



**THE MARKOFF SPECTRUM AND  
GEODESICS ON THE PUNCTURED TORUS**

DAVID J. CRISP

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## ABSTRACT

This thesis is a contribution to the area of diophantine approximation concerning the Markoff spectrum. It is based on the connection Harvey Cohn discovered between the structure of the Markoff spectrum and the behaviour of geodesics on a hyperbolic punctured torus. The torus  $\mathbf{T}$  involved is the quotient of the upper half-plane  $\mathbf{H}$  by the commutator subgroup  $\Gamma'$  of the modular group. The connection is made by associating a form  $f$  with the geodesic  $\gamma$  in  $\mathbf{H}$  whose endpoints are the roots of  $f$  and then projecting  $\gamma$  to  $\mathbf{T}$ . Cohn, [8], found that under this map the Markoff forms, that is, the forms whose Markoff values lie below 3, correspond exactly to the simple closed geodesics. Here, we study the Markoff values arising from geodesics with low self-intersection number.

Closed geodesics can be studied via their free homotopy classes. In this manner, we show that the closed geodesics with one self-intersection fall into two classes; the proper closed 1-intersectors and the improper closed 1-intersectors. The Markoff values of the improper closed 1-intersectors lie in Hall's ray and are not considered further. For open geodesics we use cutting sequences. It is known that the geodesics which correspond to forms with Markoff value 3 have aperiodic linear cutting sequences and therefore are simple and open. We show that the only other simple open geodesics have half-linear cutting sequences. This is an improvement on Haas' topological characterisation, [21], because it provides a means of calculating the associated Markoff values. We show that they all lie in Hall's ray.

By developing an understanding of how the subgroup of  $\text{Aut } \Gamma'$  which fixes Markoff values lies in  $\text{Aut } \Gamma'$  we are able to convert our characterisation of the proper closed 1-intersectors into a characterisation of the associated forms. This leads to an expression for the forms in terms of the solutions to Markoff's equation.

We demonstrate an intriguing symmetry between the Markoff values of the proper closed 1-intersectors and those of the simple closed geodesics. An examination of the spectrum near the first few values of the proper closed 1-intersectors reveals that they are isolated points of the spectrum. We conjecture that they are all isolated. Whilst we cannot prove our conjecture we do obtain some useful partial results. The techniques introduced for this purpose also allow us to describe two new families of isolated points.