,m-1) + (n_2+1)B(,n_2+1,)\\ &+ (m+1)B(,,m+1).\end{aligned} \tag{2.9b}$$

For $n_1 < d_1$, $n_2 = d_2$, $m < c$,

$$\begin{aligned}(a+n+m+\ell)B(,,,) &= a_1B(n_1-1,,) + a_2B(,n_2-1,)\\ &+ (n_1+1)B(n_1+1,,) + a_2B(,,m-1)\\ &+ (m+1)B(,,m+1).\end{aligned} \tag{2.9c}$$

For $n_1 = d_1$, $n_2 = d_2$, $m < c$,

$$\begin{aligned}(a+n+m+\ell)B(,,,) &= a_1B(n_1-1,,) + a_2B(,n_2-1,)\\ &+ a_1B(,,m-1) + a_2B(,,m-1)\\ &+ (m+1)B(,,m+1).\end{aligned} \tag{2.9d}$$

For $n_1 < d_1$, $n_2 < d_2$, $m = c$,

$$\begin{aligned}(a+n+m+\ell)B(,,,) &= a_1B(n_1-1,,) + a_2B(,n_2-1,)\\ &+ (n_1+1)B(n_1+1,,) + (n_2+1)B(,n_2+1,).\end{aligned} \tag{2.9e}$$

For $n_1 = d_1$, $n_2 < d_2$, $m = c$,

$$\begin{aligned}(a+n+m+\ell)B(,,,) &= a_1B(n_1-1,,) + a_2B(,n_2-1,)\\ &\quad + a_1B(,,m-1) + (n_2+1)B(,n_2+1,)\\ &\quad + a_1B(,,,)\\ &\quad + a_1B_{\ell_1-1,}(,,,). \end{aligned} \tag{2.9f}$$

For $n_1 < d_1$, $n_2 = d_2$, $m = c$,

$$\begin{aligned}(a+n+m+\ell)B(,,,) &= a_1B(n_1-1,,) + a_2B(,n_2-1,)\\ &\quad + (n_1+1)B(n_1+1,,) + a_2B(,,m-1)\\ &\quad + a_2B(,,,)\\ &\quad + a_2B_{,\ell_2-1}(,,,). \end{aligned} \tag{2.9g}$$

For $n_1 = d_1$, $n_2 = d_2$, $m = c$,

$$\begin{aligned}(a+n+m+\ell)B(,,,) &= a_1B(n_1-1,,) + a_2B(,n_2-1,)\\ &\quad + a_1B(,,m-1) + a_2B(,,m-1)\\ &\quad + a_1B(,,,) + a_2B(,,,)\\ &\quad + a_1B_{\ell_1-1,}(,,,) + a_2B_{,\ell_2-1}(,,,). \end{aligned} \tag{2.9h}$$

Once again, these equations could have been simplified but only at the expense of showing the patterns of changes.

Equations (2.9) describe a system of simple recursive equations in ℓ_1 and ℓ_2, whereas the state equations (2.5) are quadratic recursive equations. The transformation has 'removed' the terms involving (ℓ_1+1) and (ℓ_2+1).

If the moments are summed over all values of n_1, n_2 and m then the result, denoted by $B_{(\ell_1,\ell_2)}$ is

$$\begin{aligned}B_{(\ell_1,\ell_2)} &= \sum_{n_1=0}^{d_1}\sum_{n_2=0}^{d_2}\sum_{m=0}^{c}\sum_{k_1=\ell_1}^{\infty}\sum_{k_2=\ell_2}^{\infty}\binom{k_1}{\ell_1}\binom{k_2}{\ell_2}f(n_1,n_2,m,k_1,k_2)\\ &= \sum_{k_1=\ell_1}^{\infty}\sum_{k_2=\ell_2}^{\infty}\binom{k_1}{\ell_1}\binom{k_2}{\ell_2}\Pr\{L_1=k_1,L_2=k_2\}\\ &= E\left[\binom{L_1}{\ell_1}\binom{L_2}{\ell_2}\right]. \end{aligned} \tag{2.10}$$

Furthermore, if equations (2.9) are summed over all values of $(n_1, n_2$ and $m)$ the result gives an alternate expression for $B_{(\ell_1,\ell_2)}$, namely

$$(\ell_1+\ell_2)B_{(\ell_1,\ell_2)} = a_1 \sum_{n_2=0}^{d_2} B_{\ell_1-1,\ell_2}(d_1,n_2,c) + a_2 \sum_{n_1=0}^{d_1} B_{\ell_1,\ell_2-1}(n_1,d_2,c). \tag{2.11}$$

The derivation of (2.11) is given in Appendix A. The two equations (2.10) and (2.11) lead to expressions for the means, variances and covariances of the overflow streams in terms of the binomial moments since

$$\begin{aligned}
m_1 &= E[L_1] = B_{(1,0)} \\
m_2 &= E[L_2] = B_{(0,1)} \\
v_1 &= E[L_1^2] - E[L_1]^2 \\
&= 2E[\frac{L_1\cdot(L_1-1)}{2}] + E[L_1] - E[L_1]^2 \\
&= 2\ B_{(2,0)} + B_{(1,0)} - B^2_{(1,0)} \\
v_2 &= 2\ B_{(0,2)} + B_{(0,1)} - B^2_{(0,1)} \\
\mathrm{cov} &= E[L_1L_2] - E[L_1]\cdot E[L_2] \\
&= B_{(1,1)} - B_{(1,0)}\cdot B_{(0,1)}.
\end{aligned} \tag{2.12}$$

Therefore by (2.12) and (2.11) the five overflow statistics can be calculated if the system of equations, (2.9), can be solved for (ℓ_1,ℓ_2) equal to the values (0,0), (0,1) and (1,0).

The equations (2.9), for fixed values of ℓ_1 and ℓ_2, involve $R=(d_1+1)(d_2+1)(c+1)$ unknowns, namely the binomial moments. There are exactly the same number of equations in the system, and for (ℓ_1,ℓ_2) not equal to (0,0) the equations are linearly independent. When $\ell_1=\ell_2=0$, (2.11) reduces to the identity 0=0, indicating a linearly dependent relationship between the equations (2.9) in this case. However,

$$\begin{aligned}
B_{0,0}(n_1,n_2,m) &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} f(n_1,n_2,m,k_1,k_2) \\
&= \Pr\{N_1=n_1,N_2=n_2,M=m\}.
\end{aligned} \tag{2.13}$$

Hence, there is an additional equation which the moments must satisfy for this case, namely the normalising equation,

$$\sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \sum_{m=0}^{c} B_{0,0}(n_1,n_2,m) = 1. \qquad (2.14)$$

If any one of the equations in (2.9) is replaced by (2.14) then there are, again, exactly R linearly independent equations which the $B_{0,0}(n_1,n_2,m)$ moments must satisfy. Therefore for the three values of ℓ_1 and ℓ_2 that are of interest (and, in fact any values) the moments can be determined uniquely by solving three sets of linearly independent equations, and hence the statistics of (2.12) can be obtained.

Since the equations (2.9) are simple recursive with respect to ℓ_1 and ℓ_2, and the moments are zero when ℓ_1 or ℓ_2 is negative, the equations must first be solved for $(\ell_1,\ell_2) = (0,0)$ and then for $(\ell_1,\ell_2) = (0,1)$ and $(1,0)$.

Transformations of a similar nature have been used in many other papers. Usually they are affected using generating functions. Neal [16] uses the bivariate binomial moment generating function,

$$B(m,n_1,n_2;x_1,x_2) = \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} f(m,n_1,n_2,\ell_1,\ell_2)(1+x_1)^{\ell_1}(1+x_2)^{\ell_2}.$$

Riordan [27] and Chastang [4] use what they call factorial moment exponential generating functions which are in fact the same functions as Neal uses, defined in terms of the appropriate state probabilities, namely

$$M(\ldots,t) = \sum_{n=0}^{y} f(\ldots n)(1+t)^{n}.$$

y may be infinite or finite, and f and M may have two or three variables, but in essence the methods are the same. The transformations in these three papers could all have been obtained by the simpler method used in this model, without introducing generating functions.

A similar approach was used by Schehrer [24] for a simple overflow system, that is one with a single primary group, using factorial moments, that is

$$M_r(x) = \sum_{x=0}^{\infty} \binom{x}{r} r!\ p(x).$$

He considers both infinite and finite secondary group models, and calculates higher order moments of the overflow distribution.

The transformations not only reduce the system from quadratic recursive to simple recursive in ℓ_1 and ℓ_2, but also simplify the calculation of the statistics. For example, the formula for the mean in terms of the state probabilities is

$$E[L_1] = \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \sum_{m=0}^{c} \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \ell_1 f(n_1,n_2,m,\ell_1,\ell_2)$$

and in fact all the moments are defined in terms of infinite sums. The equivalent expressions using the binomial moments (2.12) and (2.11) are all finite sums, and in each term the summation is over one variable only.

Higher moments of the distribution of L_1 and L_2 could be found, if desired, by solving (2.9) recursively for the appropriate values of ℓ_1 and ℓ_2.

CHAPTER 3

ITERATIVE SOLUTION OF THE MODEL

3.1 Matrix formulation of the problem

The system of equations (2.9) has a unique solution since there are $R = (d_1+1)(d_2+1)(c+1)$ linearly independent equations and the same number of variables. This system can be expressed as a single matrix equation,

$$D_{\ell_1,\ell_2}\, \underset{\sim}{b}_{\ell_1,\ell_2} = \underset{\sim}{g}_{\ell_1,\ell_2} \tag{3.1}$$

where D_{ℓ_1,ℓ_2} is an R×R matrix and both the vectors have R elements. The vector $\underset{\sim}{b}_{\ell_1,\ell_2}$ consists of the moments $B_{\ell_1,\ell_2}(,,,)$ and $\underset{\sim}{g}_{\ell_1,\ell_2}$ involves $B_{\ell_1-1,\ell_2}(,,,)$ and $B_{\ell_1,\ell_2-1}(,,,)$.

It will be necessary to deviate slightly from standard matrix notation. The subscripts, ℓ_1 and ℓ_2 index a 2 parameter family of matrices (vectors) rather than indicating particular elements of a given matrix (vector).

Capital letters will still be used to indicate matrices and a lower case letter, with a tilde underneath will denote a vector. The elements of the matrices and vectors will be indicated in parenthesis and the tilde will be removed from the vector, for example

$X_{1,0}$ is the matrix X when $\ell_1 = 1$, $\ell_2 = 0$,

x(i) is the ith element of the vector x, and

X(i,j) is the element of X in the ith row and jth column.

The vector $\underset{\sim}{b}_{\ell_1,\ell_2}$ will have as its elements, the binomial moments. It is necessary to order these moments, which have a natural three dimensional representation, into a one dimensional vector. If $B_{\ell_1,\ell_2}(n_1,n_2,m)$ is the rth element of $\underset{\sim}{b}_{\ell_1,\ell_2}$, that is

$$b_{\ell_1,\ell_2}(r) = B_{\ell_1,\ell_2}(n_1,n_2,m)$$

then r is defined by

$$r = (n_1+1) + n_2(d_1+1) + m(d_1+1)(d_2+1). \tag{3.2}$$

This definition ensures a unique arrangement of the binomial moments and the three parameters (n_1,n_2,m) of the moments are arranged in the order,

$$(0,0,0),(1,0,0),(2,0,0),\ldots,(d_1,0,0),(0,1,0),(1,1,0),\ldots,$$
$$(d_1,1,0),(0,2,0),\ldots,(d_1,d_2,0),(0,0,1),(1,0,1),\ldots,(d_1,d_2,c).$$

The uniqueness of r can be verified as follows - suppose there are two values of (n_1,n_2,m) which correspond to the same value of r, namely (x,y,z) and (x',y',z')

$$(x-x') + (y-y')(d_1+1) + (z-z')(d_1+1)(d_2+1) = 0$$

Suppose $x \neq x'$ and without loss of generality $x > x'$. (The case $x = x'$ is dealt with later in the proof.) Division of the equation by (d_1+1), yields

$$\frac{x-x'}{d_1+1} + (y-y') + (z-z')(d_2+1) = 0.$$

Since x, x', y, y', z, z', d_1 and d_2 are all integers, $\dfrac{(x-x')}{d_1+1}$ is also integral and hence $(x-x')$ is a multiple of (d_1+1).

But $0 \le x' < x \le d_1$ and therefore $(x-x') < d_1+1$.
Hence $(x-x') = 0$ is the only integer solution.

Therefore

$$(y-y') + (z-z')(d_2+1) = 0$$

Repeating the argument yields $y = y'$ and hence $z = z'$, thus proving the definition of r is unique.

The vector $\underset{\sim}{g}_{\ell_1,\ell_2}$ contains moments of the form $B_{\ell_1-1,\ell_2}(n_1,n_2,m)$ and $B_{\ell_1,\ell_2-1}(n_1,n_2,m)$ and hence is a function of $\underset{\sim}{b}_{\ell_1-1,\ell_2}$ and $\underset{\sim}{b}_{\ell_1,\ell_2-1}$. Therefore $\underset{\sim}{b}_{\ell_1,\ell_2}$ can be expressed as a function of $\underset{\sim}{b}_{\ell_1-1,\ell_2}$ and $\underset{\sim}{b}_{\ell_1,\ell_2-1}$. Since $\underset{\sim}{b}_{-1,\ell_2}$ and $\underset{\sim}{b}_{\ell_1,-1}$ are zero vectors, for all values of ℓ_1 and ℓ_2, the vectors b_{ℓ_1,ℓ_2} can be calculated recursively starting with $b_{0,0}$. Note that the additional constraint $\sum_{r=1}^{N} b_{0,0}(r) = 1$ must be used to obtain a unique solution for $\underset{\sim}{b}_{0,0}$; where $N = (c+1)(d_1+1)(d_2+1)$.

3.2 The structure of the coefficient matrix

For simplicity the subscripts ℓ_1 and ℓ_2 will be omitted in this section.

The definition of $\underset{\sim}{b}$ together with equations (2.9) uniquely determine the coefficient matrix D. The matrix is sparse and there are at most seven non zero elements in any row. These elements are located in seven bands parallel to the diagonal and this gives D a highly-structured form.

The coefficient of B(,,,) appears as the diagonal element, D(r,r). The element D(r,r-1) is the coefficient of $B(n_1-1,,)$, $D(r,r-(d_1+1))$ is the coefficient of $B(,n_2-1,)$ and the coefficient of $B(,,m-1)$ is located in $D(r-(d_1+1)(d_2+1))$. The coefficients of $B(n_1+1,,)$, $B(,n_2+1,)$ and $B(,,m+1)$ are located in D(r,r+1), $D(r,r+(d_1+1))$ and $D(r,r+(d_1+1)(d_2+1))$ respectively.

The matrix may be considered as a hierarchy of tridiagonal matrices. At the highest level the matrix D is partitioned into square matrices of size $(d_1+1)(d_2+1)$. All these sub-matrices are zero matrices except the 'diagonal' matrices and those adjacent to the diagonal, as follows

$$D = \begin{bmatrix} Q_0 & U_0 & & & & 0 \\ S & Q_1 & U_1 & & & \\ & S & Q_2 & U_2 & & \\ & & \cdot & \cdot & \cdot & \\ & & & S & Q_{c-1} & U_{c-1} \\ 0 & & & & S & Q_c \end{bmatrix} \tag{3.3}$$

The matrices S and U_m are diagonal.

$$U_m = -(m+1)\, I_{(d_1+1)(d_2+1)}; \quad m=0,1,..,c-1 \tag{3.4}$$

where I_x is the identity matrix of order x.

The diagonal elements of S, are defined by

$$S(s,s) = -a_1\delta_{n_1}^{d_1} - a_2\delta_{n_2}^{d_2} \tag{3.5}$$

where $s = (n_1+1) + n_2(d_2+1)$ and δ_n^d is the Kronecker delta.

The matrices Q_m are themselves tridiagonal at the next level, in which the 'elements' are square matrices of order (d_1+1).

$$Q_m = \begin{bmatrix} Q_{0,m} & U_{0,m} & & & & 0 \\ S_1 & Q_{1,m} & U_{1,m} & & & \\ & S_1 & Q_{2,m} & U_{2,m} & & \\ & & \cdot & \cdot & \cdot & \\ & & & S_1 & Q_{d_2-1,m} & U_{d_2-1,m} \\ 0 & & & & S_1 & Q_{d_2,m} \end{bmatrix} \tag{3.6}$$

for $m=0,1,\ldots,c$.

The matrices S_1 and $U_{n_2,m}$ are again diagonal with

$$U_{n_2,m} = -(n_2+1)I_{(d_1+1)}; \quad n_2 = 0,1,\ldots, d_2-1, \quad m = 0,1,\ldots, c \tag{3.7}$$

and $$S_1 = -a_2 I_{(d_1+1)}. \tag{3.8}$$

The matrices $Q_{n_2,m}$ are again tridiagonal.

$$Q_{n_2,m} = \begin{bmatrix} q_{0,n_2,m} & -1 & & & 0 \\ -a_1 & q_{1,n_2,m} & -2 & & \\ & \cdot & \cdot & \cdot & \\ & -a_1 & & q_{d_1-1,n_2,m} & -d_1 \\ 0 & & & -a_1 & q_{d_1,n_2,m} \end{bmatrix}$$

for $n_2 = 0,1,\ldots,d$ and $m = 0,1,\ldots,c$. (3.9)

Finally,

$$q_{n_1,n_2,m} = a_1 + a_2 + n_1 + n_2 + m + \ell_1 + \ell_2 - \delta_{n_1}^{d_1}\delta_m^c a_1 - \delta_{n_2}^{d_2}\delta_m^c a_2. \tag{3.10}$$

The vector $\underset{\sim}{g}_{\ell_1,\ell_2}$ is defined by

$$g_{\ell_1,\ell_2}(r) = \delta_{n_1}^{d_1}\delta_m^c a_1 b_{\ell_1-1,\ell_2}(r) + \delta_{n_2}^{d_2}\delta_m^c a_2 b_{\ell_1,\ell_2-1}(r) \tag{3.11}$$

where r is defined by (3.2).

When $\ell_1 = \ell_2 = 0$ the last row of D is replaced by a row of ones and the last element of $\underset{\sim}{g}$ (which is in fact a zero vector) by a one. This row then corresponds to the normalising constraint, $\sum_{n_1=0}^{d_1}\sum_{n_2=0}^{d_2}\sum_{m=0}^{c} B_{0,0}(n_1,n_2,m)=1$.

The elements of the coefficient matrix, with its highly structural form and spareness, can be simply and efficiently calculated in a computer program and is highly suitable for a computational solution of equation (3.1).

3.3 Some iterative solution techniques

Theoretically the vector $\underset{\sim}{b}$ can be obtained by

$$\underset{\sim}{b} = D^{-1}\underset{\sim}{g}.$$

This involves calculating the inverse of the matrix D, but since this matrix tends to be large, and the inverse when calculated is no longer sparse, the solution is not very practical. However, there are several iterative techniques which can be used to solve the equation

$$D\underset{\sim}{b} = \underset{\sim}{g}.$$

The equation can be rewritten in the form

$$(I-A)\underset{\sim}{b} = \underset{\sim}{f},$$

which can be conveniently effected by dividing each row and the corresponding element of $\underset{\sim}{g}$ by the diagonal element. Rearrangement of this equation gives

$$\underset{\sim}{b} = A\underset{\sim}{b} + \underset{\sim}{f}. \tag{3.12}$$

If an initial estimate $\underset{\sim}{b}^0$ is chosen then successive estimates are calculated iteratively by

$$\underset{\sim}{b}^{(k+1)} = A\underset{\sim}{b}^{k} + \underset{\sim}{f} \tag{3.13}$$

until a specific error condition is fulfilled. For example, the kth and $(k+1)$th estimate of each element differ by less than 10^{-6}, that is

$$\max_{r} |b^{(k+1)}(r) - b^{k}(r)| < 10^{-6}. \tag{3.14}$$

This iterative technique is known as the Jacobi Method (see Faddeeva [8]). The criterion for convergence of this procedure, for any arbitrary initial estimate, is that all eigenvalues of A lie within the unit circle, or equivalently the spectral radius is less than one. Generally, for iterative techniques, the 'convergence criterion is of theoretical interest only, since finding the spectral radius of the iteration matrix is usually of the same order of difficulty as solving the original equations', to quote from Cooper's book [6]. A sufficient condition for convergence is strict diagonal dominance in the coefficient matrix D. That is

$$|D(r,r)| > \sum_{\substack{s \neq r \\ s=1}}^{R} |D(r,s)|. \tag{3.15}$$

This condition is not satisfied but the diagonal elements are considerably larger in absolute value than the other elements in the rows, and this suggests that convergence will occur. In fact, in every case considered convergence has occurred and for practical purposes it will be assumed that this will always happen.

An improvement on the Jacobi method is a procedure commonly known as Gauss-Siedel iteration, although the names of Liebmann and Nekrasov are also associated with the method (see Cooper [6]). In this method the elements of $\underset{\sim}{b}^{k+1}$ already obtained are used to calculate the remaining elements of the vector. The matrix A may be partitioned into a lower triangular matrix T_L and an upper triangular matrix T_U, that is

$$A = T_L + T_U.$$

Equation (3.13) would then become

$$\underset{\sim}{b}^{(k+1)} = T_L\, \underset{\sim}{b}^{k} + T_U\, \underset{\sim}{b}^{k} + \underset{\sim}{f}$$

Since the elements of $\underset{\sim}{b}^{(k+1)}$ are calculated in the order 1,2, 3,...R, at any stage of the calculation all the values of $\underset{\sim}{b}$ which have a non zero coefficient in T_L have already been found for the $(k+1)$th estimate. Since these values are presumably more accurate than the corresponding kth estimates, it is sensible to use them in the calculation of succeeding elements, that is

$$\underset{\sim}{b}^{(k+1)} = T_L\, \underset{\sim}{b}^{(k+1)} + T_U\, \underset{\sim}{b}^{k} + \underset{\sim}{f}. \qquad (3.16)$$

The criterion for convergence of this procedure is that the spectral radius of $(I-T_L)^{-1}T_U$ is less than unity, again of theoretical interest only.

Gauss-Siedel iteration can be accelerated by a technique known as successive overrelaxation (abbreviated to S.O.R.). The new estimate $\underset{\sim}{b}^{(k+1)}$ is obtained from a weighted mean of the old estimate $\underset{\sim}{b}^{k}$ and the $(k+1)$th estimate that would have been obtained using Gauss-Siedel iteration. The S.O.R. formula is

$$\underset{\sim}{b}^{(k+1)} = \theta(T_L\, \underset{\sim}{b}^{(k+1)} + T_U\, \underset{\sim}{b}^{k} + \underset{\sim}{f}) + (1-\theta)\, \underset{\sim}{b}^{k}. \qquad (3.17)$$

The parameter θ is known as the S.O.R. constant. It was found that using $\theta = 1.2$ resulted in a 10-20% improvement in the convergence rate. For example, the vector $\underset{\sim}{b}_{0,0}$ was calculated by the three methods for the case when $d_1 = 3$, $d_2 = 4$, $c = 5$, $a_1 = 6$ and $a_2 = 9$, with the stopping condition of (3.14). The Jacobi method required 113 iterations before convergence. Since there were only 120 elements in the vector this is not a significant improvement over the straightforward method of inverting the coefficient matrix $D_{0,0}$. However, for the Gauss-Siedel method only 27 iterations were needed and with an S.O.R. constant of 1.2 this was reduced to 24 iterations. The initial estimate was the 'uniform' vector with all elements set to $\frac{1}{120}$.

It may be noted that ordinary Gauss-Siedel iteration is obtained when $\theta = 1$. There are many unanswered questions about convergence in the S.O.R. method, and Cooper suggests that 'the interested reader should see Varga' [25]. Once again it is of theoretical interest only, since in the cases considered the S.O.R. technique was more efficient than ordinary Gauss-Siedel iteration.

It was suggested by Benjamin [1], that the convergence of the S.O.R. technique could be accelerated even more by using a variable value of θ. A large value of θ is used in the initial steps to 'shake up' the system, that is cause a large variation in successive estimates of $\underset{\sim}{b}$. Since the process will diverge for $\theta \geq 2$ a 'large' value would, for example, be 1.9 or 1.95. With a value of θ this large the estimates may 'overshoot' the true value of $\underset{\sim}{b}$ at each step and therefore oscillate about this true value. If the average of two successive estimates of $\underset{\sim}{b}$ is then calculated, this average should be close to the true value.

It can be shown that using a value $\theta/2$ for one iteration (the kth) is equivalent to averaging the kth and the k+1th estimates which would be calculated using the value of θ.

$$\tfrac{1}{2}(\underset{\sim}{b}^{k+1} + \underset{\sim}{b}^{k}) = \tfrac{1}{2}\{\theta(T_L\,\underset{\sim}{b}^{k+1} + T_U\,\underset{\sim}{b}^{k}) + (1-\theta)\,\underset{\sim}{b}^{k} + \underset{\sim}{b}^{k}\}$$

$$= \frac{\theta}{2}(T_L\,\underset{\sim}{b}^{k+1} + T_U\,\underset{\sim}{b}^{k}) + (1 - \frac{\theta}{2})\,\underset{\sim}{b}^{k}$$

$$= \underset{\sim}{b}^{*}.$$

The value $\underset{\sim}{b}^*$ should be a good 'initial' estimate to use in succeeding iterations with a smaller, constant, value of θ.

Hence the process may be described by the following algorithm

(1) choose an initial estimate $\underset{\sim}{b}^0$.

(2) for 5 iterations of S.O.R. method use
$\theta = \theta_L$ (a large value of θ).

(3) for one iteration use $\theta = \theta_L/2$.

(4) for remaining iterations use $\theta = \theta_0$,
some value which gives a good convergence rate.

In the S.O.R. program which was used to calculated the tables in Appendix 4, $\theta_L = 1.9$ and $\theta_0 = 1.4$. Mr. Benjamin further suggested that if convergence becomes 'slow' in step (4), steps (2) and (3) be repeated, but this was not incorporated into the program, since the convergence rate with standard S.O.R. was good, the improvement using the algorithm was slight and further 'shaking up' did not seem likely to give significant improvement.

A diagramatic representation of the iterative techniques is given in Figure 3.1. It is meant only to illustrate the comparative convergence rates and is not necessarily a true representation of any actual problem. The diagram may be considered to represent a projection of the successive estimates $\underset{\sim}{b}^k$ onto the (b(1),b(2)) plane.

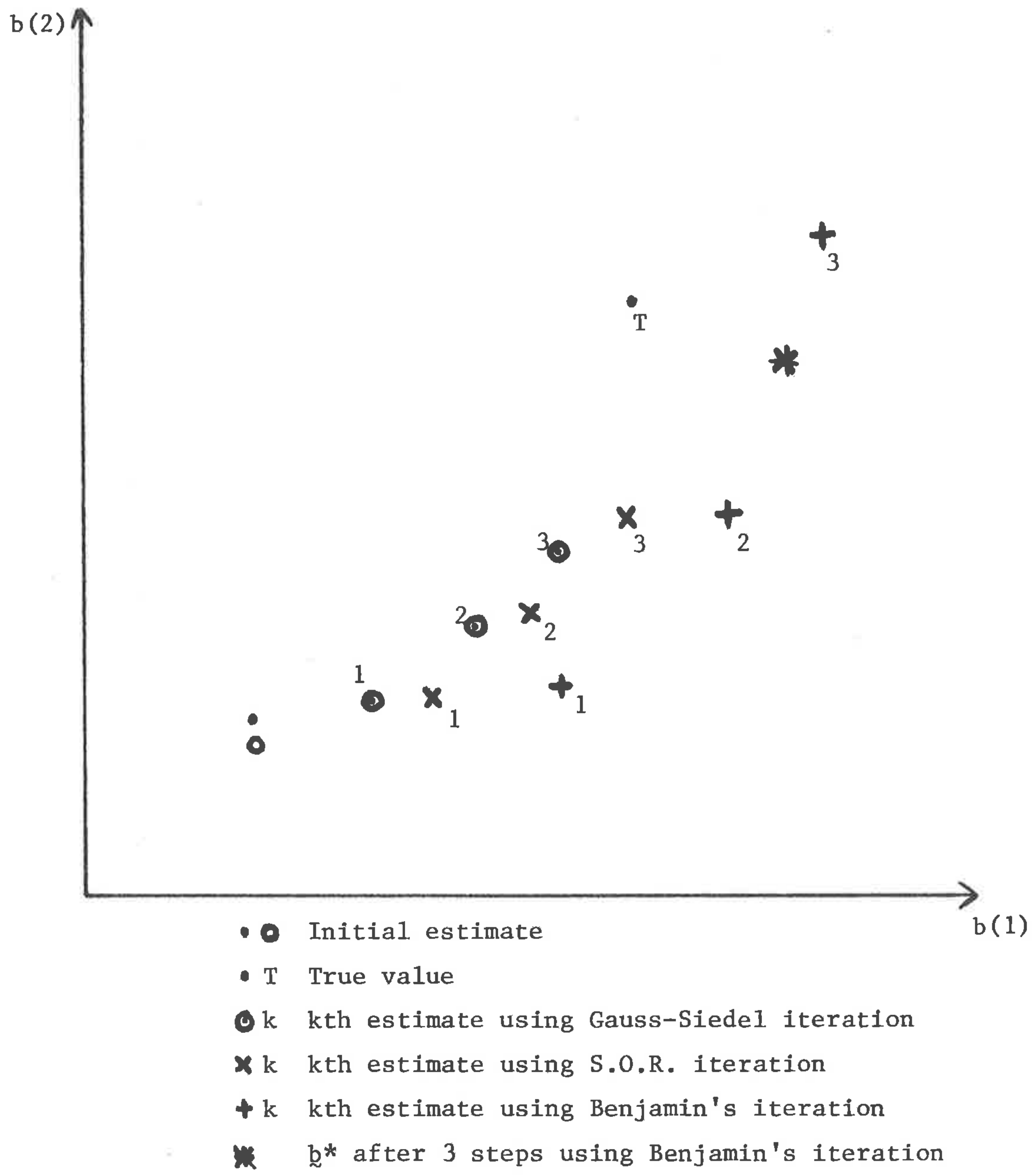


Figure 3.1 : Comparison of iterative methods.

3.4 Some aspects of programming the algorithms on a computer

Although the complete listing of the program which performs the S.O.R. iteration is given in Appendix B, there are some features worthy of special mention.

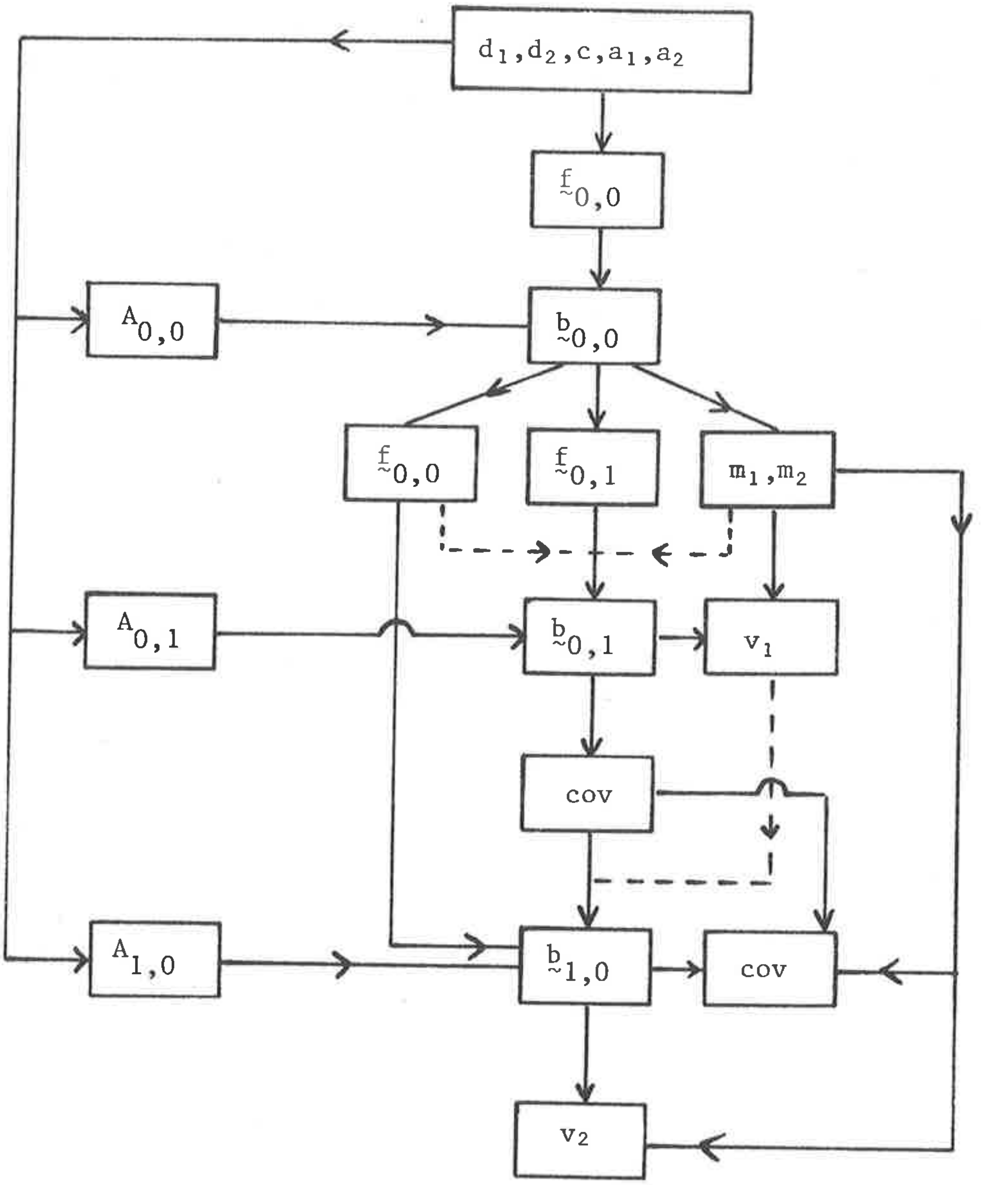
For a system with $R = (d_1+1)(d_2+1)(c+1)$ states the array which holds the values of the coefficient matrix would need R^2 elements. The 120×120 matrix needed for the case described above, ($d_1 = 3$, $d_2 = 4$, $c = 5$) is a considerably large array, although the size of the service groups is comparatively small. However, at most seven elements in each row are non-zero and these can be stored in an $R \times 7$ array, each column of the array corresponding to one 'diagonal band'. However, this still requires considerable storage space and it was found that core space was more critical than processing time in limiting the size of the systems which could be considered.

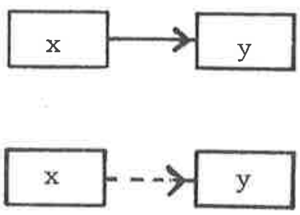
The highly structured form of D, and hence A, allowed the elements of each row of A to be calculated very quickly, and, for each iteration of the algorithm, the non-zero elements of each row were calculated when required, with the result that the storage requirements were reduced to a 1×7 array. The vector $\underset{\sim}{f}$, size $R \times 1$, also contained a majority of zero elements. In fact all non-zero elements must be in the last $(d_1+1)(d_2+1)$ positions and only these elements of $\underset{\sim}{f}$ were stored in the computer. The efficient storage of these two quantities has allowed systems with values of R up to 10,000 to be evaluated. All programs are written in FORTRAN for CDC 6000 or Cyber series computers, and require less than 60000 (octal) words of central memory to compile and execute.

The actual iterations are performed in a subroutine (SEID). Since three applications of the algorithm are required, one for each of the vectors $\underset{\sim}{b}_{0,0}$, $\underset{\sim}{b}_{0,1}$ and $\underset{\sim}{b}_{1,0}$, it is important that various parameters and arrays are calculated before the next use of the subroutine overwrites the arrays. The vector $\underset{\sim}{f}_{0,0}$ must first be calculated and this allows $\underset{\sim}{b}_{0,0}$ to be obtained. Both $\underset{\sim}{f}_{1,0}$ and $\underset{\sim}{f}_{0,1}$ are functions of $\underset{\sim}{b}_{0,0}$ and must be calculated before the next vector, $\underset{\sim}{b}_{0,1}$ is obtained. Similarly m_1 and m_2 must be calculated before $\underset{\sim}{b}_{0,0}$ is destroyed.

The vector $\underset{\sim}{b}_{0,1}$ can be obtained using $\underset{\sim}{f}_{0,1}$ and this is used to calculate v_1 and the first term of cov. Finally, $\underset{\sim}{b}_{1,0}$ is used to calculate v_2 and the second term of cov.

A 'flowchart' showing the requirements and order of calculations is given in Figure 3.2.





means x is used to calculate y

means x must be calculated before y

Figure 3.2 : Requirements and order of calculations for vectors and statistics in the program

CHAPTER 4

ANALYTIC SOLUTION OF A SPECIAL NETWORK

4.1 Description of the network

In a telephone network there may be some O-D pairs which do not warrant direct links, because of the low level of traffic between the two exchanges. When two of these O-D pairs share a common link on the first choice route, a simplified version of the model is obtained (Figure 4.1). The traffic for the two streams arrives with independent Poisson streams and the other assumptions of section 2.1 apply. This network is equivalent to the system modelled in Chapter 2, when $d_1 = 0$ and $d_2 = 0$, and the computer program which solves the model using the S.O.R. technique can still be applied.

Results from the computer program, in which d_1 and d_2 were set to zero, were tabulated and graphed. It was noted that the overflow means were proportional to the arrival rates for the particular streams and in fact,

$$m_i = a_i \, E_c(a) \tag{4.1}$$

This formula is quite well known and is a consequence of the assumption of Poisson input. (The sum of two independent Poisson streams is itself Poisson with mean equal to the sum of the means of the two component streams.) The total overflow mean and variance are

$$m = a \, E_c(a) \tag{4.2}$$

and

$$v = a \, E_c(a)(1-aE_c(a) + \frac{a}{c+1-a+aE_c(a)}). \tag{4.3}$$

These two equations are particular cases of equations (1.2) and (1.3).

Comparison of (4.1) and (4.2) indicates a similar form for the two means with one of the 'a' terms of m replaced by an 'a_1' in the expression for m_i. That is, if the function (x,y) is defined by

$$m(x,y) = x \, E_c(y) \tag{4.4}$$

then

$$m = m(a,a) \text{ and } m_i = m(a_i,a).$$

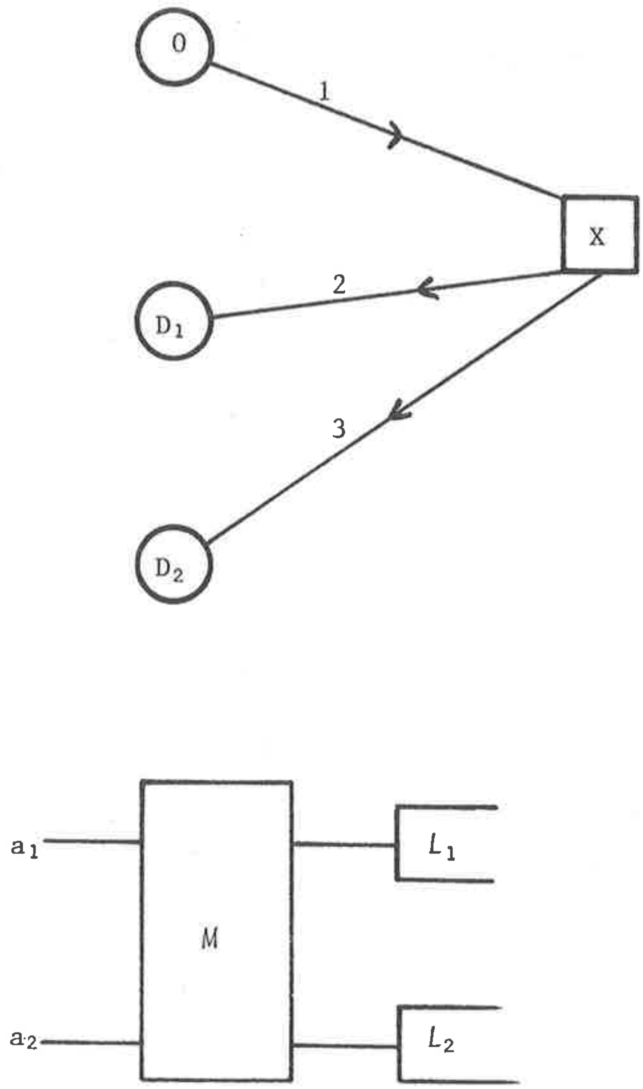


Figure 4.1 : No direct links in network

(a) Network representation;
(b) Server system representation.

It seemed possible that there may be a similar relationship holding between v_i and v. Several different expressions were evaluated, in which one or more 'a' terms in (4.3) were replaced by 'a_i' terms and these expressions were compared with the value of v_i calculated by the computer program, for some different values of a_1, a_2 and c. In all cases there was 'exact' (to the level of accuracy of the iterative solution) agreement between the results from the program and the formula

$$v_i = m_i(1 - m_i + \frac{m_i}{c+1-a+m}). \qquad (4.5)$$

This corresponds to $v = v(a,a)$ and $v_i = v(a_i,a)$ for a function defined by

$$v(x,y) = x\, E_c(y)(1 - x\, E_c(y) + \frac{x}{c+1-y+yE_c(y)}). \qquad (4.6)$$

Use of the identity,

$$\mathrm{Var}(X,Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{cov}(X,Y)$$

enable cov to be expressed explicitly by

$$\mathrm{cov} = m_2(-m_1 + \frac{a_1}{c+1-a+m}) \qquad (4.7)$$

or

$$\mathrm{cov} = m_1(-m_2 + \frac{a_2}{c+1-a+m}). \qquad (4.8)$$

These two formulas are equivalent since

$$m_1 a_2 = a_1 \cdot E_c(a) \cdot a_2 = a_1 m_2$$

(This should be expected since $\mathrm{cov}(X,Y) = \mathrm{cov}(Y,X)$.) A symmetric expression can be obtained by taking the average of the two formulas (4.7) and (4.8), namely

$$\mathrm{cov} = \tfrac{1}{2}\{m_2(-m_1 + \frac{a_1}{c+1-a+m}) + m_1(-m_2 + \frac{a_2}{c+1-a+m}) \qquad (4.9)$$

This intuitive result for the variance of the individual overflow streams, was derived analytically, following the method of Riordan [27]. The formulas were first published in the Second Progress Report to the Australian Telecommunications Commission [19] and then in a paper presented at the 8th International Teletraffic Congress [28]. They were later obtained by Pearce [17], who used a completely different technique.

The results can also be obtained using a second 'binomial' transformation similar to the transformation of (2.5) to (2.9) and in fact can be obtained directly from the moment equations. One or more of these techniques may be extended to give a solution to the general model and consideration of three different approaches gives some insight into which of these can most easily be applied to the more difficult problem.

4.2 The Analytic solution of the simple model

Although the system, illustrated in Figure 4.1, is equivalent to the general model of Chapter 2 with $d_1 = d_2 = 0$, the state equations and moment equations will again be presented.

The state of the system is described by a three parameter vector (M,L_1,L_2) corresponding to the number of busy servers in the three service groups, M, L_1 and L_2. The state probability function f is defined by

$$\begin{aligned} f(m,\ell_1,\ell_2) &= \Pr\{M=m,L_1=\ell_1,L_2=\ell_2\} \\ &\qquad \text{for } 0 \le m \le c,\ \ell_i \ge 0 \\ &= 0 \text{ otherwise.} \end{aligned} \tag{4.10}$$

The state equations for this system can be derived analogously to (2.5), and again are presented in abbreviated form.

For $m < c$

$$\begin{aligned} (a+m+\ell)f &= a\ f_1(m-1) + (m+1)\ f_1(m+1) \\ &\quad + (\ell_1+1)f_2(\ell_1+1) + (\ell_2+1)f_3(\ell_2+1) \end{aligned} \tag{4.11a}$$

and

$$\begin{aligned} (c+\ell)f &= a\ f_1(c-1) \\ &\quad + a_1 f_2(\ell_1-1) + a_2 f_3(\ell_2-1) \\ &\quad + (\ell_1+1)f_2(\ell_1+1) + (\ell_2+1)f_3(\ell_2+1). \end{aligned} \tag{4.11b}$$

The binomial moments are defined by

$$B_{\ell_1,\ell_2}(m) = \sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty} \binom{k_1}{\ell_1}\binom{k_2}{\ell_2} f(m,k_1,k_2) \tag{4.12}$$

An equivalent system involving the binomial moments can be derived by a procedure similar to the derivation of (2.5).

For $m < c$

$$\begin{aligned} (a+m+\ell)B_{\ell_1,\ell_2}(m) &= a\ B_{\ell_1,\ell_2}(m-1) \\ &\quad + (m+1)B_{\ell_1,\ell_2}(m+1), \end{aligned} \tag{4.13a}$$

and

$$\begin{aligned} (c+\ell)B_{\ell_1,\ell_2}(c) &= a\ B_{\ell_1,\ell_2}(c-1) \\ &\quad + a_1 B_{\ell_1-1,\ell_2}(c) + a_2 B_{\ell_2,\ell_1-1}(c). \end{aligned} \tag{4.13b}$$

The quantity $B_{(\ell_1,\ell_2)}$ is defined by

$$B_{(\ell_1,\ell_2)} = \sum_{m=0}^{c} B_{\ell_1,\ell_2}(m)$$

$$= E[\binom{L_1}{\ell_1}\binom{L_2}{\ell_2}] \tag{4.14}$$

and summation of (4.13) for $m=0,1,..,c$ yields

$$(\ell_1+\ell_2)B_{(\ell_1,\ell_2)} = a_1 B_{\ell_1-1,\ell_2}(c) + a_2 B_{\ell_1,\ell_2-1}(c). \tag{4.15}$$

The expressions (2.12) relating the overflow statistics to $B_{(\ell_1,\ell_2)}$ are valid for this model.

These moment equations (4.13) can now be solved by several different analytic approaches. The derivation presented in the next section is an extension of the classical approach of Riordan [27].

4.3 Solution of the model by Riordan's Method

If the constrain on m in (4.13a) is relaxed to allow m to have any non negative integer value, then new equations and new variables are introduced into the system. For each new value of $m \geq c$ a new variable $B_{\ell_1,\ell_2}(m+1)$ is introduced and defined in terms of $B_{\ell_1,\ell_2}(m)$ and $B_{\ell_1,\ell_2}(m-1)$. The introduction of these artifical variables does not affect the relationships between the physical moments corresponding to $B_{\ell_1,\ell_2}(m)$ for $m \leq c$. If the extended solution can be solved and that solution also satisfies the boundary condition (4.13b) or the normalising constraint in the case $\ell_1 = \ell_2 = 0$, then it is also a solution to the original, restricted, system of equations.

For fixed values of ℓ_1 and ℓ_2 the extended equation (4.13a) will be transformed using the generating function

$$\beta(t) = \sum_{m=0}^{\infty} B_{\ell_1,\ell_2}(m)\ t^m \qquad 0 < t < 1 \tag{4.16}$$

The derivative of β with respect to t is

$$\frac{d\beta}{dt} = \sum_{m=0}^{\infty} m\ B_{\ell_1,\ell_2}(m)\ t^{m-1} \tag{4.17}$$

Multiplication of (4.13a) by t^m and summation yields,

$$(a+\ell) \sum_{m=0}^{\infty} B_{\ell_1,\ell_2}(m) t^m + t \sum_{m=0}^{\infty} m\ B_{\ell_1,\ell_2}(m)\ t^{m-1}$$

$$= a\ t \sum_{m=0}^{\infty} B_{\ell_1,\ell_2}(m-1)\ t^{m-1}$$

$$+ \sum_{m=0}^{\infty} (m+1)\ B_{\ell_1,\ell_2}(m+1)\ t^m \tag{4.18}$$

and this simplifies to the differential equation,

$$(a+\ell)\cdot\beta + t\frac{d\beta}{dt} = at\cdot\beta + \frac{d\beta}{dt}\ . \tag{4.19}$$

Rearrangement of (4.19) gives

$$\frac{d\beta}{dt} = (a + \frac{\ell}{1-t})\beta \tag{4.20}$$

which has the solution

$$\beta(t) = \beta(0)\ \frac{e^{at}}{(1-t)^{\ell}}\ . \tag{4.21}$$

A second generating function $\sigma(t)$ is considered.

$$\sigma(t) = \sum_{m=0}^{\infty} \sigma_{\ell}(m) t^{m} = \frac{e^{at}}{(1-t)^{\ell}} \tag{4.22}$$

The σ-polynomials $\sigma_{\ell}(m)$, which are attributed to Nyquist in [27], satisfy a number of recurrence relations, including

$$\sigma_{\ell}(m) = \sigma_{\ell+1}(m) - \sigma_{\ell+1}(m-1), \tag{4.23a}$$

$$m\, \sigma_{\ell}(m) = a\, \sigma_{\ell}(m-1) + \ell\, \sigma_{\ell+1}(m-1), \tag{4.23b}$$

$$\ell\, \sigma_{\ell+1}(m) = (m+\ell-a)\, \sigma_{\ell}(m) + a\, \sigma_{\ell-1}(m), \tag{4.23c}$$

and

$$\sigma_{\ell+1}(m) = \sum_{k=0}^{m} \sigma_{\ell}(k). \tag{4.23d}$$

Substitution of (4.22) in (4.21) yields

$$\beta(t) = \beta(0) \sum_{m=0}^{\infty} \sigma_{\ell}(m)\, t^{m} \tag{4.24}$$

and equating powers of t in this equation on (4.16) yields

$$B_{\ell_1,\ell_2}(m) = \beta(0)\, \sigma_{\ell}(m). \tag{4.25}$$

Substitution of t=0 in (4.16) with the convention $0^{m} = \sigma^{0}_{m}$ implies

$$\beta(0) = B_{\ell_1,\ell_2}(0). \tag{4.26}$$

When $\ell_1 = \ell_2 = 0$ the solution (4.25) for $B_{\ell_1,\ell_2}(m)$ must satisfy the normalising constraint

$$\sum_{m=0}^{c} B_{\ell_1,\ell_2}(m) = 1$$

Therefore

$$\beta(0) \sum_{m=0}^{c} \sigma_0(m) = 1$$

and by (4.23d)

$$\beta(0) = \frac{1}{\sigma_1(c)} \tag{4.27}$$

Therefore

$$B_{0,0}(c) = \frac{\sigma_0(c)}{\sigma_1(c)} . \tag{4.28}$$

When $\ell = 0$, $\sigma(t) = e^{at}$ and hence $\sigma_0(m) = \frac{a^m}{m!}$ hence (4.29)

$$B_{0,0}(c) = \frac{\frac{a^c}{c!}}{\sum_{m=0}^{c} \frac{a^m}{m!}}$$

$$= E_c(a) \tag{4.30}$$

For $\ell > 0$ the solution (4.25) must also satisfy the boundary condition (4.13b), that is

$$(c+\ell)\ B_{\ell_1,\ell_2}(0)\ \sigma_\ell(c) = a\ B_{\ell_1,\ell_2}(0)\ \sigma_\ell(c-1)$$

$$+ a_1\ B_{\ell_1-1,\ell_2}(0)\ \sigma_{\ell-1}(c)$$

$$+ a_2\ B_{\ell_1,\ell_2-1}(0)\ \sigma_{\ell-1}(c)$$

or

$$[(c+\ell)\ \sigma_\ell(c) - a\ \sigma_\ell(c-1)]\ B_{\ell_1,\ell_2}(0)$$

$$= [a_1\ B_{\ell_1-1,\ell_2}(0) + a_2\ B_{\ell_1,\ell_2-1}(0)]\ \sigma_{\ell-1}(c) \tag{4.31}$$

But

$$(c+\ell)\ \sigma_\ell(c) - a\ \sigma_\ell(c-1)$$

$$= (c+\ell-a)\ \sigma_\ell(c) + a(\sigma_\ell(c) - \sigma_\ell(c-1))$$

$$= \ell\ \sigma_{\ell+1}(c) - a\ \sigma_{\ell-1}(m) + a\ \sigma_{\ell-1}(m)$$

$$= \ell\ \sigma_{\ell+1}(c).$$

This result is obtained using (4.23c and a) and its substitution into (4.31), gives a recursive formula for $B_{\ell_1,\ell_2}(0)$.

In detail

$$B_{1,0}(0) = \frac{a_1 B_{0,0}(0)\,\sigma_0(c)}{\sigma_2(c)} \qquad \text{from (4.32)}$$

$$\therefore \quad B_{1,0}(c) = B_{1,0}(0)\,\sigma_1(c)$$

$$= \frac{a_1 B_{0,0}(0)\,\sigma_0(c)\,\sigma_1(c)}{\sigma_2(c)}$$

$$= \frac{a_1 B_{0,0}(c)\,\sigma_1(c)}{\sigma_2(c)}$$

$$= \frac{a_1 E_c(a)\,\sigma_1(c)}{\sigma_2(c)}$$

$$= \frac{m_1}{\omega} ,$$

where $\qquad \omega = \sigma_2(c)/\sigma_1(c)$.

$$B_{\ell_1,\ell_2}(0) = \frac{1}{\ell_1+\ell_2}[a_1\, B_{\ell_1-1,\ell_2}(0) + a_2\, B_{\ell_1,\ell_2-1}(0)]\frac{\sigma_{\ell-1}(c)}{\sigma_{\ell+1}(c)} \tag{4.32}$$

This recursive formula together with (4.30) and (4.25) allows $B_{\ell_1,\ell_2}(m)$ to be calculated for any values of ℓ_1, ℓ_2 and $m \le c$, in particular, $B_{1,0}(c)$ and $B_{0,1}(c)$ which are needed to calculate v_1, v_2 and cov. [The higher moments of the overflow traffic can also be calculated using (4.15) by the recursive calculation of the appropriate binomial moments at m=c.]

Hence

$$B_{1,0}(0) = a_1\, B_{0,0}(0)\cdot \frac{\sigma_0(c)}{\sigma_1(c)}$$

and

$$B_{1,0}(c) = a_1\, E_c(a)\, \frac{\sigma_0(c)}{\sigma_{\mathbf{a}}(c)}$$

$$= \frac{m_1}{w} \qquad \text{see opposite} \tag{4.33}$$

$$\sigma_2(c) = \sum_{m=0}^{c} \sigma_1(m)$$

$$= \sum_{m=0}^{c} \sum_{r=0}^{m} \frac{a^r}{r!}$$

$$= \sum_{m=0}^{c} (c+1-m)\frac{a^m}{m!}$$

Therefore,

$$w = \frac{\sigma_2(c)}{\sigma_1(c)}$$

$$= \frac{(c+1)\sum_{m=0}^{c}\frac{a^m}{m!} - a\sum_{m=0}^{c}\frac{a^{m-1}}{(m-1)!}}{\sum_{m=0}^{c}\frac{a^m}{m!}}$$

$$= (c+1) - \frac{a\left(\sum_{m=0}^{c}\frac{a^m}{m!} - \frac{a^c}{c!}\right)}{\sum_{m=0}^{c}\frac{a^m}{m!}}$$

$$= c + 1 - a + aE_c(a)$$

$$= c + 1 - a + m.$$

Hence

$$B_{1,0}(c) = \frac{m_1}{c+1-a+m} \tag{4.34a}$$

and similarly

$$B_{0,1}(c) = \frac{m_2}{c+1-a+m}. \tag{4.34b}$$

Since equations (2.12) are valid for this model, use of (4.15), (4.30) and (4.34) allows the overflow statistics to be expressed explicitly, and the results confirm the intuitive formula.

$$m_1 = a_1 \; E_c(a)$$

$$m_2 = a_2 \; E_c(c)$$

$$v_1 = \frac{a_1 m_1}{w} + m_1 - m_1^2$$

$$= m_1(1 - m_1 + \frac{a_1}{c+1-a+m})$$

$$v_2 = m_2(1 - m_2 + \frac{a_2}{c+1-a+m})$$

$$\text{cov} = \frac{a_1 m_2}{2.w} + \frac{a_2 m_1}{2.w} - m_1 \; m_2$$

$$= m_1(-m_2 + \frac{a_2}{c+1-a+m}). \tag{4.35}$$

4.4 Solution without the use of generating functions

Equation (4.13a) is similar to (2.5) in that it is quadratic recursive in m. A second binomial transformation similar to the one applied to (4.11) can be used to simplify the model. The equations will be expressed in a new set of variables ρ_{m,ℓ_1,ℓ_2}. The subscript notation indicates that a binomial transformation has been applied to the subscripted parameters. (This is consistent with the notation $B_{\ell_1,\ell_2}(m)$.)

It is logical to relax the constraint on m for (4.13a) and allow m to take any non negative integer value. This follows from a similar relaxation in the generating function approach and suggests that ρ_{m,ℓ_1,ℓ_2} be defined by

$$\rho_{m,\ell_1,\ell_2} = \sum_{k=m}^{\infty} \binom{k}{m} B_{\ell_1,\ell_2}(k), \tag{4.36}$$

which is analogous to the definition of $B_{\ell_1,\ell_2}(m)$. The binomial transformation is achieved by replacement of the dummy variables m by k in (4.13a), multiplication of this equation by $\binom{k}{m}$ and summation over the range, $k=m$, $m+1$, ... This leads formally to the equation

$$\sum_{k=m}^{\infty} (a+k+\ell)\binom{k}{m} B_{\ell_1,\ell_2}(k) = \sum_{k=m}^{\infty} a\binom{k}{m} B_{\ell_1,\ell_2}(k-1) + \sum_{k=m}^{\infty} (k+1)\binom{k}{m} B_{\ell_1,\ell_2}(k+1), \tag{4.37}$$

which simplifies, using lemmas 1 and 2, to

$$(a+m+\ell)\rho_{m,\ell_1,\ell_2} = a\,(\rho_{m,\ell_1,\ell_2} + \rho_{m-1,\ell_1,\ell_2})$$

or

$$\rho_{m,\ell_1,\ell_2} = \frac{a}{m+\ell}\rho_{m-1,\ell_1,\ell_2}. \tag{4.38}$$

Equation (4.38) is simply recursive in m and in fact

$$\rho_{m,\ell_1,\ell_2} = \frac{a^m}{(m+\ell)_m}\rho_{0,\ell_1,\ell_2} \tag{4.39}$$

where

$$(x)_m = x(x-1)(x-2)\ldots(x+1-m) = \frac{x!}{m!}\,. \tag{4.40}$$

However, the lemmas only apply to 'sufficiently well-behaved functions' and when $m=0$, (4.37) becomes

$$\sum_{k=0}^{\infty}(a+k+\ell)\, B_{\ell_1,\ell_2}(k) = \sum_{k=0}^{\infty} a\, B_{\ell_1,\ell_2}(k-1) + \sum_{k=0}^{\infty}(k+1)\, B_{\ell_1,\ell_2}(k+1)$$

which simplifies to

$$\ell \sum_{k=0}^{\infty} B_{\ell_1,\ell_2}(k) = 0 \tag{4.41}$$

which together with (4.39) implies either $B_{\ell_1,\ell_2}(k) = 0$ for $k=0,1,\ldots$ or $\ell=0$.

In fact, when $(\ell_1,\ell_2) \neq (0,0)$ the series defining ρ_{m,ℓ_1,ℓ_2} is divergent and the artifical variables $B_{\ell_1,\ell_2}(m)$; $m>c$ are not sufficiently well-behaved. (The generating function method does not have this obstacle since the t^m factor, for $0 < t < 1$, is small enough to give a convergent series for $\beta(t)$.) When this transformation is applied the values of $B_{\ell_1,\ell_2}(c)$ obtained are

$$B_{0,0}(c) = E_c(a)$$

$$B_{1,0}(c) = \frac{m_1}{c+1-a+a\cdot z}$$

$$B_{0,1}(c) = \frac{m_2}{c+1-a+a\cdot z} \tag{4.42}$$

where

$$z = \frac{a^c/c!}{e^a - \sum_{m=0}^{c} \frac{a^m}{m!}}$$

These results are obtained using (4.40), together with the normalising constraint for $\ell=0$, and the boundary equation (4.13b) for $\ell=1$. As expected from the previous discussion,

$B_{0,0}(c)$ agrees with the correct results obtained by the generating function method, but $B_{1,0}(c)$ and $B_{0,1}(c)$ are incorrect. (z should be $E_c(a)$.)

The failure of this method is due to the implicit introduction of the artificial variables corresponding to $m>c$. This extension is not only invalid but is also unnecessary.

If the function ρ_{m,ℓ_1,ℓ_2} is defined as a finite sum then the introduction of artificial variables is avoided. That is

$$\rho_{m,\ell_1,\ell_2} = \sum_{k=m}^{c} \binom{k}{m} B_{\ell_1,\ell_2}(k). \tag{4.43}$$

Substitution in (4.43) for $B_{\ell_1,\ell_2}(k)$ gives

$$\rho_{m,\ell_1,\ell_2} = \sum_{k=m}^{c} \sum_{k_1=\ell_1}^{\infty} \sum_{k_3=\ell_3}^{\infty} \binom{k}{m}\binom{k_1}{\ell_1}\binom{k_2}{\ell_2} f(k,k_1,k_2) \tag{4.44}$$

and so ρ_{m,ℓ_1,ℓ_2} is in fact a trivariate binomial moment of m, ℓ_1 and ℓ_2.

It is convenient to use lemmas similar to Lemmas 1 and 2, but these must be adapted for finite summation. If $h(k)$ is a function defined for $k=0,1,..,c$, and $H(m)$ is defined by $H(m) = \sum_{k=m}^{c} \binom{k}{m} h(k)$ then the following results hold;

Lemma 1*

$$\sum_{k=m}^{c} k\binom{k}{m} h(k) - \sum_{k=m}^{c-1} (k+1)\binom{k}{m} h(k+1) = m\, H(m). \tag{4.45}$$

Proof

Use of the identity

$$\binom{k}{m} = \binom{k+1}{m} - \frac{m}{k+1}\binom{k+1}{m}$$

leads to

$$\text{L.H.S.} = m\, h(m)\binom{m}{m} + \sum_{k=m+1}^{c} k\binom{k}{m} h(k) - \sum_{k+1=m+1}^{c} (k+1)\binom{k+1}{m} h(k+1)$$

$$+ \sum_{k+1=m+1}^{c} m\binom{k+1}{m} h(k+1) = m \sum_{k=m}^{c} \binom{k}{m} h(k) = \text{R.H.S.}$$

Lemma 2*

$$\sum_{k=m}^{c} \binom{k}{m} h(k-1) = \left(H(m) + H(m-1) - \binom{c+1}{m} H(c)\right) \qquad (4.46)$$

Proof

Use of the identity

$$\binom{k}{m} = \binom{k-1}{m} + \binom{k-1}{m-1} \qquad (4.47)$$

leads to

$$\text{L.H.S.} = \sum_{k-1=m-1}^{c-1} \binom{k-1}{m} h(k-1) + \sum_{k-1=m-1}^{c-1} \binom{k-1}{m-1} h(k-1)$$

$$= H(m) - \binom{c}{m} h(c) + H(m-1) - \binom{c}{m-1} h(c)$$

exploiting the definition $\binom{m-1}{m} = 0$.

The result follows, since

$$H(c) = \sum_{m=c}^{c} \binom{c}{m} h(c) = h(c)$$

and

$$\binom{c}{m} + \binom{c}{m-1} = \binom{c+1}{m} \text{ by (4.47).}$$

The equations (4.13) can be rewritten in terms of the dummy variable k, with the subscripts ℓ_1 and ℓ_2 omitted, to become

$$(a+k+\ell)\, B(k) = a\, B(k-1) + (k+1)\, B(k), \qquad k>0$$

$$(a+c+\ell)\, B(c) = a\, B(c-1) + a\, B(c) + \mu \qquad (4.48)$$

where

$$\mu = \begin{cases} 0 & (\ell_1,\ell_2) = (0,0) \\ a_1\, B_{0,0}(c) & (\ell_1,\ell_2) = (1,0) \\ a_2\, B_{0,0}(c) & (\ell_1,\ell_2) = (0,1). \end{cases}$$

Multiplication of the equations by $\binom{k}{m}$ and $\binom{c}{m}$, respectively, and summation give

$$\sum_{k=m}^{c} (a+k+\ell)\binom{k}{m} B(k) = \sum_{k=m}^{c} a\binom{k}{m} B(k-1) + \sum_{k=m}^{c-1} (k+1)\binom{k}{m} B(k) +$$

$$+ \binom{c}{m} (aB(c)+\mu). \tag{4.49}$$

Application of Lemmas 1* and 2*, yields

$$(a+m+\ell)\ \rho_m = a(\rho_m + \rho_{m-1} - \binom{c+1}{m}\rho_c) + \binom{c}{m} (a\rho_c + \mu),$$

or

$$\rho_m = \frac{a}{m+\ell} (\rho_{m-1} - \binom{c}{m-1} \rho_c) + \frac{1}{m+\ell} \binom{c}{m} \mu. \tag{4.50}$$

When $\ell_1 = 0$, $\ell_2 = 0$

$$\rho_m = \frac{a}{m} (\rho_{m-1} - \binom{c}{m-1}\rho_c) \tag{4.51}$$

and $\rho_0 = \sum_{k=0}^{c} B(k) = 1$ by the normalising constraint.

Hence,

$$\rho_1 = a(1 - \binom{c}{0}\rho_c).$$

and

$$\rho_k = \frac{a^k}{k!} - \rho_c \sum_{m=1}^{k} \frac{a^m}{(k)_m} \binom{c}{k-m} \qquad k=1,2,\ldots,c. \tag{4.52}$$

Equation (4.52) can easily be obtained by induction using (4.51). Clearly, it holds for $k=1$ as basis.

In particular,

$$\rho_c = \frac{a^c}{c!} - \rho_c \sum_{m=1}^{c} \frac{a^m}{(c)_m} \binom{c}{c-m} \tag{4.53}$$

so that

$$\rho_c = \frac{\frac{a^c}{c!}}{\sum_{m=0}^{c} \frac{a^m}{m!}} = E_c(a).$$

Since $B(c) = \rho_c$

$$m_1 = a_1 B_{0,0}(c) = a_1 E_c(a)$$

and

$$m_2 = a_2 E_c(a) \tag{4.54}$$

When $(\ell_1,\ell_2) = (1,0)$ or $(0,1)$ (4.50) becomes

$$\rho_m = \frac{a}{m+1}\left(\rho_{m-1} - \binom{c}{m-1}\rho_c\right) + \frac{1}{m+1}\binom{c}{m}\mu \tag{4.55}$$

where

$$\mu = \begin{cases} m_1 & (\ell_1,\ell_2) = (1,0) \\ m_2 & (\ell_1,\ell_2) = (0,1) \end{cases}$$

$$\rho_{0,\ell_1,\ell_2} = \sum_{k=0}^{c} B_{\ell_1,\ell_2}(k)$$

$$= B_{(\ell_1,\ell_2)}$$

and $B_{(1,0)} = m_1$ and $B_{(0,1)} = m_2$.

Therefore $\rho_0 = \mu$ for both values of (ℓ_1,ℓ_2) and this leads to

$$\rho_k = \mu \sum_{m=0}^{k} \frac{a^m}{(k+1)_{m+1}}\binom{c}{k-m} - \rho_c \sum_{m=1}^{k} \frac{a^m}{(k+1)_m}\binom{c}{k-m}$$

$$k=1,2,..,c. \tag{4.56}$$

which again can be proved by induction (using (4.55)).

In particular,

$$\rho_c = \mu \sum_{m=0}^{c} \frac{a^m}{(c+1)_{m+1}}\binom{c}{c-m} - \rho_c \sum_{m=1}^{c} \frac{a^m}{(c+1)_m}\binom{c}{c-m} \tag{4.57}$$

which becomes

$$\rho_c \sum_{m=0}^{c} \frac{a^m}{m!}\frac{(c)_m}{(c+1)_m} = \mu \sum_{m=0}^{c} \frac{a^m}{m!}\frac{(c)_m}{(c+1)_{m+1}} .$$

Therefore, division by $\sum_{m=0}^{c} \frac{a^m}{m!} \cdot \frac{1}{c+1}$, yields

$$\rho_c(c+1) - \frac{\sum_{m=0}^{c} m \frac{a^m}{m!}}{\sum_{m=0}^{c} \frac{a^m}{m!}} = \mu$$

$$\sum_{m=0}^{c} m \frac{a^m}{m!} = a \sum_{m=0}^{c} \frac{a^{m-1}}{(m-1)!} = a\left(\sum_{m=0}^{c} \frac{a^m}{m!} - \frac{a^c}{c!}\right)$$

and therefore

$$\rho_c = \frac{\mu}{c+1-a+aE_c(a)} \; .$$

That is

$$B_{1,0}(c) = \frac{m_1}{c+1-a+m} \tag{4.58}$$

$$B_{0,1}(c) = \frac{m_2}{c+1-a+m} \tag{4.59}$$

which give v_1, v_2 and cov as in the Riordan method.

4.5 A direct solution of the moment equations

The overflow means and variances can be obtained directly from the moment equations (4.13), without the application of any further transformation.

When $\ell_1 = \ell_2 = 0$, (4.13b) can be rewritten as

$$aB(c-1) = cB(c)$$

or

$$B(c-1) = \frac{c!}{a^c}\,\frac{a^{c-1}}{(c-1)!}\,B(c)$$

and B(c-2), B(c-3), ... can be obtained recursively using (4.13a), to give

$$B(m) = \frac{c!}{a^c}\,\frac{a^m}{m!}\,B(c). \tag{4.60}$$

This, again, can be proved by induction. Since

$$B(c) = \frac{c!}{a^c}\,\frac{a^c}{c!}\,B(c),$$

(4.60) holds for c and c-1.

Assuming the result for $m \geq k$, for $m=k$ (4.13a) can be written,

$$\begin{aligned} B(k-1) &= B(k) - \frac{(k+1)}{a}\,B(k+1) + \frac{k}{a}\,B(k) \\ &= \frac{c!}{a^c}\,B(c)\left(\frac{a^k}{k!} - \frac{k+1}{a}\,\frac{a^{k+1}}{(k+1)!} + \frac{k}{a}\,\frac{a^k}{k!}\right) \\ &= \frac{c!}{a^c}\,\frac{a^{k-1}}{(k-1)!}\,B(c) \end{aligned}$$

as required.

Since $\sum_{m=0}^{c} B_{0,0}(m) = 1$

$$B_{0,0}(c) = \frac{\dfrac{a^c}{c!}}{\displaystyle\sum_{m=0}^{c} \frac{a^m}{m!}} = E_c(a). \tag{4.61}$$

When $(\ell_1,\ell_2) = (1,0)$ or $(0,1)$ (4.13a) becomes

$$a(B(m) - B(m-1) = (m+1)(B(m+1) - B(m)) \quad m=0,1,..,(c-1).$$

If the differences Δ_m are defined by

$$\Delta_m = B(m) - B(m-1) \quad m=1,2,..,c$$

$$\Delta_0 = B(0). \tag{4.62}$$

Then

$$\Delta_{m+1} = \frac{a}{m+1} \Delta_m$$

which implies

$$\Delta_m = \frac{a^m}{m!} \Delta_0. \tag{4.63}$$

Hence, by (4.62)

$$B(c) = \sum_{m=0}^{c} \Delta_m = \Delta_0 \sum_{m=0}^{c} \frac{a^m}{m!}.$$

Therefore

$$\Delta_0 = \frac{B(c)}{\sum_{m=0}^{c} \frac{a^m}{m!}}. \tag{4.64}$$

The boundary condition (4.13b) becomes

$$(c+1)B(c) - aB(c) = aB(c-1) + \mu - aB(c)$$

$$(c+1-a)B(c) + a\Delta_c = \mu.$$

But, by (4.63) and (4.64),

$$a\Delta_c = a \frac{a^c}{c!} \Delta_0 = a\, E_c(a)\, B(c),$$

and therefore

$$B(c) = \frac{\mu}{c+1-a+m}$$

that is,

$$B_{1,0}(c) = \frac{m_1}{c+1-a+m}$$

$$B_{0,1}(c) = \frac{m_2}{c+1-a+m} \tag{4.65}$$

Equations (4.61) and (4.65) lead to the required expressions for m_i, v_i and cov.

Both this direct solution and the binomial transformation technique of section 4.4 could have been applied to Riordan's original model, which is in fact a simplification of the model discussed in this chapter (that is $a_1 = a$ and $a_2 = 0$).

4.6 Generalisation to more than two streams

When more than two independent Poisson streams are offered to a common link (figure 4.2), the formulas for the overflow statistics can be obtained by straightforward generalisations of (4.36). Each stream has an independent Poisson arrival rate, with mean a_i; $i=1,2,..,r$, and the other assumptions of Section 2.1 apply.

The streams may be partitioned into two parts, one containing a single stream and the second containing the others. If the single stream is the ith, then the other streams may be combined into a single stream which will have a Poisson arrival rate, independent of stream i, with mean

$$a_i^* = \sum_{\substack{j=1 \\ j\neq i}}^{r} a_j \tag{4.66}$$

The total arrival rate a will be

$$a = \sum_{j=1}^{r} a_j = a_i + a_i^*.$$

The link is therefore offered two independent Poisson streams and equations (4.36) are valid. Hence

$$m_i = a_i\, E_c(a) \tag{4.67a}$$

$$v_i = m_i\,(1 - m_i + \frac{a_i}{c+1-a+m}) \tag{4.67b}$$

where

$$m = \sum_{i=1}^{r} m_i = a\, E_c(a)$$

and

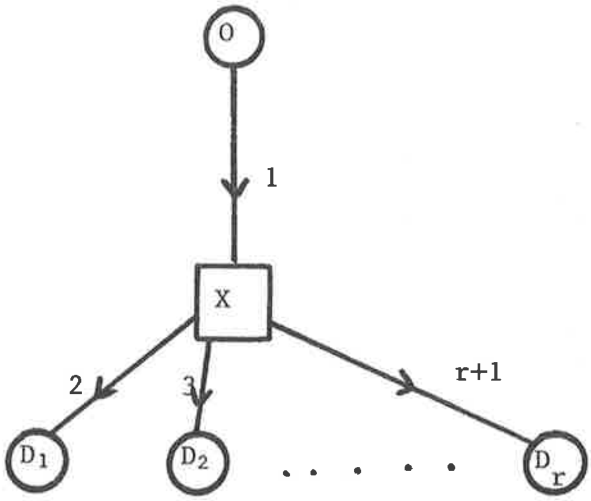
$$\mathrm{cov}_{i,i*} = m_i^*(-m_i + \frac{a_i}{c+1-a+m})$$

$$= \sum_{\substack{j=1 \\ j\neq i}}^{r} m_j\,(-m_i + \frac{a_i}{c+1-a+m}).$$

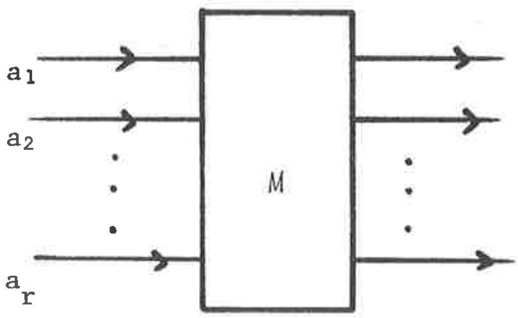
Since

$$\mathrm{cov}_{i,i^*} = \sum_{\substack{j=1 \\ j\neq i}}^{r} \mathrm{cov}_{i,j}$$

$$\mathrm{cov}_{i,j} = m_j \left(-m_i + \frac{a_i}{c+1-a+m}\right). \tag{4.67c}$$



(a)



(b)

Figure 4.2 : No direct links, r input streams

(a) Network

(b) Service system

CHAPTER 5

ANALYTIC SOLUTION OF THE GENERAL MODEL

5.1 Introduction

The first binomial transformation on the equations of state lead to a system of equations in $B_{\ell_1,\ell_2}(n_1,n_2,m)$ which was simply recursive in ℓ_1 and ℓ_2. This initial transformation is applied to the state equations in the papers by Riordan [27], Chastang [5] and Neal [16]. They affect this transformation by introducing the binomial moment generating function and then, after applying the appropriate multiplications and summations, obtain the new system of equations by equating like powers of x_1, x_2 (the carrier variables introduced in the definition of the generating function). Riordan and Neal then apply a second generating function to the 'main equation' in the new system in which the constraint on one of the parameters has been relaxed (thus introducing artificial quantities). This yields a differential equation which is solved in terms of a third generating function involving the σ-polynomials. Equating like powers of the appropriate carrier variables lead to equations relating the moments to the σ-polynomials which involve a certain number of unknown variables (introduced in a general Taylor series expansion in Neal's paper). Fortunately, the number of these unknowns is identical to the number of boundary equations and they can be found uniquely. For Riordan's model, there is only one unknown and the solution follows straightforwardly using properties of the σ-polynomials. Neal finds it necessary to introduce only three more generating functions in order to obtain his result, which effectively reduces the order of the system, by a factor of (c+1). He then solves the reduced system recursively.

Chastang, after the initial binomial transformation, simply sums the equations for which one parameter is at its upper limit, over other parameter. He deletes several terms which he claims are comparatively small and obtains an equation similar to the simple overflow case, which he had solved in the earlier part of the paper. He then postulates that a solution analogous to the solution of the simple case will be an approximate solution to the two overflow case. He admits, however, that this is not as good as the equivalent random approximation.

It was shown in chapter 4 that the simple model could be solved without the introduction of generating functions or artificial variables which are implicitly introduced by relaxing the constraint on m. A succession of these transformations can be applied to equation (2.9) to reduce the system from quadratic recursive to linear recursive in n_1, n_2 and m.

5.2 Simplification of the model by binomial transformation

The binomial transformation of (2.9) with respect to m givesa system involving the quantities

$$\theta_{m,\ell_1,\ell_2}(n_1,n_2) = \sum_{k=m}^{c} \binom{k}{m} B_{\ell_1,\ell_2}(n_1,n_2,k). \tag{5.1}$$

The new system is obtained by appropriate multiplications by $\binom{k}{m}$ or $\binom{c}{m}$ and summation, and simplified using lemmas 1* and 2*. Abbreviations similar to those applied to $B_{\ell_1,\ell_2}(n_1,n_2,m)$ will be used. Equations (2.9a) and (2.9e) become,

for $n_1 < d_1$, $n_2 < d_2$,

$$(a+n+m+\ell)\theta = a_1\,\theta(n_1-1,) + a_2\,\theta(,n_2-1)$$
$$+ (n_1+1)\theta(n_1+1,) + (n_2+1)\theta(,n_2+1). \tag{5.2a}$$

(2.9b) and (2.9f) →

for $n_1 = d_1$, $n_2 < d_2$,

$$\begin{aligned}(a+n+m+\ell)\theta = {} & a_1\,\theta(n_1-1,) + a_2\,\theta(,n_2-1) + (n_2+1)\theta(,n_2+1)\\ & + a_1\,\theta + a_1\,\theta_{m-1,,} - a_1\binom{c}{m-1}\theta_{c,,}\\ & + a_1\binom{c}{m}\,\theta_{c,\ell_1-1,\ell_2}\end{aligned} \tag{5.2b}$$

(2.9c) and (2.9g) →

for $n_1 < d_1$, $n_2 = d_2$,

$$\begin{aligned}(a+n+m+\ell)\theta = {} & a_1\,\theta(n_1-1,) + a_2\,\theta(,n_2-1) + (n_1+1)\theta(n_1+1,)\\ & + a_2\,\theta + a_2\,\theta_{m-1,,} - a_2\binom{c}{m-1}\theta_{c,,}\\ & + a_1\binom{c}{m}\,\theta_{c,\ell_1,\ell_2-1}.\end{aligned} \tag{5.2c}$$

(2.9d) and (2.9h) →

for $n_1 = d_1$, $n_2 = d_2$

$$\begin{aligned}(a+n+m+\ell)\theta = {} & a_1\,\theta(n_1-1,) + a_2\,\theta(,n_2-1)\\ & + a\,\theta + a\,\theta_{m-1} - a\binom{c}{m-1}\,\theta_c\\ & + \binom{c}{m}(a_1\,\theta_{c,\ell_1-1,\ell_2} + a_2\,\theta_{c,\ell_1,\ell_2-1})\end{aligned} \tag{5.2d}$$

A further binomial transformation, this time with respect to n_2, gives a system of equations in

$$\pi_{n_2,m,\ell_1,\ell_2}(n_1) = \sum_{j=n_2}^{d_2} \binom{j}{n_2} \theta_{m,\ell_1,\ell_2}(n_1,j) \tag{5.3}$$

(5.2a) and (5.2c) →

for $n_1 < d_1$

$$\begin{aligned}(a+n+m+\ell)\pi &= a_1\ \pi(n_1-1) + (n_1+1)\pi(n_1+1)\\ &\quad + a_2\ [\pi + \pi_{n_2-1,,,} - \binom{d_2}{n_2-1}\pi_{d_2,,,}\\ &\quad + \binom{d_2}{n_2}\{\pi_{d_2,m-1,,} - \binom{c}{m-1}\pi_{d_2,c,,} + \binom{c}{m}\pi_{d_2,c,\ell_2-1}\}]\end{aligned} \tag{5.4a}$$

(5.2b) and (5.2d) →

for $n_1 = d_1$

$$\begin{aligned}(a+n+m+\ell)\pi &= a_1\ \pi(n_1-1)\\ &\quad + a_1\ [\pi + \pi_{,m-1,,} - \binom{c}{m-1}\pi_{,c,,} + \binom{c}{m}\pi_{c,\ell_1-1,}]\\ &\quad + a_2\ [\pi + \pi_{n_2-1,,,} - \binom{d_2}{n_2-1}\pi_{d_2,m-1,,}\\ &\quad + \binom{d_2}{n_2}\{\pi_{d_2,m-1,,} - \binom{c}{m-1}\pi_{d_2,c,,} + \binom{c}{m}\pi_{d_2,c,\ell_2-1}\}].\end{aligned} \tag{5.4b}$$

Finally, a binomial transformation with respect to n_1 gives an equation in

$$\rho_{n_1,n_2,m,\ell_1,\ell_2} = \sum_{i=n_1}^{d_1} \binom{i}{n_1} \pi_{n_2,m,\ell_1,\ell_2}(i). \tag{5.5}$$

(5.4a) and (5.4b) $\rightarrow$

$$(a+n+m+\ell)\rho = a_1[\rho + \rho_{n_1-1,,,,} - \binom{d_1}{n_1-1}\rho_{d_1,,,,}$$
$$+ \binom{d_1}{n_1}\{\rho_{d_1,m-1,,} - \binom{c}{m-1}\rho_{d_1,c,,}$$
$$+ \binom{c}{m}\rho_{d_1,c,\ell_1-1,}\}]$$
$$+ a_2[\rho + \rho_{,n_2-1,,,} - \binom{d_2}{n_2-1}\rho_{,d_2,,,}$$
$$+ \binom{d_2}{n_2}\{\rho_{,d_2,m-1,,} - \binom{c}{m-1}\rho_{,d_2,c,,}$$
$$+ \binom{c}{m}\rho_{,d_2,c,\ell_2-1}\}], \tag{5.6}$$

which may be written as

$$\rho_{n_1,n_2,m,\ell_1,\ell_2} = \frac{a_1}{n+m+\ell}[\rho_{n_1-1,n_2,m,\ell_1,\ell_2} - \binom{d_1}{n_1-1}\rho_{d_1,n_2,m,\ell_1,\ell_2}$$
$$+ \binom{d_1}{n_1}\phi_{n_2,m,\ell_1,\ell_2}]$$
$$+ \frac{a_2}{n+m+\ell}[\rho_{n_1,n_2-1,m,\ell_1,\ell_2} - \binom{d_2}{n_2-1}\rho_{n_1,d_2,m,\ell_1,\ell_2}$$
$$+ \binom{d_2}{n_2}\psi_{n_1,m,\ell_1,\ell_2}] \tag{5.7}$$

where

$$\phi_{n_2,m,\ell_1,\ell_2} = \rho_{d_1,n_2,m-1,\ell_1,\ell_2} - \binom{c}{m-1}\rho_{d_1,n_2,c,\ell_1,\ell_2} +$$
$$+ \binom{c}{m}\rho_{d_1,n_2,c,\ell_1-1,\ell_2}$$

and

$$\psi_{n_1,m,\ell_1,\ell_2} = \rho_{n_1,d_2,m-1,\ell_1,\ell_2} - \binom{c}{m-1}\rho_{n_1,d_2,c,\ell_1,\ell_2} +$$
$$+ \binom{c}{m}\rho_{n_1,d_2,c,\ell_1,\ell_2-1}.$$

This equation could, in fact, have been obtained directly from the state equations (2.9) via a single penta-variate binomial transformation, since

$$\rho_{n_1,n_2,m,\ell_1,\ell_2} = \sum_{i=n_1}^{d_1} \sum_{j=n_2}^{d_2} \sum_{k=m}^{c} \sum_{k_1=\ell_1}^{\infty} \sum_{k_2=\ell_2}^{\infty} \binom{i}{n_1}\binom{j}{n_2}\binom{k}{m}\binom{k_1}{\ell_1}\binom{k_2}{\ell_2} f(i,j,k,k_1,k_2) \tag{5.8}$$

The use of four transformations was simply for convenience.

The two equivalent systems, described by (2.9) and (5.7) respectively may be compared using the 'blocks' notation of Appendix A. For fixed ℓ_1, ℓ_2 the general element $B_{\ell_1,\ell_2}(n_1,n_2,m)$ may be considered as a unit cube inside the block which contains $(d_1+1)(d_2+1)(c+1)$ such cubes. This cube is defined, according to (2.9), in terms of the six blocks which surround it. (Figure 5.1) The general element $\rho_{n_1,n_2,m,\ell_1,\ell_2}$ can be defined in terms of the two adjacent blocks which precede it in the m level of blocks, the two blocks on the faces furtherest from the axis on the same level, the two blocks immediately below these and the two blocks above these on the top face. (Figure 5.2) By solving (5.7) recursively it is possible to obtain an expression defining, for fixed ℓ_1 and ℓ_2 any cube in terms of the cubes along the two faces corresponding to $n_1 = d_1$ and $n_2 = d_2$, in the m, $m-1$ and c levels (figure 5.3). The two shaded cubes correspond to the values of ρ required to calculate the overflow statistics, namely,

$$\rho_{d_1,0,c,\ell_1,\ell_2} = \sum_{n_2=0}^{d_2} B_{\ell_1,\ell_2}(d_1,n_2,c)$$

and

$$\rho_{0,d_2,c,\ell_1,\ell_2} = \sum_{n_1=0}^{d_1} B_{\ell_1,\ell_2}(n_1,d_2,c), \tag{5.9}$$

for $(\ell_1,\ell_2) = (0,0)$, $(0,1)$ and $(1,0)$ and so on.

Equation (5.7) also contains some terms which do not correspond to any cubes in the block. These terms will be known constants if equation (5.7) is solved for (ℓ_1,ℓ_2) such that $\ell=0,1,2,\ldots$ successively.

The equation reduces to the identity 0=0 when $n_1=n_2=m=\ell_1=\ell_2=0$. However, $\rho_{0,0,0,0,0}$ is simply the sum of all the state probabilities (by 5.8) and hence

$$\rho_{0,0,0,0,0} = 1 \tag{5.10}$$

is the starting point of the recursion.

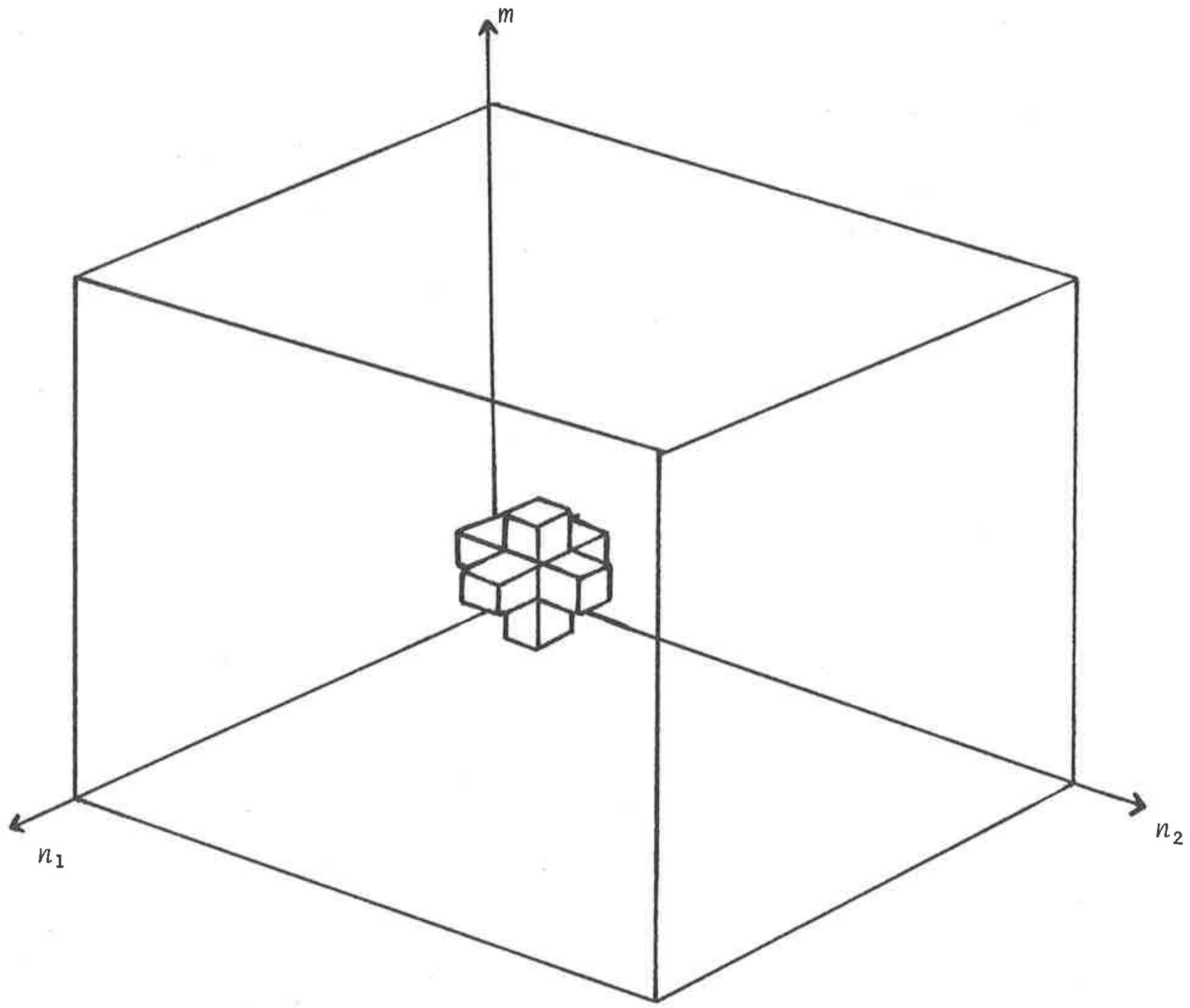


Figure 5.1 : Blocks used to calculate $B_{\ell_1,\ell_2}(n_1,n_2,m)$

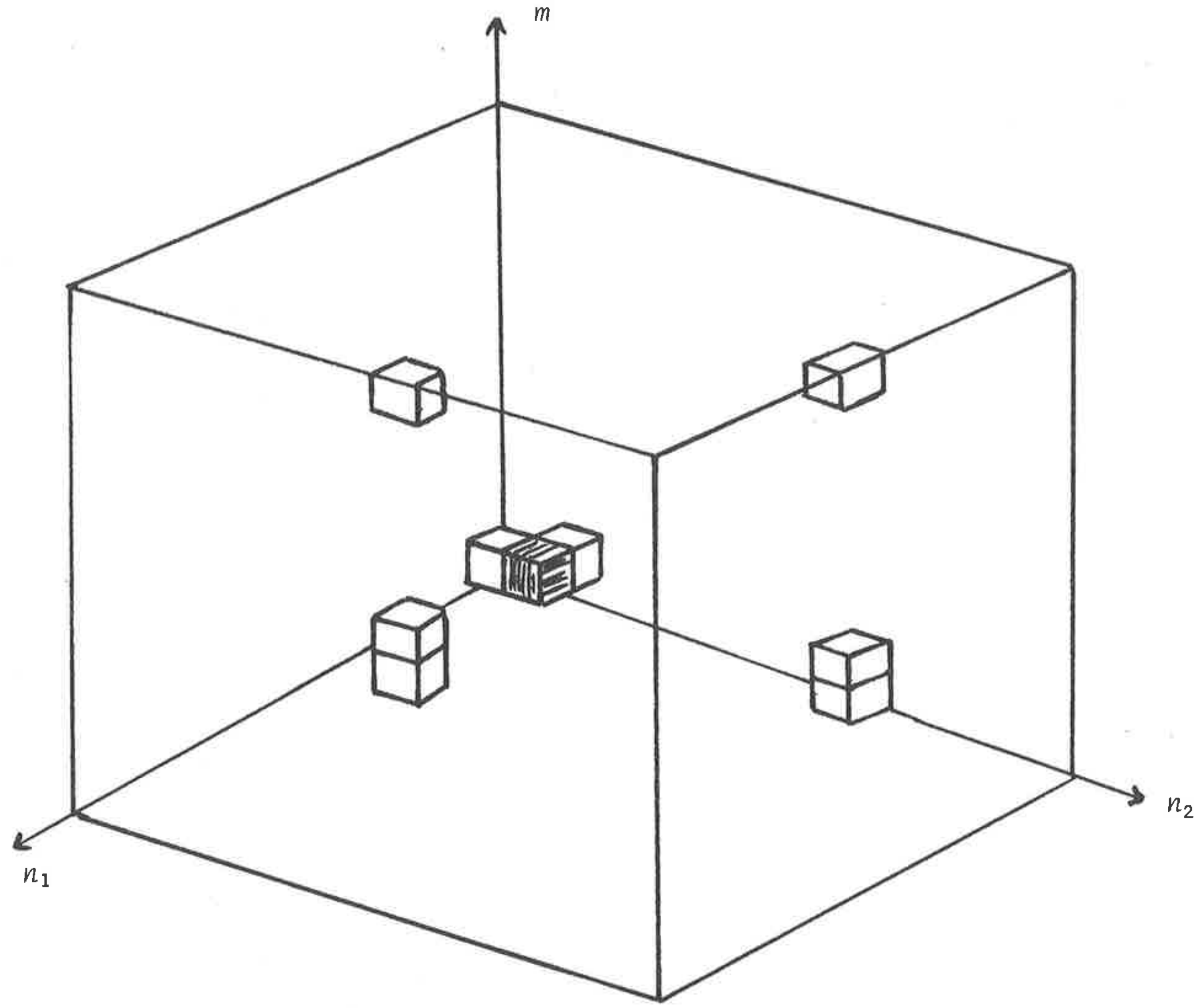


Figure 5.2 : Blocks used to calculate $\rho_{n_1,n_2,m,\ell_1,\ell_2}$

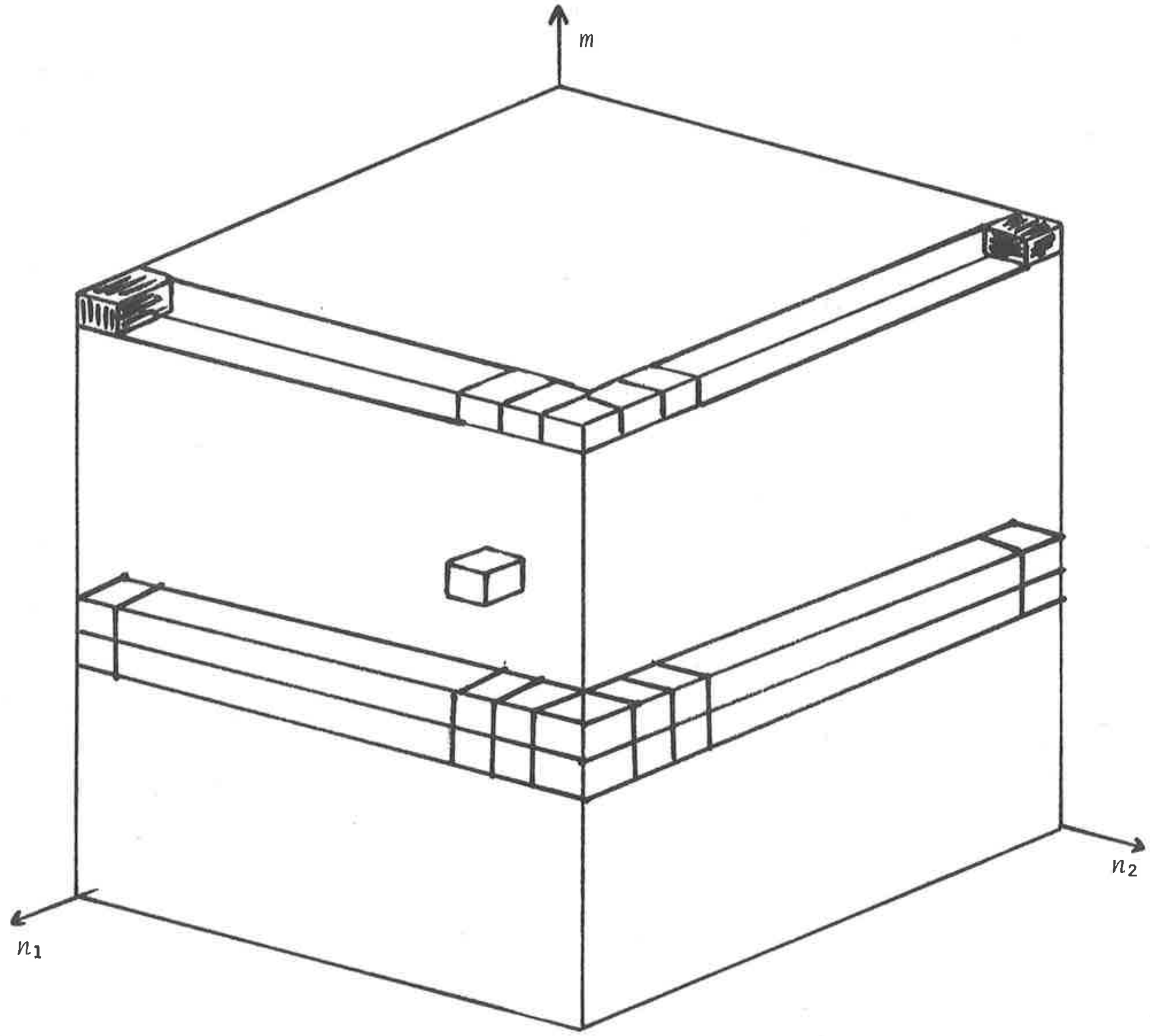


Figure 5.3 : Alternate set of blocks for calculating $\rho_{n_1,n_2,m,\ell_1,\ell_2}$.

5.3 Reduction of the order of the system of equations

The following formula for $\rho_{n_1,n_2,m}$ can be established for fixed ℓ_1 and ℓ_2 (with these subscripts omitted)

$$\rho_{n_1,n_2,m} = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \frac{a_1^{\,i+1} \, a_2^{\,n_2-j}}{(n+m+\ell)_{i+1+n_2-j}} \binom{d_1}{n_1-i} \binom{i+n_2-j}{i} \phi_{j,m}$$

$$+ \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \frac{a_1^{\,n_1-i} \, a_2^{\,j+1}}{(n+m+\ell)_{n_1-i+j+1}} \binom{d_2}{n_2-j} \binom{n_1-i+j}{j} \psi_{i,m}$$

$$- (1-\delta^0_{n_1}) \sum_{i=1}^{n_1} \sum_{j=0}^{n_2} \frac{a_1^{\,i} \, a_2^{\,n_2-j}}{(n+m+\ell)_{i+n_2-j}} \binom{d_1}{n_1-i} \binom{i+1+n_2-j}{i+1} \rho_{d_1,j,m}$$

$$- (1-\delta^0_{n_2}) \sum_{i=0}^{n_1} \sum_{j=1}^{n_2} \frac{a_1^{\,n_1-i} \, a_2^{\,j}}{(n+m+\ell)_{n_1-i+j}} \binom{d_2}{n_2-j} \binom{n_1-i+j+1}{j+1} \rho_{i,d_2,m}$$

for $m + \ell > 0$ (5.11)

and

$$\rho_{n_1,n_2,0,0,0} = \frac{a_1^{\,n_1}}{n_1!} \cdot \frac{a_2^{\,n_2}}{n_2!}$$

$$- (1-\delta^0_{n_1}) \sum_{i=1}^{n_1} \sum_{j=0}^{n_2} \frac{a_1^{\,i} \, a_2^{\,n_2-j}}{(n)_{i+n_2-j}} \binom{d_1}{n_1-i} \binom{i+1+n_2-j}{i+1} \rho_{d_1,j,0,0,0}$$

$$- (1-\delta^0_{n_2}) \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \frac{a_1^{\,n_1-i} \, a_2^{\,j}}{(n)_{n_1-i+j}} \binom{d_1}{n_2-j} \binom{n_2-i+j+1}{j+1} \rho_{i,d_2,0,0,0}$$

(5.12)

The two $(1-\delta)$ terms are included to avoid difficulties with summations from 1 to 0, since, when $n_i=0$, the corresponding sum is no longer present in the formula. The validity of this formula will be proved inductively.

For $m + \ell > 0$, $n_1 = 0$, $n_2 = 0$, (5.11) becomes

$$\rho_{0,0,m} = \frac{a_1}{m+\ell}\ \phi_{0,m} + \frac{a_2}{m+\ell}\ \psi_{0,m} \tag{5.13}$$

which agrees with (5.7) when $n_1 = n_2 = 0$, $m + \ell > 0$.

When $n_2 = 0$, $n_1 > 0$, (5.7) becomes

$$\rho_{n_1,0,m} = \frac{a_1}{n_1+m+\ell}\left[\rho_{n_1-1,0,m} - \binom{d_1}{n_1-1}\rho_{d_1,0,m} + \binom{d_1}{n_1}\phi_{0,m}\right] + \frac{a_2}{n_1+m+\ell}\psi_{n_1,m}. \tag{5.14}$$

Substitution for $\rho_{n_1-1,0,m}$, using (5.11), yields

$$\begin{aligned}
\rho_{n_1,0,m} = {} & \frac{a_1}{n_1+m+\ell}\sum_{i=0}^{n_1-1}\frac{a_1^{\,i+1}}{(n_1+m+\ell-1)_{i+1}}\binom{d_1}{n_1-1-i}\phi_{0,m} \\
& + \frac{a_1}{n_1+m+\ell}\sum_{i=0}^{n_1-1}\frac{a_1^{\,n_1-1-i}a_2}{(n+m+\ell-1)_{n_1-i}}\psi_{i,m} \\
& - (1-\delta^0_{n_1-1})\frac{a_1}{n_1+m+\ell}\sum_{i=1}^{n_1-1}\frac{a_1^{\,i}}{(n_1+m+\ell-1)_i}\binom{d_1}{n_1-1-i}\rho_{d_1,0,m} \\
& + \frac{a_1}{n_1+m+\ell}\left[-\binom{d_1}{n_1-1}\rho_{d_1,0,m} + \binom{d_1}{n_1}\phi_{0,m}\right] \\
& + \frac{a_2}{n_1+m+\ell}\ \psi_{n_1,m} \\
= {} & \sum_{i+1=1}^{n_1}\frac{a_1^{\,(i+1)+1}}{(n_1+m+\ell)_{(i+1)+1}}\binom{d_1}{n_1-(i+1)}\phi_{0,m} \\
& + \frac{a_1}{(n_1+m+\ell)_1}\binom{d_1}{n_1}\phi_{0,m} \\
& + \sum_{i=0}^{n_1-1}\frac{a_1^{\,n_1-i}a_2}{(n_1+m+\ell)_{n_1-i+1}}\psi_{i,m}
\end{aligned}$$

$$+ \frac{a_1^{n_1-n_1} \, a_2}{(n_1+m+\ell)_{n_1-n_1+1}} \psi_{n_1,m}$$

$$- (1-\delta^1_{n_1}) \sum_{i+1=2}^{n_1} \frac{a_1^{i+1}}{(n_1+m+\ell)_{i+1}} \binom{d_1}{n_1-(i+1)} \rho_{d_1,0,m}$$

$$- \binom{d_1}{n_1-1} \frac{a_1^{1}}{(n_1+m+\ell)_1} \rho_{d_1,0,m}, \tag{5.15}$$

which agrees with (5.11) evaluated at $(n_1,0,m)$. Hence, by induction, (5.11) is valid for $\{0 \le n_1 \le d_1,\ n_2=0,\ 0 \le m \le c,\ m+\ell_1+\ell_2>0\}$ and, by a similar induction, for $\{n_1=0,\ 0 \le n_2 \le d_2,\ 0 \le m \le c,\ m+\ell_1+\ell_2>0\}$.

For $n_1 > 0$, $n_2 > 0$, substitution of (5.11) in (5.7) yields,

$$\rho_{n_1,n_2,m} = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2} \frac{a_1^{i+2} \, a_2^{n_2-j}}{(n+m+\ell)_{i+2+n_2-j}} \binom{d_1}{n_1-i-1} \binom{i+n_2-j}{i} \phi_{j,m}$$

$$+ \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2} \frac{a_1^{n_1-i} \, a_2^{j+1}}{(n+m+\ell)_{n_1-i+j}} \binom{d_2}{n_2-j} \binom{n_1-i+j-1}{j} \psi_{i,m}$$

$$- (1-\delta^1_{n_1}) \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2} \frac{a_1^{i+1} \, a_2^{n_2-j}}{(n+m+\ell)_{i+n_2-j+1}} \binom{d_1}{n_1-i-1} \binom{i+n_2-j}{i+1} \rho_{d_1,j,m}$$

$$- \sum_{i=0}^{n_1-1} \sum_{j=1}^{n_2} \frac{a_1^{n_1-i} \, a_2^{j}}{(n+m+\ell)_{n_1-i+j}} \binom{d_2}{n_2-j} \binom{n_1-i+j}{j+1} \rho_{i,d_2,m}$$

$$- \frac{a_1}{n+m+\ell} \binom{d_1}{n_1-1} \rho_{d_1,n_2,m} + \frac{a_1}{n+m+\ell} \binom{d_1}{n_1} \phi_{n_2,m}$$

$$+ \sum_{i=0}^{n_1} \sum_{j=0}^{n_2-1} \frac{a_1^{i+1} \, a_2^{n_2-j}}{(n+m+\ell)_{i+1+n_2-j}} \binom{d_1}{n_1-i} \binom{i+n_2-j-1}{i} \phi_{j,m}$$

$$+ \sum_{i=0}^{n_1} \sum_{j=0}^{n_2-1} \frac{a_1^{n_1-j} \, a_2^{j+2}}{(n+m+\ell)_{n_1-i+j+2}} \binom{d_2}{n_2-j-1} \binom{n_1-i+j}{j} \psi_{i,m}$$

$$- \sum_{i=0}^{n_1} \sum_{j=0}^{n_2-1} \frac{a_1^{i} \, a_2^{n_2-j}}{(n+m+\ell)_{i+n_2-j}} \binom{d_1}{n_1-i} \binom{i+n_2-j}{i+1} \rho_{d_1,j,m}$$

$$- (1-\delta^1_{n_2}) \sum_{i=0}^{n_1} \sum_{j=1}^{n_2-1} \frac{a_1^{n_1-i} \, a_2^{j+1}}{(n+m+\ell)_{n_1-i+j+1}} \binom{d_2}{n_2-j-1} \binom{n_1-i+j+1}{j+1} \rho_{i,d_2,m}$$

$$- \frac{a_1}{n+m+\ell} \binom{d_2}{n_2-1} \rho_{n_1,d_2,m} + \frac{a_2}{n+m+\ell} \binom{d_2}{n_2} \psi_{n_1,m} \tag{5.16}$$

The terms in $\phi_{j,m}$ are

$$\sum_{i=1}^{n_1} \sum_{j=0}^{n_2} \frac{a_1^{\,i+1} a_2^{\,n_2-j}}{(n+m+\ell)_{i+1+n_2-j}} \binom{d_1}{n_1-i} \binom{i-1+n_2-j}{i-1} \phi_{j,m}$$

$$+ \sum_{i=0}^{n_1} \sum_{j=0}^{n_2-1} \frac{a_1^{\,i+1} a_2^{\,n_2-j}}{(n+m+\ell)_{i+1+n_2-j}} \binom{d_1}{n_1-i} \binom{i+n_2-j-1}{i} \phi_{j,m}$$

$$+ \frac{a_1}{n+m+\ell} \binom{d_1}{n_1} \phi_{n_2,m}. \tag{5.17}$$

Use of the identity $\binom{x-1}{y-1} + \binom{x-1}{y} = \binom{x}{y}$ in (5.17), yields

$$\sum_{i=1}^{n_1} \sum_{j=0}^{n_2-1} \frac{a_1^{\,i+1} a_2^{\,n_2-j}}{(n+m+\ell)_{i+1+n_2-j}} \binom{d_1}{n_1-i} \binom{i+n_2-j}{i} \phi_{j,m}$$

$$+ \frac{a_1}{n+m+\ell} \binom{d_1}{n_1} \phi_{n_2,m} + \sum_{i=1}^{n_1} \frac{a_1^{\,i+1}}{(n+m)_{i+1}} \binom{d_1}{n_1-i} \binom{i-1}{i-1} \phi_{d_2,m}$$

$$+ \sum_{j=0}^{n_2} \frac{a_1 a_2^{\,n_2-j}}{(n+m+\ell)_{n_2-j+1}} \binom{d_1}{n_1} \binom{n_2-j-1}{0} \phi_{j,m} \tag{5.18}$$

The last term uses the fact $\binom{n_2-n_2-1}{0} = 0$. Hence (5.18) simplifies to,

$$\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \frac{a_1^{\,i+1} a_2^{\,n_2-j}}{(n+m+\ell)_{i+1+n_2-j}} \binom{d_1}{n_1-i} \binom{i+n_2-j}{i} \phi_{j,m} \tag{5.19}$$

as required.

The terms in $\psi_{i,m}$, $\rho_{d_1,j,m}$ and $\rho_{i,d_2,m}$ also simplify to the required results and hence (5.11) is valid for all values $\{0 \le n_1 \le d_1,\ 0 \le n_2 \le d_2,\ 0 \le m \le c,\ m+\ell > 0\}$. When $m=\ell_1=\ell_2=0$, (5.12) can be proved by a similar induction starting with $\rho_{0,0,0,0,0} = 1$.

Equations (5.11) and (5.12) define the general term $\rho_{n_1,n_2,m}$ in terms of $\rho_{d_1,n_2,k}$ and $\rho_{n_1,d_2,k}$ for $k=m$, $m-1$ and c. In particular, $\rho_{d_1,n,m}$ and $\rho_{n_1,d_2,m}$ are defined in terms of

$\rho_{d_1,n_2,k}$ and $\rho_{n_1,d_2,k}$ and (since the two terms needed for calculation of the overflow statistics are $\rho_{d_1,0,c}$ and $\rho_{0,d_2,c}$) only these terms need be evaluated.

Thus the size of the system of equations needed to be solved has been reduced from $(d_1+1)(d_2+1)(c+1)$ to $(d_1+d_2+1)(c+1)$.

If $\underset{\sim}{x}_{\ell_1,\ell_2}$ is defined by

$$x_{\ell_1,\ell_2} = \begin{cases} \rho_{d_1,n_2,m,\ell_1,\ell_2} & ;\ r = (n_2+1)+(d_2+1)(m+1) \\ \rho_{n_1,d_2,m,\ell_1,\ell_2} & ;\ r = (d_2+1)(c+1)+(n_1+1)+(d_1+1)(m+1) \end{cases} \tag{5.20}$$

then (5.11) and (5.12) may be written

$$\underset{\sim}{x}_{\ell_1,\ell_2} = A_{\ell_1,\ell_2}\,\underset{\sim}{x}_{\ell_1,\ell_2} + \underset{\sim}{f}_{\ell_1,\ell_2} \tag{5.21}$$

for some matrix A_{ℓ_1,ℓ_2} and vector $\underset{\sim}{f}_{\ell_1,\ell_2}$. As before $\underset{\sim}{f}_{\ell_1,\ell_2}$ is a function of x_{ℓ_1-1,ℓ_2} and x_{ℓ_1,ℓ_2-1}. Hence these equations in ρ can be solved using the S.O.R. algorithm described in Chapter 3. The size of the matrices and vectors has been reduced, but A_{ℓ_1,ℓ_2} for this system is no longer sparse.

When c=0, ℓ=0 equation (5.12) applies, and

$$\rho_{d_1,0,0,0,0} = \frac{a_1^{d_1}}{d_1!} - \sum_{i=1}^{d_1} \frac{a_1^{\,i}}{(d_1)_i} \binom{d_1}{d_1-i} \rho_{d_1,0,0,0,0} \tag{5.22}$$

which is analogous to (4.53). Hence

$$\rho_{d_1,0,0,0,0} = E_{a_1}(d_1)$$

and

$$m_1 = M_1 = a_1 E_{a_1}(d_1) \text{ as required.}$$

Similarly

$$m_2 = a_2\, E_{a_2}(d_2).$$

When $\ell_1=1$, $\ell_2=0$, by (5.11)

$$\rho_{d_1,0,0,1,0} = \sum_{i=0}^{d_1} \frac{a_1^{i+1}}{(d_1)_{i+1}} \binom{d_1}{d_1-i} . \rho_{d_1,0,0,0,0}$$

$$- \sum_{i=1}^{d_1} \frac{a_1^{i}}{(d_1)_{i}} \binom{d_1}{d_1-i} \rho_{d_1,0,0,1,0}$$

which is similar to (4.57). Hence

$$v_1 = V_1 = m_1 \left(1 - m_1 - \frac{a_1}{d_1+1-a_1+m_1}\right), \text{ as required,}$$

and similarly $v_1 = V_2$.

Thus the model reduces to give known results in the limiting case c=0. A similar reduction will occur when $d_1=d_2=0$ to give results agreeing with those obtained in Chapter 4.

CHAPTER 6

THE ACCURACY OF SOME APPROXIMATE FORMULAS

6.1 Introduction

Several researchers in the field of teletraffic theory have suggested approximate formulas for determining the individual overflow means and variances for the network under consideration. These have generally been formulas for partitioning the total overflow mean and variance into the components corresponding to each stream and the total overflow traffic is also obtained by an approximation, such as the Equivalent Random Method.

Although some formulas give approximations for the ratio of the means of two streams (m_i:m_j) it is convenient for comparison of accuracy to consider the corresponding formula for the ratio of the ith stream to the total mean (m_i:m). The symbol $P_{m,i}$ will be used to denote the 'proportion of the total mean belonging to the ith stream', that is

$$P_{m,i} = \frac{m_i}{m},$$

(and similar abbreviations will be used to denote proportions of other parameters).

Formulas for $P_{m,i}$ have been suggested by Olsson and Wallström and for $P_{v,i}$ by Harris. The accuracy of these approximations was investigated, along with some simple approximations which have been used by traffic engineers. Since these formulas are often used to partition the total overflow mean and variance as calculated by the equivalent random method, the accuracy of this method is also discussed.

6.2 Generation of test data

A large amount of data was generated by the computer program of the S.O.R. algorithm. The convergence criterion was set at 10^{-6} and since at most four decimal places will be considered for all results in this chapter, the data may be considered to be 'exact' at this level of significance.

The networks considered corresponded to a service system in which the number of secondary group servers, c, and the mean of the total traffic offered, M, were fixed. For each value of (M,c), 30 different values of primary group servers, (d_1,d_2), were considered. The arrival rates (a_1,a_2) were chosen such that the overflow from the two primary groups had a total mean M and had fixed ratios corresponding to five values of $P_{M,1}$. Three values of c (c=2,5,8) and five values of M (M=2,4,8,16,32) were chosen to give a total of $3 \times 5 \times 30 \times 5 = 2,250$ different networks. The values of (d_1,d_2) used were (2,3), (2,4), ..., (2,10), (5,6), ..., (5,10), (8,9), (8,10) and (3,2), (4,2), ..., (10,2), (6,5), ..., (10,5), (9,8), (10,8) and $P_{M,1}$ had the values $\frac{1}{2}$, $\frac{3}{8}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$.

Since each value of (M,c) has 150 different values of $(P_{M,1},d_1,d_2)$, the effect on the overflow means and variances of small changes in the input parameters can be investigated. This sensitivity analysis has been notably lacking in other papers which compare approximate solutions to computational solutions.

Kibble [12], for example, in his comparison of the means, calculated by the equivalent random method and the 'exact' solution to the state equations, considered fewer than 200 networks and these were spread over a large range of values of (a_1,a_2,d_1,d_2,c). The results obtained gave little insight into the relationship between the overflow mean and the input parameters. Kibble was limited by the power of the computer used which he says took 7m.sec. to perform each row operation. Computer technology has improved considerably since 1968 and the same operation would take less than 150 μ sec. on the machine used by the author, a CDC Cyber 173 (that is, about 50 times faster than Kibble's computer). Even so, the data generated required hours of computing time and this indicates the impracticability of using computational solutions on an actual network which may have dozens or even hundreds of subnetworks of the type investigated in this thesis.

All results are proportions and have values between 0 and 1, and since

$$P_{x,1} + P_{x,2} = 1,$$

the absolute errors in $\hat{P}_{x,1}$ and $\hat{P}_{x,2}$ are identical. If, for example, $P_{x,2} = .9$ then a 10% error in $\hat{P}_{x,2}$ would correspond to a 90% error in $\hat{P}_{x,1}$ and hence the relative error is not a particularly useful criterion for measuring the accuracy of these formulas. Hence the worst absolute difference between the value calculated by the formula and the exact solution is used as a basis for comparison of the formulas. The results are summarised in tables which give this 'worst error' over the 150 results corresponding to each value of M and c.

The various formulas are represented graphically for the case M=8, c=8, in Appendix C.

6.3 The accuracy of the Equivalent Random Method

The equivalent random method (E.R.M.) has, in the words of Prof. Wallström [26], 'found world-wide application for the planning of alternative routing networks'. As mentioned in the previous section, Kibble [12] compared the E.R.M. with results from an iterative solution of the state equations. He does not actually compare the overflow means, but the 'probability of blocking', which he does not define. It appears that this term refers to the probability that a call will not be served on either the primary or secondary group (as distinct from the blocking probability B, defined in Chapter 1, which refers to a single group only). This would correspond to the ratios,

$$\frac{m}{a} \text{ and } \frac{m_i}{a_i},$$

for the total and ith streams respectively.

Kibble first considers networks in which $a_1 = a_2$ and $d_1 = d_2$, and the worst relative error found is 4%. When the input is asymmetric, Kibble partitioned the overflow mean in the ratio of the offered mean, (that is, $\hat{P}_{m,i} = P_{M,i}$) and errors of up to 50% were found. These errors are only partially due to the inaccuracy of the E.R.M. The assumption, $\hat{P}_{m,i} = P_{M,i}$ which is discussed in the next section, is also not particularly accurate.

The E.R.M. was used to calculate $\hat{m}$, $\hat{v}$ for the data set and the ratios $\frac{\hat{m}}{m}$ and $\frac{\hat{v}}{v}$ were calculated. The worst values of these ratios, for fixed M and c, are given in Table 6.1. It should be noted, that, although all results given correspond to $\hat{m}>m$ and $\hat{v}>v$, this is not always the case. The E.R.M. does underestimate the values in some networks but the worst errors were overestimates. It can be seen that the estimates tend to get worse as M decreases and c increases. These two trends both correspond to a decrease in the $\frac{M}{c}$ ratio, which means the servers are becoming less heavily loaded. In the worst case M=2, c=8 each server is offered an average of a quarter of an erlang of traffic. In such cases the total overflow traffic has mean of the order of 1/10 erlang or less and in practical situations this inaccuracy is not significant. There is a general trend, in all approximations suggested, for the poorest accuracy to occur in the 'underloaded' situation. Even so, for the cases where $M \geq c$, errors of 20-50% occur. This must be taken into account when considering the accuracy of the splitting formulas, since 6 decimal place accuracy in P_m or P_v is hardly necessary when m and $\hat{v}$ are accurate to only 2 or 3 significant figures.

c	M	$\hat{m}/m$	$\hat{v}/v$
2	2	1.534	1.488
	4	1.355	1.256
	8	1.168	1.085
	16	1.069	1.016
	32	1.027	1.003
5	2	1.888	1.915
	4	1.554	1.470
	8	1.233	1.135
	16	1.085	1.026
	32	1.030	1.003
8	2	2.387	2.505
	4	1.824	1.829
	8	1.326	1.230
	16	1.105	1.039
	32	1.034	1.004

Table 6.1 : Worst results, $\frac{\hat{m}}{m}$ and $\frac{\hat{v}}{v}$, for $\hat{m}$ and $\hat{v}$ calculated by the Equivalent Random Method.

6.4 Some simple approximations

One of the earliest methods of determining the overflow means was to partition the total overflow mean in the ratio of the offered mean for each stream, that is,

$$\hat{P}_{m,i} = P_{M,i} \tag{6.1}$$

as used by Kibble. This is a logical first approximation since a large overflow mean would be caused by a large offered mean. This is especially true when the total mean offered is greater than the number of junctions provided (M>c).

A second simple approximation, (which is still used by the Australian Telecommunications Commission) is

$$\hat{P}_{m,i} = P_{V,i}, \tag{6.2}$$

the overflow mean is proportional to the offered variance. This formula may be intuitively derived as follows. If M≤c and the offered variances were zero then there would be little overflow. The overflow is caused not by the magnitude of M, but by the peaks (and troughs) in the arrival streams described by the variance. Larger variances would correspond to bigger peaks and hence larger overflows. This effect holds for M>c as well but is less marked as M increases relative to c.

From the two intuitive arguments, it would be expected that (6.1) would become more accurate as $\frac{M}{c}$ increases and (6.2) would be most accurate when $\frac{M}{c}$ was near one.

The two formulas were compared with the results from the iterative solutions and are summarised in Table 6.2. The trends suggested intuitively are confirmed by these results. Graphs of $P_{m,i}$ vs $P_{M,i}$ and $P_{m,i}$ vs $P_{V,i}$ are given in Appendix C.

Table 6.3 summarises the worst errors involved when $P_{M,i}$ and $P_{V,i}$ are used as estimates of $P_{v,i}$. They both tend to overestimate the value, (as indicated in the graphs in Appendix C) because of the correlation between the overflow streams induced by the sharing of the secondary group servers and

$$P_{v,i} = \frac{v_i}{v_1+v_2+2\mathrm{cov}} \quad . \tag{6.3}$$

Since cov is largest when M_1 and M_2 are equal, this tends to cause the largest errors in these two approximations for $P_{v,i}$ to occur when $P_{M,i}$ is near ½.

6.5 Olsson's conjecture

One of the few formulas available for calculating the ratio of the means of individual overflow streams was suggested by Dr. K.M. Olsson. Dr. Olsson has been involved in research in teletraffic theory for many years and obtained his result by observation of results from simulations of teletraffic networks. This result has not been published by Olsson, but is quite well-known from private correspondence and discussions between Dr. Olsson and other researchers, for example, at the International Teletraffic Congresses.

The conjecture is that the overflow means are proportional to a combination of the offered means and variances, namely,

$$m_i \propto V_i + \frac{M_i^2}{V_i} \tag{6.4}$$

Olsson*, first considered the approximation

$$m_i \propto V_i \tag{6.5}$$

which is equivalent to (6.2). He then added a second term $\frac{M_i}{V_i}$ and found that this correction was not large enough. The term $\frac{M_i^2}{V_i}$ was found to be a better correction and this led to (6.4).

This correction term is, in fact, the inverse of a statistical quantity called the coefficient of variation. It may be noted that $\frac{M_i^2}{V_i}$ is dimensionally consistent with V_i in that both terms reduce to a_i when there are no primary groups. Thus $m_i \propto a_i$ in this case which is consistent with the results of Chapter 4.

The accuracy of Olsson's formula is summarised in Table 6.2 and again represented graphically in Appendix C. The formula tends to become more inaccurate as $\frac{M}{c}$ gets further from 1 and is especially bad when $\frac{M}{c} < 1$.

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*This information was given to the author in a discussion with Dr. Olsson at the 8th I.T.C., Melbourne 1976.

Even disregarding the cases where $M<c$ the formula has a large error and in fact the smallest of these 'worst errors' is still .0144. Olsson's conjecture is generally more accurate than (6.1) for $\frac{M}{c} \leq 1$ and better than (6.2) for $M/_c \geq 1$.

One severe criticism of this formula, (and also with the two simple approximations) is that it contains no information about the number of servers in the secondary group.

(This formula was first introduced into Australia by Pratt [20] but was incorrectly presented. This incorrect formula had been copied by several other Australian researchers before the error was corrected at the 8th I.T.C.)

6.6 A formula suggested by Wallström

Prof. B. Wallström has also been doing research in this field for many years and recently* suggested a formula for $P_{m,i}$,

$$\hat{P}_{m,i} = P_{V,i}(1-B) + B \cdot P_{M}, \tag{6.6}$$

where B is the blocking probability on the secondary group for the combined streams. That is

$$B = \frac{m}{M} \, . \tag{6.7}$$

Since m is dependent on c this formula does contain information about the number of servers in the secondary group.

It also is consistent with known results for two limiting cases. When there are no primary groups, $P_{M,i} = P_{V,i} = P_{a,i}$ and hence $P_{m,i} = P_{a,i}$ as required. When there is no secondary group, B=1 and (6.6) reduces to

$$P_{m,i} = P_{M,i}.$$

That is, the offered and overflow streams are equal as is required when there are no secondary servers.

It may be noted that disregarding the underloaded cases, $P_{M,i}$ is a more accurate estimate than $P_{V,i}$ when $\frac{M}{c}$ is large and the reverse is true when $\frac{M}{c}$ is near 1. The blocking probability, B, increases as $\frac{M}{c}$ increases and therefore Wallström's formula, which is a weighted mean of $P_{M,i}$ and $P_{V,i}$ gives a heavier weighting to whichever factor is more accurate at the appropriate value of $\frac{M}{c}$.

In general, Wallström's formula is significantly better than Olsson's conjecture, and the two are compared in Table 6.2. A graph of Wallström's formula for M=8, c=8 is given in Appendix C.

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*This formula was given to the author in a discussion with Prof. Wallström at the 8th I.T.C., Melbourne 1976.

c	M	P_M	P_v	Olsson	Wallström
2	2	.0776	.0365	.0294	.0267
	4	.0415	.0796	.0144	.0109
	8	.0191	.1045	.0314	.0116
	16	.0075	.1090	.0330	.0082
	32	.0023	.0963	.0247	.0042
5	2	.1841	.0782	.1378	.0893
	4	.1015	.0175	.0510	.0208
	8	.0483	.0768	.0130	.0204
	16	.0178	.1002	.0245	.0188
	32	.0065	.0934	.0218	.0101
8	2	.2711	.1855	.2357	.1864
	4	.1492	.0484	.1085	.0552
	8	.0768	.0472	.0296	.0169
	16	.0305	.0898	.0177	.0268
	32	.0104	.0902	.0196	.0156

The worst error $|\hat{P}_{m,i} - P_{m,i}|$ for fixed M and c is given for each of the four approximate formulas.

Table 6.2 : Comparison of formulas for overflow means.

6.7 Modifications of the formulas of Olsson and Wallström

It is unfortunate that both formulas tend to overestimate (or underestimate) the value of $P_{m,i}$ for the same network. Otherwise some weighted average of the two formulas might have given a more accurate estimate.

If Olsson's formula is considered as a corrected estimate of $P_{V,i}$, then the substitution of this factor for $P_{V,i}$ in Wallströms formula could possibly lead to an improved formula.

$$\hat{P}_{m,i} = (1-B)P^*_{V,i} + B\ P_{M,i}$$

where

$$P^*_{V,i} = \frac{V_i + M_i^2/V_i}{\sum_j V_j + M_j^2/V_j}. \tag{6.8}$$

The results of this formula are given in Table 6.3, and it can be seen that it is only more accurate than Wallströms result when M is considerably larger than c. It does however, suggest that an improved formula may be obtained by some modification of Wallström's approximation.

Olsson's formula may also be modified by weighting the correction factor $\frac{M_i^2}{V_i}$.

That is,

$$\hat{m}_i \ \alpha\ V_i + w. \frac{M_i^2}{V_i}\ . \tag{6.9}$$

The weighting factor, w, should contain some information about the secondary group, if the new formula is to be more accurate than Olsson's result.

The formula was used, with several values of w, to estimate m_i for the cases for which exact results were known. The results in Table 6.3 give the best value of w (to 1 decimal place only), which, for fixed c and M, minimise the worst absolute error over the range of (d_1,d_2 and P_M) considered and the value of that worst error.

It can be seen that w depends on both M and c, and tends to decrease as M decreases or c increases. The modified formula is considerably more accurate than the original approximation and in the majority of the cases considered was better than Wallström's results.

c	M	a)	b)	c)
2	2	.0522	.0093	0.4
	4	.0257	.0107	1.3
	8	.0113	.0099	1.8
	16	.0047	.0076	1.9
	32	.0018	.0085	1.9
5	2	.1426	.0146	-0.6
	4	.0649	.0067	0.2
	8	.0262	.0117	1.1
	16	.0111	.0108	1.6
	32	.0044	.0084	1.7
8	2	.2361	.0553	-1.1
	4	.1124	.0128	-0.4
	8	.0423	.0108	0.5
	16	.0166	.0123	1.3
	32	.0067	.0098	1.5

a) Substitution of Olsson's formula for $P_{V,i}$ in Wallström's approximation; worst absolute errors.

b) Modified Olsson's formula; worst absolute errors.

c) Weighting factor, w, used in b).

Table 6.3 : Comparison of some modified versions of Olsson's and Wallström's formulas.

6.8 Harris's formula for the overflow variances

Until recently, little research had been undertaken into the proportioning of overflow variance into its components (which include covariance terms). Neal [16] derived an analytically based recursive formula for the variances of separate streams but this was for a different system.

This research was supported in part by a contract with the Australian Telecommunications Commission (then part of the Australian Post Office). The iterative solution to the problem was first published in the Second Progress Report [19] to the Commission. Dr. R. Harris utilised this solution and corresponding program to generate results for many networks, and from observation of the results found an approximate formula.

The formula was based on some results by Descloux [7] and Lotze [15] quoted in Neal's paper [16].

$$m_i = p_i m$$

and

$$\frac{v_i}{m_i} - 1 = p_i(\frac{v}{m} - 1). \tag{6.10}$$

Harris used the formula (6.2) to estimate m_i which in fact gives

$$p_i = P_{V,i}. \tag{6.11}$$

Rearrangement of (6.10) gives

$$\hat{v}_i = p_i[p_i(v-m)+m]. \tag{6.12}$$

This formula agrees with the known results when there are no primary servers, but was found to be inaccurate in a number of cases. Improved accuracy was obtained by a small modification to give,

$$\hat{v}_i = p_i[(p_i + (1-p_i)e^{-p_i c})(v-m) + m] \tag{6.13}$$

This modified version is however, no longer accurate for a system with no primary group but is valid for other limiting cases.

When there are no secondary servers, (6.13) reduces to $\hat{v}_i = V_i$ as required and as c tends to infinity it approaches the formula of (6.12).

The values of $\hat{P}_{v,i}$ calculated using this formula are summarised in Table 6.4, and graphically represented in Appendix C. The results are generally less accurate than the corresponding estimates for the means but better overall than either of the simple approximations for $P_{v,i}$. Obviously, the assumption $P_{m,i} = P_{V,i}$ contributes appreciably to the errors.

c	M	P_M	P_V	Harris
2	2	.1987	.0916	.0600
	4	.1739	.0886	.0434
	8	.1550	.0547	.0449
	16	.1324	.0394	.0465
	32	.1064	.0220	.0390
5	2	.3081	.2022	.1444
	4	.2480	.1317	.0567
	8	.1991	.1080	.0485
	16	.1581	.0895	.0572
	32	.1181	.0528	.0527
8	2	.3802	.2946	.2204
	4	.2999	.1890	.0930
	8	.2370	.1303	.0348
	16	.1813	.1284	.0620
	32	.1309	.0812	.0516

The worst error $|\hat{P}_{v,i} - P_{v,i}|$ for fixed M and c is given for each of the three formulas.

Table 6.4 : Comparison of formulas for overflow variances.

CHAPTER 7

STATISTICALLY DERIVED APPROXIMATE FORMULAS

7.1 Introduction

The tables in Chapter 6 summarise the worst errors for various approximations for $P_{m,i}$ and $P_{v,i}$. Even disregarding the under-loaded cases, in which the estimates are almost uniformly bad, the formulas are generally not accurate to 2 decimal places. The author hoped to utilise the large amount of data generated by the iterative method, to obtain more accurate approximations.

This task was aided considerably by use of a graphic display terminal, which enabled any formula which seemed likely to be useful, to be represented visually, as a graph, in a matter of seconds. This saved a considerable amount of time plotting graphs by hand and also allowed obviously wrong formulas to be rejected quickly.

The true value of the parameter, $P_{m,i}$ or $P_{v,i}$, was plotted in the x direction and the estimates in the y direction. Both values are between 0 and 1 and a good formula would be one which gives points near to the diagonal line y=x. Graphs of the approximate formulas of Chapter 6 are given, in this form, in Appendix C.

7.2 Simple linear regression approximation

(The remarks made in this section about the overflow means are also applicable to the variances, and the i subscript will be omitted from $P_{x,i}$.)

The graph of P_M vs P_m consists of a number of sets of points which are line segments parallel to the x axis (see Appendix C). This occurs because of the way the data was generated, namely a fixed value of M and five values of $P_{M,1}$ each corresponding to 30 values of a_1 and a_2. Hence each set corresponds to 30 points which are constant with respect to the y direction.

The corresponding points in a graph of $P_{V,i}$ vs $P_{m,i}$ are also nearly linear but not parallel to the x axis. If the 30 points are represented by a staight line then, for each set, the supposition of the two graphs is similar to the diagram (figure 7.1).

If the position and slope of the $P_{V,i}$ line were known then a linear combination of P_V and P_M could be obtained which is equivalent to $P_{m,i}$.

If the P_V line passes through the point (P_M, $P_M + \beta$) and has slope θ then,

$$(P_V - P_M) = \theta(P_m - P_M) + \beta \tag{7.1}$$

or

$$P_m = P_M + \alpha(P_M - P_V + \beta) \tag{7.2}$$

where

$$\alpha = -\frac{1}{\theta}.$$

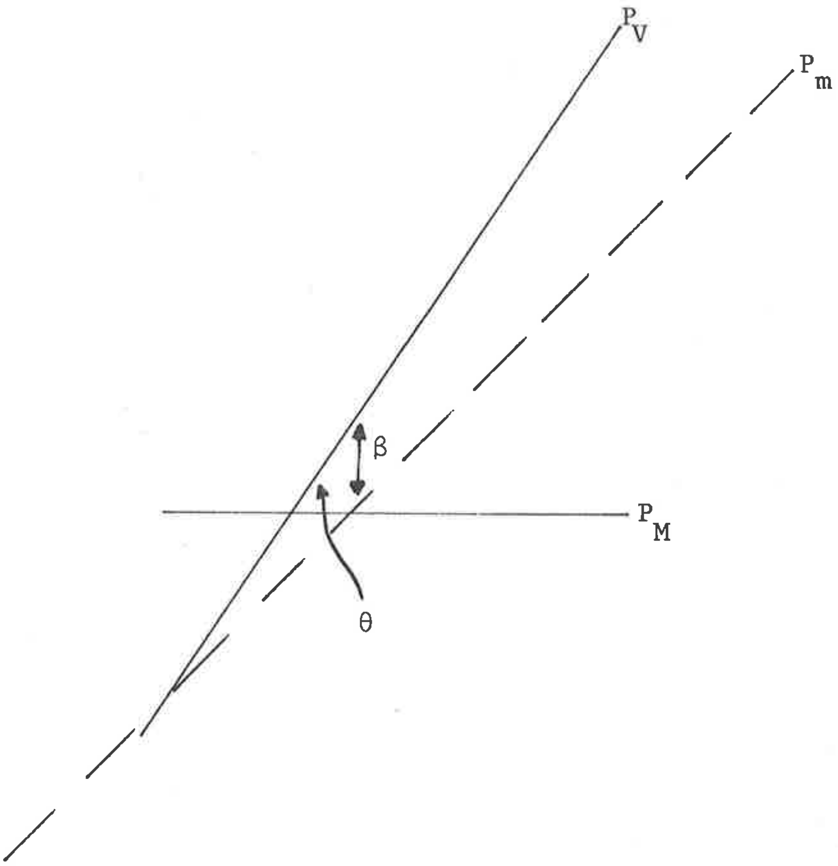


Figure 7.1 : Section of graphs of P_M and P_V against P_m.

The accuracy of this formula is limited by the accuracy of the linear approximation of P_V vs P_m, and by the accuracy of the estimates of α and β. The assumption of linearity is good except in the underloaded cases, and the estimates of α and β may be obtained using simple linear regression (S.L.R.).

If a set of data $\{(x_i, y_i);\ i=1,n\}$ is believed to satisfy a linear relationship of the form,

$$y_i = a + b\, x_i, \tag{7.3}$$

then the parameters a and b may be estimated by

$$\hat{a} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x} \tag{7.4}$$

and

$$\hat{b} = \frac{\sum_{i=1}^{n} (x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n} (x_i-\bar{x})^2} . \qquad (7.5)$$

The estimate $\hat{b}$ gives the slope of the line, passing through the point (x,y) which most closely fits the data. A close fit is one in which the sums of the squares of the distances between the actual points and the line is minimised, (that is, $\sum_{i=1}^{n} (y_i-\hat{y}_i)^2$ is minimised).

A program was written to estimate θ (and hence α) and β for the P_V vs P_m line segments.

This was used to calculate α and β and hence $\hat{P}_m$ for the test data and this estimate was compared with the exact results. The results summarised in Table 7.1 indicate the accuracy of this approach. Apart from two underloaded cases the worst errors are less than .01 and even in the two bad cases, the results are much more accurate than any other approximation. In general, the S.L.R. approximation is between 3 and 10 times better than the formula of Wallstrom.

A similar S.L.R. approximation was obtained for P_v which is summarised in Table 7.2. This is a significant improvement over Harris's formula, being between 7 and 20 times more accurate.

The two proportions have the same formula,

$$P_x = P_M + \alpha_x(P_M-P_V+\beta_x), \qquad (7.6)$$

where x is either m or v . Of course α_m and α_v, and β_m and β_v, may be quite different for the same network.

7.3 Estimation of α and β.

It is, of course, impractical to get 30 points for every value of M and c that might be needed and it was hoped that formulas could be found for α and β, in terms of M, P_M and c. The values of α and β were calculated for the data available and graphs were plotted to give a visualisation of the relationships between the parameters. Although general trends were noted and the curves appeared to belong to the same families, no simple functions could be found which were consistently accurate for the whole range of P_M, M and c*.

It seemed desirable, then to generate some tables which would give the values of α and β for a wide range of P_M, M and c, which by interpolation and extrapolation would allow calculation of α and β for intermediate values of the parameters. Again, there were practical objections to using 30 points to get each estimate and it was decided to check whether the P_V lines could be accurately generated by only two points.

Initially the values of the S.L.R. approximations for the data were calculated with α and β, 10% more and 10% less then the 'correct values'. The errors were of course greater than the correct S.L.R. approximations, but were twice as good as Wallstrom's results for the mean, and for the variances at least twice as accurate as the formula of Harris. Hence, if α and β could be estimated to 10% accuracy the S.L.R. formula would be more accurate than the other approximations. After many different pairs of points (d_1,d_2) were considered, it was found that (0,10) and (10,0) give estimates of α and β to an accuracy of better than 10%. These values were used to generate the tables in Appendix D. Nine values of c_i (c=2,4,6,8,10,15,20, 25 and 30) were chosen and for each value, seven values of M and five values of P_M were used.

---

*eg. $\beta_m = (P_M-.\frac{1}{2})(1-4(P_M-.\frac{1}{2})^2).\ln(\frac{5c}{8}).e^{-40M}$. etc. was found to be reasonably accurate but hardly simple.)

c	M	S.L.R.	(1)	(2)
2	2	.0032	.0093	.0038
	4	.0028	.0060	.0035
	8	.0016	.0030	.0021
	16	.0010	.0012	.0013
	32	.0005	.0006	.0006
5	2	.0125	.0248	.0248
	4	.0020	.0107	.0195
	8	.0029	.0059	.0055
	16	.0021	.0026	.0024
	32	.0011	.0013	.0013
8	2	.0330	.0475	.0475
	4	.0074	.0213	.0168
	8	.0024	.0082	.0071
	16	.0027	.0037	.0035
	32	.0016	.0018	.0018

(1) 10% error in α, worst results.

(2) 10% error in β, worst results.

Table 7.1 : S.L.R. approximation for P_m.

c	M	S.L.R.	(1)	(2)
2	2	.0070	.0208	.0126
	4	.0048	.0184	.0105
	8	.0023	.0165	.0065
	16	.0027	.0138	.0056
	32	.0017	.0105	.0032
5	2	.0191	.0329	.0258
	4	.0062	.0262	.0159
	8	.0032	.0212	.0122
	16	.0045	.0168	.0112
	32	.0035	.0119	.0074
8	2	.0412	.0599	.0502
	4	.0118	.0360	.0210
	8	.0039	.0247	.0157
	16	.0047	.0191	.0144
	32	.0045	.0134	.0106

(1) 10% error in α, worst results.

(2) 10% error in β, worst results.

Table 7.2 : S.L.R. Approximation for P_v.

7.4 Use and accuracy of the tables

The values of α and β in the tables are given to only three decimal places. The third significant figure, if present, should be treated with suspicion as it is probably inaccurate. However, since even a 10% fit is an improvement, this is not particularly worrisome. Generally, linear interpolation between successive values of M, P_M and c will be sufficiently accurate for all parameters α_m, β_m, α_v and β_v.

The extension of the approximations to more than 2 streams may be achieved by a method similar to section (4.6).

1) Partition the 'streams' into two groups, the ith stream and stream i*, formed by the combination of the other streams.
2) Calculate $P_{m,i}$ and $P_{v,i}$ as before.
3) Repeat 1) and 2) for all values of i.

The accuracy of this extension is difficult to evaluate but is probably comparable with similar extensions of the other approximations.

An example of calculation of means and variances using the tables is given below for a two stream case; with $(a_1,a_2,d_1,d_2,c) = (5,9,2,2,7)$. For this case the approximations for P_m and P_v are all accurate to 2 decimal places.

Example

$a_1 = 5$ $\quad$ $a_2 = 9$

$d_1 = 2$ $\quad$ $d_2 = 7$

$c = 5$

$M_1 = 3.378$ $\quad$ $M_2 = 3.254$

$V_1 = 4.220$ $\quad$ $V_2 = 5.657$

$M = 6.632$ $\quad$ $V = 9.877$

$P_{M,1} = .509$ $\quad$ $P_{M,2} = .491$

$P_{V,1} = .427$ $\quad$ $P_{V,2} = .573$

$m_1 = 1.349$ $\quad$ $m_2 = 1.543$

$v_1 = 2.071$ $\quad$ $v_2 = 2.877$

$cov = .615$

$m = 2.892$ $\quad$ $v = 6.178$

$P_{m,1} = .466$ $\quad$ $P_{m,2} = .534$

$P_{v,1} = .335$ $\quad$ $P_{v,2} = .466$

$\alpha_{m,1} = -.539$

$\beta_{m,1} = -.001$

$\alpha_{v,1} = -.888$ $\quad$ $\alpha_{v,2} = -.898$

$\beta_{v,1} = .108$ $\quad$ $\beta_{v,2} = .108$

$$\hat{P}_{m,1} = .509 - .539\ (.509 - .427 + .001)$$
$$= .464$$

and $\hat{P}_{m,2} = .536$

$$\hat{P}_{v,1} = .509 - .888\ (.509 - .427 + .108)$$
$$= .340$$

$$\hat{P}_{v,2} = .491 - .898\ (.491 - .573 + .108)$$
$$= .468$$

CHAPTER 8

CONCLUSION

The purpose of this thesis was to investigate a telephone network in which two independent overflow streams of traffic were offered to a common link. This network has been modelled as a system of service stages with two primary groups and one secondary group. Random traffic is offered to the primary groups and the overflow from these is offered to the secondary group. Correlation is induced between the two overflow streams from the link due to the sharing of the service facilities by the two streams. In order to understand the effect of sharing a common link, it was desirable to calculate the means and variances of the two overflow streams and the covariance between them. The objective of the research was to investigate methods of calculating these statistics.

The state equations of the system were derived and binomial transformations were applied to these equations in terms of the binomial moments of ℓ_1 and ℓ_2 to a finite, linear recursive, system.

Analogous transformations were made to the equations of state in papers by Riordan [27], Chastang [4] and Neal [16]. These three researchers all used a binomial moment generating function which involved the introduction of carrier variables. The new system was then found, after suitable multiplications and summations, by equating coefficients of like powers of these carrier variables.

The transformation in this thesis was effected without the use of generating functions, thus avoiding the introduction of the carrier variables. Two lemmas were proved which were convenient in the simplification of the transformation. This improved method could have been used in the papers described above.

In the third chapter, the system of equations was expressed as a single matrix equation. The highly structured form of the coefficient matrix was analysed, and suggested that a solution could be obtained iteratively. Two iterative techniques were discussed; the Jacobi and Gaus-Seidel methods. Although Gaus-Seidel iteration was better, it was shown that this could be improved by the acceleration technique known as successive over-relaxation (S.O.R.). An improved S.O.R. method with a variable S.O.R. parameter, was also discussed.

A computer program of the S.O.R. method was written and used to find the solution to a large number of networks. Some features of this program, including efficient use of storage, were discussed.

A simple system with no primary groups was considered in the next chapter. The state equations were again derived and a binomial transformation applied. The system of binomial moment equations was solved using the classical technique of Riordan [27], involving relaxing a constraint on the main equation to allow one parameter to go to infinity. This implicitly introduced an infinite number of artificial variables into the system. The extended system was solved by using a generating function, $\beta(t)$. A differential equation in β was obtained and solved in terms of σ-polynomials. The boundary condition (or normalising condition for the case $\ell_1 = \ell_2 = 0$) was used to obtain a unique solution. The means and variances thus derived, were expressed in terms of the Erlang loss formula.

An improved method of solving this system using a second binomial transformation was presented. This method does not use generating functions and does not require the relaxation of any constraints. Thus the introduction of artificial variables was avoided. Generalisations of the two lemmas were given which facilitated this transformation.

It was also shown that the system of equations could be solved directly, without any further transformations being applied. Both these techniques could have been used to solve Riordan's original problem which was a special, simple case of the model considered. The solution was generalised to allow more than two random streams to be offered to the common link. The solution also allowed calculation of higher moments, although this was not of primary interest to the research.

In the following chapter, an extension of the binomial transformation method was applied to the general model. The system of eight equations, which is quadratic recursive in n_1, n_2 and m was replaced by a single equation which was linear recursive in the variables, by three successive binomial transformations. This system was of order $(d_1+1)(d_2+1)(c+1)$ which was reduced by one dimension to order $(d_1+d_2+1)(c+1)$

using analytic techniques. The reduced problem can also be solved by iterative techniques such as the S.O.R. method.

Approximate formulas for partitioning the total overflow mean into the means of individual streams have been suggested by Olsson and Wallström, and a similar formula for the variances by Harris. These formulas, together with some simple partitioning formulas were investigated, in chapter six, using data corresponding to solutions of over 2,000 networks which was calculated by the S.O.R. method. The absolute errors of each of these formulas were evaluated and their accuracy compared. The errors in estimates calculated by Olsson's formula were generally greater than those by Wallström's formula. The worst results, for fixed c and M, were generally between .01 and .02 for Wallström's formula although in underloaded cases they were as high as .05 to .18. Harris's formula generally had worst errors of the order .03 - .06 and once again was even less accurate in underloaded situations.

Graphs in which the estimated value of the relevant parameter was plotted against the true value were produced for each formula, consisting of 150 points which corresponded to networks in which the total mean of the traffic offered to the common link was 8 erlangs and the link had 8 junctions. These enabled visual comparisons between the formulas to be made. Modifications to the formulas are suggested, one of which gives a significant improvement to Olsson's result, with worst error about .01.

Since these formula are often used to partition total overflow means and variances which are calculated by the equivalent random method, the accuracy of this approximation was also investigated. There were up to 50% errors in the results calculated by the E.R.M. for cases where $\frac{M}{c} \geq 1$ although the approximation improved as $\frac{M}{c}$ increased. In the underloaded cases the estimates were up to $2\frac{1}{2}$ times the correct values. The actual size of the parameters in these cases were very small and this large relative error corresponded to an absolute error of the order of .1 erlang.

Finally, an approximate solution based on simple linear regression was derived. This method relied on the assumption that, for fixed values of the means of the offered streams, the proprotions of the mean and variance of the overflow for each stream, have a linear relationship with the proportion of the offered variance of the corresponding stream. This assumption gave a good approximation to reality for networks in which the average erlang per server ratio was not significantly less than one, becoming a better approximation as this ratio increased. Simple linear regression was used to estimate the slope and position of the lines best approximating these relationships, and from these approximate formulas for the proportions of the mean and variances were derived. The formulas involve two parameters and Appendix D contains tables which allow these parameters to be calculated. The S.L.R. approximations, except for the underloaded cases where they were still 6 times more accurate than the other approximations, generally gave errors significantly better than .01, and even with a 10% error in one of the parameters the estimates were still twice as accurate as Wallström's formula for the means and Harris's for the variances.

This thesis has investigated the problem under consideration from three approaches; analytic, computational and approximate. Analytic techniques have been used to solve the model explicitly for a special case and to reduce the order of the problem in the general model. A technique of performing binomial transformations without the introduction of generating functions was employed, which could be used in several other models considered by other authors. A computational solution to the model was obtained and incorporated into a computer program. This program was used to generate a large amount of data which led to an investigation into the accuracy of some approximate solutions. An approximate solution which is based on simple linear regression was also developed and tables for calculating the solution by this method were provided.

APPENDIX A

Derivation of the Sum of the Moment Equations

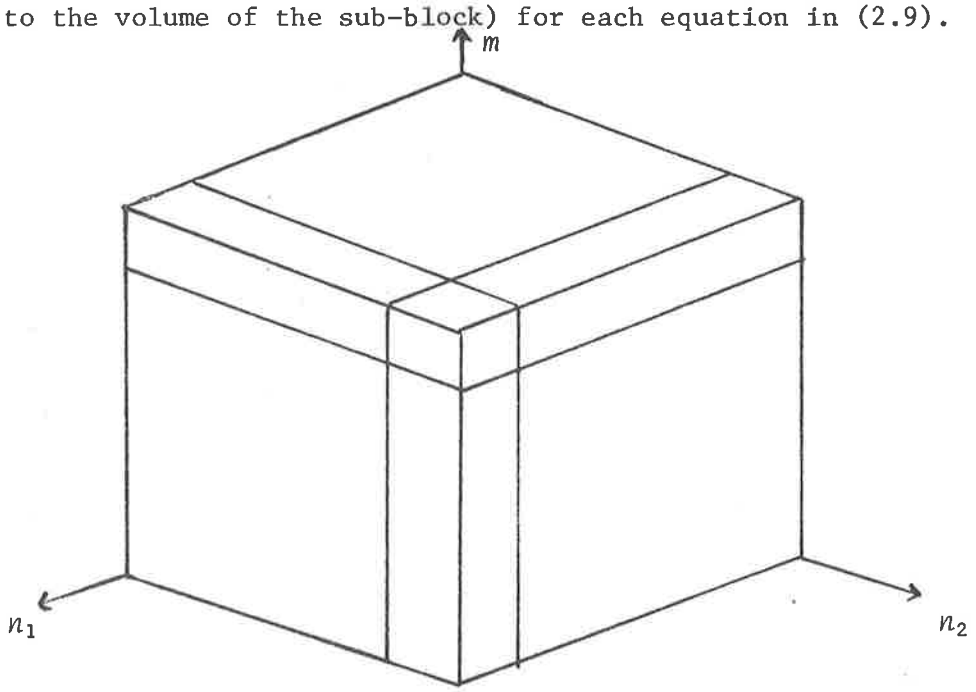
The alternate definition of $B_{(\ell_1,\ell_2)}$, equation (2.11), is obtained by summing the Moment Equations (2.9). The derivation of (2.11) may be facilitated by considering the summation symbols as operators. To simplify notation the following abbreviations will be used:

$$\Sigma\Sigma\Sigma \text{ represents } \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \sum_{m=0}^{c} .$$

If a variable is summed up to the value one less than the maximum (i.e. $d_i-1, c-1$) then the corresponding summation symbol will have a superscripted dash, that is Σ', and if the parameter is a constant then the value of the constant will replace the "Σ". For example

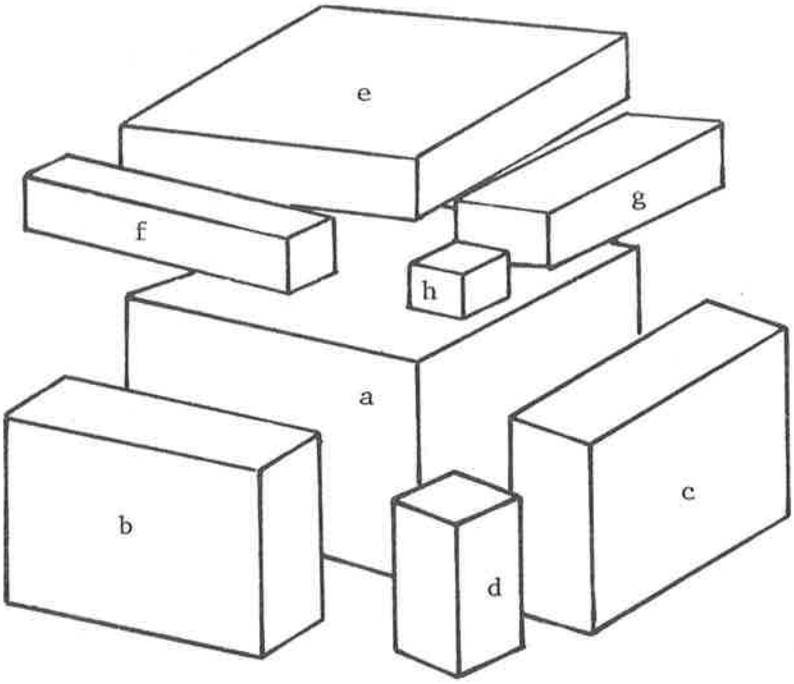
$$\Sigma\Sigma'c.g(n_1,n_2,m) \equiv \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2-1} g(n_1,n_2,c).$$

The range of summation may be represented as a cuboid in R^3. Each value of (n_1,n_2,m) corresponds to a unit cube in the block. The division of the block into 'sub-blocks' corresponding to the range of each equation is illustrated in Figure A.1, and Table A.1 lists the summation operator and number of values of (n_1,n_2,m) (equal to the volume of the sub-block) for each equation in (2.9).



a) Representation of the system of equations (2.9)

Figure A.1



b) Breakdown into sub-blocks representing each equation

Figure A.1

Equation	Operator	Number
a	$\Sigma'\Sigma'\Sigma'$	$d_1.d_2.c$
b	$d_1\ \Sigma'\Sigma'$	$d_2.c$
c	$\Sigma' d_2 \Sigma'$	$d_1.c$
d	$d_1\ d_2 \Sigma'$	Σc
e	$\Sigma'\Sigma' c$	$d_1.d_2$
f	$d_1 \Sigma' c$	d_2
g	$\Sigma' d_2 c$	d_1
h	$d_1 d_2 c$	1
Total	$\Sigma\Sigma\Sigma$	R

Table A.1

The following steps will be used to calculate the sum of (2.9).

1. Apply the appropriate operator to each equation.
2. For each term in the equations, add the operators for each equation in which the term appears.
3. Simplify this result to a single operator.
4. Add the terms together and simplify.

The term $(a+n+m+\ell)\,B(,,,)$ appears on the L.H.S. of each equation. Therefore its operator is $\Sigma\Sigma\Sigma$, the sum of the operations for equations (a) - (h).

The R.H.S. terms are

i) $a_1 B(n_1-1,,)$ which appears in every equation.

$$\mathrm{Op(i)} = \Sigma\Sigma\Sigma.$$

ii) $a_2 B(,n_2-1),)$ also appears in each equation.

$$\mathrm{Op(ii)} = \Sigma\Sigma\Sigma.$$

iii) $(n_1+1)B(n_1+1,,)$ appears in (a), (c), (e) and (g).

$$\begin{aligned}\mathrm{Op(iii)} &= \Sigma'\Sigma'\Sigma' + \Sigma' d_2 \Sigma' + \Sigma'\Sigma' c + \Sigma' d_2\, c \\ &= \Sigma'\Sigma\Sigma' + \Sigma'\Sigma c \\ &= \Sigma'\Sigma\Sigma.\end{aligned}$$

iv) $(n_2+1)B(,n_2+1,)$ appears in (a), (b), (e) and (f).

$$\mathrm{Op(iv)} = \Sigma\Sigma'\Sigma.$$

v) $(m+1)B(,,m+1)$ appears in (a), (b), (c) and (d).

$$\mathrm{Op(v)} = \Sigma\Sigma\Sigma'.$$

vi) $a_1 B(,,m-1)$ appears in (b), (d), (f) and (h).

$$\begin{aligned}\mathrm{Op(vi)} &= d_1\Sigma'\Sigma' + d_1 d_2 \Sigma' + d_1 \Sigma' c + d_1 d_2 c \\ &= d_1 \Sigma\Sigma.\end{aligned}$$

vii) $a_2 B(,,m-1)$ appears in (c), (d), (g) and (h).

$$\mathrm{Op(vii)} = \Sigma d_2 \Sigma.$$

viii) $a_1 B(,,,) + a_1 B_{\ell_1-1,}(,,,)$ appears in (f) and (h).

$$\begin{aligned}\mathrm{Op(viii)} &= d_1 \Sigma' c + d_1 d_2 c \\ &= d_1 \Sigma c.\end{aligned}$$

ix) $a_2 B(,,,) + a_2 B_{,\ell_2-1}(,,,)$ appears in (g) and (h).

$$\mathrm{Op(ix)} = \Sigma d_2 c.$$

The sum of equations (2.9) is therefore,

$$\begin{aligned}&\Sigma\Sigma\Sigma(a+n+m+\ell)B(,,,) \\ &\quad = a_1\cdot\Sigma\Sigma\Sigma\cdot B(n_1-1,,) + a_2\cdot\Sigma\Sigma\Sigma\cdot B(,n_2-1,) \\ &\quad + \Sigma'\Sigma\Sigma\cdot(n_1+1)\ B(n_1+1,,) + \Sigma\Sigma'\Sigma\cdot(n_2+1)\ B(,n_2+1,) \\ &\quad + \Sigma\Sigma\Sigma'\cdot(m+1)\ B(,,m+1) \\ &\quad + a_1\cdot d_1\Sigma\Sigma\cdot B(,,m-1) + a_2\cdot\Sigma d_2\Sigma\cdot B(,,m-1) \\ &\quad + a_1\cdot d_1\Sigma c\cdot B(,,,) + a_2\cdot\Sigma d_2 c\cdot B(,,,) \\ &\quad + a_1\cdot d_1\Sigma c\cdot B_{\ell_1-1,}(,,,) + a_2\cdot\Sigma d_2 c\cdot B_{,\ell_2-1}(,,,)\end{aligned} \qquad \text{(A.1)}$$

Now,

$$\Sigma\Sigma\Sigma\cdot B(n_1-1,,) = \Sigma'\Sigma\Sigma\cdot B(,,,),$$

$$\Sigma\Sigma\Sigma\cdot(n_1+1)B(n_1+1,,) = \Sigma\Sigma\Sigma\cdot n_1 B(,,,),$$

and similar identities are valid for other terms, therefore (A.1) becomes

$$\begin{aligned} a\Sigma\Sigma\Sigma B(,,,) + \Sigma\Sigma\Sigma nB(,,,) + \Sigma\Sigma\Sigma mB(,,,) + \ell\Sigma\Sigma\Sigma B(,,,) \\ = a_1\cdot\Sigma'\Sigma\Sigma\cdot B(,,,) + a_2\cdot\Sigma\Sigma'\Sigma\cdot B(,,,) \\ + a_1\cdot d_1\Sigma\Sigma'\cdot B(,,,) + a_2\cdot\Sigma d_2\Sigma'\cdot B(,,,) \\ + a_1\cdot d_1\Sigma c\cdot B(,,,) + a_2\cdot\Sigma d_2 c\cdot B(,,,) \\ + \Sigma\Sigma\Sigma\cdot n_1 B(,,,) + \Sigma\Sigma\Sigma\cdot n_2 B(,,,) \\ + \Sigma\Sigma\Sigma\cdot mB(,,,) \\ + a_1\cdot d_1\Sigma c\cdot B_{\ell_1-1,}(,,,) + a_2\cdot\Sigma d_2 c\cdot B_{,\ell_2-1}(,,,) \end{aligned} \tag{A.2}$$

Since,

$$\Sigma'\Sigma\Sigma + d_1\Sigma\Sigma' + d_1\Sigma c = \Sigma\Sigma\Sigma$$

$$\Sigma\Sigma'\Sigma + \Sigma d_2\Sigma' + \Sigma d_2 c = \Sigma\Sigma\Sigma$$

and

$$\Sigma\Sigma\Sigma B(,,,) = B_{(\ell_1,\ell_2)}$$

(A.2) simplifies to

$$\begin{aligned} (\ell_1+\ell_2)B_{(\ell_1,\ell_2)} &= a_1 \sum_{n_2=0}^{d_2} B_{\ell_1-1,\ell_2}(d_1,n_2,c) \\ &+ a_2 \sum_{n_2=0}^{d_1} B_{\ell_2,\ell_2-1}(d_1,d_2,c) \end{aligned}$$

which is (2.11).

APPENDIX B

COMPUTER PROGRAMS

The two main programs used in this research are S.O.R. and APPROX. The first program was used to calculate solutions to the model using Gauss-Seidel iteration with successive over-relaxation. For each value of the input parameters MT (the total mean offered to the common link) and KK (the number of junctions, c), five values of PM are considered. MT is proportioned into the two means, M1 and M2, of the offered streams. For each of these values thirty values of II and JJ (corresponding to d_1 and d_2) are considered and the corresponding arrival rates A1 and A2 are calculated such that the overflow from the direct links have means M1 and M2. For each value of (A1, A2, II, JJ and KK) the overflow means (OM1, OM2), variances (OV1, OV2) and covariance (COV) are calculated. The elements of the vector $\underset{\sim}{f}$ are calculated in the subroutine FGEN and the initial estimates of $\underset{\sim}{b}$ in EST. The vector b was stored in the array called X in the program. The iterative algorithm is performed in the subroutine SEID and the rows of the coefficient matrix A calculated as required in the subroutine and stored in the array A. The output from this program was stored on magnetic disc for later use.

The program APPROX used the data stored on disc as input. It has many subroutines, one to calculate the S.L.R. estimates (SLR), one to estimate the absolute errors (ERRCAL), one to plot graphs of the results and one for each of the approximate formulas considered. This program was used to generate results from which the tables of Chapters 6 and 7, and the graphs of Appendix C, were obtained.

Both programs were written in FORTRAN, for CDC 6000 or Cyber series computers, and use non standard, unformatted READ and PRINT statements. The plotting subroutine YPLOT (in the APPROX) uses a system of plotting subroutines, COMPLOT, available at the South Australian Institute of Technology and some other computer installations.

A modified version of SOR was used to generate the tables of Appendix D.

```
      PROGRAM SOR(INPUT,OUTPUT,DATA,TAPE3=DATA)
      DIMENSIONX(8000),F1(800),F2(800)
      DIMENSION PAR1(10),PAR2(10)
      REAL M1,M2,MT
      COMMON II,JJ,KK,IP,JP,KP,JM,KM,A1,A2,AT
C            ***************************************************************
C            *                                                             *
C            *  THIS PROGRAM SOLVES THE LINEAR EQUATIONS                   *
C            *  BY SEIDAL'S METHOD.                                        *
C            *                                                             *
C            *  THE METHOD USES THE RELATION                               *
C            *                                                             *
C            *             X = A*X + F                                     *
C            *                                                             *
C            *  IF A IS COSIDERED AS THE SUM OF A LOWER (L) AND AN UPPER   *
C            *  (U) TRIANGULAR MATRIX.I.E. A=L+U THEN                      *
C            *  STARTING WITH AN INITIAL ESTIMATE XO,XK IS CALCULATED BY   *
C            *                                                             *
C            *          XK =L*XK + U*XK-1 + F                              *
C            *                                                             *
C            *  THE ITERATION CONTINUES UNTIL THE LARGEST DIFFERENCE       *
C            *  BETWEEN THE K TH. AND K-1 TH. ESTIMATES IS LESS THAN A     *
C            *  GIVEN ERROR IN ABSOLUTE VALUE.(E.G. 1E-6)                  *
C            *                                                             *
C            *  THE PROCESS HAS BEEN ACCELERATED BY SUCCESSIVE             *
C            *  OVER-RELAXATION WITH S.O.R. CONSTANT =1.2 .                *
C            *                                                             *
C            ***************************************************************

  222 PRINT 14
   14 FORMAT(* DATA,PT1,PT2,KK  *)
      NP1=NP2=1
      READ *,PT1,PT2,KK
      IF(KK.EQ.0)GO TO 333
      CALL PARGEN(NP1,PT1,PAR1)
      IF(NP1.EQ.NP2.A.PT1.EQ.PT2)2,3
    2 DO 4 J=1,10
    4 PAR2(J)=PAR1(J)
      GO TO 5
    3 CALL PARGEN(NP2,PT2,PAR2)
    5 PRINT 40
   40 FORMAT(*1 DI   C    AI      OMI      OVI      CMI      OV/OM   CM/OM*,
     +*  ROM,ROV RCM,RVM   MI      VI      COV,RE   MT,VT*/)
      WRITE (3,49) KK,PT1,PT2
   49 FORMAT(1X,I3,2F8.4)
      DO 555 IX=2,8,3
      II=IX
      A1=PAR1(II)
      IF(A1.GT.0.)GO TO 500
      II=II+1
      A1=PAR1(II)
      IF(A1.LE.0.)GO TO 555
  500 JM=IP=II+1
      DO 554 JX=IP,10
      JJ=JX
      A2=PAR2(JJ)
      IF(A2.LE.0.)GO TO 554
      AT=A1+A2
      JP=JJ+1
      KP=KK+1
      KM=JM*JP
      NN=IP*JP*KP
      NM=IP*JP
```

```
      ERR=1E-6
      CALL EST(X,NN)
      CALL FGEN(F1,X,NN,NM,0)
      CALL SEID(X,F1,ERR,NN,NM,0)
      M1=A1*SUMJ(X,NN)
      M2=A2*SUMI(X,NN)
      MT=M1+M2
      CALL FGEN(F1,X,NN,NM,1)
      CALL FGEN(F2,X,NN,NM,2)
      CALL SEID(X,F1,ERR,NN,NM,1)
      V1=A1*SUMJ(X,NN)+M1-M1*M1
      COV=.5*A2*SUMI(X,NN)
      CALL SEID(X,F2,ERR,NN,NM,2)
      V2=A2*SUMI(X,NN)+M2-M2*M2
      COV=COV+.5*A1*SUMJ(X,NN)-M1*M2
      VT=V1+V2+2.*COV
      RE=2.*COV/VT*100.
      OM1=A1*E(II,A1)
      OM2=A2*E(JJ,A2)
      OV1=OM1*(1.-OM1+A1/(II+OM1-A1+1.))
      OV2=OM2*(1.-OM2+A2/(JJ+OM2-A2+1.))
      ROM=OM1/OM2
      ROV=OV1/OV2
      VM1=OV1/OM1
      VM2=OV2/OM2
      CM1=OM1-M1
      CM2=OM2-M2
      RCM=CM1/CM2
      CO1=CM1/OM1
      CO2=CM2/OM2
      RVM=VM1/VM2
      PRINT 41,II,KK,A1,OM1,OV1,CM1,VM1,CO1,ROM,RCM,M1,V1,COV,MT,
     +JJ,A2,OM2,OV2,CM2,VM2,CO2,ROV,RVM,M2,V2,RE,VT
  444 CONTINUE
   41 FORMAT(1X,2I3,12F8.4/1X,I3,3X,12F8.4/)
      WRITE (3,50) OV1,OV2,CM1,CM2,V1,V2,COV
   50 FORMAT(1X,7F8.4)
  554 CONTINUE
  555 CONTINUE
      PRINT 60
   60 FORMAT(10X,*ITERATIVE(S.O.R.) METHOD USED.*)
      ENDFILE 3
      GO TO 222
  333 CONTINUE
      PRINT 334
  334 FORMAT(1H ,23(1H*)/2H *,21X,1H*/24H * WARNING - PACK DATA *
     +/2H *,21X,1H*/1H ,23(1H*))
      END
      SUBROUTINE EST(X,NN)
      DIMENSION X(NN)
      COMMON II,JJ,KK,IP,JP,KP,JM,KM,A1,A2,AT
      U=1/NN
      DO 20 NR=1,NN
   20 X(NR)=U
      RETURN
      END

      SUBROUTINE FGEN(F,X,NN,NM,L)
      COMMON II,JJ,KK,IP,JP,KP,JM,KM,A1,A2,AT
      DIMENSION F(NM),X(NN)
       DO 1 N=1,NM
    1 F(N)=0.
      F(NM)=1.
      FT=0.
```

```
      JN=KK*KM
      IF(L-1)9,10,20
    9 RETURN
   10 DO 11 J=1,JP
      JL=J-1
      NR=IP+JL*JM
      XC=A1*X(NR+JN)
      F(NR)=XC
      FT=FT+XC
   11 CONTINUE
      F(NM)=FT
      RETURN
   20 DO 21 I=1,IP
      NR=I+JJ*JM
      XC=A2*X(NR+JN)
      F(NR)=XC
      FT=FT+XC
   21 CONTINUE
      F(NM)=FT
      RETURN
      END

      SUBROUTINE SEID(X,F,ERR,NN,NM,L)
      COMMON II,JJ,KK,IP,JP,KP,JM,KM,A1,A2,AT
      DIMENSION A(7)
      DIMENSION X(NN),NPOS(7),F(NM)
      NIT=0
      LL=(L+1)/2
   11 CONTINUE
      E=XS=0.
      DO 2 K=1,KP
      KL=K-1
      DO 2 J=1,JP
      JL=J-1
      DO2 I=1,IP
      IL=I-1
      NR=I+JL*JM+KL*KM
      DIAG=AT+IL+JL+KL+LL
      DO 3 M=1,7
    3 A(M)=0.
      IF(J.GT.1)A(2)=A2
      IF(I.GT.1)A(3)=A1
      IF(I.LT.IP)A(5)=I
      IF(J.LT.JP)A(6)=J
      IF(K.LT.KP)A(7)=K
      IF(J.EQ.JP)A(1)=A2
      IF(I.EQ.IP)A(1)=A(1)+A1
      IF(K.EQ.KP)A(4)=A(1)
      IF(K.EQ.1)A(1)=0
      NPOS(1)=NR-KM
      NPOS(2)=NR-JM
      NPOS(3)=NR-1
      NPOS(4)=NR
      NPOS(5)=NR+1
      NPOS(6)=NR+JM
      NPOS(7)=NR+KM
      XT=0.
      DO 5 M=1,7
      IF(A(M).EQ.0.)GOTO5
      NP =NPOS(M)
      XT=XT+A(M)*X(NP)
    5 CONTINUE
      NF=NR-KK*KM
      IF(NF.GT.0)XT=XT+F(NF)
```

```
      XT=XT/DIAG
      XT=XT*1.2-.2*X(NR)
      IF(NR.EQ.NN)XT=F(NM)-XS
      XS=XS+XT
      EN=ABS(XT-X(NR))
      X(NR)=XT
      IF(EN.LT.E)GO TO 2
      E=EN
      NE=NR
    2 CONTINUE
      NIT=NIT+1
      IF(E.LT.ERR)12,11
   12 CONTINUE
   23 FORMAT(1H0,* ERROR*F10.7* AFTER*I4* ITERATIONS AT *I4)
      RETURN $ END

      FUNCTION SUMI(X,NN)
      DIMENSION X(NN)
      COMMON II,JJ,KK,IP,JP,KP,JM,KM,A1,A2,AT
      SUMI=0.
      DO 1 I=1,IP
      NR=I+JJ*JM+KK*KM
    1 SUMI=SUMI+X(NR)
      RETURN
      ENTRY SUMJ
      SUMI=0.
      DO 2 J=1,JP
      NR=IP+(J-1)*JM+KK*KM
    2 SUMI=SUMI+X(NR)
      RETURN $ END

      FUNCTION E(N,A)
      E=1.
      IF(N.EQ.0)RETURN
      DO 1 I=1,N
      AE=A*E
    1 E=AE/(I+AE)
      RETURN $ END

      SUBROUTINE PARGEN(NPAR,PT,PAR)
      DIMENSION PAR(10)
      IF(PT.EQ.0)RETURN
      DO 106 J=1,10
      Y=PT+J
      IF(NPAR.EQ.2)Y=2*J
      X=Y/2.
      CALL CALC(X,J,PX,NPAR)
      CALL CALC(Y,J,PY,NPAR)
  102 DT=PY-PX
      EX=PT-PX
      IF(ABS(EX).LT.1E-5)GOTO104
      Z=X+(Y-X)*EX/DT
       IF(Z.LT.0)GO TO 205
      IF(Z.GT.40)GO TO 205
      CALL CALC(Z,J,PZ,NPAR)
      IF(PZ.GT.PT)GOTO 103
      X=Z $ PX=PZ
      GOTO 102
  103 Y=Z $ PY=PZ
      GOTO 102
  205 X=0.
  104 PAR(J)=X
  106 CONTINUE
      RETURN $ END
```

```
SUBROUTINE CALC(A,N,P,NP)
P=A*E(N,A)
IF(NP.EQ.1)RETURN
Q=P
P=1-Q+A/(N+1.-A+Q)
IF(NP.EQ.2)RETURN
P=Q*P
RETURN $ END
```

```
      PROGRAM APPRX(INPUT,OUTPUT,DATA,TAPE3=DATA)
      REAL M(9,2),MT(9),V(9,15,2),VT(9,15),C(9)
      REAL CM(9,15,2),CMT(9,15),OM(9,15,2),OMT(9,15)
      REAL OV(9,15,2),OVP(9,15),OVT(9,15),COV(9,15)
      REAL PM(9,2),PV(9,15,2),PCM(9,15,2),POM(9,15,2)
      REAL POVP(9,15,2),POVT(9,15,2),PCV(9,15)
      COMMON M,MT,V,VT,C,CM,CMT,OM,OMT,OV,OVP,OVT,COV,
     ,          PM,PV,PCM,POM,POVP,POVT,PCV,IMAX
      REWIND 3
      DO 10 I=1,10
      IMAX=I-1
      READ (3,49) C(I),M(I,1),M(I,2)
   49 FORMAT(F4,2F8.4)
      IF(EOF(3).NE.0)GO TO 11
      MT(I)=M(I,1)+M(I,2)
      DO 10 J=1,15
      READ (3,50) V(I,J,1),V(I,J,2),CM(I,J,1),CM(I,J,2),
     ,          OV(I,J,1),OV(I,J,2),COV(I,J)
   50 FORMAT(1X,7F8.4)
      VT(I,J)=V(I,J,1)+V(I,J,2)
      CMT(I,J)=CM(I,J,1)+CM(I,J,2)
      OMT(I,J)=MT(I)-CMT(I,J)
      OVP(I,J)=OV(I,J,1)+OV(I,J,2)
      OVT(I,J)=OVP(I,J)+2.*COV(I,J)
      DO 12 K=1,2
      OM(I,J,K)=M(I,K)-CM(I,J,K)
      PM(I,K)=M(I,K)/MT(I)
      PV(I,J,K)=V(I,J,K)/VT(I,J)
      PCM(I,J,K)=CM(I,J,K)/CMT(I,J)
      POM(I,J,K)=OM(I,J,K)/OMT(I,J)
      POVP(I,J,K)=OV(I,J,K)/OVP(I,J)
      POVT(I,J,K)=OV(I,J,K)/OVT(I,J)
      PCV(I,J)=COV(I,J)/OVT(I,J)
   12 CONTINUE
   10 CONTINUE
   11 CONTINUE
*******************************************************************
*
*  INSERT SUBROUTINE CALLS HERE
*
*******************************************************************
      STOP
      END

      SUBROUTINE SLR(PM,PV,PX,IMAX,ALPHA,BETA)
      REAL PM(9,2),PV(9,15,2),PX(9,15,2),ALPHA(9,2),BETA(9,2)
      DO 1 IA=1,IMAX
      IB=IA
      IF(IA.GT.1)IB=IA+4
      IF(IA.GT.5)IB=IA-4
      SX=SV=SXX=SXV=0.
      IF(I.GT.5)IB=I-4
      DO 2 J=1,15
      SX=SX+PX(IA,J,1)+PX(IB,J,2)
      SV=SV+PV(IA,J,1)+PV(IB,J,2)
      SXX=SXX+PX(IA,J,1)**2+PX(IB,J,2)**2
      SXV=SXV+PX(IA,J,1)*PV(IA,J,1)+PX(IB,J,2)*PV(IB,J,2)
    2 CONTINUE
      VB=SV/30.
      XB=SX/30.
      THETA=(SXV-SX*VB)/(SXX-SX*XB)
      ALPHA(IA,1)=ALPHA(IB,2)=-1./THETA
      BETA(IA,1)=BETA(IB,2)=PM(IA,1)-VB-(PM(IA,1)-XB)*THETA
```

```
 1 CONTINUE
   RETURN
   END

   SUBROUTINE WALSTM
   REAL Y(9,15,2),ERM(9,2),ERP(9,2)
   REAL M(9,2),MT(9),V(9,15,2),VT(9,15),C(9)
   REAL CM(9,15,2),CMT(9,15),OM(9,15,2),OMT(9,15)
   REAL OV(9,15,2),OVP(9,15),OVT(9,15),COV(9,15)
   REAL PM(9,2),PV(9,15,2),PCM(9,15,2),POM(9,15,2)
   REAL POVP(9,15,2),POVT(9,15,2),PCV(9,15)
   REAL BL(9,15)
   COMMON M,MT,V,VT,C,CM,CMT,OM,OMT,OV,OVP,OVT,COV,
  ,         PM,PV,PCM,POM,POVP,POVT,PCV,IMAX
   DO 1 I=1,IMAX
   DO 1 J=1,15
   B=OMT(I,J)/MT(I)
   BL(I,J)=B
   DO 1 K=1,2
   Y(I,J,K)=PV(I,J,K)*(1.-B)+PM(I,K)*B
 1 CONTINUE
   CALL ERRCAL(Y,POM,IMAX,ERM,ERP)
   PRINT 2
 2 FORMAT(/*WALLSTROM'S APPROXIMATION FOR THE MEANS*/)
   PRINT 3
 3 FORMAT(*    C   MT    PM   (-VE) ERRORS (+VE)*/)
   PRINT 4,((C(I),MT(I),PM(I,K),ERM(I,K),ERP(I,K),I=1,IMAX),K=1,1)
 4 FORMAT(2F4,F6.4,2F8.4)

   PRINT *,"PLOT WALLSTROM'S FORMULA ?"
   READ *,NAX
   IF(NAX.GE.0)CALL YPLT(Y,POM,IMAX,NAX,13HOVERFLOW MEAN,-13,
  ,         19HWALLSTROM'S FORMULA,19)
   RETURN
   END

   SUBROUTINE OLSSON
   REAL Y(9,15,2),ERM(9,2),ERP(9,2)
   REAL M(9,2),MT(9),V(9,15,2),VT(9,15),C(9)
   REAL CM(9,15,2),CMT(9,15),OM(9,15,2),OMT(9,15)
   REAL OV(9,15,2),OVP(9,15),OVT(9,15),COV(9,15)
   REAL PM(9,2),PV(9,15,2),PCM(9,15,2),POM(9,15,2)
   REAL POVP(9,15,2),POVT(9,15,2),PCV(9,15)
   REAL T(2)
   COMMON M,MT,V,VT,C,CM,CMT,OM,OMT,OV,OVP,OVT,COV,
  ,         PM,PV,PCM,POM,POVP,POVT,PCV,IMAX
77 CONTINUE
   READ *,FAC
   IF(ABS(FAC).LT.1E-4)RETURN
   DO 1 I=1,IMAX
   DO 1 J=1,15
   DO 22 K=1,2
   T(K)=V(I,J,K)+FA* *M(I,K)**2/V(I,J,K)
22 CONTINUE
   TSUM=T(1)+T(2)
   DO 1 K=1,2
   Y(I,J,K)=T(K)/TSUM
 1 CONTINUE
   CALL ERRCAL(Y,POM,IMAX,ERM,ERP)
   PRINT 2
 2 FORMAT(/*OLSSON'S APPROXIMATION FOR THE MEANS*/)
   PRINT 3
 3 FORMAT(*    C   MT    PM   (-VE) ERRORS (+VE)*/)
   PRINT 4,((C(I),MT(I),PM(I,K),ERM(I,K),ERP(I,K),I=1,IMAX),K=1,1)
```

```
    4 FORMAT(2F4,F6.4,2F8.4)
      GO TO 77

      PRINT *,"PLOT OLSSON'S FORMULA ?"
      READ *,NAX
      IF(NAX.GE.O)CALL YPLT(Y,POM,IMAX,NAX,13HOVERFLOW MEAN,-13,
     ,          16HOLSSON'S FORMULA,16)
      GO TO 77
      RETURN
      END

      SUBROUTINE HARRIS
      REAL Y(9,15,2),ERM(9,2),ERP(9,2)
      REAL M(9,2),MT(9),V(9,15,2),VT(9,15),C(9)
      REAL CM(9,15,2),CMT(9,15),OM(9,15,2),OMT(9,15)
      REAL OV(9,15,2),OVP(9,15),OVT(9,15),COV(9,15)
      REAL PM(9,2),PV(9,15,2),PCM(9,15,2),POM(9,15,2)
      REAL POVP(9,15,2),POVT(9,15,2),PCV(9,15)
      COMMON M,MT,V,VT,C,CM,CMT,OM,OMT,OV,OVP,OVT,COV,
     ,          PM,PV,PCM,POM,POVP,POVT,PCV,IMAX
      DO 1 I=1,IMAX
      DO 1 J=1,15
      DO 1 K=1,2
      RHO=PV(I,J,K)
      T=RHO+(1.-RHO)*EXP(-RHO*C(I))
      Y(I,J,K)=RHO*(T*(OVT(I,J)-OMT(I,J))+OMT(I,J))/OVT(I,J)
    1 CONTINUE
      CALL ERRCAL(Y,POVT,IMAX,ERM,ERP)
      PRINT 2
    2 FORMAT(/*HARRIS'S APPROXIMATION FOR THE VARIANCES*/)
      PRINT 3
    3 FORMAT(*    C   MT    PM   (-VE) ERRORS (+VE)*/)
      PRINT 4,((C(I),MT(I),PM(I,K),ERM(I,K),ERP(I,K),I=1,IMAX),K=1,2)
    4 FORMAT(2F4,F6.4,2F8.4)

      PRINT *,"PLOT HARRIS'S FORMULA ?"
      READ *,NAX
      IF(NAX.GE.O)CALL YPLT(Y,POVT,IMAX,NAX,17HOVERFLOW VARIANCE,-17,
     ,          16HHARRIS'S FORMULA,16)
      RETURN
      END

      SUBROUTINE SLROM
      REAL Y(9,15,2),ERM(9,2),ERP(9,2)
      REAL M(9,2),MT(9),V(9,15,2),VT(9,15),C(9)
      REAL CM(9,15,2),CMT(9,15),OM(9,15,2),OMT(9,15)
      REAL OV(9,15,2),OVP(9,15),OVT(9,15),COV(9,15)
      REAL PM(9,2),PV(9,15,2),PCM(9,15,2),POM(9,15,2)
      REAL POVP(9,15,2),POVT(9,15,2),PCV(9,15)
      COMMON M,MT,V,VT,C,CM,CMT,OM,OMT,OV,OVP,OVT,COV,
     ,          PM,PV,PCM,POM,POVP,POVT,PCV,IMAX
      REAL A(9,2),B(9,2)
      CALL SLR(PM,PV,POM,IMAX,A,B)
      DO 1 I=1,IMAX
      DO 1 J=1,15
      DO 1 K=1,2
      Y(I,J,K)=PM(I,K)+A(I,K)*(PM(I,K)-PV(I,J,K)-B(I,K))
    1 CONTINUE
      CALL ERRCAL(Y,POM,IMAX,ERM,ERP)
      PRINT 2
    2 FORMAT(/*S.L.R. APPROXIMATION FOR THE MEANS*/)
      PRINT 3
    3 FORMAT(*    C   MT    PM   (-VE) ERRORS (+VE)   A         B*/)
      PRINT 4,((C(I),MT(I),PM(I,K),ERM(I,K),ERP(I,K),A(I,K),B(I,K),I=1,I
```

```
 ,MAX)      ,K=1,1)
4 FORMAT(2F4,F6.4,4F8.4)

  PRINT *,"PLOT SLR APPROX. FOR MEANS ?"
  READ *,NAX
  IF(NAX.GE.0)CALL YPLT(Y,POM,IMAX,NAX,13HOVERFLOW MEAN,-13,
 ,         19HS.L.R.APPROXIMATION,19)
  RETURN
  END

  SUBROUTINE SLROVT
  REAL Y(9,15,2),ERM(9,2),ERP(9,2)
  REAL M(9,2),MT(9),V(9,15,2),VT(9,15),C(9)
  REAL CM(9,15,2),CMT(9,15),OM(9,15,2),OMT(9,15)
  REAL OV(9,15,2),OVP(9,15),OVT(9,15),COV(9,15)
  REAL PM(9,2),PV(9,15,2),PCM(9,15,2),POM(9,15,2)
  REAL POVP(9,15,2),POVT(9,15,2),PCV(9,15)
  COMMON M,MT,V,VT,C,CM,CMT,OM,OMT,OV,OVP,OVT,COV,
 ,         PM,PV,PCM,POM,POVP,POVT,PCV,IMAX
  REAL A(9,2),B(9,2)
  CALL SLR(PM,PV,POVT,IMAX,A,B)
  DO 1 I=1,IMAX
  DO 1 J=1,15
  DO 1 K=1,2
  Y(I,J,K)=PM(I,K)+A(I,K)*(PM(I,K)-PV(I,J,K)-B(I,K))
1 CONTINUE
  CALL ERRCAL(Y,POVT,IMAX,ERM,ERP)
  PRINT 2
2 FORMAT(/*S.L.R. APPROXIMATION FOR THE VARIANCES*/)
  PRINT 3
3 FORMAT(*   C   MT    PM   (-VE) ERRORS (+VE)   A        B*/)
  PRINT 4,((C(I),MT(I),PM(I,K),ERM(I,K),ERP(I,K),A(I,K),B(I,K),I=1,I
 ,MAX)      ,K=1,2)
4 FORMAT(2F4,F6.4,4F8.4)

  PRINT *,"PLOT SLR APPROX. FOR VARS.?"
  READ *,NAX
  IF(NAX.GE.0)CALL YPLT(Y,POVT,IMAX,NAX,17HOVERFLOW VARIANCE,-17,
 ,         19HS.L.R.APPROXIMATION,19)
  RETURN
  END

  SUBROUTINE ERRCAL(Y,X,IMAX,ERM,ERP)
  REAL X(9,15,2),Y(9,15,2),ERM(9,2),ERP(9,2)
  REAL Z(9,15,2)
  DO 1 I=1,IMAX
  DO 1 K=1,2
  EM=EP=0.
  DO 2 J=1,15
  ER=Y(I,J,K)-X(I,J,K)
  Z(I,J,K)=ER*10000
  EP=AMAX1(EP,ER)
  EM=AMIN1(EM,ER)
2 CONTINUE
  ERM(I,K)=EM
  ERP(I,K)=EP
1 CONTINUE
  PRINT 5,(((Z(I,J,K),I=1,9),K=1,2),J=1,15)
5 FORMAT(9F6,5X,9F6)
  RETURN
  END

  SUBROUTINE ERMETH
  REAL M(9,2),MT(9),V(9,15,2),VT(9,15),C(9)
```

```
      REAL CM(9,15,2),CMT(9,15),OM(9,15,2),OMT(9,15)
      REAL OV(9,15,2),OVP(9,15),OVT(9,15),COV(9,15)
      REAL PM(9,2),PV(9,15,2),PCM(9,15,2),POM(9,15,2)
      REAL POVP(9,15,2),POVT(9,15,2),PCV(9,15)
      COMMON M,MT,V,VT,C,CM,CMT,OM,OMT,OV,OVP,OVT,COV,
     ,        PM,PV,PCM,POM,POVP,POVT,PCV,IMAX
      PRINT 4
    4 FORMAT(/*E.R.APPROXIMATION*)
      PRINT 2
      EMM=EMP=EVM=EVP=1.
    2 FORMAT(/*    C   MT    PM   -VE MEAN +VE     -VE VAR. +VE*)
      DO 1 I=1,IMAX
      DO 11 J=1,15
      TV=VT(I,J)
      TM=MT(I)
      VM=TV/TM
      AE=TV+3.*VM*(VM-1.)
      DE=(AE*(TM+TV)/(TM+TV-1.))-TM-1
      EM=AE*E(C(I)+DE,AE)
      EV=EM*(1.-EM+AE/(C(I)+DE-AE+EM+1.))
      RM=EM/OMT(I,J)
      RV=EV/OVT(I,J)
      EMP=AMAX1(EMP,RM)
      EMM=AMIN1(EMM,RM)
      EVP=AMAX1(EVP,RV)
      EVM=AMIN1(EVM,RV)
   11 CONTINUE
      PRINT 3,C(I),MT(I),PM(I,1),EMM,EMP,EVM,EVP
    3 FORMAT(2F4,5F8.4)
    1 CONTINUE
      RETURN
      END

      SUBROUTINE OFFMN
      REAL Y(9,15,2),ERM(10,2),ERP(10,2)
      REAL M(9,2),MT(9),V(9,15,2),VT(9,15),C(9)
      REAL CM(9,15,2),CMT(9,15),OM(9,15,2),OMT(9,15)
      REAL OV(9,15,2),OVP(9,15),OVT(9,15),COV(9,15)
      REAL PM(9,2),PV(9,15,2),PCM(9,15,2),POM(9,15,2)
      REAL POVP(9,15,2),POVT(9,15,2),PCV(9,15)
      COMMON M,MT,V,VT,C,CM,CMT,OM,OMT,OV,OVP,OVT,COV,
     ,        PM,PV,PCM,POM,POVP,POVT,PCV,IMAX
      PRINT 2
      PRINT 3
      DO 1 I=1,IMAX
      DO 1 J=1,15
      DO 1 K=1,2
      Y(I,J,K)=PM(I,K)
    1 CONTINUE
      CALL ERRCAL(Y,POM,IMAX,ERM,ERP)
    2 FORMAT(/*OFFERED MEAN APPROXIMATION FOR THE MEANS*/)
    3 FORMAT(*    C   MT    PM      (-VE) ERRORS (+VE)*/)
      PRINT 4,((C(I),MT(I),PM(I,K),ERM(I,K),ERP(I,K),I=1,IMAX),K=1,1)
    4 FORMAT(2F4,3F8.4)

      PRINT *,"PLOT OFFERED MEAN ?"
      READ *,NAX
      IF(NAX.GE.0)CALL YPLT(Y,POM,IMAX,NAX,13HOVERFLOW MEAN,-13,
     ,        12HOFFERED MEAN,12)
      RETURN
      END

      SUBROUTINE OFFVAR
      REAL Y(9,15,1),ERM(10,2),ERP(10,2)
```

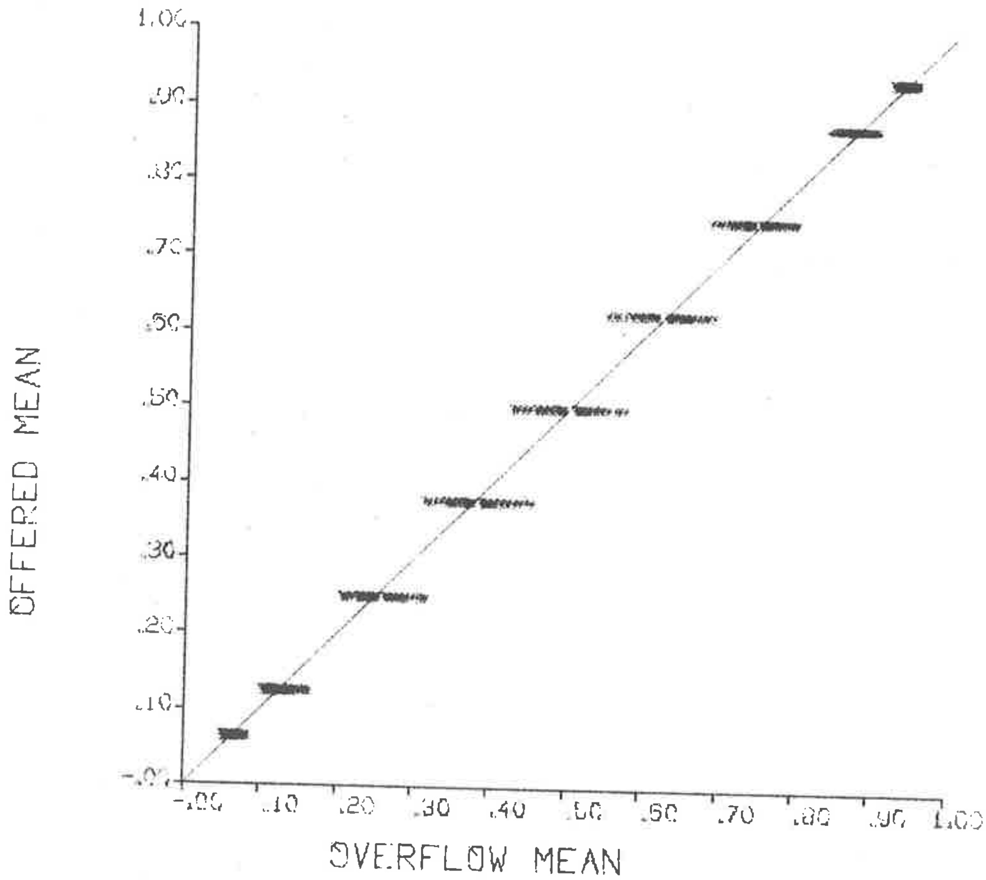
```
      REAL M(9,2),MT(9),V(9,15,2),VT(9,15),C(9)
      REAL CM(9,15,2),CMT(9,15),OM(9,15,2),OMT(9,15)
      REAL OV(9,15,2),OVP(9,15),OVT(9,15),COV(9,15)
      REAL PM(9,2),PV(9,15,2),PCM(9,15,2),POM(9,15,2)
      REAL POVP(9,15,2),POVT(9,15,2),PCV(9,15)
      COMMON M,MT,V,VT,C,CM,CMT,OM,OMT,OV,OVP,OVT,COV,
     ,          PM,PV,PCM,POM,POVP,POVT,PCV,IMAX
      CALL ERRCAL(PV,POM,IMAX,ERM,ERP)
      PRINT 2
    2 FORMAT(/*OFFERED VAR. APPROXIMATION FOR THE MEANS*/)
      PRINT 3
    3 FORMAT(*    C   MT    PM     (-VE) ERRORS (+VE)*/)
      PRINT 4,((C(I),MT(I),PM(I,K),ERM(I,K),ERP(I,K),I=1,IMAX),K=1,2)
    4 FORMAT(2F4,3F8.4)

      PRINT *,"PLOT OFFERED VARIANCE ?"
      READ *,NAX
      IF(NAX.GE.0)CALL YPLT(PV,POM,IMAX,NAX,13HOVERFLOW MEAN,-13,
     ,          16HOFFERED VARIANCE,16)
      RETURN
      END

      FUNCTION E(D,A)
      E=1.
      IF(D.EQ.0.)RETURN
      NL=D
      NP=NL+1
      DO 1 I=1,NL
      AE=A*E
      E=AE/(I+AE)
    1 CONTINUE
      EP=A*E/(NP+A*E)
      E=E+(D-NL)*(EP-E)
      RETURN
      END

      SUBROUTINE YPLT(Y,X,IMAX,NAX,IX,NX,IY,NY)
      REAL X(9,15,2),Y(9,15,2)
      REAL XX(270),YY(270)
      DO 1 I=1,IMAX
      DO 1 J=1,15
      DO 1 K=1,2
      N=J+(I-1)*15+(K-1)*IMAX*15
      XX(N)=X(I,J,K)
      YY(N)=Y(I,J,K)
    1 CONTINUE
      CALL PLOTS(7HTEK4010)
      IF(NAX.EQ.0)GO TO 2
      CALL PLOT(0.,5.,-3)
      DO 7 L=1,4
      CALL AXIS(0.,0.,IX,NX,10.,0.,0.,.1,0)
      CALL AXIS(0.,0.,IY,NY,10.,90.,0.,.1,-1)
    7 CONTINUE
    2 NMAX=IMAX*30
      DO 3 N=1,NMAX
    3 CALL NSSYMB(10.*XX(N),10.*YY(N),.01,7,0.)
      CALL PLOT(0.,0.,3)
      CALL PLOT(10.,10.,2)
      CALL PLOTE
      RETURN
      END

      SUBROUTINE OFFMNV
      REAL Y(9,15,2),ERM(10,2),ERP(10,2)
```

```
      REAL M(9,2),MT(9),V(9,15,2),VT(9,15),C(9)
      REAL CM(9,15,2),CMT(9,15),OM(9,15,2),OMT(9,15)
      REAL OV(9,15,2),OVP(9,15),OVT(9,15),COV(9,15)
      REAL PM(9,2),PV(9,15,2),PCM(9,15,2),POM(9,15,2)
      REAL POVP(9,15,2),POVT(9,15,2),PCV(9,15)
      COMMON M,MT,V,VT,C,CM,CMT,OM,OMT,OV,OVP,OVT,COV,
     ,          PM,PV,PCM,POM,POVP,POVT,PCV,IMAX
      PRINT 2
      PRINT 3
      DO 1 I=1,IMAX
      DO 1 J=1,15
      DO 1 K=1,2
      Y(I,J,K)=PM(I,K)
    1 CONTINUE
      CALL ERRCAL(Y,POVT,IMAX,ERM,ERP)
    2 FORMAT(/*OFFERED MEAN APPROXIMATION FOR THE VARIANCES*/)
    3 FORMAT(*   C  MT    PM     (-VE) ERRORS (+VE)*/)
      PRINT 4,((C(I),MT(I),PM(I,K),ERM(I,K),ERP(I,K),I=1,IMAX),K=1,1)
    4 FORMAT(2F4,3F8.4)

      PRINT *,"PLOT OFFERED MEAN ?"
      READ *,NAX
      IF(NAX.GE.0)CALL YPLT(Y,POVT,IMAX,NAX,17HOVERFLOW VARIANCE,-17,
     ,          12HOFFERED MEAN,12)
      RETURN
      END

      SUBROUTINE OFFVARV
      REAL Y(9,15,1),ERM(10,2),ERP(10,2)
      REAL M(9,2),MT(9),V(9,15,2),VT(9,15),C(9)
      REAL CM(9,15,2),CMT(9,15),OM(9,15,2),OMT(9,15)
      REAL OV(9,15,2),OVP(9,15),OVT(9,15),COV(9,15)
      REAL PM(9,2),PV(9,15,2),PCM(9,15,2),POM(9,15,2)
      REAL POVP(9,15,2),POVT(9,15,2),PCV(9,15)
      COMMON M,MT,V,VT,C,CM,CMT,OM,OMT,OV,OVP,OVT,COV,
     ,          PM,PV,PCM,POM,POVP,POVT,PCV,IMAX
      CALL ERRCAL(PV,POVT,IMAX,ERM,ERP)
      PRINT 2
    2 FORMAT(/*OFFERED VAR. APPROXIMATION FOR THE VARIANCES*/)
      PRINT 3
    3 FORMAT(*   C  MT    PM     (-VE) ERRORS (+VE)*/)
      PRINT 4,((C(I),MT(I),PM(I,K),ERM(I,K),ERP(I,K),I=1,IMAX),K=1,2)
    4 FORMAT(2F4,3F8.4)

      PRINT *,"PLOT OFFERED VARIANCE ?"
      READ *,NAX
      IF(NAX.GE.0)CALL YPLT(PV,POVT,IMAX,NAX,17HOVERFLOW VARIANCE,-17,
     ,          16HOFFERED VARIANCE,16)
      RETURN
      END
```

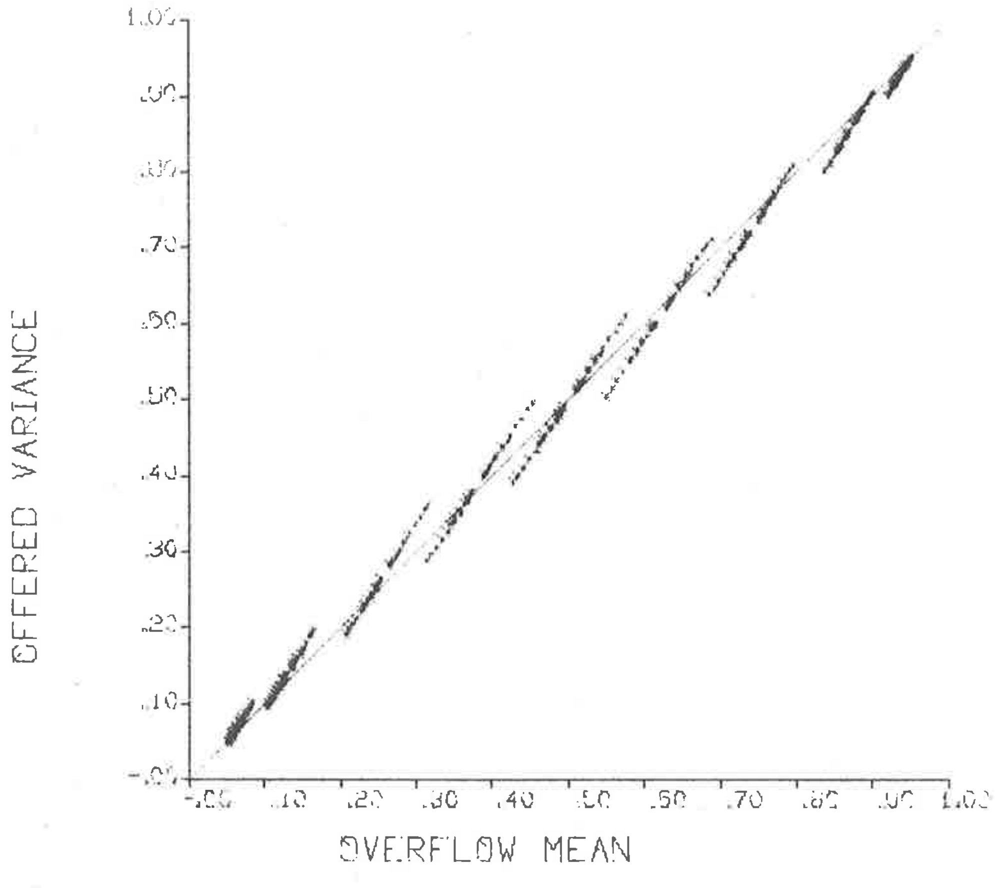
APPENDIX C

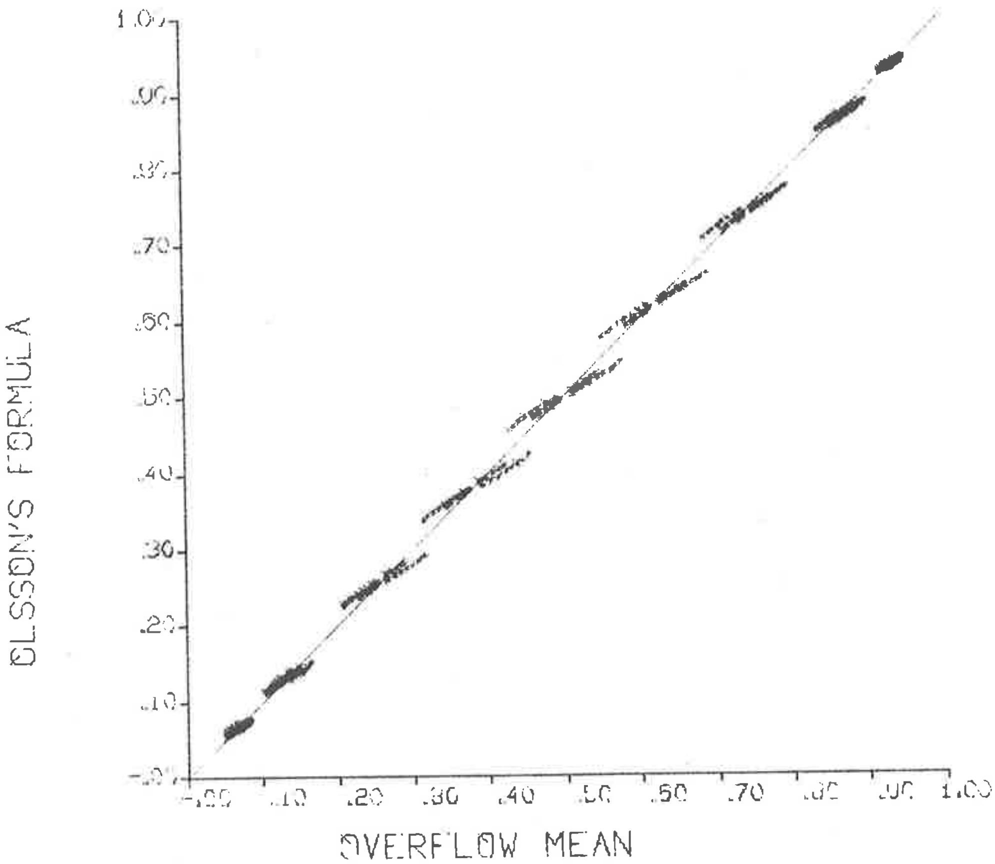
GRAPHS OF THE APPROXIMATE FORMULAS

The graphs presented in this appendix all correspond to the case M=8, c=8. The value of P_m or P_v as calculated by the program SOR is plotted in the x direction and the estimate of this value, as calculated by the approximation indicated by the labelling, in the y direction. The 'points' are represented by asterisks one m.m. in height.

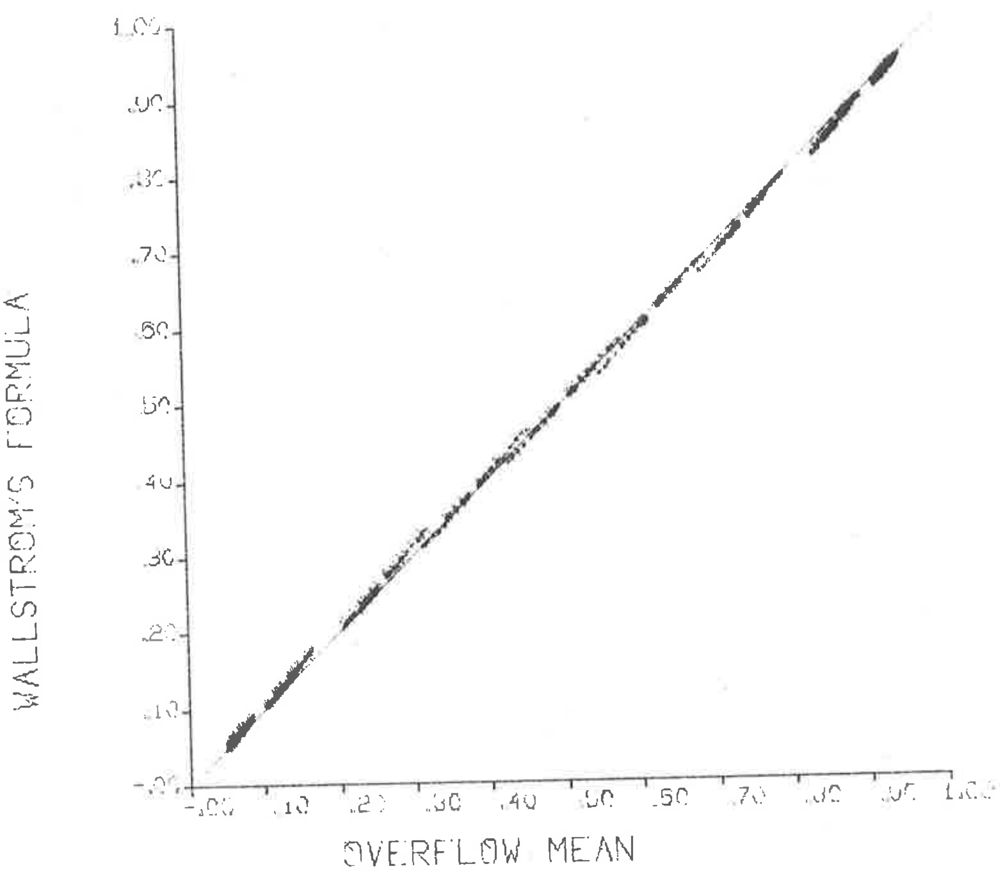


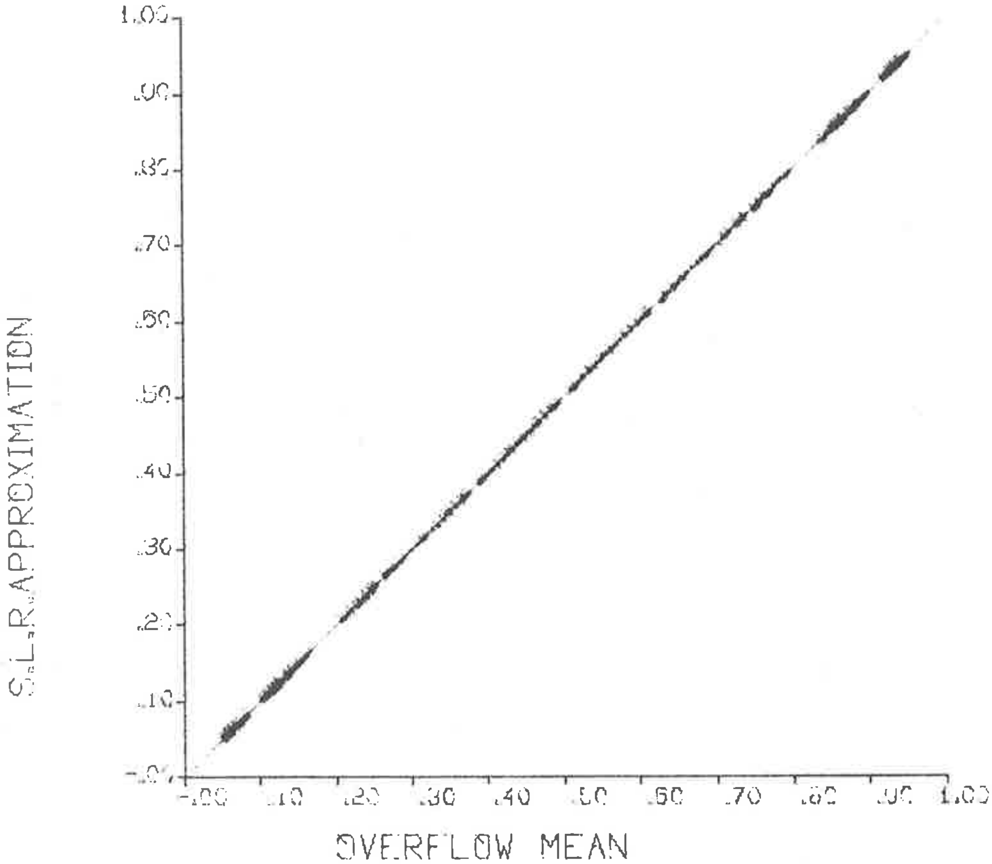
OFFERED MEAN
OVERFLOW MEAN
1.00
.90
.80
.70
.60
.50
.40
.30
.20
.10
-.00
.10
.20
.30
.40
.50
.60
.70
.80
.90
1.00

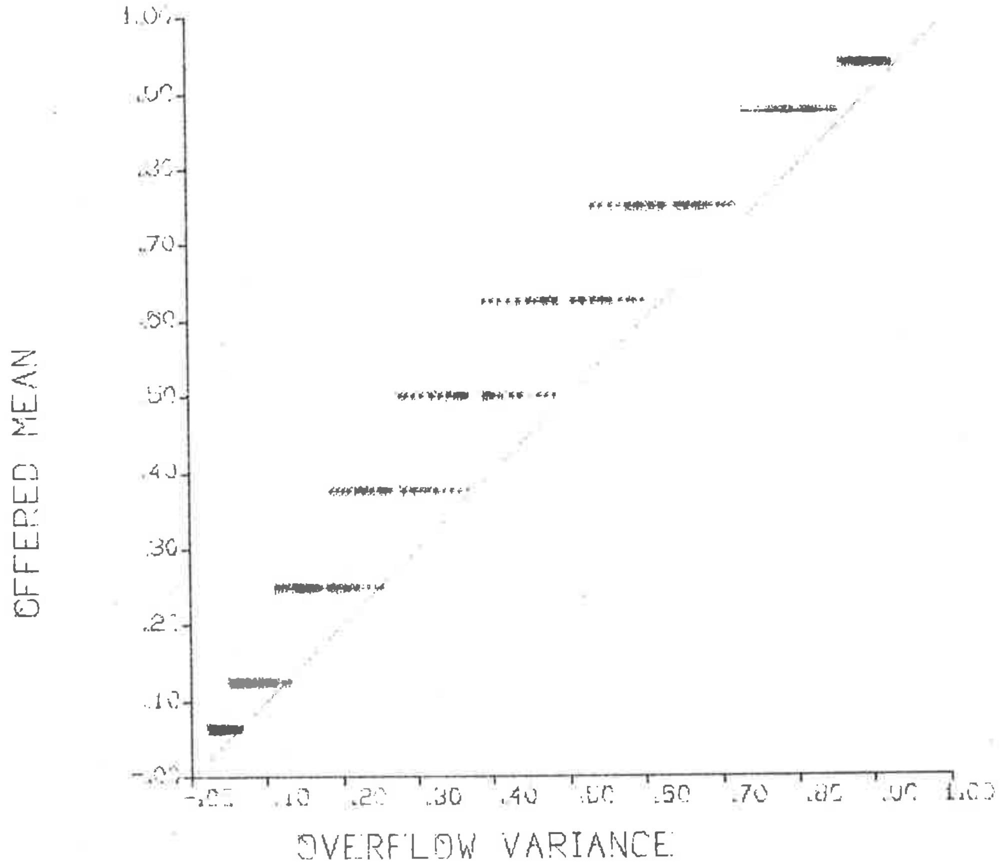

OFFERED VARIANCE
OVERFLOW MEAN

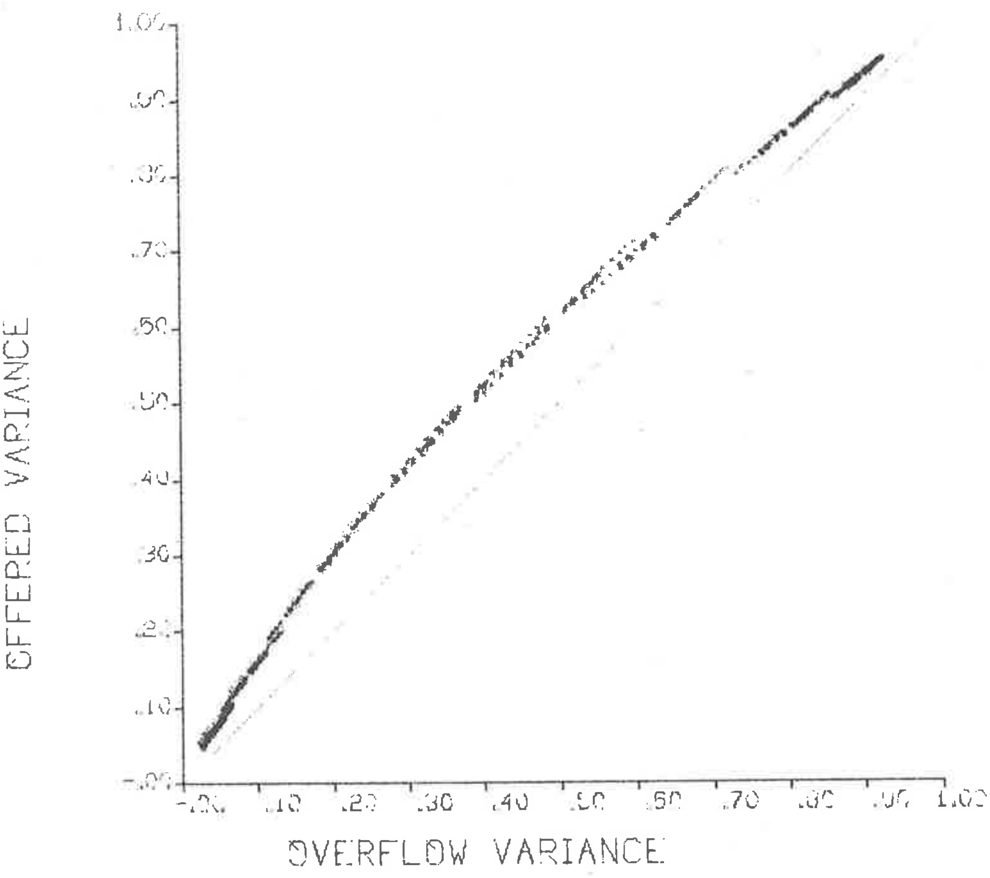


OLSSON'S FORMULA
OVERFLOW MEAN

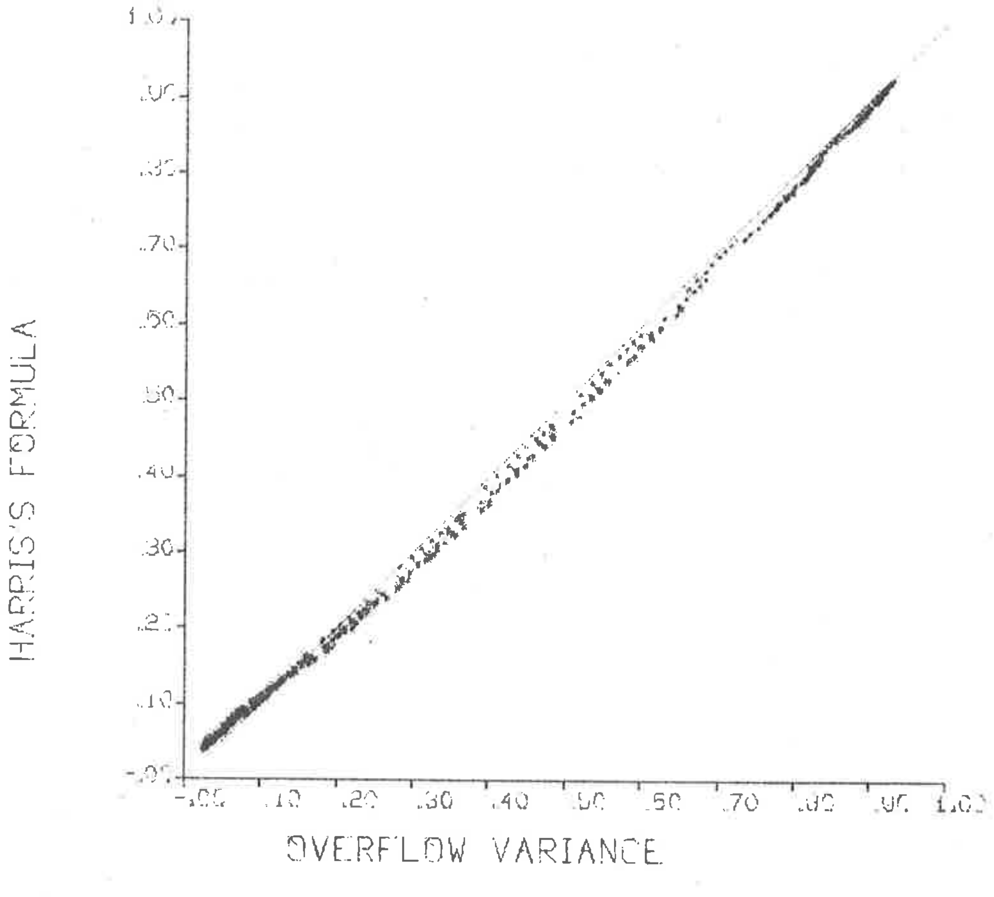

1.00
.90
.80
.70
.60
.50
.40
.30
.20
.10
.00
.10
.20
.30
.40
.50
.60
.70
.80
.90
1.00
WALLSTROM'S FORMULA
OVERFLOW MEAN

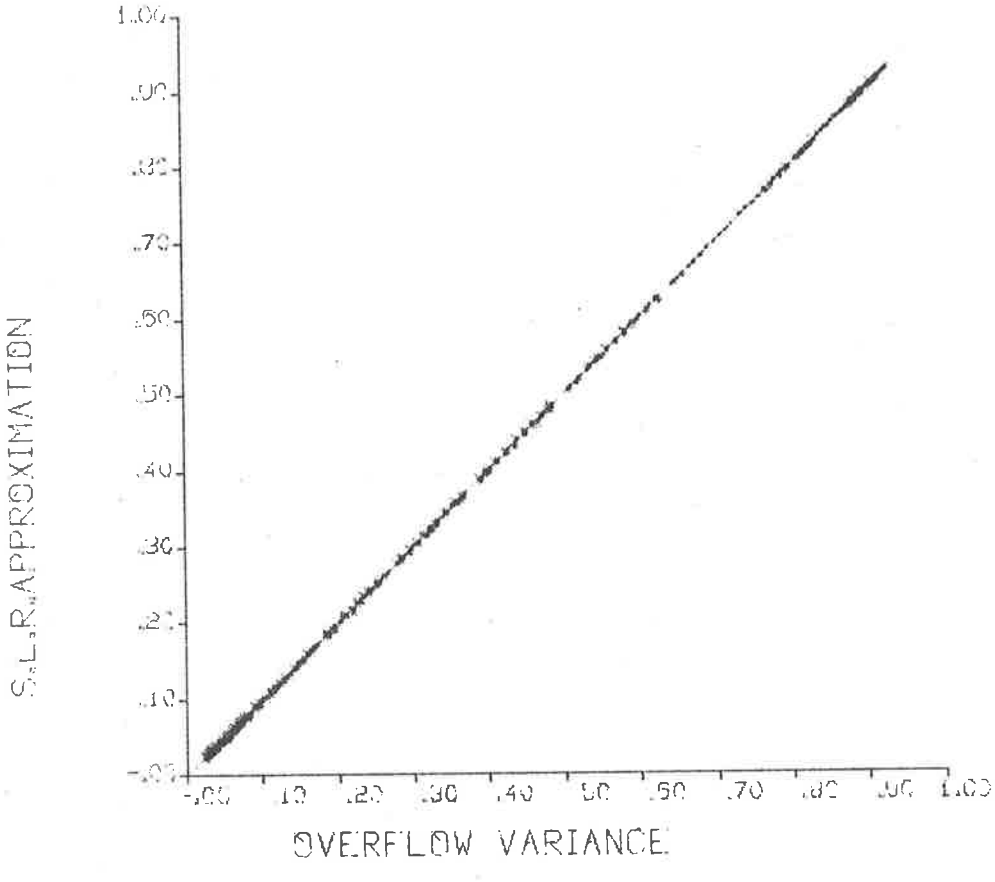

S.L.R.APPROXIMATION
OVERFLOW MEAN


OFFERED MEAN
OVERFLOW VARIANCE



OFFERED VARIANCE
OVERFLOW VARIANCE


HARRIS'S FORMULA
OVERFLOW VARIANCE


S.L.R.APPROXIMATION
OVERFLOW VARIANCE

APPENDIX D

TABLES FOR THE S.L.R. APPROXIMATION

Each table corresponds to a single value of c (the number of secondary group servers). Seven values of total offered mean are considered and these are listed under MT. The total mean is split into proportions corresponding to .1, .2, ..., .9, as indicated by PM. Since α_m has the same value at p and 1-p only values of PM $\leq$ .5 are given. Similarly only values of PM < .5 are considered for β_m. If p > .5 then $\beta_m|_{PM=p} = -\beta_m|_{PM=1-P}$ and $\beta_m = 0$ for $P_M = 0$, .5 or 1.

The values of α and β not represented on the table may be obtained by linear interpolation between the values given. For large values of $\frac{M}{c}$, P_m should be approximated by P_M.

NUMBER OF JUNCTIONS 2

\ PM MT \	ALPHA - MEAN .1	.2	.3	.4	.5	BETA - MEAN .1	.2	.3	.4
1.	-1.355	-1.338	-1.331	-1.329	-1.328	-.004	-.004	-.003	-.002
2.	-.708	-.719	-.734	-.746	-.750	.013	.019	.017	.010
4.	-.350	-.356	-.366	-.373	-.375	.027	.038	.033	.019
6.	-.221	-.227	-.234	-.239	-.241	.033	.046	.040	.022
10.	-.117	-.123	-.129	-.133	-.134	.039	.051	.043	.024
15.	-.068	-.074	-.079	-.083	-.084	.041	.051	.042	.023
20.	-.046	-.051	-.056	-.059	-.060	.041	.049	.040	.022

\ PM MT \	ALPHA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
1.	-1.823	-1.752	-1.703	-1.687	-1.703	-1.753	-1.841	-1.970	-2.134
2.	-1.198	-1.200	-1.205	-1.216	-1.233	-1.259	-1.294	-1.337	-1.386
4.	-.951	-.955	-.958	-.963	-.969	-.977	-.988	-1.001	-1.016
6.	-.902	-.902	-.902	-.902	-.903	-.905	-.909	-.915	-.921
10.	-.890	-.888	-.886	-.883	-.882	-.881	-.882	-.885	-.889
15.	-.900	-.899	-.896	-.894	-.892	-.891	-.893	-.896	-.902
20.	-.911	-.911	-.909	-.907	-.906	-.905	-.907	-.912	-.919

\ PM MT \	BETA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
1.	-.004	-.001	.007	.018	.028	.036	.039	.035	.022
2.	.012	.024	.035	.043	.048	.049	.045	.036	.021
4.	.020	.036	.049	.057	.061	.060	.054	.042	.024
6.	.020	.035	.047	.054	.058	.058	.053	.042	.025
10.	.016	.028	.037	.043	.047	.047	.044	.036	.021
15.	.012	.021	.028	.033	.036	.036	.034	.027	.017
20.	.010	.017	.022	.026	.028	.028	.026	.022	.013

NUMBER OF JUNCTIONS 4

MT \ PM	ALPHA - MEAN .1	.2	.3	.4	.5	BETA - MEAN .1	.2	.3	.4
2.	-1.442	-1.384	-1.342	-1.317	-1.309	-.010	-.014	-.012	-.007
4.	-.656	-.681	-.706	-.722	-.728	.010	.014	.012	.007
6.	-.411	-.435	-.459	-.474	-.480	.021	.028	.024	.014
10.	-.221	-.239	-.256	-.267	-.271	.031	.040	.034	.019
15.	-.131	-.146	-.158	-.166	-.169	.035	.044	.036	.020
20.	-.089	-.102	-.112	-.119	-.121	.037	.044	.036	.019
25.	-.065	-.077	-.086	-.092	-.094	.037	.043	.034	.018

MT \ PM	ALPHA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
2.	-1.659	-1.562	-1.507	-1.498	-1.532	-1.610	-1.741	-1.941	-2.221
4.	-.968	-.984	-1.003	-1.030	-1.063	-1.106	-1.157	-1.219	-1.287
6.	-.832	-.846	-.861	-.878	-.898	-.922	-.950	-.984	-1.020
10.	-.791	-.796	-.800	-.804	-.810	-.819	-.832	-.851	-.872
15.	-.806	-.808	-.808	-.807	-.808	-.812	-.820	-.834	-.852
20.	-.827	-.828	-.827	-.826	-.825	-.828	-.835	-.848	-.866
25.	-.845	-.847	-.846	-.844	-.844	-.846	-.853	-.866	-.883

MT \ PM	BETA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
2.	-.001	.007	.021	.036	.048	.056	.056	.048	.029
4.	.027	.050	.068	.080	.085	.082	.072	.055	.031
6.	.035	.062	.082	.094	.098	.094	.082	.061	.034
10.	.032	.057	.076	.087	.092	.090	.080	.062	.035
15.	.026	.045	.060	.069	.074	.073	.066	.052	.030
20.	.021	.036	.048	.055	.059	.059	.053	.042	.024
25.	.017	.030	.039	.045	.048	.048	.044	.035	.020

NUMBER OF JUNCTIONS 6

\ PM MT \	ALPHA - MEAN .1	.2	.3	.4	.5	BETA - MEAN .1	.2	.3	.4
4.	-1.021	-1.012	-1.003	-.998	-.996	-.004	-.006	-.005	-.003
6.	-.614	-.643	-.669	-.685	-.690	.009	.011	.009	.005
8.	-.428	-.461	-.489	-.507	-.514	.017	.022	.019	.010
10.	-.323	-.353	-.380	-.397	-.403	.022	.029	.024	.013
15.	-.191	-.216	-.236	-.249	-.254	.030	.037	.030	.017
20.	-.130	-.151	-.168	-.179	-.182	.033	.039	.032	.017
25.	-.096	-.114	-.129	-.138	-.141	.034	.039	.031	.017

\ PM MT \	ALPHA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
4.	-1.121	-1.110	-1.115	-1.143	-1.194	-1.269	-1.374	-1.519	-1.706
6.	-.822	-.847	-.875	-.910	-.954	-1.005	-1.065	-1.136	-1.215
8.	-.736	-.758	-.781	-.807	-.838	-.873	-.914	-.959	-1.008
10.	-.712	-.728	-.744	-.762	-.783	-.808	-.838	-.872	-.910
15.	-.721	-.729	-.735	-.741	-.749	-.761	-.779	-.804	-.834
20.	-.748	-.753	-.755	-.757	-.760	-.768	-.782	-.804	-.832
25.	-.774	-.778	-.778	-.779	-.780	-.786	-.799	-.819	-.846

\ PM MT \	BETA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
4.	.020	.042	.062	.077	.085	.084	.076	.059	.033
6.	.041	.074	.097	.111	.114	.107	.091	.067	.037
8.	.048	.085	.110	.124	.126	.118	.100	.074	.040
10.	.048	.085	.110	.124	.128	.121	.103	.077	.042
15.	.040	.071	.093	.107	.112	.108	.094	.071	.040
20.	.033	.057	.075	.087	.092	.089	.079	.061	.034
25.	.027	.047	.062	.071	.076	.074	.066	.051	.029

NUMBER OF JUNCTIONS 8

\ PM / MT \	ALPHA - MEAN .1	.2	.3	.4	.5	BETA - MEAN .1	.2	.3	.4
4.	-1.482	-1.335	-1.242	-1.191	-1.175	-.015	-.019	-.017	-.009
6.	-.854	-.857	-.858	-.857	-.856	-.001	-.002	-.002	-.001
8.	-.581	-.612	-.637	-.651	-.656	.008	.010	.008	.004
10.	-.432	-.469	-.498	-.517	-.523	.015	.018	.015	.008
15.	-.253	-.285	-.313	-.330	-.336	.024	.030	.024	.013
20.	-.172	-.200	-.223	-.238	-.243	.029	.034	.027	.015
25.	-.127	-.151	-.171	-.184	-.188	.031	.036	.028	.015

\ PM / MT \	ALPHA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
4.	-1.428	-1.304	-1.254	-1.258	-1.307	-1.402	-1.559	-1.814	-2.230
6.	-.893	-.908	-.934	-.975	-1.032	-1.106	-1.202	-1.328	-1.487
8.	-.720	-.751	-.786	-.828	-.877	-.934	-.999	-1.073	-1.157
10.	-.661	-.689	-.719	-.753	-.791	-.834	-.881	-.934	-.991
15.	-.649	-.664	-.679	-.695	-.714	-.737	-.766	-.800	-.839
20.	-.677	-.686	-.694	-.701	-.711	-.726	-.748	-.777	-.812
25.	-.707	-.714	-.718	-.722	-.728	-.739	-.757	-.784	-.818

\ PM / MT \	BETA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
4.	.007	.025	.047	.065	.077	.080	.075	.060	.035
6.	.038	.070	.095	.109	.113	.107	.092	.068	.037
8.	.055	.096	.124	.137	.138	.127	.106	.077	.041
10.	.061	.106	.135	.149	.150	.138	.115	.084	.045
15.	.056	.097	.126	.142	.145	.136	.116	.086	.046
20.	.046	.081	.105	.120	.124	.119	.103	.077	.042
25.	.038	.067	.087	.100	.104	.101	.088	.067	.037

NUMBER OF JUNCTIONS 10

\ PM MT \	ALPHA - MEAN .1	.2	.3	.4	.5	BETA - MEAN .1	.2	.3	.4
6.	-1.140	-1.068	-1.018	-.990	-.981	-.009	-.012	-.011	-.006
8.	-.758	-.767	-.772	-.773	-.773	.000	-.000	-.000	-.000
10.	-.555	-.587	-.610	-.624	-.628	.008	.009	.007	.004
15.	-.317	-.356	-.387	-.407	-.413	.019	.023	.019	.010
20.	-.214	-.249	-.277	-.295	-.301	.025	.029	.023	.013
25.	-.158	-.189	-.214	-.229	-.234	.028	.032	.025	.013
30.	-.123	-.151	-.173	-.186	-.191	.029	.033	.025	.014

\ PM MT \	ALPHA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
6.	-1.049	-1.016	-1.017	-1.049	-1.108	-1.198	-1.328	-1.523	-1.815
8.	-.758	-.786	-.823	-.871	-.931	-1.005	-1.096	-1.211	-1.354
10.	-.645	-.681	-.721	-.768	-.821	-.880	-.947	-1.023	-1.109
15.	-.590	-.615	-.641	-.669	-.701	-.736	-.775	-.818	-.865
20.	-.612	-.628	-.643	-.659	-.677	-.700	-.729	-.764	-.805
25.	-.644	-.656	-.666	-.675	-.686	-.702	-.726	-.758	-.798
30.	-.675	-.685	-.691	-.697	-.704	-.717	-.738	-.769	-.808

\ PM MT \	BETA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
6.	.027	.057	.083	.100	.107	.103	.090	.067	.037
8.	.054	.095	.123	.137	.138	.127	.106	.077	.041
10.	.068	.117	.147	.161	.159	.144	.119	.085	.045
15.	.071	.123	.156	.172	.171	.157	.131	.095	.051
20.	.061	.105	.136	.152	.155	.145	.123	.091	.049
25.	.050	.088	.114	.129	.133	.127	.109	.081	.044
30.	.042	.074	.096	.109	.114	.109	.095	.071	.039

NUMBER OF JUNCTIONS 15

\ PM / MT \	ALPHA - MEAN .1	.2	.3	.4	.5	BETA - MEAN .1	.2	.3	.4
8.	-1.306	-1.125	-1.032	-.988	-.975	-.013	-.016	-.013	-.007
10.	-.934	-.881	-.847	-.829	-.824	-.006	-.008	-.006	-.003
15.	-.506	-.540	-.562	-.574	-.577	.007	.008	.007	.004
20.	-.331	-.376	-.409	-.428	-.434	.015	.018	.014	.007
25.	-.240	-.284	-.317	-.337	-.344	.020	.023	.018	.009
30.	-.185	-.226	-.257	-.276	-.283	.023	.025	.020	.010
35.	-.149	-.187	-.216	-.233	-.239	.025	.027	.020	.011

\ PM / MT \	ALPHA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
8.	-1.082	-.990	-.975	-1.004	-1.064	-1.159	-1.302	-1.537	-1.970
10.	-.772	-.779	-.810	-.860	-.927	-1.014	-1.128	-1.291	-1.537
15.	-.518	-.564	-.615	-.670	-.729	-.792	-.859	-.933	-1.017
20.	-.492	-.527	-.565	-.606	-.649	-.693	-.739	-.787	-.838
25.	-.514	-.540	-.567	-.596	-.626	-.658	-.694	-.734	-.780
30.	-.546	-.567	-.586	-.606	-.628	-.653	-.684	-.722	-.768
35.	-.578	-.596	-.611	-.625	-.641	-.662	-.689	-.726	-.773

\ PM / MT \	BETA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
8.	.027	.063	.093	.112	.118	.112	.096	.071	.038
10.	.055	.102	.134	.150	.149	.136	.113	.080	.042
15.	.099	.164	.198	.209	.201	.178	.144	.102	.053
20.	.100	.166	.204	.218	.212	.190	.155	.111	.059
25.	.087	.147	.184	.201	.199	.182	.151	.109	.058
30.	.073	.126	.160	.177	.178	.165	.139	.101	.054
35.	.063	.107	.138	.154	.157	.147	.124	.091	.049

NUMBER OF JUNCTIONS 20

\ PM MT \	ALPHA - MEAN .1	.2	.3	.4	.5	BETA - MEAN .1	.2	.3	.4
10.	-1.379	-1.123	-1.014	-.965	-.951	-.014	-.016	-.013	-.007
15.	-.741	-.722	-.710	-.703	-.701	-.002	-.002	-.001	-.001
20.	-.471	-.508	-.530	-.541	-.545	.007	.008	.006	.003
25.	-.334	-.383	-.416	-.434	-.440	.013	.015	.011	.006
30.	-.255	-.304	-.339	-.360	-.367	.017	.019	.014	.008
35.	-.203	-.251	-.285	-.306	-.313	.020	.021	.016	.008
40.	-.168	-.212	-.246	-.265	-.272	.021	.022	.017	.009

\ PM MT \	ALPHA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
10.	-1.089	-.957	-.938	-.964	-1.025	-1.117	-1.260	-1.501	-2.013
15.	-.570	-.610	-.662	-.724	-.795	-.876	-.973	-1.099	-1.277
20.	-.439	-.492	-.550	-.611	-.673	-.737	-.803	-.872	-.950
25.	-.425	-.469	-.516	-.565	-.615	-.664	-.712	-.760	-.811
30.	-.445	-.481	-.517	-.555	-.594	-.632	-.671	-.712	-.758
35.	-.475	-.504	-.533	-.562	-.592	-.623	-.657	-.696	-.743
40.	-.506	-.521	-.554	-.577	-.601	-.627	-.658	-.696	-.745

\ PM MT \	BETA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
10.	.027	.068	.100	.119	.124	.116	.098	.072	.038
15.	.092	.156	.191	.202	.194	.171	.137	.096	.049
20.	.128	.203	.240	.247	.233	.204	.164	.115	.059
25.	.126	.204	.244	.254	.242	.214	.174	.123	.065
30.	.111	.184	.224	.239	.232	.208	.171	.122	.065
35.	.096	.169	.199	.216	.213	.194	.161	.116	.061
40.	.082	.132	.175	.192	.192	.177	.148	.107	.057

NUMBER OF JUNCTIONS 25

\ PM / MT \	ALPHA - MEAN .1	.2	.3	.4	.5	BETA - MEAN .1	.2	.3	.4
10.	-1.831	-1.314	-1.130	-1.055	-1.036	-.019	-.021	-.016	-.008
15.	-1.006	-.881	-.825	-.800	-.793	-.008	-.008	-.006	-.003
20.	-.639	-.636	-.635	-.634	-.634	.001	.001	.001	.001
25.	-.446	-.484	-.507	-.519	-.523	.007	.008	.006	.003
30.	-.334	-.384	-.417	-.435	-.441	.012	.013	.010	.005
35.	-.264	-.316	-.353	-.373	-.380	.015	.016	.012	.006
40.	-.216	-.268	-.304	-.326	-.333	.017	.018	.014	.007

\ PM / MT \	ALPHA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
10.	-1.463	-1.121	-1.055	-1.056	-1.099	-1.193	-1.356	-1.679	-2.688
15.	-.734	-.716	-.744	-.796	-.865	-.952	-1.069	-1.246	-1.574
20.	-.465	-.518	-.580	-.648	-.721	-.798	-.885	-.989	-1.131
25.	-.385	-.444	-.506	-.571	-.636	-.700	-.763	-.828	-.898
30.	-.377	-.428	-.482	-.537	-.591	-.643	-.692	-.738	-.786
35.	-.395	-.438	-.482	-.527	-.572	-.614	-.655	-.694	-.738
40.	-.422	-.458	-.495	-.532	-.568	-.604	-.639	-.677	-.722

\ PM / MT \	BETA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
10.	.008	.042	.070	.091	.099	.096	.084	.063	.035
15.	.063	.122	.158	.173	.170	.152	.123	.086	.043
20.	.124	.200	.236	.242	.227	.198	.157	.109	.055
25.	.154	.238	.274	.277	.259	.225	.179	.126	.065
30.	.151	.237	.276	.283	.266	.233	.188	.133	.070
35.	.135	.217	.258	.269	.257	.229	.187	.133	.070
40.	.117	.192	.233	.248	.241	.216	.178	.128	.067

NUMBER OF JUNCTIONS 30

\ PM / MT \	ALPHA - MEAN .1	.2	.3	.4	.5	BETA - MEAN .1	.2	.3	.4
15.	-1.236	-1.009	-.915	-.876	-.864	-.011	-.012	-.009	-.005
20.	-.823	-.753	-.722	-.709	-.706	-.004	-.004	-.002	-.001
25.	-.574	-.582	-.587	-.591	-.592	.002	.003	.003	.002
30.	-.426	-.465	-.489	-.502	-.506	.007	.008	.006	.003
35.	-.332	-.384	-.417	-.435	-.441	.011	.012	.009	.005
40.	-.269	-.325	-.361	-.382	-.389	.013	.014	.011	.006
45.	-.225	-.280	-.318	-.340	-.347	.015	.016	.012	.006

\ PM / MT \	ALPHA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
15.	-.923	-.830	-.823	-.866	-.925	-1.011	-1.136	-1.347	-1.784
20.	-.563	-.587	-.638	-.701	-.774	-.858	-.959	-1.099	-1.339
25.	-.398	-.460	-.528	-.599	-.673	-.748	-.827	-.917	-1.034
30.	-.345	-.408	-.475	-.542	-.609	-.673	-.735	-.794	-.857
35.	-.341	-.397	-.456	-.515	-.573	-.627	-.676	-.720	-.763
40.	-.357	-.405	-.456	-.507	-.556	-.601	-.642	-.680	-.719
45.	-.381	-.423	-.466	-.509	-.551	-.590	-.626	-.662	-.703

\ PM / MT \	BETA - VARIANCE .1	.2	.3	.4	.5	.6	.7	.8	.9
15.	.040	.089	.126	.141	.143	.131	.108	.076	.039
20.	.097	.168	.206	.216	.206	.180	.143	.098	.049
25.	.154	.238	.272	.274	.254	.219	.173	.120	.061
30.	.179	.269	.303	.303	.280	.241	.192	.135	.070
35.	.174	.267	.304	.307	.286	.249	.201	.142	.075
40.	.158	.247	.288	.295	.278	.245	.199	.142	.075
45.	.138	.221	.263	.275	.263	.235	.192	.138	.073

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