Constructions with Bundle Gerbes

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Abstract

This thesis develops the theory of bundle gerbes and examines a number of useful constructions in this theory. These allow us to gain a greater insight into the structure of bundle gerbes and related objects. Furthermore they naturally lead to some interesting applications in physics.

Statement of Originality

This thesis contains no material which has been accepted for the award of any other degree or diploma at any other university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being made available for loan and photocopying.

Stuart Johnson
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Chapter 1

Introduction

Bundle gerbes were introduced by Murray [36] as geometric realisations of classes in $H^3(M,\mathbb{Z})$ on a manifold M. By geometric realisation we mean an equivalence class of geometric objects which is isomorphic to $H^3(M,\mathbb{Z})$. An example of this in lower degree is the relationship between isomorphism classes of principal $U(1)$-bundles, or equivalently line bundles, over M and the Chern class in $H^2(M,\mathbb{Z})$. The interest in $H^3(M,\mathbb{Z})$ was motivated by the appearance of such integral cohomology classes in a number of situations including central extensions of structure groups of principal bundles and Wess-Zumino-Witten (WZW) theory. There were already a number of other realisations of $H^3(M;\mathbb{Z})$, of particular interest are the gerbes of Giraud as described in Brylinski's book [5]. These are, from a rather simplistic view, defined as sheaves of groupoids. The idea of bundle gerbe theory was to define realisations which did not involve sheaves. Instead it is possible to build a realisation out of principal $U(1)$-bundles.

Essentially a bundle gerbe over a manifold M consists of a submersion $Y \xrightarrow{\pi} M$ and a $U(1)$-bundle $P \to Y^{[2]}$ over the fibre product

$$Y^{[2]} = Y \times_\pi Y = \{(y,y') \in Y^2 | \pi(y) = \pi(y')\}$$

The fibres of P are required to carry a certain associative product structure which is called the bundle gerbe product.

In [36] bundle gerbe connections and curving were defined. Together these form a higher analogue of connections on $U(1)$-bundles. They also correspond to connective structures and curvings on gerbes [5]. A bundle gerbe connection is a connection, A, on the bundle P which is compatible with the bundle gerbe product. Denote the curvature of this connection by F. Let π_1 and π_2 be the projections of each component $Y^{[2]} \to Y$. A curving is a 2-form, η, on Y which satisfies $\delta(\eta) = F$ where $\delta(\eta) = \pi_2^*\eta - \pi_1^*\eta$. A bundle gerbe with connection and curving defines a class in the Deligne cohomology group $H^3(M,\mathbb{Z}(3)_D)$ which may be thought of as the hypercohomology of a complex of sheaves, $H^2(M, \underline{U(1)}_M \to \Omega^1(M) \to \Omega^2(M))$.

The bundle gerbe construction has also been used to consider higher degree Čech and Deligne classes [13]. In particular Stevenson has developed a theory of bundle 2-gerbes ([44],[43]) which have an associated class in $H^4(M,\mathbb{Z})$, and when given a higher analogue of connection and curving give rise to a class in $H^4(M,\mathbb{Z}(4)_D)$.

There is a cup product in Deligne cohomology which was described by Esnault and Viehweg [17] and which has been given a geometric interpretation by Brylinski and McLaughlin ([5],[8]).

A generalisation of the concept of holonomy to bundle gerbes was first considered in [36]. Just as the holonomy of a $U(1)$-bundle with connection associates an element of $U(1)$ to every loop in the base, the holonomy of a bundle gerbe with connection and curving associates an element of $U(1)$ to every closed surface in the base. In one particular case bundle gerbe holonomy has been used to describe the WZW action [12].

These considerations lead to the transgression formulae derived by Gawedski [23] for dealing with such actions in general settings. These formulae generalise bundle holonomy and parallel transport to higher degree Deligne classes. Completely general transgression formulae have been given by Gomi and Terashima ([25],[26]).

The relevance of Čech and Deligne cohomology classes to applications in physics has been well established. For example Dijkgraaf and Witten [15] have used differential characters to find a general Chern-Simons Theory, Gomi [24] has considered the relation between gerbes and Chern-Simons theory and Freed and Witten [21] have considered the role of Deligne classes in anomaly cancellation in D-brane theory. As well as the WZW case we have already alluded to there are a number of other examples discussed in [12].

The basic aim of this thesis is to provide further development of the theory of bundle gerbes. The goal has been to develop this theory in a way which keeps in mind the need for a balance between an abstract approach which readily accommodates generalisation and an approach which more easily allows application of the theory and which could be of interest to a wider audience. For the first factor the most important feature is the bundle gerbe hierarchy principle which is a guiding principle for relating bundle gerbe type constructions corresponding to Deligne cohomology in various degrees. For the second factor we show how various constructions may be described in geometric terms, often allowing manipulation of diagrammatic representations of bundle gerbes to take the place of complicated calculations.

With these factors as a guide we have described a number of constructions involving bundle gerbes. Some of these which have already been developed elsewhere are given in different forms or with a different emphasis to demonstrate the hierarchy principle or to relate more easily to our applications. We also describe some constructions which are new to bundle gerbe theory. Finally we show how these constructions are useful in applications of bundle gerbe theory to physics.

We begin with a review of the basic features of Deligne cohomology and introduce the bundle gerbe family of geometric realisations via bundle 0-gerbes as an alternative to $U(1)$-bundles. This would appear to be a complicated approach however it simplifies the transition from bundles to bundle gerbes and allows us to develop some features of bundle gerbe theory in a setting which is still relatively familiar. We then define bundle gerbes and explain their role as representatives of degree 3 Deligne cohomology.

In Chapter 3 we consider some important examples of bundle gerbes. Tautological bundle gerbes [36] are introduced by first defining a bundle 0-gerbe, helping to gain a feel for the bundle gerbe hierarchy. Trivial bundle gerbes are discussed in some detail since they play an important role in many constructions. In particular we give a detailed account of the distinction between trivial bundle gerbes which by definition have trivial Čech class and D-trivial bundle gerbes which have trivial Deligne class. We then consider torsion bundle gerbes which are defined as bundle gerbes with a torsion Čech class. We describe bundle gerbe modules which were introduced in [3] and derive corresponding local data. We briefly describe the example of the lifting bundle gerbe [36] and then describe bundle gerbes representing cup products of Deligne classes. Each of these examples is useful in subsequent constructions and applications.

In Chapter 4 the bundle gerbe hierarchy principle is introduced via comparison with some other geometric realisations of Deligne cohomology. The correspondence between bundle gerbes and gerbes which has previously been described in [36] and [38] is put in the context of the hierarchy. We then define bundle 2-gerbes following [44], however we consider the product structures as members of the hierarchy and define the Deligne class using the language of D-obstruction forms which we established in Chapter 3. We prove the isomorphism between bundle 2-gerbes with connection and curving and $H^4(M, \mathbb{Z}(4)_D)$. Next we go in the other direction and define $\mathbb{Z}$-bundle gerbes which lie at the bottom of the hierarchy. These would seem rather trivial however it is of interest to see how the various aspects of the higher theory appear here, in particular the $\mathbb{Z}$-bundle gerbe connection naturally motivates classifying theory, our next topic for consideration. This is a generalisation of classifying theory for bundles. We present a number of results of Gajer [22] relating to Deligne classes and of Murray and Stevenson relating specifically to bundle gerbes [38]. Chapter 4 concludes with a table which catalogues the various realisations of Deligne cohomology which we have dealt with.

Chapter 5 begins with an account of the holonomy of $U(1)$-bundles which differs somewhat from the standard treatment of the subject. The reason for this is that we need a theory of holonomy which relates directly to the Deligne class rather than concepts such as horizontal lifts which are not easily generalised to bundle gerbes and beyond. We then define the holonomy of bundle gerbes with an explanation of how it relates to the holonomy of $U(1)$-bundles and with details of how local formulae are obtained. The concept of holonomy is extended to bundle 2-gerbes and to general Deligne classes.

In Chapter 6 we describe the extension of the notion of parallel transport from $U(1)$-bundles to bundle gerbes, bundle 2-gerbes and general Deligne classes and provide detailed derivations of local formulae. In particular we discuss how to obtain a $U(1)$-bundle on the loop space LM from a bundle gerbe on M and bundle 2-gerbe generalisations of this construction.

The basic properties of bundle gerbe holonomy are described in section 7.1. The motivation for considering these particular properties comes from those of line bundles obtained via transgression ([5],[20]). We consider the example of the tautological bundle gerbe which motivates holonomy reconstruction, that is, reconstructing a bundle gerbe with connection and curving from its holonomy on closed surfaces. We show how the holonomy reconstruction for bundle gerbes relates to that of gerbes as described in [31]. Transgression formulae are then used to give an alternative approach to holonomy reconstruction which allows us to consider the case of bundle 2-gerbes. We conclude the chapter with the gauge invariance properties of holonomy.

Finally in Chapter 8 we use bundle gerbe theory to examine applications in physics. Constructions in Wess-Zumino-Witten and Chern-Simons (CS) theories are shown to follow naturally from various constructions in bundle gerbe theory. In the WZW case we just interpret the standard results (see the Appendix A of [20]) in terms of bundle gerbes. In the case of Chern-Simons theory we define a bundle 2-gerbe whose holonomy gives a general definition of the Chern-Simons action. We can then interpret Chern-Simons lines, gauge invariance and the relationship with the central extension of the loop group in terms of bundle gerbe theory. All of these results are quite straightforward from this point of view. Bundle gerbes also prove useful in studying anomaly cancellation in D-branes as described in [11]. Here we emphasise local aspects which were not discussed in detail in that paper. We also add some comments on the potential for the application of bundle gerbes to the problem of anomalies involving C-fields and higher dimensional generalisations of Chern-Simons theory. We conclude with comments on the relationship between bundle gerbes and the axiomatic approach to topological quantum field theory.

It is necessary here for a brief comment on terminology. We shall refer to line bundles and their associated principal bundles interchangeably. Also we shall usually work in the Hermitian setting, so our bundles are $U(1)$-bundles. Since we deal almost exclusively with these bundles we shall often simply refer to bundles, in situations where we require a principal bundle with a more general structure group we refer to it as a G-bundle.

Chapter 2

Bundle Gerbes and Deligne Cohomology

In this chapter we discuss a number of geometric realisations of low degree Deligne cohomology, in particular bundle gerbes.

2.1 A Review of Sheaf Cohomology

We begin with some background material regarding sheaf cohomology. We mostly follow Brylinski's book [5], though some material is drawn from Bott and Tu [2].

We assume that the reader is familiar with the definitions of sheaves and related concepts such as morphisms of sheaves. We provide the minimum amount of detail necessary to define sheaf cohomology, for further details see [5]. Let A be a sheaf of Abelian groups on a manifold M. Recall that this means that associated with every open $U \subseteq M$ there is an Abelian group $A(U)$ which satisfies certain axioms with respect to restrictions. We always assume that manifolds are paracompact, that is, every open cover has a locally finite subcover. We shall be interested in the following examples:

$\mathbb{Z}_M, \mathbb{R}_M, U(1)_M$:	sheaf of locally constant functions on a manifold M
$\underline{\mathbb{R}}_M, \underline{U(1)}_M$:	sheaf of smooth $\mathbb{R}$ or $U(1)$- valued functions on a manifold M
Ω^p_M:	sheaf of real differential p-forms on M

A *complex of sheaves* $K^\bullet$ is a sequence

$$\cdots \xrightarrow{d^{n-1}} K^n \xrightarrow{d^n} K^{n+1} \xrightarrow{d^{n+1}} \cdots$$

where $n \in \mathbb{Z}$ and $d^n : K^n \to K^{n+1}$ are morphisms of sheaves of Abelian groups satisfying $d^n \circ d^{n-1} = 0$. The map d^n is called the *differential* of the complex. We always assume that $K^p = 0$ for $p < 0$. A *morphism of complexes of sheaves* $\phi : K^\bullet \to L^\bullet$ consists of a family of morphisms of sheaves $\phi^n : K^n \to L^n$ such that $\phi^{n+1} \circ d^n_K = d^n_L \circ \phi^n$. Given two morphisms ϕ and ψ from $K^\bullet$ to $L^\bullet$ a *homotopy* H from ϕ to ψ consists of a series of morphisms $H^n : K^n \to L^{n-1}$ such that $d^{n-}_L H^n + H^{n+1} d^n_K = \phi^n - \psi^n$. A morphism of complexes of sheaves $\phi : K^\bullet \to L^\bullet$ is a *homotopy equivalence* if there exists a morphism $\psi : L^\bullet \to K^\bullet$ and $\phi\psi$ and $\psi\phi$ are both homotopic to the identity map. A complex of sheaves is called *acyclic* if $Ker(d^n) = Im(d^{n-1})$ for all n. The *cohomology sheaves* $\underline{H}^p(K^\bullet)$ are defined by the presheaf $Ker(d^j)/Im(d^{j-1})$. For an acyclic complex all of the cohomology sheaves are zero. A sheaf I is called *injective* if given any morphism $f : A \to I$ and an injective morphism $i : A \to B$ there exists a morphism $g : B \to I$ such that $g \circ i = f$. An *injective resolution* of A is a complex of injective sheaves $I^\bullet$ such that $A \to I^\bullet$ is an acyclic complex of sheaves. Injective resolutions always exist and are unique up to homotopy equivalence. Let $\Gamma(M, .)$ be the functor of global sections which takes the sheaf A to the Abelian group $\Gamma(M; A)$ defined by $A(M)$. The *sheaf cohomology groups* $H^p(M, A)$ are defined as the p-th cohomology of the complex

$$\cdots \to \Gamma(M, I^j) \to \Gamma(M, I^{j+1}) \to \cdots$$

where $I^\bullet$ is an injective resolution of A. Given a short exact sequence of sheaves

$$0 \to A \to B \to C \to 0$$

there is a long exact sequence in sheaf cohomology

$$\cdots \to H^n(M, A) \to H^n(M, B) \to H^n(M, C) \to H^{n+1}(M, A) \to \cdots$$

If a morphism of complexes induces an isomorphism of cohomology sheaves $\underline{H}^n(K^\bullet) \cong \underline{H}^n(L^\bullet)$ then it is called a *quasi-isomorphism.*

A useful example of a resolution is the *Čech resolution.* Let $\mathcal{U}$ be an open cover of M and let $U_{i_0,\ldots,i_p}$ denote an intersection $U_{\iota_0} \cap \ldots \cap U_{i_p}$ of open sets in this cover. The Čech resolution is a complex $\check{C}^\bullet(\mathcal{U}, A)$ which is defined by $\check{C}^p(\mathcal{U}, A) = \Pi_{i_0,\ldots,i_p} A(U_{i_0,\ldots,i_p})$. and $\delta : \check{C}^p(\mathcal{U}, A) \to \check{C}^{p+1}(\mathcal{U}, A)$ is defined by

$$\delta(\underline{\alpha})_{i_0,\ldots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j (\alpha_{i_0,\ldots,i_{j-1},i_{j+1},\ldots,i_{p+1}})_{|U_{i_0,\ldots,i_{p+1}}}$$

where we have introduced the notation $\underline{\alpha} \in \check{C}^p(\mathcal{U}, A)$ where the underline denotes a family $\alpha_{i_0,\ldots,i_p} \in A(U_{i_0,\ldots,i_p})$. In general the resolution and the resulting cohomology groups should depend on the choice of open cover however we shall be interested only in spaces which are manifolds and these always admit a good cover, that is, a cover in which all non-empty intersections are contractible. In this case the construction is independent of the choice of cover and the Čech resolution computes the sheaf cohomology. It is not an injective resolution, however there exists a morphism with the Čech resolution which induces an isomorphism in cohomology. If A is the sheaf $\mathbb{Z}$ then we recover the usual Čech cohomology groups $\check{H}^p(M, \mathbb{Z})$, for example a class $\check{H}^1(M, \mathbb{Z})$ consists of a family of $\mathbb{Z}$ valued constants c_{ij} associated with double intersections U_{ij} such that $c_{jk} - c_{ik} + c_{ij} = 0$ on U_{ijk} and under the equivalence relation $c_{ij} \sim c_{ij} + b_j - b_i$ where b_i and b_j are from a family of constants defined on single open sets.

A *double complex* of sheaves, $K^{\bullet\bullet}$, consists of sheaves $K^{p,q}$ and two differential maps $d : K^{p,q} \to K^{p+1,q}$ and $\delta : K^{p,q} \to K^{p,q+1}$ such that $dd = 0$, $\delta\delta = 0$ and $d\delta = \delta d$. A double complex may be represented diagrammatically as follows:

$$\begin{array}{ccccccc}
 & & \uparrow d & & \uparrow d & & \\
\xrightarrow{\delta} & & K^{p,q} & \xrightarrow{\delta} & K^{p+1,q} & \xrightarrow{\delta} & \\
 & & \uparrow d & & \uparrow d & & \\
\xrightarrow{\delta} & & K^{p,q-1} & \xrightarrow{\delta} & K^{p+1,q-1} & \xrightarrow{\delta} & \\
 & & \uparrow d & & \uparrow d & &
\end{array}$$

Each row and column defines a complex of sheaves denoted $K^{\bullet,q}$ and $K^{p,\bullet}$ respectively. The *total complex* of a double complex $K^{\bullet\bullet}$ is an ordinary complex $K^\bullet$ which is defined by $K^n = \bigoplus_{p+q=n} K^{p,q}$ with differential $D = \delta + (-1)^p d$.

Let $K^\bullet$ be a complex of sheaves with differential d_K. An *injective resolution* of $K^\bullet$ is a double complex $I^{\bullet\bullet}$ with differentials d and δ such that for each q the complex $I^{\bullet,q}$ with differential δ is an injective resolution of K^q, the complex $d(I^{\bullet,q-1}) \subseteq I^{\bullet,q}$ is an injective resolution of $d_K(K^{q-1})$, the complex of sheaves $Ker(d) \subseteq I^{\bullet,q}$ is an injective resolution of $Ker(d_K : K^{q-1} \to K^q)$ and complex of cohomology sheaves of the rows, $\underline{H}^{\bullet,q}$ is an injective resolution of $\underline{H}^q(K^\bullet)$.

For all of our examples an injective resolution of a complex of sheaves exists and is unique up to homotopy. Given a complex $K^\bullet$ and an injective resolution $I^{\bullet\bullet}$ the *hypercohomology group* $H^p(M, K^\bullet)$ is defined to be the p-th cohomology of the double complex $\Gamma(M, I^{\bullet\bullet})$. Given a short exact sequence of complexes of sheaves there is a long exact sequence in hypercohomology. Quasi-isomorphisms $\phi : K^\bullet \to L^\bullet$ induce isomorphisms in sheaf hypercohomology, $H^n(M, K^\bullet) \cong H^n(M, L^\bullet)$. The Čech resolution may be extended to the case of a complex of sheaves by taking the usual Čech resolution for each sheaf in the complex.

We shall describe a specific example of this. Let $K^\bullet = \underline{U(1)}_M \xrightarrow{d\log} \Omega^1_M$. The Čech resolution looks like

$$\begin{tikzcd}
0 & 0 & 0 & \\
\Omega^1_M \arrow[u] \arrow[r] & \check{C}^0(\mathcal{U}, \Omega^1) \arrow[u] \arrow[r, "\delta"] & \check{C}^1(\mathcal{U}, \Omega^1) \arrow[u] \arrow[r, "\delta"] & {} \\
U(1)_M \arrow[u, "d\log"] \arrow[r] & \check{C}^0(\mathcal{U}, \underline{U(1)}) \arrow[u, "d\log"] \arrow[r, "\delta"] & \check{C}^1(\mathcal{U}, \underline{U(1)}) \arrow[u, "d\log"] \arrow[r, "\delta"] & {}
\end{tikzcd}$$

A class in $H^0(M, \underline{U(1)} \to \Omega^1)$ consists of $\underline{f} \in \check{C}^0(\mathcal{U}, \underline{U(1)})$ such that $f_\beta f_\alpha^{-1} = 1$ and $d\log f_\alpha = 0$. This is a locally constant $U(1)$-valued function on M.

A class in $H^1(M, \underline{U(1)} \to \Omega^1)$ consists of a pair $(\underline{g}, \underline{A}) \in \check{C}^1(\mathcal{U}, \underline{U(1)}) \oplus \check{C}^0(\mathcal{U}, \Omega^1)$ such that $g_{\beta\gamma} g_{\alpha\gamma}^{-1} g_{\alpha\beta} = 1$ and $d\log g_{\alpha\beta} = A_\beta - A_\alpha$ and is defined modulo exact cocycles of the form $(h_\alpha^{-1} h_\beta, d\log h_\alpha)$ for some $\underline{h} \in \check{C}^0(\mathcal{U}, \underline{U(1)})$. Classes of higher degree are defined in a similar way.

2.2 $U(1)$-Functions

We examine $U(1)$-valued functions as the starting point for our geometric objects corresponding to Deligne cohomology classes, focusing in particular on features which are of interest when we move on to geometric realisations of higher degree classes.

Let M be a smooth manifold. We shall consider smooth functions $f : M \to U(1)$. Such functions are elements of the sheaf cohomology group $H^0(M, \underline{U(1)}_M)$.

Our interest in $U(1)$-valued functions is due to their role as representatives of the smooth Deligne cohomology group $H^1(M, \mathbb{Z}(1)_D)$.

Definition 2.1. [5] Let $\mathbb{Z}(p) = (2\pi\sqrt{-1})^p \cdot \mathbb{Z}$. Define a complex of sheaves $\mathbb{Z}(p)_D$, for $p > 0$ by

$$\mathbb{Z}(p)_M \xrightarrow{i} \underline{\Omega}^0_M \xrightarrow{d} \underline{\Omega}^1_M \xrightarrow{d} \cdots \xrightarrow{d} \underline{\Omega}^{p-1}_M$$

Where Ω^k_M is the sheaf of real differential k-forms on M and the map i is the inclusion. We define $\mathbb{Z}(0)_D$ to be $\mathbb{Z}_M$. The Deligne cohomology groups of M are defined as the hypercohomology groups $H^q(M, \mathbb{Z}(p)_D)$.

Deligne classes are realised explicitly by first using a quasi-isomorphism of sheaves [5] [1],

$$\begin{array}{ccccccccccc} 0 & \to & \mathbb{Z}(p) & \to & \mathbb{R}_M & \xrightarrow{d} & \underline{\Omega}^1_M & \to & \cdots & \to & \underline{\Omega}^{p-1} \\ & & & & \downarrow & & \downarrow & & & & \downarrow \\ & & & & \underline{U(1)}_M & \xrightarrow{d\log} & \underline{\Omega}^1_M & \to & \cdots & \to & \underline{\Omega}^{p-1} \end{array} \tag{2.1}$$

which induces an isomorphism

$$H^q(M, \mathbb{Z}(p)_D) \cong H^{q-1}(M, \underline{U(1)}_M \xrightarrow{d\log} \underline{\Omega}^1_M \to \cdots \to \underline{\Omega}^{p-1}) \tag{2.2}$$

In general we shall denote the complex

$$\underline{U(1)}_M \xrightarrow{d\log} \underline{\Omega}^1_M \to \cdots \to \underline{\Omega}^{p-1}$$

by $\mathcal{D}^{p-1}$ so we can write (2.2) as

$$H^q(M, \mathbb{Z}(p)_D) \cong H^{q-1}(M, \mathcal{D}^{p-1})$$

We shall usually deal directly the groups $H^{q-1}(M, \mathcal{D}^{p-1})$ so we shall also refer to these as Deligne cohomology. The original definition is still required for certain purposes such as cup products. To get concrete expressions for these sheaf cohomology classes they are represented in terms of the Čech resolution relative to a good open cover as discussed in the example at the end of the previous section.

There are exact sequences [22]

$$0 \to H^{p-1}(M, U(1)) \to H^p(M, \mathbb{Z}(p)_D) \to \Omega^p_0(M) \to 0 \tag{2.3}$$

which for $p = 1$ becomes

$$0 \to H^0(M, U(1)) \xrightarrow{i} H^1(M, \mathbb{Z}(1)_D) \xrightarrow{d\log} \Omega^1_0(M) \to 0$$

where $\Omega^p_0(M)$ denotes the set of closed p-forms on M which have periods in $\mathbb{Z}(1)$ (that is, the integral over a closed p-cycle is in $\mathbb{Z}(1)$) . We shall refer to forms satisfying this integrality requirement as 2π-*integral.* Given $\{f_\alpha\} \in H^0(M, \underline{U(1)}) \cong H^1(M, \mathbb{Z}(1)_D)$ we have $d\log f \in \Omega^1(M)_0$. This is the globally defined since $d(\log_\alpha f - \log_\beta f) = 0$ and may be thought of as a lower dimensional version of the curvature of a connection. If $\{f_i\} \in \ker(d\log)$ then there are $U(1)$-valued constants $c_i = f_i$ which are classes in $H^0(M, U(1))$.

[1] This quasi-isomorphism is derived from the exact sequence

$$0 \to \mathbb{Z}_M \to \mathbb{R}_M \to \underline{U(1)}_M \to 0$$

which may be replaced by

$$0 \to \mathbb{Z}_M \to \underline{\mathbb{C}}_M \to \underline{\mathbb{C}}^\times_M \to 0$$

giving an equivalent theory in terms of $\mathbb{C}^\times$ rather than $U(1)$.

2.3 $U(1)$-Bundles

We present basic material on the relationship between principal $U(1)$-bundles and Deligne cohomology to further develop the theory of geometric realisations of Deligne classes.

We follow the detailed treatment of the role of line bundles as geometric realisations of degree 2 Deligne cohomology in Brylinski's book [5]. Let P denote a principal $U(1)$ bundle over M. It is well known that the isomorphism classes of $U(1)$ bundles corresponds to the sheaf cohomology group $H^1(M, \underline{U(1)}_M)$. A representative, $g_{\alpha\beta}$, of $H^1(M, \underline{U(1)}_M)$ corresponds to the transition functions of the bundle. T here is an isomorphism with Čech cohomology $H^1(M, \underline{U(1)}_M) \cong H^2(M, \mathbb{Z})$. The image of $g_{\alpha\beta}$ under this isomorphism is the Chern class, $n_{\alpha\beta\gamma} = -\log(g_{\beta\gamma}) + \log(g_{\alpha\gamma}) - \log(g_{\alpha\beta})$. Note that there is also an isomorphism with Deligne cohomology $H^1(M, \underline{U(1)}_M) \cong H^2(M, \mathbb{Z}(1)_D)$, where the Deligne class corresponding to $g_{\alpha\beta}$ is $(n_{\alpha\beta\gamma}, \log(g_{\alpha\beta}))$.

It is also well known that isomorphism classes of bundles with connection lie in the hypercohomology group $H^1(M, \underline{U(1)}_M \to \underline{\Omega}^1_M) \equiv H^1(M, \mathcal{D}^1)$. We use the Čech resolution of the complex to produce explicit representatives of these hypercohomology classes. If $(g_{\alpha\beta}, A_\alpha)$ is a class in $H^1(M, \mathcal{D}^1)$, then it represents a $U(1)$-bundle with transition functions $g_{\alpha\beta}$ and local connection 1-forms A_α.

The space of bundles with connection is related to the space of bundles via the exact sequence [22]

$$0 \to \Omega^1(M)/\Omega^1(M)_0 \to H^2(M, \mathbb{Z}(2)_D) \to H^2(M, \mathbb{Z}) \to 0 \tag{2.4}$$

The quasi-isomorphism of complexes of sheaves (2.1) induces the usual isomorphism

$$H^1(M, \mathcal{D}^1) \cong H^2(M, \mathbb{Z}(2)_D)$$

Substituting $p = 2$ into the exact sequence (2.3) gives the exact sequence

$$0 \to H^1(M, U(1)) \xrightarrow{i} H^1(M, \mathcal{D}^1) \xrightarrow{d} \Omega^2(M)_0 \to 0$$

where d is the map which applies d to the component of $H^1(M, \mathcal{D}^1)$ with the highest d-degree. Geometrically, d maps a bundle with connection to its curvature 2-form. This implies that $H^1(M, U(1))$ represents the set of flat bundles on M. This can be seen explicitly in the following way [28], let $(g_{\alpha\beta}, A_\alpha)$ represent a flat bundle. Thus we have $dA_\alpha = 0$. Each element of a good cover is contractible so Poincare's Lemma applies and there exist $U(1)$-valued functions a_α such that $d\log a_\alpha = A_\alpha$. Now we have

$$\begin{aligned} d\log g_{\alpha\beta} &= A_\alpha - A_\beta \\ &= d\log a_\alpha - d\log a_\beta \\ d\log(g_{\alpha\beta} \cdot a_\alpha^{-1} \cdot a_\beta) &= 0 \end{aligned}$$

so we have constants $c_{\alpha\beta} = g_{\alpha\beta} \cdot a_\alpha^{-1} \cdot a_\beta$ which represent a cocycle in $H^1(M, U(1))$. We shall refer to the cocycle $c_{\alpha\beta}$ as the flat holonomy of the bundle represented by $(g_{\alpha\beta}, A_\alpha)$.

The space of flat bundles may also be represented by the Deligne cohomology groups $H^1(M, \mathcal{D}^p)$ for $p > 1$. To see why this is so, consider what happens when the Deligne differential, D, is applied to a class $(g_{\alpha\beta}, A_\alpha) \in H^1(M, \mathcal{D}^1)$.

$$D(g_{\alpha\beta}, A_\alpha) = (\delta(g)_{\alpha\beta\gamma}, d\log(g_{\alpha\beta}) + \delta(A)_{\alpha\beta})$$

This leads to the usual requirements for $(g_{\alpha\beta}, A_\alpha)$ to represent a bundle with connection. If we truncate the Deligne complex at a higher value of p then the third component of $D(g_{\alpha\beta}, A_\alpha)$ will be dA_α. This means that a Deligne cycle will represent a flat bundle.

2.4 Bundle 0-Gerbes

These were introduced by Murray [35]. The objects described here should actually be called $U(1)$-bundle 0-gerbes, however since we only use this type we omit the $U(1)$ prefix. Initially it may seem that bundle 0-gerbes are just a more complicated way of looking at line bundles, certainly if one was interested only in line bundles then there would be little point in studying them. Our motivation is that we are working towards bundle gerbes and bundle 2-gerbes. In this situation there are two advantages to considering bundle 0-gerbes. Several properties of these higher objects also appear in the bundle 0-gerbe case so it is useful to become familiar with them in a simpler setting. Secondly we are interested in viewing all of these objects as part of a hierarchy and it will become clear that the lower dimensional geometric realisation of Deligne cohomology should be a bundle 0-gerbe rather than a line bundle. In this way bundle 0-gerbes will be useful in gaining an understanding of this hierarchy.

Definition 2.2. Let $Y \to M$ be a submersion. Let $Y^{[2]}$ denote the fibre product

$$Y^{[2]} = Y \times_\pi Y = \{(y, y') \in Y^2 | \pi(y) = \pi(y')\}$$

and let

$$g : Y^{[2]} \to U(1)$$

be a $U(1)$-function satisfying the cocycle identity

$$g(y_1, y_2)g(y_2, y_3) = g(y_1, y_3).$$

The triple (g, Y, M) defines a $U(1)$ *bundle 0-gerbe.*

Note that the cocycle identity implies that $g(y, y) = 1$ and $g(y_1, y_2) = g^{-1}(y_2, y_1)$.

Recall that a submersion is an onto map with onto differential. It admits local sections and all fibrations are submersions, however there exist submersions which are not fibrations.

Bundle 0-gerbes may be represented diagrammatically in the following way:

$$\begin{tikzcd} & U(1) \\ Y^{[2]} \arrow[ur, "g"] \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[d] \\ & M \end{tikzcd}$$

A bundle 0-gerbe is called *trivial* if there exists a $U(1)$-function h on Y satisfying

$$g(y_1, y_2) = h(y_1)^{-1}h(y_2).$$

In this case we write $g = \delta(h)$. The dual of a bundle 0-gerbe (g, Y, M) is defined as (g^{-1}, Y, M). Given bundle 0-gerbes (g, Y, M) and (g', Y', M) we can take the product

$$(g, Y, M) \otimes (g', Y', M) = (g \cdot g', Y \times_\pi Y', M)$$

which is easily verified to be a bundle 0-gerbe.

A *bundle 0-gerbe morphism* is a smooth map $\phi : Y \to Y'$ such that $\pi' \circ \phi = \pi$ and $g = g' \circ \phi^{[2]}$ where $\phi^{[2]} : Y^{[2]} \to Y'^{[2]}$ is induced by the fibre product.

We say that two bundle 0-gerbes (g, Y, M) and (g', Y', M) are stably isomorphic if there exists a trivial bundle 0-gerbe $(\delta(h), X, M)$ and a bundle 0-gerbe morphism

$$(g, Y, M) \cong (g', Y', M) \otimes (\delta(h), X, M).$$

Since $(g, Y, M) \otimes (g^{-1}, Y, M)$ is canonically trivial then this condition is equivalent to requiring that $(g, Y, M) \otimes (g'^{-1}, Y', M)$ is trivial.

An example of a stable isomorphism may be defined in the following way. Let (g, Y, M) and (g', Y', M) be two bundle 0-gerbes and suppose there exists $\phi : Y' \to Y$ such that $\pi_{Y'} = \pi_Y \circ \phi$ and $g' = g \circ \phi^{[2]}$. Then (g, Y, M) and (g', Y', M) are stably isomorphic.

To see this consider the product bundle 0-gerbe

$$\begin{array}{ccc} & & U(1) \\ & \overset{g'^{-1}g}{\nearrow} & \\ (Y' \times Y)^{[2]} & \rightrightarrows & Y' \times Y \\ & & \downarrow \\ & & M \end{array}$$

The function $g'^{-1}g : (Y' \times Y)^{[2]} \to U(1)$ is defined by

$$\begin{aligned} g'^{-1}g(y_1', y_2', y_1, y_2) &= g'^{-1}(y_1', y_2')g(y_1, y_2) \\ &= g^{-1}(\phi(y_1'), \phi(y_2'))g(y_1, y_2) \\ &= g^{-1}(\phi(y_1'), y)g^{-1}(y, \phi(y_2'))g(y_1, y)g(y, y_2) \\ &= g^{-1}(\phi(y_1'), y_1)g(\phi(y_2'), y_2) \\ &= \delta(g(\phi(y'), y)) \end{aligned}$$

and hence the two bundle 0-gerbes are stably isomorphic.

Lemma 2.1. *The set of stable isomorphism classes of bundle 0-gerbes forms a group.*

The associativity of the product is clear. The identity element is the equivalence class of trivial bundle 0-gerbes and the inverse of (g, Y, M) is (g^{-1}, Y, M).

Proposition 2.1. *The group of stable isomorphism classes of bundle 0-gerbes over M is isomorphic to $H^1(M, \underline{U(1)})$.*

Proof. Let (g, Y, M) be a bundle 0-gerbe. Define a Čech cycle on M by

$$g_{\alpha\beta}(x) = g(s_\alpha, s_\beta)$$

where s_α and s_β are local sections on M. Independence of the choice of cover follows from the standard argument in the case of the Chern class of a bundle, for example see Theorem 2.1.3 of Brylinski [5].

Suppose we choose different sections and define

$$g'_{\alpha\beta}(m) = g(s'_\alpha(m), s'_\beta(m)).$$

Using the cocycle identity for g,

$$\begin{aligned} g_{\alpha\beta}(m) &= g(s_\alpha(m), s_\beta(m)) \\ &= g(s_\alpha(m), s'_\alpha(m))g(s'_\alpha(m), s'_\beta(m))g(s'_\beta(m), s_\beta(m)) \\ &= g'_{\alpha\beta}(m)\delta(h(m))_{\alpha\beta} \end{aligned}$$

where $h_\alpha(m) = g(s'_\alpha(m), s_\alpha(m))$. Therefore $g_{\alpha\beta}$ is a well defined Čech cochain on M. Furthermore the cocycle identity on g makes $g_{\alpha\beta}$ a Čech cocycle.

Consider the cocycle corresponding to the product bundle 0-gerbe $g \otimes g'$. Clearly

$$(g \otimes g')_{\alpha\beta}(m) = g_{\alpha\beta}(m)g'_{\alpha\beta}(m)$$

Thus we have a homomorphism from the set of bundle 0-gerbes to $Z^1(M, \underline{U(1)})$. Suppose (g, Y, M) is a trivial bundle 0-gerbe with trivialisation $h : Y \to \overline{U(1)}$. In this case

$$\begin{aligned} g_{\alpha\beta}(m) &= h^{-1}(s_\alpha(m))h(s_\beta(m)) \\ &= \delta(h(m))_{\alpha\beta} \end{aligned}$$

where $h_\alpha(m) = h(s_\alpha(m))$. This ensures that stable equivalence classes map to Čech cohomology classes and hence we have a homomorphism from the set of stable equivalence classes of bundle 0-gerbes to $H^1(M, \underline{U(1)})$.

Suppose that for a bundle 0-gerbe (g, Y, M), $g_{\alpha\beta}$ is trivial. Then there is a bundle 0-gerbe trivialisation given by

$$h(y) = g(s_\alpha(\pi(y)), y)h_\alpha(\pi(y)).$$

It is easily verified that this is independent of α. This proves injectivity of the homomorphism. To prove surjectivity we construct a bundle 0-gerbe corresponding to a class in $H^2(M, \mathbb{Z})$ by following the method of theorem 2.1.3 of [5].

Let $g_{\alpha\beta} \in H^1(M, \underline{U(1)})$. We define a bundle 0-gerbe (g, Y, M) where $Y = \coprod_{\alpha \in A} U_\alpha$ and $g(y_1, y_2) = g_{\alpha\beta}(\pi(y_1))$ where $y_1 \in U_\alpha \subset Y$ and $y_2 \in U_\beta \subset Y$. Since $g(s_\alpha(m), s_\beta(m)) = g_{\alpha\beta}(m)$ this construction proves surjectivity. □

Corollary 2.1. *The group of stable isomorphism classes of bundle 0-gerbes on M is isomorphic to $H^2(M, \mathbb{Z})$.*

Corollary 2.2. *The group of stable isomorphism classes of bundle 0-gerbes on M is isomorphic to the group of isomorphism classes of bundles on M.*

The $U(1)$ bundle corresponding to a bundle 0-gerbe is defined by letting the total space be $Y \times S^1$ with the equivalence relation

$$(y_1, g(y_1, y_2)) \sim (y_2, 1).$$

Conversely, given a bundle (P, M) the corresponding bundle 0-gerbe is (g, P, M) where g is defined by $p_1 g(p_1, p_2) = p_2$.

We can describe an explicit correspondence between bundle isomorphisms and bundle 0-gerbe stable isomorphisms. Let (g, Y, M) and (h, X, M) be two bundle 0-gerbes and suppose $\phi : Y \to X$ is a stable isomorphism. We claim that $\tilde{\phi} : Y \times S^1/\sim \to X \times S^1/\sim$ defined by $\tilde{\phi}([y, \theta]) = [\phi(y), \theta]$ is an isomorphism of the corresponding bundles. First we check that this map is well defined on equivalence classes. Consider

$$\begin{aligned}\tilde{\phi}([y_1, g(y_1, y_2)]) &= [\phi(y_1), g(y_1, y_2)] \\ &= [\phi(y_1), h(\phi(y_1), \phi(y_2))] \\ &= [\phi(y_2), 1] \\ &= \tilde{\phi}([y_2, 1])\end{aligned}$$

Clearly the S^1 action is preserved by this map. Now suppose that $\tilde{\phi}([y_1, \theta_1]) = \tilde{\phi}([y_2, \theta_2])$. Then

$$\begin{aligned}[\phi(y_1), \theta_1] &= [\phi(y_2), \theta_2] \\ &= [\phi(y_1), h(\phi(y_1), \phi(y_2))\theta_2] \\ &= [\phi(y_1), g(y_1, y_2)\theta_2]\end{aligned}$$

so $\theta_1 = g(y_1, y_2)\theta_2$ and thus $[y_1, \theta_1] = [y_1, g(y_1, y_2)\theta_2] = [y_2, \theta_2]$. We define an inverse of ϕ by

$$\tilde{\phi}^{-1}([x, \theta]) = [y, h(\phi(y)x)\theta] \tag{2.5}$$

where y is any element of Y. It is independent of this choice since given $y_1, y_2 \in Y$ we have

$$\begin{aligned}[y_2, h(\phi(y_2), x)\theta] &= [y_1, g(y_1, y_2)h(\phi(y_2), x)\theta] \\ &= [y_1, h(\phi(y_1), \phi(y_2))h(\phi(y_2), x)\theta] \\ &= [y_1, h(\phi(y_1), x)\theta]\end{aligned} \tag{2.6}$$

Thus the map $\tilde{\phi}$ is a bundle isomorphism. It is easy to check that a bundle isomorphism defines a stable isomorphism on the associated bundle 0-gerbe.

Let π_1 and π_2 be the maps from $Y^{[2]}$ to Y defined by

$$\begin{aligned}\pi_1(y_1, y_2) &= y_2 \\ \pi_2(y_1, y_2) &= y_1\end{aligned}$$

This notation may appear counter intuitive. The idea is that the subscript on π indicates which element will be omitted. This allows the maps π_i to be generalised to $p_i : Y^{[p]} \to Y^{[p-1]}$ for $i = 1 \ldots p$.

Let δ be the pull back $\pi_1^* - \pi_2^*$.

Definition 2.3. A *bundle 0-gerbe connection*, A, is a 1-form on Y satisfying

$$\delta(A) = g^{-1}dg.$$

A bundle 0-gerbe (g, Y, M) with connection A may be written as $(g, Y, M; A)$ or $(g; A)$ where there is no ambiguity.

The existence of bundle 0-gerbe connections is established by considering the following complex which has no cohomology [36]

$$\Omega^q(M) \xrightarrow{\pi^*} \Omega^q(Y) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Omega^q(Y^{[p-1]}) \xrightarrow{\delta} \Omega^q(Y^{[p]}) \xrightarrow{\delta} \Omega^q(Y^{[p+1]}) \xrightarrow{\delta} \cdots \tag{2.7}$$

The general δ map from $\Omega^q(Y^{[p]})$ to $\Omega^q(Y^{[p+1]})$ is defined by $\delta(f) = \sum_{i=1}^{p} \pi_i^* f$. The cocycle condition on g implies that $\delta(g) = 0 \in \Omega^0(Y^{[3]})$. Since d and δ commute this means that $\delta(d\log(g)) = 0 \in \Omega^1(Y^{[3]})$ and so the exactness of the complex 2.7 implies the existence of $A \in \Omega^1(Y)$ such that $\delta(A) = d\log(g)$.

It was established in [36] that if we have $\delta(dA) = 0$ for $A \in \Omega^q(Y)$ then there exists a unique $F \in \Omega^{q+1}(M)$ satisfying $dA = \pi^*(F)$. In this case we have a two-form F which we call the *bundle 0-gerbe curvature.* It is easily shown that changing the choice of connection does not change the de Rham class of the curvature.

All of the operations which we have described on bundle 0-gerbes are possible for bundle 0-gerbes with connection as well,

$$\begin{aligned} (g, A)^* &= (g^{-1}, -A) \\ (g_1, A_1) \otimes (g_2, A_2) &= (g_1 g_2, A_1 + A_2) \end{aligned}$$

Now we show that corresponding to the bundle 0-gerbe connection is a connection for the corresponding bundle. Define a one-form on $Y \times S^1$ by

$$\tilde{A} = A + \theta^{-1} d\theta.$$

Consider this form at two equivalent points $(y_1, g(y_1, y_2))$ and $(y_2, 1)$. The difference is given by

$$A_{y_1} + d\log(g(y_1, y_2)) - A_{y_2}$$

which is equal to $d\log(g) - \delta(A)$ on $Y^{[2]}$. Since this is zero by the definition of the bundle 0-gerbe connection this 1-form is well defined on equivalence classes. Furthermore it can be easily shown that it satisfies the conditions for a connection 1-form.

Suppose A is a connection 1-form on a bundle $L \to M$. We claim that A is also a connection on the corresponding bundle 0-gerbe. This is true because

$$\begin{aligned} \delta(A)_{(y_1, y_2)} &= A_{y_2} - A_{y_1} \\ &= A_{(y_1 g(y_1, y_2))} - A_{y_1} \\ &= A_{y_1} + d\log(g)_{(y_1, y_2)} - A_{y_1} \\ &= d\log(g)_{(y_1, y_2)} \end{aligned}$$

Thus we have a correspondence between bundle 0-gerbes with connection and bundles with connection, however it is not yet clear whether this carries over to an isomorphism with Deligne cohomology. Recall that in the case without connections the role of isomorphism classes for bundles was taken by stable isomorphism classes for bundle 0-gerbes. We must define a slightly different notion of stable isomorphism for bundle 0-gerbes with connection. This is because a trivialisation of a bundle 0-gerbe corresponds to a trivialisation in Čech cohomology of the Chern class. Explicitly this is a cochain $\underline{h}$

satisfying $\delta(\underline{h}) = \underline{g}$. When we have a choice of connection there is a further requirement on $\underline{h}$ since we consider it as a Deligne cochain and require that $D(\underline{h}) = (\underline{g}, \underline{A})$. This means that h must satisfy

$$\begin{aligned} \delta(\underline{h}) &= \underline{g} \quad \text{and} & (2.8) \\ d\log(\underline{h}) &= \underline{A}. & (2.9) \end{aligned}$$

The geometric realisation of the cochain $\underline{h}$ is a function $h : Y \to S^1$ such that $\delta(h) = g$ and $d\log h = A$. We shall refer to a cochain satisfying (2.8) as a trivialisation and to one satisfying both (2.8) and (2.9) as a *D-trivialisation* and a bundle 0-gerbe with connection which has a D-trivialisation is called *D-trivial.*

Definition 2.4. Let $(g_1; A_1)$ and $(g_2; A_2)$ be bundle 0-gerbes with connection. We say that they are *D-stably isomorphic* if there exists a D-trivial bundle gerbe with connection $(\tau; C)$ and an isomorphism

$$(g_1; A_1) \cong (g_2; A_2) \otimes (\tau; C).$$

It is easy to verify that the set of D-stable isomorphism classes of bundle 0-gerbes with connection forms a group.

Proposition 2.2. *The group of D-stable isomorphism classes of bundle 0-gerbes with connection is isomorphic to $H^1(M, \mathcal{D}^1)$.*

Proof. The proof is a simple extension of that for Proposition 2.1. □

2.5 Bundle Gerbes

Bundle gerbes were introduced in [36] as a geometric realisation of classes in $H^3(M, \mathbb{Z})$. They are the key object of interest in this thesis, here we present the basic theory.

Definition 2.5. Let $Y \xrightarrow{\pi} M$ be a submersion and let $P \xrightarrow{\pi_P} Y^{[2]}$ be a $U(1)$ bundle. A *$U(1)$-bundle gerbe* is a triple (P, Y, M) together with a $U(1)$-bundle isomorphism $P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \to P_{(y_1,y_3)}$ which is called the bundle gerbe product. Associativity is required whenever triple products are defined.

The bundle gerbe (P, Y, M) is represented diagrammatically by

$$\begin{array}{ccc} P & & \\ \downarrow & & \\ Y^{[2]} & \rightrightarrows & Y \\ & & \downarrow \\ & & M \end{array}$$

Since we only deal with $U(1)$-bundle gerbes we shall refer to them simply as bundle gerbes. Often we will say that (P, Y) or P is a bundle gerbe over M when there is no ambiguity. Given a map $\phi : N \to M$ we may define the pullback $\phi^{-1}P$ which is a bundle gerbe on M. Given two bundle gerbes (P, Y, M) and (Q, X, M) there is a product bundle gerbe $(P \otimes Q, Y \times_M X, M)$. For any bundle gerbe P there exists a dual bundle gerbe P^*. For details of these constructions see [36].

In the definition of a bundle gerbe the bundle over $Y^{[2]}$ may be replaced with a bundle 0-gerbe $(\rho, X, Y^{[2]})$. In this case the product is no longer a morphism since X is not acted on by S^1. Since we are dealing with bundle 0-gerbes rather than bundles it is not surprising that morphisms should be replaced by stable morphisms. A choice of stable morphism $\rho_{(y_1,y_2)} \otimes \rho_{(y_2,y_3)} \to \rho_{(y_1,y_3)}$ is equivalent to a choice of trivialisation

$$\rho_{(y_1,y_2)} \otimes \rho_{(y_2,y_3)} \otimes \rho^*_{(y_1,y_3)} \cong \delta(m_{123})$$

This trivialisation represents a bundle gerbe product if it satisfies the associativity condition

$$m_{123} \cdot m_{134} = m_{124} \cdot m_{234}.$$

Definition 2.6. A bundle gerbe (P, Y, M) is called *trivial* if there exists a bundle $J \to Y$ such that there is a bundle isomorphism

$$P \cong \pi_1^{-1}(J) \otimes \pi_2^{-1}(J)^*$$

where π_1 and π_2 are the projections of each component of $Y^{[2]}$ onto Y. The product $\pi_1^{-1}(J) \otimes \pi_2^{-1}(J)^*$ is also denoted by $\delta(J)$.

Definition 2.7. A *bundle gerbe morphism* between (P, Y, M) and (Q, X, N) is a triple of maps (α, β, γ) where $\beta : Y \to X$ is a fibre preserving map covering $\gamma : M \to N$ and $\alpha : P \to Q$ is a bundle morphism covering the induced map $\beta^{[2]} : Y^{[2]} \to X^{[2]}$. Furthermore α must commute with the bundle gerbe product. An *isomorphism of bundle gerbes* is a bundle gerbe morphism with $M = N$ and where γ is the identity map. Two bundle gerbes P and Q are *stably isomorphic* if $P \cong Q \otimes \delta(J)$.

Proposition 2.3. *[36] The set of stable isomorphism classes of bundle gerbes over M is isomorphic to $H^3(M, \mathbb{Z})$.*

We construct a class $g_{\alpha\beta\gamma} \in H^2(M, \underline{U(1)})$ corresponding to a bundle gerbe (P, Y, M). The standard isomorphism gives a corresponding class in $H^3(M, \mathbb{Z})$ which is known as the Dixmier-Douady class. We shall also refer to $g_{\alpha\beta\gamma}$ as the Dixmier-Douady class, or by analogy with the local data associated with a bundle we shall also call these transition functions. Let s_α and s_β be two local sections of $Y \to M$ defined on $U_\alpha \subset M$ and $U_\beta \subset M$ respectively. These define a section $(s_\alpha, s_\beta) : U_{\alpha\beta} \to Y^{[2]}$. Use this section to form the pull-back bundle $P_{\alpha\beta} = (s_\alpha, s_\beta)^* P$ over $U_{\alpha\beta}$. Since $U_{\alpha\beta}$ is contractible $P_{\alpha\beta}$ is trivial and so admits a global section which we shall denote by $\sigma_{\alpha\beta} : U_{\alpha\beta} \to P_{\alpha\beta}$. Over the triple intersection $U_{\alpha\beta\gamma}$ the bundle gerbe product gives a bundle isomorphism $P_{\alpha\beta} \otimes P_{\beta\gamma} \cong P_{\alpha\gamma}$. Thus we can define $g_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to U(1)$ by

$$\sigma_{\alpha\beta} \otimes \sigma_{\beta\gamma} = \sigma_{\alpha\gamma} g_{\alpha\beta\gamma}.$$

To get a class in Deligne cohomology we will also need to define connections and curvings on bundle gerbes.

Definition 2.8. Let (P, Y, M) be a bundle gerbe. A *bundle gerbe connection*, A, is a connection on the bundle $P \to Y^{[2]}$ which commutes with the bundle gerbe product.

Definition 2.9. Let (P, Y, M) be a bundle gerbe with connection A. Let $F_A \in \Omega^2(Y^{[2]})$ be the curvature of A considered as a bundle connection on $P \to Y^{[2]}$. A *curving* is a 2-form η on Y satisfying $\delta(\eta) = F_A$.

A bundle gerbe (P, Y, M) with connection A and curving η may also be referred to as $(P, Y, M; A, \eta)$ or $(P; A, \eta)$. We may now define the Deligne class associated to a bundle gerbe (P, Y, M) with connection, A, and curving, η. Given local sections $s_\alpha : U_\alpha \to Y$ we may define the *local curvings*

$$\eta_\alpha = s_\alpha^* \eta.$$

We have already defined the bundles $P_{\alpha\beta}$. The pull back by (s_α, s_β) induces connections on each of these bundles which may be pulled back to $U_{\alpha\beta}$ using the sections $\sigma_{\alpha\beta}$ to give a collection of 1-forms $A_{\alpha\beta}$ on double intersections of open sets on M. We call these *local connections.*

Proposition 2.4. *[36] Let (P, Y, M) be a bundle gerbe with connection and curving. Let $g_{\alpha\beta\gamma}$ be the Dixmier-Douady class, $A_{\alpha\beta}$ be the local connections and η_α be the local curvings. Then $(g_{\alpha\beta\gamma}, A_{\alpha\beta}, \eta_\alpha)$ defines a class in the sheaf cohomology group $H^2(M, \mathcal{D}^2)$.*

For $(g_{\alpha\beta\gamma}, A_{\alpha\beta}, \eta_\alpha)$ to be a class in $H^2(M, \mathcal{D}^2)$ it must satisfy

$$\begin{aligned} g_{\beta\gamma\delta} - g_{\alpha\gamma\delta} + g_{\alpha\beta\delta} - g_{\alpha\beta\gamma} &= 0 && (2.10)\\ A_{\beta\gamma} - A_{\alpha\gamma} + A_{\alpha\beta} &= d\log(g_{\alpha\beta\gamma}) && (2.11)\\ \eta_\alpha - \eta_\beta &= dA_{\alpha\beta} && (2.12) \end{aligned}$$

As in the previous cases there is an isomorphism

$$H^3(M, \mathbb{Z}(3)_D) \cong H^2(M, \mathcal{D}^2)$$

so each bundle gerbe with connection and curving gives rise to an element of $H^3(M, \mathbb{Z}(3)_D)$. Explicitly the Deligne class is given by

$$(n_{\alpha\beta\gamma\delta}, \log(g_{\alpha\beta\gamma}), A_{\alpha\beta}, \eta_\alpha)$$

where $n_{\alpha\beta\gamma\delta} = \delta(\log(g))_{\alpha\beta\gamma\delta}$ though we will often refer to the class $(g_{\alpha\beta\gamma}, A_{\alpha\beta}, \eta_\alpha)$ as the Deligne class.

As with the case of bundle 0-gerbes it is necessary to introduce D-trivialisations for bundle gerbes with connection and curving. A D-trivialisation of a Deligne class $(\underline{g}, \underline{A}, \underline{\eta})$ is a cochain $(\underline{h}, \underline{B})$ which satisfies

$$\begin{aligned} \delta(\underline{h}) &= \underline{g} && (2.13)\\ d\log(\underline{h}) - \delta(\underline{B}) &= \underline{A} && (2.14)\\ d\underline{B} &= \underline{\eta} && (2.15) \end{aligned}$$

Geometrically a D-trivialisation of $(P; A, \eta)$ is a bundle J with connection B such that $\delta(J; B) \cong (P; A)$ as bundle gerbes with connection, where $\delta(J; B)$ is the bundle $\delta(J)$ with connection induced from B by δ. Furthermore, in order to satisfy (2.15) the curvature of $(J; B)$ must be equal to the curving η. We may define D-stable isomorphisms in the obvious way and state a bundle gerbe version of Proposition 2.2:

Proposition 2.5. *The group of D-stable isomorphism classes of bundle gerbes with connection and curving are isomorphic to $H^2(M, \mathcal{D}^2)$.*

Proof. First we show independence of the choice of sections. There are two different types of section involved in the construction of the Deligne class. Suppose the sections $\sigma_{\alpha\beta}$ are replaced by $\tilde{\sigma}_{\alpha\beta}$. We have two choices of section of a principal bundle so they differ by functions $f_{\alpha\beta}$ and the corresponding change in transition functions is

$$\tilde{g}_{\alpha\beta\gamma} = g_{\alpha\beta\gamma} f_{\alpha\beta} f_{\beta\gamma} f_{\alpha\gamma}^{-1}$$

The local connections are related by the usual change of connection formula

$$\tilde{A}_{\alpha\beta} = A_{\alpha\beta} + d \log f_{\alpha\beta}$$

and the local curvings are unaffected so the overall contribution is the trivial cocycle $D(f_{\alpha\beta}, 0)$.

Now suppose we change the sections s_α to s'_α. In general these are not sections of a principal bundle so they do not differ by a function. Using the bundle gerbe product we have an isomorphism

$$P_{(s'_\alpha, s'_\beta)} = P_{(s'_\alpha, s_\alpha)} \otimes P_{(s_\alpha, s_\beta)} \otimes P_{(s_\beta, s'_\beta)} \tag{2.16}$$

Let $\sigma_{\alpha\beta}$, $\sigma'_{\alpha\beta}$, δ_α and δ_β be sections of the trivial bundles $P_{(s_\alpha, s_\beta)}$, $P_{(s'_\alpha, s'_\beta)}$, $P_{(s'_\alpha, s_\alpha)}$ and $P_{(s'_\beta, s_\beta)}$ respectively. We have two sections, $\sigma'_{\alpha\beta}$ and $\delta_\alpha \sigma_{\alpha\beta} \delta_\beta^{-1}$ of isomorphic bundles so they differ by functions $h_{\alpha\beta}$. When comparing the transition functions defined using $\sigma_{\alpha\beta}$ or $\sigma'_{\alpha\beta}$ the δ sections all cancel out and we have essentially the previous case. Equation (2.16) also leads to an equation involving local connections,

$$A'_{\alpha\beta} = k_\alpha + A_{\alpha\beta} - k_\beta \tag{2.17}$$

where k_α is defined by pulling back the bundle gerbe connection to $P_{(s'_\alpha, s_\alpha)}$ and then pulling this connection back to U_α using the section δ_α. Consider what happens to the local curvings. Since η satisfies $\delta(\eta) = F$ then ${s'_\alpha}^*\eta - {s_\alpha}^*\eta$ is equal to the curvature of $P_{(s'_\alpha, s_\alpha)}$ which is dk_α, so

$$\eta'_\alpha = \eta_\alpha + dk_\alpha \tag{2.18}$$

so we have added a trivial cocycle $D(1, k_\alpha)$. Hence the Deligne class is independent of all choices of sections.

The homomorphism property is a straightforward consequence of the definition of the tensor product of bundle gerbes and the Deligne class so we omit details.

The result that a bundle gerbe is trivial if and only if it has a trivial Čech class has been discussed in detail elsewhere ([36],[44]). Essentially it comes down to the fact that for a trivial bundle gerbe the sections $\sigma_{\alpha\beta}$ are of the form $\delta_\alpha^* \otimes \delta_\beta$. The inclusion of connections and curvings does not add any significant complications.

Finally we need to describe a bundle gerbe which is classified by a particular Deligne class $(g_{\alpha\beta\gamma}, A_{\alpha\beta}, \eta_\alpha)$. Let $Y = \amalg_\alpha U_\alpha$, the disjoint product of all of the elements of the open cover of M. Let $P \to Y^{[2]}$ be the trivial bundle. An element of $Y^{[2]}$ is of the form (m_α, m_β) where $m \in U_{\alpha\beta}$ and m_α is m considered as an element of $U_\alpha \in Y$. The define the bundle gerbe product by

$$(m_\alpha, m_\beta, z_1) \cdot (m_\beta, m_\gamma, z_2) = (m_\alpha, m_\gamma, z_1 z_2 g_{\alpha\beta\gamma}) \tag{2.19}$$

where $z_1, z_2 \in U(1)$. Since P is trivial then we can define the connection as a 1-form on $Y^{[2]}$. At $(m_\alpha, m_\beta) \in Y^{[2]}$ the connection 1-form is given by $A_{\alpha\beta}$ at m. Define the curving on $U_\alpha \in Y$ by η_α.

□

We have only considered bundle gerbes with connection and curving. It is easily seen that bundle gerbes with a choice of connection but no choice of curving are classified by $H^3(M, \mathbb{Z}(2)_D) \equiv H^2(M, \mathcal{D}^1)$. It is a standard result (see [5]) that $H^p(M, \mathbb{Z}(q)_D) \cong H^p(M, \mathbb{Z}(1)_D)$ whenever $p > q$, thus the stable isomorphism class of bundle gerbe with connection is invariant under a change of connection.

As in the previous cases the exact sequence (2.3)

$$0 \to H^2(M, U(1)) \xrightarrow{i} H^2(M, \mathcal{D}^2) \xrightarrow{d} \Omega^3(M)_0 \to 0$$

gives the curvature and flat holonomy. Explicitly the curvature is $\omega \in \Omega^3(M)$ satisfying

$$\pi^*\omega = d\eta$$

and is guaranteed to exist since $\delta(d\eta) = 0$. In terms of local curvings the 3-curvature is defined in terms of a collection of local 3-forms $\omega_\alpha = d\eta_\alpha$ which agree on overlaps since $\delta(d\eta_\alpha) = 0$.

The flat holonomy is calculated in the following way [28]. Suppose $\omega = 0$. Then $d\eta_\alpha = 0$ so there exist local 1-forms B_α satisfying $dB_\alpha = \eta_\alpha$. Furthermore

$$\eta_\beta - \eta_\alpha = dA_{\alpha\beta} = d(B_\beta - B_\alpha)$$

so there exists functions $a_{\alpha\beta}$ which are defined on double intersections and satisfy

$$A_{\alpha\beta} - B_\beta + B_\alpha = d\log(a_{\alpha\beta}).$$

It follows that

$$d\log(a_{\alpha\beta} \cdot a_{\beta\gamma} \cdot a_{\alpha\gamma}^{-1} \cdot g_{\alpha\beta\gamma}^{-1}) = 0$$

and the flat holonomy is

$$c_{\alpha\beta\gamma} = a_{\alpha\beta} \cdot a_{\beta\gamma} \cdot a_{\alpha\gamma}^{-1} \cdot (g_{\alpha\beta\gamma}^{-1})$$

We conclude our discussion of flat bundle gerbes with the observation that the Deligne cohomology groups $H^2(M, \mathbb{Z}(p)_D)$ represent flat bundle gerbes for any $p > 2$. A class in this cohomology group is the same as a class

$$(g_{\alpha\beta\gamma}, A_{\alpha\beta}, \eta_\alpha) \in H^2(M, \mathcal{D}^2)$$

with the additional condition that $d\eta_\alpha = 0$, therefore the class represents a flat bundle gerbe.

Chapter 3

Examples of Bundle Gerbes

We define and present the basic properties of a number of examples of bundle gerbes which shall be of use to us.

3.1 Tautological Bundle Gerbes

The tautological bundle gerbe was introduced in [36] as a way to construct a bundle gerbe with any given closed, 2π-integral 3-form as its 3-curvature. Our approach will be similar to that in [44] however we use bundle 0-gerbes rather than bundles.

Let M be a 1-connected manifold with distinguished base point m_0. Denote by $\mathcal{P}_0M$ the space of paths in M which are based at m_0. An element of $\mathcal{P}_0M$ is a map $\mu : [0,1] \to M$ such that $\mu(0) = m_0$. There is a fibration $\mathcal{P}_0M \to M$ defined by the projection $\pi : \mu \mapsto \mu(1)$. The fibre product $\mathcal{P}_0M^{[2]}$ over $m \in M$ consists of pairs of paths between m_0 and m. By reversing the orientation of one of the paths this pair may be identified with a loop based at m_0. Thus we can identify $P_0M^{[2]}$ with $\mathcal{L}_0M$, the space of smooth loops in M which are based at m_0. There is a technical point that needs to be dealt with here. When two paths are joined together the resultant loop may not be smooth at the two points where the paths are joined. To overcome this problem we follow Caetano and Picken [10] and re-parametrise the paths around these points such that there is a sitting instant at each of these points, that is, an interval of length ϵ around a point $t_0 \in [0,1]$ such that the loop is constant in the interval $(t_0 - \epsilon, t_0 + \epsilon)$. The obvious adjustment is made when $t_0 = 0$ (or equivalently $t_0 = 1$). The structure above $\mathcal{L}_0M$ is defined in terms of an integral which is invariant under such reparametrisations.

Let F be a closed, 2π-integral 2-form. Let $\rho : \mathcal{L}_0M \to U(1)$ be defined by

$$\rho(\gamma) = \exp(\int_\Sigma F) \tag{3.1}$$

where Σ is any surface such that $\partial\Sigma = \gamma$. Equivalently we may write

$$\rho(\mu_1, \mu_2) = \exp(\int_{I^2} H^*F) \tag{3.2}$$

where $\mu_1, \mu_2 \in \mathcal{P}M^{[2]}$ and H is a homotopy between μ_1 and μ_2. To see that ρ is independent of the choice of Σ note that if we choose a different surface, $\bar{\Sigma}'$, where the bar indicates that the opposite orientation is induced on the boundary, and let

$$\rho'(\gamma) = \exp(\int_{\Sigma'} F)$$

then we have

$$\begin{aligned}\rho(\gamma)/\rho'(\gamma) &= \exp(\int_{\Sigma \cup \bar{\Sigma}'} F) \\ &= 1\end{aligned}$$

Let $\mu_1, \mu_2, \mu_3 \in \pi^{-1}(m) \subset \mathcal{P}_0 M$ and let γ_{ij} denote the loop identified with $(\mu_i, \mu_j) \in \mathcal{P}_0 M^{[2]}$. Then

$$\begin{aligned}\rho(\gamma_{12})\rho(\gamma_{23}) &= \exp(\int_{\Sigma_{12}} F + \int_{\Sigma_{23}} F) \\ &= \exp(\int_{\Sigma_{12} \cup \Sigma_{23}} F).\end{aligned}$$

Note that γ_{12} and γ_{23} are connected along μ_2 and hence the surface $\Sigma_{12} \cup \Sigma_{23}$ has boundary γ_{13} and the cocycle condition

$$\rho(\gamma_{12})\rho(\gamma_{23}) = \rho(\gamma_{13})$$

is satisfied. Therefore $(\rho, \mathcal{P}_0 M, M)$ defines a bundle 0-gerbe. Furthermore if we let the connection form on $\mathcal{P}_0 M$ be given by

$$A = \int_I ev^* F$$

where ev is the evaluation map $ev : \mathcal{P}_0 M \times I \to M$ then it may be shown that A satisfies

$$\begin{aligned}\delta(A) &= d \log \rho \\ dA &= \pi^* F\end{aligned}$$

If M is not connected then we may carry out this construction on each connected component.

Lemma 3.1. *The tautological bundle 0-gerbe is independent (up to stable isomorphism) of the choice of base point in M.*

Proof. Suppose we have a curvature form F and two choices of base point, m_0 and m_1. We shall show that the resulting tautological bundle 0-gerbes are stably isomorphic. Over M we can form two different path fibrations, $\mathcal{P}_0 M$ and $\mathcal{P}_1 M$ using the two choices of base point. Form the corresponding tautological bundle 0-gerbes and take the fibre product,

$$\begin{array}{ccc} & & S^1 \\ & \overset{\rho_0^{-1}\rho_1}{\nearrow} & \\ \mathcal{L}_0 M \times_\pi \mathcal{L}_1 M & \rightrightarrows & \mathcal{P}_0 \times_\pi \mathcal{P}_1 \\ & & \downarrow \\ & & M \end{array} \tag{3.3}$$

An element of $\pi^{-1}(m) \subset \mathcal{P}_0 \times_\pi \mathcal{P}_1$ is a path from m_0 to m_1 passing through m. An element of $\mathcal{L}_0 M \times_\pi \mathcal{L}_1 M$ is a figure eight with m at the centre with each loop passing through either m_0 or m_1. To define a trivialisation of this bundle 0-gerbe we need to choose a path q from m_0 to m_1. The trivialisation is then given by the function $h(\mu, \eta) = \exp \int_\Sigma F$ where Σ is a surface bounded by $q^{-1} \star \mu^{-1} \star \eta$. Taking δ of h gives the integral of F over a surface with boundary $\eta_1^{-1} \star \mu_1 \star q \star q^{-1} \star \mu_2^{-1} \star \eta_2$. After eliminating the $q \star q^{-1}$ component this is equal to $\rho_0^{-1}\rho_1$, therefore the two bundle 0-gerbes are stably isomorphic. Calculation of $d \log h$ at $(X_0, X_1) \in T(\mathcal{P}_0 M \times_\pi \mathcal{P}_1 M)$ gives $\int_{\mu_1} F(\mu_1', X_1) - \int_{\mu_0} F(\mu_0', X_0)$ which is equal to $A_1 - A_0$, the difference of the connections corresponding to each choice of base point, so h defines a D-stable morphism. Since the construction depends on the choice of q this is not a canonical stable isomorphism. □

Example 3.1. Let G be a compact simply connected semisimple Lie group. Let M be the loop group $\mathcal{L}G$ and let the curvature 2-form be $\int_{S^1} ev^* < g^{-1}dg \wedge [g^{-1}dg \wedge g^{-1}dg] >$ where ev is the evaluation map $ev : \mathcal{L}G \times S^1 \to G$, $<,>$ is the Killing form and $[,]$ is the Lie bracket. We may then construct a tautological bundle 0-gerbe. Since G is simply connected, discs in G may be thought of as paths of loops based at a constant loop and may be recentred as in lemma 3.1. This means that we may consider the fibre over γ to consist of discs bounded by γ. The bundle obtained by the standard construction from this tautological bundle 0-gerbe is the central extension of the loop group $\widetilde{\mathcal{L}G} \to \mathcal{L}G$ as described by Mickelsson [32].

Now suppose that we have a closed, 2π-integral 3-form, ω on a 2-connected manifold M. Let $Q[F] \to (\mathcal{P}_0 M)^{[2]}$ be the tautological bundle over $(\mathcal{P}_0 M)^{[2]}$ with curvature $F = \int_{S^1} ev^*\omega$. Here we have identified $(\mathcal{P}_0 M)^{[2]}$ with $\mathcal{L}_0 M$ and used the evaluation map $ev : \mathcal{L}_0 M \times S^1 \to M$. The tautological bundle on $\mathcal{L}_0 M$ may be defined since M is 2-connected. To avoid the need for a base point in $\mathcal{L}_0 M$ we shall use a slightly different definition of tautological bundle. In fact the tautological construction is more natural over a fibre product space, the introduction of a base point when the base is not a fibre product compensates for this. This construction of the tautological bundle follows the approach of [13]. Over $\mathcal{L}_0 M$ we have the space $\Sigma^\partial M$ of 2-surfaces, such that the fibre over γ is a surface with γ as its boundary. Elements of the fibre product may be considered as elements of ΣM, the space of smooth maps of closed 2-surfaces into M (with possible reparametrisation to deal with any problems with smoothness along γ) and we may define the tautological function in the usual way to give a tautological bundle $Q[F] \to \mathcal{L}_0 M$. We can now construct a bundle gerbe over M,

$$\begin{array}{ccc} Q[F] & & \\ \downarrow & & \\ \mathcal{L}_0 M & \rightrightarrows & \mathcal{P}_0 M \\ & & \downarrow \\ & & M \end{array}$$

To define the product we observe that for any Y the fibration $\mathcal{P}_0(Y^{[2]}) \to Y^{[2]}$, where the base point lies in the diagonal subset of $Y^{[2]}$, admits a product covering $(y_1, y_2) \times (y_2, y_3) \to (y_1, y_3)$ which is defined by composition of paths. Strictly speaking the composition of paths is not associative, however we do have associativity up to reparametrisations which do not affect the overall structure of the bundle gerbe. In general this will be the bundle gerbe product for any bundle gerbes which we define in terms of a bundle on the loop space. The connection is given by the connection on the tautological bundle, which in this case may be written as

$$A = \int_{\Sigma^\theta} ev^* \omega$$

If the curving is defined by

$$\eta = \int_I ev^* \omega$$

then the curvature is ω.

As with the tautological bundle 0-gerbe, the two tautological bundle gerbes obtained by a change of base point are stably isomorphic. The trivialisation over $\mathcal{P}_0 \times_\pi \mathcal{P}_1$ is defined by $J_{(\mu,\eta)} = Q[F]_{(\mu \star q^{-1}, \eta)}$ where $\mu \in \mathcal{P}_0 M$, $\eta \in \mathcal{P}_1 M$, q is a path from m_0 to m_1 and $Q[F]$ is the tautological bundle over $\mathcal{L}_1 M$. Using the product on $Q[F]$ it can be shown that

$$\delta(J)_{(\mu_1,\eta_1,\mu_2,\eta_2)} = Q[F]^*_{(\mu_1 \star q^{-1}, \mu_2 \star q^{-1})} \otimes Q[F]_{(\eta_1,\eta_2)} \tag{3.4}$$

Consider the fibre in $Q[F]$ over $(\mu_1 \star q^{-1}, \mu_2 \star q^{-1})$. This consists of surfaces bounded by $\mu_2 \star q^{-1} \star q \star \mu_1$ and may be identified with $Q[F]_{(\mu_1,\mu_2)}$ over $\mathcal{L}_0 M$. Thus we see that J is a trivialisation.

We sometimes abbreviate the tautological bundle as $Q[F] \to M$ and the tautological bundle gerbe as $Q[\omega] \Rightarrow M$.

3.2 Trivial Bundle Gerbes

In the previous chapter we defined what it means for a bundle gerbe to be trivial or D-trivial. In this section we examine the properties of these classes of bundle gerbe.

Lemma 3.2. **([36])** *Let (P, Y, M) be a bundle gerbe. Suppose the projection $Y \xrightarrow{\pi} M$ admits a global section, s. Then (P, Y, M) is a trivial bundle gerbe.*

The trivialisation is $(s \circ \pi, 1)^{-1} P$. The converse of this proposition is not true. To see this, consider the following counterexample. Let $Y \to M$ be a projection which admits local sections, but has no global section. Let $Y^{[2]} \times S^1 \to Y^{[2]}$ be the trivial bundle. We make $(Y^{[2]} \times S^1, Y, M)$ into a bundle gerbe with the product $(y_1, y_2, \theta) \times (y_2, y_3, \phi) = (y_1, y_3, \theta\phi)$. There are sections

$$\sigma_{\alpha\beta} : U_{\alpha\beta} \to (s_\alpha, s_\beta)^{-1}(Y^{[2]} \times S^1)$$

which are given by

$$\sigma_{\alpha\beta}(m) = (s_\alpha(m), s_\beta(m), 1)$$

and which clearly satisfy the cocycle identity

$$\sigma_{\alpha\beta}\sigma_{\beta\gamma} = \sigma_{\alpha\gamma}.$$

Thus the Dixmier-Douady class is 1 and the bundle gerbe is trivial.

As a special case of lemma 3.2 we may consider restricting a bundle gerbe over M to an open set U_α. This admits a section s_α and so we may construct a trivialisation as described above.

We now review the geometric realisation of a trivial Dixmier-Douady class which was originally given in [36], and described in greater detail in [44]. Let (P, Y, M) be a bundle gerbe with Dixmier-Douady class $\underline{g}$ and let $\underline{h}$ be a trivialisation. Let J_α be the bundle on $s_\alpha(U_\alpha) \subset Y$ defined by

$$J_\alpha = (1, s_\alpha \circ \pi)^{-1} P.$$

Define isomorphisms $\phi_{\alpha\beta} : J_\alpha \to J_\beta$ by

$$\phi_{\alpha\beta}(u) = m(\sigma_{\alpha\beta} h_{\alpha\beta}^{-1} \otimes u)$$

where $u \in J_\alpha$. The bundle J obtained by gluing together the J_α using the standard clutching construction with the isomorphisms $\phi_{\alpha\beta}$ is a trivialisation of P.

Conversely, if we are given a trivialisation J then we can recover the trivialisation of the Dixmier-Douady class in the following way. Let J_α be defined by $s_\alpha^{-1} J$. Since this is a bundle over U_α it must be trivial and admits a global section δ_α. Since $\delta(J) \cong P$ then there exist functions $h_{\alpha\beta} : U_{\alpha\beta} \to S^1$ such that

$$\sigma_{\alpha\beta}(m) = (\delta_\alpha^{-1}(m) \otimes \delta_\beta(m)) h_{\alpha\beta}(m). \tag{3.5}$$

It may be shown that the $h_{\alpha\beta}$ trivialise the Dixmier-Douady class.

As an example we may calculate the local data for the canonical trivialisation over an open set U_0. The trivialisation is defined by $J_y^0 = P_{(s_0(\pi(y)),y)}$ where $s_0 : U_0 \to Y$ is a section. Over any U_α restricted to U_0 we can pull back J^0 by a section s_α to get $s_\alpha^{-1} J_m^0 = P_{(s_0(m), s_\alpha(m))}$. These have sections $\delta_\alpha = \sigma_{0\alpha}$. The local data for the trivialisation, $h_{\alpha\beta}$ is then defined by

$$\sigma_{\alpha\beta} = \sigma_{0\alpha}^{-1} \otimes \sigma_{0\beta} h_{\alpha\beta} \tag{3.6}$$

so $h_{\alpha\beta} = g_{0\alpha\beta}$.

Next we consider the relationship between trivial bundle gerbes and D-trivial bundle gerbes. To do this we first consider the relationship between bundle 0-gerbes and bundle 0-gerbes with connection which is given by the exact sequence 2.4,

$$0 \to \Omega^1(M)/\Omega_0^1(M) \to H^1(M, \mathcal{D}^1) \to \check{H}^2(M, \mathbb{Z}) \to 0.$$

The space $\Omega^1(M)/\Omega_0^1(M)$ may be interpreted as equivalence classes of connections on the trivial bundle 0-gerbe. This implies that a trivial bundle gerbe with connection $A \in \Omega_0^1(M)$ is D-trivial. We can define the D-trivialisation in the following way. Let $h = \pi^* \rho$, where

$$\rho(m) = \exp(\int_\mu A)$$

where $\mu \in \mathcal{P}_0 M$ for some base point m_0 and $\mu(1) = m$. The 2π-integrality of A ensures that ρ is independent of the choice of path and applying the Deligne differential to the class representing ρ gives the bundle 0-gerbe $(1, A)$. This construction is essentially the same as that used for the tautological bundle 0-gerbe and bundle gerbe, so we may refer to ρ as the tautological function. In fact the function defining the tautological bundle 0-gerbe is the tautological function. Note that the construction relies on the assumption that M is connected. If M is not connected then the construction may be repeated for each connected component.

Suppose the bundle 0-gerbe with connection $(g; A)$ is D-trivial. Furthermore suppose we have a particular choice of trivialisation, $\underline{h}$, which is not necessarily a D-trivialisation. We would like to see how this trivialisation differs from a D-trivialisation. Using the fact that $\delta(\underline{h}) = \underline{g}$ and applying $d\log$ gives

$$\begin{aligned} d\log(\delta(\underline{h})) &= d\log(\underline{g}) \\ \delta(d\log(\underline{h})) &= \delta(A) \end{aligned}$$

so there exists a 1-form χ such that

$$d\log(h_\alpha) = A_\alpha - \chi$$

If we had not assumed that $(g; A)$ is D-trivial then χ would represent the obstruction in $\Omega^1(M)/\Omega^1_0(M)$ to a trivial bundle 0-gerbe being D-trivial as well. This is true since a change in choice of trivialisation changes χ by an element of $\Omega^1_0(M)$ which is the 1-curvature of the function defined by the difference between two trivialisations. Furthermore

$$\begin{aligned} d\chi &= dA_\alpha \\ &= F \end{aligned}$$

where F is the bundle 0-gerbe curvature. We shall refer to $\chi \in \Omega^1(M)/\Omega^1_0(M)$ as the *D-obstruction form.*

Now we return to the case where $(g; A)$ is D-trivial, hence it is flat and A is locally exact. Thus χ is closed. If it is also 2π-integral then we may construct the tautological function ρ on M with curvature χ. Finally we define a D-trivialisation by the product $h \cdot \pi^*\rho$. To check that it is indeed a D-trivialisation observe that

$$\begin{aligned} D(h_\alpha \cdot \rho) &= (\delta(h)_{\alpha\beta}, d\log(h_\alpha)) + (1, \chi) \\ &= (\delta(h)_{\alpha\beta}, d\log(h_\alpha) + \chi) \\ &= (g_{\alpha\beta}, A_\alpha) \end{aligned}$$

The bundle gerbe case is very similar to that for bundle 0-gerbes and was described in [38]. Let $(\underline{g}, \underline{A}, \underline{\eta})$ be the Deligne class of a bundle gerbe with connection and curving, $(P; A, \eta)$. Suppose we have a trivialisation J which is represented by a Deligne cochain $\underline{h}$. Since there is an isomorphism between bundle gerbes and bundle gerbes with connection we may choose a connection on B such that $(J; B)$ trivialises $(P; A)$, however we may not assume that it trivialises $(P; A, \eta)$. In terms of cochains this means that we have

$$\begin{aligned} \underline{g} &= \delta(\underline{h}) && (3.7) \\ \underline{A} &= d\log(\underline{h}) - \delta(\underline{B}) && (3.8) \end{aligned}$$

but it is not true that $\underline{\eta} = d\underline{B}$. We can, however, deduce from (3.8) that

$$\eta_\alpha - dB_\alpha = \chi$$

If this D-obstruction form is closed and 2π-integral then we can construct the tautological bundle $(Q[\chi]; \int_I ev^*\chi)$ with curvature χ and define a D-trivialisation by

$$(P; A, \eta) = D((J; B) \otimes \pi^{-1}(Q[\chi]; \int_I ev^*\chi)).$$

An extension of D-obstruction theory which is useful is the situation where we have two trivialisations $\delta(L)$ and $\delta(J)$ of the same bundle gerbe. If they have D-obstruction forms χ_L and χ_J respectively which satisfy $d\chi_L = d\chi_J$ then $D(L) = D(J \otimes \pi^{-1}K)$ where K is the tautological bundle with curvature $\chi_L - \chi_J = dB_J - dB_L$ which is closed and 2π-integral since it is the difference of two curvatures. We now consider the situation of a bundle gerbe with two different trivialisations.

Proposition 3.1. *[38] Let (P, Y, M) be a bundle gerbe and let L and J be two trivialisations. Then there exists a bundle (K, M) such that $L = J \otimes \pi^{-1}K$ as bundles over Y.*

Proposition 3.2. *Let $(P, Y, M; A, \eta)$ be a bundle gerbe and let L and J be two D-trivialisations. Then there exists a flat bundle (K, M) such that $L = J \otimes \pi^{-1}K$ as bundles with connection over Y.*

This follows from the previous proposition together with the observation that the curvatures of L and J must both be equal to η.

Proposition 3.3. *Suppose $\delta(L) = \delta(K)$ and $F_L = F_K$. Furthermore suppose that $\chi_K - \chi_L$ is closed and 2π-integral. Then there exists a bundle J with curvature $\chi_K - \chi_L$ such that $L = K \otimes \pi^{-1}J$.*

This result shall be useful for studying bundle 2-gerbes in the next chapter.

Proof. Suppose $\delta(L) = \delta(K) = P$. Then if T is the canonically trivial bundle $P^* \otimes P$ then $T = \delta(L) \otimes \delta(K)$. Since the D-obstruction of the left hand side is trivial we have $T = D(L^* \otimes K \otimes \pi^{-1}E)$ where E has curvature $\chi_K - \chi_L$. There exists a flat bundle, R such that $\pi^{-1}R = L^* \otimes K \otimes \pi^{-1}E$, so $L = K \otimes \pi^{-1}J$ where $\pi^{-1}J$ is the bundle $\pi^{-1}E \otimes \pi^{-1}R$ which has curvature $\chi_K - \chi_L$. □

There is also a local theory of trivialisations with connection. First consider a bundle gerbe that is δ-trivial but not necessarily D-trivial. Given a trivialisation with connection $(J; B)$ define 1-forms k_α by $\delta_\alpha^* B_\alpha$ where δ_α is a section of $s_\alpha^{-1}J_\alpha$ and B_α is the pulled back connection on J_α. Using the definition of $h_{\alpha\beta}$ (3.5) it immediately follows that

$$d \log h_{\alpha\beta} - k_\beta + k_\alpha = A_{\alpha\beta} \tag{3.9}$$

In the example of the bundle gerbe over U_0 we have $k_\alpha = A_{0\alpha}$. To be D-trivial there is the additional requirement that $dk_\alpha = \eta_\alpha$ which is satisfied if $F_{J^0} = \eta$. Note that in the U_0 example we have $dk_\alpha = dA_{0\alpha} = \eta_\alpha - \eta_0$ so the D-obstruction is η_0.

3.3 Lifting Bundle Gerbes

The lifting bundle gerbe was introduced in [36] as one of the motivating examples of the theory of bundle gerbes. Let

$$0 \to U(1) \to \tilde{G} \xrightarrow{p} G \to 0$$

be a central extension of groups and let $P_G \to M$ be a principal G bundle. The lifting bundle gerbe is defined by the following diagram:

$$\begin{tikzcd} & \tilde{G} \arrow[d] \\ & G \\ P_G^{[2]} \arrow[ur, "g"] \arrow[r, shift left] \arrow[r, shift right] & P_G \arrow[d] \\ & M \end{tikzcd}$$

The map $g : P_G^{[2]} \to G$ is defined such that $g(p_1, p_2)$ is the element of G which satisfies $p_2 = p_1 g(p_1, p_2)$. Alternatively $P_G^{[2]}$ may be identified with $P \times G$ via $(p_1, p_2) \mapsto (p_1, g(p_1, p_2))$ and $(p, g) \mapsto (p, pg)$ in which case let $g : P_G \times G \to G$ be the projection of the second factor. It is to be understood that $\tilde{G} \to G$ is pulled back to a bundle over $P_G^{[2]}$ by g and the bundle gerbe product is induced from the group product on $\tilde{G}$.

Proposition 3.4. *[36] The lifting bundle gerbe associated with a G bundle $P_G \to M$ and a central extension $\tilde{G}$ is trivial if and only if P_G lifts to a $\tilde{G}$ bundle.*

In general a connection A on $\tilde{G} \to G$ does not define a bundle gerbe connection. This is because the corresponding curvature form, $g^* F_A$, for the bundle $g^{-1}\tilde{G}$ may not satisfy the condition $\delta(g^* F_A)$ on $P^{[3]}$. It is shown in [37] that in general there exists a 1-form ϵ on $P_G^{[2]}$ such that $g^*(A) - \epsilon$ is a bundle gerbe connection.

3.4 Torsion Bundle Gerbes

If the Dixmier-Douady class of a bundle gerbe is torsion then we refer to it as a *torsion bundle gerbe.* These bundle gerbes naturally arise in applications to physics and there are two particular aspects which are of interest: the canonical bundle gerbe of a class in $H^2(M, \mathbb{Z}_p)$ and bundle gerbe modules.

Associated to the short exact sequence

$$\mathbb{Z} \xrightarrow{p\times} \mathbb{Z} \to \mathbb{Z}_p$$

is a Bockstein operator $\beta : H^2(M, \mathbb{Z}_p) \to H^3(M, Z)$. This indicates that given a class $w \in H^2(M, \mathbb{Z}_p)$ we may define a bundle gerbe with Dixmier-Douady class $\beta(w)$. We would like to demonstrate that there is a canonical choice of Deligne class arising from w.

The class $\beta(w)$ must satisfy $p \cdot \beta(w) = 0$ so we have a torsion bundle gerbe. Consider the exact sequence

$$0 \to H^2(M, U(1)) \to H^2(M, \mathcal{D}^2) \to \Omega_0^3(M) \to 0$$

Recall that this may be interpreted as

flat holonomy class $\to$ bundle gerbe with connection and curving $\to$ curvature

If the bundle gerbe is torsion then since the curvature is the image of the Dixmier-Douady class in de Rham cohomology it must be an exact form. The possible D-stable isomorphism class of bundle gerbes with a particular choice of curvature are given by flat holonomy classes. The map $H^2(M, \mathbb{Z}_p) \to H^2(M, U(1))$ allows us to consider w as a flat holonomy class. This in turn defines a bundle gerbe with Deligne class $(\underline{w}, 0, \underline{dB})$. The fact that the transition functions are constant and the curvature is exact mean that this is a Deligne cocycle. The Dixmier-Douady class of this bundle gerbe is $w_{\alpha\beta\gamma} = -\log w_{\beta\gamma} + \log w_{\alpha\gamma} - \log w_{\alpha\beta}$. The class $pw_{\alpha\beta\gamma}$ is trivial in $H^3(M, \mathbb{Z})$, with trivialisation $p \log w_{\alpha\beta}$. The mod p reduction then gives $w_{\alpha\beta} \in H^2(M, \mathbb{Z})$, so the transition functions are given by $\beta(w)$ as desired. Trivially the flat holonomy class of this bundle gerbe is $\underline{w}$. A canonical choice of such a bundle gerbe is given by setting $\underline{dB} = 0$. Thus associated with a torsion class $w \in H^2(M, \mathbb{Z}_p)$ we have a canonical torsion Deligne class $(\underline{w}, 0, 0)$. This construction extends to Deligne classes of arbitrary degree.

An interesting class of examples of torsion bundle gerbe is given by the lifting bundle gerbes associated with a central extension

$$\mathbb{Z}_p \to \tilde{G} \to G$$

We may use the map $H^2(M, \mathbb{Z}_p) \to H^2(M, U(1))$ and the discussion above to see that this is a torsion bundle gerbe. A particular example is given by the obstruction to lifting a projective unitary bundle to a unitary bundle [11], which is defined in terms of the central extension

$$\mathbb{Z}_n \to U(n) \to PU(n)$$

Next we define bundle gerbe modules.

Definition 3.1. [3] Let (P, Y, M) be a bundle gerbe. Let $E \to Y$ be a finite rank hermitian vector bundle such that there exists a hermitian bundle isomorphism

$$\phi : P \otimes \pi_1^{-1} E \cong \pi_2^{-1} E$$

We require that this isomorphism is compatible with the bundle gerbe product in the sense that the maps

$$P_{(y_1, y_2)} \otimes (P_{(y_2, y_3)} \otimes E_{y_3}) \to P_{(y_1, y_2)} \otimes E_{y_2} \to E_{y_1}$$

and

$$(P_{(y_1, y_2)} \otimes P_{(y_2, y_3)}) \otimes E_{y_3} \to P_{(y_1, y_3)} \otimes E_{y_3} \to E_{y_1}$$

are the same. Call E a *bundle gerbe module* and say that the bundle gerbe P acts on E.

A rank one bundle gerbe module is a trivialisation. Given a rank r bundle gerbe module the product P^r acts on the rank one bundle $\Lambda^r(E)$ and hence we have

Proposition 3.5. *[3] Let (P, Y, M) be a bundle gerbe with Dixmier-Douady class* $\mathrm{dd}(P)$. *Suppose (P, Y, M) has a bundle gerbe module E of rank r. Then P is a torsion bundle gerbe with* $r\mathrm{dd}(P) = 0$.

A connection on a bundle gerbe module is called a *bundle gerbe module connection* if the isomorphism ϕ is an isomorphism of bundles with connection.

A bundle gerbe module with connection may also be defined in terms of local data. Suppose we have a bundle gerbe represented locally in Deligne cohomology by $(\underline{g}, \underline{A}, \underline{f})$. Let E be a bundle gerbe module, and define a set of local bundles on M by $E_\alpha = s_\alpha^{-1}E$. These bundles are trivial with sections δ_α. We consider the isomorphism $P \otimes \pi_1^{-1}E \cong \pi_2^{-1}E$ at the local level. In terms of sections we may define local matrix valued functions $h_{\alpha\beta}$ such that

$$\sigma_{\alpha\beta} \otimes \delta_\alpha = \delta_\beta h_{\alpha\beta} \tag{3.10}$$

Consider the section $\sigma_{\alpha\beta} \otimes \sigma_{\beta\gamma} \otimes \delta_\gamma$ associated with $P_{\alpha\beta} \otimes P_{\beta\gamma} \otimes E_\gamma$. This can be simplified in two different ways,

$$\begin{aligned} \sigma_{\alpha\beta} \otimes \sigma_{\beta\gamma} \otimes \delta_\gamma &= \sigma_{\alpha\beta} g_{\alpha\beta\gamma} \otimes \delta_\gamma \\ &= \delta_\alpha h_{\alpha\gamma} g_{\alpha\beta\gamma} 1 \end{aligned} \tag{3.11}$$

where 1 is the identity matrix of the same rank as E, or

$$\begin{aligned} \sigma_{\alpha\beta} \otimes \sigma_{\beta\gamma} \otimes \delta_\gamma &= \sigma_{\alpha\beta} \otimes \delta_\beta h_{\beta\gamma} \\ &= \delta_\alpha h_{\alpha\beta} h_{\beta\gamma} \end{aligned} \tag{3.12}$$

and hence we have

$$h_{\alpha\beta} h_{\beta\gamma} = h_{\alpha\gamma} g_{\alpha\beta\gamma} 1 \tag{3.13}$$

To get a local expression for the connection let ∇_P and ∇_E be the connections on P and E respectively. Using the sections $\sigma_{\alpha\beta}$, δ_α and δ_β we have two choices of connection on bundles $P_{\alpha\beta} \otimes E_\alpha$ and E_β over U_α. These are $\sigma_{\alpha\beta}^{-1}\nabla_P + \delta_\alpha^{-1}\nabla_E$ and $\delta_\beta^{-1}\nabla_E$. The isomorphism of bundles is given by the local functions $h_{\alpha\beta}$ and so, letting $\delta_\alpha^{-1}\nabla_E = k_\alpha$, the two choices of connection are related by the usual formula for connections under a change of section,

$$A_{\alpha\beta} 1 + k_\alpha = h_{\alpha\beta}^{-1} k_\beta h_{\alpha\beta} + h_{\alpha\beta}^{-1} dh_{\alpha\beta} \tag{3.14}$$

3.5 Cup Product Bundle Gerbes

There is a cup product in Deligne cohomology (see [17] or [5]). The correspondence between Deligne cohomology and geometric objects implies that the cup product may be used to construct new examples of geometric objects which realise Deligne classes [8]. We shall demonstrate how to construct bundle gerbes corresponding to various cup products. First we consider the bundle obtained by taking the cup product of two functions ([5],[17]) as a bundle 0-gerbe. This helps us to find geometric realisations of various bundle gerbes which may be obtained by taking cup products. Let f,g and h be $U(1)$-valued functions on M and let $L \to M$ be a $U(1)$-bundle. We consider the cup products $f \cup L$, $L \cup f$ and $f \cup g \cup h$.

The cup product is induced by a product in the Deligne complex

$$\cup : \mathbb{Z}(p)_D \otimes \mathbb{Z}(q)_D \to \mathbb{Z}(p+q)_D$$

which is defined by

$$x \cup y = \begin{cases} x \cdot y & \text{if deg } x = 0, \\ x \wedge dy & \text{if deg } x > 0 \text{ and deg } y = q, \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

It is a standard result that this product is associative and induces a product of Deligne cohomology groups. Furthermore it is anticommutative, that is, for $\alpha \in H^q(M, \mathbb{Z}(p)_D)$ and $\beta \in H^{q'}(M, \mathbb{Z}(p')_D)$ the cup product satisfies $\alpha \cup \beta = (-1)^{qq'} \beta \cup \alpha$. We shall calculate some specific examples and construct corresponding geometric objects.

The cup product of two functions was described in [17] and [5]. We review it in detail as it provides the basis for all of our subsequent examples. Let f and g be $U(1)$-valued functions on M. As we have seen f and g may be represented by the Deligne classes $(n_{\alpha\beta}, \log_\alpha(f)) \in H^1(M, \mathbb{Z}(1)_D)$ and $(m_{\alpha\beta}, \log_\alpha(g)) \in H^1(M, \mathbb{Z}(1)_D)$ respectively, where $\log_\alpha$ is a branch of the logarithm function which is defined on $U_\alpha \subset M$, and the integers $n_{\alpha\beta}$ and $m_{\alpha\beta}$ are defined by the differences $\log_\beta(f) - \log_\alpha(f)$ and $\log_\beta(g) - \log_\alpha(g)$ respectively. The cup product $f \cup g$ is the Deligne class $(n_{\alpha\beta} m_{\beta\gamma}, n_{\alpha\beta} \log_\beta(g), \log_\alpha(f) d\log(g)) \in H^2(M, \mathbb{Z}(2)_D)$. Under the isomorphism $H^2(M, \mathbb{Z}(2)_D) \cong H^1(M, \mathcal{D}^1)$ this becomes $(g^{n_{\alpha\beta}}, \log_\alpha(f) d\log(g))$ which represents a bundle 0-gerbe with connection which may be described explicitly.

We construct a bundle 0-gerbe over $S^1 \times S^1$ and pull it back to M via the map $(f, g) : M \to S^1 \times S^1$. Let the bundle 0-gerbe over $S^1 \times S^1$ be defined by the following diagram:

$$\begin{array}{ccc} & & S^1 \\ & \overset{\rho_U}{\nearrow} & \\ (\mathbb{R} \times \mathbb{Z}) \times (\mathbb{R} \times \mathbb{Z}) & \rightrightarrows & \mathbb{R} \times \mathbb{R} \\ & & \downarrow \\ & & S^1 \times S^1 \end{array}$$

where the projection to the base is given by two copies of the exponential and the map ρ_U is defined by $\rho_U(r, n, s, m) = \exp(sn)$. We have used the identification $\mathbb{R}^{[2]} = \mathbb{R} \times \mathbb{Z}$ for the $\mathbb{Z}$-bundle $\mathbb{R} \to S^1$ as discussed for a general G-bundle in §3.3. It is easily shown that the 1-form rds is a connection for this bundle 0-gerbe.

Proposition 3.6. *The bundle 0-gerbe $(f, g)^{-1}(\rho_U, \mathbb{R} \times \mathbb{R}, S^1 \times S^1)$ with connection rds has Deligne class $(g^{n_{\alpha\beta}}, \log_\alpha(f) d\log(g))$.*

Proof. Define local sections s_α of $\mathbb{R} \times \mathbb{R} \to M$ by $s_\alpha(\theta, \phi) = (\log_\alpha(\theta), \log_\alpha(\phi))$. Then $(s_\alpha, s_\beta) = (\log_\alpha(\theta), \log_\beta(\theta) - \log_\alpha(\theta), \log_\beta(\phi), \log_\beta(\phi) - \log_\alpha(\phi))$ and $\rho(s_\alpha, s_\beta) = \exp(\log_\beta(\phi)(\log_\beta$ $\log_\alpha(\theta)))$. To get the pull back to M we simply replace θ and ϕ with $f(m)$ and $g(m)$ respectively to get the required transition functions, $g^{n_{\alpha\beta}}$. To complete the proof observe that $s_\alpha^*(rds) = \log_\alpha(\theta) d\log(\phi)$, so under the pull back we get $\log_\alpha(f) d\log(g)$. □

Clearly the bundle 0-gerbe $g \cup f$ is obtained by replacing ρ_U with $\rho_U^* : (r, n, s, m) \mapsto \exp(rm)$. By anticommutativaty the product bundle 0-gerbe $\rho_U \otimes \rho_U^*$ should be trivial. We may demonstrate this directly by defining r_1, r_2, s_1 and s_2 such that $(r, n, s, m) =$

$(r_1, s_1, r_2 - r_1, s_1, s_2 - s_1)$ and considering

$$
\begin{aligned}
(\rho_\cup \rho_\cup^*)(r, n, s, m) &= \exp(sn + rm) \\
&= \exp(s_1(r_2 - r_1) + (r_2 - n)m) \\
&= \exp(s_1 r_2 - s_1 r_1 + r_2(s_2 - s_1) - nm) \\
&= \exp(s_1 r_2 - s_1 r_1 + r_2 s_2 - r_2 s_1) \\
&= \exp(r_2 s_2 - r_1 s_1) \\
&= \delta(\exp(rs))
\end{aligned}
$$

There are three ways of obtaining a bundle gerbe via cup products. We shall calculate the Deligne class and provide a geometric construction for each one.

The Bundle Gerbe $f \cup L$

Let f be a $U(1)$-valued function on M and let L be a bundle 0-gerbe over M. Let f and L have Deligne class $(n_{\alpha\beta}, \log_\alpha(f)) \in H^1(M, \mathbb{Z}(1)_D)$ and $(m_{\alpha\beta\gamma}, \log(g_{\alpha\beta})) \in H^2(M, \mathbb{Z}(1)_D)$ respectively. Then $m_{\alpha\beta\gamma} = -\log(g_{\beta\gamma}) + \log(g_{\alpha\gamma}) - \log(g_{\alpha\beta})$. The product $f \cup L$ is

$$(n_{\alpha\beta} m_{\beta\gamma\delta}, n_{\alpha\beta} \log(g_{\beta\gamma}), \log_\alpha(f) d\log(g_{\alpha\beta})) \in H^3(M, \mathbb{Z}(2)_D)$$

which, under the usual isomorphism, becomes $(g_{\beta\gamma}^{n_{\alpha\beta}}, \log_\alpha(f) d\log(g_{\alpha\beta}))$. We define the corresponding bundle gerbe with the following diagram:

$$
\begin{array}{ccc}
 & & K_\cup \\
 & & \downarrow \\
 & & S^1 \times S^1 \\
 & \overset{\rho}{\nearrow} & \\
\mathbb{R} \times \mathbb{Z} \times L \times S^1 & \rightrightarrows & \mathbb{R} \times L \\
 & & \downarrow \\
M & \xrightarrow{(f,m)} & S^1 \times M
\end{array}
$$

where $K_\cup$ is the cup product bundle described in the previous section, $m : M \to M$ is the identity map and the pullback to M by (f, m) is implied. Local sections are defined by $(\log_\alpha f, s_\alpha)$ where s_α is a local section of L. The sections $\sigma_{\alpha\beta}$ are given by sections of $\rho^{-1}K_\cup$ over $(\log_\alpha f, n_{\alpha\beta}, s_\alpha, g_{\alpha\beta})$. Essentially the fibres of $\rho^{-1}K_\cup$ look like $\mathbb{R} \times \mathbb{R} \times S^1$ with an equivalence relation $(a + n, b + m, z) \sim (a, b, z e^{nb})$ where $a, b \in \mathbb{R}$, $n, m \in \mathbb{Z}$ and $z \in S^1$. In the fibre there are also copies of $\mathbb{Z}$ and L but we omit these to simplify the expressions. The projection takes (a, b, z) to $(a, n, l, e^b) \in \mathbb{R} \times \mathbb{Z} \times L \times S^1$. We need an expression for the bundle gerbe product. It must satisfy two conditions: it must cover a particular product on the base and it must respect the equivalence relation in the definition of $\rho^{-1}K_\cup$. To find the map on the base which must be covered by the bundle gerbe product consider it in the form $(\mathbb{R} \times L)^{[2]}$ in which case the product is

$$(r_1, r_2, l_1, l_2) \times (r_2, r_3, l_2, l_3) \to (r_1, r_3, l_1, l_3) \tag{3.16}$$

Under the standard identification with $\mathbb{R} \times \mathbb{Z} \times L \times S^1$ given by $(r, n, l, \theta) = (r, r+n, l, l\theta)$ this becomes

$$(r_1, n_1, l_1, \theta_1) \times (r_2, n_2, l_2, \theta_2) \to (r_1, n_1 + n_2, l_1, \theta_1 \theta_2) \tag{3.17}$$

so the bundle gerbe product must be of the form

$$(a_1, b_1, z_1) \times (a_2, b_2, z_2) = (a_1, b_1 + b_2, z_1 z_2 \Pi) \tag{3.18}$$

where Π is some function $\Pi(a_1, b_1, z_1, a_2, b_2, z_2)$.

To determine an expression for Π we consider what happens to this product under the equivalence relation. First we replace (a_1, b_1, z_1) by $(a_1 + n, b_1 + m, z_1 e^{-nb})$.

$$\begin{aligned}(a_1 + n, b_1 + m, z_1 e^{-nb_1}) \times (a_2, b_2, z_2) &= (a_1 + n, b_1 + b_2 + m, z_1 z_2 e^{-nb_1} \Pi) \\ &= (a_1, b_1 + b_2, z_1 z_2 e^{-nb_1} e^{n(b_1+b_2)} \Pi) \\ &= (a_1, b_1 + b_2, z_1 z_2 e^{nb_2} \Pi)\end{aligned} \tag{3.19}$$

so $\Pi(a_1 + n, b_1 + m, z_1 e^{-nb_1}, a_2, b_2, z_2) = e^{-nb_2} \Pi(a_1, b_1, z_1, a_2, b_2, z_2)$. Now we consider the second factor,

$$\begin{aligned}(a_1, b_1, z_1) \times (a_2 + n, b_2 + m, z_2 e^{-nb_2}) &= (a_1, b_1 + b_2 + m, z_1 z_2 e^{-nb_2} \Pi) \\ &= (a_1, b_1 + b_2, z_1 z_2 e^{-nb_2} \Pi)\end{aligned} \tag{3.20}$$

so $\Pi(a_1, b_1, z_1, a_2 + n, b_2 + m, z_2 e^{-nb_2}) = e^{nb_2} \Pi(a_1, b_1, z_1, a_2, b_2, z_2)$.

Let $\Pi(a_1, b_1, z_1, a_2, b_2, z_2) = e^{b_2(a_2 - a_1)}$. Under the transformation $a_1 \to a_1 + n$ we have $\Pi \to e^{b_2(a_2 - a_1 - n)} = \Pi e^{-nb_2}$. Under the transformation $(a_2, b_2) \to (a_2 + n, b_2 + m)$ we have $\Pi \to e^{b_2(a_2 - a_1)} e^{nb_2} e^{m(a_2 - a_1)} = \Pi e^{nb_2}$ since $m(a_2 - a_1) \in \mathbb{Z}$.

Define sections $\sigma_{\alpha\beta} = (\log_\alpha f, \log g_{\alpha\beta}, 1)$. We may now calculate the product $\sigma_{\alpha\beta}\sigma_{\beta\gamma}$,

$$\begin{aligned}\sigma_{\alpha\beta}\sigma_{\beta\gamma} &= (\log_\alpha f, \log g_{\alpha\beta}, 1) \times (\log_\beta f, \log g_{\beta\gamma}, 1) \\ &= (\log_\alpha f, \log g_{\alpha\beta} + \log g_{\beta\gamma}, e^{n_{\alpha\beta} \log g_{\beta\gamma}}) \\ &= (\log_\alpha f, \log g_{\alpha\gamma} + m_{\alpha\beta\gamma}, g_{\beta\gamma}^{n_{\alpha\beta}}) \\ &= (\log_\alpha f, \log g_{\alpha\gamma}, g_{\beta\gamma}^{n_{\alpha\beta}}) \\ &= \sigma_{\alpha\gamma} g_{\beta\gamma}^{n_{\alpha\beta}}\end{aligned} \tag{3.21}$$

This gives the required transition functions.

The local connections $\log_\alpha(f) d\log(g_{\alpha\beta})$ are induced by a bundle gerbe connection adb at $(a, b, z) \in \mathbb{R} \times \mathbb{R} \times S^1/\sim$, the total space of $K_\cup$.

We may also consider the cup product bundle gerbe $f \cup L$ where L is a bundle with connection A. The Deligne class of the product is $(g_{\beta\gamma}^{n_{\alpha\beta}}, n_{\alpha\beta} A_\beta, \log_\alpha(f) dA)$. The Dixmier-Douady class is the same as in the previous case however the connection is different and we also have a choice of curving. The connection form is $n(A_l + db)$ at a point (r, n, l, θ, z) in the total space of $\rho^{-1}K$, $\mathbb{R} \times \mathbb{Z} \times L \times \mathbb{R} \times S^1$. The curving is (rdA) on $\mathbb{R} \times L$. It is not difficult to see that these give the appropriate local expressions.

There is an alternate geometric representation of the bundle gerbe $f \cup L$ which is of interest. Define it with the following diagram:

$$\begin{array}{ccc} L^{-n} & & \\ \downarrow & & \\ \mathbb{R} \times \mathbb{Z} \times M & \rightrightarrows & \mathbb{R} \times M \\ & & \downarrow \\ M & \xrightarrow{(f,m)} & S^1 \times M \end{array}$$

where L^{-n} is the bundle with fibre at (r, n, m) equal to the fibre of the n-th tensor product bundle of L^* at m. Sections of $Y^{[2]}$ over $U_{\alpha\beta}$ are given by $(\log_\alpha f, n_{\alpha\beta}, m)$, so we define the sections $\sigma_{\alpha\beta}$ by

$$\sigma_{\alpha\beta} = s_\beta^{-n_{\alpha\beta}} \tag{3.22}$$

where s_α is a local section of L. Using these to calculate the transition function we get

$$\begin{aligned}\sigma_{\alpha\beta}\sigma_{\beta\gamma} &= s_\beta^{-n_{\alpha\beta}} s_\gamma^{-n_{\beta\gamma}} \\ &= s_\gamma^{-n_{\alpha\beta}} g_{\beta\gamma}^{n_{\alpha\beta}} s_\gamma^{-n_{\beta\gamma}} \\ &= s_\gamma^{-n_{\alpha\gamma}} g_{\beta\gamma}^{n_{\alpha\beta}} \\ &= \sigma_{\alpha\gamma} g_{\beta\gamma}^{n_{\alpha\beta}}\end{aligned} \tag{3.23}$$

The connection at (r, n, m) is given by $-nA$ at m, the connection on L^{-n} induced in the natural way from that of L, and the curving at (r, m) is rF where F is the curvature of L.

This representation allows for a much simpler calculation of the Deligne class, however from the point of view of bundle gerbe theory it is not as general as the previous case since it depends on L as a bundle rather than a bundle 0-gerbe. The significance of the first method is that the cup product bundle appears in the definition of the cup product bundle gerbe. This is related to the bundle gerbe hierarchy which we shall consider in the next chapter. We shall find both methods useful in considering cup products and bundle 2-gerbes in §4.3. Also it is not obvious how to approach the bundle gerbe $L \cup f$ using the second method.

The Bundle Gerbe $L \cup f$

The Deligne class for this bundle gerbe with connection is $(f^{m_{\alpha\beta\gamma}}, \log(g_{\alpha\beta}) d\log f)$. By the commutativity of the cup product this bundle gerbe should be stably isomorphic to $f \cup L$. We define it by replacing K with the dual bundle obtained by swapping the two functions in the cup product. The result is that the equivalence relation on $\mathbb{R} \times \mathbb{R} \times S^1$ becomes $(a, b, z) \sim (a + n, b + m, ze^{ma})$. The product is still of the form

$$(a_1, b_1, z_1) \times (a_2, b_2, z_2) = (a_1, b_1 + b_2, z_1 z_2 \Pi)$$

Changing representatives of the equivalence class gives

$$\begin{aligned}(a_1 + n, b_1 + m, z_1 e^{-ma_1}) \times (a_2, b_2, z_2) &= (a_1 + n, b_1 + b_2 + m, z_1 z_2 e^{-ma_1}\Pi) \\ &= (a_1, b_1 + b_2, z_1 z_2 \Pi) \\ (a_1, b_1, z_1) \times (a_2 + n, b_2 + m, z_2 e^{-ma_2}) &= (a_1, b_1 + b_2, z_1 z_2 e^{m(a_1 - a_2)}\Pi) \\ &= (a_1, b_1 + b_2, z_1 z_2 \Pi)\end{aligned} \tag{3.24}$$

since $m(a_1 - a_2) \in \mathbb{Z}$. This means we may define a bundle gerbe product by $\Pi = 1$. The transition functions may now be calculated,

$$\begin{aligned}\sigma_{\alpha\beta}\sigma_{\beta\gamma} &= (\log_\alpha f, \log g_{\alpha\beta}, 1) \times (\log_\beta f, \log g_{\beta\gamma}, 1) \\ &= (\log_\alpha f, \log g_{\alpha\beta} + \log g_{\beta\gamma}, 1) \\ &= (\log_\alpha f, \log g_{\alpha\gamma} - m_{\alpha\beta\gamma}, 1) \\ &= (\log_\alpha f, \log g_{\alpha\gamma}, e^{m_{\alpha\beta\gamma}\log_\alpha f}) \\ &= \sigma_{\alpha\gamma} f^{m_{\alpha\beta\gamma}}\end{aligned} \tag{3.25}$$

The connection is given by the 1-form bda at $(a, b, z) \in \mathbb{R} \times \mathbb{R} \times S^1/\sim$.

When the line bundle L has connection A the Deligne class of the cup product is $(f^{m_{\alpha\beta\gamma}}, \log(g_{\alpha\beta})d\log f, A_\alpha \wedge d\log(f))$. The connection is still the same and the curving is defined by $A \wedge dr$.

The Triple Cup Product Bundle Gerbe

The triple cup product bundle gerbe is defined by $f \cup g \cup h$ where f, g and h are all $U(1)$ valued functions on M. The Deligne class of this cup product is

$$(h^{n_{\alpha\beta}m_{\beta\gamma}}, n_{\alpha\beta}\log_\beta(g)d\log h, \log_\alpha(f)d\log g \wedge d\log h) \in H^3(M, \mathbb{Z}(3)_D)$$

where $n_{\alpha\beta} = \log_\beta(f) - \log_\alpha(f)$ and $m_{\beta\gamma} = \log_\gamma(g) - \log_\beta(g)$. This bundle gerbe could be represented geometrically by a combination of the cup product bundle and either of the cup product bundle gerbes already discussed. There also a simpler representation which we discuss here. Define a bundle gerbe by the following diagram:

$$\begin{array}{ccc} T & & \\ \downarrow & & \\ (\mathbb{R} \times \mathbb{Z})^3 & \rightrightarrows & \mathbb{R}^3 \\ & & \downarrow \\ M & \stackrel{(f,g,h)}{\rightarrow} & S^1 \times S^1 \times S^1 \end{array}$$

where T is the trivial bundle. We define a bundle gerbe product on T by

$$\begin{aligned}(r_1, n_1, s_1, m_1, t_1, k_1, z_1) &\times (r_2, n_2, s_2, m_2, t_2, k_2, z_2) \\ &= (r_1, n_1 + n_2, s_1, m_1 + m_2, t_1, k_1 + k_2, z_1 z_2 e^{t_1 n_1 m_2})\end{aligned} \tag{3.26}$$

where for $i = 1, 2$, $r_i, s_i, t_i \in \mathbb{R}$, $n_i, m_i, k_i \in \mathbb{Z}$ and $z_i \in S^1$ for $i = 1, 2$. The sections $\sigma_{\alpha\beta}$ may be defined by

$$\sigma_{\alpha\beta} = (\log_\alpha(f), n_{\alpha\beta}, \log_\alpha(g), m_{\alpha\beta}, \log_\alpha(h), k_{\alpha\beta}, 1) \tag{3.27}$$

Using $n_{\alpha\beta} + n_{\beta\gamma} = n_{\alpha\gamma}$ and similar results for m and k we calculate the product $\sigma_{\alpha\beta}\sigma_{\beta\gamma}$,

$$(\log_\alpha(f), n_{\alpha\gamma}, \log_\alpha(g), m_{\alpha\gamma}, \log_\alpha(h), k_{\alpha\gamma}, e^{\log_\alpha(h)n_{\alpha\beta}m_{\alpha\beta}}) = \sigma_{\alpha\gamma}h^{n_{\alpha\beta}m_{\alpha\beta}} \tag{3.28}$$

giving the required transition functions. Observe that this bundle has been constructed using a similar method to the canonical bundle associated with a Deligne class which was described in the proof of proposition 2.5. The connection at the point $(r, n, s, m, t, k, z) \in T$ is $nsdt$ and the curving at $(r, s, t) \in \mathbb{R}^3$ is $rds \wedge dt$.

Chapter 4

Other Geometric Realisations of Deligne Cohomology

In this chapter we consider the bundle gerbe hierarchy of geometric realisations of Deligne cohomology. First we review what we have considered so far, then we consider the relationship of bundle gerbes with gerbes to clarify this picture of the hierarchy. We then extend the hierarchy by considering bundle 2-gerbes. Following this we use $\mathbb{Z}$-bundle 0-gerbes to complete our catalogue of realisations and to motivate a comparison with the theory of B^pS^1-bundles.

4.1 The Bundle Gerbe Hierarchy

We summarise the results on geometric realisations of Deligne cohomology with a table:

Table 4.1: Low Dimensional Realisations of Deligne Cohomology

Deligne Cohomology Group	Geometric Realisation
$H^0(M,\underline{U(1)})$	$U(1)$-functions
$H^0(M,\mathcal{D}^p), p > 0$	constant $U(1)$-functions
$H^1(M,\underline{U(1)})$	$U(1)$-bundles
	$U(1)$-bundle 0-gerbes
$H^1(M,\mathcal{D}^1)$	$U(1)$-bundles with connection
	$U(1)$-bundle 0-gerbes with connection
$H^1(M,\mathcal{D}^p), p > 1$	flat $U(1)$-bundles
	flat $U(1)$-bundle 0-gerbes
$H^2(M,\underline{U(1)})$	$U(1)$-bundle gerbes
$H^2(M,\mathcal{D}^1)$	$U(1)$-bundle gerbes with connection
$H^2(M,D^2)$	$U(1)$-bundle gerbes with connection and curving
$H^3(M,\mathcal{D}^p), p > 2$	flat $U(1)$-bundle gerbes

It is to be understood that the right hand column of the above table refers to equivalence classes of geometric object, that is, isomorphism classes in the case of bundles and stable isomorphism classes in the case of bundle 0-gerbes and bundle gerbes.

Recall that bundle 0-gerbes are defined as functions on $Y^{[2]}$ and bundle gerbes are defined as bundles over $Y^{[2]}$. Since there is an equivalence between bundles and bundle 0-gerbes we could also think of bundle gerbes as bundle 0-gerbes over $Y^{[2]}$. Thus there is a hierarchy

$$\begin{array}{c}\text{functions}\\ \downarrow\\ \text{bundle 0-gerbes}\\ \downarrow\\ \text{bundle gerbes}\end{array}$$

where each object is built out of the preceding object and a submersion $Y \to M$.

Note also the similarity in the definitions of bundle 0-gerbe connections and bundle gerbe curvings. In both cases we have a differential form on Y satisfying the condition that applying δ gives the curvature of the object over $Y^{[2]}$.

4.2 Gerbes

Gerbes are the most well known geometric realisation of $H^3(M, \mathbb{Z}(p)_D)$. We shall review some relevant results about gerbes, for a detailed account see [5]. It will suffice for us to think of a gerbe as a sheaf of groupoids. Isomorphism classes of gerbes are represented by classes in $H^3(M, \mathbb{Z}(1)_D)$. To each bundle gerbe there is an associated gerbe and equivalence classes of gerbes are in bijective correspondence with stable isomorphism classes of bundle gerbes [38]. It is possible to define certain differential geometric structures on gerbes which are called a connective structure and a choice of curving. Under the bijective correspondence these are equivalent to a connection and curving on a bundle gerbe. Gerbes with connective structure are classified by $H^3(M, \mathbb{Z}(2)_D)$ and gerbes with connective structure and a choice of curving are classified by $H^3(M, \mathbb{Z}(3)_D)$.

We wish to demonstrate that gerbes are to bundle gerbes what bundles are to bundle 0-gerbes and thus remove the ambiguity in our bundle gerbe hierarchy. We briefly review the construction of a gerbe from a bundle gerbe in [38]. Since a gerbe is a sheaf of groupoids we need to define a category over each open set. The objects corresponding to $U \subset M$ are bundle gerbe trivialisations over U. The morphisms are morphisms of bundle gerbe trivialisations over $Y_U = \pi^{-1}(U)$. A gerbe is a bundle of groupoids in the sense that over each m we have the fibre of a bundle gerbe which is a groupoid with objects defined by trivialisations.

Consider a bundle 0-gerbe (g, Y, M). Over $U \subset M$ there exists a trivialisation $h : \pi^{-1}(U) \to S^1$. Given h', a second trivialisation over U, there exists a function $q_U : U \to S^1$ such that $h' = h \cdot \pi^* q_U$. We may define a bundle with trivialisations over U given by $\sigma_U(x) = (x, h(s_u(x)))$. The transition functions will be identical to those of the original bundle 0-gerbes. If we replace h with h' it is clearly seen that on overlaps $q_U^{-1} q_V = 1$ and so we have a function $q : M \to S^1$ which defines an automorphism of bundles. The fibre of the bundle may be considered as made up of bundle 0-gerbe trivialisations with any two differing by $q(m) \in S^1$.

This analysis suggests a refinement of the bundle gerbe hierarchy described above.

$$\begin{array}{ccc} & \text{functions} & \\ \swarrow & & \searrow \\ \text{bundle 0-gerbes} & & \text{bundles} \\ \downarrow & & \downarrow \\ \text{bundle gerbes} & & \text{gerbes} \end{array}$$

There is no simple diagrammatic representation of gerbes such as we have for bundle gerbes.

4.3 Bundle 2-Gerbes

We would like to construct a geometric realisation of $H^3(M, \mathcal{D}^p)$. This leads to the notion of a bundle 2-gerbe which has been developed in [44]. We use a slightly different approach which is more suited to the bundle gerbe hierarchy. We shall require bundle 2-gerbes for some applications in Chapter 8.

First we must deal with a matter of notation. Consider the fibre product spaces $X^{[2]}$ and $X^{[3]}$ associated with a submersion $X \to M$. We may define three projection maps $\pi_i : X^{[3]} \to X^{[2]}, i \in \{1, 2, 3\}$ by omission of the ith component of $X^{[3]}$. For example $\pi_1(x_1, x_2, x_3) = (x_2, x_3)$. If there is a bundle 0-gerbe or bundle gerbe P over $X^{[2]}$ then this may be pulled back by each of these projections. We will use the notation

$$\begin{aligned} P_{12} &= \pi_3^{-1} P \\ P_{13} &= \pi_2^{-1} P \\ P_{23} &= \pi_1^{-1} P \end{aligned}$$

Using this notation the bundle gerbe product can be written as a stable morphism of bundle 0-gerbes

$$P_{12} \otimes P_{23} \cong P_{13}. \tag{4.1}$$

This notation extends to projections $X^{[p+1]} \to X^{[p]}$ for any positive integer p.

We now examine what happens if we replace the bundle 0-gerbes in (4.1) by bundle gerbes. A choice of such a stable isomorphism is equivalent to a choice of bundle gerbe trivialisation such that there is a bundle gerbe isomorphism

$$P_{12} \otimes P_{23} = P_{13} \otimes \delta(J_{123}) \tag{4.2}$$

The collection of trivialisations J_{123} will be referred to as the *bundle 2-gerbe product*, J. Observe that over $X^{[4]}$

$$P_{12} \otimes P_{23} \otimes P_{34} = P_{14} \otimes \delta(J_{123}) \otimes \delta(J_{134}) = P_{14} \otimes \delta(J_{124}) \otimes \delta(J_{234}). \tag{4.3}$$

We would like to write $\delta(J_{123}) \otimes \delta(J_{134}) = \delta(J_{123} \otimes J_{134})$ but this is not true since the symbol $\otimes$ represents the bundle 0-gerbe product which is a contracted tensor product of bundle 0-gerbes which have the same base space. Instead, given bundle 0-gerbes (L, X) and (J, Y), with projections $X \to M$ and $Y \to M$ we define a product $(L \otimes_\delta J, X \times_M Y)$ as the bundle 0-gerbe with fibre over (x, y) given by $L_x \otimes J_y$. It is easy to show that given trivial bundle gerbes $(\delta(L), X, M)$ and $(\delta(J), Y, M)$ there is an isomorphism $\delta(L) \otimes \delta(J) = \delta(L \otimes_\delta J)$. We shall refer to this product as the *trivialisation product.*

We can now express (4.3) in terms of trivialisation products as

$$P_{12} \otimes P_{23} \otimes P_{34} = P_{14} \otimes \delta(J_{123} \otimes_\delta J_{134}) = P_{14} \otimes \delta(J_{124} \otimes_\delta J_{234}). \tag{4.4}$$

Since we have two trivialisations of the same bundle gerbe there exists a bundle 0-gerbe A_{1234} on $X^{[4]}$, called the *associator bundle 0-gerbe*, satisfying

$$\pi^{-1}A_{1234} \otimes (J_{123} \otimes_\delta J_{134}) = (J_{124} \otimes_\delta J_{234})$$

There is a technical point to be dealt with here. Up to this point we have not needed to know anything about the base spaces for the trivialisations when considered as bundle 0-gerbes. For the formula above to make sense we need the bundle 0-gerbes on both sides to have the same base. There is no reason for this to be true in general, however since we are really only interested in A_{1234} on $X^{[4]}$ we can get around this problem easily. We just take the fibre product over $X^{[4]}$ of the base spaces from each side and assume that we are actually dealing with the pullbacks to this product by the appropriate projection maps. The resultant bundle 0-gerbes still define trivialisations and A_{1234} is well defined. Throughout the rest of our definition of bundle 2-gerbes we shall assume this construction is used and will not specify base spaces for trivialisations.

Now suppose that there is a trivialisation a_{1234} of A_{1234} which we call the *associator function.* Recall that if A_{1234} is a bundle 0-gerbe over $X^{[4]}$,

$$\begin{array}{ccc} & & S^1 \\ & \overset{g}{\nearrow} & \\ A^{[2]}_{1234} & \rightrightarrows & A_{1234} \\ & & \downarrow \\ & & X^{[4]} \end{array}$$

then a_{1234} is a function $A_{1234} \to S^1$ satisfying $\delta(a_{1234}) = g$. Furthermore we require that a_{1234} satisfies a coherency condition over $X^{[5]}$. Consider the series of bundle 0-gerbe isomorphisms given by each of the embeddings $X^{[4]} \to X^{[5]}$

$$\pi^{-1}A_{1234} \otimes (J_{123} \otimes_\delta J_{134}) = (J_{124} \otimes_\delta J_{234}) \tag{4.5}$$
$$\pi^{-1}A_{1235} \otimes (J_{123} \otimes_\delta J_{135}) = (J_{125} \otimes_\delta J_{235}) \tag{4.6}$$
$$\pi^{-1}A_{1245} \otimes (J_{124} \otimes_\delta J_{145}) = (J_{125} \otimes_\delta J_{245}) \tag{4.7}$$
$$\pi^{-1}A_{1345} \otimes (J_{134} \otimes_\delta J_{145}) = (J_{135} \otimes_\delta J_{345}) \tag{4.8}$$
$$\pi^{-1}A_{2345} \otimes (J_{234} \otimes_\delta J_{245}) = (J_{235} \otimes_\delta J_{345}) \tag{4.9}$$

Consider the trivialisation product of (4.6) and (4.8)

$$(\pi^{-1}A_{1235} \otimes (J_{123} \otimes_\delta J_{135})) \otimes_\delta (\pi^{-1}A_{1345} \otimes (J_{124} \otimes_\delta J_{145})).$$

This is isomorphic to $(J_{125} \otimes_\delta J_{235}) \otimes_\delta (J_{135} \otimes_\delta J_{345})$

$$J_{125} \otimes_\delta J_{235} \otimes_\delta J_{135} \otimes_\delta J_{345}$$

and using (4.9) this is isomorphic to

$$J_{125} \otimes_\delta J_{135} \otimes_\delta (\pi^{-1} A_{2345} \otimes (J_{234} \otimes_\delta J_{245}))$$

If we continue this process using the remaining isomorphisms (4.5) and (4.7) we end up with

$$\pi^{-1} A_{2345} \otimes_\delta \pi^{-1} A_{1245} \otimes_\delta \pi^{-1} A_{1234} \otimes_\delta (J_{123} \otimes_\delta J_{135}) \otimes_\delta (J_{134} \otimes_\delta J_{145}).$$

Using the trivialisations of the associator bundle 0-gerbes this implies that there exists a function, f_{12345}, on $X^{[5]}$ such that

$$a_{1234} \otimes a_{1245} \otimes a_{2345} = a_{1235} \otimes a_{1345} \otimes \pi^{-1} f_{12345}$$

We call f_{12345} the *coherency function.*
We now return to our definition of a higher bundle gerbe. Consideration of the bundle gerbe hierarchy leads to the following

Definition 4.1. [44] Let $X \to M$ be a submersion. Let $(P, Y, X^{[2]})$ be a bundle gerbe. Then the quadruple (P, Y, X, M) is a *bundle 2-gerbe* if there is a bundle gerbe stable isomorphism

$$P_{12} \otimes P_{23} \cong P_{13}.$$

such that the corresponding associator bundle 0-gerbe is trivial, and the coherency function is identically 1. These last two conditions are called the associator trivialisation and the coherency condition respectively. The stable isomorphism together with the associator trivialisation and the coherency condition is called the bundle 2-gerbe product.

This definition corresponds to Stevenson's definition of a stable bundle 2-gerbe [44]. The bundle 2-gerbe (P, Y, X, M) may be represented diagrammatically in the following way:

$$\begin{array}{ccccc}
P & & & & \\
\downarrow & & & & \\
Y^{[2]} & \rightrightarrows & Y & & \\
& & \downarrow & & \\
& & X^{[2]} & \rightrightarrows & X \\
& & & & \downarrow \\
& & & & M
\end{array}$$

We may define pullbacks, products, duals, morphisms and trivial bundle 2-gerbes by analogy with the definitions for bundle 0-gerbes and bundle gerbes.

By following the lower dimensional cases we may define connections and curvings by choosing a bundle gerbe connection and curving on $(P, Y, X^{[2]})$ and a 3-form $\nu \in \Omega^3(X)$ such that $\delta(\nu) = \omega$ where ω is the 3-curvature of $(P, Y, X^{[2]})$. The 3-form is called the 3-curving, the curving on $(P, Y, X^{[2]})$ is called the 2-curving and the connection on $(P, Y, X^{[2]})$ is also referred to as the connection on the bundle 2-gerbe. We shall sometimes refer to a bundle 2-gerbe with connection, 2-curving and 3-curving simply as a bundle 2-gerbe with curvings. There is essentially no difference between the connection and the curvings, a connection could be referred to as a 1-curving however we shall continue to use the familiar terminology. The *4-curvature* of a bundle 2-gerbe is a 4-form $\Theta \in \Omega^4(M)$ satisfying $\pi^*\Theta = d\nu$. A bundle 2-gerbe is *flat* if the curvature is zero. For bundle 2-gerbes with curvings we also require that the bundle 2-gerbe product J and the associator trivialisation a are D-trivialisations.

Proposition 4.1. *Associated to every bundle 2-gerbe with curvings is a class in* $H^3(M, \mathcal{D}^3)$.

Proof. See [44]. □

All constructions and operations involving bundle gerbes can be carried out for bundle 2-gerbes. We describe some which are relevant here (see [44] for more detail).

If there exists a bundle gerbe $R \Rightarrow X$ such that there is a bundle gerbe morphism $\delta(R) \cong P$ over $X^{[2]}$ which is compatible with the bundle 2-gerbe product and associator function then the bundle 2-gerbe is called *trivial.*

The set of D-stable isomorphism classes of bundle 2-gerbes with 2-curving form a group under the tensor product, which is defined by analogy with the bundle gerbe case.

A flat bundle 2-gerbe has a flat holonomy which is a class in $H^3(M, U(1))$.

A D-trivial bundle 2-gerbe with curvings has a trivialisation with connection and curving such that the curvature 3-form of the trivialisation is equal to the 3-curving of the bundle 2-gerbe.

A trivial bundle 2-gerbe with curvings, $(P, Y, X, M; A, \eta, \nu)$ has a D-obstruction 3-form χ. If $\chi \in \Omega^3_0(M)$ then for any bundle gerbe $(Q, X; B, \zeta)$ which trivialises $(P; A, \eta)$ there exists a bundle gerbe $(R, M; C, \mu)$ such that $Q \otimes \pi^{-1}R$ is a D-trivialisation of $(P; A, \eta, \nu)$.

Example 4.1. [44] There exists a tautological bundle 2-gerbe associated with any closed, 2π-integral 4-form Θ on a 3-connected base M. This is defined by the following diagram

$$\begin{array}{ccc} Q[\omega] & & \\ \Downarrow & & \\ \mathcal{L}_0 M & \rightrightarrows & \mathcal{P}_0 M \\ & & \downarrow \\ & & M \end{array}$$

where $Q[\omega] \Rightarrow \mathcal{L}_0$ is the tautological bundle gerbe with curvature 3-form $\omega = \int_{S^1} ev^*\Theta$. The 3-curving is $\int_I ev^*\Theta$ and the product is defined by composition of paths in $\Omega_0 M$ as in the bundle gerbe case. We need M to be 3-connected so that $\Omega_0(M)$ is 2-connected and hence the tautological bundle gerbe is well defined.

Example 4.2. [44] The bundle 2-gerbe associated with a principal G-bundle $P_G \to M$, where G is a compact, simply connected, simple Lie group, is defined by the following diagram:

$$\begin{array}{ccc} & & Q[\omega] \\ & & \Downarrow \\ & & G \\ & \overset{g}{\nearrow} & \\ P_G^{[2]} & \rightrightarrows & P_G \\ & & \downarrow \\ & & M \end{array}$$

where g is defined as in the bundle gerbe case (see §3.3) and $Q \Rightarrow G$ is the tautological bundle gerbe with Dixmier-Douady class given by the canonical generator of $H^3(G, \mathbb{Z}) = \mathbb{Z}$. The main result regarding such bundle 2-gerbes is that the Čech 4-class is equal to the first Pontryagin class of the bundle P.

Classes in $H^3(M, \mathcal{D}^3)$ may also be represented by *2-gerbes* . We shall not give a proper definition of these here since it is quite complicated and is not of direct relevance. Full definitions may be found in [7] or [44]. Essentially if we think of a gerbe as a sheaf of groupoids then a 2-gerbe is a sheaf of 2-groupoids. These are defined in terms of higher categories. A 2-category consists of objects, 1-arrows (morphisms) and 2-arrows (transformations between morphisms) with a number of axioms relating to composition, associativity and identity. A 2-groupoid is a 2-category with invertible 2-arrows and 1-arrows which are invertible up to 2-arrows. A 2-gerbe is a sheaf of 2-groupoids with a number of gluing and descent axioms. Given a bundle 2-gerbe the objects of the 2-groupoid associated with an open set are defined by the trivialisations of the bundle 2-gerbe over the set. The 1-arrows are morphisms between the trivialisations. Since trivialisations of bundle 2-gerbes are bundle gerbes over some space their morphisms may be thought of as bundle gerbe trivialisations. The 2-arrows are morphisms of these. It is a result of Stevenson [44] that this construction gives a 2-gerbe with the same class in $H^4(M, \mathbb{Z})$ as the original bundle 2-gerbe. There are differential geometric structures on 2-gerbes which may be used to obtain a full Deligne class in $H^3(M, \mathcal{D}^3)$ [6]. We do not have a direct relationship between these and bundle 2-gerbes with connection and curvings however we shall see that such a relationship may be established indirectly via the Deligne class.

Bundle 2-gerbes and Deligne Cohomology

We prove here the main result on bundle 2-gerbes which places them in the bundle gerbe hierarchy.

Proposition 4.2. *The group of D-stable isomorphism classes of bundle 2-gerbes with 2-curving is isomorphic to $H^3(M, \mathcal{D}^3)$.*

This extends the results of Stevenson [44] which state that a bundle 2-gerbe with connection and curving defines a Deligne class and that a trivial bundle 2-gerbe has a trivial Čech class.

Proof. We shall first describe the element of $H^3(M, \mathcal{D}^3)$ representing a bundle 2-gerbe. Suppose we have a bundle 2-gerbe (P, Y, X, M), with connection, A, 2-curving η and 3-curving ν. On $U_\alpha \subset M$ define

$$\nu_\alpha = s_\alpha^* \nu.$$

Now consider the family of bundle gerbes obtained by pulling back the bundle gerbe $(P, Y, X^{[2]})$ with

$$(s_\alpha, s_\beta) : U_{\alpha\beta} \to X^{[2]}.$$

Denote these pullback bundle gerbes by $(P_{\alpha\beta}, Y_{\alpha\beta}, U_{\alpha\beta})$. They have induced connection and curving $A_{\alpha\beta}$ and $\eta_{\alpha\beta}$ respectively. Since each base space $U_{\alpha\beta}$ is contractible, each bundle gerbe $P_{\alpha\beta}$ is trivial. Thus associated with each $P_{\alpha\beta}$ is a D-obstruction 2-form

$\chi_{\alpha\beta}$. Recall that if we choose trivialisations with connections, $L_{\alpha\beta} \to Y_{\alpha\beta}$ and let $F_{L_{\alpha\beta}}$ denote the curvature of the connection on $L_{\alpha\beta}$, then $\chi_{\alpha\beta}$ is defined by $F_{L_{\alpha\beta}} = \eta_{\alpha\beta} - \pi^*\chi_{\alpha\beta}$. Also recall from §3.2 that the D-obstruction form satisfies $d\chi_{\alpha\beta} = \omega_{\alpha\beta}$ where $\omega_{\alpha\beta}$ is the 3-curvature of $P_{\alpha\beta}$. The 3-curving ν is defined such that $\delta(\nu) = \omega$ and it follows that

$$d\chi_{\alpha\beta} = \omega_{\alpha\beta} = \nu_\beta - \nu_\alpha$$

Consider the isomorphism over $U_{\alpha\beta\gamma}$,

$$P_{\alpha\beta} \otimes P_{\beta\gamma} = P_{\alpha\gamma} \otimes D(J_{\alpha\beta\gamma})$$

Using the trivialisations L_{ij} this becomes

$$\delta(L_{\alpha\beta}) \otimes \delta(L_{\beta\gamma}) = \delta(L_{\alpha\gamma}) \otimes D(J_{\alpha\beta\gamma}).$$

The D obstruction form for the left hand side is $\chi_{\alpha\beta} + \chi_{\beta\gamma}$, and for the right hand side is $\chi_{\alpha\gamma}$. We have assumed without loss of generality that the D-obstruction form of $D(J_{\alpha\beta\gamma})$ is zero rather than a general closed, 2π-integral form. If this were not the case then we could redefine it as described in §3.2. Comparing the curvatures of both sides we have

$$d\chi_{\alpha\beta} + d\chi_{\beta\gamma} = d\chi_{\alpha\gamma} = \nu_\gamma - \nu_\alpha \tag{4.10}$$

Also in terms of the definition of χ we have

$$\begin{aligned} \chi_{\alpha\beta} + \chi_{\beta\gamma} - \chi_{\alpha\gamma} &= \eta_{\alpha\beta} + \eta_{\beta\gamma} - \eta_{\alpha\gamma} - F_{L_{\alpha\beta}} - F_{L_{\beta\gamma}} + F_{L_{\alpha\gamma}} \\ &= F_{J_{\alpha\beta\gamma}} - F_{L_{\alpha\beta}} - F_{L_{\beta\gamma}} + F_{L_{\alpha\gamma}} \end{aligned} \tag{4.11}$$

so this difference is closed and 2π-integral. Hence we may apply proposition 3.3 and there exists a bundle 0-gerbe $K_{\alpha\beta\gamma}$ with curvature $-\chi_{\alpha\beta} - \chi_{\beta\gamma} + \chi_{\alpha\gamma}$ such that

$$L_{\alpha\beta} \otimes_\delta L_{\beta\gamma} = L_{\alpha\gamma} \otimes_\delta J_{\alpha\beta\gamma} \otimes_\delta \pi^{-1} K_{\alpha\beta\gamma} \tag{4.12}$$

Since $K_{\alpha\beta\gamma}$ is a bundle 0-gerbe on $U_{\alpha\beta\gamma}$ it is trivial and has a D-obstruction 1-form $\kappa_{\alpha\beta\gamma}$ which satisfies

$$d\kappa_{\alpha\beta\gamma} = -\chi_{\alpha\beta} - \chi_{\beta\gamma} + \chi_{\alpha\gamma}.$$

Using (4.12) we get

$$\begin{aligned} L_{\alpha\beta} \otimes_\delta L_{\beta\gamma} \otimes_\delta L_{\gamma\delta} &= L_{\alpha\delta} \otimes_\delta J_{\beta\gamma\delta} \otimes_\delta \pi^{-1} K_{\beta\gamma\delta} \otimes_\delta J_{\alpha\beta\delta} \otimes_\delta \pi^{-1} K_{\alpha\beta\delta} \\ &= L_{\alpha\delta} \otimes_\delta J_{\alpha\gamma\delta} \otimes_\delta \pi^{-1} K_{\alpha\gamma\delta} \otimes_\delta J_{\alpha\beta\gamma} \otimes_\delta \pi^{-1} K_{\alpha\beta\gamma}. \end{aligned}$$

Furthermore, using the definition of the associator bundle we have

$$\begin{aligned} L_{\alpha\delta} \otimes_\delta J_{\alpha\gamma\delta} \otimes_\delta J_{\alpha\beta\gamma} \otimes_\delta \pi^{-1} A_{\alpha\beta\gamma\delta} \otimes_\delta \pi^{-1} K_{\beta\gamma\delta} \otimes_\delta \pi^{-1} K_{\alpha\beta\delta} = \\ L_{\alpha\delta} \otimes_\delta J_{\alpha\gamma\delta} \otimes_\delta J_{\alpha\beta\gamma} \otimes_\delta \pi^{-1} K_{\alpha\gamma\delta} \otimes_\delta \pi^{-1} K_{\alpha\beta\gamma}. \end{aligned} \tag{4.13}$$

Thus over $U_{\alpha\beta\gamma\delta}$

$$A_{\alpha\beta\gamma\delta} \otimes K_{\beta\gamma\delta} \otimes K_{\alpha\beta\delta} = K_{\alpha\gamma\delta} \otimes K_{\alpha\beta\gamma} \tag{4.14}$$

Let $h_{\alpha\beta\gamma}$ be the trivialisation of $K_{\alpha\beta\gamma}$. Using these trivialisations together with $A_{\alpha\beta\gamma\delta} = D(a_{\alpha\beta\gamma\delta})$, equation (4.14) becomes

$$D(a_{\alpha\beta\gamma\delta}) \otimes \delta(h_{\beta\gamma\delta}) \otimes \delta(h_{\alpha\beta\delta}) = \delta(h_{\alpha\gamma\delta}) \otimes \delta(h_{\alpha\beta\gamma}) \tag{4.15}$$

The D-obstruction of the right hand side is $\kappa_{\alpha\gamma\delta} + \kappa_{\alpha\beta\gamma}$ and for the left hand side is $\kappa_{\beta\gamma\delta} + \kappa_{\alpha\beta\delta}$. The curvature of each of these is $-\chi_{\alpha\beta} - \chi_{\beta\gamma} - \chi_{\gamma\delta} + \chi_{\alpha\delta}$, and by a version of proposition 3.3 for bundles there is a function $g_{\alpha\beta\gamma\delta}$ on $U_{\alpha\beta\gamma\delta}$ which satisfies

$$d\log(g_{\alpha\beta\gamma\delta}) = -\kappa_{\alpha\beta\gamma} + \kappa_{\alpha\beta\delta} - \kappa_{\alpha\gamma\delta} + \kappa_{\beta\gamma\delta} \tag{4.16}$$

and

$$a_{\alpha\beta\gamma\delta} \otimes_\delta h_{\beta\gamma\delta} \otimes_\delta h_{\alpha\beta\delta} = \pi^* g_{\alpha\beta\gamma\delta} \otimes_\delta h_{\alpha\gamma\delta} \otimes_\delta h_{\alpha\beta\gamma} \tag{4.17}$$

on $A_{\alpha\beta\gamma\delta} \times_M K_{\beta\gamma\delta} \times_M K_{\alpha\beta\delta} \times_M K_{\alpha\gamma\delta} \times_M K_{\alpha\beta\gamma}$ as a fibre product of total spaces of the respective bundle 0-gerbes. Furthermore it may be shown that the coherency condition implies that

$$g_{\beta\gamma\delta\epsilon} \cdot g_{\alpha\beta\delta\epsilon} \cdot g_{\alpha\beta\gamma\delta} = g_{\alpha\gamma\delta\epsilon} \cdot g_{\alpha\beta\gamma\epsilon}. \tag{4.18}$$

on $U_{\alpha\beta\gamma\delta\epsilon}$.
The Deligne class is given by

$$(g_{\alpha\beta\gamma\delta}, \kappa_{\alpha\beta\gamma}, \chi_{\alpha\beta}, \nu_\alpha).$$

In trivialisations h could be replaced by sections since these notions are equivalent for bundles. Similarly the D-obstruction form $\kappa_{\alpha\beta\gamma}$ could be replaced by the pull back of the connection on $K_{\alpha\beta\gamma}$ by this section. We have used the more general terms above to highlight the role of the hierarchy, and because we believe that the language of trivialisations and D-obstructions has potential use in dealing with higher objects.

It appears that the connection form A was not used in this derivation, while it does not appear explicitly it is involved. When we trivialise $P_{\alpha\beta}$ information about A is carried in the connections on the trivialisations $L_{\alpha\beta}$. The local one forms κ depend on both the connection of the bundle gerbe and the connections on the bundles J_{123}.

Suppose we have a bundle 2-gerbe $(P, Y, X, M; A, \eta, \nu)$ represented by $(g_{\alpha\beta\gamma\delta}, \kappa_{\alpha\beta\gamma}, \chi_{\alpha\beta}, \nu_\alpha)$. To prove that this gives an isomorphism we first show that the Deligne class is independent of all choices in the construction. Suppose we were to replace $h_{\alpha\beta\gamma}$ by $\tilde{h}_{\alpha\beta\gamma}$. These are both trivialisations of the bundle 0-gerbe $K_{\alpha\beta\gamma} \to U_{\alpha\beta\gamma}$ so they differ by a function $p_{\alpha\beta\gamma}$. Comparing the two versions of equation (4.17) obtained from the two choices we have

$$a_{\alpha\beta\gamma\delta} \otimes_\delta a^{-1}_{\alpha\beta\gamma\delta} \otimes_\delta \pi^* p_{\beta\gamma\delta} \otimes_\delta \pi^* p_{\alpha\beta\delta} = \pi^* \tilde{g}_{\alpha\beta\gamma\delta} \otimes_\delta \pi^* g^{-1}_{\alpha\beta\gamma\delta} \otimes_\delta \pi^* p_{\alpha\gamma\delta} \otimes_\delta \pi^* p_{\alpha\beta\gamma} \tag{4.19}$$

and so we have

$$\tilde{g}_{\alpha\beta\gamma\delta} = g_{\alpha\beta\gamma\delta}\delta(p)_{\alpha\beta\gamma\delta} \tag{4.20}$$

Recall that the local connections $\kappa_{\alpha\beta\gamma}$ were defined as D-obstruction forms for the bundle 0-gerbes $K_{\alpha\beta\gamma}$. Changing the choice of trivialisation changes this D-obstruction form by the 1-curvature of the function defined by the difference of the two trivialisations so we have

$$\tilde{\kappa}_{\alpha\beta\gamma} = \kappa_{\alpha\beta\gamma} + dp_{\alpha\beta\gamma} \tag{4.21}$$

Together equations (4.20) and (4.21) change the Deligne class by a trivial cocycle.

Now suppose that we change $L_{\alpha\beta}$ to $\tilde{L}_{\alpha\beta}$. These differ by a bundle with connection $T_{\alpha\beta} \to U_{\alpha\beta}$. Comparing the two versions of equation (4.12) we have

$$\pi^{-1}T_{\alpha\beta} \otimes_\delta \pi^{-1}T_{\beta\gamma} = \pi^{-1}T_{\alpha\gamma} \otimes_\delta J_{\alpha\beta\gamma} \otimes_\delta \pi^{-1}\tilde{K}_{\alpha\beta\gamma} \otimes_\delta J^*_{\alpha\beta\gamma} \otimes_\delta \pi^{-1}K^*_{\alpha\beta\gamma} \tag{4.22}$$

so the connection on $K_{\alpha\beta\gamma}$ changes by $B_{\alpha\beta} + B_{\beta\gamma} - B_{\alpha\gamma}$, where $B_{\alpha\beta}$ is the connection on $T_{\alpha\beta}$ and hence the D-obstruction form $\kappa_{\alpha\beta\gamma}$ is changed in the same way,

$$\tilde{\kappa}_{\alpha\beta\gamma} = \kappa_{\alpha\beta\gamma} + B_{\alpha\beta} + B_{\beta\gamma} - B_{\alpha\gamma} \tag{4.23}$$

Under the change from $L_{\alpha\beta}$ to $\tilde{L}_{\alpha\beta}$ the D-obstruction forms $\chi_{\alpha\beta}$ will change by the curvature of $T_{\alpha\beta}$,

$$\tilde{\chi}_{\alpha\beta} = \chi_{\alpha\beta} + dB_{\alpha\beta} \tag{4.24}$$

Together equations (4.23) and (4.24) change the Deligne class by a trivial cocycle.

The final choice that we have made is of the sections s_α. For each choice of section there is a trivialisation R_α, a bundle gerbe over $\pi^{-1}(U_\alpha)$ which is defined by $R_\alpha = (1, s_\alpha)^{-1}P$, where P is considered as a bundle gerbe on $Y^{[2]}$. Then $\pi^{-1}P_{\alpha\beta} = R^*_\alpha \otimes R_\beta$ and a different choice of section, $\tilde{s}_\alpha$ defines a bundle gerbe ξ_α on U_α satisfying $\tilde{R}_\alpha = R_\alpha \otimes \pi^*\xi_\alpha$. Thus a change of section changes $P_{\alpha\beta}$ by

$$\tilde{P}_{\alpha\beta} = P_{\alpha\beta} \otimes \xi^*_\alpha \otimes \xi_\beta \tag{4.25}$$

If ξ_α has curving μ_α then the 2-forms $f_{\alpha\beta}$ and hence the D-obstruction forms $\chi_{\alpha\beta}$ change by $\mu_\beta - \mu_\alpha$. The local 3-curvings may be thought of as a D-obstruction form for the trivialisation R_α so they change by $d\mu_\alpha$ and once again we have a trivial contribution to the Deligne class.

Next we claim that this assignment of a Deligne class to a bundle 2-gerbe is a homomorphism. Since the 3-curving of the tensor product of two bundle 2-gerbes is the sum of their respective 3-curvings, and the local 3-curvings are defined by pullback, then it is clear that the local two curvings will be additive under tensor products. The local 2-curvings and connections are both defined as D-obstruction forms for a bundle gerbe and bundle 0-gerbe respectively. Since D-obstructions are additive under tensor products in both of these cases then so will the local 2-curvings and connections. The transition functions may be thought of in similar terms as a D-obstruction defined in terms of two trivialisations of a bundle 0-gerbe and hence the assignment of a Deligne class preserves the tensor product of bundle 2-gerbes with curvings. To see that this gives a homomorphism between equivalence classes we show that a D-trivial bundle 2-gerbe has a trivial Deligne class. Let $Q \to Y$ be a D-trivialisation of the bundle 2-gerbe (Q, X, Y, M). The 3-curvature ω_Q of Q satisfies $\omega_Q = \nu$, where ν is the 3-curving of P. Using a section $s_\alpha : U_\alpha \to Y$ we pull back Q to a bundle gerbe Q_α. This bundle gerbe must be trivial so let q_α be the D-obstruction form. This satisfies

$$dq_\alpha = s^*_\alpha \omega_Q = s^*_\alpha \nu = \nu_\alpha \tag{4.26}$$

If R_α is a trivialisation of Q_α then the isomorphism

$$P_{\alpha\beta} = Q^*_\alpha \otimes Q_\beta \tag{4.27}$$

induces an isomorphism $\delta(L_{\alpha\beta}) = \delta(R^*_\alpha) \otimes \delta(R_\beta)$. This means we may define bundles $N_{\alpha\beta}$ which satisfy

$$L_{\alpha\beta} = R^*_\alpha \otimes R_\beta \otimes \pi^{-1} N_{\alpha\beta} \tag{4.28}$$

We may now find an expression for the D-obstruction form for $P_{\alpha\beta}$ which is defined by $\pi^*\chi_{\alpha\beta} = f_{\alpha\beta} - F_{L_{\alpha\beta}}$. Since Q is a D-trivialisation then from equation (4.27) we have $f_{\alpha\beta} = f_{Q_\beta} - f_{Q_\beta}$ where the terms on the right are the curvings induced on Q_α and Q_β from that of Q. Equation (4.28) gives an equation for the curvature of $L_{\alpha\beta}$, $F_{L_{\alpha\beta}} = F_{Q_\beta} - F_{Q_\alpha} + \pi^{-1}F_{N_{\alpha\beta}}$. Since $N_{\alpha\beta}$ is trivial it has a D-obstruction form $n_{\alpha\beta}$ which satisfies $dn_{\alpha\beta} = F_{N_{\alpha\beta}}$. Putting these together we get

$$\begin{aligned} \pi^*\chi_{\alpha\beta} &= f_{Q_\beta} - f_{Q_\beta} - F_{Q_\beta} + F_{Q_\alpha} - \pi^* dn_{\alpha\beta} \\ &= \pi^* q_\beta - \pi^* q_\alpha - \pi^* dn_{\alpha\beta} \end{aligned} \tag{4.29}$$

and so

$$\chi_{\alpha\beta} = q_\beta - q_\alpha - dn_{\alpha\beta} \tag{4.30}$$

Next we need a D-obstruction form for $K_{\alpha\beta\gamma}$. First observe that we may express the bundle 2-gerbe product in terms of Q, using $P_{\alpha\beta} \otimes P_{\beta\gamma} = P_{\alpha\gamma} \otimes D(J_{\alpha\beta\gamma})$ to get $Q^*_\alpha \otimes Q_\beta \otimes Q^*_\beta \otimes Q_\alpha = Q^*_\alpha \otimes Q_\gamma \otimes D(J_{\alpha\beta\gamma})$. Using the trivialisations R_α we may express this as a difference of two trivialisation and define a bundle $M_{\alpha\beta\gamma}$ such that

$$R^*_\alpha \otimes R_\beta \otimes R^*_\beta \otimes R_\gamma = R^*_\alpha \otimes R_\gamma \otimes J_{\alpha\beta\gamma} \otimes \pi^{-1} M_{\alpha\beta\gamma} \tag{4.31}$$

The combination of R terms is actually a D-trivialisation since the sum of the D-obstructions cancels so we may assume that $M_{\alpha\beta\gamma}$ is flat.

Substituting into equation (4.12) gives

$$K_{\alpha\beta\gamma} = N_{\alpha\beta} \otimes N_{\beta\gamma} \otimes N^*_{\alpha\beta} \otimes M_{\alpha\beta\gamma} \tag{4.32}$$

The D-obstruction form is defined by $\pi^*\kappa_{\alpha\beta\gamma} = A_{K_{\alpha\beta\gamma}} - d\log h_{\alpha\beta\gamma}$. A formula for the connection may be obtained from equation (4.32),

$$A_{K_{\alpha\beta\gamma}} = A_{N_{\alpha\beta}} + A_{N_{\beta\gamma}} - A_{N_{\alpha\beta}} + A_{M_{\alpha\beta\gamma}} \tag{4.33}$$

where the terms on the right are the connections on the respective bundles. Let $\zeta_{\alpha\beta}$ be a trivialisation of $N_{\alpha\beta}$ and let $\epsilon_{\alpha\beta\gamma}$ be a trivialisation of $M_{\alpha\beta\gamma}$. Then using (4.32) we may define two trivialisations and hence define functions $\rho_{\alpha\beta\gamma}$ on $U_{\alpha\beta\gamma}$ which satisfy

$$h_{\alpha\beta\gamma} = \zeta_{\alpha\beta}\zeta_{\beta\gamma}\zeta^{-1}_{\alpha\gamma}\epsilon_{\alpha\beta\gamma}\pi^*\rho_{\alpha\beta\gamma} \tag{4.34}$$

Thus we have

$$d\log h_{\alpha\beta\gamma} = d\log\zeta_{\alpha\beta} + d\log\zeta_{\beta\gamma} - d\log\zeta_{\alpha\gamma} + d\log\epsilon_{\alpha\beta\gamma} - \pi^* d\log\rho_{\alpha\beta\gamma} \tag{4.35}$$

Observe that $A_{N_{\alpha\beta}} - d\log\zeta_{\alpha\beta} = \pi^* n_{\alpha\beta}$ by definition. Also, since $M_{\alpha\beta\gamma}$ is flat and is defined on a contractible set then we may assume that we have a flat trivialisation so $A_{M_{\alpha\beta\gamma}} = d\log\epsilon_{\alpha\beta\gamma}$. Combining all of this we have

$$\kappa_{\alpha\beta\gamma} = n_{\alpha\beta} + n_{\beta\gamma} - n_{\alpha\gamma} + d\log\rho_{\alpha\beta\gamma} \tag{4.36}$$

The final step follows the same pattern however the arguments will be simpler as we are dealing with functions. First use (4.14) to compare a with ζ and ϵ. Substituting this expression for a and the expressions (4.34) for the h terms into (4.17) will lead to the cancellation of all ζ and ϵ terms leaving

$$g_{\alpha\beta\gamma\delta} = \delta(\rho)_{\alpha\beta\gamma\delta} \tag{4.37}$$

thus we have a trivial Deligne class

$$(g_{\alpha\beta\gamma\delta}, \kappa_{\alpha\beta\gamma}, \chi_{\alpha\beta}, \nu_\alpha) = (\delta(\rho)_{\alpha\beta\gamma\delta}, \delta(n)_{\alpha\beta\gamma} + d\log\rho_{\alpha\beta\gamma}, \delta(q)_{\alpha\beta} - dn_{\alpha\beta}, dq_\alpha) \tag{4.38}$$

and the assignment of a Deligne class to a D-stable isomorphism class of bundle 2-gerbes is a homomorphism.

To show injectivity of this homomorphism we shall show that the Deligne class of a bundle 2-gerbe is trivial only if the bundle 2-gerbe is D-trivial. The corresponding result for δ-trivialisations and Čech classes has already been given by Stevenson [44]. This means that if a bundle 2-gerbe is not δ-trivial then its Deligne class is not trivial, so we may assume without loss of generality that our bundle 2-gerbe is δ-trivial, but not D-trivial. Furthermore by standard arguments this trivialisation may be given connection and curving which are compatible with the trivialisation. In this case there exists a D-obstruction form $\zeta \in \Omega^3(M)$ which is defined by $\pi^*\zeta = \nu - \omega_Q$, where ν is the 3-curving of the bundle 2-gerbe and ω_Q is the curvature of the trivialisation Q. By assumption the D-obstruction form is non-trivial, this implies that $\zeta \notin \Omega^3_0(M)$. To find the Deligne class of this bundle 2-gerbe the arguments above, where the Deligne class of a D-trivial bundle 2-gerbe was calculated, still apply with the exception of the local 3-curving, which is now given by $\nu_\alpha = dq_\alpha + \zeta$. Thus we have

$$\begin{aligned}(g_{\alpha\beta\gamma\delta}, \kappa_{\alpha\beta\gamma}, \chi_{\alpha\beta}, \nu_\alpha) &= (\delta(\rho)_{\alpha\beta\gamma\delta}, \delta(n)_{\alpha\beta\gamma} + d\log\rho_{\alpha\beta\gamma}, \delta(q)_{\alpha\beta} - dn_{\alpha\beta}, dq_\alpha + \zeta) \\ &= D(\rho, n, q) + (1, 0, 0, \zeta)\end{aligned} \tag{4.39}$$

Since $\zeta \notin \Omega^3_0(M)$ the class $(1, 0, 0, \zeta)$ is not D-exact, so we have shown that a bundle 2-gerbe which is not D-trivial cannot have a trivial Deligne class.

Finally we need to show that there exists a bundle 2-gerbe with connection and curvings which is represented by any given Deligne class $(g_{\alpha\beta\gamma\delta}, \kappa_{\alpha\beta\gamma}, \chi_{\alpha\beta}, \nu_\alpha)$. Define this bundle 2-gerbe by

$$\begin{array}{ccccc} T^* \otimes T & & T & & \\ \downarrow & & \downarrow & & \\ \amalg U_{ij} & \rightrightarrows & \amalg U_{ij} & & \\ & & \downarrow & & \\ & & \amalg U_{ij} & \rightrightarrows & \amalg U_i \\ & & & & \downarrow \\ & & & & M \end{array}$$

where the projection $\amalg U_{ij} \to \amalg U_{ij}$ is the identity and T is the flat D-trivial bundle. This basic structure was suggested by Danny Stevenson. The 3-curving at $m \in U_i$ is ν_i. Clearly $s_\alpha^* \nu = \nu_\alpha$. The 2-curving on $\amalg U_{ij}$ is χ_{ij}. The local 2-curving is the D-obstruction form for $P_{\alpha\beta}$. In this example $P_{\alpha\beta}$ is simply the restriction of the trivial bundle gerbe over $\amalg U_{ij}$ to $U_{\alpha\beta}$. A trivialisation is given by $T_{\alpha\beta}$. Since $F_{T_{ij}} = 0$ the D-obstruction form for $P_{\alpha\beta}$ is $\chi_{\alpha\beta}$. To define the bundle 2-gerbe product we need to consider the following trivial bundle gerbe

$$\begin{array}{ccc} T^* \otimes T & & T \\ \downarrow & & \downarrow \\ \amalg U_{ijk} & \rightrightarrows & \amalg U_{ijk} \\ & & \downarrow \\ & & \amalg U_{ijk} \end{array}$$

The bundle 2-gerbe product is defined by a D-trivialisation of this bundle gerbe, J_{ijk}. We define this to be the trivial bundle on $\amalg U_{ijk}$ with connection at $m \in U_{ijk}$ given by κ_{ijk}.

Now we would like to find the local connection 1-forms. It might appear that these would have to be trivial since the bundle 2-gerbe connection is, however the product also carries information on the local 1-connections. We know from equation (4.12) that there is an isomorphism of bundles with connection

$$T_{\alpha\beta} \otimes T_{\beta\gamma} = T_{\alpha\gamma} \otimes J_{\alpha\beta\gamma} \otimes K^*_{\alpha\beta\gamma} \tag{4.40}$$

where we have used the fact that the projection is just the identity to pull back all of these to $U_{\alpha\beta\gamma}$. The D-obstruction form for $K_{\alpha\beta\gamma}$ is now just $\kappa_{\alpha\beta\gamma}$. This is because the T's have zero connections and flat trivialisations so their D-obstructions are zero.

Finally we define the associator function on $\amalg U_{ijkl}$ by g_{ijkl} which satisfies the coherency condition and so we have a bundle 2-gerbe which, by construction, has Deligne class $(g_{\alpha\beta\gamma\delta}, \kappa_{\alpha\beta\gamma}, \chi_{\alpha\beta}, \nu_\alpha)$.

□

It is sometimes possible, for example when $Y \to X^{[2]}$ is a fibration, to calculate the transition functions of a bundle 2-gerbe by an easier method as described by Stevenson [43]. We shall give a brief outline of this method. It applies when the trivial bundle gerbes $(P_{\alpha\beta}, Y_{\alpha\beta}, U_{\alpha\beta})$ admit a section $\sigma_{\alpha\beta} : U_{\alpha\beta} \to Y_{\alpha\beta}$. Recall that unlike bundles, trivial bundle gerbes do not necessarily admit a section (see comments after lemma 3.2). If they do then we have a map $\sigma_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to Y^{[2]}_{\alpha\beta\gamma}$ given by $(\sigma_{\alpha\gamma}, \sigma_{\beta\gamma} \circ \sigma_{\alpha\beta})$ where $\sigma_{\beta\gamma} \circ \sigma_{\alpha\beta}$ is the map $Y_{\beta\gamma} \times Y_{\alpha\beta} \to Y_{\alpha\gamma}$ which is implicitly defined by the bundle 2-gerbe product. Use $\sigma_{\alpha\beta\gamma}$ to pull back the bundle $P \to Y^{[2]}$ to $P_{\alpha\beta\gamma} \to U_{\alpha\beta\gamma}$. These bundles play the role of $K_{\alpha\beta\gamma}$ in the general method. We now continue as in the general case by choosing sections $h_{\alpha\beta\gamma}$ (which are equivalent to trivialisations for bundles) of $P_{\alpha\beta\gamma}$ which then satisfy equation (4.17) (with notation adjusted for sections),

$$a_{\alpha\beta\gamma\delta} h_{\beta\gamma\delta} h_{\alpha\beta\delta} = g_{\alpha\beta\gamma\delta} h_{\alpha\gamma\delta} h_{\alpha\beta\gamma}$$

With the presence of sections rather than trivialisations the D-obstruction forms used to define the full Deligne class may be replaced by the pullbacks of the relevant connections and curvings by the sections.

We conclude our discussion of bundle 2-gerbes with two constructions involving the cup product in Deligne cohomology which provide concrete examples and demonstrate the usefulness of the geometric picture of Deligne cohomology which bundle gerbes provide.

The Cup Product of Two Bundles

Let L and J be two bundles over M. Then there is a bundle 2-gerbe defined by the cup product $L \cup J$. If L and J both have connections then $L \cup J$ has a 2-curving. In this case the Deligne class is $(h_{\gamma,\delta}^{m_{\alpha\beta\gamma}}, m_{\alpha\beta\gamma}B_\gamma, \log(g_{\alpha\beta})F_B, A_\alpha \wedge F_B)$, where the Deligne classes of L and J are $(g_{\alpha\beta}, A_\alpha)$ and $(h_{\alpha\beta}, B_\alpha)$ respectively, $m_{\alpha\beta\gamma} = \log(g_{\alpha\gamma}) - \log(g_{\beta\gamma}) - \log(g_{\alpha\beta})$ is the Chern class of L and F_B is the curvature of J. We define the bundle 2-gerbe $L \cup J$ by the following diagram:

$$\begin{array}{ccc} & & g \cup J \\ & & \downarrow \\ & & S^1 \times M \\ & \overset{(g,\pi_J)}{\nearrow} & \\ (L \times S^1) \times_\pi (J \times S^1) & \rightrightarrows & L \times_\pi J \\ & & \downarrow \\ & & M \end{array}$$

where the map (g, π_J) is defined by $(g, \pi_J)(l, \theta, j, \phi) = (\theta, \pi_J(j))$. The fibre product bundle $L \times_\pi J$ with structure group $S^1 \times S^1$ is often written as $L \oplus J$. We take the bundle gerbe product to be the trivial and the associator function to be identically 1. The bundle gerbes $P_{\alpha\beta}$ are given by $g \cup J$ over $(g_{\alpha\beta}, m)$. We may define sections of these bundle gerbes and so use the simpler method for calculating the Deligne class. Recall that we may write $g \cup J$ as

$$\begin{array}{ccc} J^{-n} & & \\ \downarrow & & \\ \mathbb{R} \times \mathbb{Z} \times M & \rightrightarrows & \mathbb{R} \times M \\ & & \downarrow \\ & & S^1 \times M \end{array}$$

Define sections $\sigma_{\alpha\beta}$ by

$$\sigma_{\alpha\beta} = (\log(g_{\alpha\beta}), m) \tag{4.41}$$

The section $\sigma_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to (\mathbb{R} \times \mathbb{Z} \times M)_{\alpha\beta}$ is then given by

$$\sigma_{\alpha\beta\gamma} = (\log(g_{\alpha\gamma}), -m_{\alpha\beta\gamma}, m) \tag{4.42}$$

Sections $h_{\alpha\beta\gamma}$ of the bundle $P_{\alpha\beta\gamma}$ are $t_\gamma^{m_{\alpha\beta\gamma}}$ where t_γ is a local section of J. We can now calculate the transition functions

$$\begin{aligned} g_{\alpha\beta\gamma\delta} h_{\alpha\gamma\delta} h_{\alpha\beta\gamma} &= h_{\beta\gamma\delta} h_{\alpha\beta\delta} \\ g_{\alpha\beta\gamma\delta} t_\delta^{m_{\alpha\gamma\delta}} t_\gamma^{m_{\alpha\beta\gamma}} &= t_\delta^{m_{\beta\gamma\delta}} t_\delta^{m_{\alpha\beta\delta}} \\ g_{\alpha\beta\gamma\delta} t_\delta^{m_{\alpha\gamma\delta}+m_{\alpha\beta\gamma}} h_{\gamma\delta}^{-m_{\alpha\beta\gamma}} &= t_\delta^{m_{\beta\gamma\delta}+m_{\alpha\beta\delta}} \\ g_{\alpha\beta\gamma\delta} &= h_{\gamma\delta}^{m_{\alpha\beta\gamma}} \end{aligned} \tag{4.43}$$

If the connection is $-nB$ on J^{-n} then the local connection forms given by the pullback by h are $m_{\alpha\beta\gamma}B_\gamma$ as required. If the 2-curving is $f = rF_B$ on $\mathbb{R} \times M$ then $\sigma^*_{\alpha\beta} f = \log(g_{\alpha\beta})F_B$, and the 3-curving is given by $A \wedge B$ on $L \times_\pi J$.

The general structure of this bundle 2-gerbe demonstrates a hierarchy principle for cup product structures. Recall that the cup product $f \cup L$ may be constructed in a similar way in terms of the cup product bundle, so contained within this cup product bundle 2-gerbe is a cup product bundle gerbe and within that a cup product bundle.

A second point to note is that this bundle 2-gerbe to some extent resembles the associated bundle gerbe for a G-bundle. In this case the G-bundle would be the $S^1 \times S^1$ bundle $L \oplus J$. The 4-curvature of this bundle 2-gerbe is given in terms of the curvatures of the two bundles, $F_L \wedge F_J$. This is the image in real cohomology of the first Pontryagin class of $L \oplus J$, which also be the case with a bundle 2-gerbe associated to a G-bundle. Since this structure group is not simply connected such an associated bundle is not actually defined, the obstruction being the fact that the tautological bundle gerbe on $S^1 \times S^1$ is not well defined. In this particular case we are able to build a similar bundle 2-gerbe by replacing the tautological bundle with the cup product bundle.

The Cup Product of a Function and a Bundle Gerbe

Let f be a $U(1)$-function and let (P, Y, M) be a bundle gerbe with connection A, curving η and 3-curvature ω. The cup product $f \cup P$ has Deligne class

$$(g_{\beta\gamma\delta}^{n_{\alpha\beta}}, n_{\alpha\beta}A_{\beta\gamma}, n_{\alpha\beta}\eta_\beta, \log_\alpha(f)\omega)$$

The second realisation of $f \cup L$ in §3.5 suggests that this bundle 2-gerbe should be realised geometrically with the following diagram:

$$\begin{array}{ccc}
P^{-n} & & \\
\downarrow & & \\
\mathbb{R} \times \mathbb{Z} \times M & \rightrightarrows & \mathbb{R} \times M \\
 & & \downarrow \\
M & \xrightarrow{(f,m)} & S^1 \times M
\end{array}$$

where the fibre over $(r, n, m) \in \mathbb{R} \times \mathbb{Z} \times M$ is the n-fold tensor product of P with the trivial bundle 2-gerbe product and associator function. The 3-curving is $r\omega$, the 2-curving $-n\eta$ and connection $-nA$. Pulling back by the section $(\log_\alpha(f), n_{\alpha\beta}, m)$ we have a bundle gerbe $P^{-n_{\alpha\beta}}$ over $U_{\alpha\beta}$. In this case the local constructions may once again be simplified however this time the construction is slightly different.

There exist trivialisations J_α of P_α over U_α, so over U_{ij} we have trivialisations $J_\beta^{n_{\alpha\beta}}$ of $P^{-n_{\alpha\beta}}$. On double overlaps $J_\beta \cong J_\alpha \otimes \pi^{-1}L_{\alpha\beta}$, for some bundle $L_{\alpha\beta} \to U_{\alpha\beta}$, so on triple overlaps the local bundles are obtained by comparing $J_\beta^{-n_{\alpha\beta}} \otimes J_\gamma^{-n_{\beta\gamma}}$ and $J_\gamma^{-n_{\alpha\gamma}}$,

$$\begin{aligned}
J_\beta^{-n_{\alpha\beta}} \otimes J_\gamma^{-n_{\beta\gamma}} \otimes J_\gamma^{n_{\alpha\gamma}} &= \pi^{-1}L_{\beta\gamma}^{-n_{\alpha\gamma}} \otimes J_\gamma^{-n_{\alpha\beta}} \otimes J_\gamma^{-n_{\beta\gamma}} \otimes J_\gamma^{n_{\alpha\gamma}} \\
&= \pi^{-1}L_{\beta\gamma}^{-n_{\alpha\gamma}}
\end{aligned} \tag{4.44}$$

Each $L_{\alpha\beta}$ has a section $l_{\alpha\beta}$ over $U_{\alpha\beta}$, this allows us to find a section $l_{\beta\gamma}^{n_{\alpha\beta}}$. Moreover the sections $l_{\alpha\beta}$ are the sections which determine the Dixmier-Douady class of P, so they satisfy $l_{\alpha\beta}l_{\beta\gamma} = l_{\alpha\gamma}g_{\alpha\beta\gamma}$. Using this it is possible to calculate $\delta(l_{\beta\gamma}^{n_{\alpha\beta}})_{\alpha\beta\gamma\delta}$ over $U_{\alpha\beta\gamma\delta}$. This gives the correct transition functions and the local connections and curvings may also be obtained without difficulty and agree with what is expected.

4.4 $\mathbb{Z}$-Bundle 0-Gerbes

Consider once again the bundle gerbe hierarchy as given in table 4.1. A general method for constructing geometric realisations may be approached in the following way. Begin with some geometric representation, R, of a Deligne cohomology group H^p. Build representations of higher dimensional Deligne cohomology in the following way

$$
\begin{array}{llll}
H^p: & R & & \\
 & \vdots & & \\
H^{p+1}: & R & & \\
 & \downarrow & & \\
 & Y^{[2]} & \rightrightarrows Y & \\
 & & \downarrow & \\
 & & M & \\
 & \vdots & & \\
H^{p+2}: & R & & \\
 & \downarrow & & \\
 & Y^{[2]} & \rightrightarrows Y & \\
 & & \downarrow & \\
 & & X^{[2]} & \rightrightarrows X \\
 & & & \downarrow \\
 & & & M
\end{array}
$$

The examples which we have already dealt with are where R is either a function or a bundle. Furthermore we have shown that when we start with a function the next object in the hierarchy is a bundle 0-gerbe which is equivalent to a bundle. Continuing this method gives the basic structure of bundle gerbes and bundle 2-gerbes. Attempts to generalise to higher degree meet difficulties due to the increasingly complicated nature of product structures and related associativity conditions. In this section we wish to address the question of whether there is a starting point in the hierarchy below $U(1)$ functions. This leads to consideration of $\mathbb{Z}$-bundle 0-gerbes.

Since $U(1)$-functions represent $H^1(M, \mathbb{Z}(1)_D)$ then the only lower object in the hierarchy would be a representative of $H^0(M, \mathbb{Z}(0)_D)$, that is, the set of $\mathbb{Z}$ valued functions on M. Using these as a basis for a hierarchy we get a new family of objects, $\mathbb{Z}$-bundle n-gerbes. A *$\mathbb{Z}$-bundle 0-gerbe* is defined as a bundle 0-gerbe (λ, Y, M) where the function λ takes values in $\mathbb{Z}$.

Proposition 4.3. *The group of stable isomorphism classes of $\mathbb{Z}$-bundle 0-gerbes is isomorphic to $H^1(M, \mathbb{Z}(0)_D)$.*

Proof. The arguments of Proposition 2.1 still apply in this case giving an isomorphism with $H^1(M, \mathbb{Z})$. Furthermore $H^1(M, \mathbb{Z}) \cong H^1(M, \mathbb{Z}(0)_D)$ since $\mathbb{Z}(0)_D = \mathbb{Z}$. □

We wish to find an analogy with the equivalence of bundle 0-gerbes and bundles and of bundle gerbes and gerbes. This leads us to consider the Deligne cohomology group

$H^1(M, \mathbb{Z}(1)_D)$ and the isomorphisms

$$H^1(M, \mathbb{Z}(1)_D) \cong H^1(M, \mathbb{Z}(1) \to \underline{\mathbb{R}}) \cong H^0(M, \underline{U(1)}).$$

To relate $\mathbb{Z}$-bundle 0-gerbes to our usual geometric representation of degree 1 Deligne cohomology, $U(1)$-functions, we shall require some extra structure.

Definition 4.2. Let (λ, Y, M) be a $\mathbb{Z}$-bundle 0-gerbe. A *$\mathbb{Z}$-curving* on (λ, Y, M) is a map $f : Y \to \mathbb{R}$ which satisfies $\delta(f) = \lambda$.

Proposition 4.4. *For each $\mathbb{Z}$-bundle 0-gerbe there exists a $\mathbb{Z}$-curving which is unique up to the pull back of a globally defined $\mathbb{R}$-valued function on the base.*

Proof. Let (λ, Y, M) be a $\mathbb{Z}$-bundle 0-gerbe. Choose an open cover $\{U_\alpha\}$ of M. Let $f_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{Z}$ be the family of functions defined by

$$f_\alpha(y) = \lambda(s_\alpha(\pi(y)), y).$$

Let $\{\phi_\alpha\}$ be a partition of unity on M and let $f : Y \to \mathbb{R}$ be defined by

$$f(y) = \sum_\alpha \phi_\alpha(\pi(y)) f_\alpha(y)$$

Let $(y_1, y_2) \in Y^{[2]}$ with $\pi(y_1) = \pi(y_2) = m$. Then

$$\begin{aligned}
\delta(f)(y_1, y_2) &= f(y_2) - f(y_1) \\
&= \sum_\alpha [\phi_\alpha(\pi(y_2)) f_\alpha(y_2) - \phi_\alpha(\pi(y_1)) f_\alpha(y_1)] \\
&= \sum_\alpha \phi_\alpha(m)[\lambda(s_\alpha(m), y_2) - \lambda(s_\alpha(m), y_1)] \\
&= \sum_\alpha \phi_\alpha(m) \lambda(y_1, y_2) \\
&= \lambda(y_1, y_2)
\end{aligned}$$

Thus we see that f is a $\mathbb{Z}$-curving for (λ, Y, M).

Suppose there exists another $\mathbb{Z}$-curving, g. Then $\delta(f - g) = 0$ so $f - g$ descends to a function on M. □

The correspondence between $U(1)$-bundle 0-gerbes and $U(1)$-bundles also applies with $U(1)$ replaced by $\mathbb{Z}$, so we may replace stable isomorphism classes of bundle 0-gerbes with isomorphism classes of $\mathbb{Z}$-bundles.

4.5 B^pS^1-Bundles

The correspondence between B^pS^1-bundles and Deligne cohomology was established by Gajer [22]. This was proven abstractly using sheaf theoretical arguments and also given in terms of explicit classifying maps using the bar resolution to obtain a realisation of the classifying spaces. We shall show that this correspondence is suggested by consideration of the classifying theory of bundles together with our discussion of $\mathbb{Z}$-bundle 0-gerbes and the bundle gerbe hierarchy.

We recall some well known results on the classification of bundles (see, for example, [29] or [16]). Let $P_G \to M$ be a principal G-bundle. There exists a G-bundle $EG \to BG$ such that $P_G = \psi^{-1}EG$ where ψ, called the classifying map, is unique up to homotopy. The space BG is called the universal classifying space and $EG \to BG$ is called the universal G-bundle. The classifying bundle is a G bundle with a contractible total space which is unique up to homotopy equivalence.

The results of the previous section may be interpreted in terms of classifying theory. Since $\mathbb{R} \to S^1$ defines a $\mathbb{Z}$ bundle and $\mathbb{R}$ is contractible then $B\mathbb{Z} = S^1$. The equivalence between $\mathbb{Z}$ bundles and homotopy classes of S^1 valued functions corresponds to the classifying theory of $\mathbb{Z}$ bundles. Given a $\mathbb{Z}$-bundle 0-gerbe with $\mathbb{Z}$-curving $(\lambda, Y, M; f)$ we define an S^1-function by $\hat{f}(m) = \exp f(y)$, where $y \in \pi^{-1}(m)$. This is independent of the choice of y since for $y, y' \in \pi^{-1}(m)$

$$\begin{aligned}(\exp f(y))(\exp f(y'))^{-1} &= \exp(f(y) - f(y')) \\ &= \exp(\lambda(y', y)) \\ &= 1\end{aligned}$$

since $\lambda(y', y) \in \mathbb{Z}$. The existence of a $\mathbb{Z}$-curving up to a global $\mathbb{R}$-function is equivalent to the existence of a classifying map up to homotopy.

We now have the following equivalences of geometric realisations of Deligne cohomology:

$$\begin{array}{ccc}\mathbb{Z}\text{-bundle 0-gerbes} & \longleftrightarrow & B\mathbb{Z}\text{-functions} \\ \text{bundle 0-gerbes} & \longleftrightarrow & BS^1\text{-functions}\end{array}$$

where the left hand side consists of stable isomorphism classes and the right hand side are homotopy classes of maps. The case of higher dimensional objects is dealt with by the following

Proposition 4.5. *[22] The group $H^p(M, \mathbb{Z})$ is isomorphic to the group of isomorphism classes of smooth principal $B^{p-2}S^1$-bundles over M.*

Smoothness of classifying spaces is defined in terms of a differentiable space structure. For more details see [22] or [33]. In general the iterated classifying spaces B^pG are only defined if G is Abelian. When this is the case each of the spaces B^pG is also an Abelian group. Consider this result for low values of p. When $p = 2$ we have the usual correspondence between S^1-bundles and $H^2(M, \mathbb{Z})$ given by the Chern class. When $p = 3$ we have $H^3(M, \mathbb{Z})$ and BS^1-bundles. Our usual geometric realisation of $H^3(M, \mathbb{Z})$ is stable isomorphism classes of bundle gerbes so the following result is not surprising,

Proposition 4.6. *[38] The set of all stable isomorphism classes of bundle gerbes on M is in bijective correspondence with the set of all isomorphism classes of BS^1 bundles on M.*

Since BS^1 is an Abelian group we could replace BS^1-bundles with BS^1-bundle 0-gerbes. The bundle gerbe corresponding to a BS^1 bundle which is referred to in the proposition is the lifting bundle gerbe

$$\begin{array}{ccc} & & ES^1 \\ & & \downarrow \\ & & BS^1 \\ & \nearrow & \\ P_{BS^1}^{[2]} & \rightrightarrows & P_{BS^1} \\ & & \downarrow \\ & & M \end{array}$$

Given a principal BS^1-bundle on M it is possible to construct a classifying map $M \to BBS^1$. This implies that BBS^1 is a classifying space for bundle gerbes [38]. This gives a series of equivalent realisations

$$\text{bundle gerbes} \quad \longleftrightarrow \quad BS^1\text{-bundles} \quad \longleftrightarrow \quad BBS^1\text{-functions}$$

Note the similarity with the bundle gerbe hierarchy.

Now consider the case $p = 4$. Since there is an isomorphism between $H^4(M, \mathbb{Z})$ and the stable isomorphism class of bundle 2-gerbes on M then proposition 4.5 implies that there is an isomorphism between stable isomorphism classes of bundle 2-gerbe and isomorphism classes of BBS^1 bundles. The cases which we have already considered suggest that the bundle 2-gerbe corresponding to a BBS^1 bundle (P_{BBS^1}, M) should be the following bundle 2-gerbe associated to a BBS^1-bundle:

$$\begin{array}{ccc} & & ES^1 \\ & & \downarrow \\ & & BS^1 \\ & \nearrow & \\ EBS^{1[2]} & \rightrightarrows & EBS^1 \\ & & \downarrow \\ & & BBS^1 \\ & \nearrow & \\ P_{BBS^1}^{[2]} & \rightrightarrows & P_{BBS^1} \\ & & \downarrow \\ & & M \end{array}$$

Since the relevant local data may not be easily calculated then to prove this we would need to consider the theory of iterated classifying spaces and the constructions of Gajer [22] in much more detail. This leads us away from our principal concerns here and so we have not done this.

Other realisations are obtained by noting that BBS^1 bundles are classified by $BBBS^1$ functions, and that BS^1 bundle gerbes are equivalent to BBS^1 bundles , so we have the following series of realisations:

$$\text{bundle 2-gerbes} \quad \leftrightarrow \quad BS^1\text{-bundle gerbes} \quad \leftrightarrow \quad B^2S^1 \text{ bundles} \quad \leftrightarrow \quad B^3S^1 \text{ functions}$$

where it is to be understood that we are dealing the appropriate equivalence classes in each case, that is, stable isomorphism for bundle gerbes, isomorphism for bundles and homotopy equivalence for functions.

4.6 Comparing the Various Realisations

We now present a table comparing the various geometric realisations of Deligne cohomology which we have discussed. We include for completeness the differential characters of Cheeger and Simons ([14], [5]). Since the relationship between these and bundle gerbes is closely related to the theory of holonomy we postpone a definition and further discussion until Chapter 7.

Table 4.2: Geometric Realisations of Deligne Cohomology

$H^0(M, \mathbb{Z}(0)_D)$	$\mathbb{Z}$-functions				
$H^0(M, \mathbb{Z}(1)_D)$	constant $\mathbb{Z}$-functions				
$H^1(M, \mathbb{Z}(1)_D)$	$\mathbb{Z}$-bundle 0-gerbes	S^1-functions			deg 1 differential characters
$H^1(M, \mathbb{Z}(2)_D)$	Flat $\mathbb{Z}$-bundle 0-gerbes	constant S^1-functions			
$H^2(M, \mathbb{Z}(1)_D)$	$\mathbb{Z}$-bundle gerbes	bundle 0-gerbes	BS^1-functions	bundles	
$H^2(M, \mathbb{Z}(2)_D)$	$\mathbb{Z}$-bundle gerbes with curving	bundle 0-gerbes with connection		bundles with connection	deg 2 differential characters
$H^2(M, \mathbb{Z}(3)_D)$	flat $\mathbb{Z}$-bundle gerbes	flat bundle 0-gerbes		flat bundles	
$H^3(M, \mathbb{Z}(1)_D)$	$\mathbb{Z}$-bundle 2-gerbes	bundle gerbes	BS^1-bundles	gerbes	
$H^3(M, \mathbb{Z}(3)_D)$	$\mathbb{Z}$-bundle 2-gerbes with 2-curving	bundle gerbes with curving	BS^1-bundles with connection	gerbes with curving	deg 3 differential characters
$H^3(M, \mathbb{Z}(4)_D)$	flat $\mathbb{Z}$-bundle 2-gerbes	flat bundle gerbes	flat BS^1-bundles	flat gerbes	
$H^4(M, \mathbb{Z}(1)_D)$		bundle 2-gerbes	BBS^1-bundles	2-gerbes	
$H^4(M, \mathbb{Z}(4)_D)$		bundle 2-gerbes with 2-curving		2-gerbes with 2-curving	deg 4 differential characters
$H^4(M, \mathbb{Z}(5)_D)$		flat bundle 2-gerbes		flat 2-gerbes	

Chapter 5

Holonomy and Transgression

In this chapter we consider the generalisation of the holonomy of a bundle around a loop to bundle gerbes. We show how this relates to local formulae which define a transgression map in Deligne cohomology. The key to this generalisation is to consider holonomy as a property of a Deligne class. We have already defined the flat holonomy of a flat Deligne class. The holonomy of a general class in $H^p(M, \mathcal{D}^p)$ associated with a map $\psi : X \to M$ for some closed p-manifold X is defined to be the flat holonomy of the pullback of the class to X. We shall see how this approach relates to the usual construction of holonomy for bundles, and then go on to consider holonomy for bundle gerbes, bundle 2-gerbes and general Deligne classes.

5.1 Holonomy of $U(1)$-Bundles

We review the holonomy of principal $U(1)$-bundles with an emphasis on Deligne cohomology which is useful for generalisation to bundle gerbes.

Recall that flat bundles and bundle 0-gerbes have a flat holonomy which is a class in $H^1(M, U(1))$. It is useful to review the equations which define the Deligne cohomology class in general and in the particular cases of flat and trivial bundles.

The Deligne class $(\underline{g}, \underline{A})$ satisfies

$$d \log g_{\alpha\beta} = A_\beta - A_\alpha \tag{5.1}$$

If it is flat then we can find $U(1)$-valued functions satisfying $d \log a_\alpha = A_\alpha$ and we have

$$c_{\alpha\beta} = g_{\alpha\beta}^{-1} a_\alpha^{-1} a_\beta \tag{5.2}$$

The functions $c_{\alpha\beta}$ are constant and define the flat holonomy class. If the Deligne class has a trivialisation $\underline{h}$ then

$$\begin{aligned} g_{\alpha\beta} &= h_\alpha^{-1} h_\beta & (5.3)\\ d \log h_\alpha - A_\alpha &= d \log h_\beta - A_\beta & (5.4) \end{aligned}$$

Proposition 5.1. *[5] The assignment of a flat holonomy to a flat bundle gives an isomorphism $H^1(M, \mathcal{D}^p) = H^1(M, U(1))$ for $p > 1$.*

Proof. Suppose we have a flat bundle represented by a Deligne class $(g_{\alpha\beta}, A_\alpha)$ with flat holonomy $c_{\alpha\beta}$. We first show that the class in $H^1(M, U(1))$ is independent of the choices of a_α. Choose a'_α satisfying $d\log a'_\alpha = A_\alpha$. Then $a'_\alpha = a_\alpha + K_\alpha$ where K_α are $U(1)$-valued constants. Thus $c'_{\alpha\beta} = c_{\alpha\beta} + \delta(K)_{\alpha\beta}$ and so $\underline{c}'$ and $\underline{c}$ define the same class in $H^1(M, U(1))$.

Clearly the map $(\underline{g}, \underline{A}) \mapsto \underline{c}$ is a homomorphism. Let $(\delta(\underline{h}), d\log \underline{h})$ represent a trivial class in $H^1(M, \mathcal{D}^p)$. The corresponding flat holonomy is given by $\delta(\underline{h})\delta(\underline{a})^{-1} = \delta(\underline{h}.\underline{a}^{-1})$. Since $d\log \underline{h} = d\log \underline{a}$ this represents a trivial class in $H^1(M, U(1))$.

Given a class $\underline{c} \in H^1(M, U(1))$ define a class in $H^1(M, \mathcal{D}^p)$ by $(-\underline{c}, 0)$. Observe that if $\underline{c}$ is the flat holonomy of $(\underline{g}, \underline{A})$ then the two Deligne classes $(-\underline{c}, 0)$ and $(\underline{g}, \underline{A})$ differ by a trivial class $(\delta(\underline{a}), d\log \underline{a})$. Therefore the map $(\underline{g}, \underline{A}) \mapsto \underline{c}$ is onto. Also it is clear that a trivial class in $H^1(M, U(1))$ leads to a trivial class in $H^1(M, \mathcal{D}^p)$. Therefore we have an isomorphism. □

Mostly we shall be interested in the flat holonomy of a bundle over S^1. All bundles with connection over S^1 are flat so they all have a flat holonomy $\underline{c} \in H^1(S^1, U(1)) = S^1$. We shall demonstrate how to calculate this element of S^1 for a given bundle with connection. Let $(g_{\alpha\beta}, A_\alpha)$ represent a flat bundle on S^1 with flat holonomy $c_{\alpha\beta}$. Since $H^2(S^1, \mathbb{Z}) = 0$ there exists a trivialisation $\delta(\underline{h}) = \underline{g}$. Using this the flat holonomy becomes $c_{\alpha\beta} = h_\alpha \cdot h_\beta^{-1} \cdot a_\alpha a_\beta^{-1}$. By considering $\log(c_{\alpha\beta})$ as a representative of a class in the Čech cohomology of S^1 we can use the following diagram to calculate the isomorphism with de Rham cohomology:

$$\begin{array}{ccc} A_\alpha - d\log(h_\alpha) & & \\ \uparrow{\scriptstyle d} & & \\ \log(a_\alpha) - \log(h_\alpha) & \xrightarrow{\delta} & \log(c_{\alpha\beta}) \end{array}$$

Thus the 1-form $A_\alpha - d\log(h_\alpha)$ of equation (5.4) is the de Rham representative of the flat holonomy. It is globally defined on S^1 and is well defined modulo 2π-integral forms since the original Čech class was defined modulo $\mathbb{Z}(1)$. To evaluate it as an element, $H(c_{\alpha\beta})$, of S^1 we integrate,

$$H(c_{\alpha\beta}) = \exp \int_{S^1} A_\alpha - d\log(h_\alpha). \tag{5.5}$$

Since the flat holonomy class is isomorphic to the Deligne class we should be able to write (5.5) as a function of the Deligne class, $H(g_{\alpha\beta}, A_\alpha)$. To do this we would like to separate the two terms in the integral into separate integrals however they are not independently defined globally so this is not possible. We shall have to break up the integral into a sum of integrals on intervals where $d\log(h_\alpha)$ and A_α are defined, to do this we use the method used by Gawedski [23].

At the moment we have a Deligne class in terms of some open cover of S^1, denoted by subscripts α and β. Let t be a triangulation of S^1 consisting of edges, e, and vertices, v such that each edge is contained wholly within U_α for at least one α. Such a triangulation is said to be subordinate to the open cover and is guaranteed to exist since compactness implies the existence of a Lebesgue number [34, p179], we simply triangulate the circle such that all edges have length which is less than this number and therefore are contained within a set in the open cover. We can express S^1 as a sum $\sum e$ over all $e \in t$, so this should allow us to break up the integral of the global 1-form $A_\alpha - d\log(h_\alpha)$ into a sum over these edges, but we need to choose an open set that covers each edge first. For each e let $U_{\rho(e)}$ be an element of the open cover of S^1 such that $e \subset U_{\rho(e)}$. Here $\rho : t \to \mathcal{A}$ is an *index map* from the triangulation to the index set for the open cover of M [1]. We can now split up the terms in the integral,

$$
\begin{aligned}
H(c_{\alpha\beta}) &= \exp \sum_e [\int_e A_{\rho(e)} - \int_e d\log(h_{\rho(e)})] \\
&= \exp[\sum_e \int_e A_{\rho(e)} + \sum_{v,e} \log(h^{-1}_{\rho(e)})(v)]
\end{aligned}
$$

where we use the convention that $\sum\limits_{v,e}$ represents a sum over all edges and all vertices bounding each edge such that the sign is reversed for vertices which inherit the opposite orientation to the corresponding edge. This means that for each vertex there are two terms with opposite sign, one each for each of the edges bounded by that vertex. Observe that the following equality follows from (5.3),

$$
\sum_{e,v} \log(h^{-1}_{\rho(e)})(v) = \sum_{e,v} \log(g_{\rho(e)\rho(v)})(v) - \log(h_{\rho(v)})(v) \tag{5.6}
$$

Furthermore the second term on the right hand side is equal to zero since each vertex bounds exactly two edges which give two equal terms with opposite signs in the summation. The flat holonomy is now

$$
\begin{aligned}
H(c_{\alpha\beta}) &= \prod_e \exp \int_e A_{\rho(e)} \cdot \prod_{e,v} g_{\rho(v)\rho(e)}(v) \qquad (5.7) \\
&= H(g_{\alpha\beta}, A_\alpha)
\end{aligned}
$$

This construction is independent of the choice of triangulation. Suppose we choose another triangulation, $\hat{t}$. Since this triangulation must also be subordinate to the open cover we may assume without loss of generality that $\rho(\hat{e}) = \rho(e)$. Denote the flat holonomy corresponding to $\hat{t}$ by $\hat{H}$, and denote the two components $\exp \hat{H}_g$ and $\exp \hat{H}_A$. For this calculation is advantageous to expand the sum over the pair e, v as a sum over v in the following way,

$$
\begin{aligned}
\sum_{v,e} \log(g_{\rho(e)\rho(v)})(v) &= \sum_v \log(g_{\rho(e^+(v))\rho(v)})(v) - \log(g_{\rho(e^-(v))\rho(v)}) \qquad (5.8) \\
&= \sum_v \log(g_{\rho(e^+(v))\rho(e^-(v))})(v) \qquad (5.9)
\end{aligned}
$$

where $e^+(v)$ (resp. $e^-(v)$) is the edge bounded by v such that it inherits a positive (negative) orientation. Now the difference between the terms corresponding to the two triangulations is

$$
H_g - \hat{H}_g = \sum_v \log(g_{\rho(e^+(v))\rho(e^-(v))})(v) - \sum_{\hat{v}} \log(g_{\rho(e^+(\hat{v})\rho(e^-(\hat{v}))})(\hat{v})
$$

[1] Gawedski did not use index maps explicitly though they were implicit in his construction. They were used in this context by Brylinski [5] and the terminology appears to be due to Gomi and Terashima [25]

Since both triangulations are subordinate to the open cover we may consider pairs $(v, \hat{v})$ which are the unique vertices from each triangulation that lie within a particular double intersection of open sets. We can replace both summations in the expression above by a summation over such pairings. Furthermore given such a pairing we have $\rho(e^+(v)) = \rho(e^+(\hat{v}))$ and $\rho(e^-(v)) = \rho(e^-(\hat{v}))$. The difference now becomes

$$\begin{aligned} H_g - \hat{H}_g &= \sum_{(v,\hat{v})} \log(g_{\rho(e^+(v))\rho(e^-(v))})(v) - \log(g_{\rho(e^+(v))\rho(e^-(v))})(\hat{v}) \\ &= \sum_{(v,\hat{v})} \int_{v-\hat{v}} d\log g_{\rho(e^+(v))\rho(e^-(v))} \\ &= \sum_{(v,\hat{v})} \int_{v-\hat{v}} A_{\rho(e^-(v))} - A_{\rho(e^+(v))} \end{aligned}$$

Now consider the difference

$$H_A - \hat{H}_A = \sum_e \int_e A_{\rho(e)} - \sum_{\hat{e}} \int_{\hat{e}} A_{\rho(\hat{e})}$$

As with the vertices we can pair the edges $(e, \hat{e})$ such that $\rho(e) = \rho(\hat{e})$ and replace both sums with a sum over these pairings to get

$$\begin{aligned} H_A - \hat{H}_A &= \sum_{(e,\hat{e})} \int_e A_{\rho(e)} - \int_{\hat{e}} A_{\rho(e)} \\ &= \sum_{(e,\hat{e})} \int_{e-\hat{e}} A_{\rho(e)} \end{aligned}$$

Each difference $e - \hat{e}$ consists of two components (in terms of vertices), $e^+ - \hat{e}^+$ and $e^- - \hat{e}^-$. Using this to split up the integral into two terms we get

$$\begin{aligned} H_A - \hat{H}_A &= \sum_{(e,\hat{e})} \int_{e^+-\hat{e}^+} A_{\rho(e)} + \int_{e^--\hat{e}^-} A_{\rho(e)} \\ &= \sum_{(v,\hat{v})} \int_{v-\hat{v}} A_{\rho(e^+(v))} - A_{\rho(e^-(v))} \end{aligned}$$

where we have changed to a summation over vertices and used the fact that $v - \hat{v}$ is equal to one component each from $e^+(v) - \hat{e}^+(v)$ and $\hat{e}^-(v) - e^-(v)$. This term is the opposite of $H_g - \hat{H}_g$ therefore $H = \hat{H}$.

Since the 1-forms $A_{\rho(e)} - d\log h_{\rho(e)}$ are global then the integral defining the holonomy must be independent of the choice of index map ρ. This implies that (5.7) should also be independent of the choice of ρ. This may be easily verified. Suppose we have two such choices, ρ_0 and ρ_1. Then the difference is given by

$$
\begin{aligned}
\prod_e \exp \int_e A_{\rho_1(e)} - A_{\rho_0(e)} \cdot \prod_{v,e} g_{\rho_1(e)\rho_1(v)} g^{-1}_{\rho_0(e)\rho_1(e)} &= \prod_e \exp \int_e d \log g_{\rho_0(e)\rho_1(e)} \\
&\quad \cdot \prod_{v,e} g_{\rho_1(e)\rho_1(v)} g^{-1}_{\rho_0(e)\rho_1(e)} \\
&= \prod_{v,e} g_{\rho_0(e)\rho_1(e)} g_{\rho_1(e)\rho_1(v)} g^{-1}_{\rho_0(e)\rho_0(v)} \\
&= \prod_{v,e} g_{\rho_0(v)\rho_1(v)} \\
&= 1
\end{aligned}
\tag{5.10}
$$

since for each v there are two identical terms with opposite signs corresponding to the two edges which share v as a bounding vertex. Note that the global version is not explicitly independent of the choice of trivialisation, h, however we may deduce this from the explicit independence of the local version (5.7). It may also be calculated directly, this calculation is quite similar to the one described above.

We have defined an element of S^1 associated with every isomorphism class of flat bundle over S^1. It is given by equation (5.7) and is well defined. We would like now to show how this relates to the usual concept of the holonomy of a bundle with connection around a loop. Let $(L, M; A)$ be a bundle with connection (not necessarily flat) over M. Let γ be a loop in M, that is, γ is a smooth map $S^1 \to M$. Use γ to pull L back to S^1. Let $H(\gamma^{-1}(L; A))$ be the flat holonomy of the pull back bundle. We define this to be the holonomy of $(L; A)$ around γ.

We would like to give an explicit formula for the holonomy. To do these we need to examine the Deligne class of a pull back bundle. Once we have this we can apply equation (5.7).

Suppose we have a map $N \xrightarrow{\phi} M$ between compact manifolds. Let $\{U_\alpha\}_{\alpha \in A}$ be a good cover on M. The set A is finite since M is compact. There is a cover $\{V_{\phi(\alpha)} = \phi^{-1}(U_\alpha)\}_{\alpha \in A}$ on N called the induced cover. If we have a bundle with connection $(L; A)$ on M, then we can calculate the Deligne class of $(\phi^{-1}L, N; \phi^* A)$ in terms of the induced cover.

Lemma 5.1. *Let $(g_{\alpha\beta}, A_\alpha)$ be the Deligne class of $(L, M; A)$. Then the Deligne class of $(\phi^{-1}L, N; \phi^* A)$ with respect to the induced cover is $(g_{\phi(\alpha)\phi(\beta)}, A_{\phi(\alpha)})$ where*

$$
\begin{aligned}
g_{\phi(\alpha)\phi(\beta)}(n) &= g_{\alpha\beta}(\phi(n)) \qquad \textit{and} \\
A_{\phi(\alpha)} &= \phi^* A_\alpha
\end{aligned}
$$

Putting this together with (5.7) we get

Proposition 5.2. *[5][23] The holonomy of a bundle with Deligne class $(\underline{g}, \underline{A})$ around a loop γ is given by*

$$
H((\underline{g}, \underline{A}); \gamma) = \prod_e \exp \int_e \gamma^* A_{\rho(e)} \cdot \prod_{v,e} g_{\rho(e)\rho(v)}(\gamma(v)) \tag{5.11}
$$

Now recall the usual definition of the holonomy of a bundle

Definition 5.1. Let (L, M) be a bundle with connection A. Any path in M has a unique lift through each element of the fibre over the starting point which is horizontal with respect to A. In particular each loop γ has a unique horizontal lift $\tilde{\gamma}$ which defines an automorphism of the fibre over $\gamma(0)$. The *holonomy of the connection A around γ* is the element of S^1 defined by $\tilde{\gamma}(1) = \tilde{\gamma}(0) \cdot H(\gamma)$.

Proposition 5.3. *[23] The holonomy of Proposition 5.2 is the same as the holonomy of definition 5.1.*

We shall relate these two concepts of holonomy by considering *parallel transport.* Given a path $\mu \in \mathrm{Map}(I, M)$ the horizontal lift $\tilde{\mu}$ defines a morphism of fibres $P_{\mu(0)} \to P_{\mu(1)}$. This is called parallel transport. Two paths μ and μ' such that $\mu(1) = \mu'(0)$ may be composed and the horizontal lift of the composition defines a composition of parallel transports. If we consider a loop as a composition of a number of paths then the holonomy is defined by the composition of the parallel transports along each path. By breaking up the loop into components $\gamma([t_i, t_{i+1}])$ over which P admits sections s_i then there is an explicit formula for parallel transport over each component:

$$s_i(\gamma(t_i)) \mapsto s_i(\gamma(t_{i+1})) \exp(\int_{t_i}^{t_{i+1}} s_i^* A) \tag{5.12}$$

Composition then gives a product of terms which combine to give the local formula (5.11). This suggests that we have used a rather long and complicated method for calculating the holonomy of a bundle, however it turns out that our method is useful as it generalises to bundle gerbes and to higher degrees. In addition to this it allowed us to demonstrate certain features of the higher theory in a relatively simple setting.

5.2 Holonomy of Bundle Gerbes

To define the holonomy of a bundle gerbe we follow the procedure used in the previous section. The standard technique for deriving a formula for holonomy of a bundle (as described at the end of the previous section) cannot be used here for two main reasons. One is that it turns out that a bundle gerbe has a holonomy over a surface rather than a loop, so we cannot just choose a direction to integrate around as is the case with a loop. Secondly it is not clear what a horizontal lift or parallel transport map would be in this situation. This motivates us to define the holonomy of a bundle gerbe by first considering the holonomy of a Deligne class corresponding to a bundle gerbe.

The Deligne class $(\underline{g}, \underline{A}, \underline{\eta})$ of a bundle gerbe satisfies

$$\begin{aligned} d \log g_{\alpha\beta\gamma} &= -A_{\beta\gamma} + A_{\alpha\gamma} - A_{\alpha\beta} && (5.13) \\ dA_{\alpha\beta} &= \eta_\beta - \eta_\alpha && (5.14) \end{aligned}$$

If the bundle gerbe is flat then

$$\begin{aligned} \eta_\alpha &= dB_\alpha && (5.15) \\ A_{\alpha\beta} &= B_\beta - B_\alpha + d \log a_{\alpha\beta} && (5.16) \\ c_{\alpha\beta\gamma} &= g_{\alpha\beta\gamma}^{-1} a_{\beta\gamma}^{-1} a_{\alpha\gamma} a_{\alpha\beta}^{-1} && (5.17) \end{aligned}$$

and $c_{\alpha\beta\gamma}$ is the flat holonomy class. If the bundle gerbe has trivialisation $\underline{h}$ then

$$\begin{aligned} g_{\alpha\beta\gamma} &= h_{\beta\gamma}h_{\alpha\gamma}^{-1}h_{\alpha\beta} && (5.18)\\ d\log h_{\alpha\beta} &= -A_{\alpha\beta} + k_\beta - k_\alpha && (5.19)\\ \eta_\alpha - dk_\alpha &= \eta_\beta - dk_\beta && (5.20)\end{aligned}$$

Now consider the particular case of a bundle gerbe over Σ, a 2-manifold without boundary. In this case the bundle gerbe is not only flat, but also trivial. The Čech-de Rham isomorphism is given by the following diagram:

$$\begin{tikzcd} \eta_\alpha - dk_\alpha & & \\ B_\alpha - k_\alpha \arrow[u, "d"] \arrow[r, "\delta"] & 0 & \\ & -\log a_{\alpha\beta} - \log h_{\alpha\beta} \arrow[u, "-d"] \arrow[r, "\delta"] & \log c_{\alpha\beta\gamma} \end{tikzcd}$$

Thus the globally defined 2-form $\eta - dk$ is the de Rham representative of the flat holonomy of the bundle gerbe. Since $H^2(\Sigma, U(1)) = U(1)$ we may evaluate this class as an element of the circle by integrating over the surface Σ and taking the exponential.

Thus in terms of bundle gerbes holonomy is defined in the following way.

Definition 5.2. [12] Let $(P, Y, M; A, \eta)$ be a bundle gerbe with connection and curving and let $\psi : \Sigma \to M$ be a map of a surface into M. The *holonomy of* $(P, Y, M; A, \eta)$ *over* Σ is the flat holonomy of ψ^*P.

To see that this is well defined consider that when we pull back the bundle gerbe P to Σ using ψ the resulting bundle gerbe has an induced curving which we denote $\psi^*\eta$ and for dimensional reasons has a trivialisation L. Denote the curvature of this trivialisation (given some connection which is compatible with the bundle gerbe connection) by F_L. The 2-form $\psi^*\eta - F_L$ descends to Σ and its integral over Σ defines the flat holonomy which is an element of $H^2(\Sigma, U(1)) = U(1)$. This is independent of the choice of trivialisation since a different choice just changes F_L by a closed 2-form which descends to Σ. We shall also see this when we calculate a formula for the holonomy which is explicitly independent of this choice.

Proposition 5.4. *The holonomy of a bundle gerbe with Deligne class $(\underline{g}, \underline{A}, \underline{\eta})$ on M over a surface $\psi : \Sigma \to M$ is given by the following formula of Gawedski [23]:*

$$H((\underline{g}, \underline{A}, \underline{\eta}); \psi) = \prod_b \exp \int_b \psi^*\eta_{\rho(b)} \cdot \prod_{e,b} \exp \int_e \psi^* A_{\rho(b)\rho(e)} \cdot \prod_{v,e,b} g_{\rho(b)\rho(e)\rho(v)}(\psi(v)) \tag{5.21}$$

Proof. To evaluate the holonomy in terms of the original Deligne class we shall need to triangulate Σ. This triangulation, t, will consist of vertices, v, edges, e, and faces b and is required to be subordinate to the open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$. Thus there exists an index map $\rho : t \to \mathcal{A}$ such that $b \subset U_{\rho(b)}$, $e \subset U_{\rho(e)}$ and $v \subset U_{\rho(v)}$ for all $b, e, v \in t$. The integral over Σ can be broken up into a sum of integrals over b,

$$\begin{aligned} H(c_{\alpha\beta\gamma}) &= \exp \sum_b \int_b (\eta - dk) \\ &= \exp \sum_b \left(\int_b \eta_{\rho(b)} + \int_b -dk_{\rho(b)} \right) \end{aligned}$$

Applying Stokes' theorem to the second term gives

$$H(c_{\alpha\beta\gamma}) = \exp\left(\sum_b \int_b \eta_{\rho(b)} + \sum_b \int_{\partial b} -k_{\rho(b)}\right)$$

In the second term we have a sum $\sum_b \int_{\partial b}$. If we break ∂b into a sum of edges we can write this as $\sum\limits_{e,b} \int_e$ where the convention is that the sum is over all faces and all edges bounding each face, and the integral is given the corresponding induced orientation.

$$\begin{aligned}
H(c_{\alpha\beta\gamma}) &= \exp\left(\sum_b \int_b \eta_{\rho(b)} + \sum_{e,b} \int_e -k_{\rho(b)}\right) \\
&= \exp\left(\sum_b \int_b \eta_{\rho(b)} + \sum_{e,b} \int_e \left(A_{\rho(b)\rho(e)} + d\log(h_{\rho(b)\rho(e)}) - k_{\rho(e)}\right)\right) \\
&= \exp\left(\sum_b \int_b \eta_{\rho(b)} + \sum_{e,b} \int_e (A_{\rho(b)\rho(e)} + \int_{\partial e} \log(h_{\rho(b)\rho(e)})\right) \\
&= \exp\left(\sum_b \int_b \eta_{\rho(b)} + \sum_{e,b} \int_e A_{\rho(b)\rho(e)} + \sum_{v,e,b} \log(h_{\rho(b)\rho(e)}(v))\right) \\
&= \exp(\sum_b \int_b \eta_{\rho(b)} + \sum_{e,b} \int_e A_{\rho(b)\rho(e)} + \sum_{v,e,b} \log(g_{\rho(b)\rho(e)\rho(v)}(v)) \\
&\qquad - \log(h_{\rho(e)\rho(v)}(v)) + \log(h_{\rho(b)\rho(v)}(v))) \\
&= \exp\left(\sum_b \int_b \eta_{\rho(b)} + \sum_{e,b} \int_e A_{\rho(b)\rho(e)} + \sum_{v,e,b} \log(g_{\rho(b)\rho(e)\rho(v)}(v))\right)
\end{aligned}$$

We have claimed in this calculation that certain terms cancel out. Let $I(e)$ denote a term depending only on e, $I(e,b)$ a term depending only on e and b and so on. Then we have used the following results:

$$\sum_{e,b} I(e) = 0 \tag{5.22}$$

$$\sum_{v,e,b} I(v,e) = 0 \tag{5.23}$$

$$\sum_{v,e,b} I(v,b) = 0 \tag{5.24}$$

The first two are true because for each edge there are exactly two faces with that edge as boundary and they have opposite induced orientations. The third is true since given a face and a vertex of that face there are exactly two edges which bound the face and have the vertex as a boundary component. Furthermore the vertex inherits opposite orientations from each of these edges. Note that the first two results would no longer hold if we triangulate a surface with boundary. We shall deal with this situation in the next chapter.

We now have a formula for the holonomy of a flat bundle gerbe in terms of its

Deligne class,

$$H(\underline{g},\underline{A},\underline{\eta}) = \prod_b \exp\int_b \eta_{\rho(b)} \cdot \prod_{e,b} \exp\int_e A_{\rho(b)\rho(e)} \cdot \prod_{v,e,b} g_{\rho(b)\rho(e)\rho(v)}(v) \tag{5.25}$$

As in the previous section this formula may be adapted to define the holonomy of a general bundle gerbe with curving, $(P, Y, M; A, \eta)$, associated with a smooth map of surface into M, $\psi : \Sigma \to M$.

This leads us to the required formula

$$H((\underline{g},\underline{A},\underline{\eta});\psi) = \prod_b \exp\int_b \psi^*\eta_{\rho(b)} \cdot \prod_{e,b} \exp\int_e \psi^* A_{\rho(b)\rho(e)} \cdot \prod_{v,e,b} g_{\rho(b)\rho(e)\rho(v)}(\psi(v))$$

$\square$

5.3 Holonomy of Bundle 2-Gerbes

For the case of a bundle 2-gerbe we must first establish the notation associated with the flat holonomy and with trivialisations.

The Deligne class $(\underline{g},\underline{A},\underline{\eta},\underline{\nu})$ of a bundle 2-gerbe satisfies the following equations:

$$d\log g_{\alpha\beta\gamma\delta} = A_{\beta\gamma\delta} - A_{\alpha\gamma\delta} + A_{\alpha\beta\delta} - A_{\alpha\beta\gamma} \tag{5.26}$$
$$dA_{\alpha\beta\gamma} = -\eta_{\beta\gamma} + \eta_{\alpha\gamma} - \eta_{\alpha\beta} \tag{5.27}$$
$$d\eta_{\alpha\beta} = \nu_\beta - \nu_\alpha \tag{5.28}$$

If we assume that the bundle 2-gerbe is flat then we have the following set of equations

$$\nu_\alpha = dq_\alpha \tag{5.29}$$
$$\eta_{\alpha\beta} = q_\beta - q_\alpha + dB_{\alpha\beta} \tag{5.30}$$
$$A_{\alpha\beta\gamma} = -B_{\beta\gamma} + B_{\alpha\gamma} - B_{\alpha\beta} + d\log a_{\alpha\beta\gamma} \tag{5.31}$$
$$c_{\alpha\beta\gamma\delta} = g^{-1}_{\alpha\beta\gamma\delta} a_{\beta\gamma\delta} a^{-1}_{\alpha\gamma\delta} a_{\alpha\beta\gamma} a^{-1}_{\alpha\beta\gamma} \tag{5.32}$$

The constants $c_{\alpha\beta\gamma\delta}$ define the flat holonomy class.

If we have a bundle 2-gerbe with trivialisation $\underline{h}$ then we have the following:

$$g_{\alpha\beta\gamma\delta} = h_{\beta\gamma\delta} h^{-1}_{\alpha\gamma\delta} h_{\alpha\beta\gamma} h^{-1}_{\alpha\beta\gamma} \tag{5.33}$$
$$d\log h_{\alpha\beta\gamma} = A_{\alpha\beta\gamma} - k_{\beta\gamma} + k_{\alpha\gamma} - k_{\alpha\beta} \tag{5.34}$$
$$\eta_{\alpha\beta} = -dk_{\alpha\beta} + j_\beta - j_\alpha \tag{5.35}$$
$$\nu_\alpha - dj_\alpha = \nu_\beta - dj_\beta \tag{5.36}$$

If we have a bundle 2-gerbe over a 3-manifold without boundary, X, then it is both flat and trivial. In this case we have a Čech - de Rham isomorphism as described by the following diagram:

$$\begin{array}{cccccc}
\nu_\alpha - dj_\alpha & & & & & \\
\uparrow d & & & & & \\
q_\alpha - j_\alpha & \xrightarrow{\delta} & 0 & & & \\
& & \uparrow -d & & & \\
& & -k_{\alpha\beta} - B_{\alpha\beta} & \xrightarrow{\delta} & 0 & \\
& & & & \uparrow d & \\
& & & & \log a_{\alpha\beta\gamma} - \log h_{\alpha\beta\gamma} & \xrightarrow{\delta} \log c_{\alpha\beta\gamma\delta}
\end{array}$$

This tells us that the flat holonomy may be realised as an element of S^1 by the following formula

$$H(c_{\alpha\beta\gamma\delta}) = \exp \int_X \nu_\alpha - dj_\alpha \tag{5.37}$$

This suggests the following

Definition 5.3. Let $(P, M; A, \eta, \nu)$ be a bundle 2-gerbe with connection and curvings. The *holonomy of* $(P, M; A, \eta, \nu)$ *over a closed 3-manifold* X with $\psi : X \to M$, is the flat holonomy of $\psi^* P$.

Over X the bundle 2-gerbe $\psi^* P$ is trivial. We choose a trivialisation with connection and curving. The 3-form defined by the difference between the 3-curving induced by the pullback and the 3-curvature of the trivialisation may be integrated over X to define the holonomy.

Once again to find a corresponding formula in terms of the Deligne class we shall need a triangulation, t, of X which is subordinate to the open cover used to define the Deligne class. This triangulation consists of tetrahedrons, faces, edges and vertices which are denoted by w, b, e and v respectively. As usual we choose an index map ρ with respect to the triangulation t and the open cover of M.

Replacing the integral over X with a sum of integrals over w,

$$\begin{aligned}
H(c_{\alpha\beta\gamma\delta}) &= exp \sum_w \int_w (\nu_{\rho(w)} - dj_{\rho(w)}) \\
&= \exp \sum_w \int_w \nu_{\rho(w)} + \int_{\partial w} -j_{\rho(w)} \quad ,
\end{aligned}$$

$$\begin{aligned}
\exp \sum_w \int_{\partial w} -j_{\rho(w)} &= \exp \sum_{b,w} \int_b -j_{\rho(w)} \\
&= \exp \sum_{b,w} \int_b \eta_{\rho(w)\rho(b)} + dk_{\rho(w)\rho(b)} - j_\rho(b) \\
&= \exp \sum_{b,w} \int_b \eta_{\rho(w)\rho(b)} + \int_{\partial b} k_{\rho(w)\rho(b)}
\end{aligned}$$

where $\sum_{b,w} \int_b -j_{\rho(b)} = 0$ since each face bounds exactly two tetrahedrons with opposite orientations.

$$\begin{aligned}
\exp \sum_{b,w} \int_{\partial b} k_{\rho(w)\rho(b)} &= \exp \sum_{e,b,w} \int_e k_{\rho(w)\rho(b)} \\
&= \exp \sum_{e,b,w} \int_e A_{\rho(w)\rho(b)\rho(e)} - d \log h_{\rho(w)\rho(b)\rho(e)} + k_{\rho(w)\rho(e)} - k_{\rho(b)\rho(e)} \\
&= \exp \sum_{e,b,w} \int_e A_{\rho(w)\rho(b)\rho(e)} - \int_{\partial e} \log h_{\rho(w)\rho(b)\rho(e)}
\end{aligned}$$

where $\sum_{e,b,w} \int_e k_{\rho(w)\rho(e)} - k_{\rho(b)\rho(e)} = 0$ since each edge of a particular tetrahedron bounds exactly two faces of that tetrahedron and each edge of a particular face is an edge of exactly two tetrahedrons and in both cases the corresponding orientations are opposite. Finally,

$$\begin{aligned}
\exp \sum_{e,b,w} \int_{\partial e} - \log h_{\rho(w)\rho(b)\rho(e)} &= \exp \sum_{v,e,b,w} - \log h_{\rho(w)\rho(b)\rho(e)}(v) \\
&= \exp \sum_{v,e,b,w} \log g_{\rho(w)\rho(b)\rho(e)\rho(v)}(v) - \log h_{\rho(w)\rho(b)\rho(v)}(v) \\
&\qquad + \log h_{\rho(w)\rho(e)\rho(v)}(v) - \log h_{\rho(b)\rho(e)\rho(v)}(v) \\
&= \exp \sum_{v,e,b,w} \log g_{\rho(w)\rho(b)\rho(e)\rho(v)}(v)
\end{aligned}$$

where once again we get cancellation of terms due to opposite contributions as we sum over missing indices.

Collecting these results we have

$$\begin{aligned}
H(\underline{g}, \underline{A}, \underline{\eta}, \underline{\nu}) = \prod_w \exp \int_w \nu_{\rho(w)} \cdot \prod_{b,w} \exp \int_b \eta_{\rho(w)\rho(b)} \cdot \prod_{e,b,w} \exp \int_e A_{\rho(w)\rho(b)\rho(e)} \\
\cdot \prod_{v,e,b,w} g_{\rho(w)\rho(b)\rho(e)\rho(v)}(v)
\end{aligned} \tag{5.38}$$

and the corresponding formula for an embedding of a closed 3-manifold $\psi : X \to M$ is

$$\begin{aligned}
H((\underline{g}, \underline{A}, \underline{\eta}, \underline{\nu}); \psi) = \prod_w \exp \int_w \psi^* \nu_{\rho(w)} \cdot \prod_{b,w} \exp \int_b \psi^* \eta_{\rho(w)\rho(b)} \cdot \prod_{e,b,w} \exp \int_e \psi^* A_{\rho(w)\rho(b)\rho(e)} \\
\cdot \prod_{v,e,b,w} g_{\rho(w)\rho(b)\rho(e)\rho(v)}(\psi(v))
\end{aligned} \tag{5.39}$$

5.4 A General Holonomy Formula

Using the results from the previous sections we can find a formula for the holonomy of a class in $H^p(M, \mathcal{D}^p)$ associated with an embedding of a closed p-manifold X. Since we do not necessarily have a geometric realisation of this Deligne class in general, here holonomy is not meant in the traditional sense. It is defined purely in terms of the Deligne class, specifically it is the flat holonomy class of the pullback Deligne class on X, evaluated over X as an element of S^1. This formula gives a particular example of the even more general transgression formula given by Gomi and Terashima ([26], [25]). The key feature of our derivation is that it clearly generalises the geometric notion of holonomy as we have defined it in the low degree cases.

Definition 5.4. Denote a Deligne class on X by $(\underline{g}, \underline{A}^1, \ldots, \underline{A}^p)$. Think of this class as the pull back of a class on M. It is flat and trivial so there exists a cochain $(\underline{h}, \underline{B}^1, \ldots, \underline{B}^{p-1})$ such that

$$\begin{aligned} \underline{g} &= \delta(\underline{h}) \\ \underline{A}^q &= \delta(\underline{B}^q) + (-1)^{p-q} d\underline{B}^{q-1} \\ \delta(\underline{A}^p - d\underline{B}^{p-1}) &= 0 \end{aligned} \tag{5.40}$$

The *holonomy of the Deligne class* is defined by

$$\exp \int_X \underline{A}^p - d\underline{B}^{p-1} \tag{5.41}$$

This expression is not satisfactory since it depends explicitly on $\underline{B}$. To deal with this we triangulate X with $t : |K| \to M$, where K is a p-dimensional simplicial complex, and let ρ be an index map for this triangulation. In terms of the triangulation the holonomy is

$$\exp \left[\sum_{\sigma^p} \int_{\sigma^p} A^p_{\rho(\sigma^p)} + \sum_{\sigma^p} \int_{\sigma^p} -dB^{p-1}_{\rho(\sigma^p)} \right] \tag{5.42}$$

Consider the second term:

$$\sum_{\sigma^p} \int_{\sigma^p} -dB^{p-1}_{\rho(\sigma^p)} = \sum_{\sigma^p} \int_{\partial\sigma^p} -B^{p-1}_{\rho(\sigma^p)} \tag{5.43}$$

In this expression we may express the combination of the sum and the integral in terms of flags of simplices:

$$\begin{aligned} \sum_{\sigma^p} \int_{\partial\sigma^p} &= \sum_{\sigma^p} \sum_{\sigma^{p-1} \subset \sigma^p} \int_{\sigma^{p-1}} \\ &\equiv \sum_{\underline{\sigma}^{p-1}} \int_{\sigma^{p-1}} \end{aligned} \tag{5.44}$$

where we have defined a new notation $\underline{\sigma}$. In general this denotes a flag of simplices,

$$\underline{\sigma}^q = \{(\sigma^q, \sigma^{q+1}, \ldots, \sigma^p) | \sigma^q \subset \cdots \subset \sigma^p\} \tag{5.45}$$

All subsimplices inherit relative orientations. A similar notation was used in [25] to generalise transgression formulae.

Returning to the holonomy formula, we now have

$$\sum_{\underline{\sigma}^{p-1}} \int_{\sigma^{p-1}} -B^{p-1}_{\rho(\sigma^p)} \tag{5.46}$$

Now use equation (5.40),

$$\delta(\underline{B}^q) = \underline{A}^q - (-1)^{p-q} d\underline{B}^{q-1} \tag{5.47}$$

to get

$$-B^{p-1}_{\rho(\sigma^p)} = -B^{p-1}_{\rho(\sigma^{p-1})} + A^{p-1}_{\rho(\sigma^p)\rho(\sigma^{p-1})} - dB^{p-2}_{\rho(\sigma^p)\rho(\sigma^{p-1})} \tag{5.48}$$

Using the fact that each $(p-1)$-face in the simplicial complex bounds exactly two p-faces we have

$$\sum_{\underline{\sigma}^{p-1}} \int_{\sigma^{p-1}} -B^{p-1}_{\rho(\sigma^{p-1})} = 0 \tag{5.49}$$

since the two terms inherit opposite orientations from σ^p. Thus

$$\sum_{\underline{\sigma}^{p-1}} \int_{\sigma^{p-1}} -B^{p-1}_{\rho(\sigma^p)} = \sum_{\underline{\sigma}^{p-1}} \int_{\sigma^{p-1}} A^{p-1}_{\rho(\sigma^p)\rho(\sigma^{p-1})} + dB^{p-2}_{\rho(\sigma^p)\rho(\sigma^{p-1})} \tag{5.50}$$

The next step would be to extract the A^{p-1} term for the final answer and proceed as above to deal with the dB^{p-2} term. This suggests an inductive approach with respect to $k = p - q$.

Lemma 5.2. *For every q such that $1 \leq q \leq p$*

$$\sum_{\underline{\sigma}^q} \int_{\sigma^q} dB^{q-1}_{\rho(\sigma^p)\ldots\rho(\sigma^q)} = \sum_{\underline{\sigma}^{q-1}} \int_{\sigma^{q-1}} (-1)^{p-q+1} A^{q-1}_{\rho(\sigma^p)\ldots\rho(\sigma^{q-1})} - dB^{q-2}_{\rho(\sigma^p)\ldots\rho(\sigma^{q-1})} \tag{5.51}$$

where we use the conventions $\underline{A}^0 = \log \underline{g}$, $\underline{B}^0 = \log \underline{h}$ and $\underline{B}^{-1} = 0$.

Proof. We have already proved the particular case $p = q$. More generally

$$\begin{aligned}
\sum_{\underline{\sigma}^q} \int_{\sigma^q} dB^{q-1}_{\rho(\sigma^p)\ldots\rho(\sigma^q)} &= \sum_{\underline{\sigma}^q} \int_{\partial\sigma^q} B^{q-1}_{\rho(\sigma^p)\ldots\rho(\sigma^q)} \\
&= \sum_{\underline{\sigma}^q} \sum_{\sigma^{q-1}\subset\sigma^q} \int_{\sigma^{q-1}} B^{q-1}_{\rho(\sigma^p)\ldots\rho(\sigma^q)} \\
&= \sum_{\underline{\sigma}^{q-1}} \int_{\sigma^{q-1}} B^{q-1}_{\rho(\sigma^p)\ldots\rho(\sigma^q)}
\end{aligned} \tag{5.52}$$

Next we claim that

$$\sum_{\underline{\sigma}^{q-1}} \int_{\sigma^{q-1}} B^{q-1}_{\rho(\sigma^p)\ldots\rho(\sigma^q)} = \sum_{\underline{\sigma}^{q-1}} \int_{\sigma^{q-1}} (-1)^{p-q+1} (\delta B^{q-1})_{\rho(\sigma^p)\ldots\rho(\sigma^{q-1})} \tag{5.53}$$

The right hand side consists of all terms of the form

$$\sum_{\underline{\sigma}^{q-1}} \int_{\sigma^{q-1}} (-1)^{p-q+1} B^{q-1}_{\rho(\sigma^p)\ldots\widehat{\rho(\sigma^k)}\ldots\rho(\sigma^q)} \tag{5.54}$$

for all $q-1 \leq k \leq p$ and where the hat symbol denotes that a subscript should be omitted. The case $k = q-1$ corresponds to the left hand side of (5.53).
Now consider $q-1 < k < p$. Suppose in the summation we have a flag $(\sigma^{q-1}, \ldots, \sigma^p)$, with the summand depending on all simplices in the flag except for σ^k. This leads to a number of identical terms corresponding to all flags which agree in all degrees except for k. There can only be two such flags. This is because such flags must satisfy

$$\sigma^k \subset \sigma^{k+1} \tag{5.55}$$

$$\sigma^{k-1} \subset \sigma^k \tag{5.56}$$

This means that σ^k is defined by $k+1$ of the $k+2$ vertices of σ^{k+1} and σ^{k-1} is defined by k of these. Since σ^{k+1} and σ^{k-1} are fixed then there are only two choices for σ^k as there are two vertices in σ^{k+1} which are not in σ^{k-1}. Furthermore the two possible choice of flags will lead to opposite induced orientations of σ^{q-1}. The induced orientations are derived from the orientation of σ^p. The orientations of all the simplices from σ^p to σ^{k+1} must be the same since they are all identical. The two choices for σ^k must give opposite orientations for σ^{k-1}. This condition is equivalent to the basic result $\partial^2 = 0$ for the boundary operator in the theory of simplicial complexes. From σ^{k-1} down to σ^{q-1} all of the simplices are equal so there can be no further change in the relative orientations of the two choices.

Finally we consider the case $k = p$. In this case we once again have only two choices of flag corresponding to the two choices of orientation and these contribute terms of opposite sign. This proves the claim.

The lemma now follows from equation (5.40).

□

This lemma leads to the following

Proposition 5.5. *For all $p \geq 1$ the holonomy of the Deligne class $(\underline{g}, \underline{A}^1, \ldots, \underline{A}^p)$ is given by the following formula:*

$$\exp \int_X \underline{A}^p - d\underline{B}^{p-1} = \exp \sum_{n=0}^{p} \sum_{\underline{\sigma}^{p-n}} \int_{\sigma^{p-n}} A^{p-n}_{\rho(\sigma^p)\ldots\rho(\sigma^{p-n})} \tag{5.57}$$

As before we let $\underline{A}^0 = \log \underline{g}$.

Proof. It is easily verified that the formulae obtained in the previous sections of this chapter prove the result for $p = 1$, 2 and 3. To prove the more general case we use the following intermediate result:

$$\begin{aligned} \exp \int_X \underline{A}^p - d\underline{B}^{p-1} = \exp(&\sum_{n=0}^{k} \sum_{\underline{\sigma}^{p-n}} \int_{\sigma^{p-n}} A^{p-n}_{\rho(\sigma^p)\ldots\rho(\sigma^{p-n})}) \\ &\cdot \exp \sum_{\underline{\sigma}^{p-k}} \int_{\sigma^{p-k}} (-1)^{k+1} dB^{p-k-1}_{\rho(\sigma^p)\ldots\rho(\sigma^{p-k})} \end{aligned} \tag{5.58}$$

For $k = 0$ this is simply rewriting the integral over X in terms of the triangulation. We prove the general case, $0 < k \leq p$ by induction. Suppose (5.58) is true for some

$k < p$. Applying Lemma 5.2 to the dB term gives

$$\begin{aligned}
\exp \sum_{\underline{\sigma}^{p-k}} \int_{\sigma^{p-k}} (-1)^{k+1} dB^{p-k-1}_{\rho(\sigma^p)\dots\rho(\sigma^{p-k})} &= \exp \sum_{\underline{\sigma}^{p-k-1}} \int_{\sigma^{p-k-1}} (-1)^{k+1}(-1)^{k+1} A^{p-k-1}_{\rho(\sigma^p)\dots\rho(\sigma^{p-k-1})} \\
&\quad - (-1)^{k+1} dB^{p-k-2}_{\rho(\sigma^p)\dots\rho(\sigma^{p-k-1})} \\
&= \exp \sum_{\underline{\sigma}^{p-(k+1)}} \int_{\sigma^{p-(k+1)}} A^{p-(k+1)}_{\rho(\sigma^p)\dots\rho(\sigma^{p-(k+1)})} \\
&\quad + (-1)^{(k+1)+1} dB^{p-(k+1)-1}_{\rho(\sigma^p)\dots\rho(\sigma^{p-(k+1)})}
\end{aligned} \tag{5.59}$$

Substituting (5.59) back into (5.58) gives

$$\begin{aligned}
\exp \int_X \underline{A}^p - d\underline{B}^{p-1} &= \exp(\sum_{n=0}^{k+1} \sum_{\underline{\sigma}^{p-n}} \int_{\sigma^{p-n}} A^{p-n}_{\rho(\sigma^p)\dots\rho(\sigma^{p-n})}) \\
&\quad \cdot \exp \sum_{\underline{\sigma}^{p-(k+1)}} \int_{\sigma^{p-(k+1)}} (-1)^{(k+1)+1} dB^{p-(k+1)-1}_{\rho(\sigma^p)\dots\rho(\sigma^{p-(k+1)})}
\end{aligned} \tag{5.60}$$

thus the statement is true for $k+1$ and therefore by induction is true for all $1 \leq k \leq p$. In particular the case $k = p$ is equivalent to the statement of the proposition since $\underline{B}^{-1} = 0$, thus this is sufficient to prove the proposition.

□

5.5 Transgression for Closed Manifolds

Consider the constructions of the previous sections of this chapter. In each case we start with a bundle $(n-1)$-gerbe with curving ($n = 1$, 2 or 3). Then we construct an element of S^1 corresponding to a smooth mapping of a closed manifold of dimension n. Furthermore for $n > 3$ we can carry out this construction purely in terms of the Deligne class. We would like to consider the holonomy as a smooth function on the infinite dimensional manifold $\mathrm{Map}(X, M)$. We give this mapping space the compact-open smooth topology [27, p34]. Since the holonomy is defined in terms of sums, integrals and pull backs it will define a smooth, continuous function on $\mathrm{Map}(X, M)$. To see that it defines a class in Deligne cohomology consider the following open cover of the mapping space:

Definition 5.5. Let $\mathcal{U} \equiv \{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of M. Let t be a triangulation of X consisting of simplices σ and suppose we have an index map $\rho : t \to \mathcal{A}$. Then the set $V_{(t,\rho)}$ is defined by

$$V_{(t,\rho)} = \{\phi \in \mathrm{Map}(X, M) | \quad \phi(\sigma) \subset U_{\rho(\sigma)}\} \tag{5.61}$$

Denote open cover defined by these sets by $\mathcal{V}$

These sets are open in the compact-open smooth topology since they are made up of smooth maps of simplices (which are compact) into open sets in M. Following [23] we use $\mathcal{V}$ as our open cover of $\mathrm{Map}(X, M)$. We have already used this cover to calculate the holonomy, so we may think of the holonomy as a collection of S^1 functions defined on open sets in $\mathcal{V}$, that is, a cochain in $C^0(\mathrm{Map}(X,M),\underline{U(1)})$. The fact that our construction was independent of the choice of the pair (t,ρ) implies that this cochain is actually a cocycle in $H^0(\mathrm{Map}(X,M),\underline{U(1)})$. Following [23] and [5] we define the *transgression* homomorphism $\tau_X : H^n(M,\mathcal{D}^n) \to H^0(\mathrm{Map}(X,M),\underline{U(1)})$.

This homomorphism has been interpreted ([5],[25]) as a composition of an evaluation map

$$ev^* : H^n(M,\mathcal{D}^n) \to H^n(\mathrm{Map}(X,M)\times X,\mathcal{D}^n)$$

and a fibre integration map

$$\int_X : H^n(\mathrm{Map}(X,M)\times X,\mathcal{D}^n) \to H^0(\mathrm{Map}(X,M),\underline{U(1)}).$$

This homomorphism is compatible with the corresponding map on curvatures, that is, if the curvature of the Deligne class on M is ω then the curvature of the transgressed class on $\mathrm{Map}(X,M)$ is $\int_X ev^*\omega$.

To see that this agrees with our constructions of the preceding sections suppose that $(\underline{g},\underline{A}^1,\ldots,\underline{A}^n)\in H^n(M,\mathcal{D}^n)$. Pulling back by the evaluation map gives the class $(ev^*\underline{g},ev^*\underline{A}^1,\ldots,ev^*\underline{A}^n)$. The pull back of the evaluation map gives a homomorphism in cohomology. Restricted to a fixed $\psi\in\mathrm{Map}(X,M)$ this class is equal to $(\psi^*\underline{g},\psi^*\underline{A}^1,\ldots,\psi^*\underline{A}^n)$ which represents a flat bundle $(n-1)$-gerbe on X. The fibre integration map evaluates the flat holonomy for each value of ψ. It was proven in [25] that the fibre integration map is also a homomorphism.

In conclusion, we have developed the geometric notion of holonomy from the familiar case of line bundles to the case of bundle gerbes and bundle 2-gerbes. The generalisation was guided by the consideration of holonomy as a property of the Deligne class, specifically as the evaluation of the flat holonomy class of the pullback of the Deligne class to a closed manifold of appropriate dimension. The relationship between these cohomological and geometric concepts was demonstrated. As a property of Deligne cohomology holonomy could be extended to higher degree classes and also considered as an example of the more general notion of a transgression homomorphism.

Chapter 6

Parallel Transport and Transgression with Boundary

In this chapter we investigate what happens to the constructions of the previous chapter when we consider manifolds with boundary. This leads to generalised notions of parallel transport. These results may also be viewed in terms of an extension of the transgression homomorphism to manifolds with boundary.

We shall see that parallel transport may be thought of as a section of a trivial bundle. Let $ev_t : \mathcal{P}M \to M$ be the evaluation map that takes $\mu \in \mathcal{P}M$ to $\mu(t) \in M$. The the parallel transport map, which is a map between the fibres over $\mu(0)$ and $\mu(1)$ of a bundle L may be thought of as an element of $L^*_{\mu(0)} \otimes L_{\mu(1)}$. This is the same as a section of the bundle $(ev_0^* L)^* \otimes (ev_1^* L)$ on $\mathcal{P}M$. We shall see that such a section arises from the extension of holonomy from loops to paths.

This approach to parallel transport will lead to a similar interpretation in the case of bundle gerbes. Here we consider a surface with boundary made up of loops. The parallel transport is now defined by pulling back a bundle on the loop space to the space $\mathrm{Map}(\Sigma, M)$ of maps of the surface into M. The holonomy of a closed surface generalises to give a section of this bundle. For the example of the cylinder this construction gives a map between fibres over the two end loops, a situation similar to parallel transport for bundles, however there is no problem considering surfaces with different topologies. This construction will give a geometric interpretation to Gawedski's results on holonomy of classes in $H^2(M, \mathcal{D}^2)$ over surfaces with boundary [23].

6.1 Parallel Transport for Bundles

We now attempt to calculate the holonomy of a bundle over a path in the same way in which we calculated the holonomy over a loop. Once again it may seem that we are using a long and unnecessarily complicated method, however there are good reasons for this. Firstly it is not an unreasonable assumption that a method of describing holonomy which generalises to higher cases should be a good starting point for generalising parallel transport. It turns out that there are a number of different ways of approaching this, and since these appear in the literature it is worthwhile seeing how they arise in this context and how they relate to each other. We give as much detail as possible at the level of bundles since the key features of the theory are present, but relatively easy to deal with compared with the bundle gerbe case.

We consider a path as a smooth map $\mu : I \to M$ where I is the unit interval $[0,1] \in \mathbb{R}$. The path space $\mathcal{P}M$ is the space which consists of all such maps, we give it the compact-open smooth topology (see §5.5).

As in the previous case the pull back of any bundle to I is flat and trivial. The flat holonomy class is an element of $H^1(I, U(1)) = 0$, thus we cannot evaluate a holonomy in the same sense as the case without boundary. If we cannot define holonomy then we would like to define something which is as close as possible to holonomy. This turns out to be parallel transport. For motivation let us consider the case of a principal bundle. Parallel transport assigns to each path μ a $U(1)$-equivariant map between the fibres over $\mu(0)$ and $\mu(1)$. When we have a loop this gives an equivariant map from a fibre to itself which is of the form $p \mapsto pz$ for some $z \in U(1)$ which is the holonomy. In general the parallel transport map takes p to $\tilde{\mu}(1)$ where $\tilde{\mu}$ is a horizontal lift of μ satisfying $\tilde{\mu}(0) = p$. Another view is that parallel transport satisfies the condition that given any two paths which may be joined to form a loop then composing the respective parallel transports gives the holonomy. In our case the holonomy is given by $\exp \int_\gamma \chi$ where χ is a D-obstruction form given by $A_\alpha - d\log h_\alpha$. We have shown that this is equivalent to a formula $H(\underline{g}, \underline{A})$ in terms of the Deligne class. These two definitions of the function on the loop space lead to two equivalent ways of defining a function on the path space which satisfies the required criteria.

Given two paths μ_1, μ_2 with the same endpoints we may define a loop, by convention we define this loop associated with a pair (μ_1, μ_2) to be the composition $\mu_1 \star \mu_2^{-1}$. We may define a map on the path space by $H_B(\mu) = \exp \int_\mu \chi$, so $H_B(\mu_1) H_B^{-1}(\mu_2) = H(\mu_1 \star \mu_2^{-1})$. It is important to remember that in this case χ is no longer a D-obstruction form so we cannot be sure that the construction is independent of the choice of the trivialisation h. This is because in the construction of holonomy the value at a particular loop is given by the flat holonomy which is a property of flat bundle 0-gerbes. The triviality is only used to express it as a differential form, which turns out to be the D-obstruction, a property of trivial bundles. In fact we find that the map H_B does depend on the choice of trivialisation:

$$\begin{aligned} H_B(\mu) &= \exp \sum_e \int_e \mu^* A_{\rho(e)} \cdot \prod_{v,e} g_{\rho(e)\rho(v)}(\mu(v)) \cdot \prod_{v,e} h^{-1}_{\rho(v)}(\mu(v)) \\ &= \exp \sum_e \int_e \mu^* A_{\rho(e)} \cdot \prod_{v,e} g_{\rho(e)\rho(v)}(\mu(v)) \cdot h_{\rho(v_0)}(\mu(0)) h^{-1}_{\rho(v_1)}(\mu(1)) \end{aligned} \tag{6.1}$$

where v_0 and v_1 are the endpoint vertices of the triangulation of I. The final term fails to cancel this time because of contributions from $\mu(0)$ and $\mu(1)$. This expression is independent of the choice of ρ, but the dependence on a choice of trivialisation causes difficulties. We would prefer to have a ρ dependence instead, this could be achieved by restricting to a particular ρ and then choosing a trivialisation, however this is rather technical. It is easy enough to choose a trivialisation over each element of the triangulation using the canonical trivialisation over an element of the open cover which was described in §3.2. The problem is that these trivialisations then need to be glued together in some way. This is possible but will become even more complicated in higher degrees (see for example the proof of Proposition 6.5.1 in [5]), so is not suitable for our purposes.

Instead, let us turn to another expression for the holonomy, the transgression formula of proposition 5.2,

$$H_{(t_0,\rho_0)}(\mu) = \prod_e \exp \int_e \mu^* A_{\rho_0(e)} \cdot \prod_{v,e} g_{\rho_0(e)\rho_0(v)}(\mu(v)) \tag{6.2}$$

It is easily shown that on paths with boundary this function is not independent of the choice of ρ_0 so it is not globally defined on the path space. This dependence on the triangulation means that we have to be careful about describing the open covers on the loop space and path space. Given a decomposition of a loop into two paths with the same boundary where the loop is considered as an element of some $V_{(t_0,\rho_0)} \subset \mathcal{L}M$ there is an inherited triangulation and corresponding element of an open cover of $\mathcal{P}M$ for each path. In general the two paths do not lie in the same open set in terms of this cover, however it turns out that there is a more appropriate cover on the path space in this situation. Suppose that when $\gamma \in V_{(t_0,\rho_0)}$ is split into paths μ_1 and μ_2 the induced open sets on $\mathcal{P}M$ are $W_{(t_1,\rho_1)}$ and $W_{(t_2,\rho_2)}$ respectively. Since they are both induced from the same cover on the loop space they must satisfy the condition $\rho_1(v) = \rho_2(v)$ for $v \in \partial\mu_1 (= \partial\mu_2)$. In this case we have

$$H(\mu_1 \star \mu_2^{-1}) = H_{(t_1,\rho_1)}(\mu_1) H^{-1}_{(t_2,\rho_2)}(\mu_2) \tag{6.3}$$

where we have just broken up the expression for the holonomy (5.11) into the parts corresponding to each path.

Since we are using the same formula for both cases we use H for for both loops and paths. The distinction should be clear in all cases from the argument and the fact that the function on the path space has a local dependence indicated by a subscript.

Consider now the case where we are given two paths which share a boundary. When is equation (6.3) satisfied? The fact that H is independent of the open cover suggests that this equation is satisfied whenever the two covers on the path space combine to form a cover on the loop space, that is, precisely whenever $\rho_1(v) = \rho_2(v)$ on the boundary of the paths. Suppose that μ_2 also lies in the open set $W_{(t_3,\rho_3)}$ and $\rho_3(v) = \rho_2(v)$ on the boundary. Then using (6.3)

$$\begin{aligned} H_{(t_2,\rho_2)}(\mu_2) &= H^{-1}(\mu_1 \star \mu_2^{-1}) H_{(t_1,\rho_1)}(\mu_1) \\ &= H^{-1}_{(t_1,\rho_1)}(\mu_1) H_{(t_3,\rho_3)}(\mu_2) H_{(t_1,\rho_1)}(\mu_1) \\ &= H_{(t_3,\rho_3)}(\mu_2) \end{aligned} \tag{6.4}$$

This result may also be seen by considering what happens when we change $\rho_0(e)$ to $\rho_0'(e)$ for any e in the triangulation for μ in the formula for $H_0(\mu)$. The two expressions for $h_0(\mu)$ agree except on the terms corresponding to e where the difference is

$$\int_e (A_{\rho(e)} - A_{\rho'(e)}) \cdot \prod_{v \subset \partial e} g_{\rho(e)\rho(v)} g^{-1}_{\rho'(e)\rho'(v)} = \prod_{v \subset \partial e} g_{\rho'(e)\rho(e)} g_{\rho(e)\rho(v)} g^{-1}_{\rho'(e)\rho'(v)} = 1 \tag{6.5}$$

Thus we may use a coarser open cover on $\mathcal{P}M$, the cover induced by the projection to $M \times M$. An open set $\tilde{W}_{\rho_0,t_0}$ in this cover consists of all paths μ with triangulation t_0 such that the endpoints of μ are vertices v_0 and v_1 satisfying $\mu(0) \in U_{\rho_0(v_0)}$ and $\mu(1) \in U_{\rho_0(v_1)}$. In terms of this cover we shall denote the functions by H_0, H_1 and so on. We now have a set of locally defined functions on $\mathcal{P}M$ from which we may recover the holonomy in the required manner. It is natural to ask what the obstruction is to these defining a global function. Another way of viewing this is that given a set of local functions, on overlaps we may define transition functions for a trivial bundle, with trivialisations (or equivalently sections) defined by the original functions. Using a similar calculation to those performed in the previous chapter we find

$$
(H^{-1}_{(t_0,\rho_0)}H_{(t_1,\rho_1)})(\mu) = \prod_{v,\partial e} g_{\rho_0(v)\rho_1(v)}(\mu(v)) \tag{6.6}
$$

Since the right hand side depends only on the boundary of μ it may be written as $r^*G_{(t_0,\rho_0)(t_1,\rho_1)}$ where r is the restriction to the boundary $\partial\mu$ and $G_{(t_0,\rho_0)(t_1,\rho_1)}$ are defined on $\mathrm{Map}(\partial I, M)$. By applying $d\log$ to $H_{(t_0,\rho)}$ it is possible to obtain a formula for local connection 1-forms. To distinguish the differential on $\mathcal{P}M$ from d on M we shall denoted it $\tilde{d}$ (this notation will carry over to other spaces of maps into M as well). Let $\xi \in T_\mu(\mathcal{P}M)$.

$$
\begin{aligned}
(\tilde{d}\log H_0)(\xi) &= \sum_e \int_e \mu^*(d\iota_\xi A_{\rho_0(e)} + \iota_\xi dA_{\rho_0(e)}) + \sum_{v,e} \iota_\xi(v) d\log g_{\rho_0(e)\rho_0(v)}(\mu(v)) \\
&= \sum_{v,\partial e} \mu^* \iota_{\xi(v)} A_{\rho_0(v)}
\end{aligned} \tag{6.7}
$$

Since this depends only on the boundary we may write it as r^*B_0 where B_0 is a local one form on $\mathrm{Map}(\partial I, M)$. We denote the trivial bundle with connection on $\mathcal{P}M$ by $D(\tau_I L)$ (so $\tau_I L$ is the trivialisation H_0) and the bundle with connection represented by local data (G_{01}, B_0) is denoted by $\tau_{\partial I} L$.

We would like to consider a more global description of the bundles $D(\tau_I L)$ and $\tau_{\partial I}L$. This is given by the following diagram:

$$
\begin{array}{ccccc}
 & & S^1 & & D(\tau_I L) \\
 & \overset{H}{\nearrow} & & \swarrow & \\
\mathcal{L}M & \rightrightarrows & \mathcal{P}M & & \\
 & & \downarrow \partial & & \\
 & & M \times M & &
\end{array}
$$

Here we are claiming that the bundle 0-gerbe described locally by the construction of G_{01} is the same as that represented by the above diagram. We shall consider this as a particular example of a general result. Let (λ, Y, M) be a bundle 0-gerbe and let H_α be locally defined functions on $\pi^{-1}(U_\alpha) \subset Y$ such that $H_\alpha(y_1)H_\alpha^{-1}(y_2) = \lambda(y_1, y_2)$. Then the transition functions of the bundle 0-gerbe are given by $g_{\alpha\beta}(m) = H_\alpha^{-1}(y)H_\beta(y)$ for any $y \in \pi^{-1}(m)$. Observe that

$$
\begin{aligned}
\lambda(s_\alpha(m), s_\beta(m)) &= H_\alpha(s_\alpha)H_\alpha^{-1}(s_\beta) \\
&= H_\alpha(s_\alpha)H_\beta^{-1}(s_\beta)H_\beta(s_\beta)H_\alpha^{-1}(s_\beta) \\
&= H_\alpha(s_\alpha)H_\beta^{-1}(s_\beta)g_{\alpha\beta}(m) \\
&= \delta(s_\alpha^* H_\alpha)(m) g_{\alpha\beta}(m)
\end{aligned} \tag{6.8}
$$

The bundle 0-gerbe defined by this diagram is $\tau_{\partial I}L$ and as we have seen it is D-stably isomorphic to $m_0^{-1}L^* \otimes m_1^{-1}L$ where m_0 and m_1 are the projections of the two components of $M \times M$ onto M. This is related to the the holonomy reconstruction theorem as described in [31], which says that a bundle with connection may be reconstructed, up to isomorphism, from the function on the loop space defined by holonomy. We shall consider this theorem and bundle gerbe generalisations of it in the next chapter.

The bundle $\tau_I L$ is canonically trivial, with local trivialisation functions H_ρ such that $H_\rho(\mu_1)H_\rho^{-1}(\mu_2) = H(\mu_1 \star \mu_2^{-1})$. It is also the canonically trivial bundle obtained by pulling back $\tau_{\partial I} L$ to $\mathcal{P}M$ with the map ∂, the restriction to the boundary. The connection on $\tau_I L$ is given locally on $\mathcal{P}M$ by $d \int_\mu \chi$.

There is another approach to this problem. Following Hitchin [28] we may consider the space of trivialisations of L over $\mathrm{Map}(I, M)$, which we shall denote $\mathfrak{T}_I$. Each element of $\mathfrak{T}_I$ has a particular path associated with it, giving a projection map onto $\mathrm{Map}(I, M)$. Using this we may calculate the function $\delta(H) : {\mathfrak{T}_I}^{[2]} \to S^1$ in the following way,

$$\delta(H)(H^0_{(t,\rho)}, H^1_{(t,\rho)}; \mu) = \sum_{v,e} \log H^1_{\rho(v)}(\mu(v)) - \log H^0_{\rho(v)}(\mu(v)) \tag{6.9}$$

Recall that any two trivialisations of a bundle differ by a function. Equation (6.9) gives the 'holonomy' of the function defined by trivialisations h^0_ρ and h^1_ρ over $\partial\mu$. Thus once again we see that this bundle 0-gerbe is pulled back from $\mathrm{Map}(\partial I, M)$. Over $\mathrm{Map}(\partial I, M)$ we have a space of trivialisations $\mathfrak{T}_{\partial I}$ and (6.9) gives a function on ${\mathfrak{T}_{\partial I}}^{[2]}$ which defines the bundle 0-gerbe. Note that this function is no longer of the form $\delta(H)$ since H is not defined on $\mathfrak{T}_{\partial I}$. By forming the bundle corresponding to this bundle gerbe we obtain the lower dimensional version of the moduli space of flat trivialisations ([28]) as the total space. The difficulty with this approach is having to deal with the space of trivialisations. The space of trivialisations of a bundle is the infinite dimensional set $\mathrm{Map}(M, U(1))$, however the space of trivialisations of a bundle gerbe is a collection of all line bundles on M which is not a set and would have to be considered in terms of category theory.

In [26] it was proved that the local transgression formulae that we have described above correspond to the usual notion of parallel transport for bundles. We would like to describe how the local functions $H_{(t_0,\rho_0)}$ on the loop space lead to the parallel transport map.

Recall that given a bundle with connection $L \to M$ and a path μ in M, parallel transport is a $U(1)$-equivariant map of fibres $PT : L_{\mu(0)} \to L_{\mu(1)}$ which is defined by the unique lifting of μ to L which is horizontal with respect to the connection. Transgression defines a trivialisation of $r^{-1}L = L^*_{\mu(0)} \otimes L_{\mu(1)}$. This means we have a global section on $\mathrm{Map}(I, M)$ which assigns an element of $L^*_{\mu(0)} \otimes L_{\mu(1)}$ to each path, but this may also be interpreted as a $U(1)$- equivariant map $L_{\mu(0)} \to L_{\mu(1)}$ which defines the parallel transport.

6.2 Loop Transgression of Bundle Gerbes

We consider what happens to the holonomy of a bundle gerbe over M when we allow surfaces with boundary. We obtain a section of a trivial line bundle over $\mathrm{Map}(\Sigma^\partial, M)$ which gives a generalisation of parallel transport. Furthermore this is the pullback via restriction to the boundary of a possibly non-trivial line bundle over $\mathrm{Map}(\partial\Sigma, M)$. This may be related to a line bundle over the loop space. We derive a local formula which gives a transgression homomorphism $\tau_{S^1} : H^2(M, \mathcal{D}^2) \to H^1(\mathcal{L}M, \mathcal{D}^1)$. Throughout we abbreviate $\mathrm{Map}(X, M)$ by XM for various manifolds X.

We already have a formula (5.21) which gives an S^1-function over the space of smooth mappings of closed surfaces into M. This corresponds to pulling back the bundle gerbe and evaluating the flat holonomy as an element of $H^2(\Sigma, U(1)) = S^1$. In the case with boundary we proceed as in the case of parallel transport.

Starting with $\exp \int_\Sigma \eta - dk$ and following the same procedure as for closed surfaces we find that the terms involving the trivialisation $(\underline{h}, \underline{k})$ do not cancel out on the boundary components. Given a choice ρ_0 we could use the canonical trivialisation over each $U_{\rho_0(v)}$ and $U_{\rho_0(e)}$. The problem here, unlike in the previous case where boundary components were just points, is that each boundary component is a loop and will consist of a number of edges and vertices, so to define a trivialisation over the whole loop it would be necessary to glue together each of these in some way. This is possible (see [5] for a description of this in the gerbe case), however it is not a method that will be suitable for generalisation to higher degree as it becomes very complicated. Instead we turn to the second method that was developed in the previous section.

We define a function on $\Sigma^\partial M$ by $H_{(t_0,\rho_0)}(\psi)$,

$$H_{(t_0,\rho_0)}((\underline{g}, \underline{A}, \underline{\eta}); \psi) = \prod_b \exp \int_b \psi^* \eta_{\rho_0(b)} \cdot \prod_{e,b} \exp \int_e \psi^* A_{\rho_0(b)\rho_0(e)} \cdot \prod_{v,e,b} g_{\rho_0(b)\rho_0(e)\rho_0(v)}(\psi(v)) \tag{6.10}$$

the usual holonomy formula which is well-defined but not globally defined on $\Sigma^\partial M$. This is the approach taken by Gawedski [23]. Clearly for two surfaces with the same boundary this 'trivialises' the holonomy function, but it is not a proper trivialisation since it is not globally defined. These local functions define a D-trivial bundle $D(\tau_{\Sigma^\partial} P) \to \Sigma^\partial M$. As in the previous case we can define a bundle 0-gerbe $\tau_{\partial\Sigma} P$ on the space of mappings of the boundary,

$$\begin{array}{ccccc} & & S^1 & & D(\tau_{\Sigma^\partial} P) \\ & \overset{H}{\nearrow} & & \swarrow & \\ \Sigma M & \rightrightarrows & \Sigma^\partial M & & \\ & & \downarrow & & \\ & & \partial\Sigma M & & \end{array} \tag{6.11}$$

The transition functions of $D(\tau_{\Sigma^\partial})$ descend to $G_{(t_0,\rho_0)(t_1,\rho_1)}$ on $\partial\Sigma M$, the transition functions of $\tau_{\partial\Sigma}P$. We may calculate these explicitly,

$$\begin{aligned}
G_{(t_0,\rho_0)(t_1,\rho_1)} &= H^{-1}_{(t_0,\rho_0)}H_{(t_1,\rho_1)} \\
&= \exp(\sum_b \int_b (\eta_{\rho_1(b)} - \eta_{\rho_0(b)}) + \sum_{e,b}(A_{\rho_1(b)\rho_1(e)} - A_{\rho_0(b)\rho_0(e)})) \\
&\qquad \prod_{v,e,b} g_{\rho_1(b)\rho_1(e)\rho_1(v)} g^{-1}_{\rho_0(b)\rho_0(e)\rho_0(v)} \\
&= \exp(\sum_b \int_b dA_{\rho_0(b)\rho_1(b)} + \sum_{e,b}(A_{\rho_1(b)\rho_1(e)} - A_{\rho_0(b)\rho_0(e)})) \\
&\qquad \prod_{v,e,b} g_{\rho_1(b)\rho_1(e)\rho_1(v)} g^{-1}_{\rho_0(b)\rho_0(e)\rho_0(v)} \\
&= \exp \sum_{e,b}(A_{\rho_0(b)\rho_1(b)} + A_{\rho_1(b)\rho_1(e)} - A_{\rho_0(b)\rho_0(e)}) \prod_{v,e,b} g_{\rho_1(b)\rho_1(e)\rho_1(v)} g^{-1}_{\rho_0(b)\rho_0(e)\rho_0(v)} \\
&= \exp \sum_{e,b}(A_{\rho_0(e)\rho_1(e)} - d\log g_{\rho_0(b)\rho_1(b)\rho_1(e)} + d\log g_{\rho_0(b)\rho_0(e)\rho_1(e)}) \\
&\qquad \prod_{v,e,b} g_{\rho_1(b)\rho_1(e)\rho_1(v)} g^{-1}_{\rho_0(b)\rho_0(e)\rho_0(v)} \\
&= \exp(\sum_{e,b} A_{\rho_0(e)\rho_1(e)}) \prod_{v,e,b} g^{-1}_{\rho_0(b)\rho_1(b)\rho_1(e)} g_{\rho_0(b)\rho_0(e)\rho_1(e)} g_{\rho_1(b)\rho_1(e)\rho_1(v)} g^{-1}_{\rho_0(b)\rho_0(e)\rho_0(v)} \\
&= \exp(\sum_{e,b} A_{\rho_0(e)\rho_1(e)}) \prod_{v,e,b} g^{-1}_{\rho_0(e)\rho_0(v)\rho_1(v)} g_{\rho_0(e)\rho_1(e)\rho_1(v)}
\end{aligned} \tag{6.12}$$

where the last step involves repeated applications of the cocycle condition on g and the elimination of terms depending only on b and v. All interior terms will cancel due to the summation over b, leaving only edges in the boundary which we denote ∂e. In terms of $\partial\psi$, the restriction of the map ψ to the boundary, we have

$$\begin{aligned}
G_{(t_0,\rho_0)(t_1,\rho_1)}(\partial\psi) &= \exp \sum_{\partial e} \int_e \partial\psi^* A_{\rho_0(e)\rho_1(e)} \cdot \prod_{v,\partial e} g^{-1}_{\rho_0(e)\rho_0(v)\rho_1(v)} g_{\rho_0(e)\rho_1(e)\rho_1(v)}(\partial\psi(v)) \\
&= \exp \sum_{\partial e} \int_e \partial\psi^* A_{\rho_0(e)\rho_1(e)} \cdot \prod_{v,\partial e} g^{-1}_{\rho_1(e)\rho_0(v)\rho_1(v)} g_{\rho_0(e)\rho_1(e)\rho_0(v)}(\partial\psi(v))
\end{aligned} \tag{6.13}$$

where the second line is an alternate formula obtained using the cocycle condition which is equivalent to that given by Brylinski [5].

It is possible to relate this directly with the approach of Hitchin [28] where the holonomy of a gerbe over a loop is defined in terms of the holonomy of a bundle defined by the difference between two trivialisations. In this case we have trivialisations $(\underline{k}^0, \underline{h}^0)$ and $(\underline{k}^1, \underline{h}^1)$ defined over loops in $V_{(t_0,\rho_0)}$ and $V_{(t_1,\rho_1)}$ respectively. In this case transition functions would be

$$\hat{G}_{(t_0,\rho_0)(t_1,\rho_1)} = \exp \sum_e \int_e (k^0_{\rho(e)} - k^1_{\rho(e)}) \cdot \prod_{v,e} h^0_{\rho(e)\rho(v)} {h^1}^{-1}_{\rho(e)\rho(v)} \tag{6.14}$$

where (t,ρ) is any index of the open cover of the loop space which is defined on $V_{(t_0,\rho_0)(t_1,\rho_1)}$. The transition functions are independent of this choice since they are defined as a holonomy. In the following calculation we shall choose this to be (t_0,ρ_0). We now directly compare $G_{(t_0,\rho_0)(t_1,\rho_1)}$ and $\hat{G}_{(t_0,\rho_0)(t_1,\rho_1)}$:

$$\begin{aligned}
G_{(t_0,\rho_0)(t_1,\rho_1)}\hat{G}^{-1}_{(t_0,\rho_0)(t_1,\rho_1)} &= \exp\sum_e\int_e(A_{\rho_0(e)\rho_1(e)} - k^0_{\rho_0(e)} + k^1_{\rho_0(e)}) \\
&\quad\cdot\prod_{v,e} g^{-1}_{\rho_0(e)\rho_0(v)\rho_1(v)} g_{\rho_0(e)\rho_1(e)\rho_1(v)} h^{0\,-1}_{\rho_0(e)\rho_0(v)} h^1_{\rho_0(e)\rho_0(v)} \\
&= \exp\sum_e\int_e(A_{\rho_0(e)\rho_1(e)} - k^0_{\rho_0(e)} + k^1_{\rho_0(e)}) \\
&\quad\cdot\prod_{v,e} h^1_{\rho_0(e)\rho_1(e)} h^1_{\rho_1(e)\rho_1(v)} h^{0\,-1}_{\rho_0(e)\rho_0(v)} \\
&= \exp\sum_e\int_e(k^1_{\rho_1(e)} - k^0_{\rho_0(e)})\cdot\prod_{v,e} h^1_{\rho_1(e)\rho_1(v)} h^{0\,-1}_{\rho_0(e)\rho_0(v)}
\end{aligned} \tag{6.15}$$

This is a trivial Deligne class so $\underline{G}$ and $\underline{\hat{G}}$ define isomorphic bundles.

To get the local connection 1-form on $V_{(t_0,\rho_0)}$ we calculate $\tilde{d}\log H_{(t_0,\rho_0)}$.

$$\begin{aligned}
\tilde{d}\log H_{(t_0,\rho_0)}(\xi) &= \sum_b\int_b(\iota_\xi d\eta_{\rho_0(b)} + d\iota_\xi\eta_{\rho_0(b)}) + \sum_{e,b}\int_e(\iota_\xi dA_{\rho_0(b)\rho_0(e)} + d\iota_\xi A_{\rho_0(b)\rho_0(e)}) \\
&\quad + \sum_{v,e,b}(\iota_\xi d\log g_{\rho_0(b)\rho_0(e)\rho_0(v)}) \\
&= \sum_b\int_b\iota_\xi\omega + \sum_{e,b}\int_e\iota_\xi(\eta_{\rho_0(b)} - \eta_{\rho_0(b)} + \eta_{\rho_0(e)}) \\
&\quad + \sum_{v,e,b}\iota_\xi(A_{\rho_0(b)\rho_0(e)} - A_{\rho_0(b)\rho_0(e)} + A_{\rho_0(b)\rho_0(v)} - A_{\rho_0(e)\rho_0(v)}) \\
&= (\int_\Sigma ev^*\omega)(\xi) + \sum_{\partial e}\int_e\iota_\xi\eta_{\rho_0(e)} + \sum_{v,\partial e} -\iota_\xi A_{\rho_0(e)\rho_0(v)}
\end{aligned} \tag{6.16}$$

The 1-form $\tilde{d}\log H_{(t_0,\rho_0)}$ does not descend to the boundary however $\tilde{d}\log H_{(t_0,\rho_0)} - \int_\Sigma ev^*\omega$ is an equivalent connection on the trivial bundle $D(\tau_{\Sigma^\theta})$ which does descend, so the local connection forms are given by

$$B_{(t_0,\rho_0)} = \sum_{\partial e}\int_e\iota_\xi\eta_{\rho_0(e)} + \sum_{v,\partial e} -\iota_\xi A_{\rho_0(e)\rho_0(v)} \tag{6.17}$$

If the map $\partial\psi$ consists of a number of components $\partial\psi_i$ then

$$G_{(t_0,\rho_0)(t_1,\rho_1)}(\partial\psi) = \prod_i G_{(t_0,\rho)(t_1,\rho_1)}(\partial\psi_i) \tag{6.18}$$

and

$$B_{(t_0,\rho_0)}(\xi) = \sum_i B_{(t_0,\rho_0)}(\xi_i) \tag{6.19}$$

thus if we consider maps, ∂_i, of the boundary components into the free loop space LM then the bundle described by $\underline{G}$ is a product of bundles over each component loop, $\bigotimes_i \partial_i^{-1} L$, where L is the line bundle over the loop space defined by the transgression formula (6.13). The bundle was described in this way in [5].

Let us summarise what we know about transgression of bundle gerbes. Given a bundle gerbe with connection and curving, P, over M we obtain a trivial bundle $D(\tau_{\Sigma^\partial} P)$ over $\mathrm{Map}(\Sigma^\partial, M)$. A section of this trivial bundle defines a generalised notion of parallel transport. Also we have seen that $D(\tau_{\Sigma^\partial} P)$ is the pullback of a bundle on the boundary, $\tau_{\partial\Sigma} P$. This bundle may be defined in terms of the transgression to the loop space which is a bundle $\tau_{S^1} P$.

It follows that whenever M is 1-connected, we have a simple geometric picture of the bundle over the loop space (as a bundle 0-gerbe),

$$\begin{array}{ccc} & & S^1 \\ & \overset{H}{\nearrow} & \\ S^2(M) & \rightrightarrows & D^2(M) \\ & & \downarrow \\ & & \mathcal{L}M \end{array}$$

where $S^2(M)$ is the space of smooth maps of the 2-sphere into M and $D^2(M)$ is the space of smooth maps of the 2-disc into M. We know from our previous discussion that the transition functions of this bundle gerbe are $G_{(t_0,\rho_0)(t_1,\rho_1)}$. Previously we considered the bundle 0-gerbe defined by the holonomy function on ΣM where Σ is any surface on M however in this case spheres are sufficient to obtain all possible loops in M since every loop bounds a disc. For more general M there exist loops which do not bound a disc, in which case there is no simple geometric picture of the bundle on the loop space, however, the local transition functions are still well defined. There is an analogy between this situation and the holonomy of bundles over M. When M is 1-connected the holonomy may be defined in simple terms as an integral of the curvature over some surface, however more generally if we want to define the holonomy as a function on M we use a local formula.

6.3 A General Formula for Parallel Transport

In this section we take a similar approach to that taken in §5.4. The formulae obtained are examples of the more general fibre integration formulae of Gomi and Terashima ([25][26]). The difference with our approach is that rather than starting with a formula and proving that it satisfies the requirements for parallel transport, we are deriving formulae by first generalising the notion of parallel transport. Suppose we have a class in $H^p(M, \mathcal{D}^p)$ and a p-manifold, X^∂, with boundary ∂X. Then we may define a set of local functions $H_{(t_0,\rho_0)}$ on $X^\partial M$ such that given any two mappings ψ_1, ψ_2 which have the same boundary then $H^{-1}_{(t_0,\rho_0)}(\psi_1) H_{(t_0,\rho_0)}(\psi_2) = H(\psi_1 \# \psi_2)$ where H is the holonomy function on XM, $\#$ is the connected sum which replaces the composition for paths and loops and the open cover is induced by the restriction to ∂X. We may define transition functions $G_{(t_0,\rho_0)(t_1,\rho_1)} = H^{-1}_{(t_0,\rho_0)} H_{(t_1,\rho_1)}$ which define a trivial bundle on $X^\partial M$ and descend to define a possibly non-trivial bundle on $\partial X M$. We calculate the formula for these transition functions using our general formula for holonomy (5.57).

$$G_{(t_0,\rho_0)(t_1,\rho_1)} = \exp \sum_{n=0}^{p} \sum_{\underline{\sigma}^{p-n}} \int_{\sigma^{p-n}} [A^{p-n}_{\rho_1(\sigma^p)\ldots\rho_1(\sigma^{p-n})} - A^{p-n}_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-n})}] \tag{6.20}$$

We would like to express this in a form which explicitly relies only on the restriction to the boundary. Consider the first term in the summation over n,

$$\begin{aligned}\sum_{\underline{\sigma}^p} \int_{\sigma^p} [A^p_{\rho_1(\sigma^p)} - A^p_{\rho_0(\sigma^p)}] &= \sum_{\underline{\sigma}^p} \int_{\sigma^p} \delta(A^p)_{\rho_0(\sigma^p)\rho_1(\sigma^p)} \\ &= \sum_{\underline{\sigma}^p} \int_{\sigma^p} dA^{p-1}_{\rho_0(\sigma^p)\rho_1(\sigma^p)} \\ &= \sum_{\underline{\sigma}^{p-1}} \int_{\sigma^{p-1}} A^{p-1}_{\rho_0(\sigma^p)\rho_1(\sigma^p)}\end{aligned} \tag{6.21}$$

Combining this with the term corresponding to $n = 1$ gives

$$\begin{gathered}\sum_{\underline{\sigma}^{p-1}} \int_{\sigma^{p-1}} A^{p-1}_{\rho_0(\sigma^p)\rho_1(\sigma^p)} + A^{p-1}_{\rho_1(\sigma^p)\rho_1(\sigma^{p-1})} - A^{p-1}_{\rho_0(\sigma^p)\rho_0(\sigma^{p-1})} = \\ \sum_{\underline{\sigma}^{p-1}} \int_{\sigma^{p-1}} A^{p-1}_{\rho_0(\sigma^{p-1})\rho_1(\sigma^{p-1})} + \delta(A^{p-1})_{\rho_0(\sigma^p)\rho_1(\sigma^p)\rho_1(\sigma^{p-1})} - \delta(A^{p-1})_{\rho_0(\sigma^p)\rho_0(\sigma^{p-1})\rho_1(\sigma^{p-1})}\end{gathered} \tag{6.22}$$

We would now like to iterate this process. First we shall prove a formula which will simplify some of the terms,

Lemma 6.1. *For* $1 \le n \le p$

$$\begin{aligned}&\sum_{r=0}^{n}(-1)^r \delta(A^{p-n})_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-r})\rho_1(\sigma^{p-r})\ldots\rho_1(\sigma^{p-n})} = \\ &\quad (-1)^{n+1}[\sum_{r=0}^{n-1}(-1)^r A^{p-n}_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-r})\rho_1(\sigma^{p-r})\ldots\rho_1(\sigma^{p-n+1})}] \\ &\qquad + \sum_{r=1}^{n}(-1)^r A^{p-n}_{\rho_0(\sigma^{p-1})\ldots\rho_0(\sigma^{p-r})\rho_1(\sigma^{p-r})\ldots\rho_1(\sigma^{p-n})} + A^{p-n}_{\rho_1(\sigma^{p-n})\ldots\rho_1(\sigma^p)} \\ &\qquad\qquad - A^{p-n}_{\rho_0(\sigma^{p-n})\ldots\rho_0(\sigma^p)} + I\end{aligned} \tag{6.23}$$

where I consists of a sum of terms in which one subscript $\rho_i(\sigma^m)$ is omitted, for $p-n < m < p$ and $i \in \{0,1\}$. These are exactly the terms which cancel out under the sum $\sum_{\underline{\sigma}^{p-n}} \int_{\sigma^{p-n}} I$.

Proof. Most of the terms on the left hand side are absorbed into the term I on the right hand side. Consider the remaining terms. There are three distinct types:

1. Those for which the subscript $\rho_1(\sigma^{p-n})$ is omitted. There is one term of this type for each r in the summation. In each case a factor of $(-1)^{n+1}$ is introduced by the definition of δ.

2. Those for which the subscript $\rho_0(\sigma^p)$ is omitted. There is one term of this type for each r in the summation.

3. Terms for which the subscripts include σ^m for all $p-n \leq m \leq p$. This is only possible if the subscript omitted by δ is either $\rho_0(\sigma^{p-r})$ or $\rho_1(\sigma^{p-r})$. Note that the term obtained by omitting $\rho_0(\sigma^{p-r})$ is the same as that obtained by omitting the $\rho_1(\sigma^{p-r-1})$ subscript in the $(r-1)$-st term of the summation over r. In both of these cases we are eliminating the subscript immediately after $\rho_0(\sigma^{p-r-1})$ so the signs arising from the δ map are equal, however the two terms get opposite signs from the factor $(-1)^r$, and hence they cancel out.
There are two terms which do not cancel out in this way. One is obtained by omitting the subscript $\rho_0(\sigma^p)$ from the $r=0$ term in the summation. The factor $(-1)^r$ is equal to 1 and the coefficient from δ is also 1 since we are omitting the first term.
The other term which does not cancel out is the one obtained by omitting the subscript $\rho_1(\sigma^{p-n})$ from the $r=n$ term.

□

Proposition 6.1. *The formula for the transition functions given in equation* (6.20) *is equivalent to*

$$\begin{aligned} &\exp\sum_{n=1}^{k-1}\sum_{\underline{\sigma}^{p-n}}\int_{\sigma^{p-n}} -\sum_{r=1}^{n}(-1)^r A^{p-n}_{\rho_0(\sigma^{p-1})\ldots\rho_0(\sigma^{p-r})\rho_1(\sigma^{p-r})\ldots\rho_1(\sigma^{p-n})} \\ &\cdot\exp\sum_{\underline{\sigma}^{p-k}}\int_{\sigma^{p-k}}(-1)^{k+1}[\sum_{r=0}^{k-1}(-1)^r A^{p-k}_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-r})\rho_1(\sigma^{p-r})\ldots\rho_1(\sigma^{p-k+1})}] \\ &\cdot\exp\sum_{n=k}^{p}\sum_{\underline{\sigma}^{p-n}}\int_{\sigma^{p-n}}[A^{p-n}_{\rho_1(\sigma^p)\ldots\rho_1(\sigma^{p-n})} - A^{p-n}_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-n})}] \end{aligned} \tag{6.24}$$

for each k such that $1 \leq k \leq p$.

Proof. Equation (6.21) shows that the result is true for $k=1$. We now proceed by induction on k. Assume the result is true for $k=l$. Define terms $A(n)$, $B(n)$ and $C(n)$ such that equation (6.24) becomes

$$\exp[\sum_{n=1}^{k-1}A(n) + B(k) + \sum_{n=k}^{p}C(n)] \tag{6.25}$$

To prove that the equation holds for $k=l+1$ we need to show

$$\exp[\sum_{n=1}^{l-1}A(n) + B(l) + \sum_{n=l}^{p}C(n)] = \exp[\sum_{n=1}^{l}A(n) + B(l+1) + \sum_{n=l+1}^{p}C(n)] \tag{6.26}$$

Clearly if $B(l)+C(l) = A(l)+B(l+1)$ then the result holds since all other terms in

(6.26) are identical. Consider

$$
\begin{aligned}
B(l)+C(l) &= \sum_{\underline{\sigma}^{p-l}} \int_{\sigma^{p-l}} (-1)^{l+1} \sum_{r=0}^{l-1} (-1)^r A^{p-l}_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-r})\rho_1(\sigma^{p-r})\ldots\rho_1(\sigma^{p-l+1})} \\
&\quad + \sum_{\underline{\sigma}^{p-l}} \int_{\sigma^{p-l}} [A^{p-l}_{\rho_1(\sigma^p)\ldots\rho_1(\sigma^{p-l})} - A^{p-l}_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-l})}] \\
&= \sum_{\underline{\sigma}^{p-l}} \int_{\sigma^{p-l}} \sum_{r=0}^{l} (-1)^r \delta(A^{p-l})_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-r})\rho_1(\sigma^{p-r})\ldots\rho_1(\sigma^{p-l})} \\
&\quad - \sum_{\underline{\sigma}^{p-l}} \int_{\sigma^{p-l}} \sum_{r=1}^{l} (-1)^r A^{p-l}_{\rho_0(\sigma^{p-1})\ldots\rho_0(\sigma^{p-r})\rho_1(\sigma^{p-r})\ldots\rho_1(\sigma^{p-l})}
\end{aligned}
$$

by Lemma 6.1,

$$
\begin{aligned}
&= \sum_{\underline{\sigma}^{p-l}} \int_{\sigma^{p-l}} \sum_{r=0}^{l} (-1)^r (-1)^l dA^{p-l-1}_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-r})\rho_1(\sigma^{p-r})\ldots\rho_1(\sigma^{p-l})} \\
&\quad - \sum_{\underline{\sigma}^{p-l}} \int_{\sigma^{p-l}} \sum_{r=1}^{l} (-1)^r A^{p-l}_{\rho_0(\sigma^{p-1})\ldots\rho_0(\sigma^{p-r})\rho_1(\sigma^{p-r})\ldots\rho_1(\sigma^{p-l})}
\end{aligned}
$$

since $\delta(\underline{A}^{p-n}) = (-1)^n d\underline{A}^{p-n-1}$ for a Deligne class,

$$
\begin{aligned}
&= \sum_{\underline{\sigma}^{p-l-1}} \int_{\sigma^{p-l-1}} (-1)^l \sum_{r=0}^{l} (-1)^r A^{p-l-1}_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-r})\rho_1(\sigma^{p-r})\ldots\rho_1(\sigma^{p-l})} \\
&\quad - \sum_{\underline{\sigma}^{p-l}} \int_{\sigma^{p-l}} \sum_{r=1}^{l} (-1)^r A^{p-l}_{\rho_0(\sigma^{p-l})\ldots\rho_0(\sigma^{p-r})\rho_1(\sigma^{p-r})\ldots\rho_1(\sigma^{p-1})} \\
&= B(l+1) + A(l)
\end{aligned}
\tag{6.27}
$$

□

This proposition gives a general formula for parallel transport by considering the case $k = p$. In this case the transition functions are given by

$$
\begin{aligned}
G_{(t_0,\rho_0)(t_1,\rho_1)} &= \exp \sum_{n=1}^{p-1} \sum_{\underline{\sigma}^{p-n}} \int_{\sigma^{p-n}} - \sum_{r=1}^{n} (-1)^r A^{p-n}_{\rho_0(\sigma^{p-1})\ldots\rho_0(\sigma^{p-r})\rho_1(\sigma^{p-r})\ldots\rho_1(\sigma^{p-n})} \\
&\quad \cdot \exp(B(p) + C(p))
\end{aligned}
\tag{6.28}
$$

It is not difficult to show that $B(p) + C(p) = A(p)$. This follows from the proof that $B(l) + C(l) = A(l) + B(l+1)$. Recall that the $B(l+1)$ term appears after applying Stokes' theorem to a dA^{p-l-1} term which in turn comes from applying the Deligne cocycle condition to a $\delta(A^{p-l})$ term. In this case we have $\delta(A^0) = 0$ so the $B(l+1)$ term is absent, leaving the required result.

Thus we have

Corollary 6.1. *[25, 26] The transition functions for the parallel transport bundle associated with a Deligne $(p+1)$-class over M and a smooth map of a p-manifold with boundary into M are given by*

$$G_{(t_0,\rho_0)(t_1,\rho_1)} = \exp \sum_{n=1}^{p} \sum_{\underline{\sigma}^{p-n}} \int_{\sigma^{p-n}} \sum_{r=1}^{n} (-1)^{r+1} A^{p-n}_{\rho_0(\sigma^{p-1})\ldots\rho_0(\sigma^{p-r})\rho_1(\sigma^{p-r})\ldots\rho_1(\sigma^{p-n})} \tag{6.29}$$

We now derive a formula for the connection. Once again we start with the local functions

$$H_{(t_0,\rho_0)} = \exp \sum_{n=0}^{p} \sum_{\underline{\sigma}^{p-n}} \int_{\sigma^{p-n}} A^{p-n}_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-n})} \tag{6.30}$$

We apply $\tilde{d}\log$ and evaluate at $\xi \in T(\mathrm{Map}(\partial\Sigma, M))$ to get

$$\begin{aligned}
\tilde{d}\log(H_{(t_0,\rho_0)})(\xi) - \Big(\int_\Sigma ev^*\omega\Big)(\xi) &= \sum_{n=0}^{p} \sum_{\underline{\sigma}^{p-n}} \int_{\sigma^{p-n}} [d\iota_\xi A^{p-n}_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-n})} + \iota_\xi dA^{p-n}_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-n})}] \\
&= \sum_{n=0}^{p} [\sum_{\underline{\sigma}^{p-n-1}} \int_{\sigma^{p-n-1}} \iota_\xi A^{p-n}_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-n})} \\
&\qquad + \sum_{\underline{\sigma}^{p-n}} \int_{\sigma^{p-n}} (-1)^{n+1} \iota_\xi \delta(A^{p-n+1})_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-n})}] \\
&= \sum_{n=0}^{p-1} \sum_{\underline{\sigma}^{p-n-1}} \int_{\sigma^{p-n-1}} \iota_\xi A^{p-n}_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-n})} \\
&\qquad + \sum_{n=1}^{p} \sum_{\underline{\sigma}^{p-n}} \int_{\sigma^{p-n}} (-1)^{n+1} \iota_\xi \delta(A^{p-n+1})_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-n})}
\end{aligned}$$

where we have used $\iota_\xi \underline{A}^0 = 0$ and $\underline{A}^{p+1} = 0$,

$$\begin{aligned}
&= \sum_{n=0}^{p-1} \sum_{\underline{\sigma}^{p-n-1}} \int_{\sigma^{p-n-1}} \iota_\xi A^{p-n}_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-n})} \\
&\qquad + (-1)^n \iota_\xi \delta(A^{p-n})_{\rho_0(\sigma^p)\ldots\rho_0(\sigma^{p-n-1})}
\end{aligned} \tag{6.31}$$

Only two terms from the δ part survive under the sum over $\underline{\sigma}^{p-n-1}$. These are the ones which omit the subscripts $\rho_0(\sigma^p)$ and $\rho_0(\sigma^{p-n-1})$ respectively. Consider the first of these. Since it is the first term in the δ expansion it is positive, thus we have $(-1)^n \iota_\xi A^{p-n}_{\rho_0(\sigma^{p-1})\ldots\rho_0(\sigma^{p-n-1})}$. The term omitting the subscript $\rho_0(\sigma^p)$ in the δ expansion will have the additional coefficient $(-1)^{n+1}$ which makes equal to the negative of the first term in (6.31). Thus we have shown that the connection on the bundle over $\mathrm{Map}(\partial\Sigma, M)$ representing parallel transport is given by

$$B_{(t_0,\rho_0)}(\xi) = \sum_{n=0}^{p-1} \sum_{\underline{\sigma}^{p-n-1}} \int_{\sigma^{p-n-1}} (-1)^n \iota_\xi A^{p-n}_{\rho_0(\sigma^{p-1})\ldots\rho_0(\sigma^{p-n-1})} \tag{6.32}$$

Example 6.1. Using these general formulae we can calculate the transition functions and local connections corresponding to the parallel transport of a bundle 2-gerbe on M over a smooth map of 3-manifold with boundary, X into M. Let the bundle 2-gerbe be represented by the Deligne class $(\underline{g}, \underline{A}, \underline{\eta}, \underline{\nu})$, let X be triangulated by 3-faces w, 2-faces b, edges e and vertices v. Denote the map $X \to M$ by ψ. Then the transition functions are given by

$$G_{(t_0,\rho_0)(t_1,\rho_1)}(\psi) = \exp\left(\sum_{b,w}\int_b \psi^*\eta_{\rho_0(b)\rho_1(b)} + \sum_{e,b,w}\int_e \psi^*(A_{\rho_0(b)\rho_1(b)\rho_1(e)} - A_{\rho_0(b)\rho_0(e)\rho_1(e)})\right)$$
$$\cdot \prod_{v,e,b,w} g_{\rho_0(b)\rho_1(b)\rho_1(e)\rho_1(v)} g^{-1}_{\rho_0(b)\rho_0(e)\rho_1(e)\rho_1(v)} g_{\rho_0(b)\rho_0(e)\rho_0(v)\rho_1(v)}(\psi(v)) \tag{6.33}$$

and the local connections are

$$B_{(t_0,\rho_0)}(\xi) = \sum_{b,w}\int_b \iota_\xi \nu_{\rho_0(b)} - \sum_{e,b,w}\int_e \iota_\xi \eta_{\rho_0(b)\rho_0(e)} + \sum_{v,e,b,w} \iota_\xi A_{\rho_0(b)\rho_0(e)\rho_0(v)} \tag{6.34}$$

In the case where M is 2-connected then we may, by analogy with $\tau_{S^1}P$ in the bundle gerbe case, define a bundle 0-gerbe $\tau_{S^2}P$ which may be represented in the following way:

$$\begin{array}{ccc} & & S^1 \\ & \overset{H}{\nearrow} & \\ S^3(M) & \rightrightarrows & D^3(M) \\ & & \downarrow \\ & & S^2(M) \end{array}$$

Gomi and Terashima [25, 26] suggest that it would be of interest to find geometric realisations of transgression in higher degrees. This construction gives such a realisation in terms of bundle gerbes.

6.4 Loop Transgression of Bundle 2-Gerbes

Recall that so far we have considered a generalised form of parallel transport. This has involved transgression of a bundle gerbe to a bundle 0-gerbe (or, equivalently, a bundle) over the loop space and the transgression of a bundle 2-gerbe to a bundle 0-gerbe on the space ∂XM of smooth maps of boundaries of 3-manifolds in M. As in the case of loop transgression of a bundle gerbe the fact that the transgression formula may be broken up into a product of factors over each boundary component implies that this bundle 0-gerbe may be realised as a product $\bigotimes_i \partial_i^{-1} L$ where L is a bundle 0-gerbe on ΣM which is defined locally by the transgression formula.

We may now proceed as in the case of loop transgression of bundle gerbes. To do this we apply the hierarchy principle, replacing functions with bundles. This means that we want to find a bundle on $\Sigma^\partial M$ that locally trivialises $\tau_{\partial X}P$. To do this simply apply the formula for $G_{(t_0,\rho_0)(t_1,\rho_1)}$. If this fails to be consistent on triple intersections then it will define a trivial bundle gerbe. This is quite a long calculation involving the repeated application of the various cocycle conditions which define the Deligne class of the bundle 2-gerbe. We start with the highest term, η and apply the appropriate cocycle condition and then use Stokes' Theorem to move down to the next level. This process is repeated at each level with terms such as $\sum A$ and $\prod g$ used as an abbreviation of all of the terms at the other levels. We also write G_{01} for $G_{(t_0,\rho_0)(t_1,\rho_1)}$ and so on.

$$\begin{aligned}
G_{01}G_{12}G_{02}^{-1} &= \exp(\sum_b \int_b \eta_{\rho_0(b)\rho_1(b)} + \eta_{\rho_1(b)\rho_2(b)} - \eta_{\rho_0(b)\rho_2(b)}) \cdot \exp \sum A \cdot \prod g \\
&= \exp(\sum_{e,b} \int_e -A_{\rho_0(b)\rho_1(b)\rho_2(b)} + A_{\rho_0(b)\rho_1(b)\rho_1(e)} - A_{\rho_0(b)\rho_0(e)\rho_1(e)} + A_{\rho_1(b)\rho_2(b)\rho_2(e)} \\
&\qquad - A_{\rho_1(b)\rho_1(e)\rho_2(e)} - A_{\rho_0(b)\rho_2(b)\rho_2(e)} + A_{\rho_0(b)\rho_0(e)\rho_2(e)}) \cdot \prod g \\
&= \exp(\sum_{e,b} \int_e -A_{\rho_0(e)\rho_1(e)\rho_2(e)} + d\log g_{\rho_0(b)\rho_1(b)\rho_2(b)\rho_1(e)} + d\log g_{\rho_1(b)\rho_2(b)\rho_1(e)\rho_2(e)} \\
&\qquad - d\log g_{\rho_0(b)\rho_2(b)\rho_1(e)\rho_2(e)} + d\log g_{\rho_0(b)\rho_0(e)\rho_1(e)\rho_2(e)}) \cdot \prod g \\
&= \exp(\sum_{e,b} \int_e -A_{\rho_0(e)\rho_1(e)\rho_2(e)}) \cdot \prod_{v,e,b} g^{-1}_{\rho_0(e)\rho_0(v)\rho_1(v)\rho_2(v)} g_{\rho_0(e)\rho_1(e)\rho_1(v)\rho_2(v)} \\
&\qquad g^{-1}_{\rho_0(e)\rho_1(e)\rho_2(e)\rho_2(v)}(v)
\end{aligned} \tag{6.35}$$

All of the interior terms will cancel in the sum over b so these transition functions descend to $\partial\Sigma M$. We also have a canonical choice of connection on this bundle gerbe which is given by the D-trivial local connection forms on $\Sigma^\partial M$,

$$(B_1 - B_0 - \tilde{d}\log G_{01})(\xi) = \sum_{e,b} \int_e -\iota_\xi \eta_{\rho_0(e)\rho_1(e)} + \sum_{v,e,b} \iota_\xi (A_{\rho_0(e)\rho_1(e)\rho_1(v)} - A_{\rho_0(e)\rho_0(v)\rho_1(v)}) \tag{6.36}$$

The details here are similar to those of previous calculations and have been omitted.

The canonical choice of curving is defined by

$$\begin{aligned}
\tilde{d}B_0(\xi, \mu) &= \sum_b \int_b \iota_\mu \iota_\xi d\nu_{\rho_0(b)} - d\iota_\mu \iota_\xi \nu_{\rho_0(b)} + \sum_{e,b} \int_e -\iota_\mu \iota_\xi d\eta_{\rho_0(b)\rho_0(e)} + d\iota_\mu \iota_\xi \eta_{\rho_0(b)\rho_0(e)} \\
&\qquad + \sum_{v,e,b} \iota_\mu \iota_\xi dA_{\rho_0(b)\rho_0(e)\rho_0(v)} \\
&= \sum_b \int_b \iota_\mu \iota_\xi \omega + \sum_{e,b} \int_e -\iota_\mu \iota_\xi \nu_{\rho_0(e)} + \sum_{v,e,b} -\iota_\mu \iota_\xi \eta_{\rho_0(e)\rho_0(v)}
\end{aligned} \tag{6.37}$$

The 2-form $\tilde{d}B_0 - \int_\Sigma ev^* \omega$ descends to give local curving 2-forms

$$\zeta_0(\xi, \mu) = \sum_{e,b} \int_e -\iota_\mu \iota_\xi \nu_{\rho_0(e)} + \sum_{v,e,b} -\iota_\mu \iota_\xi \eta_{\rho_0(e)\rho_0(v)} \tag{6.38}$$

This local bundle gerbe data splits into a product of terms over each component of $\partial\Sigma M$ and defines a bundle gerbe over the loop space. This situation is more complicated than the usual concept of parallel transport. We shall give a direct comparison for the example of a cylinder. For a bundle gerbe there is a line bundle L on the loop space, so associated with each boundary loop of the cylinder is a fibre of this bundle. Over the cylinder there is a trivial bundle with fibres given by the product of the fibres of L at the end loops. A trivial bundle has a section, which in this case defines a $U(1)$-equivariant map between the fibres at the end loops. If, on the other hand, we start with a bundle 2-gerbe then we have seen that a bundle gerbe on the loop space is obtained. Thus associated with each boundary loop is the fibre of a bundle gerbe, which is a $U(1)$-groupoid. Associated with a cylinder is a trivial bundle gerbe however this does not necessarily define a section.

This is similar to what happens when we consider a bundle gerbe over a path. Starting with the line bundle on the loop space obtained by transgression we obtain a trivialisation of a bundle gerbe on $\mathcal{P}M$. The trivial bundle gerbe is the pull back of a bundle gerbe on $\mathrm{Map}(\partial I, M)$. The fibre over each component of $\mathrm{Map}(\partial I, M)$ is simply the fibre of the original bundle gerbe over the relevant point in M. This example leads to holonomy reconstruction which shall be discussed in the next chapter, detailed calculations shall be given there.

Chapter 7

Further Results on Holonomy and Transgression

We briefly discuss the terminology used in this chapter. Strictly speaking holonomy is the $U(1)$-valued map corresponding to a bundle n-gerbe over a closed $(n+1)$ manifold. Corresponding to $(n+1)$-manifolds there are sections of trivial bundles which generalise parallel transport. Corresponding to closed n-manifolds there is the transgression bundle, named after a homomorphism in Deligne cohomology. Holonomy and parallel transport may also both be viewed as arising from such transgression maps so we sometimes use the term transgression to apply to all of these cases. It is also possible to view all cases as a generalisation of the notion of holonomy, sometimes the term holonomy is used in this context.

We present some basic properties of holonomy and transgression. We consider the holonomy of some of the examples we have encountered and discuss consequences for the theory of holonomy reconstruction. We also consider gauge invariance properties which are relevant to applications.

7.1 Some General Results

In Chapter 5 we considered generalisations of holonomy to bundle gerbes and higher objects. Here we present some basic properties of bundle holonomy and show that they also apply to bundle gerbe holonomy. We then go on to see how these results apply to the more general notion of transgression which was discussed in Chapter 6. Throughout the final two chapters hol will denote the holonomy map as defined in Chapter 5, with the bundle n-gerbe ($n = 0,1,2$), connections and curvings and the closed $n+1$ manifold indicated where necessary.

Functoriality

Let $P \to M$ be a bundle with connection A and suppose we have a map $\phi : N \to M$. Then

$$\mathrm{hol}(\phi^{-1}P; \phi^* A) = \phi^* \mathrm{hol}(P; A) \tag{7.1}$$

as functions on LN.

Now suppose we have a bundle gerbe $(P, Y, M; A, f)$ and map $\phi : N \to M$. Then we can form the pullback bundle gerbe over N and use this to define a holonomy function on $\text{Map}(\Sigma, N)$. For an element ψ of $\text{Map}(\Sigma, N)$ the holonomy function is defined by the evaluation of the flat holonomy class of the bundle gerbe $\psi^{-1}(\phi^{-1}P)$. This is the same as the bundle gerbe over Σ obtained by the pullback of P by $\phi \circ \psi \in \text{Map}(\Sigma, M)$. The flat holonomy class of this bundle gerbe is the holonomy function on $\text{Map}(\Sigma, M)$ evaluated at $\phi \circ \psi$, so it follows that

$$\text{hol}(\phi^{-1}P, \phi^{-1}Y, N; \phi^*A, \phi^*f) = \phi^*\text{hol}(P, Y, M) \tag{7.2}$$

as functions on $\text{Map}(\Sigma, N)$. Furthermore a similar result clearly holds for bundle 2-gerbes and maps of 3-manifolds.

Now consider the transgression of the bundle gerbe $\phi^{-1}P$ to the loop space $\mathcal{L}N$. The map ϕ induces a map $\mathcal{L}_\phi : \mathcal{L}N \to \mathcal{L}M$ such that given any $\mu \in \mathcal{L}N$ the map $L_\phi \mu \in \mathcal{L}M$ is defined by

$$\mathcal{L}_\phi \mu(\theta) = \phi(\mu(\theta)) \tag{7.3}$$

Now consider the transgression of P to the loop space $\mathcal{L}M$. This gives a bundle 0-gerbe which may be pulled back to $\mathcal{L}N$ by the map $\mathcal{L}_\phi$. We claim that this is stably isomorphic to the bundle 0-gerbe over $\mathcal{L}N$ mentioned above. First consider the case where there are no non-trivial loops and there is a geometric picture of this bundle 0-gerbe. At each level we have a map between surfaces induced by ϕ. In particular given $\psi \in \Sigma N$ then we can map to the surface given pointwise by $\phi \circ \psi$ in M. The holonomy map on ΣN gives the flat holonomy class of $\psi^{-1}\phi^{-1}P$ which is clearly equal to the holonomy map on ΣM corresponding to the surface $\phi \circ \psi$. Furthermore since the connection is derived from the holonomy function then we have a D-stable isomorphism. This result also extends to the bundle gerbe over the loop space associated with a bundle 2-gerbe since once again we have an induced map between smooth maps of manifolds at each level which is invariant under the holonomy map. Since the transgression formulae are obtained by this construction we claim that the functoriality applies to transgression in the general case, for example, functoriality of the holonomy of a bundle gerbe extends also to the local functions on surfaces with boundary and the transition functions are then defined in terms of these.

Orientation

We consider the effect of a change of orientation of the manifold over which the holonomy is evaluated. In the case without boundary then it is clear that the holonomy function changes sign under a change of orientation for bundles, bundle gerbes and bundle 2-gerbes. Recall that the function at a particular point of the relevant mapping space is the evaluation of the flat holonomy class of the corresponding pullback. This evaluation involves integrating the D-obstruction form, so a change of orientation will reverse the sign of this integral and lead to an overall reversal of the sign of the holonomy function.

This argument easily extends to the case with boundary. Our usual approach here is to proceed as in the case without boundary to obtain a function which locally trivialises the holonomy function on the fibre product. As above, a reverse in orientation leads to a reverse in sign of this function. The transition functions and local connections of the transgression bundle are derived from the local expressions for this function, so they too have signs reversed. Thus the bundle 0-gerbes with connection obtained by transgression change to their duals under a reverse of orientation of the embedded manifold. In the case of a bundle 2-gerbe over a surface with boundary then we have a bundle 0-gerbe which locally trivialises the bundle 0-gerbe on ΣM. Since this has a change of orientation then so do the trivialisations and thus the bundle gerbe on the loop space has a change of orientation.

Multiplicativity

Suppose $\Sigma_1 \sqcup \Sigma_2$ is a disjoint union of closed n-manifolds, for $n = 1, 2, 3$. If P is a bundle $(n-1)$-gerbe then

$$\mathrm{hol}_{\Sigma_1 \sqcup \Sigma_2}(P) = \mathrm{hol}_{\Sigma_1}(P) \cdot \mathrm{hol}_{\Sigma_2}(P) \tag{7.4}$$

The left hand side is equal to $\exp \int_{\Sigma_1 \sqcup \Sigma_2} \chi$ where χ is the D-obstruction form for the flat, trivial bundle $(n-1)$-gerbe $(\Sigma_1 \sqcup \Sigma_2)^{-1}P$. This integral separates into the two holonomies on the right hand side.
In the case with boundary then we find a similar relation however the functions are only defined locally. Taking D of this to get the bundle on the boundary, L we have

$$L_{\Sigma_1 \sqcup \Sigma_2} = L_{\Sigma_1} \otimes L_{\Sigma_2}$$

an isomorphism of bundle with connection. Furthermore since the sections of the pull backs to the mapping space of the manifold with boundary are given by the local function these also satisfy an additivity property.

Suppose we have a disjoint union of a closed manifold, Σ_1, and one with boundary, Σ_2. In this case the union is a manifold with boundary so there is a function H_ρ on $\Sigma^\partial M$ leading to a bundle on the restriction to the boundary, however since the H_ρ term corresponding to Σ_1 has no boundary then it is globally defined and so does not contribute to the transition function and thus the bundle does not depend on Σ_1 and is simply L_{Σ_2}. The difference from the case where we just have Σ_2 is the section of the trivial bundle defined by H_ρ. The term corresponding to Σ_1 gives a different choice of trivialisation of the bundle $D(H_\rho)$ than if we just have the Σ_2 term.

These results are easily extended to bundle 2-gerbes. The functions, bundles and bundle gerbes obtained by the various transgressions satisfy the obvious additivity conditions. In the situation where we have one component with boundary and one closed then the result of transgression is the transgression corresponding to the component with boundary, with the closed component contributing to the canonical section/trivialisation.

Gluing

Let Σ_1 and Σ_2 be two n manifolds where $n = 1, 2$ or 3. Suppose that $\partial\Sigma_1 = X_1 \cup X$ and $\partial\Sigma_2 = X_2 \cup X$. In this case we may glue Σ_1 and Σ_2 along X to get a new manifold Σ with $\partial\Sigma = X_1 \cup -X_2$. When evaluating the holonomy of Σ with respect to a bundle $(n-1)$-gerbe we get functions $H_{(t_0,\rho_0)}$ which separate into a boundary component and an interior component which descends to the mapping space of the boundary. We claim that this will be equal to the product of $H_{\Sigma_1(t_0,\rho_0)}$ and $-H_{\Sigma_2(t_0,\rho_0)}$. The change in sign is due to the need to reverse orientation for gluing. Clearly the claim holds for the interior components since these are not affected by gluing. Since the boundary components are summations over a triangulation then we may break these down to get $H^{\partial}_{X_1(t_0,\rho_0)} + H^{\partial}_{X(t_0,\rho_0)} - H^{\partial}_{X(t_0,\rho_0)} - H^{\partial}_{X_2(t_0,\rho_0)}$ which is equal to the boundary term for Σ. Thus we have

$$\tau_{\partial\Sigma}P = \tau_{\partial X_1}P \otimes (\tau_{\partial X_2}P)^* \tag{7.5}$$

Now consider the special case where we have two n-manifolds Σ_1 and Σ_2, such that $\partial\Sigma_1 = -\partial\Sigma_2$. Then it is possible to glue them together to obtain a closed manifold Σ. This time all of the boundary terms cancel leaving only the interior terms. Putting these together we get the holonomy function on $\mathrm{Map}(\Sigma, M)$. This function corresponds to the product of the two sections over $\mathrm{Map}(\Sigma, M)$.

In a similar way, given a bundle 2-gerbe P and surfaces Σ_1 and Σ_2 then the same results apply in terms of bundle gerbes.

7.2 The Holonomy of the Tautological Bundle Gerbe

We begin by considering the holonomy of the tautological bundle 0-gerbe. Recall that given a closed, 2π-integral 2-form, F, on a 1-connected manifold, M, this is defined by the diagram

$$\begin{array}{ccc} & & U(1) \\ & \overset{\rho}{\nearrow} & \\ \mathcal{L}_0 M & \rightrightarrows & \mathcal{P}_0 M \\ & & \downarrow \\ & & M \end{array}$$

where the map ρ is defined by

$$\rho(\gamma) = \exp \int_{\Sigma} F \tag{7.6}$$

and Σ is any surface which is bounded by γ. The connectedness requirement ensures that we can choose such a Σ and we have already established that ρ is independent of the choice of Σ.

Now consider the holonomy of this bundle 0-gerbe. Since the curvature is F then we know that the holonomy must also satisfy the condition on ρ in (7.6). This condition completely characterises the function and thus the holonomy of the tautological bundle 0-gerbe is the function $\rho : \mathcal{L}_0(M) \to U(1)$. This is a rather trivial fact however it will serve as an indication of what to expect as we move up the bundle gerbe hierarchy.

Now consider the tautological bundle gerbe, with the tautological bundle over $\mathcal{L}_0(M)$ considered as a bundle 0-gerbe,

$$\begin{array}{ccccc} & & U(1) & & \\ & \overset{\rho}{\nearrow} & & & \\ S^2(M) & \rightrightarrows & D^2(M) & & \\ & & \downarrow & & \\ & & \mathcal{L}_0 M & \rightrightarrows & \mathcal{P}_0 M \\ & & & & \downarrow \\ & & & & M \end{array} \tag{7.7}$$

By our connectedness assumption on M all closed 2-surfaces in M are of the form ∂X for some 3-manifold X. The holonomy satisfies $H(\partial X) = \exp \int_X \omega = \rho(\partial X)$. Since by the usual arguments this is independent of the choice of X we can conclude that the holonomy function on $S^2(M)$ is equal to ρ.

Recall that we can also transgress a bundle gerbe to get a bundle over $\mathcal{L}M$. We wish to compare this with the bundle over $\mathcal{L}_0M$ in the tautological bundle gerbe described above. Recall Lemma (3.1) tells us that the tautological construction is independent of the choice of base point, so without loss of generality we may replace $\mathcal{L}M$ with $\mathcal{L}_0M$. Now the fact that ρ is equal to the holonomy map shows that the bundle 0-gerbe obtained by transgression of the tautological bundle gerbe is the same as the bundle 0-gerbe on the loop space which is used to define the tautological bundle gerbe in (7.7). This implies the following

Proposition 7.1. *The transgression to the loop space $\mathcal{L}G$ of the tautological bundle gerbe over a compact, simply connected, semi-simple Lie group G with curvature 3-form $\mathrm{Tr} < g^{-1}dg \wedge [g^{-1}dg \wedge g^{-1}dg] >$ is the bundle associated with the central extension $\widetilde{\mathcal{L}G} \to \mathcal{L}G$ (see example 3.1).*

Next consider the tautological bundle 2-gerbe,

$$\begin{tikzcd}
 & S^1 & & \\
S^3(M) \arrow[ur, "\rho"] \arrow[r, shift left] \arrow[r, shift right] & D^3(M) \arrow[d] & & \\
 & S^2(M) \arrow[r, shift left] \arrow[r, shift right] & D^2(M) \arrow[d] & \\
 & & \mathcal{L}_0M \arrow[r, shift left] \arrow[r, shift right] & \mathcal{P}_0M \arrow[d] \\
 & & & M
\end{tikzcd} \tag{7.8}$$

where M is 3-connected. By the assumption on M, every element of $D^3(M)$ is the boundary of a 4-manifold, W and thus the holonomy function is uniquely determined by the property $H(\partial W) = \exp \int_W \Theta$ where Θ is the 4-curvature. This is exactly the definition of the function ρ, so it is clear that the transgression bundle τ_{S^2} is the bundle over $S^2(M)$ in (7.8) and the transgression bundle gerbe $\tau_{\mathcal{L}M}$ is the bundle gerbe over $\mathcal{L}M$ in (7.8).

7.3 Holonomy Reconstruction

In this section we consider the theory of holonomy reconstruction in the bundle gerbe context.

Let us consider the implications of our connectedness assumptions in the tautological case. This will lead to an understanding of the more general case. Let M be 1-connected so that the tautological bundle 0-gerbe is defined. Then we also have $H_1(M) = 0$. Recall that there is an exact sequence defined by the map of a Deligne class to its curvature,

$$0 \to H^1(M, U(1)) \to H^1(M, \mathcal{D}^1) \to A_0^2(M) \to 0$$

First we use the Universal Coefficient Theorem for cohomology ([2]);

$$H^q(X,G) \cong \mathrm{Hom}(H_q(X),G) \oplus \mathrm{Ext}(H_{q-1},G) \tag{7.9}$$

Since we have $H_1(M) = 0$ and $\mathrm{Ext}(H_0(M),U(1)) = 0$ then

$$H^1(M,U(1)) \cong \mathrm{Hom}(0,U(1)) \cong 0 \tag{7.10}$$

Together with (2.3) this gives the result

$$H^1(M,\mathcal{D}^1) \cong A_0^2(M) \tag{7.11}$$

If we think of this Deligne class as a bundle then this tells us that when $\pi_1(M) = 0$ all bundle gerbes with connection on M are completely determined by their curvature. The tautological construction gives us a bundle with connection over a 1-connected base M which has a particular curvature, so this tells us that the tautological bundle is in fact the unique (up to isomorphism) bundle with connection satisfying these requirements. Rather than constructing a bundle from its curvature we shall construct it from its holonomy function. In the tautological case the holonomy function is completely determined by the curvature, so this approach does make sense as a generalisation of the tautological case.

We now relax the requirement that M be 1-connected. Let us start with a bundle 0-gerbe with connection defined by a class in $H^1(M,\mathcal{D}^1)$. There is now no map ρ, however in the previous section we found that for the tautological bundle ρ is equal to the holonomy, which does exist in the general case. An explicit construction of the Deligne class from the holonomy is described in [31]. We shall construct a bundle 0-gerbe using a similar approach.

We use the tautological construction replacing ρ with the holonomy, giving the following bundle 0-gerbe:

$$\begin{array}{ccc} & & U(1) \\ & \overset{H}{\nearrow} & \\ \mathcal{L}_0 M & \rightrightarrows & \mathcal{P}_0 M \\ & & \downarrow \\ & & M \end{array} \tag{7.12}$$

We define the connection to be $\int_I ev^* F$ as in the tautological case. The inverse of the holonomy map follows from the relationship between transition functions and the flat holonomy class, see the proof of proposition 5.1 for an example of this.

Now we outline the method for finding local expressions of a bundle with connection from [31]. The first step is to define a section over each $U_i \subset M$ which gives a path in U_i for each which ends at $m_i \in U_i$. These are composed with paths p_i which connect them to the base point and so define local sections of the path fibration. The sections over U_i and U_j are then composed and the holonomy is evaluated over the resulting loop. Observe that this gives precisely the transition functions of the bundle 0-gerbe (7.12). In particular note that the inverse is present since the direction of the path is positive on the lift over U_i and negative over U_j. Furthermore it is shown that the resulting bundle is independent of the choice of base point.

Next we show that our connection is the same as that obtained in [31]. Since our connection is defined on the path space we need to pull it back using a local connection for comparison. Let s_i be the local section over U_i and let v be a vector at $x \in U_i$ which may be represented by $q'(0)$ for some curve $q \subset U_i$. The vector $s_{i*}X$ is a vector in the path space which is based at the path $s_i(x)$. It is given by $\frac{d}{dk}s_i(q(k))|_{k=0}$, where for each k the path $s_i(q_k)$ is from m_i, the centre of U_i, to $q(k)$. Strictly speaking the paths actually begin at m_0, but since $s_i(q(k))$ is constant between m_0 and m_i we are only interested in what happens between m_i and $q(k)$. The local form of our connection evaluated at v is $\int_I \iota_{s_{i*v}} ev^*F = \int_I F(s_{i*v}(t), dt)$ where $s_{i*v}(t)$ is the vector on U_i given by the element of the vector field on the path $s_i(x)$ at $s_i(x)(t)$.

In [31] the connection is defined by first taking the holonomy of a particular loop associated with a vector v as above. Omitting the trivial part again, this loop consists of three components, first the lift $s_i(x)$, then the curve q and finally the lift $s_i(q(k))$ in the reverse direction. The local one form evaluated at v is then defined by taking minus the k derivative of log of this holonomy at $k = 0$. Note that the loop around which the holonomy is taken corresponds to the boundary of a surface, Σ_k, defined by the collection of all of the loops $s_i(q(s))$ from 0 to k. This means that we can express this holonomy as an integral of the curvature, $\exp \int_{\Sigma_k} F$. The integral over Σ_k may be parametrised by $I \times [0, k]$ where I parametrises the curves from m_i to $q(s)$ and $[0, k]$ parametrises the curves. We can now calculate the connection:

$$\frac{d}{dk}\int_I \int_0^k F(s_i(q'(s))(t), dt)|_{k=0} = \int_I ev^*F(s_{i*v}(t), dt) \tag{7.13}$$

as required.

Now consider a general bundle 0-gerbe with connection $(g, Y, M; A)$. If we calculate the holonomy and then use it to reconstruct the bundle 0-gerbe then we have a new bundle 0-gerbe $(H, \mathcal{P}_0 M, M; B)$ which is D-stably isomorphic to $(g, Y, M; A)$. This may be shown explicitly by considering the product bundle 0-gerbe

$$\begin{tikzcd} & U(1) \\ Y^{[2]} \times_\pi \mathcal{L}_0 M \arrow[r, shift left] \arrow[r, shift right] \arrow[ur, "g^{-1}H"] & Y \times_\pi \mathcal{P}_0 M \arrow[d] \\ & M \end{tikzcd} \tag{7.14}$$

The transition functions for the bundle 0-gerbe are given by

$$g_{\alpha\beta}^{-1} \cdot \exp \int_{\mu_\alpha \star \mu_\beta^{-1}} (A_i - d\log h_i) \tag{7.15}$$

where μ_α and μ_β are paths given by the local sections of the path fibration at $m \in U_{\alpha\beta}$. Normally we deal with the term in the integral by breaking it down into a sum of edges in a triangulation of the loop. Using the sections of the path space described in [31] we can break down $\mu_\alpha \star \mu_\beta^{-1}$ into a sum of four components. First there is a path from m_0 to m_α, the "centre" of U_α. This is independent of m and so contributes a constant factor K_α to the transition function. Next there is a path, $\tilde{\mu}_\alpha$, from m_α to m. This is contained within U_α so the integral corresponding to this component is $\int_{\tilde{\mu}_\alpha} A_\alpha - d\log h_\alpha$. The remaining components are from m to m_β and from m_β to m_0 and contribute similar terms to give

$$g_{\alpha\beta}^{-1}(m) \cdot K_\alpha K_\beta^{-1} \cdot \exp \int_{\tilde{\mu}_\beta} (-A_\beta + d\log h_\beta) \cdot \exp \int_{\tilde{\mu}_\alpha} (A_\alpha - d\log h_\alpha) \tag{7.16}$$

Applying Stokes' theorem to the h terms gives

$$g_{\alpha\beta}^{-1}(m) \cdot K_\alpha K_\beta^{-1} \cdot \exp\left(\int_{\bar{\mu}_\beta} -A_\beta + \int_{\bar{\mu}_\alpha} A_\alpha\right) \cdot h_\beta^{-1}(m_\beta) h_\beta(m) h_\alpha^{-1}(m) h_\alpha(m_\alpha) \tag{7.17}$$

We may now cancel out $g_{\alpha\beta}^{-1}(m)$ with $h_\beta(m)h_\alpha^{-1}(m)$ and incorporate the other h terms into the constants since they don't depend on m. This leaves

$$\delta\left(K_\alpha^{-1} \cdot \exp\left(\int_{\bar{\mu}_\alpha} -A_\alpha\right)\right) \tag{7.18}$$

proving that the two bundle 0-gerbes are stably isomorphic. To see that this extends to a D-trivialisation it remains to observe that applying $d\log$ gives the pull back of the connection form, $-A + \int_I ev^*F$ by a local section.

Using these results we now have a canonical representative of the D-stable isomorphism class of any bundle 0-gerbe with connection which we shall call the *holonomy representative.* Given any bundle 0-gerbe with connection this is obtained by taking the holonomy and then reconstructing a bundle 0-gerbe.

We now consider the case of bundle gerbes. We assume for now that M is 1-connected. We shall postpone discussion of this requirement until the next section. Given a function representing the holonomy of a bundle gerbe we reconstruct the bundle gerbe in the following way

$$\begin{tikzcd} & U(1) & \\ S^2(M) \arrow[ur, "H"] \arrow[r, shift left] \arrow[r, shift right] & D^2(M) \arrow[d] & \\ & \mathcal{L}_0 M \arrow[r, shift left] \arrow[r, shift right] & \mathcal{P}_0 M \arrow[d] \\ & & M \end{tikzcd} \tag{7.19}$$

Over U_{ij} we have the sections s_i and s_j. Use these to define the pullback bundle 0-gerbe

$$\begin{tikzcd} & U(1) \\ S^2(M)_{ij} \arrow[ur, "H_{ij}"] \arrow[r, shift left] \arrow[r, shift right] & D^2(M)_{ij} \arrow[d] \\ & U_{ij} \end{tikzcd} \tag{7.20}$$

where elements of $D^2(M)_{ij}$ which lie in the fibre over m are surfaces bounded by the 1-cycle $s_i(m) \star s_j(m)^{-1}$. Since U_{ij} is contractible this bundle 0-gerbe is trivial. Let the trivialisation be defined by a function $e_{ij} : D^2(M)_{ij} \to U(1)$ which is a homomorphism with respect to the gluing of surfaces and is equal to H_{ij} for surfaces with no boundary. The function e_{ij} plays the same role as the section σ_{ij} in the usual construction of the transition functions of a bundle gerbe. In a similar way we may define e_{jk} and e_{ik} and form the products of these over U_{ijk}. Consider the product $e_{ij}e_{jk}e_{ik}$. This is defined on the fibre product $D^2(M)_{ij} \times_\pi D^2(M)_{ij} \times_\pi D^2(M)_{ij}$. Consider an a general element of this space. First you glue together a surface with boundary $s_i(m) \star s_j(m)^{-1}$ and one with boundary $s_j(m) \star s_k(m)^{-1}$ This forms a new surface with boundary

$s_i(m) \star s_k(m)^{-1}$. Finally we glue this to another surface with the same boundary, thus giving a surface with no boundary. Using the gluing properties this means that the result of the product is the holonomy around this surface. We claim that this is equivalent to the construction in [31].

The bundle gerbe connection on $D^2(M)$ is $A = \int_\Sigma ev^*\omega$ where $\Sigma \in D^2(M)$. To get a local expression we consider again the bundle 0-gerbe on U_{ij} obtained by pulling back with (s_i, s_j). To get a local formula we pull back using a section $s_{ij} : U_{ij} \to D^2(M)_{ij}$. This section takes $x \in U_{ij}$ to a surface Σ_{ij} with boundary $s_i \star s_j^{-1}$. Given a vector $v \in T_x(M)$, the vector $s_{ij*}v$ is a vector field on $s_{ij}(x)$. Suppose we parameterise Σ_{ij} with s and t. Then the pull back is $\int_{\Sigma_{ij}} \omega(s_{ij*}v(s,t), ds, dt)$. The construction in [31] involves taking the holonomy around the surface Σ_{ij}. We write this as $\int_{X_{ij}} \omega$ where X_{ij} is the 3-manifold bounded by Σ_{ij} which is defined by the family of surfaces given by lifting the curve, q, defining v with s_{ij}. Taking the derivative with respect to k, the parameter giving the endpoint of q, we get

$$\frac{d}{dk}\int_{\Sigma_{ij}}\int_{u=0}^{k} \omega(s_{ij}(q'(u))(s,t), ds, dt) = \int_{\Sigma_{ij}} \omega(s_{ij*}v(s,t), ds, dt) \tag{7.21}$$

and hence our connection is the same as that in [31].

The curving may be dealt with in a similar way. We start with the 2-form on $\mathcal{P}_0 M$, $\int_I ev^*\omega$ and pull this back to U_i with s_i, and then evaluate at a pair of vectors $v = q'(0)$ and $w = r'(0)$. This gives $\int_I \omega(s_{i*v}(t), s_{i*w}(t), dt)$. In [31] the approach is to take the holonomy over a surface Σ_i which is defined in the following way. Locally the vectors v and w are extended to vector fields defined by commuting flows. These flows consist of families of curves, q_m and r_m where $q_m'(0)$ and $r_m'(0)$ give the elements in the respective vector fields at the point m. Thus we have $q_x'(0) = v$ and $r_x'(0) = w$. Furthermore these flows commute, that is, $q_{r(l)}(k) = r_{q(k)}(l)$. The vector $s_{i*}v$ is defined by a path of paths which forms a surface bounded by $s_i(x)$, q_x and $s_i(q_x(k))$. Similarly associated with $s_{i*}w$ is a surface bounded by $s_i(x)$, r_x and $s_i(r_x(l))$. To get a closed surface we make similar constructions at the point $q_{r(l)}(k) = r_{q(k)}(l)$. This gives a surface which is a cone from m_i to the surface l_{q_k, r_l} (to use the notation of [31]) bounded by $q_x, r_x, q_{r(l)}$ and $r_{q(k)}$ with appropriate orientations. This cone defines the surface Σ_i and we denote the enclosed volume X_i. The holonomy over Σ_i may be expressed as $\int_{X_i}\omega$. We may parametrise X_i by $I \times [0,k] \times [0,l]$ where the last two give a parametrisation of the discs in the cross section of the cone and I parametrises the length. Taking the partial derivatives in k and l then gives

$$\frac{\partial}{\partial k \partial l}\int_I \int_0^l \int_0^k \omega(s_i(q_x'(s))(t), s_i(r_x'(u))(t), dt) = \int_I \omega(s_{i*v}(t), s_{i*w}(t), dt) \tag{7.22}$$

Thus our definition of a bundle gerbe reconstructed from holonomy agrees with the definition of a gerbe given in [31].

7.4 Reconstruction via Transgression

The techniques of the previous section do not easily extend to the case of bundle 2-gerbes since the construction of the local data from sections becomes quite complicated. Another problem is that we have only been able to deal with base manifolds M

which are 1-connected. Instead we use transgression formulae to approach holonomy reconstruction.

Recall that in §6.1 we noted that the parallel transport of a bundle 0-gerbe gives a bundle over M which is D-stably isomorphic to the original bundle. This gives us an alternative way of calculating local data using transgression formulae. Let us consider the bundle gerbe case first, assuming for now that M is 1-connected. Given a bundle gerbe on M with corresponding holonomy map $H : S^2(M) \to U(1)$ we wish to show that the following bundle gerbe, with connection $\int_{D^2} ev^*\omega$ and curving $\int_I ev^*\omega$ is D-stably isomorphic to the original one with curvature ω:

$$\begin{array}{ccccc} & & U(1) & & \\ & \overset{H}{\nearrow} & & & \\ S^2(M) & \rightrightarrows & D^2(M) & & \\ & & \downarrow & & \\ & & \mathcal{L}_0 M & \rightrightarrows & \mathcal{P}_0 M \\ & & & & \downarrow \\ & & & & M \end{array}$$

We do this by using transgression to find the local data for this bundle gerbe. First we consider only the transition functions. The transition functions for the bundle 0-gerbe $(H, D^2(M), \mathcal{L}_0 M)$ are those obtained by transgression to the loop space, $G_{(t_0,\rho_0)(t_1,\rho_1)}$. We define functions on $\mathcal{P}_0 M$ by the same formula, and denote these by $h_{(t_i,\rho_i)(t_j,\rho_j)}$ or just h_{ij}. These functions satisfy

$$h_{ij}(\mu_1) h_{ij}^{-1}(\mu_2) = G_{01}(\mu_1 \star \mu_2^{-1}) \tag{7.23}$$

whenever ρ_i and ρ_j agree with ρ_0 and ρ_1 on the boundary. Thus by similar arguments to those used in the bundle 0-gerbe case we may use the open cover on $\mathcal{P}_0 M$ which is induced by the projection to M. Next consider what happens on $\pi^{-1}(U_{\alpha\beta\gamma})$ by calculating $h_{\alpha\beta} h_{\beta\gamma} h_{\alpha\gamma}^{-1}$. Let (ρ_α, t_α) denote any choice of (ρ, t) such that $\rho_\alpha(v) = \alpha$ where v is the endpoint of the path. Then we have

$$\begin{aligned} h_{\alpha\beta} h_{\beta\gamma} h_{\alpha\gamma}^{-1} &= \exp \sum_e \int_e A_{\rho_\alpha(e)\rho_\beta(e)} + A_{\rho_\beta(e)\rho_\gamma(e)} - A_{\rho_\alpha(e)\rho_\gamma(e)} \cdot \prod_{v,e} g^{-1}_{\rho_\alpha(e)\rho_\alpha(v)\rho_\beta(v)} \\ &\qquad g_{\rho_\alpha(e)\rho_\beta(e)\rho_\beta(v)} g^{-1}_{\rho_\beta(e)\rho_\beta(v)\rho_\gamma(v)} g_{\rho_\beta(e)\rho_\gamma(e)\rho_\gamma(v)} g_{\rho_\alpha(e)\rho_\alpha(v)\rho_\gamma(v)} g^{-1}_{\rho_\alpha(e)\rho_\gamma(e)\rho_\gamma(v)}(v) \\ &= \prod_{v,e} g^{-1}_{\rho_\alpha(e)\rho_\beta(e)\rho_\gamma(e)} g^{-1}_{\rho_\alpha(e)\rho_\alpha(v)\rho_\beta(v)} g_{\rho_\alpha(e)\rho_\beta(e)\rho_\beta(v)} g^{-1}_{\rho_\beta(e)\rho_\beta(v)\rho_\gamma(v)} g_{\rho_\beta(e)\rho_\gamma(e)\rho_\gamma(v)} \\ &\qquad\qquad g_{\rho_\alpha(e)\rho_\alpha(v)\rho_\gamma(v)} g^{-1}_{\rho_\alpha(e)\rho_\gamma(e)\rho_\gamma(v)}(v) \\ &= \prod_{v,e} g^{-1}_{\rho_\alpha(v)\rho_\beta(v)\rho_\gamma(v)}(v) \end{aligned} \tag{7.24}$$

where the last line is obtained by repeated application of the cocycle identity on $\underline{g}$. It is not difficult to see that this descends to M, so we have $h_{\alpha\beta}^{-1} h_{\beta\gamma}^{-1} h_{\alpha\gamma}(\mu) = g_{\alpha\beta\gamma}(\pi(\mu))$. We need to show that these are transition functions for the bundle gerbe described above. So far we have

$$G_{01}(\mu_1^{-1} \star \mu_2) = h_{\alpha\beta}^{-1}(\mu_1) h_{\alpha\beta}(\mu_2) \tag{7.25}$$

$$h_{\alpha\beta}^{-1} h_{\beta\gamma}^{-1} h_{\alpha\gamma}(\mu_1) = g_{\alpha\beta\gamma}(\pi(\mu_2)) \tag{7.26}$$

Since we wish to avoid using local sections we shall prove directly that this gives the obstruction to this bundle gerbe being trivial. First suppose that there exists a bundle gerbe trivialisation. This means that there exist functions $q_{\alpha\beta}$ such that

$$G_{01}(\mu_1^{-1} \star \mu_2) = q_{\alpha\beta}^{-1}(\mu_1)q_{\alpha\beta}(\mu_2) \tag{7.27}$$
$$q_{\alpha\beta}q_{\beta\gamma}q_{\alpha\gamma}^{-1}(\mu_1) = 1 \tag{7.28}$$

Consider the functions $h_{\alpha\beta}^{-1}q_{\alpha\beta}$ on $\mathcal{P}_0M$. Since $h_{\alpha\beta}^{-1}q_{\alpha\beta}(\mu_1)h_{\alpha\beta}q_{\alpha\beta}^{-1}(\mu_2) = G_{01}G_{01}^{-1} = 1$ these functions descend to M. On M we have $\delta(h^{-1}q^{-1})_{\alpha\beta\gamma} = \delta(h^{-1})_{\alpha\beta\gamma} = g_{\alpha\beta\gamma}$, thus $\underline{g}$ is a trivial cocycle. Conversely if $\underline{g}$ is trivial then let $g_{\alpha\beta\gamma} = g_{\alpha\beta}g_{\beta\gamma}g_{\alpha\gamma}^{-1}$. On $\mathcal{P}_0M$ the functions $h_{\alpha\beta}^{-1}(\mu)g_{\alpha\beta}(\pi(\mu))$ are globally defined and are transition functions for a bundle which trivialises the bundle defined by $\underline{G}$.

Next we show that the connection may also be reconstructed by this method. We need to show that the connection $\int_{D^2} ev^*\omega$ corresponds to the original connection A. Transgression gives a formula for B_{t_0,ρ_0}, which are local connection 1-forms on $\Omega^1(M)$. These are defined by

$$\pi^*B_0 = \tilde{d}\log h_0 - \int_{D^2} ev^*\omega \tag{7.29}$$

Note that the term $\tilde{d}\log h_0$ is trivial when considering this as a connection on the bundle over $\mathcal{L}_0M$, so we see that the one forms $-B_0$ are local representatives for the connection. Over $\mathcal{P}_0M$ we have 1-forms $-k_\alpha$ induced by the extension of B_0 from loops to paths. These satisfy $\delta(k_\alpha) = B_0$ whenever $\rho_0(v) = \alpha$ where v is the endpoint of the path. The bundle gerbe connection is trivial if these form a connection on $\mathcal{P}_0M$, that is, if $k_\alpha - k_\beta = \tilde{d}\log h_{\alpha\beta}$. If these are not equal then they differ by a 1-form which descends to M which is the local representative $A_{\alpha\beta}$ of the bundle gerbe connection.

In terms of transgression formulae we have

$$\begin{aligned}
k_\alpha(\xi) &= \sum_e \int_e \iota_\xi f_{\rho_\alpha(e)} + \sum_{v,e} -\iota_\xi A_{\rho_\alpha(e)\rho_\alpha(v)} && (7.30)\\
\tilde{d}\log h_{\alpha\beta}(\xi) &= \sum_e \int_e \iota_\xi (f_{\rho_\beta(e)} - f_{\rho_\alpha(e)}) && (7.31)\\
&\quad + \sum_{v,e} \iota_\xi (A_{\rho_\alpha(e)\rho_\alpha(v)} + A_{\rho_\alpha(v)\rho_\beta(v)} - A_{\rho_\beta(e)\rho_\beta(v)}) && (7.32)\\
(\tilde{d}\log h_{\alpha\beta} - k_\beta + k_\alpha)(\xi) &= \sum_{v,e} \iota_\xi A_{\rho_\alpha(v)\rho_\beta(v)} && (7.33)\\
&= A_{\alpha\beta}(\xi) && (7.34)
\end{aligned}$$

Thus we have reconstructed the local representative of the original connection.

The local curving is given by $f - \tilde{d}k_\alpha$, the extent to which the curving fails to be a curvature for the connection k_α.

$$
\begin{aligned}
\tilde{d}k_\alpha(\xi,\nu) &= \tilde{d}(k_\alpha(\nu))(\xi) - \tilde{d}(k_\alpha(\xi))(\nu) - \iota_{[\xi,\nu]}k_\alpha \\
&= \sum_e \int_e d\iota_\xi \iota_\nu f_{\rho(e)} + \iota_\xi d\iota_\nu f_{\rho(e)} - d\iota_\nu \iota_\xi f_{\rho(e)} - \iota_\nu d\iota_\xi f_{\rho(e)} - \iota_{[\xi,\nu]} f_{\rho(e)} \\
&\qquad\qquad + \sum_{v,e} -\iota_\xi d\iota_\nu A_{\rho(e)\rho(v)} + \iota_\nu d\iota_\xi A_{\rho(e)\rho(v)} + \iota_{[\xi,\nu]} A_{\rho(e)\rho(v)} \\
&= \sum_e \int_e d\iota_\xi \iota_\nu f_{\rho(e)} + \mathcal{L}_\xi \iota_\nu f_{\rho(e)} - d\iota_\xi \iota_\nu f_{\rho(e)} - \iota_\nu \mathcal{L}_\xi f_{\rho(e)} + \iota_\nu \iota_\eta \omega - \iota_{[\xi,\nu]} f_{\rho(e)} \\
&\qquad\qquad + \sum_{v,e} -\iota_\nu \iota_\eta f_{\rho(e)} - \mathcal{L}_\xi \iota_\nu A_{\rho(e)\rho(v)} + \iota_\nu \mathcal{L}_\xi A_{\rho(e)\rho(v)} - \iota_\nu \iota_\xi f_{\rho(v)} \\
&\qquad\qquad\qquad + \iota_\nu \iota_\xi f_{\rho(e)} + \iota_{[\xi,\nu]} A_{\rho(e)\rho(v)} \\
&= \sum_e \int_e \iota_\nu \iota_\eta \omega - \sum_{v,e} \iota_\nu \iota_\xi f_{\rho(v)} \\
&= (\int_I ev^*\omega)(\xi,\nu) - \pi^* f_\alpha(\xi,\nu)
\end{aligned}
\tag{7.35}
$$

therefore the local curving is given by f_α as required.

In trying to deal with the case where M is not 1-connected we still have the problem that $D^2(M) \to \mathcal{L}_0 M$ is not well defined. It has been noted [31] that in this case the holonomy map may not be used to reconstruct the bundle gerbe, instead reconstruction is given in terms of a parallel transport structure. We may think of this as equivalent to a transgression line bundle on the loop space. If we start with a bundle gerbe P then there is a transgression bundle L on the loop space regardless of whether M is 1-connected. Using this we define a bundle gerbe

$$
\begin{array}{ccc}
L & & \\
\downarrow & & \\
\mathcal{L}_0 M & \rightrightarrows & \mathcal{P}_0 M \\
 & & \downarrow \\
 & & M
\end{array}
$$

with curving defined once again by $\int_I ev^*\omega$. The same arguments used in the previous case apply to prove that this is equivalent to the original bundle gerbe.

7.5 Reconstruction of Bundle 2-Gerbes

Let P be a bundle 2-gerbe on M, a 2-connected manifold, with holonomy function $H : S^3(M) \to U(1)$ and curvature Θ. We may apply the techniques of the previous section to prove that the original bundle 2-gerbe may be reconstructed from the holonomy using the following diagram:

$$
\begin{tikzcd}
 & S^1 & & \\
S^3(M) \arrow[ur, "H"] \arrow[r, shift left] \arrow[r, shift right] & D^3(M) \arrow[d] & & \\
 & S^2(M) \arrow[r, shift left] \arrow[r, shift right] & D^2(M) \arrow[d] & \\
 & & \mathcal{L}_0M \arrow[r, shift left] \arrow[r, shift right] & \mathcal{P}_0M \arrow[d] \\
 & & & M
\end{tikzcd}
$$

with connection, 2-curving and 3-curving given by

$$A = \int_{D^3} ev^*\Theta \tag{7.36}$$

$$\eta = \int_{D^2} ev^*\Theta \tag{7.37}$$

$$\nu = \int_I ev^*\Theta \tag{7.38}$$

We know from the previous results on reconstruction that the bundle gerbe over $\mathcal{L}_0M$ is that obtained by transgression in §6.4. We recall the local data on $\mathcal{L}_0M$:

$$
\begin{aligned}
G_{012} &= \exp(\sum_{e,b} \int_e -A_{\rho_0(e)\rho_1(e)\rho_2(e)}) \cdot \prod_{v,e,b} g^{-1}_{\rho_0(e)\rho_0(v)\rho_1(v)\rho_2(v)} g_{\rho_0(e)\rho_1(e)\rho_1(v)\rho_2(v)} \\
&\qquad g^{-1}_{\rho_0(e)\rho_1(e)\rho_2(e)\rho_2(v)}(v) \\
B_{01} &= \sum_{e,b} \int_e -\iota_\xi \eta_{\rho_0(e)\rho_1(e)} + \sum_{v,e,b} \iota_\xi (A_{\rho_0(e)\rho_1(e)\rho_1(v)} - A_{\rho_0(e)\rho_0(v)\rho_1(v)}) \\
\zeta_0 &= \sum_{e,b} \int_e -\iota_\nu \iota_\xi \nu_{\rho_0(e)} + \sum_{v,e,b} -\iota_\nu \iota_\xi \eta_{\rho_0(e)\rho_0(v)}
\end{aligned}
$$

We extend these to $\mathcal{P}_0M$ where they are locally defined. As usual they are independent of the choice of ρ up to the choice on the boundary so we may express them in terms of the open cover induced from the base, $(G_{\alpha\beta\gamma}, B_{\alpha\gamma}, \zeta_\alpha)$. The D-trivial bundle 2-gerbe obtained from applying $-D$ to this local data descends to M to give the local data for the bundle 2-gerbe described in the diagram above. We now calculate this data to show that it is the same as that for the original bundle 2-gerbe with holonomy H.

$$\begin{aligned}
G^{-1}_{\beta\gamma\delta}G_{\alpha\gamma\delta}G^{-1}_{\alpha\beta\delta}G_{\alpha\beta\gamma} = \exp(\sum_e \int_e & A_{\rho_\beta(e)\rho_\gamma(e)\rho_\delta(e)} - A_{\rho_\alpha(e)\rho_\gamma(e)\rho_\delta(e)} + A_{\rho_\alpha(e)\rho_\beta(e)\rho_\delta(e)} \\
& - A_{\rho_\alpha(e)\rho_\beta(e)\rho_\gamma(e)}) \cdot \prod g \\
&= \prod_{v,e} g_{\rho_\alpha(e)\rho_\beta(e)\rho_\gamma(e)\rho_\delta(e)} \delta(g_{\rho_\alpha(e)\rho_\alpha(v)\rho_\beta(v)\rho_\gamma(v)} g^{-1}_{\rho_\alpha(e)\rho_\beta(e)\rho_\beta(v)\rho_\gamma(v)} \\
& \qquad g_{\rho_\alpha(e)\rho_\beta(e)\rho_\gamma(e)\rho_\gamma(v)})_{\alpha\beta\gamma\delta} \\
&= \prod_{v,e} g_{\rho_\alpha(v)\rho_\beta(v)\rho_\gamma(v)\rho_\delta(v)} \\
&= g_{\alpha\beta\gamma\delta}
\end{aligned} \tag{7.39}$$

$$\begin{aligned}
(-\tilde{d}\log G_{\alpha\beta\gamma} - \delta(B)_{\alpha\beta\gamma})(\xi) &= \sum_e \iota_\xi dA_{\rho_\alpha(e)\rho_\beta(e)\rho_\gamma(e)} + d\iota_\xi A_{\rho_\alpha(e)\rho_\beta(e)\rho_\gamma(e)} + \sum_{v,e} g \\
& + \sum_e \int_e \iota_\xi \eta_{\rho_\beta(e)\rho_\gamma(e)} - \iota_\xi \eta_{\rho_\alpha(e)\rho_\gamma(e)} + \iota_\xi \eta_{\rho_\alpha(e)\rho_\beta(e)} + \sum_{v,e} A \\
&= \sum_{v,e} \iota_\xi (A_{\rho_\alpha(e)\rho_\beta(e)\rho_\gamma(e)} + d\log g_{\rho_\alpha(e)\rho_\alpha(v)\rho_\beta(v)\rho_\gamma(v)} \\
& \quad - d\log g_{\rho_\alpha(e)\rho_\beta(e)\rho_\beta(v)\rho_\gamma(v)} + d\log g_{\rho_\alpha(e)\rho_\beta(e)\rho_\gamma(e)\rho_\gamma(v)} \\
& \quad - \delta(A_{\rho_\alpha(e)\rho_\beta(e)\rho_\beta(v)} - A_{\rho_\alpha(e)\rho_\alpha(v)\rho_\beta(v)})_{\alpha\beta\gamma}) \\
&= \sum_{v,e} \iota_\xi A_{\rho_\alpha(v)\rho_\beta(v)\rho_\gamma(v)}
\end{aligned} \tag{7.40}$$

$$\begin{aligned}
(\tilde{d}B_{\alpha\beta} - \delta(\zeta)_{\alpha\beta})(\xi,\nu) &= \sum_e \int_e -\iota_\xi d\iota_\nu \eta_{\rho_\alpha(e)\rho_\beta(e)} - d\iota_\xi \iota_\nu \eta_{\rho_\alpha(e)\rho_\beta(e)} + \iota_\nu d\iota_\xi \eta_{\rho_\alpha(e)\rho_\beta(e)} \\
& + d\iota_\nu \iota_\xi \eta_{\rho_\alpha(e)\rho_\beta(e)} + \iota_{[\xi,\nu]}\eta_{\rho_\alpha(e)\rho_\beta(e)} + \sum_{v,e} A \\
& - \sum_e \int_e -\iota_\nu \iota_\xi \nu_{\rho_\beta(e)} - \iota_\nu \iota_\xi \nu_{\rho_\alpha(e)} + \sum_{v,e} \eta \\
&= \sum_{v,e} \iota_\nu \iota_\xi \eta_{\rho_\alpha(e)\rho_\beta(e)} + \iota_\nu \iota_\xi \dot{d} A_{\rho_\alpha(e)\rho_\beta(e)\rho_\beta(v)} - \iota_\nu \iota_\xi \dot{d} A_{\rho_\alpha(e)\rho_\alpha(v)\rho_\beta(v)} \\
& \quad + \iota_\nu \iota_\xi \eta_{\rho_\beta(e)\rho_\beta(v)} - \iota_\nu \iota_\xi \eta_{\rho_\alpha(e)\rho_\alpha(v)} \\
&= \sum_{v,e} \iota_\nu \iota_\xi \eta_{\rho_\alpha(v)\rho_\beta(v)}
\end{aligned} \tag{7.41}$$

$$\begin{aligned}
-\tilde{d}\zeta_\alpha(\xi,\nu,\mu) &= \sum_e \int_e \iota_\mu \iota_\nu \iota_\xi d\nu_{\rho_\alpha(e)} + d\iota_\mu \iota_\nu \iota_\xi \nu_{\rho_\alpha(e)} + \sum_{v,e} + \iota_\nu \iota_\mu \iota_\xi d\eta_{\rho_\alpha(e)\rho_\alpha(v)} \\
&= \sum_e \int_e \iota_\mu \iota_\nu \iota_\xi \Theta + \sum_{v,e} \iota_\mu \iota_\nu \iota_\xi \nu_{\rho_\alpha(v)}
\end{aligned} \tag{7.42}$$

If M is only 1-connected then we may reconstruct the bundle 2-gerbe from the transgression bundle on $S^2(M)$. If M is not 1-connected then the bundle 2-gerbe may only be reconstructed from the transgression bundle gerbe on the loop space. These are both proven using the same calculations as above.

We would like to briefly comment on *Cheeger-Simons Differential Characters* and their relation to holonomy reconstruction. Let $Z_p(M)$ denote the group of smooth singular p-cycles on M. A degree p differential character ([14], [5]) is a $U(1)$-valued homomorphism on $Z_{p-1}(M)$, c, together with a p-form, α, on M which satisfy the condition

$$c(\partial\gamma) = \exp\int_\gamma \alpha \tag{7.43}$$

for $\gamma \in Z_p(M)$. These are classified by Deligne cohomology so there must be a one to one correspondence between degree 3 differential characters and D-stable isomorphism classes of bundle gerbes with connection and curving. The holonomy map satisfies equation (7.43) on p-manifolds which are the boundary of a $(p+1)$-manifold, and the additivity property is similar to the homomorphism property of differential characters. We have seen that unlike differential characters a holonomy map is only sufficient to reconstruct the Deligne class under certain assumptions regarding the topology of the base. The difference appears to correspond to the distinction between smooth mapping spaces and simplicial complexes. Differential characters may be useful in further investigation of holonomy and Deligne cohomology however given that they are not needed for our applications we have not studied this in any further detail.

7.6 Geometric Transgression

In this section we consider some examples where we are able to give a geometric interpretation of transgression. These include lifting bundle gerbes and the bundle 2-gerbe associated to a principal bundle.

We begin with the more general case where rather than a fibre bundle over M we only have a fibration, $Y \to M$.

Proposition 7.2. *Let $Y \to M$ be a fibration with M 1-connected. Then the transgression of a bundle gerbe (P, Y, M) to LM is the bundle 0-gerbe described by the following diagram:*

$$\begin{array}{ccc} & & S^1 \\ & \overset{\mathrm{hol}(P)}{\nearrow} & \\ LY^{[2]} & \rightrightarrows & LY \\ & & \downarrow \\ & & LM \end{array}$$

The function $\mathrm{hol}(P)$ *is evaluated with respect to the bundle gerbe connection A which is also a bundle connection on P. The connection 1-form on LY is given by $\int_{S^1} ev^* f$ where f is the curving 2-form on Y.*

This is a well-defined bundle 0-gerbe since the cocycle condition is satisfied due to the gluing property of holonomy.

Proof. We need to show that the bundle 0-gerbe described above is equivalent to

$$\begin{array}{ccc} & & S^1 \\ & \nearrow & \\ S^2(M) & \rightrightarrows & D^2(M) \\ & & \downarrow \\ & & \mathcal{L}M \end{array}$$

We do this by showing that the following bundle 0-gerbe is trivial:

$$\begin{array}{ccc} & & S^1 \\ & \overset{\Lambda}{\nearrow} & \\ S^2(M) \times_\pi LY^{[2]} & \rightrightarrows & D^2(M) \times LY \\ & & \downarrow \\ & & \mathcal{L}M \end{array}$$

where the map $\Lambda : S^2(M) \times_\pi LY^{[2]} \to S^1$ is defined by

$$\Lambda(\Sigma_1, \Sigma_2, \mu_1, \mu_2) = \mathrm{hol}^{-1}_{\Sigma_1 * \Sigma_2}(P, Y, M) \cdot \mathrm{hol}_{(\mu_1,\mu_2)}(P, Y^{[2]}) \tag{7.44}$$

where $\Sigma_1, \Sigma_2 \in S^2(M)$ and $\mu_1, \mu_2 \in \mathcal{L}Y$. Note that the first factor is a bundle gerbe holonomy and the second is a bundle 0-gerbe holonomy.

Define a trivialisation of Λ by

$$l(\Sigma, \mu) = \exp \int_\Sigma (-f + dk_J) \cdot \mathrm{hol}_\mu(J) \tag{7.45}$$

where $J \to Y_\Sigma$ is a trivialisation of the bundle gerbe P over Σ. The second factor is well defined since μ is a lift of $\gamma = \partial\Sigma$ to Y (this is where the assumption that $Y \to M$ is a fibration is required). This is independent of the choice of trivialisation since the difference is

$$\begin{aligned} \exp \int_\Sigma (dk_J - dk_{J'}) \cdot \mathrm{hol}_\mu(J^* \otimes J') &= \mathrm{hol}^{-1}_{\partial\Sigma}(L) \cdot \mathrm{hol}_\mu(\pi^{-1}L) \\ &= \mathrm{hol}^{-1}_\gamma(L) \cdot \mathrm{hol}_\gamma(L) \end{aligned} \tag{7.46}$$

where $\gamma = \partial\Sigma = \pi(\mu)$ and $\pi^{-1}L = J^* \otimes J'$.

If we let J be a trivialisation over $\Sigma_1 \# \Sigma_2$ with restrictions J_1 and J_2 to Σ_1 and Σ_2 respectively then we have

$$\begin{aligned} l^{-1}(\Sigma_1, \mu_1) l(\Sigma_2, \mu_2) &= \exp \int_{\Sigma_1 \# \Sigma_2} (-f + dk_J) \cdot \mathrm{hol}_{\mu_1}(J_1) \cdot \mathrm{hol}_{\mu_2}(J_2) \\ &= \mathrm{hol}^{-1}_{\Sigma_1 \# \Sigma_2}(P, Y, M) \cdot \mathrm{hol}_{(\mu_1,\mu_2)}(P, Y^{[2]}) \end{aligned} \tag{7.47}$$

We now show that the connection of the theorem is a bundle 0-gerbe connection.

$$\begin{aligned} \delta\Big(\int_{S^1} ev^* f\Big) &= \int_{S^1} ev^* \delta(f) \\ &= \int_{S^1} ev^* F \end{aligned}$$

$$
\begin{aligned}
d\log(\mathrm{hol}(L)) &= d\Big(\int_{S^1} ev^* A\Big) \\
&= \int_{S^1} ev^* F.
\end{aligned}
$$

□

We would like to use this result to examine the holonomy of a lifting bundle gerbe (see §3.3) when M is 1-connected,

$$
\begin{tikzcd}
 & \hat{G} \arrow[d] \\
 & G \\
P_G^{[2]} \arrow[ur, "\rho"] \arrow[r, shift left] \arrow[r, shift right] & P_G \arrow[d] \\
 & M
\end{tikzcd}
$$

If we assume that this bundle gerbe has a connection and curving then immediately we see that the transgression to the loop space is given by the following diagram:

$$
\begin{tikzcd}
 & S^1 \\
\mathcal{L}P_G^{[2]} \arrow[ur, "\mathrm{hol}(\rho^* \hat{G})"] \arrow[r, shift left] \arrow[r, shift right] & \mathcal{L}P_G \arrow[d] \\
 & \mathcal{L}M
\end{tikzcd}
$$

It is tempting to apply the functorial property of holonomy here however the bundle gerbe connection may not be equal to the pullback of the connection on $\hat{G}$ since, in general, this does not give a bundle gerbe connection however there is a 1-form ϵ such that $\phi^* A - \epsilon$ is a connection. Some explicit calculations of such 1-forms are given in [37]. In terms of holonomy we have

$$
\begin{aligned}
\mathrm{hol}(\phi^{-1}\hat{G}; \phi^* A - \epsilon) &= \mathrm{hol}(\phi^{-1}\hat{G}; \phi^* A) \cdot I(\epsilon) \\
&= \phi^* \mathrm{hol}(\hat{G}; A) \cdot I(\epsilon)
\end{aligned}
\tag{7.48}
$$

where $I : \mathcal{L}P^{[2]} \to S^1$ is defined by

$$
I_\gamma(\epsilon) = \exp \int_{S^1} \gamma^* \epsilon \tag{7.49}
$$

This is well defined since ϵ is the difference between two choices of connection and thus descends to $P^{[2]}$. The connection on this bundle 0-gerbe depends on the curving chosen. Unlike the bundle gerbe connection there is no canonical choice.

Now we turn to the case of bundle 2-gerbes.

Proposition 7.3. *The transgression to the mapping space $S^2(M)$ of a bundle 2-gerbe $(P, Y, X, M; A, \eta, \nu)$ such that $Y \to M$ is a fibration with simply connected fibres and*

M is 2-connected is given by the following diagram:

$$\begin{array}{ccc} & & S^1 \\ & \overset{\mathrm{hol}(P;\eta,\nu)}{\nearrow} & \\ S^2(X^{[2]}) & \rightrightarrows & S^2(X) \\ & & \downarrow \\ & & S^2(M) \end{array}$$

The connection is given by $\int_\Sigma ev^*\nu$.

Proof. This is proven using an argument that is similar to that used for proposition 7.2. The stable isomorphism is given by a function $l : D^3(M) \times_\pi S^2(X) \to U(1)$ which is defined by

$$l(\Delta, \Sigma) = \exp \int_\Delta (-\nu + dj_R) \cdot \mathrm{hol}_\Sigma(R) \tag{7.50}$$

where R is a trivialisation of the bundle 2-gerbe over $\Delta \in D^3(M)$, which defines a bundle gerbe over X_Δ and the second term is the holonomy of this bundle gerbe over a closed surface. We require the connected condition on the fibres so that lifts of surfaces to X are well defined. The proof now follows that of Proposition 7.2. $\square$

Now we would like to apply this to the bundle 2-gerbe of a principal G-bundle [44] where G is simply connected. Recall that this bundle 2-gerbe is defined in the following way:

$$\begin{array}{ccc} & & (R,Y) \\ & & \Downarrow \\ & & G \\ & \overset{\rho}{\nearrow} & \\ P_G^{[2]} & \rightrightarrows & P_G \\ & & \downarrow \\ & & M \end{array}$$

where $P_G \to M$ is a principal G-bundle, $\rho : P_G^{[2]} \to G$ is the usual map to the group element which acts on p_2 to give p_1 and (R, Y) is a bundle gerbe over G. On the pullback bundle gerbe $\rho^{-1}R$ the pull back connection may be used as the bundle 2-gerbe connection, however the curving may not be a bundle 2-gerbe 2-curving. It is a result of Stevenson [44] that given a curving on a bundle gerbe over $Y^{[2]}$ it is possible to make it into a bundle 2-gerbe 2-curving. This involves subtraction of $\pi^*\epsilon$ where ϵ is some 2-form on $Y^{[2]}$. We will provide a detailed calculation of such an ϵ for a specific example in §8.2.

Proposition 7.4. *Let $(\rho^{-1}R, \rho^{-1}Y, P_G, M)$ be a bundle 2-gerbe associated with a principal G-bundle $P_G \to M$ over a 2-connected base M and a bundle gerbe with connection and curving $(R, Y, G; \eta, A)$ Then the transgression to $S^2(M)$ is the bundle 0-gerbe given by the following diagram:*

$$\begin{array}{ccc} & & S^1 \\ & \overset{\rho^*\mathrm{hol}(R;A,\eta)\cdot I(\epsilon)}{\nearrow} & \\ S^2(P_G^{[2]}) & \rightrightarrows & S^2(P_G) \\ & & \downarrow \\ & & S^2(M) \end{array} \tag{7.51}$$

where $I(\epsilon)$ is the S^1-valued function on $S^2(P_G^{[2]})$ defined by $I(\epsilon)(\psi) = \exp \int \psi^* \epsilon$.

Proof. This follows easily from the previous discussion and the bundle gerbe case. $\square$

7.7 Gauge Transformations

We would like to define gauge transformations for bundle gerbes and consider their effect on holonomy. Since a gauge transformation of a bundle gerbe is basically a stable isomorphism of a bundle gerbe with itself it is no surprise that it turns out to be invariant under holonomy however it is of interest to see how this invariance arises in the bundle gerbe context. This will be of interest in subsequent applications of bundle gerbe theory.

Bundles and Bundle 0-Gerbes

We recall some basic facts about gauge transformations of $U(1)$-bundles.

Definition 7.1. A *gauge transformation* of a principal G-Bundle is an automorphism of the total space which covers the identity on the base space.

The automorphism property guarantees that a gauge transformation preserves fibres, hence for any gauge transformation $\phi : P \to P$ we have a map $g_\phi : P \to G$ defined by

$$\phi(p) = pg_\phi(p).$$

Since $\phi(pg) = \phi(p)g$ then we have $g_\phi(pg) = g^{-1}g_\phi(p)g$. We are interested in the case where $G = U(1)$, so this becomes $g_\phi(pg) = g_\phi(p)$, that is, g_ϕ is constant on fibres, so it induces a map $\hat{g}_\phi : M \to G$.

Suppose we have a connection A on the $U(1)$-bundle $P \to M$. Pulling back by the gauge transformation ϕ gives

$$\phi^* A = A + g_\phi^{-1} dg_\phi. \tag{7.52}$$

To generalise to the bundle gerbe case we first consider bundle 0-gerbes. Recall that to each bundle $P \to M$ we can associate the following bundle 0-gerbe which has the same Deligne class:

$$\begin{array}{ccc} & & U(1) \\ & \overset{\rho}{\nearrow} & \\ P^{[2]} & \rightrightarrows & P \\ & & \downarrow \\ & & M \end{array} \tag{7.53}$$

where the map $\rho : P^{[2]} \to S^1$ is defined by $\rho(p_1, p_2) = g$ where $p_2 = p_1 g$. Equivalently we can identify $P^{[2]}$ with $P \times S^1$ in which case ρ is simply the identity map on S^1.

Given a general bundle 0-gerbe (λ, Y, M) a gauge transformation should obviously be a map $\phi : Y \to Y$ such that $\pi \circ \phi = \pi$ however the condition $\phi(pg) = \phi(p)g$ cannot be used here as in general we don't have a group action on Y. Instead we need a condition on the map $\phi^{[2]} : Y^{[2]} \to Y^{[2]}$ and $\lambda : Y^{[2]} \to S^1$. Consider again the bundle 0-gerbe (ρ, P, M). Applying $\phi^{[2]}$ to $P^{[2]}$ we have

$$\begin{aligned}
\rho(\phi^{[2]}(p_1, p_2)) &= \rho(\phi(p_1), \phi(p_2)) \\
&= \rho(\phi(p_1), \phi(p_1\rho(p_1, p_2))) \\
&= \rho(\phi(p_1), \phi(p_1)\rho(p_1, p_2)) \\
&= \rho(p_1, p_2)
\end{aligned}$$

This suggests the following

Definition 7.2. Let (λ, Y, M) be a bundle 0-gerbe. A *gauge transformation* of (λ, Y, M) is a smooth map $\phi : Y \to Y$ which satisfies the following conditions:

$$\pi \circ \phi = \pi \tag{7.54}$$

$$\lambda \circ \phi^{[2]} = \lambda \tag{7.55}$$

Let A be a bundle 0-gerbe connection 1-form on Y. This means that A satisfies the equation $\delta(A) = d \log \lambda$. The gauge transformation ϕ may be used to construct a map $(1, \phi) : Y \to Y^{[2]}$. Use this map to pull back $\delta(A)$,

$$\begin{aligned}
(1, \phi)^*\delta(A) &= (1, \phi)^*(\pi_2^* - \pi_1^*)A \\
&= (\pi_2 \circ (1, \phi))^*A - (\pi_1 \circ (1, \phi))^*A \\
&= \phi^*A - A
\end{aligned} \tag{7.56}$$

thus we have

$$\phi^*A = A + (1, \phi)^*\delta(A) = A + (1, \phi)^*d \log \lambda \tag{7.57}$$

We would like to compare this result with equation (7.52). By definition $\rho(p, \phi(p)) = g_\phi(p)$, so this immediately shows the equivalence of the two equations.
The function $(1, \phi)$ may also be used to define an S^1-function $(1, \phi)^*\lambda$ on Y.

$$\begin{aligned}
\delta((1, \phi)^*\lambda)(y_1, y_2) &= \lambda^{-1}(y_1, \phi(y_1))\lambda(y_2, \phi(y_2)) \\
&= \lambda(\phi(y_1), y_1)\lambda(y_2, \phi(y_2)) \\
&= \lambda(\phi(y_1), \phi(y_2))\lambda(\phi(y_2), y_1)\lambda(y_2, \phi(y_2)) \\
&= \lambda(\phi(y_1), \phi(y_2))\lambda(y_2, y_1) \\
&= \lambda(y_1, y_2)\lambda^{-1}(y_1, y_2) \\
&= 1
\end{aligned} \tag{7.58}$$

therefore the function $(1, \phi)^*\lambda$ descends to M. Note that once again this function has similar properties to g_ϕ. We shall denote the function on M by λ_ϕ.

Finally we calculate the holonomy of the bundle 0-gerbe (λ, Y, M) with respect to the transformed connection ϕ^*A. To do this we need the D-obstruction form. In general for bundle 0-gerbes this is given by $A_\alpha - d \log h_\alpha$ where A_α are the local connection forms and h_α is a trivialisation. In this case to get the connection we use a local section of $Y \to M$ to pull back the right hand side of (7.57) giving

$$s_\alpha^*(A + \pi^*d \log \lambda_\phi) = A_\alpha + d \log \lambda_\phi \tag{7.59}$$

The holonomy is then given by

$$\begin{aligned} H((\underline{g}, \underline{\phi^* A}); \gamma) &= \exp \int_\gamma A_\alpha - d\log h_\alpha + d\log \lambda_\phi \\ &= H((\underline{g}, \underline{A}); \gamma) \cdot \exp \int_\gamma d\log \lambda_\phi \\ &= H((\underline{g}, \underline{A}); \gamma) \end{aligned} \tag{7.60}$$

So the gauge transformation leaves the holonomy unchanged. This result is not so surprising if we consider that the difference of the Deligne classes $(\underline{g}, \underline{A})$ and $(\underline{g}, \underline{\phi^* A})$ is $\underline{(1, d\log \lambda_\phi)} = D(\underline{\lambda_\phi})$.

Now we consider parallel transport. If μ is an open path in M then recall that the holonomy function now depends on the choice of trivialisation,

$$\begin{aligned} H((\underline{h}, \underline{\phi^* A}); \mu) &= \exp \int_\mu A_\alpha - d\log h_\alpha + d\log \lambda_\phi \\ &= H((\underline{h}, \underline{A}); \mu) \cdot \lambda_\phi^{-1}(\mu(0))\lambda_\phi(\mu(1)) \end{aligned} \tag{7.61}$$

Thus the gauge transformation contributes an extra term to the Deligne cochain on $\mathrm{Map}(I; M)$ obtained by transgression. If we apply D to this cochain to get a bundle then the extra term will cancel out as it doesn't depends on the choice of trivialisation. The local connections on this bundle will pick up an extra term $\iota_{\xi(\mu(1))} d\log \lambda_\phi - \iota_{\xi(\mu(0))} d\log \lambda_\phi$, however, just as on the original bundle this difference is D-trivial.

Bundle Gerbes and Bundle 2-gerbes

We shall extend the concept of gauge transformation to bundle gerbes and bundle 2-gerbes. We shall refer to these collectively as bundle n-gerbes where it is to be understood that $n = 0, 1$ or 2.

Definition 7.3. Let (P, Y, M) be a bundle gerbe. A *gauge transformation* of (P, Y, M) is a smooth map $\phi : Y \to Y$ which satisfies the following conditions:

$$\pi \circ \phi = \pi \tag{7.62}$$

$$\phi^{[2]*}(P, Y^{[2]}) = (P, Y^{[2]}) \tag{7.63}$$

with the second condition involving a choice of isomorphism of bundles, $\tilde{\phi}$, over $Y^{[2]}$ which preserves the bundle gerbe product.

Similarly,

Definition 7.4. Let (P, Y, X, M) be a bundle 2-gerbe. A *gauge transformation* of (P, Y, X, M) is a smooth map $\phi : X \to X$ which satisfies the following conditions:

$$\pi \circ \phi = \pi \tag{7.64}$$

$$\phi^{[2]*}(P, Y, X^{[2]}) = (P, Y, X^{[2]}) \tag{7.65}$$

with the second condition being a choice of stable isomorphism of bundle gerbes, $\tilde{\phi}$ over $X^{[2]}$, which preserves the bundle 2-gerbe product.

The next step is too see how the various connections and curvings transform. We begin with the 2-curving for a bundle gerbe. This situation is similar to that of the bundle 0-gerbe connection. If we denote the curving by η then following (7.56) we have

$$(1,\phi)^*\delta(\eta) = \phi^*\eta - \eta \tag{7.66}$$

so

$$\phi^*\eta = \eta + (1,\phi)^*F \tag{7.67}$$

where F is the curvature of the bundle $P \to Y^{[2]}$. Next we consider the bundle $(1,\phi)^{-1}P$ over Y.

$$\begin{aligned}\delta((1,\phi)^{-1}P)_{(y_1,y_2)} &= P^*_{(y_1,\phi(y_1))} \otimes P_{(y_2,\phi(y_2))} \\ &= P_{(\phi(y_1),\phi(y_2))} \otimes P_{(\phi(y_2),y_1)} \otimes P_{(y_2,\phi(y_2))} \\ &= P_{(\phi(y_1),\phi(y_2))} \otimes P^*_{(y_1,y_2)}\end{aligned} \tag{7.68}$$

Thus we see that $(1,\phi)^{-1}P$ is a trivialisation of the bundle gerbe $P^* \otimes \phi^{[2]*}P$. More importantly observe that we have an isomorphism of bundles,

$$\begin{aligned}\delta((1,\phi)^{-1}P) &= \phi^{[2]*}P \otimes P^* \\ &= P \otimes P^*\end{aligned} \tag{7.69}$$

therefore $(1,\phi)^{-1}P$ descends to a bundle on M which we shall call P_ϕ.
There is a connection $\nabla_{(1,\phi)^{-1}P}$ on $(1,\phi)^{-1}P$ which satisfies

$$\delta(\nabla_{(1,\phi)^{-1}P}) = \nabla_{\phi^{[2]*}P} \otimes \nabla^*_P \tag{7.70}$$

On the right hand sides we have two choices of connection on isomorphic bundles, so they differ by a 1-form α on $Y^{[2]}$ such that $d\alpha = \phi^{[2]*}F - F$. Over Y we may compare the connections on $\pi^{-1}P_\phi$ and $(1,\phi)^{-1}P$. These differ by a 1-form β on Y. Furthermore since $\delta(\nabla_{\pi^{-1}P_\phi}) = 0$ we have $\delta(\beta) = \alpha$. The curvatures of these two bundles are related by $\pi^*F_\phi + d\beta = (1,\phi)^*F$. We may now express (7.67) in terms of F_ϕ,

$$\phi^*\eta = \eta + \pi^*F_\phi + d\beta \tag{7.71}$$

We also have the relationship between the connections, where ϕ^*A refers to the induced connection on $\phi^{[2]*}P$,

$$\phi^*A = A + \pi_P^*\delta(\beta) \tag{7.72}$$

We may interpret these results in terms of D-obstructions. The bundle gerbe $\phi^{[2]*}P\otimes P$ has a trivialisation $(1,\phi)^{-1}P$ which is not necessarily a D-trivialisation. The obstruction is given by a 2-form χ on M such that locally $\pi^*\chi = \eta - F_L$ where F_L is the curvature of the connection on the trivialisation. In this case $F_L = d\beta$, so the D-obstruction form is F_ϕ. This means that the D-obstruction form is trivial and so the two bundle gerbes $\phi^{[2]*}P$ and P are D-stably isomorphic. Thus we expect the holonomy to be invariant, as we shall see from explicit calculations.

We can now work out the holonomy corresponding to the new connection and curving by considering the local formula. We may assume without loss of generality that the transition functions are equal, since introducing an extra factor from the stable isomorphism will be cancelled by an additional term in the connection.

Substituting $\phi^*\eta$ and ϕ^*A into the local formula for the holonomy of a bundle gerbe over a closed surface Σ gives

$$\begin{aligned}
H(\phi^{-1}P;\phi^*A,\phi^*\eta) &= H(p;A,\eta)\cdot\exp\left(\sum_b\int_b F_\phi + d\beta_{\rho(b)}\sum_{e,b}\int_e \beta_{\rho(e)}-\beta_{\rho(b)}\right)\\
&= H(p;A,\eta)\cdot\exp\sum_b\int_b F_\phi \qquad (7.73)\\
&= H(p;A,\eta)\cdot\exp\int_\Sigma F_\phi
\end{aligned}$$

where the β terms cancel due to Stokes' theorem and the usual combinatorial arguments. Since F_ϕ is a curvature we have

$$H(\phi^{-1}P;\phi^*A,\phi^*\eta) = H(p;A,\eta) \tag{7.74}$$

so the holonomy is an invariant of the gauge transformation of a bundle gerbe.

In the case where Σ has boundary, there are extra terms in the function H on the space of trivialisations,

$$\exp\int_\Sigma F_\phi + \sum_{e,b}\int_e \beta_{\rho(e)} \tag{7.75}$$

These are independent of the choice of trivialisation so they do not affect the transition functions on the bundle over $\partial\Sigma$. The local connections of this bundle gain an extra term $\int_{\partial\Sigma}\iota_\xi F_\phi$ which is D-trivial.

For bundle 2-gerbes the situation is very similar. Let $(P,Y,X,M;A,\eta,\nu)$ be a bundle 2-gerbe and let ϕ be a gauge transformation. There is an isomorphism of bundle 2-gerbes

$$\phi^{[2]*}P\otimes P^* = \delta((1,\phi)^{-1}P) \tag{7.76}$$

by the same arguments as in the lower cases. We may give the trivialisation a connection and curving which are compatible with δ. The 3-curvings satisfy

$$\phi^*\nu = \nu + (1,\phi)^*\omega \tag{7.77}$$

where ω is the three curvature of the bundle gerbe $(P,Y,X^{[2]})$ which satisfies $\omega = \delta(\nu)$. By similar arguments to the bundle gerbe case above, the bundle gerbe $(1,\phi)^{-1}P$ descends to a bundle gerbe P_ϕ on M with 3-curvature ω_ϕ which satisfies

$$(1,\phi)^*\omega = \pi^*\omega_\phi + d\beta \tag{7.78}$$

where β is the 2-curving of the trivialisation. Thus we see that ω_ϕ is the D-obstruction. Since it is a curvature then the D-obstruction is trivial and the holonomy of bundle 2-gerbes is invariant under transgression, though as in the previous terms there will be a different choice of section of the bundle on the mapping space in the case with boundary.

G-Gauge Transformations

We have seen examples of bundle gerbes and bundle 2-gerbes, namely lifting bundle gerbes and the bundle 2-gerbe associated with a principal G-bundle, which have the following general form:

$$\begin{array}{ccc} & & Q \\ & & \downarrow / \Downarrow \\ & & G \\ & \overset{\rho}{\nearrow} & \\ P_G \times G & \rightrightarrows & P_G \\ & & \downarrow \\ & & M \end{array} \tag{7.79}$$

where $\downarrow / \Downarrow$ indicates that Q may be a bundle or bundle gerbe and ρ is projection onto the second factor of $P \times G$ which may also be thought of as the map $P^{[2]} \to G$ defined by $p_2 = p_1\rho(p_1, p_2)$. We shall refer to a gauge transformation of the G-bundle $P \to M$ as a *G-gauge transformation.*

Proposition 7.5. *Let Q be a bundle (2-)gerbe as described above. A G-gauge transformation of P defines a gauge transformation of Q.*

Proof. Let $\psi : P \to P$ be a G-gauge transformation. By definition we have $\pi \circ \psi = \pi$ so we need only verify that there is an isomorphism of line bundles (or stable isomorphism of bundle gerbes) $\psi^{[2]*}(\rho^{-1}Q) = \rho^{-1}Q$. On $P^{[2]}$ we have $\psi^{[2]}(p_1, p_2) = \psi^{[2]}(p_1, p_1g_{12}) = (\psi(p_1), \psi(p_1g_{12})) = (\psi(p_1), \psi(p_1)g_{12})$. So on $P \times G$ we have $\psi^{[2]}(p, g) = (\psi(p), g)$ and $\rho(\psi^{[2]}(p, g)) = g = \rho(p, g)$, therefore $\psi^{[2]*}(\rho^{-1}Q) = \rho^{-1}Q$ as required. $\square$

Chapter 8

Applications

We consider some applications of the various constructions in bundle gerbe theory which we have discussed to physics.

8.1 The Wess-Zumino-Witten Action

We shall review the bundle gerbe model of the Wess-Zumino-Witten (WZW) action as described in [12]. This example serves as motivation for the use of bundle gerbe holonomy to study topological actions. Furthermore the WZW theory plays a role in the discussion of Chern-Simons theory which follows.

Following [12] the WZW action is defined as a function on the space of maps from a Riemann surface Σ to a compact Lie group G, which we denote by ΣG. This function is defined by the equation

$$WZW(\psi) = \exp \int_X \hat{\psi}^* \omega \tag{8.1}$$

where X is a 3-manifold with boundary Σ, $\hat{\psi}$ is an extension of $\psi \in \Sigma M$ to XM and ω is a closed 3-form which generates the integral cohomology of G. This is well defined as long as such a $\hat{\psi}$ exists, for example if G is simply connected. In this case (8.1) is the holonomy of the tautological bundle gerbe with curvature ω. When such a $\hat{\psi}$ does not exist we may replace (8.1) with the holonomy of any bundle gerbe with curvature ω, though, as is observed in [15] where similar constructions are made using differential characters, this bundle gerbe is not uniquely determined by ω. Any two choices differ by a flat bundle gerbe which is classified by $H^2(G, U(1))$. To eliminate this ambiguity the action must be defined in terms of a full Deligne class rather than just the Dixmier-Douady class. This leads to

Definition 8.1. Let $\alpha \in H^2(G, \mathcal{D}^2)$ be a Deligne class. The WZW action evaluated on a map $\psi : \Sigma \to G$ is the holonomy of the class α, that is, the flat holonomy of $\psi^*\alpha$.

If we represent α by a bundle gerbe with connection and curving $(P, Y, G; A, \eta)$ then the action may be written as [12]

$$\exp \int_\Sigma \psi^* \eta - F_L$$

where F_L is the curvature of a trivialisation of ψ^*P. In this way each bundle gerbe with connection and curving over G defines a WZW action. This general form of the WZW action was given in terms of a transgression formula by Gawedski [23].

From [20] and [23] we see that when we attempt to define this action for surfaces with boundary we need to consider line bundles over the boundary maps, just as with holonomy. Recall that we start with the holonomy function on ΣG, which for each $\phi \in \Sigma G$ may be thought of as the evaluation of the corresponding flat holonomy class χ. This function is extended to surfaces with boundary in such a way that it is multiplicative with respect to unions so that given two surfaces with the same boundary the product is equal to the holonomy of the combined surface. It turns out that such a function can in general only be defined locally on $\Sigma^\partial G$. These local functions are used to define a trivial bundle with connection which pulls back to $\partial\Sigma G$ to give a possibly non-trivial bundle with connection. The local data corresponding to this line bundle as derived in Chapter 6 agrees with the formulae given by Gawedski [23].

In the case where G is simply connected then the theory of transgression of tautological bundle gerbes tells us that the bundle on the loop space is just the bundle over $\mathcal{L}_0 G$ in the definition of the bundle gerbe. This is of the form

$$
\begin{array}{ccc}
 & & S^1 \\
 & \overset{\rho}{\nearrow} & \\
S^2(G) & \rightrightarrows & D^2(G) \\
 & & \downarrow \\
 & & \mathcal{L}_0 G
\end{array}
$$

Furthermore if G has an integral bilinear form $< .,. >$ then we may write $\omega = -\frac{1}{6} < \theta \wedge [\theta \wedge \theta] >$ where θ is the Maurer-Cartan form and where we have used the same normalisation as [20]. When G is simple we have $H^3(G, \mathbb{Z}) = \mathbb{Z}$ and it is generated by this ω. We know the connection and curvature of L since these come from the corresponding objects on the tautological bundle gerbe. The connection on $D^2(G)$ is $\int_D ev^*\omega$ and the curvature is $\int_{S^1} ev^*\omega$. We would like to get more concrete expressions of a similar nature to those in [20]. To do this recall that for the Maurer-Cartan form, θ, we have $\iota_\xi \theta = \xi$. Consider the connection evaluated at $\Xi \in T_\phi(D^2(G))$,

$$
\begin{aligned}
\int_D \iota_\Xi \omega &= -\frac{1}{6}\int_D \iota_\Xi < \phi^{-1}d\phi \wedge [\phi^{-1}d\phi \wedge \phi^{-1}d\phi] > \\
&= -\frac{1}{2}\int_D < \Xi \wedge [\phi^{-1}d\phi \wedge \phi^{-1}d\phi] >
\end{aligned}
\tag{8.2}
$$

For the curvature evaluated at vectors $\xi_1, \xi_2 \in T_\gamma(\mathcal{L}_0 G)$ we have

$$
\begin{aligned}
\int_{S^1} \iota_{\xi_1}\iota_{\xi_2}\omega &= -\frac{1}{6}\int_{S^1} \iota_{\xi_1}\iota_{\xi_2} < \gamma^{-1}d\gamma \wedge [\gamma^{-1}d\gamma \wedge \gamma^{-1}d\gamma] > \\
&= -\frac{1}{2}\int_{S^1} \iota_{\xi_1} < \xi_2 \wedge [\gamma^{-1}d\gamma \wedge \gamma^{-1}d\gamma] > \\
&= -\int_{S^1} < [\xi_1, \xi_2] \wedge \gamma^{-1}d\gamma >
\end{aligned}
\tag{8.3}
$$

Proposition 7.1 implies that the hermitian lines over the loop space defined by the transgression of the WZW bundle gerbe give the standard central extension of the loop group [20].

8.2 The Chern-Simons Action

Our description of basic Chern-Simons (CS) theory follows Freed [20] and Dijkgraaf-Witten [15]. We show that there is a bundle gerbe interpretation of the cases they deal with and see that it is useful for generalisation to more general theories. We show that the bundle gerbe CS theory reproduces the expected results when restricted to specific cases (usually relying on restriction of the group G such that it satisfies particular properties).

Let G be a compact Lie group and X an oriented 3-manifold. Let $P_G \to X$ be a principal G-bundle with connection 1-form A. Define a 3-form on P_G, called the *Chern-Simons form* by

$$CS(A) = \mathrm{Tr}\,(A \wedge dA) + \frac{2}{3}\mathrm{Tr}\,(A \wedge A \wedge A) \tag{8.4}$$

If the bundle $P_G \to X$ is trivial, with section s, then the *Chern-Simons action* associated a 3-manifold X, is defined by

$$\exp \int_X s^* CS(A) \tag{8.5}$$

Ideally this should be independent of the choice of section. A change of section is given by a gauge transformation $\phi : P_G \to P_G$, or alternatively $g_\phi : P_G \to G$. Under such a gauge transformation the CS form transforms as follows,

$$\phi^* CS(A) = CS(A) + d\mathrm{Tr}\,(g_\phi^{-1} A g_\phi \wedge g_\phi^{-1} dg_\phi) - \frac{1}{3}\mathrm{Tr}\,(g_\phi^{-1} dg_\phi)^3 \tag{8.6}$$

This suggests that for the action to be independent of the choice of section we should require that the trace be normalised to make $\frac{1}{3}\mathrm{Tr}\,(g_\phi^{-1} dg_\phi)^3$ a 2π-integral form.

If A extends over a 4-manifold W such that $\partial W = X$ then the action is

$$\exp \int_W \mathrm{Tr}\,(F \wedge F) \tag{8.7}$$

Note that if the bundle is non-trivial then this definition of the action still makes sense as long as the bundle and connection extend over W.

This situation very closely parallels the problem of defining the WZW action for general G (§8.1) and the problem of generalising the tautological bundle gerbe to get holonomy reconstruction (§7.3). This suggests that a general definition of the CS action may be obtained by considering it as the holonomy of a bundle 2-gerbe. Furthermore the dependence on a principal G-bundle with connection suggests that we are interested in particular in a bundle 2-gerbe associated with a G-bundle.

The Chern-Simons Bundle 2-Gerbe

This construction is based on the bundle 2-gerbe associated with the principal bundle P_G. Thus we require that G is connected, simply connected and simple. A 2-gerbe of a similar nature was described in [7]. Our basic geometric structure is given in the following diagram

$$\begin{array}{ccc} \rho^{-1}Q[\omega] & & \\ \Downarrow & & \\ P_G^{[2]} & \rightrightarrows & P_G \\ & & \downarrow \\ & & X \end{array}$$

where $\rho : P_G^{[2]} \to G$ satisfies $p\rho(p,q) = q$ and $Q[\omega]$ is a tautological bundle gerbe associated with a curvature 3-form ω on G. Following [7] we let $\omega = k\mathrm{Tr}\,(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg)$. Here the trace replaces the more general bilinear form we considered in the case of WZW theory. In this case we can set $\beta = 3k\mathrm{Tr}\,(g_1^{-1}dg_1 \wedge dg_2g_2^{-1})$ and we have $\delta(\omega) = d\beta$ and $\delta(\beta) = 0$. Consider the following diagram:

$$\begin{array}{ccc} P_G^{[3]} & \to & G \times G \\ \downarrow & & \downarrow \\ P_G^{[2]} & \to & G \end{array}$$

We have $\omega \in \Omega^3(G)$ and $\beta \in \Omega^2(G \times G)$. We can pull these back to $\rho^*\omega \in \Omega^3(P_G^{[2]})$ and $\rho^*\beta \in \Omega^2(P_G^{[3]})$. Now we have $\delta\rho^*\beta = \rho^*\delta\beta = 0$ therefore there exists $\epsilon \in \Omega^2(P_G[2])$ such that $\delta\epsilon = \rho^*\beta$. On $\Omega^3(P_G^{[2]})$ we have the equation

$$\delta(\rho^*\omega - d\epsilon) = \rho^*(\delta\omega - d\beta) = 0$$

so we may define $\alpha \in \Omega^3(P_G)$ such that

$$\delta\alpha = \rho^*\omega - d\epsilon.$$

When we pull back the tautological bundle gerbe by ρ to $P_G^{[2]}$, the curvature pulls back to $\rho^*\omega$, however this is not adequate as the 3-curvature on our bundle 2-gerbe (meaning the 3-curvature of the bundle gerbe $\rho^{-1}Q[\omega]$) since this is not δ-exact for $\delta : P_G \to P_G^{[2]}$. As we have shown above, subtraction of $d\epsilon$ will result in δ-exactness. This is a specific example of the general ϵ referred to in proposition 7.4. The 2-curving of the bundle gerbe is given by

$$\begin{aligned} d\tilde{\eta} &= \pi^*\rho^*\omega - \pi^*d\epsilon \\ &= \rho^*\pi^*\omega - d\pi^*\epsilon \\ &= \rho^*d\eta - d\pi^*\epsilon \end{aligned}$$

where η is the curving of the tautological bundle gerbe. So let $\tilde{\eta} = \eta - \pi^*\epsilon$.

Now we will find solutions for ϵ and α. To do this we will identify $P_G \times G$ and $P_G^{[2]}$ via the map $(p,g) \mapsto (p,pg)$. Similarly we have a map from $P_G \times G \times G$ to $P_G^{[3]}$ given by $(p,g_1,g_2) \mapsto (p,pg_1,pg_1g_2)$. This will change the δ maps. We want the following diagram to commute

$$\begin{array}{ccc} P_G^{[3]} & \to & P_G \times G \times G \\ \downarrow & & \downarrow \\ P_G^{[2]} & \to & P_G \times G \end{array}$$

For each map π_i of δ we will have a diagram which shows what the induced map from $P_G \times G \times G$ to $P_G \times G$ should be. For π_0 we have

$$\begin{array}{ccc} (p,pg_1,pg_1g_2) & \to & (p,g_1,g_2) \\ \downarrow & & \downarrow \\ (pg_1,pg_1g_2) & \to & (pg_1,g_2) \end{array}$$

For π_1,

$$\begin{array}{ccc} (p,pg_1,pg_1g_2) & \to & (p,g_1,g_2) \\ \downarrow & & \downarrow \\ (p,pg_1g_2) & \to & (p,g_1g_2) \end{array}$$

For π_2,

$$\begin{array}{ccc} (p, pg_1, pg_1g_2) & \to & (p, g_1, g_2) \\ \downarrow & & \downarrow \\ (p, pg_1) & \to & (p, g_1) \end{array}$$

These diagrams give us the following equations:

$$\begin{aligned} \pi_0(p, g_1, g_2) &= (pg_1, g_2) \\ \pi_1(p, g_1, g_2) &= (p, g_1g_2) \\ \pi_2(p, g_1, g_2) &= (p, g_1) \end{aligned}$$

Lemma 8.1. *Let $\epsilon \in \Omega^2(P_G \times G)$ be defined by $\epsilon = 3k\mathrm{Tr}\,(A \wedge dgg^{-1})$ where A is a connection for $P_G \to X$. Then $\delta\epsilon = \beta$ and thus $\delta(\rho^*\omega - d\epsilon) = 0$.*

Proof. We will omit $3k$ since it appears in all expressions. We need to evaluate $\delta\epsilon = \pi_0^*\epsilon - \pi_1^*\epsilon + \pi_2^*\epsilon$. There are three types of pullback map that we will need. Let $(Z, \xi) \in TP_G \times TG$. Then $p_*Z = Z$ and $g_*\xi = \xi$. This leaves pg_*Z. Applying the chain rule gives the result $pg_*Z = R_{g*}Z + g^{\#}$ where R_g is the right action of g and $g^{\#}$ is the fundamental field of g. We can now use the two defining properties of A, which are $R_{g*}A = g^{-1}Ag$ and $A(g^{\#}) = g$. Now we can write

$$\begin{aligned} \pi_0^*\epsilon &= \mathrm{Tr}\,((g_1^{-1}Ag_1 - g_1^{-1}dg_1) \wedge dg_2g_2^{-1}) \\ \pi_1^*\epsilon &= \mathrm{Tr}\,(A \wedge d(g_1g_2)(g_1g_2)^{-1}) \\ &= \mathrm{Tr}\,(A \wedge (dg_1g_1^{-1} + g_1dg_2g_2^{-1}g_1^{-1})) \\ \pi_2^*\epsilon &= \mathrm{Tr}\,(A \wedge dg_1g_1^{-1}) \end{aligned}$$

Putting these together we have

$$\delta\epsilon = \mathrm{Tr}\,(g_1^{-1}dg_1 \wedge dg_2g_2^{-1})$$

where Ad invariance of the trace has been used to eliminate the other terms. □

Now we can write down an expression for $\rho^*\omega - d\epsilon$. The map $\rho : P_G \times G \to G$ is defined by $(p, g) \mapsto g$, so $\rho^*\omega = \omega$. Applying d to ϵ yields

$$\begin{aligned} d\mathrm{Tr}\,(A \wedge dgg^{-1}) &= \mathrm{Tr}\,(dA \wedge dgg^{-1}) - \mathrm{Tr}\,(A \wedge d(dgg^{-1})) \\ &= \mathrm{Tr}\,(dA \wedge dgg^{-1}) - \ Tr(A \wedge dgg^{-1} \wedge dgg^{-1}) \end{aligned}$$

Thus we have

$$\begin{aligned} \rho^*\omega - d\epsilon = k\mathrm{Tr}\,(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg) - 3k\mathrm{Tr}\,(dA \wedge dgg^{-1}) \\ + 3k\mathrm{Tr}\,(A \wedge dgg^{-1} \wedge dgg^{-1}). \end{aligned}$$

Proposition 8.1. *Let $\alpha \in \Omega^3(P_G)$ be defined by the Chern-Simons form*

$$\mathrm{Tr}\,(A \wedge dA) + \frac{2}{3}\mathrm{Tr}\,(A \wedge A \wedge A).$$

Then

$$\delta(-3k\alpha) = \rho^*\omega - d\epsilon.$$

and α is a 3-curving for the bundle 2-gerbe described above.

Proof. The map δ is given by $\pi_2^* - \pi_1^*$ where $\pi_1 : (p, g) \mapsto p$ and $\pi_2 : (p, g) \mapsto pg$. First we will calculate $\pi_2^* Tr(A \wedge dA)$. Recall that $pg_* Z = R_{g*} Z + g^{\#}$. Throughout the following calculations we will make use of the Ad-invariance and the cyclic symmetry of the trace. We also omit the symbol $\wedge$.

$$\begin{aligned}
pg^* \mathrm{Tr}\,(A \wedge dA) &= \mathrm{Tr}\,((g^{-1}Ag + g^{-1}dg) \wedge d(g^{-1}Ag + g^{-1}dg)) \\
&= \mathrm{Tr}\,((g^{-1}Ag + g^{-1}dg) \\
&\quad \wedge(-g^{-1}dgg^{-1}Ag + g^{-1}dAg - g^{-1}Adg - g^{-1}dgg^{-1}dg)) \\
&= -\mathrm{Tr}\,(AAdgg^{-1}) + \mathrm{Tr}\,(AdA) - \mathrm{Tr}\,(AAdgg^{-1}) \\
&\quad -\mathrm{Tr}\,(Adgg^{-1}dgg^{-1}) - \mathrm{Tr}\,(Adgg^{-1}dgg^{-1}) + \mathrm{Tr}\,(dAdgg^{-1}) \\
&\quad -\mathrm{Tr}\,(Adgg^{-1}dgg^{-1}) - \mathrm{Tr}\,(dgg^{-1}dgg^{-1}dgg^{-1}) \\
&= \mathrm{Tr}\,(AdA) - 2\mathrm{Tr}\,(AAdgg^{-1}) + \mathrm{Tr}\,(dAdgg^{-1}) \\
&\quad -3\mathrm{Tr}\,(Adgg^{-1}dgg^{-1}) - \mathrm{Tr}\,(dgg^{-1}dgg^{-1}dgg^{-1})
\end{aligned}$$

$$\begin{aligned}
pg^* \mathrm{Tr}\,(A \wedge A \wedge A) &= \mathrm{Tr}\,((g^{-1}Ag + g^{-1}dg) \wedge (g^{-1}Ag + g^{-1}dg) \\
&\quad \wedge(g^{-1}Ag + g^{-1}dg)) \\
&= \mathrm{Tr}\,(AAA) + 3\mathrm{Tr}\,(AAdgg^{-1}) + 3\mathrm{Tr}\,(Adgg^{-1}dgg^{-1}) \\
&\quad +\mathrm{Tr}\,(dgg^{-1}dgg^{-1}dgg^{-1})
\end{aligned}$$

$$p^* \mathrm{Tr}\,(A \wedge dA) = \mathrm{Tr}\,(AdA)$$

$$p^* \mathrm{Tr}\,(A \wedge A \wedge A) = \mathrm{Tr}\,(AAA)$$

Putting all of this together we get

$$\begin{aligned}
&(pg^* - p^*)\mathrm{Tr}\,(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) = \\
&\quad \mathrm{Tr}\,(AdA) - 2\mathrm{Tr}\,(AAdgg^{-1}) + \mathrm{Tr}\,(dAdgg^{-1}) \\
&\quad -3\mathrm{Tr}\,(Adgg^{-1}dgg^{-1}) - \mathrm{Tr}\,(dgg^{-1}dgg^{-1}dgg^{-1}) + \frac{2}{3} Tr(AAA) \\
&\quad +2\mathrm{Tr}\,(AAdgg^{-1}) + 2\mathrm{Tr}\,(Adgg^{-1}dgg^{-1}) + \frac{2}{3}\mathrm{Tr}\,(dgg^{-1}dgg^{-1}dgg^{-1}) \\
&\quad -\mathrm{Tr}\,(AdA) - \frac{2}{3}\mathrm{Tr}\,(AAA)
\end{aligned}$$

Collecting terms gives

$$\mathrm{Tr}\,(dAdgg^{-1}) - \mathrm{Tr}\,(Adgg^{-1}dgg^{-1}) - \frac{1}{3}\mathrm{Tr}\,(dgg^{-1}dgg^{-1}dgg^{-1})$$

which is the desired result. □

We call this bundle 2-gerbe with connection and curvings the *Chern-Simons bundle 2-gerbe.* Its holonomy satisfies the properties of the Chern-Simons action, this shows

Proposition 8.2. *The Chern-Simons action associated with a principal G-bundle, where G is connected, simply connected and simple, may be realised as the holonomy of the Chern-Simons bundle 2-gerbe over a closed 3-manifold.*

The real purpose of using bundle gerbe theory to approach this problem is to understand how it generalises when we relax the requirements on G in the proposition above. So far we have removed only the requirement that the bundle P be trivial in the original definition of the action. This is possible because defining the holonomy only requires that the bundle 2-gerbe be trivial, which is always true over a 3-manifold. The existence of a section of P implies this triviality however there exist trivial bundle 2-gerbes for which such a section does not exist.

Suppose we wish to allow G to be only semi-simple instead of simple. In this case we can still define the CS form using the Killing form on the Lie algebra. The difference with the simple case is that we may have $H^3(G,\mathbb{Z}) \neq \mathbb{Z}$. We can still define the CS-bundle 2-gerbe as above, the only difference is that the possible bundle gerbes over G in the construction do not necessarily account for all bundle gerbes over G. To allow non semi-simple groups we may replace the trace with an invariant quadratic polynomial on the Lie algebra, as in [15]. If G is not simply connected then the tautological bundle gerbe on G must be replaced with a general bundle gerbe with curvature $\mathrm{Tr}\,(g^{-1}dg)^3$. This may require additional data (such as a full Deligne class) since the bundle gerbe is no longer determined completely by its curvature.

Using this interpretation we may consider further aspects of CS theory in terms of the theory of holonomy of bundle 2-gerbes.

Chern-Simons Lines and Gauge Invariance

For the purposes of this section we shall follow [20] and set $\omega = -\frac{1}{3}\mathrm{Tr}\,(g^{-1}dg)^3$ and $\epsilon = -Tr(Adgg^{-1})$ so that the curving is precisely $CS(A)$.

It is a standard fact ([20],[15]) that given a 3-manifold, X, with non-empty boundary ∂X, the CS action cannot be defined as a function, rather it is a section of a line bundle called a *Chern-Simons line*. This is, of course, exactly what we would expect since we have interpreted the CS action as the holonomy of a bundle 2-gerbe. We shall give arguments as to why the line bundle corresponding to the transgression of a bundle 2-gerbe as described in Chapter 6 above is the same as the CS lines described in [20] and [15].

Recall that Proposition 7.4 tells us that when M is 2-connected and G is simply connected the transgression of the bundle 2-gerbe associated with a principal bundle is described by the following diagram:

$$\begin{array}{ccc} & & S^1 \\ & \overset{\rho^*\mathrm{hol}(Q[\omega])\cdot I(\epsilon)}{\nearrow} & \\ S^2(P_G) \times S^2(G) & \rightrightarrows & S^2(P_G) \\ & & \downarrow \\ & & S^2(M) \end{array}$$

For purposes of comparison it is useful to describe the fibres of the line bundle which may be obtained from this bundle 0-gerbe. Over $\Sigma \in S^2(M)$ the fibre consists of elements of an equivalence class $[\tilde{\Sigma},\theta]$, where $\tilde{\Sigma}$ is a lift of Σ to P_G and $\theta \in S^1$, and the equivalence is given by $[\tilde{\Sigma}_1, \mathrm{hol}_{(\tilde{\Sigma}_1,\tilde{\Sigma}_2)}(\rho^{-1}Q[\omega])] \sim [\tilde{\Sigma}_1, 1]$. The trivial bundle over X, where $\partial X = \Sigma$ is obtained by pulling back this bundle using the restriction to the boundary. A trivialisation is given by the extension of the holonomy function on $S^3(M)$.

We now recall some earlier results on gauge transformations which are relevant here. By Proposition 7.5 any G-gauge transformation on $P_G \to X$ is also a bundle 2-gerbe gauge transformation, so we may apply the results of §7.7 to examine gauge invariance of the CS action. In the case of a closed 3-manifold the gauge invariance of the holonomy implies the same for the CS action. In the case with boundary there are two additional terms in the section of the trivial line bundle,

$$\int_X \omega_\phi - \sum_{b,w} \int_b \beta_{\rho(b)} \tag{8.8}$$

Recall that ω_ϕ is the 3-curvature of the bundle gerbe $(1,\phi)^{-1}\rho^{-1}Q[\omega]$. Since $\rho \circ (1,\phi) = g_\phi$ this is just $\mathrm{Tr}\,(g_\phi^{-1}dg_\phi)^3$. Recall also that the 2-form β arises from the failure of the curvature of $(1,\phi)^{-1}P_G$ to descend to a curvature on M. In this case the former is $(1,\phi)^*(\rho^*\omega - d\epsilon) = g_\phi^*\omega - d(1,\phi)^*\epsilon$. Observe that $\delta(g_\phi^*\omega) = \mathrm{Ad}_g\omega - \omega = 0$ by the invariance of the trace, so this part descends and $\beta = (1,\phi)^*\epsilon$. For $\epsilon = \mathrm{Tr}\,(Adgg^{-1})$ we have $(1,\phi)^*\epsilon = \mathrm{Tr}\,(Adg_\phi g_\phi^{-1})$, so the section changes by

$$\exp\left(\int_X -\frac{1}{3}\mathrm{Tr}\,(g_\phi^{-1}dg_\phi)^3 + \sum_{b,w}\int_b \mathrm{Tr}\,(A_{\rho(b)}dg_\phi g_\phi^{-1})\right) \tag{8.9}$$

where we have used the fact that $\delta(dg_\phi g_\phi^{-1}) = 0$, so only A need be expressed in local form. If the G-bundle $P_G \to M$ is trivial with section s then we recover proposition 2.10 of [20], where the section changes by

$$\exp\left(\int_X -\frac{1}{3}\mathrm{Tr}\,(g_\phi^{-1}dg_\phi)^3 + \int_{\partial X}\mathrm{Tr}\,(s^*Adg_\phi g_\phi^{-1})\right) \tag{8.10}$$

We may now compare our construction of the CS lines with that of Freed [20]. Suppose the bundle $P_G \to Y$ is trivial where Y is a closed 2-manifold. We think of Y as the image in M of an element of $\mathrm{Map}(\Sigma, M)$. Each choice of a section $Y \to P_G$ gives a lift $\tilde{Y}$. Let s_1 and s_2 be two such choices with corresponding lifts $\tilde{Y}_1$ and $\tilde{Y}_2$. There is a G-gauge transformation, ϕ, which gives the difference between these two sections. The pair $(\tilde{Y}_1, \tilde{Y}_2) \in (\Sigma P_G)^{[2]}$ is then equivalent to $(\tilde{Y}_1, g_\phi(\tilde{Y}_1)) \in \Sigma P_G \times \Sigma G$. Thus we have $\rho(\tilde{Y}_1, \tilde{Y}_2) = g_\phi(\tilde{Y}_1)$, though it should be kept in mind that the $\tilde{Y}_2$ dependence is contained in the definition of ϕ. If we let $Y = \partial X$ then the equivalence relation in the definition of the line bundle is given by the function (8.10), which is used in an analogous way in the construction of the line in [20].

Viewing CS theory from a bundle gerbe point of view it is no surprise that the WZW action arises when we apply gauge transformations. The CS bundle 2-gerbe includes a bundle gerbe over G with curvature $Tr(g^{-1}dg)^3$ in its definition, this is the bundle gerbe which produces the most common form of the WZW action (that is, the one obtained when G is simple) via its holonomy. That the holonomy of this bundle gerbe should be relevant here follows from the results on the effects of gauge transformations on holonomy.

Relationship with the Central Extension of the Loop Group

In the previous section we considered the transgression of the Chern-Simons bundle 2-gerbe to a line bundle on $S^2(M)$. We have also seen (§6.4) that it is possible to transgress a bundle 2-gerbe to a bundle gerbe on the loop space. We are interested here in the case where G is simply connected, so the only bundle gerbe (up to D-stable isomorphism) is the tautological one. In the case of the CS bundle 2-gerbe we get the following bundle gerbe:

$$\begin{tikzcd} & \tau_{S^1}(Q[\omega]) \arrow[d] \\ & \mathcal{L}G \\ \mathcal{L}P \times \mathcal{L}G \arrow[ur, "\mathcal{L}\rho"] \arrow[r, shift left] \arrow[r, shift right] & \mathcal{L}P \arrow[d] \\ & \mathcal{L}M \end{tikzcd}$$

where $\tau_{S^1}(Q[\omega])$ is the loop space transgression of the tautological bundle gerbe on G. Recall (see proposition 7.1) that this transgression is isomorphic to the bundle corresponding to the central extension $\widetilde{\mathcal{L}G} \to \mathcal{L}G$, so we have

$$\begin{tikzcd} & \widetilde{\mathcal{L}G} \arrow[d] \\ & \mathcal{L}G \\ \mathcal{L}P_G \times \mathcal{L}G \arrow[ur, "\mathcal{L}\rho"] \arrow[r, shift left] \arrow[r, shift right] & \mathcal{L}P_G \arrow[d] \\ & \mathcal{L}M \end{tikzcd}$$

This is the lifting bundle gerbe which describes the obstruction to lifting the structure group of the bundle $\mathcal{L}P_G \to \mathcal{L}M$ from $\mathcal{L}G$ to $\widetilde{\mathcal{L}G}$. This result is given in terms of gerbes in [7] and [24].

In conclusion, we have seen that the standard Chern-Simons action may be interpreted as the holonomy of a bundle 2-gerbe. Just as with WZW theory, this allows us to understand the failure of the action to be well defined in the general case, that is, when there is no section of the G-bundle or it cannot be defined as an integral of a 4-curvature. These features are key characteristics of holonomy. A number of other features of the theory have been explained in bundle gerbe terms. In [15] more general theories are discussed in terms of general WZW theories. In terms of bundle gerbes these could be interpreted as a generalisation of the associated bundle gerbe to a case where the bundle gerbe on G is not tautological (an example that was similar to this, $L \cup J$ was described in section 4.3). Even more generally differential characters are used, since these correspond to classes in Deligne cohomology this suggests that bundle 2-gerbes can play the same role.

8.3 D-Branes and Anomaly Cancellation

In [11] it is shown how to use bundle gerbes to cancel anomalies in D-brane theory. Here we concentrate on the local aspects of this approach as an application of the holonomy of bundle gerbes.

The basic situation described in [11] is that we have actions which are functions associated with maps of a surface with boundary, Σ, into a manifold M with submanifold Q such that $\partial\Sigma \subset Q$. The submanifold Q is referred to as the brane. The action turns out to be a section of a trivial line bundle on ΣM. This should come as no surprise by now since we have seen examples of actions which behave as holonomies and this is precisely the behaviour we would expect from the holonomy of a bundle gerbe. This failure of the action to be a well defined function is called the anomaly. Anomaly cancellation involves the introduction of an extra term (or terms) such that together they define a function. Our approach is guided by the knowledge that two trivialisations differ by a global function, so to cancel the anomaly we need to find another trivialisation of the bundle on ΣM. The general technique for doing this is as follows. Recall that if we transgress a bundle gerbe to the loop space then we can obtain the trivial line bundle over ΣM by pulling back the line bundle on the loop space with the map $\partial : \Sigma M \to \mathcal{L}M$, which is induced from the restriction to the boundary. Suppose we have a term in the action which corresponds to a section of the trivial bundle over Σ corresponding to the transgression, $L \to \mathcal{L}M$, of a bundle gerbe, P on M. If we can find another bundle L' on the loop space which is isomorphic to L then the product $L \otimes L'^*$ will be trivial, and this trivialisation will induce a trivialisation of $\partial^{-1}L \otimes \partial^{-1}L'^*$. Thus the combination of the usual trivialisation of the pull back of L' to ΣM and and the trivialisation of the product bundle on $\mathcal{L}M$ will cancel the anomaly. Furthermore the functoriality of transgression tells us that a suitable bundle L' may be found via the transgression to $\mathcal{L}M$ of a bundle gerbe P' with the same Dixmier-Douady class as P. This bundle gerbe P' is known as a B-field in the physics literature and the requirement that $\mathrm{dd}(P) = \mathrm{dd}(P')$ leads to a natural division of the anomaly cancellation problem into three distinct cases.

First we consider the situation described by Freed and Witten [21] as interpreted in [11]. In this case the first term in the action is derived from the transgression of a torsion bundle gerbe, that is, a bundle gerbe with a torsion Dixmier-Douady class. The Deligne class of this bundle gerbe is determined by the second Stiefel-Whitney class, $w_2 \in H^2(M, \mathbb{Z}_2)$, of the normal bundle to Q. This class determines a Deligne class $(w_{\alpha\beta\gamma}, 0, 0)$ where $w_{\alpha\beta\gamma} \in H^2(M, U(1))$ is induced by the inclusion $\mathbb{Z}_2 \subset U(1)$. Let P_{w_2} be a bundle gerbe which is classified by this Deligne class.

The B-field is defined as a triple $(g_{\alpha\beta\gamma}, k_{\alpha\beta}, B_\alpha)$ which defines a Deligne cohomology class and hence a D-stable isomorphism class of bundle gerbes (note that B is a 2-form field). Let P_B be a representative of this class. In this case we specify that the Dixmier-Douady class of the B-field is equal to that of the torsion bundle gerbe described above. Thus the two transgression bundles on the loop space are isomorphic and there exists a section which may be pulled back to ΣM to cancel the anomaly. We wish to get a local expression for this term.

The product $P_w^* \otimes P_B$ is represented locally by the Deligne class $(g_{\alpha\beta\gamma} w_{\alpha\beta\gamma}^{-1}, k_{\alpha\beta}, B_\alpha)$. The local formula for the transition functions of the transgression to the loop space is obtained by applying equation (6.13),

$$G_{01} = \exp \sum_e \int_e k_{\rho_0(e)\rho_1(e)} \cdot \prod_{v,e} g^{-1}_{\rho_0(e)\rho_0(v)\rho_1(v)} g_{\rho_0(e)\rho_1(e)\rho_1(v)} w_{\rho_0(e)\rho_0(v)\rho_1(v)} w^{-1}_{\rho_0(e)\rho_1(e)\rho_1(v)}(v) \tag{8.11}$$

Since P_w and P_B have the same Dixmier-Douady classes then by functoriality of the transgression (see §7.1) the line bundles $\tau(P_w)$ and $\tau(P_B)$ have the same Chern class, so there exists a trivialisation $(h_{\alpha\beta}, A_\alpha)$ satisfying

$$g_{\alpha\beta\gamma} w_{\alpha\beta\gamma}^{-1} = h_{\beta\gamma} h_{\alpha\gamma}^{-1} h_{\alpha\beta} \tag{8.12}$$

$$k_{\alpha\beta} = -d \log h_{\alpha\beta} + A_\alpha - A_\beta \tag{8.13}$$

The pair $(\underline{h}, \underline{A})$ defines an A-field [21]. Substituting into (8.11) and using the usual combinatorial arguments we get

$$G_{01} = \exp \sum_e \int_e (A_{\rho_1(e)} - A_{\rho_0(e)}) \cdot \prod_{v,e} h_{\rho_1(e)\rho_1(v)} h_{\rho_0(e)\rho_0(v)}^{-1}(v) \tag{8.14}$$

and thus we have local functions

$$\Gamma_0 = \exp \sum_e \int_e A_{\rho_0(e)} \cdot \prod_{v,e} h_{\rho_0(e)\rho_0(v)}(v) \tag{8.15}$$

satisfying $\Gamma_0^{-1}\Gamma_1 = G_{01}$. These local functions may be pulled back to give local functions (or equivalently sections of a trivial bundle) on ΣM and cancel the anomaly.

We would like to indicate how this local picture relates to the global version given in [11]. Denote the transgressions of P_w and P_B by L_w and L_B respectively. The original term in the action from which the anomaly arises is the Pfaffian of the Dirac operator on the world sheet, denoted Pfaff, which is a section of L_w. We refer to [21] for further details since this term does not arise from bundle gerbe considerations.

The bundle L_B corresponds to the following bundle 0-gerbe,

$$\begin{array}{ccc} & & S^1 \\ & \overset{\mathrm{hol}(P_B)}{\nearrow} & \\ S^2(M) & \rightrightarrows & D^2(M) \\ & & \downarrow \\ & & \mathcal{L}M \end{array}$$

so the bundle $\partial^{-1}L_B$ is given by

$$\begin{array}{ccc} & & S^1 \\ & \overset{\mathrm{hol}(P_B)}{\nearrow} & \\ \Sigma M \times_\pi S^2(M) & \rightrightarrows & \Sigma M \times_\pi D^2(M) \\ & & \downarrow \\ & & \Sigma M \end{array}$$

The section of $\partial^{-1}L_B$ may be defined by $\phi_B(\Sigma, \sigma) = \mathrm{hol}(P_B; \Sigma\#\sigma)$ where $\sigma \in D(M)$ satisfies $\partial\sigma = \partial\Sigma$. The gluing property of holonomy ensures that this is a bundle 0-gerbe trivialisation on $\Sigma M \times_\pi D(M)$, so it defines a section of the line bundle $\partial^{-1}L_B$.

To get a local expression let χ_B be a D-obstruction form for P_B over Σ corresponding to a trivialisation T_B, then

$$\begin{aligned} \mathrm{hol}(P_B; \Sigma\#\sigma) &= \exp \int_{\Sigma\#\sigma} \chi_B \\ &= H_{int}(B;\Sigma) H_{int}^{-1}(B;\sigma) H_\partial(T_B;\partial\Sigma) H_\partial^{-1}(T_B;\Sigma) \\ &= H_{int}(B;\Sigma) H_{int}^{-1}(B;\sigma) \end{aligned} \tag{8.16}$$

where we have used the fact that the local expression for $\exp \int_{\Sigma\#\sigma} \chi_B$ splits into terms on the interior of $\Sigma\#\sigma$, $H_\int$, which are the extension of the holonomy formula on closed manifolds, and terms on the boundary, H_∂, which depend on a choice of trivialisations. See Chapter 5 for further details.

The section corresponding to the trivialisation of $L_w \otimes L_B$ may be defined as a function λ on $D(Q)$ such that $\lambda(\sigma_2) = \lambda(\sigma_1)\mathrm{hol}(\sigma_1\#\sigma_2)$, so it is a trivialisation of the transgression bundle 0-gerbe

$$\begin{tikzcd} & S^1 \\ S^2(Q) \arrow[ur, "\mathrm{hol}(P_B\otimes P_w^*)"] \arrow[r, shift left] \arrow[r, shift right] & D^2(Q) \arrow[d] \\ & \mathcal{L}(Q) \end{tikzcd}$$

Note that this bundle is only defined on Q since it is only on the brane that the Dixmier-Douady classes of P_w and P_B agree. The standard section of this bundle is obtained by extending the holonomy function to discs. Let A be a trivialisation of $P_w^* \otimes P_B$ (the trivialisation defined by the A-field). Then in terms of the corresponding D-obstruction form χ_A we have

$$\begin{aligned} \lambda_A(\sigma) &= \exp \int_\sigma \chi_A \\ &= H_{int}(B - w; \sigma)H_\partial(A; \Sigma) \\ &= H_{int}(B; \sigma)H_{int}^{-1}(w; \sigma)H_\partial(A; \Sigma) \end{aligned} \tag{8.17}$$

Now we combine ϕ_B and λ_A to get

$$H_{int}(B; \Sigma)H_{int}^{-1}(w; \sigma)H_\partial(A; \Sigma) \tag{8.18}$$

In this context the anomaly corresponds to σ dependence, so while we have cancelled some σ terms there is still one left. This is because we have not yet incorporated the section of $\partial^{-1}L_w$. Consider this as a bundle 0-gerbe,

$$\begin{tikzcd} & S^1 \\ \Sigma_Q(M) \times S^2(Q) \arrow[ur, "\mathrm{hol}(P_w)"] \arrow[r, shift left] \arrow[r, shift right] & \Sigma_Q(M) \times_\pi D^2(Q) \arrow[d] \\ & \Sigma_Q(M) \end{tikzcd} \tag{8.19}$$

Observe that in this case we cannot use the same approach that we used to define the section ϕ_B since P_w is only defined on Q and in general elements of Σ_Q may not lie entirely in Q. Due to the definition of P_w is turns out that there is a section of this bundle called Pfaff [21]. Given any such section we may find a $\mathbb{C}$-valued function (since the section may vanish) on $\Sigma_Q \times_\pi D(Q)$ via the corresponding local functions, p_0. This function is defined by $\pi^* p_0(\Sigma)H_{int}(w; \sigma)$. It is easily verified that this is a section and is a globally defined function since the local dependence of the two terms cancels. Thus when we incorporate this term into (8.18) the anomaly is cancelled and we are left with terms derived from the Pfaffian, the B-field and the A-field.

The second case, which appears in [30], involves a B-field which has a different Dixmier-Douady class to P_w, but it is still required to be torsion. In this case the line bundle obtained by transgressing $P_w^* \otimes P_B$ is no longer trivial so we need some further structure to cancel the anomaly. Before we introduce this we would like to see the nature of the obstruction from a local point of view. The following arguments follow [30] closely. We have a torsion bundle gerbe $P_{(w,B)} = P_w^* \otimes P_B$ with Deligne class $(\underline{gw}^{-1}, \underline{k}, \underline{B})$. Since the image of $H^3(M,\mathbb{Z})$ in $H^3(M,\mathbb{R})$ is zero then the curvature is exact, so denote it by $d\tilde{B}$. We now have a series of equations

$$\begin{aligned} dB_\alpha &= d\tilde{B} && (8.20)\\ B_\alpha - dm_\alpha &= B_\beta - dm_\beta && (8.21)\\ k_{\alpha\beta} &= m_\beta - m_\alpha + d\log q_{\alpha\beta} && (8.22)\\ g_{\alpha\beta\gamma} w_{\alpha\beta\gamma}^{-1} &= q_{\beta\gamma} q_{\alpha\gamma}^{-1} q_{\alpha\beta} \zeta_{\alpha\beta\gamma} && (8.23) \end{aligned}$$

where m_α are 1-forms, $q_{\alpha\beta}$ are $U(1)$-valued functions and $\zeta_{\alpha\beta\gamma}$ are $U(1)$-valued constants. These constants correspond to the torsion class which measures the obstruction to the equality of $\mathrm{dd}(P_w)$ and $\mathrm{dd}(P_B)$. Since $g_{\alpha\beta\gamma} w_{\alpha\beta\gamma}^{-1}$ represents a torsion bundle gerbe it admits a bundle gerbe module, so from a local point of view there exist matrix valued functions $\lambda_{\alpha\beta}$ satisfying

$$g_{\alpha\beta\gamma} w_{\alpha\beta\gamma}^{-1} = \lambda_{\beta\gamma} \lambda_{\alpha\gamma}^{-1} \lambda_{\beta\gamma} \tag{8.24}$$

and so we have a sort of 'non-Abelian trivialisation' of $\underline{\zeta}$,

$$\zeta_{\alpha\beta\gamma} = \lambda_{\beta\gamma} q_{\beta\gamma}^{-1} \lambda_{\alpha\gamma}^{-1} q_{\alpha\gamma} \lambda_{\beta\gamma} q_{\alpha\gamma}^{-1} \tag{8.25}$$

where it is assumed that all scalar functions are multiplied by the unit matrix of the appropriate dimension so that this expression makes sense. We may view this in terms of a more general problem: if we are given a bundle gerbe with a trivialisation then we may find a trivialisation of the transgression bundle on the loop space, so if we have a bundle gerbe module represented locally by $(\lambda_{\alpha\beta}, \theta_\alpha)$ then we want to know to what extent we can use this to trivialise the bundle on the loop space. The answer is that in general we cannot trivialise the bundle, this would violate functoriality since the original bundle gerbe is non trivial, however we can find a $\mathbb{C}$-valued function which 'trivialises' it. The distinction is analogous to that between a non-vanishing section of a line bundle and a section in general. It is a result of Kapustin [30] that this is section is given by the trace of the holonomy of the bundle gerbe module [1]. We may realise this locally in terms of the holonomy of a non-Abelian bundle [11]. Over a disc the bundle gerbe which acts on the module is trivial, choose a trivialisation J. Let J be represented locally by the pair $(K_\alpha, j_{\alpha\beta})$. The bundle $E \otimes J^*$ then descends to the disc. The trace of the holonomy of this bundle can be calculated over the boundary of the disc. To eliminate the J dependence we must introduce another term, $\exp \int_\sigma \chi_J$, where χ_J is a D-obstruction form for the bundle gerbe ζ and trivialisation J. It is easily shown [11] that this defines a section, as a $\mathbb{C}$ valued function on $D(Q)$.

To examine the anomaly cancellation from a local point of view we must be careful as we cannot use the usual holonomy formula in the non-Abelian case. When the boundary loop $\partial\sigma$ is triangulated the holonomy breaks down in to an ordered product

[1] Kapustin dealt with Azumaya modules which have the same local data as bundle gerbe modules.

of parallel transport terms along edges and jumps at vertices. Following Kapustin [30] we denote the parallel transport for the bundle with connection $\theta - K$ along the edge e by $\mathrm{hol}_e(\theta_{\rho(e)} - K_{\rho_e})$. The jumps are given by terms of the form $\lambda_{\rho(e)\rho(v)} j^{-1}_{\rho(e)\rho(v)}$. The trace of the holonomy is then given by

$$\mathrm{Tr}\left[\mathrm{hol}_{e_0}(\theta - K)_{\rho(e_0)} \cdot (\lambda j^{-1})_{\rho(e_0)\rho(v_{01})} \cdot (j\lambda^{-1})_{\rho(v_{01})\rho(e_1)} \mathrm{hol}_{e_1}(\theta - K)_{\rho(e_1)} \ldots\right] \tag{8.26}$$

The Abelian parts may be pulled out of the trace leaving the trace of holonomy term of Kapustin, which we denote simply by $\mathrm{Tr}\,\mathrm{hol}(\theta; \partial\sigma)$. The Abelian terms maybe be then dealt with by the usual combinatorial methods to give the term $H_\partial^{-1}(J;\sigma)$. Combining all terms corresponding to the bundle gerbe ζ and module E we now have

$$\mathrm{Tr}\,\mathrm{hol}(\theta) H_\partial^{-1}(J;\partial\sigma) H_{int}(\zeta;\sigma) H_\partial(J;\partial\sigma) = \mathrm{Tr}\,\mathrm{hol}(\theta;\partial\sigma) H_{int}(\zeta;\sigma) \tag{8.27}$$

The contributions from the B-field and the torsion class w are as in the previous case,

$$H_{int}(B;\Sigma) H_{int}^{-1}(B;\sigma) \mathrm{Pfaff} H_{int}^{-1}(w;\sigma) \tag{8.28}$$

The A-field now trivialises $P_w^* \otimes P_B \otimes P_\zeta^*$, so the corresponding terms are the opposites of all of the H_{int} terms in the previous expressions as well as $H_\partial(A;\partial\Sigma)$. Thus combining all terms and using $\partial\sigma = \partial\Sigma$ we gain a combination of terms,

$$H_{int}(B;\Sigma) \cdot \mathrm{Pfaff} \cdot \mathrm{Tr}\,\mathrm{hol}(\theta;\partial\Sigma) \cdot H_\partial(A;\partial\Sigma) \tag{8.29}$$

which is independent of σ and so the anomaly is cancelled.

The third case is where the B-field is non-torsion, so the class ζ is non-torsion and so does not represent a bundle gerbe which admits a bundle gerbe module. To get around this it is possible to define bundle gerbe modules with infinite dimensional fibres [3] which are acted on by non-torsion bundle gerbes. These may be used to define a $\mathrm{Tr}\,\mathrm{hol}$ term which cancels the anomaly [11]. The details of this approach are not particularly relevant here, however we make note of it since it shows that the bundle gerbe theoretical approach of the simpler cases described above leads to a way of dealing with the general case.

***C*-Fields.** It seems likely that bundle gerbes could be useful in other string theory applications. In particular it has been noted ([42], [41]) that C-fields in five-brane theory may be represented locally by the following data:

$$\begin{aligned} C_\alpha - C_\beta &= dB_{\alpha\beta} && (8.30)\\ B_{\alpha\beta} + B_{\beta\gamma} + B_{\gamma\alpha} &= dA_{\alpha\beta\gamma} && (8.31)\\ A_{\beta\gamma\delta} - A_{\alpha\gamma\delta} + A_{\alpha\beta\gamma} - A_{\alpha\beta\gamma} &= d\log h_{\alpha\beta\gamma\delta} && (8.32)\\ \delta h_{\alpha\beta\gamma\delta} &= 1 && (8.33) \end{aligned}$$

This data defines a class in $H^3(M, \mathcal{D}^3)$ or an equivalence class of bundle 2-gerbes. The actions which are defined using C-fields are not the holonomy of this bundle 2-gerbe since they are usually defined in seven or eleven dimensions rather than three ([46],[18]). These actions are higher dimensional generalisations of Chern-Simons theory, and while we do not have a theory of higher bundle gerbes that applies in such dimensions the actions may still be interpreted in terms of Deligne cohomology. If the curvature of the

C-field is G then the seven dimensional Chern-Simons term is defined on a 7-manifold M by its extension to an 8-dimensional manifold X as

$$CS_7(C) = \exp \int_X G \wedge G \tag{8.34}$$

We may think of this as the holonomy of a Deligne class in $H^7(M, \mathcal{D}^7)$ with curvature $G \wedge G$. Such a Deligne class may be constructed via a cup product. Let $[C] \in H^3(M, \mathcal{D}^3)$ be the Deligne class of the C-field. Then $[C] \cup [C]$ is a class in $H^7(M, \mathcal{D}^7)$ [2] with curvature $G \wedge G$. The action may then be defined without the extension X as the holonomy of this class over M. If the local 3-curving forms C_α are actually globally defined (corresponding to G being de-Rham trivial and the C-field representing a torsion bundle 2-gerbe) then this may be expressed as

$$CS_7(C) = \exp \int_X C \wedge G \tag{8.35}$$

In the general case it would be necessary to use the formula for the cup product (definition 3.15) and to substitute the resulting Deligne class into the general formula for holonomy given by proposition 5.5. Given a 6-manifold W then it is possible to construct a line bundle by transgression, a local formula for the transition functions would be given by equation (6.29).

An 11-dimensional Chern-Simons theory may be defined in a similar way. This time the Deligne class is given by the triple cup product $[C] \cup [C] \cup [C]$ so that the curvature is $G \wedge G \wedge G$. The holonomy is defined as an integral of this curvature over a 12-manifold, an integral of $C \wedge G \wedge G$ over an 11-manifold or more generally by a transgression formula. There is a transgression line bundle obtained by considering the holonomy over 10-manifolds.

These observations give only a starting point for a bundle gerbe analysis of C-fields and 5-brane theories. We have not analysed anomaly cancellation for this theory however it is possible that our approach to anomaly cancellation in the D-brane case could also apply here to some extent.

8.4 Axiomatic Topological Quantum Field Theory

We would like to relate the properties of bundle gerbe holonomy and transgression to the axioms of topological quantum field theory (TQFT) ([1],[39]). This arises from the relationship between holonomy and topological actions that has been demonstrated in the previous sections, however it should be noted that we have considered only classical actions. It is possible to proceed to topological quantum field theories using the technique of path integration (see [45] for a discussion of the case of Chern-Simons theory), however this is not generally well defined and will not be discussed here. The axiomatic definition of TQFT is of interest however since TQFTs may be derived from classical theories satisfying similar axioms.

As additional motivation we cite some relevant literature. It has been noted that the line bundle obtained via transgression of a gerbe [7] and the Chern-Simons lines [20],

[2] Recall that the cup product of two classes in $H^p(M, \mathcal{D}^p)$ gives a class in $H^{2p+1}(M, \mathcal{D}^{2p+1})$ since the cup product is actually defined on $H^{p+1}(M, \mathbb{Z}(p+1)_D) \cong H^p(M, \mathcal{D}^p)$.

which we have derived via transgression satisfy certain axioms which are closely related to those of TQFT. This approach to quantum Chern-Simons theory has been taken by Freed [19] in the case where the gauge is group is finite since in this case the path integral reduces to a finite sum which is well defined. In this instance the properties of the classical theory carry over to give the axiomatic properties of the quantum theory. Another approach is to consider *homotopy quantum field theories* [9] which in certain cases are closely related to gerbes. Generalisations relating to higher categories have also been considered [4]. Also Segal [40] considers an axiomatic approach to B-fields in string theory, following the axiomatic definition of conformal field theory (CFT). Thus the link between gerbes, topological field theories and axiomatic definition of such theories has arisen in a number of different ways.

We shall consider first the axioms given by Atiyah and then examine the extent to which they relate to bundle gerbe theory.

Definition 8.2. [1] A *topological quantum field theory* (TQFT) in dimension d defined over a ground ring Λ, consists of a finitely generated Λ-module $Z(\Sigma)$ associated to each oriented closed smooth d-manifold Σ, and an element $Z(X) \in Z(\partial X)$ associated to each oriented smooth $(d+1)$-manifold X. These are required to satisfy the following axioms:

1. Z is functorial with respect to orientation preserving diffeomorphism of Σ and X,

2. Z is involutory, that is, reversing orientation of the manifold gives the dual module,

3. Z is multiplicative under disjoint unions and gluing of manifolds.

We also note some further explanation from [1] about each of these axioms.

- **Functoriality.** Let $\phi : \Sigma \to \Sigma'$ be an orientation preserving diffeomorphism. Then there is an isomorphism of modules $Z(\phi) : Z(\Sigma) \to Z(\Sigma')$ such that $Z(\psi \circ \phi) = Z(\psi) \circ Z(\phi)$ where $\psi : \Sigma' \to \Sigma''$. When ϕ extends to an orientation preserving diffeomorphism $X \to X'$, with $\partial X = \Sigma$, $\partial X' = \Sigma'$ then the isomorphism $Z(\phi)$ maps $Z(X)$ to $Z(X')$.

- **Involution.** In general a reverse in orientation gives a 'dual' module. When Λ is a field then a reverse of orientation gives dual vector spaces. We need not be concerned here with details of the general case.

- **Multiplication.** In the case where Σ and Σ' are disjoint we have $Z(\Sigma \cup \Sigma') = Z(\Sigma) \otimes Z(\Sigma')$. If X has boundary $\Sigma_1 \cup \Sigma_2$ and we cut X along Σ_3 to get two components such that $\partial X_1 = \Sigma_1 \cup \Sigma_3$ and $\partial X_2 = \Sigma_2 \cup \Sigma_3$ then

$$Z(X) =< Z(X_1), Z(X_2) > \tag{8.36}$$

which is defined to be the natural pairing

$$Z(\Sigma_1) \otimes Z(\Sigma_3) \otimes Z(\Sigma_3)^* \otimes Z(\Sigma_2) \to Z(\Sigma_1) \otimes Z(\Sigma_2) \tag{8.37}$$

If we set $Z(\emptyset_d) = \Lambda$, where $\emptyset_d$ denotes the empty d-manifold then we may extend this to the case where X is closed and may be cut along Σ to make $X = X_1 \cup_\Sigma X_2$. In this case we also get (8.36) however this time it represents a pairing

$$Z(\Sigma) \otimes Z(\Sigma)^* \to \Lambda \tag{8.38}$$

If $\emptyset_{d+1}$ is considered as the empty $(d+1)$-manifold then we let $Z(\emptyset_{d+1}) = 1$.

Our model for relating these axioms to bundle gerbe theory will be the case where the module is a vector space defined as the fibre of a vector bundle. To stay consistent with the preceding work we shall allow $\Lambda = U(1)$, so instead of a vector space we have a principal $U(1)$-space. This should be thought of in the same terms as the equivalence between principal bundles and associated vector bundles. Holonomy and transgression will not define TQFTs in this sense, however we have given the axioms in this form since they are well known in the literature. Instead we consider the "classical" TQFTs which Quinn [39] uses in a study of Chern-Simons theory in terms of axiomatic TQFT. These theories differ from the TQFTs defined above in that there is extra data associated with the manifolds on which the theory is defined. It is required that there is a topology on this extra data, for example it may consist of a space of mappings. To be specific, if the holonomy of a $U(1)$ bundle $L \to M$ with connection A was considered to be a theory associated with a closed 1-manifold Γ (a disjoint union of loops) then the problem is that the theory does not just depend on Γ itself, it also depends on the map of the Γ into M. Recall that to define the holonomy we pull L back using this map, so the extra data could be considered either as an appropriate equivalence class of maps of γ into M or alternatively as a space of isomorphism classes of bundles with connection over Γ. Since all such bundles are trivial then this is actually a space of gauge equivalent connections which arises in the path integral. Note that these theories are not to be confused with classical theories which take values in a field (for example $\mathbb{R}$) and which differ significantly from the quantum theory in that the multiplicative property involves a scalar product rather than a tensor product. See [39] for a discussion of the importance of this difference.

Allowing for extra data as discussed above, the following examples all satisfy the axioms by the results discussed in §7.1.

Holonomy and Parallel Transport of Line Bundles. In terms of definition 8.2 we are dealing with a 0-dimensional theory where we think of a 0-dimensional manifold as a point. Let (L, M) be a line bundle with connection, then associated with any point $m \in M$ we have a group defined by L_m, the fibre of L at m. Given a path μ in M with $\partial\mu = \{m_0, m_1\}$ then there is an element $Z(\mu) \in Z(\partial\mu) = L^*_{m_0} \otimes L_{m_1}$ which is defined by parallel transport. Given a closed loop γ then there is an element $Z(\gamma) \in Z(\emptyset) = U(1)$ which is defined by the holonomy around γ.

Holonomy and Parallel Transport of Bundle Gerbes. These define 1-dimensional theories. Let (P, Y, M) be a bundle gerbe with connection and curving. For any $\gamma \in \mathcal{L}M$ let $Z(\gamma)$ be the fibre of the transgression bundle $L = \tau_{S^1}P$ at γ. Given a 2-manifold Σ with boundary $\partial\Sigma = \bigcup_i \gamma_i$ then there is an element $Z(\Sigma) \in Z(\partial\Sigma) = \bigotimes_i L^{\sigma(i)}_{\gamma_i}$, where $\sigma(i)$ is the orientation of the boundary component, defined by the section of $\partial^{-1}L \to \Sigma M$ which is derived from the holonomy. For a closed 2-manifold the element of $Z(\emptyset) = U(1)$ is defined by the holonomy.

Holonomy and Parallel Transport of Bundle 2-Gerbes. These define 2-dimensional theories in precisely the same way as the previous two examples so we omit details.

The bundle gerbe hierarchy and the properties of holonomy and transgression imply the existence of more general theories where $Z(\Sigma)$ is no longer a vector space but which essentially satisfy the same axioms. We have used fibres of bundles in the place of the modules $Z(\Sigma)$, so the next step in the hierarchy is to use the fibre of a bundle gerbe. Such a fibre is a $U(1)$-groupoid in the sense of [36]. We shall review this construction here, the important point being that all operations on modules which are required in definition 8.2 have analogous constructions in the groupoid setting.

Definition 8.3. A $U(1)$-*groupoid* with base X is a principal $U(1)$-bundle $P \to X^2$ which has a product which is a bundle morphism covering the map $((x,y),(y,z)) \to (x,z)$. This product is required to be associative.

We may denote this groupoid by the pair (P, X). A $U(1)$-groupoid has an identity, given by a section of P over the diagonal $(x,x) \in X^2$, and an inverse which is given by taking the dual bundle $P^* \to X^2$. The existence of the identity and inverse is implied by the definition (see [36] for details). A morphism of $U(1)$-groupoids is a morphism of $U(1)$-bundles which respects the product structure. It is clear from the definition of a bundle gerbe that the fibre over a point in the base has the structure of a $U(1)$-groupoid. Given a point m in the base then the objects are all $y \in Y$ such that $\pi(y) = m$ and the morphism between two objects y_0 and y_1 is given by $P_{(y_0,y_1)}$. Composition of morphisms is given by the groupoid product. The tensor product of two groupoids is defined by analogy with the tensor product of bundle gerbes. Given groupoids defined by (P, X) and (Q, Y) the product is the groupoid defined by the tensor product bundle $P \otimes Q \to X^{[2]} \times Y^{[2]}$. A trivialisation of a $U(1)$-groupoid (P, X) is a $U(1)$-bundle $L \to X$ such that there is a bundle isomorphism $P_{(x_1,x_2)} \cong L^*_{x_1} \otimes L_{x_2}$.

We now consider theories for which the modules $Z(\Sigma)$ are replaced by groupoids and elements of modules $Z(M) \in Z(\partial M)$ are replaced by trivialisations. The reason for this is that when $Z(\partial M)$ is the fibre of a bundle then an element of the fibre is determined by a section which is equivalent to a trivialisation. All operations involved in the axioms are replaced by those described above. If we are dealing with a d-dimensional theory then we define $Z(\emptyset_d)$ to be the groupoid with one object, that is (P, x) where x is a single point, which may also be viewed as a fibre of a bundle given by the trivial morphism $P_{(x,x)}$. This ensures consistency of the multiplicative property. Similar ideas have been explored by Freed [19] using actions which take values in torsors.

Fibres of a bundle gerbe. Let (P, Y, M) be a bundle gerbe with connection and curving. We use this to define a 0-dimensional groupoid theory. For any point $m \in M$ let $Z(m)$ be the fibre of the bundle gerbe over m. Given a path μ from m_0 to m_1 there is a trivialisation of $P^*_{m_0} \otimes P_{m_1}$ defined by the extension of the loop space transgression to paths. Recall that when using the transgression approach to holonomy reconstruction the transition functions of the bundle over the loop space extend to a trivialisation of a bundle gerbe on $\mathcal{P}M$ which is isomorphic to the pull back of the original bundle gerbe by the boundary restriction map. Given a closed loop γ we have a fibre of the transgression bundle L_γ which we consider as an element of the trivial groupoid.

The loop space transgression of a bundle 2-gerbe If we have a bundle 2-gerbe (P, Y, X, M) then we may define a 1-dimensional theory by applying the previous example to the bundle gerbe on $\mathcal{L}M$ which is obtained by transgression (see §6.4).

In theory this approach could be extended to an even more abstract setting by moving further up the bundle gerbe hierarchy. By considering fibres of a bundle 2-gerbe over a point one would obtain a theory where the modules are replaced by 2-groupoids.

Finally we comment on the fact the theories which we have described here all correspond to theories involving modules (or groupoids) which are one dimensional vector spaces. This is because there is not currently a satisfactory theory of non-Abelian bundle gerbes, so we only have generalisations of line bundles and not vector bundles of higher rank.

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