EFFECTIVE ACTIONS AND CHARGES OF D-BRANES IN CURVED SPACE-TIME

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Abstract

This research concentrates on aspects of D-branes in curved space-time. D-branes are multi-dimensional objects predicted by string theory to be part of the basic building blocks of force and matter in the universe.

As part of a wider movement in string theory to investigate the Maldacena conjecture relating string theory in $AdS_m \times S^n$ curved space-time to Conformal Field Theory on $m-1$ Minkowski space, the conformal boundary of AdS_m, the earlier research in this thesis investigates κ-invariant and supersymmetric actions of D-branes in such curved space-time.

In particular, actions for D1 and D5-branes lying totally in $AdS_3 \times S^3$ are found, as well as the action of a D5-brane in an $AdS_5 \times S^5$ background. Some progress is made towards finding the action for the D5-brane free to move in the whole of $AdS_3 \times S^3 \times T^4$ space-time.

Following this, research into charges of D-branes in a group manifold are studied. In particular the charge groups are determined for the symmetry preserving (or untwisted) D-branes on a compact, simple, connected, simply connected group manifold. The purpose of this research is to determine these charge groups in order that they can be compared to the charge groups predicted by twisted K-theory for D-branes in a group manifold, thus providing a future important check to the theorem that D-brane charges are determined by twisted K-theory, which is one of the most important recent ideas in string theory.

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

Signed,

Peter Dawson.

Papers resulting from the work carried out in my PhD:

- "D1 and D5-Brane Actions in $AdS_m \times S^n$,"
  Peter Dawson,
  Int. J. Mod. Phys. **A16** (2001) 267-284, [hep-th/0002030]

- "D-branes on Group Manifolds and Fusion Rings,"
  Peter Bouwknegt, Peter Dawson and David Ridout,
  J. High Energy Physics, **0212** (2002) 065, [hep-th/0210302]

- "K-matrices for 2D conformal field theories,"
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Contents

Chapter 1

Introduction

D-branes are one of the pillars on which string theory stands. They are solitonic objects which greatly enhance the richness and viability of the theory.

D-branes arose through the study of the boundary conditions of the two dimensional string world-sheets [105]. In particular, the ends of the world-sheets contain two types of boundary conditions, Neumann and Dirichlet boundary conditions. A Neumann boundary condition allows the end of the world-sheet to move in the dimension to which it is applied, and a Dirichlet condition restricts the movement of the end of the world-sheet in that dimension.

To see the consequences of this, take the case of an open string existing in 3+1 dimensional spacetime. If an end of the open string had a Dirichlet condition in one dimension, and Neumann conditions in the others, then the end of the string would be constrained to lie on a 2+1 dimensional world-volume in spacetime.

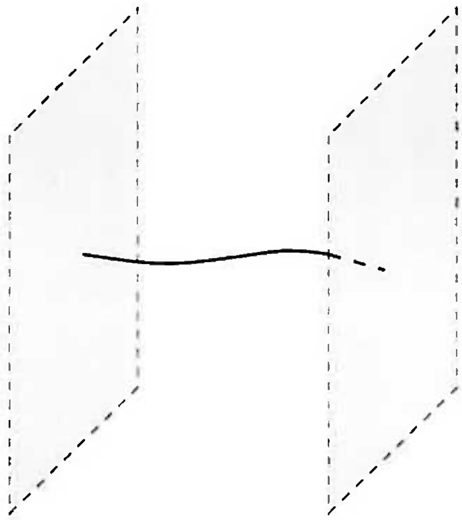


Figure 1.1: *Open string stretched between two D-branes.*

The sheet on which the string end is allowed to move is an example of a two dimensional D-brane. In general in a $D+1$ dimensional spacetime, if the end of the world-sheet is subjected to $p+1$ Neumann and $D-p$ Dirichlet conditions, the end of the brane is then constrained to a $p+1$ dimensional sheet, a Dp-brane.

Dp-branes were known about for quite some time and were thought to be of little importance, similar to the p-branes of supergravity[1]. However is was realised by Polchinski [105] that these D-branes have a $U(1)$ gauge field and have a Ramond-Ramond (R-R) charge. The fact that D-branes contain a R-R charge was a revolution. It was known that string theory should contain something physical that possessed the R-R charge of the theory, but it was also known that the fundamental strings do not carry the charge, and thus the discovery that D-branes carried this charge filled in a large hole in the theory.

Further more, while Neumann boundary conditions prevent momentum flowing off the end of the strings/world-sheets, Dirichlet conditions did not, and thus strings could impart and receive momentum to and from D-branes, making the D-branes dynamic objects. Thus these topological, dynamical objects, which possess a gauge field are solitons in the string theory.

Whenever open strings are possible in string theory, closed strings are also possible, because an open string can move in a closed loop, forming a tubular world-sheet which is equivalent to a closed string world-sheet. Thus if Dirichlet conditions are imposed on the open strings, the closed string world-sheets can have ends lying on D-branes as well, as in Figure 1.2. This is interpreted as a closed string being emitted and absorbed

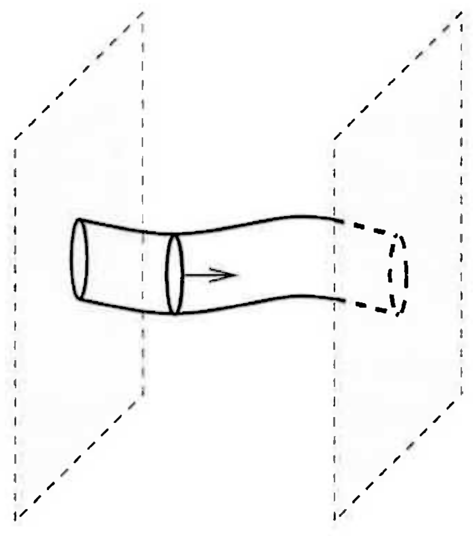


Figure 1.2: *Closed string being emitted by a D-brane and absorbed by another.*

by the D-branes.

Currently it is unknown how to derive the Standard Model from string theory. One likely and popular theory regarding how to do this is that while gravity is transmitted

[1] Supergravity is the low energy limit of super string theory, and indeed in the low energy limit, Dp-branes are mapped to p-branes of supergravity

via exchange of closed strings between D-branes, the other forces of nature are fields and excitations on the D-branes themselves. Fermionic matter particles would also have this same source.

Some cosmologists take this idea to the next step, and conjecture that our subjective universe lies on a D3-brane (or multiple parallel and coincident D3-branes) (see for eg [48, 67, 75, 82]). If this theory was to be true the dimensions transverse to the D3-brane would need to be very small (by small it is meant that the extra six dimensions are curved in upon themselves, such as for a six dimensional torus, and have a small radius), in order to not effect the Newtonian behaviour of gravity, as gravity would be free to move in these extra dimensions, but the $1/r^2$ behaviour of gravity suggests that at the scale of centimetres or larger, the universe is four dimensional (ie the extra dimensions are very much smaller than centimetres).

One of the beauties of this idea is that if there are other D-branes parallel to ours, then while our universe would interact with matter of these other branes via gravity, there could be no interaction via the other forces, and as such matter in other parallel D-branes could be a candidate for dark matter.

The research in this thesis is split up into two topics. The first is the study of supersymmetric and κ symmetric D-brane actions in a $AdS_m \times S^n$ spacetime, covered in Chapter 2, and the second is a study of the D-brane charge algebra of condensing stacks of D-branes in group manifolds, and its use as a test of whether or not D-brane charge is given by a twisted K-theory. This is covered in Chapter 3.

The motivation for the work in Chapter 2 is that since late 1997 there has been a lot of interest in superstring theory in $AdS_m \times S^n$ space, as according to the Maldacena conjecture [85, 134] string theory on $AdS_m \times S^n$ can be compared to conformal field theory (CFT) in $m-1$ dimensions. Initially, the greatest interest fell to *IIB* superstring theory on $AdS_5 \times S^5$, due to $AdS_5 \times S^5$ being a maximally supersymmetric vacuum of *IIB* supergravity and its CFT counterpart being $\mathcal{N} = 4$, $D = 4$ Super Yang Mills Theory. However the problems of quantising this theory so that it can be compared to its CFT counterpart and of our poor understanding of four dimensional CFT has led to much effort going into the study of string theory on $AdS_3 \times S^3 \times \mathcal{M}^4$, which should be simpler to compare to CFT.

The $AdS_3 \times S^3 \times \mathcal{M}^4$ background is the near horizon geometry of a *IIB* D1 - D5-brane system. Finding the action of a D1 or D5-brane in $AdS_3 \times S^3 \times \mathcal{M}^4$ is therefore equivalent to finding the action of one of the parallel and coincident D1 or D5-branes which cause the space to deform into $AdS_3 \times S^3 \times \mathcal{M}^4$, if the relevant D1 or D5-brane has been slightly extracted from the rest of the D-branes. Here $\mathcal{M}^4$ is a Ricci-flat four dimensional compact manifold, such as $K3$ or T^4 [85, 134].

Despite this being a much studied system there had been a gap in the research, ie: finding the full κ-invariant and supersymmetric *IIB* effective actions of the D1 and D5-branes in $AdS_3 \times S^3$ (although some papers had either: proposed a method for finding the D1-brane action [101]; given just the DBI-action [28]; given a bosonic action and performed an energy analysis of the branes [23, 56, 113]; or used the BPS method to find a D5-brane BPS solution of supergravity in $AdS_5 \times S^5$ [29]).

In this thesis we determine the action of the D1 and D5-branes in an $AdS_3 \times S^3$ background, where both the D1 and D5-branes lie entirely in $AdS_3 \times S^3$ and the $\mathcal{M}^4$ space has been compactified to zero volume. This D5-brane does not correspond to one of the D5-branes that are deforming the background to $AdS_3 \times S^3 \times \mathcal{M}^4$, as such branes have four dimensions in $\mathcal{M}^4$. The method used was based on the method of Metsaev and Tseytlin in [91]. The actions are described by constructions of Cartan forms on a coset superspace. The supersymmetric DBI-action is found first, and then κ-symmetry and supersymmetry are used to determine what WZ term is needed to complete the action [3, 10, 14, 25–27]. The coset superspace of $AdS_3 \times S^3$ is $\frac{SU(1,1|2)^2}{SO(2,1)\times SO(3)}$ which describes the spacetime supersymmetries the action should display.

The D5-brane found above is of interest due to the fact that as the brane must lie in $AdS_3 \times S^3$, instead of the larger space $AdS_3 \times S^3 \times \mathcal{M}^4$, this can be interpreted as a constraint imposed on the fields of a D5 action in $AdS_3 \times S^3 \times \mathcal{M}^4$, such that the solution must lie in $AdS_3 \times S^3$ and with the $\mathcal{M}^4$ compactified away. This constraint on the solutions can be thought of as describing a brane that is lying in the background of another D5-brane (or branes), with the brane dimensions in $\mathcal{M}^4$ compactified away and with the branes intersecting in one spatial dimension. Such a configuration is S-dual to the NS5-NS5 configuration described in [130], which is BPS and thus such a D5-D5 system is also BPS.

Next the action of the D5-brane in $AdS_5 \times S^5$ was found using the background coset superspace of $AdS_5 \times S^5$, $\frac{SU(2,2|4)}{SO(4,1)\times SO(5)}$. This was initially done as an exercise in preparation for handling a D5-brane in $AdS_3 \times S^3 \times \mathcal{M}^4$, however it is of interest in its own right. This brane, described by the BPS method in [29], is related to the much studied D5-brane of [23, 56, 68, 135] (and others), which describes a D5-brane in the presence of N D3-branes, and connected to them by N fundamental strings. Such a brane describes a baryon vertex in the corresponding four dimensional gauge theory. To transform the D5-brane action found here to the D5 action of [23, 29, 56, 68, 135] restrictions [56, 68] must be placed on which dimensions the D5-brane can lie in and some of the fermionic degrees of freedom must be projected out to accommodate the presence of the fundamental strings as well as the D5-brane.

Finally an attempt was made to find the most general D5-brane action in $AdS_3 \times S^3 \times T^4$.

The work in Chapter 3 is based on investigations into the D-brane charge group on a group manifold.

There is some connection between this topic and the previous one. D-branes in an $AdS_m \times S^n$ space are equivalent to D-branes in a $\frac{SO(m-1,2)\times SO(n+1)}{SO(m-1,1)\times SO(n)}$ coset space and thus such D-branes have a description as boundary states in an open string WZW model, specifically string theory on an AdS_3 space is described by an open string WZW model for $SL(2;\mathbf{R})_{\mathbf{k+2}}$, and a $SU(2)_{k-2}$ WZW model for the S^3 space. Unlike WZW models for compact, simple, connected, simply-connected group manifolds G, the $SL(2;\mathbf{R})_{\mathbf{k+2}}$ WZW model has an infinite number of integrable representations, and thus presumably an infinite number of positions in which D-branes can exist. The relations between Green-Schwarz/DBI actions of D-branes on such coset spaces and predictions from WZW theory on the coset spaces were investigated and have been found to agree (see [17] and [62] for a list of references).

Rather than a direct extension of the above work, the research in this chapter investigates brane condensation in the simpler case of WZW models for compact, simple, connected, simply-connected group manifolds G[2]. The main reason for choosing this background is that it provides a good test for an important conjecture in string theory, that the D-brane charge group is given by a twisted K-group. These D-brane charges are the R-R charges of the D-branes.

It was until recently believed that the charge group for D-branes should naturally be a de Rham cohomology. However upon the elucidation of the theory of stable and unstable non-BPS D-branes (which carry no R-R charge with respect to a de Rham cohomology) and D-brane condensation in string theory [12,64,118,119,121] it became clear that the D-brane charge group must be a more complicated group.

It was shown in [21] that in type IIA and IIB string theory on a background manifold X with a 3-form $[H] \in H^3(X, \mathbb{Z})$ (where $H = dB$ locally and $[H]$ is the cohomology class of H) then the the charge group should be a twisted K-group, $K^*(X, [H])$. This was a generalisation of earlier work [136] that found that for the less general case of a vanishing three form H, the charge group should be an untwisted K-group, $K^*(X)$, and if H is torsion, ie an integer multiple of $[H]$ vanishes in $H^3(X, \mathbb{Z})$, then the charge group is a twisted K-group, but of a simpler realisation than for the more general, not necessarily torsion case analysed in [21].

String theory on group manifolds are ideal testing grounds for this idea firstly because they have nontrivial, non-torsion H 3-forms (H is the integrand of the WZ term in the WZW action) and as such will have differing torsion components for K-groups and de Rham cohomologies, which will thus give different predictions for the charge group for these two possibilities. Also, the great amount of symmetry present on a group manifold increases the tractability of calculating D-brane charges and their charge group in WZW models [40].

In [40] the process of a stack of D-branes on a group manifold condensing into decay product D-branes predicted by the fusion rules of the system is derived, and D-brane charge conservation relations modulo some integer x are arrived at. Using this formula the authors of [40] managed to derive a formula for the D-brane charge, and find the

[2] Let it be understood that whenever mention of a group G is made, G is compact, simple, connected and simply-connected

charge group for the untwisted D-branes on a manifold $\hat{A}_{N,k}$, and find some constraints on what the charge groups for the twisted D-branes should be.

The research in Chapter 3 investigates the charge groups $\mathbb{Z}_x$ of untwisted D-branes of the group manifolds G, equivalent to deriving the value of x for each $\hat{\mathfrak{g}}_k$ algebra. Several methods are employed in this investigation. The first is directly solving the charge conservation relations modulo x of [40]. The other methods rely on the realisation that the constraints of the charge conservation relations of [40] on the values of x are equivalent to taking the greatest common divisor of all the polynomials of dimensions gained from substituting the dimensions of representations for their characters in the generators of the fusion ideals.

After obtaining the x values that determine the charge groups for the untwisted affine Lie algebras being studied, the symmetries of the D-brane charge lattice were studied, yielding a surprising and previously unsuspected symmetry, which when applied to the charge groups as a constraint greatly increases simplicity of the formulas for x, giving a formula that has the same form in terms of the Coxeter number and exponents of the untwisted affine Lie algebra, for all algebras being considered.

This thesis has been structured as follows.

Chapter 2 reviews the theory of D-branes in supersymmetric string theory, their effective actions and the Maldacena conjecture relating IIB $AdS_m \times S^n$ spacetime, warped by the gravity of D-branes, to $m-1$ dimensional super conformal field theory. An in-depth discussion of the motivation for researching supersymmetric and κ symmetric D1 and D5-branes in $AdS_m \times S^n$ space is then given before outlining the research itself.

The superalgebras and notation used in Chapter 2 are outlined in Appendix A. Appendix B gives many of the relations regarding Cartan forms necessary for finding the effective actions in Chapter 2. Appendix C outlines the derivation of a Fierz identity needed to investigate D5-brane effective actions, and Appendix D contains much of the calculations needed for deriving the D5 actions.

The κ symmetry of the actions is broken by the Killing gauge in Appendix E. This is done to simplify the actions to explicitly remove the half of the supersymmetry generators that κ symmetry renders non-independent to the other half.

Chapter 3 begins by discussing research relating D-branes in IIB $AdS_m \times S^n$ spacetime to WZW models, and then goes on to review WZW models, the related fusion rules, D-branes in WZW models and D-brane condensation on a group manifold, before discussing in more depth the reasons for investigating the D-brane charge group of untwisted affine Lie algebras for Lie groups G.

The investigation into deriving the charge groups for the affine Lie algebras in question is covered, as well as the consequent analysis of the D-brane charge lattice symmetries.

Appendix F outlines the notation used in Chapter 3, as well as key information about the simple Lie algebras. Appendix G summarises the subjects of Weyl groups and affine Weyl groups, and their relations to the highest weights of irreducible representations, their characters and their dimensions. The relationship between Weyl transformations and the outer automorphisms of the affine Lie algebra (symmetries of the affine Dynkin diagram) are also discussed.

More detailed discussions of the contents of each chapter are given at the beginnings of each chapter, and the discussion of the results of both chapters is delayed until the Conclusion (Chapter 4).

Chapter 2

Supersymmetric and κ-symmetric D-branes Actions in $AdS_m \times S^n$

As mentioned in the introduction, this chapter presents research into supersymmetric and κ symmetric D1-brane and D5-brane actions in $AdS_m \times S^n$ space for IIB string theory. While a brief account of the motivation for studying this problem was given in the introduction, a more in-depth discussion of the motivation can be found in §2.10. It is presented after the background information to better enable understanding of the reasons.

This chapter is arranged as follows. In §2.1 the basic theory of D-branes is reviewed briefly. This information is also useful for Chapter 3. In this review of D-branes Dirichlet and Neumann boundary conditions are reviewed, as well as T-duality in closed (§2.1.1) and open (§2.1.2) string theory and T-duality's effects on the D-branes defined by the boundary conditions. §2.1.3 extends the theory of D-branes to supersymmetry, including reviewing the R-R D-brane charge that arises from this, and the symmetry breaking of D-branes. §2.1.4 reviews the BPS nature of D-branes.

The actions investigated in this chapter are the effective D-brane actions, which consist of a Dirac-Born-Infeld (DBI) action with a Wess-Zumino (WZ) term. §2.2 gives a brief, qualitative explanation of the DBI action. §2.3 reviews the symmetries of bosonic spatial coordinates in $AdS_m \times S^n$ space, and the extension of this to supersymmetry.

The actions in this chapter are constructed out of Cartan forms, in order to make the actions inherently supersymmetric. Cartan forms are explained in §2.4. This is followed by a review of κ symmetry invariance, which reduces the amount of independent spacetime supersymmetry by half (see §2.5).

The methods used to find the effective D-brane actions in this chapter are based upon the methods reviewed in §2.6 and §2.7. The κ symmetry of the D-brane actions can be fixed via the Killing gauge to give a simpler form, which is reviewed in section §2.8.

The motivation for studying the D-brane actions in $AdS_m \times S^n$ space comes from the Maldacena conjecture, which is reviewed in §2.9. An in-depth discussion following from this regarding the motivation for studying D1 and D5-brane actions in $AdS_3 \times S^3$ and $AdS_5 \times S^5$ spacetime backgrounds in particular is presented in (§2.10).

Next the original work of this chapter is discussed. The notation used, as well as the supersymmetry algebra in the $AdS_m \times S^n$ backgrounds is outlined in Appendix A. The application of this supersymmetry algebra to the Cartan forms and the Maurer-Cartan relations is outlined in §2.11 and Appendix B respectively. §2.11.2 outlines the derivation of the gauge invariant and supersymmetric two form $\mathcal{F}$. The D1-brane in a $AdS_3 \times S^3$ action is analysed in §2.12. The D5 brane action is analysed in the same background in §2.13. Much of the analysis for this can be found in Appendices C and D. §2.14 analyses the D5-brane in a $AdS_5 \times S^5$ with respect to the interpretation as a Baryon Vertex in the Super-Yang-Mills theory that the IIB string theory corresponds to via the Maldacena conjecture. The fixing of the κ symmetry with the Killing gauge for these actions is delayed until Appendix E. The D5-brane is analysed for the full ten dimensional space $AdS_3 \times S^3 \times T^4$ in §2.15.

The results from this research, and future direction are discussed in the Conclusions (Chapter 4)

2.1 Review of D-branes

As previously mentioned D-branes arise through the effects of boundary conditions on the end of strings. To understand this, it is necessary to first look at the action of the strings. Let us take the bosonic string action (see [60,107] for reviews):

$$S = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{\frac{1}{2}}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu, \tag{2.1.1}$$

where M is the two dimensional world-sheet of the string, τ and σ are the coordinates on the world-sheet, $X^\mu(\tau,\sigma)$ is the mapping of the world-sheet into the target spacetime (which has 26 dimensions for bosonic string theory), γ^{ab} is the world-sheet metric, $\gamma = \det \gamma_{ab}$ and α' is the Regge slope and has units of length-squared.

To study the boundary conditions we must take the variation of this action. Choosing a world-sheet such that τ is the timelike variable with $-\infty < \tau < \infty$ and σ is the spacelike variable with $0 \le \sigma \le l$, varying X yields:

$$\delta S = \frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty} d\tau \int_0^l d\sigma(-\gamma)^{\frac{1}{2}}\delta X^\mu \partial_a\partial^a X_\mu - \frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty} d\tau(-\gamma)^{\frac{1}{2}}\delta X^\mu \partial_\sigma X_\mu \,|_{\sigma=0}^{\sigma=l} . \tag{2.1.2}$$

The boundary conditions sufficient to eliminate the second term are:

$$\begin{aligned}
&\text{closed string} && X^\mu(\tau,l) = X^\mu(\tau,0), \quad \partial_\sigma X^\mu(\tau,l) = \partial_\sigma X^\mu(\tau,0) \\
& && \&\ \gamma^{ab}(\tau,l) = \gamma^{ab}(\tau,0) && (2.1.3)\\
&\text{open string} && \partial_\sigma X^\mu(\tau,0) = \partial_\sigma X^\mu(\tau,l) = 0 && (2.1.4)\\
&\text{open string} && \delta X^\mu = 0 && (2.1.5)
\end{aligned}$$

These are respectively the periodic, Neumann and Dirichlet boundary conditions.

The periodic boundary condition eqn (2.1.3) is the condition that the string loops back upon itself, thus forming a closed string.

The Neumann condition lets the ends of the strings move freely. It can be rewritten as:

$$n^a \partial_a X_\mu = 0 \quad \text{on} \quad \partial M, \tag{2.1.6}$$

where n^a is the normal to the boundary of M, ∂M. The Neumann condition is equivalent to not allowing momentum to flow off of the boundary of the world-sheet (transverse momentum is zero at the boundary).

The Dirichlet condition in the μ direction fixes the position of the string end in the target space field X^μ in that direction [105,106].

2.1.1 T-duality

Dirichlet conditions and Neumann conditions are linked through T-duality of strings.

Before the T-duality between these conditions for open strings can be observed, it is first necessary to describe T-duality for closed strings. Let us begin by studying periodic conditions in closed dimensions, for example a closed string wrapped around S^1.

Consider periodic compactification of coordinate X^{25} onto S^1, in 26 dimensional bosonic space, with a closed string wrapped in this dimension:

$$X^{25}(\sigma + 2\pi) = X^{25}(\sigma) + 2\pi R w \quad w \in \mathbb{Z}, \tag{2.1.7}$$

$$X^{y}(\sigma + 2\pi) = X^{y}(\sigma), \tag{2.1.8}$$

where $y \in \{0, \ldots, 24\}$ and w, the 'winding number,' is an integer describing the number of times the closed string is wrapped around X^{25}. This is for Euclidean coordinates σ^1 and σ^2 where the closed string is of length 2π and the open string is of length π.

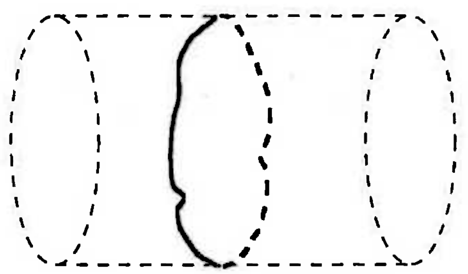


Figure 2.1: *A closed string wrapped around a cylinder (an example of a string wrapped in the periodic direction of a $R \times S^1$ space).*

The momentum in the X^{25} direction is quantised as the dimension is finite and periodic.

$$p = \frac{n}{R}, \quad n \in \mathbb{Z} \tag{2.1.9}$$

Now consider the net momentum in X^{25}, $p = p_R - p_L$, where p_L and p_R are the momentum in the two directions of the string:

$$p^{25} = \frac{n}{R} = \frac{1}{2\pi\alpha'}\oint_c (dz\partial X^{25} - d\bar{z}\bar{\partial}X^{25}) = (2\alpha')^{-\frac{1}{2}}(\alpha_0^{25} + \tilde{\alpha}_0^{25}), \tag{2.1.10}$$

where the contour c is any closed contour in z coordinates going anti-clockwise once around the origin (the $\bar{z}$ integration follows the same contour in the opposite direction), $z = e^{-i(\sigma-\tau)}$ and $\bar{z} = e^{i(\sigma+\tau)}$ (functions of z are 'right' movers and functions of $\bar{z}$ are 'left' movers) and the field X is decomposed in terms of its left and right excitation operators ($\tilde{\alpha}_i^\mu$ and α_i^μ respectively) as:

$$X^\mu(z,\bar{z}) = x^\mu - i\frac{\alpha'}{2}p^\mu \ln|z|^2 + i\left(\frac{\alpha'}{2}\right)^2 \sum_{m\neq 0}\frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m} + \frac{\tilde{\alpha}_m^\mu}{\bar{z}^m}\right). \tag{2.1.11}$$

The total change in coordinate X^{25} for $\sigma \to \sigma + 2\pi$ is:

$$\Delta X^{25} = 2\pi R w = \oint (dz\partial X^{25} + d\bar{Z}\bar{\partial}X^{25}) = \pi(2\alpha')^{\frac{1}{2}}(-\alpha_0^{25} + \tilde{\alpha}_0^{25}). \tag{2.1.12}$$

Using the eqn's (2.1.9), (2.1.10) and (2.1.12) it is possible to show:

$$p_L^{25} = \left(\frac{2}{\alpha'}\right)^{\frac{1}{2}} \tilde{\alpha}_0^{25} = \frac{n}{R} + \frac{wR}{\alpha'}, \tag{2.1.13}$$

$$p_R^{25} = \left(\frac{2}{\alpha'}\right)^{\frac{1}{2}} \alpha_0^{25} = \frac{n}{R} - \frac{wR}{\alpha'}. \tag{2.1.14}$$

The mass shell condition is:

$$m^2 = -p^y p_y - p^{25}p_{25} = \frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2), \tag{2.1.15}$$

$$0 = nw + N - \tilde{N}, \tag{2.1.16}$$

where N and $\tilde{N}$ are the levels of excitation in the closed string in left and right directions. Massless states exist for $n = w = 0$ and $N = \tilde{N} = 1$.

For the limit $R \to \infty$ the winding states become infinitely massive and are thus excluded from the state spectrum. On the other hand, the energy difference between momentum quanta reduces to zero and thus the momentum becomes a continuous degree of freedom.

For the limit $R \to 0$ the $n \neq 0$ momentum states become infinitely massive and disappear. However the energy difference between winding states reduces to zero and thus the winding number becomes a continuous degree of freedom.

This behaviours is very suggestive of a duality and closer analysis shows that the theory for X^{25} radius R is dual/equivalent to the theory with radius R':

$$R \to R' = \frac{\alpha'}{R} \quad n \leftrightarrow w. \tag{2.1.17}$$

This is a symmetry of the mass shell condition but it is true more generally as well.

Under this transformation, which is called *T-duality*, the momentum transforms as:

$$p_L^{25} \to p_L^{25} \quad p_R^{25} \to -p_R^{25} \tag{2.1.18}$$

and the spatial field transforms as:

$$X^{25}(z,\bar{z}) = X_L^{25}(\bar{z}) + X_R^{25}(z) \to X'^{\,25}(z,\bar{z}) = X_L^{25}(\bar{z}) - X_R^{25}(z). \tag{2.1.19}$$

T-duality relates different two different states of a single theory and thus is a $\mathbb{Z}_2$ symmetry.

During the majority of work in this thesis, we will be working with the mathematics of the D-branes of the theory, and rarely deal directly with the mathematics of strings, such as for the above equations. A general knowledge of the basics of string theory is assumed, however the interested reader would find the texts [60,107] excellent reviews of the subject. In particular see [107] for more information on D-branes.

Also see §3.4.3 for a more detailed discussion of boundary conditions in curved space, in particular for string theory on group manifolds (WZW models).

2.1.2 T-Duality and Open Strings

Having studied T-duality on closed strings, the duality between Neumann and Dirichlet conditions can be explored for open strings. This gives rise to D-branes [105,106].

Starting with a flat background space infinite in every direction, T-dualise along the X^{25} direction:

$$X^{25} \to X'^{25}(z,\bar{z}) = X_L^{25}(\bar{z}) - X_R^{25}(z). \tag{2.1.20}$$

Unlike closed strings, for an open string with a Neumann condition acting on a string in the direction of a periodic dimension, there is no winding number (as an open string can always be unwrapped from a periodic dimension) and thus when reducing R from ∞ to 0, no extra continuous degree of freedom is created from the winding number. The dimension appears compactified to zero volume. Now consider the open string away from its ends. This region of the string has the same excitations as the closed string and thus the bulk of the open string should behave the same as the closed string. The bulk of the open string should then 'see' the T-dual space of the compactified space and the newly created continuum a closed string would see. It is only the open string ends that feel the effects of the dimension being compactified to zero volume.

To make this point clearer, consider the action of the T-duality on the Neumann conditions:

$$\partial_n X^{25} = 0 \;\rightarrow\; -i\partial_t X'^{25} = 0, \tag{2.1.21}$$

where ∂_n is the derivative normal to the world-sheet boundary and ∂_t is the tangential derivative. So in the closed string perspective, via the compactification of X^{25} the ends of the open string (in the dual space of infinite radius) are now restricted via the condition $\partial_t X'^{\,25} = 0$ which restricts the movement of the string in the X'^{25} direction. Thus T-duality transforms the Neumann condition into the Dirichlet condition of eqn (2.1.5).

Consider the difference of positions of string ends:

$$\begin{aligned} X'^{\,25}(\pi) - X'^{25}(0) &= \int_0^\pi d\sigma^1 \partial_1 X'^{\,25} = -i\int_0^\pi d\sigma^1 \partial_2 X^{25}, \\ &= -2\pi\alpha' p^{25} = -2\pi n R', \quad n \in \mathbb{Z}, \end{aligned} \tag{2.1.22}$$

for Euclidean coordinates $\sigma = \sigma^1$, $\tau = -i\sigma^2$. n here acts as a winding number for the number of times a string is wrapped around the periodic dimension (see Figure 2.2).

Figure 2.2 illustrates how the ends of a string are restricted to stay on the plane due to the Dirichlet condition (for a periodic dimension).

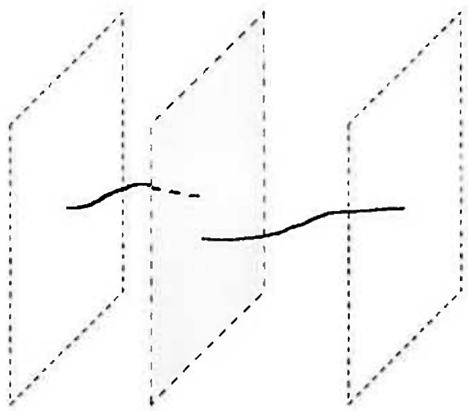


Figure 2.2: *Open string wrapped around a periodic dimension, with both ends ending on the same D-brane.*

If we take an open string in D dimensional spacetime and apply Dirichlet conditions in $D-p-1$ dimensions (and Neumann conditions in the rest), the ends then are restricted in $D-p-1$ dimensions and free to move in the subspace defined by the dimensions with Neumann conditions. This is known as a Dp-brane (often abbreviated to D-brane or brane).

There can be multiple D-branes in a particular direction. Consider an open string with Chan-Paton factors on its ends. Then:

$$X'^{\,25}(\pi) - X'^{\,25}(0) = -(2\pi n - \theta_j + \theta_i)R', \tag{2.1.23}$$

where the θ_i's parametrise the positions of various Dirichlet conditions in the X^{25} direction. This means the string ends can lie on multiple planes at positions:

$$X'^{\,25} = \theta_i R'. \tag{2.1.24}$$

For $U(N)$ Chan-Paton factors there are N different D-branes.

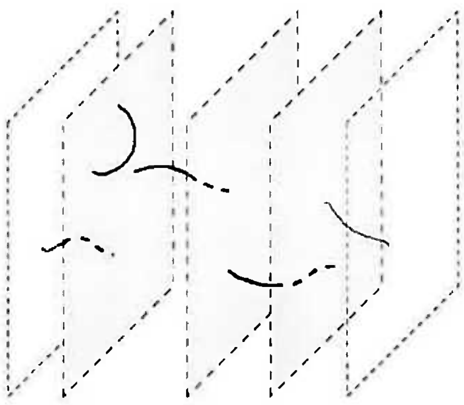


Figure 2.3: *Three D-branes in a periodic direction, allow open strings to end and start on any of the D-branes.*

Now consider closed strings between two D-branes. In a theory with no D-branes, closed string world-sheets cannot have boundaries, but if D-branes are present, it is valid to have the cylindrical world-sheet terminate with a boundary, and for the Dirichlet conditions to be applied to this circular boundary, such that this closed string boundary is fixed spatially by the Dirichlet conditions, thus being restrained to lie on the D-brane. This is equivalent to saying that D-branes can absorb and emit closed strings. The condition $\partial_t X^\mu = 0$ applies just as well to the edges of the closed string world-volume actions in Figure 2.4 as to open string world-volumes. Indeed, as can be seen in Figure 2.4 a closed string stretched between two branes is equivalent to an open string stretched between the two branes and rotated in a vacuum loop.

These D-branes are dynamical objects in the theory. To see this consider the open string states for a string with both ends on a D-brane at $X^{25} = \theta_i R'$.

The massless, first excited string states are:

$$\alpha^y_{-1}|k, ii\rangle \tag{2.1.25}$$

$$\alpha^{25}_{-1}|k, ii\rangle \tag{2.1.26}$$

$|k, ii\rangle$ denotes both ends of the string ending on brane at position $X^{25} = \theta_i R'$. $\alpha^{25}_{-1}|p, ii\rangle$ is a state transverse to the brane and is a transverse motion collective coordinate of the brane, and the states $\alpha^y_{-1}|p, ii\rangle$ tangential to the brane define a $U(1)$ gauge field on the brane. This is because the massless state vector $\alpha^\mu_{-1}|k\rangle$ couples to a conserved current

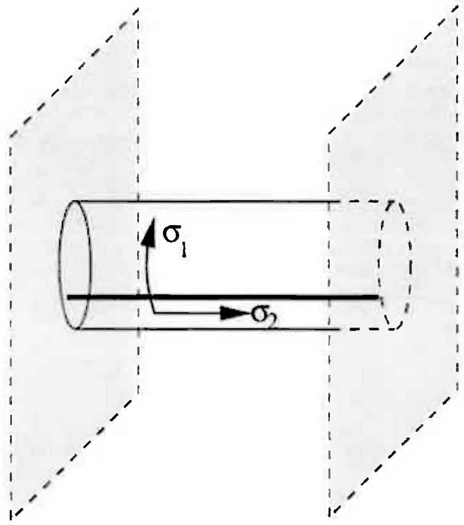


Figure 2.4: *A closed string emitted from one D-brane and absorbed by another (with time parametrised by σ_2) is equivalent to the vacuum loop of an open string (denoted by the dark line) stretched between the two D-branes (with time parametrised by σ_1).*

and has a gauge invariance, giving rise to a $U(1)$ gauge field at each end of the string. In this situation there is a separate $U(1)$ gauge field for each brane, corresponding to the states $\alpha^{y}_{-1}|p,ii\rangle$, $i \in \{0,\ldots,N\}$. When m branes are parallel and coincident, this gauge symmetry is enhanced to $U(m)$ as the massless states of strings ending and starting on the same brane are now enhanced by the massless states possible due to some of the massive states of strings stretched between branes becoming massless. This is because the mass of states of open strings stretched between two branes is given by[1]:

$$m^2 = \frac{(2\pi n - \theta_j + \theta_i)^2}{2\pi^2 R^2} + \frac{1}{\alpha'}(N-1). \tag{2.1.27}$$

This shows that open string states on strings stretched between branes can only be massless if the string starts and ends on coincident branes. Thus when m D-branes are parallel and coincident there is a richer host of massless states. Now considering that the gravity of closed strings warps spacetime, these D-branes must be able to move, as an x dependent background would produce x dependent boundary conditions on the strings. Also, Dirichlet conditions allow momentum to not be conserved at the ends of strings, and thus momentum can be exchanged between the strings and branes.

2.1.3 Extension to Supersymmetry

The interested reader can refer to [60, 106, 107] for the details of the large subject of supersymmetric string theory. Here only the points relevant to this work will be highlighted, with a minimum of background.

[1]This is shown by studying the conjugate momenta in the presence of the D-branes, which is derived from the string action with the gauge fields of the D-brane present. [105]

In the generalisation of the bosonic string action to a supersymmetric action, the action picks up fermionic terms:

$$S = \frac{1}{4\pi}\int d^2z\left(\frac{2}{\alpha'}\partial X^\mu\bar{\partial}X_\mu + \psi^\mu\bar{\partial}\psi_\mu + \tilde{\psi}^\mu\partial\tilde{\psi}_\mu\right), \tag{2.1.28}$$

where $\psi(z)$ and $\tilde{\psi}(\bar{z})$ are the right and left moving fermions on the string world-sheet respectively and the background/target space is now ten dimensional.

The supersymmetry transformations are:

$$\begin{aligned}
\delta_\epsilon X^\mu &= \left(\frac{\alpha'}{2}\right)^{\frac{1}{2}}(-\epsilon(z)\psi^\mu(z) - \epsilon(\bar{z})^*\tilde{\psi}^\mu(\bar{z})), && (2.1.29)\\
\delta_\epsilon\psi^\mu(z) &= \left(\frac{\alpha'}{2}\right)^{\frac{1}{2}}\epsilon(z)\partial X^\mu(z), && (2.1.30)\\
\delta_\epsilon\tilde{\psi}^\mu(\bar{z}) &= \left(\frac{\alpha'}{2}\right)^{\frac{1}{2}}\epsilon(\bar{z})^*\bar{\partial}X^\mu(\bar{z}), && (2.1.31)
\end{aligned}$$

where η is a Grassmann variable.

The fermionic fields have two boundary conditions for closed strings, the Ramond (R), and Neveu-Schwarz (NS) conditions:

$$\begin{aligned}
&(\mathrm{R})\ \ \psi^\mu(\sigma^1+2\pi,\sigma^2) = \psi^\mu(\sigma^1,\sigma^2), && (2.1.32)\\
&(\mathrm{NS})\ \ \psi^\mu(\sigma^1+2\pi,\sigma^2) = -\psi^\mu(\sigma^1,\sigma^2). && (2.1.33)
\end{aligned}$$

Due to the fact that the boundary conditions in the left and right moving directions can be different, the closed strings have four different types of string excitations, those with R-R boundary conditions, R-NS, NS-R and NS-NS (where the left condition is for the left movers and the right condition for the right movers).

For open strings the boundary conditions are:

$$\begin{aligned}
&(\mathrm{R})\ \ \psi^\mu(0,\sigma^2) = \tilde{\psi}^\mu(0,\sigma^2),\ \ \psi^\mu(\pi,\sigma^2) = \tilde{\psi}^\mu(\pi,\sigma^2),\\
&(\mathrm{NS})\ \ \psi^\mu(0,\sigma^2) = -\tilde{\psi}^\mu(0,\sigma^2),\ \ \psi^\mu(\pi,\sigma^2) = \tilde{\psi}^\mu(\pi,\sigma^2).
\end{aligned} \tag{2.1.34}$$

D-branes contain the charge that arises from the R-R sector of the theory. It was shown in [105] that Dp-branes have a $p+2$ dimensional field strength F_{p+2}, corresponding to a $p+1$ dimensional "Electric" field A_{p+1} ($dA = F$). The effective action of the effects of the D-brane on string theory contains the terms [105]:

$$S_p = \frac{\alpha_p}{2}\int F^*_{p+2}F_{p+2} + i\mu_p\int_{V_p} A_{p+1}, \tag{2.1.35}$$

where α_p and μ_p are constants determined by the flux of the system to be quantised, and V_p is the Dp-brane world volume (the first term is integrated over the entire background spacetime). The second term defines the R-R charge.

The vertex operator corresponding to this R-R charge is [105]:

$$\bar{\mathcal{V}}\Gamma^{[\mu_1}\ldots\Gamma^{\mu_p]}\tilde{\mathcal{V}}F_{\mu_1\ldots\mu_p}(X). \tag{2.1.36}$$

where Γ^μ are the Clifford algebra matrices. $\mathcal{V}$ and $\tilde{\mathcal{V}}$ are spin fields with weights $(0,1)$ and $(1,0)$, and are also the vertex operators of the R ground states. As such they are the world-sheet currents for spacetime supersymmmetry associated with the left and right movers respectively.

Type IIA and IIB string theory are defined respectively as theories with opposite chirality between the left and right spinors, and the same chirality between the spinors. This means that to satisfy these chirality conditions, only the R-R vertex operators (2.1.36) with odd p belong in IIB, and even p in IIA.

From this it is directly implied that IIB string theory contains only Dp-branes of odd p, and IIA contains the even p branes. Type I string theory contains both.[2]

Applying T-duality in direction X^ν in supersymmetric theory has the same effect on X as for the bosonic theory, $X^\nu(\bar{z}) = X^\nu(\bar{z})$ and $X^\nu(z) = -X^\nu(z)$. The T-duality acts on the fermionic coordinates as:

$$\psi'^\nu(z) = -\psi^\nu(z). \tag{2.1.37}$$

The left movers $\tilde{\psi}^\nu(\bar{z})$ are not altered. This has the effect of changing the chirality of the right movers of the theory. Thus type IIA and type IIB string theory are switched through T-duality (see [106, 107] for details). This implies that T-duality transforms R-R charges in IIB to R-R charges in IIA and vice versa. This occurs through the action of T-duality on the spinors:

$$\begin{aligned} \mathcal{V}'_\alpha(z) &= \beta^\mu_{\alpha\gamma}\mathcal{V}'_\alpha(z), \\ \tilde{\mathcal{V}}'_\alpha(\bar{z}) &= \tilde{\mathcal{V}}'_\alpha(\bar{z}), \end{aligned} \tag{2.1.38}$$

for T-duality in direction μ. β^μ is the parity transformation in this direction, and thus commutes with $\Gamma^{\nu\neq\mu}$ and anti-commutes with Γ^μ. Thus $\beta^\mu = \Gamma^\mu\Gamma$

Therefore the β^μ in this transformation either adds to or cancels a Γ^ν in eqn (2.1.36), depending on if a $\Gamma^{\nu=\mu}$ was already present. If $\Gamma^{\nu=\mu}$ is already present:

$$F_{\mu\nu_1\ldots\nu_p} \to F_{\nu_1\ldots\nu_p},$$

and if it is not present:

$$F_{\nu_1\ldots\nu_p} \to F_{\mu\nu_1\ldots\nu_p}.$$

For T-duality in multiple directions:

$$\beta_\perp = \prod_m \beta^m. \tag{2.1.39}$$

[2]Heterotic string theory contains no D-branes as there is only supersymmetry in one direction on the string, whereas the R-R charge vertex operator involves both left and right moving supersymmetric spinors.

Now consider applying T-duality in type I superstring theory in a infinite, flat dimension, thus $R = \infty \to R = 0$. From the same arguments as for the bosonic string, a D-brane is formed perpendicular to this dimension. Open strings are constrained to end on this D-brane, so far away from the brane, only closed strings exist, as in a type II superstring theory. This indeed implies that string theory away from the brane is type II, and type IIA in particular as the equal chirality of left and right spinors in type I is changed into opposite chirality (eqn (2.1.37)). However as will be seen, to preserve the numbers of supersymmetry generators, half of the type II supersymmetry is broken.

To see why this occurs, consider a type I open string with ends lying on D-branes, the bosonic fields are constrained by the normal Neumann and Dirichlet boundary conditions (eqn (2.1.4) and eqn (2.1.5) respectively). The fermionic boundary conditions (2.1.34) are only invariant for type $\mathcal{N} = 1$ supersymmetry (as only type I superstring theory contains open strings). Thus D-branes are natural in type I. For such an open string world-sheet ending on a brane the left moving supersymmetry current $\tilde{\mathcal{V}}_\alpha(\bar{z})$ flows into the boundary, and right moving current $\mathcal{V}_\alpha(z)$ flows out, such that only the total left and right supersymmetric charge $Q_\alpha + \tilde{Q}_\alpha$ is conserved, where Q_α is the left movers supersymmetry charge, and $\tilde{Q}_\alpha$ is the right movers supersymmetry charge. (The supersymmetric charges Q and $\tilde{Q}$ are of course the supersymmetry operators that the supersymmetry algebra shall be constructed out of shortly.)

T-duality from type I in the X^μ direction converts this condition to $\beta^\mu Q' + \tilde{Q}'$. This condition is imposed on the type IIA supersymmetry of the T-dual theory, and the relation of the left to right movers breaks half of the supersymmetries, thus preserving the same number of supersymmetries as for type I.

Applying more T-dualities in various directions can be used to construct all D-branes in type IIA and IIB string theory. A more general D-brane with Dirichlet conditions in the set of directions A conserves the charge relation $\beta_\perp Q' + \tilde{Q}'$, $\beta_\perp = \prod_{m \in A} \beta^m$.

$\mathcal{N} = 2$ supersymmetry in the presence of a D-brane takes on the form [106]:

$$\begin{aligned}
\{Q_\alpha, \bar{Q}_\beta\} &= -2[P_\mu + (2\pi\alpha')^{-1} Q^{NS}_\mu]\Gamma^\mu_{\alpha\beta}, \\
\{\tilde{Q}_\alpha, \bar{\tilde{Q}}_\beta\} &= -2[P_\mu - (2\pi\alpha')^{-1} Q^{NS}_\mu]\Gamma^\mu_{\alpha\beta}, \\
\{Q_\alpha, \bar{\tilde{Q}}_\beta\} &= -2\sum_p \frac{T_p}{p!} Q^R_{\mu_1 \ldots \mu_p} (\beta^{\mu_1} \ldots \beta^{\mu_p})_{\alpha\beta},
\end{aligned} \tag{2.1.40}$$

where T_p is the Dp-brane tension, Q^{NS}_μ is the NS-NS charge, $Q^R_{\mu_1 \ldots \mu_p}$ is this the R-R charge of a Dp-brane, Q_α acts on the right movers, and $\tilde{Q}$ acts on the left movers.

2.1.4 D-branes and the BPS bound

BPS (Bogomolnyi-Prasad-Sommerfield) states of a soliton describe limits of the soliton configurations when the mass energy of the soliton is equal to the energy contributions from the soliton's charge. In these limits the Hamiltonian and thus the energy of the soliton is minimised.

In supergravity, black p-branes are the BPS limits of the extended solitons in the theory. These p-branes break half of the supersymmetry of the theory. When considering full supersymmetric string theory, rather than supergravity, its low energy limit, these black-p-branes are seen to be low energy effective descriptions of D-branes in the theory. However in string theory, the D-branes are not solitons, as in supergravity, and thus they are not shown to be BPS states through minimising their Hamiltonian. It was mentioned earlier that black-p-branes break half of the supersymmetry in supergravity theory. Similarly in string theory, when considering the limit where the mass of the D-branes is equal to the brane charge, the supersymmetry algebra is simplified such that half of the supersymmetry decouples from the theory, as will be shown below.[3]

To see this behaviour, consider the supersymmetry of a particle[4] eqn (2.1.40) in the presence of D0-branes of mass m for type $\mathcal{N}$ supersymmetry:

$$\{Q^A_\alpha, \bar{Q}^B_\beta\} = -2P_\mu \delta^{AB}\Gamma^\mu_{\alpha\beta} - 2iZ^{AB}\delta_{\alpha\beta}, \tag{2.1.41}$$

$$\tag{2.1.42}$$

$A,\ B \in\ \{1, 2, \ldots, \mathcal{N}\}$. Z is given by:

$$Z^{AB} = \begin{pmatrix} 0 & q_1 & & & \ldots \\ -q_1 & 0 & & & \\ 0 & 0 & 0 & q_2 & \\ 0 & 0 & -q_2 & 0 & \\ \vdots & & & & \end{pmatrix}, \tag{2.1.43}$$

where q_i are the R-R charges.

In the rest frame, $P_\mu\Gamma^\mu = m\Gamma^0$, this becomes:

$$\{Q^A_\alpha, Q^{\dagger B}_\beta\} \ = \ 2m\delta^{AB}\delta_{\alpha\beta} - 2iZ^{AB}\Gamma^0_{\alpha\beta}. \tag{2.1.44}$$

Considering that the LHS is non negative, and $\Gamma^0_{\alpha\beta}$ has half its eigenvalues equal 1 and half equal -1, the RHS has half its eigenvalues equal to $m - q_i$ and the other half equal to $m + q_i$, and thus a bound on the mass is found:

$$m \geq |q_i|. \tag{2.1.45}$$

This is the BPS bound.

Now if $m = q_i = q$ for all i, then the $m - q_i$ eigenvalues are zero and the $m + q_i$ eigenvalues are $2q$.

[3] A similar simplification happens for the supersymmetry algebra of the heterotic string (without the need for D-branes to be present) when considering the ground state excitations. [106, 107]

[4] For a particle, unlike a string, we do not have to consider separate algebras acting on left and right movers.

This in turn decouples half of the supersymmetry generators, with anti-commutators of zero eigenvalues, from the algebra, and thus the size of the supersymmetry algebra is halved. The extension of this to D-branes in string theory is similar. The supersymmetry algebra of string theory with a D-brane saturates this bound and thus the superalgebra of string theory with a D-brane is said to be BPS.

Collections of D-branes can also satisfy BPS bounds. Strings acting between branes create forces, but for some combinations and orientations of branes the forces (which are due to the attraction of the graviton and dilaton, and the repulsion of the R-R tensor field) cancel. They cancel when the superalgebra is BPS.

Such BPS combinations will generally obey more restricted supersymmetry algebra than for single branes, as there exists a separate supersymmetry conservation condition

$$Q + \beta_{\perp} Q \tag{2.1.46}$$

for each brane. $\beta_{\perp}$ is different for each brane, unless they are of the same dimension and orientation. Thus if the multiple branes are of the same type and orientation, then supersymmetry restrictions are the same as for a lone brane.

More details can be found in [106, 107].

2.2 Review of DBI actions

Let us consider the Dirac-Born-Infeld action (DBI). To understand it properly it is necessary to first understand the effective string action when background perturbing effects of the massless string states have been taken into account [22].

The massless closed string states are:

$$e^{\mu\nu}|p\rangle = \alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|p\rangle, \qquad \text{for } p^2 = 0. \tag{2.2.1}$$

$e^{\mu\nu}$ is a reducible $SO(26)$ tensor and it reduces to a symmetric traceless term, the graviton $G^{\mu\nu}$, an antisymmetric term, the two form gauge boson $B^{\mu\nu}$ and a scalar term, the dilaton Φ.

$$e^{\mu\nu} = \frac{1}{2}(e^{\mu\nu} + e^{\nu\mu} - \frac{2}{D}\delta^{\mu\nu}e^{\gamma\gamma}) + \frac{1}{2}(e^{\mu\nu} - e^{\nu\mu}) + \frac{1}{D}\delta^{\mu\nu}e^{\gamma\gamma}. \tag{2.2.2}$$

The open string massless states are: $\alpha^{\mu}_{-1}|p\rangle$. This massless vector is the 'photon' and couples to a conserved current and has a gauge invariance (which is $U(1)$ unless Chan-Paton factors are introduced).

The string action generalised from the flat space action by perturbing the flat metric with the effects of the closed string fields $G^{\mu\nu}$, $B^{\mu\nu}$ and Φ is:

$$S = \frac{1}{4\pi\alpha'}\int_M d^2\sigma g^{\frac{1}{2}}\left[(g^{ab}G_{\mu\nu}(X) + i\varepsilon^{ab}B_{\mu\nu}(X))\partial_a X^{\mu}\partial_b X^{\nu} + \alpha' R\Phi(X)\right], \tag{2.2.3}$$

where R is the Ricci scalar. This is confirmed through checking it is symmetric under its gauge variations.

The low energy effective action of eqn (2.2.3) is:

$$\begin{aligned} S &= \frac{1}{2\kappa_0^2}\int d^Dx(-G)^{\frac{1}{2}}e^{-2\Phi}\left(-\frac{2(D-26)}{3\alpha'}+R-\frac{1}{12}H_{\mu\nu\gamma}H^{\mu\nu\gamma}\right. \\ &\quad \left.+4\partial_\mu\Phi\partial^\mu\Phi+O(\alpha')\right), \end{aligned} \tag{2.2.4}$$

where G is the determinant of $G_{\mu\nu}$ and $H_{\mu\nu\gamma}=\partial_\mu B_{\nu\gamma}+\partial_\nu B_{\gamma\mu}+\partial_\gamma B_{\mu\nu}$.

The rigorous way to define the DBI action is to take the low energy limit of open and closed string amplitudes but this is difficult [83]. A brief, inexact, qualitative argument for the DBI shall be provided here.

Firstly it is expected that the brane should resemble the Nambu-Goto (NG) action of an extended object, which integrates the world-volume and describes fluctuations of the D-brane away from the flat world-volume:

$$S_{NG}=\int_{M_p} d^{p+1}\xi\sqrt{-\det(G_{ab})}. \tag{2.2.5}$$

Here ξ^a $a\in$ $\{0,\ldots,p\}$ are the brane coordinates, M_p is the brane world-volume and $G_{ab}=\frac{\partial X^\mu}{\partial\xi^a}\frac{\partial X^\nu}{\partial\xi^b}G_{\mu\nu}$.

The action has a dilaton dependence due the self-interactions of the open strings and their interactions with closed strings, which can all be conformally mapped to operators on the disk. The precise form of the dilaton dependence in the effective D-brane action should be the same as that of the effective string action, eqn (2.2.4). Thus the DBI action has a $e^{-\Phi}$ dependence.

The $U(1)$ gauge field on the brane should make an appearance. The standard explanation of its appearance is to consider a constant gauge field F_{12} in the X^1 and X^2 dimensions, with Neumann conditions in these dimensions. In the gauge:

$$A_2=X^1F_{12},\quad A_1=X^2F_{21}, \tag{2.2.6}$$

observe the term (∂_1X^2) from the NG action (where $\xi^1=X^1$ and $\xi^2=X^2$). Considering that the positions of D-branes given by eqn (2.1.24) can be rewritten in terms of the gauge fields as:

$$X'^\mu=-2\pi\alpha' A_\mu,$$

under T-duality in the X^2 direction, this becomes:

$$X'^{\,2}=-2\pi\alpha' X^1F_{12},\quad (\partial_1X^2)=-2\pi\alpha' F_{12} \tag{2.2.7}$$

Therefore the action contains $(2\pi\alpha' F_{12})$ contributions.

Finally it can be shown that $B_{\mu\nu}+2\pi\alpha' F_{\mu\nu}$ is gauge invariant and thus B and F should likely appear in the action in combination with each other.

All these influences combine to make plausible the action [83]:

$$S=-T_p\int_{M_p} d^{p+1}\xi\; e^{-\Phi}\sqrt{-\det(G_{ab}+B_{ab}+2\pi\alpha' F_{ab})}, \tag{2.2.8}$$

where T_p is the brane tension.

See §2.6 for the generalisation of this to supersymmetry.

2.3 Supersymmetry in $AdS_m \times S^n$

Anti-de Sitter space in $D = p + 2$ dimensions can be expressed in terms of the coordinates X_i, $i \in \{-1, 0, 1, \ldots, p{+}1\}$ of $D{+}1$ dimensional flat space by the hyperboloid [85]:

$$-X_{-1}^2 - X_0^2 + X_1^2 + \ldots + X_{p+1}^2 = -R^2. \tag{2.3.1}$$

The $D+1$ flat space metric is: $\eta = \mathrm{Diag}(-1, -1, 1, \ldots, 1)$. The isometry group of this hyperboloid is $SO(D-1, 2)$.

The isotropy group of AdS_D is $SO(D-1, 1)$.

This is most easily seen through analogy to S^D space. This hypersphere can be described by an embedding into a $D+1$ dimensional Euclidean space:

$$X_0^2 + X_1^2 + \ldots + X_D^2 = R^2.$$

The symmetry group is $SO(D+1)$. Any point on the sphere S^D is invariant under rotations around the axis passing through that point. The rotations around such an axis describe the group $SO(D)$, thus the isotropy group is $SO(D)$ and the coset space of S^D is $\frac{SO(D+1)}{SO(D)}$.

Take S^2 as an explicit example. The symmetry group is $SO(3)$. Consider the "north pole," point (0,0,1). This point is invariant by rotations around the axis passing through the north and south poles. Ie. it is invariant under the action of the circle group S^1.

In analogy, the isotropy group of AdS_D is $SO(D-1, 1)$, where the isotropy group has been chosen to lie in a direction that excludes a time timelike dimension.

The coset space of $AdS_m \times S^n$ for bosonic spacetime is thus: $\frac{SO(m-1,2)\times SO(n+1)}{SO(m-1,1)\times SO(n)}$.

For $AdS_3 \times S^3$ this is: $\frac{SO(2,2)\times SO(4)}{SO(2,1)\times SO(3)} \sim \frac{(SO(2,1)_L\times SO(3)_L)\times(SO(2,1)_R\times SO(3)_R)}{SO(2,1)\times SO(3)}$. The equivalence to the second form can be seen through the isomorphisms listed in Table 9 of [45].

In this thesis we are interested in the supersymmetric extensions of $AdS_3 \times S^3$ and $AdS_5 \times S^5$.

$$\begin{aligned} AdS_3 \times S^3 : \ & \frac{(SO(2,1) \times SO(3))_L \times (SO(2,1) \times SO(3))_R}{SO(2,1) \times SO(3)} \\ & \rightarrow \frac{SU(1,1|2)_L \times SU(1,1|2)_R}{SO(2,1) \times SO(3)} \end{aligned} \tag{2.3.2}$$

$$AdS_5 \times S^5 : \ \frac{SO(4,2) \times SO(6)}{SO(4,1) \times SO(5)} \rightarrow \frac{SU(2,2|4)}{SO(4,1) \times SO(5)} \tag{2.3.3}$$

where the notation $SU(a,b|c)$ refers to a special unitary group in $C^{a,b|c}$. $C^{a,b|c}$ is a complex vector space with $a+b$ bosonic and c fermionic dimensions, and the signature of the hermitian form in the bosonic directions has a + signs and b - minus signs.

2.4 Cartan Forms

Consider a Lie algebra $\mathfrak{g}$:

$$[M_a, M_b] = f^c_{ab} M_c, \tag{2.4.1}$$

with elements of the group G (exponentiation of $\mathfrak{g}$) being $g = e^{x^a M_a}$.

The Cartan forms are given by (see for eg [77]):

$$g^{-1} dg = L^a M_a. \tag{2.4.2}$$

These Cartan one forms L^a are important, as they are inherently symmetric under the left action of G. Consider transforming the position dependent $g(x)$ by $h \in G$, $g' = hg$, for h position independent. Under this transformation the Cartan forms become:

$$g^{-1} h^{-1} d(hg) = g^{-1} dg = L^a M_a. \tag{2.4.3}$$

Using Cartan forms, metrics that are symmetric under the transformation of the group can be quickly written:

$$g^{-1} dg \otimes g^{-1} dg. \tag{2.4.4}$$

Similarly, any action composed entirely from Cartan forms is left invariant under the action of the group.

This is generalised to supersymmetric cases in §'s 2.7 and 2.11.

2.5 κ Symmetry Review

When describing supersymmetric effective actions using the Green-Schwarz (GS) formulation, for strings, p-branes or Dp-branes, as is done in the analysis in this chapter, a special symmetry specific to the GS formulation must be considered.

Take the example of type II supersymmetry. The Green Schwarz formulation of effective actions, described in §2.6 and §2.7, imposes the supersymmetry as a symmetry of the spacetime coordinates, and the string or brane that is being considered has a bosonic world-volume mapped to this supersymmetric space (see for eg [3, 10, 13, 14, 25, 27, 77, 90, 91, 100, 104, 108, 122]). In a ten dimensional space there are thus two 32 dimensional spinors. The Majorana-Weyl conditions restrict each of these spinors to having 16 fermionic degrees of freedom, giving a total of 32. Now string theory in 10 dimensional spacetime has 8 bosonic degrees of freedom in the light cone gauge, $\mathcal{N} = 2$ should map this to 16 fermionic degrees of freedom.

Such a formulation of string theory must therefore have an extra symmetry which maps half of the fermionic degrees of freedom into the other half, thus reducing the number of independent variables by a factor of 2. This extra symmetry is the κ symmetry.

The extra symmetry resembles:

$$\delta_\kappa \theta^\alpha = \kappa^\alpha, \quad \frac{1}{2}(1+\Gamma)\kappa = \kappa, \quad \delta_\kappa X^a = -i\bar{\theta}\Gamma^a\kappa, \quad (\Gamma)^2 = 1. \tag{2.5.1}$$

where θ^α is a spinor describing the spacetime fermionic degrees of freedom (a Majorana-Weyl 32 component spinor in this example) and X^a are the bosonic degrees of freedom. The Γ present here is a matrix constructed from the both Clifford algebra Γ^a matrices and the stress energy tensor. $\frac{1}{2}(1+\Gamma)$ is a projector that eliminates half of the Grassmann degrees of freedom (explicitly expressing the breaking of half of the supersymmetry) [13, 27, 122].

Therefore the number of fermionic degrees of freedom has thus been reduced from 32 to 16, the right amount for a ten dimensional type II theory.

If the action that is being found is that of a D-brane (or p-brane), then the presence of the D-brane breaks a further half of the supersymmetry. Thus in the example being considered earlier the number of fermionic degrees of freedom is reduced to 8, as in §2.1.3 and §2.1.4, allowing type I open strings to be present in the theory as well.

κ symmetry also plays an inportant part in formulating the actions of doubly supersymmetric D-branes (branes with a supersymmetric world volume embedded into a supersymmetric spacetime). See for example: [4, 72, 84, 122].

2.6 κ Symmetry in 10 Dimensional Curved Superspace

The case of describing D-branes using the GS method was done for the general type IIB case in [26, 27]. In this section the main results from these papers will be summarised.

The first ingredient needed to describe the brane action is the supersymmetric Dirac-Born-Infeld action for a bosonic Dp-brane world-volume embedded in a general 10 dimensional target space:

$$I_{DBI} = -\int_{M_{p+1}} d^{p+1}\sigma \; e^{\frac{p-3}{4}\phi}\sqrt{-det(G_{ij} + e^{-\frac{1}{2}\phi}F_{ij})}, \tag{2.6.1}$$

where M_{p+1} is the world-volume of the Dp-brane, ϕ is the dilaton field, $F_{ij} = \mathcal{F}_{ij} - B_{ij}$ is the field strength of the brane's gauge field minus the NS-NS 2-form B_{ij}. See §2.2 for more information on B_{ij}.

I_{DBI} is supersymmetric. The metric is:

$$G_{ij} = L_i^a L_j^b \eta_{ab}. \tag{2.6.2}$$

i & j run over $\{0, \ldots, p\}$ and are the world-volume indices and a & b are the indices correspond to the bosonic components of the target space. L_i^a and L_j^b are pullbacks from the world-volume to the bosonic components of the target space,[5] and η_{ab} is the Minkowski metric for the bosonic components of the target space. The metric is supersymmetric as it is expressed in terms of L_i^a and as seen in §2.4. such a pullback is symmetric under the target space symmetry, which is a supersymmetry in this case.

$F = \mathcal{F} - B$ is also supersymmetric. $\mathcal{F}$ is constructed in terms of Cartan forms (see the next section for an example) and $B = dA$ is supersymmetric for a suitably chosen gauge field.

The vielbein is part of the larger super vielbein pulling back the whole super target space to the world-volume.

$$L^A = d\sigma^i L_i^A = d\sigma^i \partial_i X^M L_M^A, \tag{2.6.3}$$

where A is a super index that runs over all the bosonic and fermionic target space coordinates, σ^i are the world-volume coordinates, X^M are the target space super coordinates and L^A are the Cartan forms for the target space.

If κ symmetry did not have to be taken into account, this supersymmetric DBI action would be sufficient to describe the brane dynamics, but due to the necessary presence of κ symmetry, the appropriate Wess-Zumino (WZ) term must be found. The first step to this is defining the κ symmetry correctly.

The general form of the κ symmetry of the system is described by [27, 117]:

$$\begin{aligned} \delta_\kappa X^M &= \kappa^\alpha L_\alpha^M, & (2.6.4)\\ \kappa^a &= 0, & (2.6.5)\\ \delta_\kappa A &= i_\kappa B_{(2)}, & (2.6.6)\\ \delta_\kappa \phi &= \kappa^\alpha \partial_\alpha \phi, & (2.6.7)\\ \Gamma\kappa &= \kappa, & (2.6.8)\end{aligned}$$

α are the indices labelling the fermionic superspace coordinates. $\delta_\kappa = \{d, \imath_\kappa\}$ is the Lie derivative. The contraction $\imath_w$ in a Lie derivative δ_w defines the variation with respect to some vector or spinor w^j:

$$\imath_w \nu_{ij} dx^i \wedge dx^j = \nu_{ij} w^i dx^j - \nu_{ij} dx^i w^j = 2\nu_{ij} w^i dx^j. \tag{2.6.9}$$

While this determines the general form of the κ symmetry, it is necessary to determine the correct Γ matrix used in the projection. It has to have half of the rank of the Clifford algebra (in ten dimensions 16 down from 32). This is achieved by making a Γ matrix that is traceless and has eigenvalues of $\pm$ 1. Its square must also be unity: $(\Gamma)^2 = \mathbf{1}$. This last is required such that $\frac{1}{2}(\mathbf{1} + \Gamma)$ is a projector.

[5]The Cartan forms here are typically dependent on the bosonic and fermionic coordinates of the target space. The exact form of this dependence is determined by the details of the supersymmetry algebra. The solution for the Cartan forms in eqn (2.7.3) subject to the algebra (2.7.1) is found in Appendix B.

It was conjectured in [26] that Γ in ten dimensional IIB superspace is defined by:

$$d^{p+1}\sigma\ \Gamma \;=\; -\frac{e^{\frac{1}{4}(p-3)\phi}}{\mathcal{L}_{DBI}}\exp(e^{-\frac{1}{2}\phi}F)\wedge(\bigoplus_n \Gamma_{(2n)}\mathcal{K}^n\mathcal{E})\Big|_{M_{p+1}}, \tag{2.6.10}$$

$$= -\frac{e^{\frac{1}{4}(p-3)\phi}}{\mathcal{L}_{DBI}}\sum_{j=0}^{\frac{p+1}{2}}(e^{-\frac{1}{2}\phi}F)^j\Gamma_{(p+1-2j)}\mathcal{K}^{\frac{p+1-2j}{2}}\mathcal{E} \tag{2.6.11}$$

$$\Gamma_{(m)} \;=\; \frac{1}{m!}L^{a_m}\wedge\ldots\wedge L^{a_1}\Gamma_{a_1\ldots a_n},\qquad \Gamma_{a_1\ldots a_n}=\Gamma_{[a_1}\ldots\Gamma_{a_N]} \tag{2.6.12}$$

where p is odd and $\mathcal{K}$ and $\mathcal{E}$ act on the $\mathcal{N}=2$ supersymmetry labels (A.1.4). The proof that this matrix embodies the correct properties is contained in [27].

The WZ action should be of the form [26,27,34,58]:

$$I_{WZ} \;=\; \int_{M_{p+1}} e^F\wedge C,\qquad C=\bigoplus_{n=0}^{10}C_{(n)}, \tag{2.6.13}$$

$$= \int_{M_{p+1}}\sum_{j=0}^{\frac{p+1}{2}}\frac{1}{j!}F^jC_{(p+1-2j)} \tag{2.6.14}$$

where $C_{(n)}$ are the R-R n-form potentials pulled back to the world volume (a $C_{(n)}$ form charge couples to a D$(n-1)$-brane, so a C_{10} couples to the space filling D9-brane, and a C_0 form couples to a D(-1)-brane, an instanton). These R-R forms are the R-R vertex operators of §2.1.3. In general the R-R n-forms in terms of the target space variables are unknown and need to be derived for particular backgrounds.

Observation of the κ transformation of the DBI action yields that the necessary κ variation of the WZ action to make the entire action κ invariant is:

$$\delta_\kappa S_{WZ}=\int_M e^F\wedge\imath_\kappa R, \tag{2.6.15}$$

where the same notation for an integration of a form is used as in eqn (2.6.14), and R for IIB supersymmetry is given by [27]:

$$R \;=\; e^{B_{(2)}}\wedge d(e^{-B_{(2)}}\wedge C)=\bigoplus_{n=1}^{10}R_{(n)}, \tag{2.6.16}$$

$$R_{(n)a_1\ldots a_{n-2}\alpha\beta} \;=\; 2ie^{\frac{n-5}{4}\phi}(\Gamma_{a_1\ldots a_{n-2}}\mathcal{K}^{\frac{n-1}{2}}\mathcal{E})_{\alpha\beta}, \tag{2.6.17}$$

$$R_{(n)a_1\ldots a_{n-1}\alpha} \;=\; -\frac{n-5}{4}e^{\frac{n-5}{4}\phi}(\Gamma_{a_1\ldots a_{n-1}\alpha\beta}\mathcal{K}^{\frac{n-1}{2}}\mathcal{E}\partial_\beta\phi). \tag{2.6.18}$$

The expression for R was originally derived in [66].

From this data, for any given background, it should be possible (after considerable calculation) to construct for any D-brane in any IIB supersymmetric ten dimensional background a super-symmetric and κ symmetric action. This is what is done in the research in §2.15 §2.14, for the ten dimensional backgrounds $AdS_3\times S^3\times S^4$ and

$AdS_5 \times S^5$. The technique is then adjusted to the six dimensional background: $AdS_3 \times S^3$ (§2.12 and §2.13).

In this section only the results from [26, 27] for IIB supersymmetry were summarised, as that is what is applicable for this research, but their analysis was also done for IIA supersymmetry.

2.7 D3-Brane in $AdS_5 \times S^5$

The case of a supersymmetric and κ symmetric action for a D3-brane in IIB $AdS_5 \times S^5$ space was done by Metsaev and Tseytlin in [91]. The warping of the supersymmetric background into $AdS_5 \times S^5$ is due to the presence of many parallel and coincident D3-branes, and thus a D3-brane action in this background acts as a probe into this background, ie, the action of one of the D3-branes in the presence of many other D3-branes generating the background, but free to move in the entire space. The main motivation for investigating such an action was that due to the Maldacena Conjecture (see §2.9) in the large N limit the D3-brane action should coincide with the dominant IR components of the $\mathcal{N} = 4$ $D = 4$ Super Yang Mills theory (SYM) (for the case when the $U(1)$ $\mathcal{N} = 4$ vector multiplet is kept as an external background and integrating out the massive SYM fields). In particular fixing the static and κ symmetry gauges should make the D3-brane action equivalent to a SYM effective action.

To find such an action it is best to use the GS method outlined in §2.6. The first step to this is to construct the superalgebra. The super coset space for $AdS_5 \times S^5$ is $\frac{SU(2,2|4)}{SO(4,1)\times SO(5)}$ and has superalgebra:

$$\begin{aligned}
&[P_a, P_b] = J_{ab},\\
&[P_{a'}, P_{b'}] = -J_{a'b'},\\
&[P_a, J_{bc}] = \eta_{ab}P_c - \eta_{ac}P_b,\\
&[P_{a'}, J_{b'c'}] = \eta_{a'b'}P_{c'} - \eta_{a'c'}P_{b'},\\
&[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}, \qquad (2.7.1)\\
&[J_{a'b'}, J_{c'd'}] = \eta_{b'c'}J_{a'd'} + \eta_{a'd'}J_{b'c'} - \eta_{a'c'}J_{b'd'} - \eta_{b'd'}J_{a'c'},\\
&[Q, P_{\hat{a}}] = \frac{i}{2}Q\mathcal{E}\sigma_+\Gamma_{\hat{a}},\\
&[Q, J_{\hat{a}\hat{b}}] = -\frac{1}{2}Q\Gamma_{\hat{a}\hat{b}},\\
&\{Q_{\hat{\alpha}}, Q_{\hat{\beta}}\} = -2i(\hat{C}\Gamma^{\hat{a}}\pi_+)_{\hat{\alpha}\hat{\beta}}P_{\hat{a}} + \mathcal{E}[(\hat{C}\Gamma^{ab}\sigma_-)_{\hat{\alpha}\hat{\beta}}J_{ab} - (\hat{C}\Gamma^{a'b'}\sigma_-)_{\hat{\alpha}\hat{\beta}}J_{a'b'}].(2.7.2)
\end{aligned}$$

The notation used is outlined in Appendix A.3.

The next step is to express the Cartan forms and the Maurer-Cartan relations. The Cartan forms are described by:

$$G^{-1}dG = L^{\hat{a}}P_{\hat{a}} + \frac{1}{2}L^{\hat{a}\hat{b}}J_{\hat{a}\hat{b}} + L^{\hat{\alpha}}Q_{\hat{\alpha}}, \qquad (2.7.3)$$

where $L^A = dX^M L_M^A$ are the Cartan forms, for the target space coordinates $X^M = (x^{\hat{a}}, \theta^{\hat{\alpha}})$. The Cartan forms are inherently supersymmetric (§2.4). In [91] the gauge choice $G = g(x)e^{\theta Q}$ is made.

The Maurer Cartan relations and the method used to derive them can be found in Appendix B. The equations appearing here are derived for $AdS_5 \times S^5$ [91], however due to the intentional similarity of notation in this thesis to [91], these background calculations also apply to $AdS_3 \times S^3$ and $AdS_3 \times S^3 \times T^4$ with only small changes to the ranges of the indices, and the structure of the spinors, as explained in the appendix.

In the same appendix the parametrized Cartan forms $L_t^A(x, \theta) = L^A(x, t\theta)$ are defined, and the useful formulas describing $\partial_t L_t^A$ are derived.

With these useful formulas derived, it is next important to use eqn (2.6.10) to identify the projection matrix Γ defining the κ symmetry:

$$\Gamma = \frac{\varepsilon^{i_1 \ldots i_4}}{\mathcal{L}_{DBI}} \left(\frac{\Gamma_{i_1 \ldots i_4}}{4!} \mathcal{E} + \frac{\Gamma_{i_1 i_2} F_{i_3 i_4}}{4} \mathcal{J} + \frac{F_{i_1 i_2} F_{i_3 i_4}}{2^3} \mathcal{E} \right), \tag{2.7.4}$$

where $\Gamma_{i_1 \ldots i_n} = \widehat{L}_{[i_1} \ldots \widehat{L}_{i_n]}$.

This can be used to find the κ variations of the Cartan forms, as done in Appendix B.

As described in §2.6 the action of the D3-brane takes the form of a Dirac-Born-Infeld action with a Wess-Zumino term:

$$\begin{aligned} S &= S_{DBI} + S_{WZ}, & (2.7.5)\\ S_{DBI} &= -\int_{M_4} d^4\sigma \sqrt{-det(G_{ij} + F_{ij})}, & (2.7.6)\\ S_{WZ} &= \int_{M_4} \Omega_4 = \int_{M_5} H_5 \qquad H_5 = d\Omega_4, & (2.7.7) \end{aligned}$$

where Ω_4 is some four form, G_{ij} is the world-volume metric ($i, j \in \{0, 1, 2, 3\}$), M_5 is a manifold whose boundary is the bosonic world volume M_4 and F is the supersymmetric extension of the field strength.

The dilaton field ϕ is an arbitrary constant in $AdS_m \times S^n$ space [2,97] and thus can be set to zero and does not appear in the action here (cf. eqn (2.6.1)).

The next step is to find an explicit form for F:

$$F = dA + 2i \int_0^1 dt L_t^{\hat{a}} \wedge \bar{\theta} \Gamma^{\hat{a}} \mathcal{K} L_t. \tag{2.7.8}$$

In §2.11.2 it is outlined how to prove this to be supersymmetric. First act on F with the exterior derivative d, and then use the Maurer Cartan formulas to expand the integrand of dF. After transforming with the Fierz Identity eqn (2.13.8), use the $\partial_t L_t^A$ relations to express dF totally in terms of Cartan forms, which are inherently supersymmetric. Thus dF is supersymmetric. F is also supersymmetric for an appropriate choice of the transformation of the gauge field A [91].

G_{ij} is also supersymmetric as it is expressed in terms of Cartan forms, and thus the DBI is supersymmetric.

With the DBI action found, the Wess-Zumino term can be found by finding the κ variation of the DBI action, and then finding an appropriate supersymmetric WZ term whose κ variation cancels the DBI action's contribution. The five form H_5 (see eqns (2.7.7)) is found to be given by:

$$\begin{aligned} H_5 &= i\bar{L} \wedge (\frac{1}{6}\widehat{L} \wedge \widehat{L} \wedge \widehat{L}\mathcal{E} + F \wedge \widehat{L}\mathcal{J}) \wedge L \\ &+ \frac{1}{30}(\varepsilon^{a_1 \ldots a_5} L^{a_1} \wedge \ldots \wedge L^{a^5} + \varepsilon^{a'_1 \ldots a'_5} L^{a'_1} \wedge \ldots \wedge L^{a'^5}). \end{aligned} \tag{2.7.9}$$

H_5 and thus S_{WZ} is supersymmetric as it is expressed entirely in terms of supersymmetric Cartan forms. For more detail on how to do this calculation, the calculation is similar to that in §'s 2.12, 2.14 and 2.15.

In order to find Ω_4, which allows the WZ term to be expressed as an integral over the same world-volume as the DBI term, employ the t parametrisation mentioned earlier (and summarised in Appendix B). Let:

$$H_5 = \int_0^1 dt \partial_t H_{5,t} + H_{5,t=0}, \tag{2.7.10}$$

and use the formulas (B.21), (B.22) to expand the $\partial_t L^A$ terms in the integrand. Then after transforming some of the terms using Fierz Identities (2.11.11) and (2.13.8), $\partial_t H_{5,t}$ is then in a form in which (B.1) and (B.2) can be used to re-express it as a total exterior derivative:

$$\partial_t H_{5,t} = 2id(\frac{1}{6}\bar{\theta}(\bar{L}_t)^3 \mathcal{E} L_t + \bar{\theta}\bar{L}_t \wedge F_t \wedge \mathcal{J} L_t). \tag{2.7.11}$$

$H_{5,t}$ at $t = 0$ (equivalent to $\theta = 0$) has purely bosonic contributions, as at $t = 0$ the super Cartan forms reduce to the Cartan forms of just the bosonic parts of the algebra (see Appendix B). Thus:

$$H_{5,t=0} = \frac{1}{30}(\varepsilon^{a_1 \ldots a_5} e^{a_1} \wedge \ldots \wedge e^{a^5} + \varepsilon^{a'_1 \ldots a'_5} e^{a'_1} \wedge \ldots \wedge e^{a'^5}). \tag{2.7.12}$$

As $d\Omega_4 = H_5$, it is thus shown that the Wess-Zumino action can be expressed as:

$$\begin{aligned} S_{WZ} &= 2i \int_{M_4} \int_0^1 dt (\frac{1}{6}\bar{\theta}(\bar{L}_t)^3 \mathcal{E} L_t + \bar{\theta}\bar{L}_t \wedge F_t \wedge \mathcal{J} L_t) \\ &+ \int_{M_5} \frac{1}{30}(\varepsilon^{a_1 \ldots a_5} e^{a_1} \wedge \ldots \wedge e^{a^5} + \varepsilon^{a'_1 \ldots a'_5} e^{a'_1} \wedge \ldots \wedge e^{a'^5}). \end{aligned} \tag{2.7.13}$$

2.8 Killing Gauge

Actions of the supersymmetric form $S = S_{DBI} + S_{WZ}$ seen in §'s 2.6 and 2.7 are both κ symmetric and reparametrization symmetric (the latter because the actions are expressed in terms of reparametrization invariant Cartan forms). As both of these symmetries are local symmetries, they can be gauge fixed to simplify the action, and also act to express the action in terms of supersymmetric coordinates on the world-volume.

κ symmetry maps half of the fermionic degrees of freedom to the other half, thereby halving the number of independent fermionic degrees of freedom. Gauge fixing the κ symmetry eliminates the redundant fermionic variables. The most common gauge fixing for the κ symmetry is the Killing gauge [13,74,76,78,101].

The Killing gauge is a gauge that is based on the background isometry. When applied to actions in $AdS_m \times S^n$ space it is determined by the alignment of the parallel D-branes whose gravity is warping the spacetime. This is done using the projection:

$$\mathcal{P}^{IJ}_{\pm} = \frac{1}{2}(\delta^{IJ} \pm \Gamma_* \varepsilon^{IJ}), \tag{2.8.1}$$

where the projection matrix Γ_* is of the form:

$$\Gamma_* = (i)^n \Gamma_{0\ldots p}. \tag{2.8.2}$$

This is for a background generated by Dp-branes lying in dimensions $\{0, \ldots, p\}$, and I & J label which supersymmetry a spinor belongs to in type $\mathcal{N}$ supersymmetry. ($\Gamma_{0\ldots p}$ is defined in eqn (2.11.14)). The Killing gauge defined as:

$$\mathcal{P}^{IJ}_{-}\theta^J = 0. \tag{2.8.3}$$

It should be noted that in eqn (2.8.2), $n = 0$ or 1, and it depends on p. This is an essential property of the projection matrix because when the Killing Gauge is inserted into formulas involving the contraction of spinors it does not alter the hermitian/antihermitian behaviour, for example:

$$(\bar{\theta}\Gamma^{\hat{a}}\psi)^\dagger = -(\bar{\psi}\Gamma^{\hat{a}}\theta). \tag{2.8.4}$$

Inserting the projections into the left hand side of (2.8.4) gives:

$$(\theta^\dagger \Gamma^0 \mathcal{P}_- \Gamma^{\hat{a}} \mathcal{P}_+ \psi)^\dagger \quad = \quad -(\psi^\dagger \mathcal{P}^\dagger_+ \Gamma^0 \Gamma^{\hat{a}} \Gamma^0 \mathcal{P}^\dagger_- (\Gamma^0)^3 \theta). \tag{2.8.5}$$

This in turn gives the right hand side of (2.8.4) if $\mathcal{P}_\pm$ are hermitian and $\mathcal{P}_\pm \Gamma_0 = \Gamma_0 \mathcal{P}_\mp$. For this to be true, Γ_* must be antihermitian and it must anticommute with Γ_0. For the $AdS_5 \times S^5$ background (for which $p = 3$ as generated by D3-branes), this is satisfied for $n = 0$ [78]. See Appendix E for details on Γ_* for general p.

Examples of this gauge being used to simplify the fundamental string action in $AdS_5 \times S^5$ can be found in [76,78]. It is applied to the actions found in the work of this thesis in Appendix E.

As an aside, the most common way to fix the reparametrization symmetry is to use the static gauge. This involves choosing the brane to lay in specific target space coordinates, for example for a D3-brane in ten dimensional target space one static gauge would be:

$$X^a = \sigma^a, \quad \forall\, a \in \{0,1,2,3\}, \quad X^a = \text{const}, \quad \forall\, a \in \{4, \ldots, 9\}. \tag{2.8.6}$$

2.9 AdS/CFT conjecture

It has been conjectured by J. Maldacena that IIB string theory in $AdS_{d+1} \times S^q \times M^{D-q-d-1}$ space is equivalent to conformal field theory (CFT) on the d dimensional conformal boundary of AdS_{d+1} [35, 85] (D is the total number of spacetime dimensions and $M^{D-q-d-1}$ is a compact manifold). This is a very powerful conjecture, as problems intractable in one mathematical context may be quite tractable in the other, potentially allowing a whole new understanding of string theory. For example, IIB string theory in $AdS_5 \times S^5$ should be equivalent to a $\mathcal{N} = 4$ $D = 4$ Super-Yang-Mills theory, a mathematical framework which does not yield much to current mathematical analysis. Therefore such a correspondence could lead to greater understanding on how to work with these higher dimensional CFT's. Another important example is string theory on $AdS_3 \times S^3 \times M^4$ (M^4 is a Ricci flat four dimensional compact manifold), as it corresponds to a two dimensional supersymmetric CFT (the details of which are contained in [55]). Two dimensional super CFT is very well understood, and thus could yield many new results for string theory.

A brief outline of Maldacena's AdS/CFT conjecture shall be given here. The conjecture arose through looking at Planck scale black holes. In particular, Planck scale black holes are thought to be generated by the gravity of many parallel D-branes. This is because Planck scale black hole singularities can be extended objects, just as D-branes are, and D-branes have the same long range fields as black holes [65]. A bonus of describing black holes via D-branes is that D-branes in string theory have known degrees of freedom, which allows accurate calculations of the brane's Hawking entropy to be possible.

There are two options for describing these D-brane black holes. The first is via supergravity [65], and the second is via the super conformal field theory present on the parallel and coincident D-branes [61, 85].

2.9.1 IIB Supergravity D3-brane Black Holes

Let us first look at the case of black holes generated by D3-branes. Consider N D3-branes, each brane separated by a distance r from the next [65].

In a supergravity description, the presence of the D3-branes is encapsulated by the spacetime warping denoted by the metric. The metric is found by solving the supergravity equations of motion [65]:

$$ds^2 = f^{-\frac{1}{2}} dx^\mu dx^\nu \eta_{\mu\nu} + f^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2), \tag{2.9.1}$$

$$f(r) = 1 + \frac{\lambda l_s^4}{r^4} \qquad \lambda = 4\pi g_s N, \tag{2.9.2}$$

where g_s is the string coupling and l_s is the string length scale. The first term corresponds to the brane coordinates (ie $\mu \in \{0, \ldots, 3\}$) and the $d\Omega_5^2$ term corresponds to

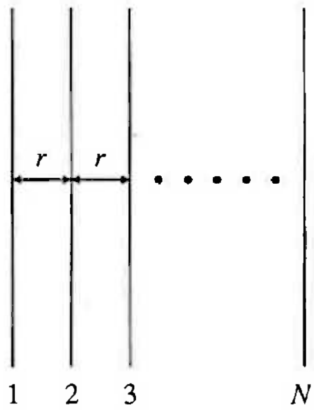


Figure 2.5: *N parallel D-branes separated by a distance r.*

the S^5 metric.

In the small r limit this metric becomes:

$$ds^2 = \frac{r^2}{\sqrt{\lambda l_s^4}} dx^\mu dx^\nu \eta_{\mu\nu} + \frac{\sqrt{\lambda} l_s^2}{r^2} dr^2 + \sqrt{\lambda} l_s^2 d\Omega_5^2. \tag{2.9.3}$$

This exactly corresponds to the $AdS_5 \times S^5$ metric, with radius $R = \lambda^{\frac{1}{4}} l_s$ (see §2.3 for more details about AdS space). The first two terms are the AdS_5 metric, and the last term is the S^5 metric. The metric is singular in the limit $r \to 0$, the limit in which the metric describes a black hole.

The metric is invariant under $SO(4,2) \times SO(6)$ and each point in the space is invariant under $SO(4,1) \times SO(5)$, so the coset space of $AdS_5 \times S^5$ is $\frac{SO(4,2)\times SO(6)}{SO(4,1)\times SO(5)}$. The supersymmetric extension of this is $\frac{SU(2,2|4)}{SO(4,1)\times SO(5)}$ (see §2.3).

Note that earlier we discussed the limit of the metric for $r \to 0$, however supergravity is usually only an accurate approximation to string theory for $r \gg l_s$. If r is smaller the stringy and quantum effects between the branes need to be taken into account. However if the string coupling g_s is taken to zero at the same time as $r \to 0$, the quantum effects and the closed string modes can be ignored and supergravity is still valid.

From this it is possible to conclude that the supergravity description of D-brane Planck scale black holes is valid for $r \ll R$, $g_s \to 0$, $R \gg l_s$ and all energies (E) of interest satisfy: $E\ l_s \ll 1$. And the description is still valid for: $r \ll l_s$.

This is equivalent to: $g_s N \gg 1$, $g_s \ll 1$ and $N \gg 1$.

2.9.2 SCFT: $\mathcal{N} = 4$ $4D$ $U(N)$ SYM Description of D3-brane Black Holes

The other way of describing the Planck scale black hole generated via the D3-branes in Figure 2.5 is via the gauge theories existing on the branes [61, 85]. Each D-brane has a $U(1)$ gauge field on it, originating from the $U(1)$ gauge field on each end of an open string, so N parallel and non-coincident D-branes are described by a $U(1)^N$ gauge theory. Of course, there are stringy and quantum physical effects in the bulk away from and between the branes, but in the limit $r \to 0$, with $\frac{r}{\alpha'}$ =fixed it is possible to ignore these effects. In this limit, as described in §2.1.2, the massless gauge fields of $U(1)^N$ are enhanced by the presence of new massless states possible due to the ground states of supersymmetric open strings stretched between different D-branes becoming massless, enhancing the Chan-Paton gauge symmetry from $U(1)^N$ to $U(N)$.

Thus for the IIB D3-branes the $\mathcal{N}=4$ $D=4$ SYM theory is obtained on the flat space of the coincident branes, which is the conformal boundary of the AdS_5 space.

Like the supergravity theory, the super coset space of $\mathcal{N}=4$ $D=4$ SYM is $\frac{SU(2,2|4)}{SO(4,1)\times SO(5)}$.

This description is valid for $r \ll l_s$, finite N, $g_s \ll 1$ and $g_s N \ll 1$.

2.9.3 The Conjecture

The Maldacena conjecture arises when comparing the descriptions of the D-brane black holes.

The IIB supergravity theory is valid for:

$$g_s N \gg 1, \quad g_s \ll 1, \quad N \gg 1. \tag{2.9.4}$$

The $\mathcal{N}=4$ $D=4$ SYM theory is valid for:

$$g_s N \ll 1, \quad g_s \ll 1, \quad N \text{ finite}. \tag{2.9.5}$$

Now if the SYM theory is extrapolated to large N the two theories are valid for the same conditions. When this fact is combined with the consideration that they have the same supersymmetry group the following conjecture arises: that IIB supergravity on an $AdS_5 \times S^5$ background is equivalent to the super conformal field theory (SCFT) $\mathcal{N}=4$ $D=4$ SYM for large N.

A second conjecture arises for considering N not necessarily $N \gg 1$ for the supergravity theory. In this case the supergravity approximation is no longer valid, and the full IIB string theory needs to be considered (which still has the same coset structure in the $AdS_5 \times S^5$ background) [85].

In this case a second conjecture arises that IIB string theory on $AdS_5 \times S^5$ is equivalent to $\mathcal{N}=4$ $D=4$ SYM for all N.

Similar conjectures exist between IIB string theory on $AdS_3 \times S^3 \times M^4$ and 2 dimensional SCFT (for D1 and D5 branes generating the background), M-theory in $AdS_4 \times S^7$ and 3 dimensional SCFT (for M-theory membranes generating the background) and M-theory in $AdS_7 \times S^4$ and 5 dimensional SCFT (for M-theory 5-branes generating the background) [85].

2.9.4 The $AdS_3 \times S^3 \times M^4$ Case

Considering the AdS_3 is of primary concern in this research, let us briefly focus on this case [85].

The background metric is generated by the gravity of parallel D5 and D1-branes, all separated by distance r. Four of the D5-brane world-volume dimensions are wrapped on the M^4 compact manifold, such that the remaining two dimensions lay in the AdS_3 space and are parallel to the D1-branes. The branes dimensions are distributed as described in Table 2.1.

Dimension	0	1	2	3	4	5	6	7	8	9
D1-brane	×	×								
D5-brane	×	×					×	×	×	×

Table 2.1: *Orientation of Background D1-D5 branes in $AdS_3 \times S^3 \times M^4$. A D-brane world-volume is extended in each of the dimensions marker by a cross.*

The first three dimensions belong to the AdS_3 space, the second three to S^3, and the last four to M^4.

If M^4 is compactified away (and the Kaluza Klein modes are ignored) we can study the IIB string theory on the truncated $AdS_3 \times S^3$ space.

The supergravity metric due to the gravity of these D3-branes, when considering only these six dimensions, is found to be:

$$\begin{aligned} ds^2 &= f_1^{-\frac{1}{2}} f_5^{-\frac{1}{2}} dx^\mu dx^\nu \eta_{\mu\nu} + f_1^{\frac{1}{2}} f_5^{\frac{1}{2}} (dr^2 + r^2 + r^2 d\Omega_3^2), & (2.9.6)\\ f_1(r) &= 1 + \frac{g_s \alpha' Q_1}{v r^2}, \qquad f_5(r) = 1 + \frac{g_s \alpha' Q_5}{r^2}, & (2.9.7)\\ v &= \frac{\text{volume}(M^4)}{(2\pi)^4 (\alpha')^2}, & (2.9.8) \end{aligned}$$

where Q_1 and Q_5 are the number of D1 and D5-branes respectively.

In the small r limit this becomes the metric of $AdS_3 \times S^3$, where the radii of these spaces are determined by Q_1 and Q_5.

The super coset space of this theory is: $\frac{SU(1,1|2) \times SU(1,1|2)}{SO(2,1) \times SO(3)}$ (see §2.3).

To check the conjecture that this supergravity theory corresponds to the two dimensional SCFT with the same super coset (as well as for the other AdS/SCFT conjectures) it is necessary to identify phenomena in the string theory with phenomena in the SCFT. $AdS_3 \times S^3$ and its corresponding SCFT is often studied for this purpose as two dimensional SCFT is much more tractable than greater dimensional SCFT.

2.10 Motivation for studying the D1-D5 branes in $AdS_m \times S^n$

As mentioned previously, the original motivation for the research in this chapter arises from the Maldacena conjecture.

For a curved spacetime generated by the gravity of many parallel and coincident Dp-branes, finding an effective action of the Dp-brane in the background of these parallel and coincident branes acts as a probe into the system of D-branes generating the background. Ie the probe Dp-brane can be considered the action of one of the Dp-branes generating the spacetime curvature, teased a little way away from the other branes.

More importantly, considering that: D3-branes generate an $AdS_5 \times S^5$ background; D1 and D5-branes generate an $AdS_3 \times S^3 \times M^4$ background; M5-branes generates an $AdS_7 \times S^4$ background and M2-branes generates an $AdS_4 \times S^7$, the Maldacena conjecture predicts that IIB string theory in such backgrounds should be equivalent to a $p+1$ super conformal field theory for $AdS_{p+2} \times S^q \times M^{D-p-q-2}$ (where D is the number of dimensions). It was realised [91] that these probe brane effective actions should be equivalent to the leading IR terms of the super conformal field theories, when any massive fields have been integrated out.

The D3-brane effective action in $AdS_5 \times S^5$ was found in [91]. In §2.12 the effective action of a D1-brane in $AdS_3 \times S^3$ is found. The M^4 part of spacetime was not considered here, as most of the interesting physics should occur in $AdS_3 \times S^3$, rather than the M^4, which can be compactified away.

The $AdS_3 \times S^3$ background is generated by both D1 and D5 branes, lying in directions listed in Table 2.1. It can be seen there that four of the D5-brane spatial directions lie in the M^4, and thus when considering just $AdS_3 \times S^3$ it is sufficient to just consider D1-branes.

However in §2.13 the D5-brane in $AdS_3 \times S^3$ is analysed. This is equivalent to one of the background D5-branes rotated until it lies fully in this space. While this action will probably not turn out to be as relevant for finding an effective action describing the IR components of the two dimensional superconformal field theory, it is still physically interesting as it was found in [130] that a configuration of NS5-branes, with two groups of parallel and coincident branes where the the two groups share only one dimension is a stable BPS configuration. S duality [63] maps this to a system of D5-branes with the same orientation, and as S-duality preserves the BPS condition, the D5-brane system is also BPS. Therefore, a D5-brane filling the space $AdS_3 \times S^3$ sharing a single direction with the background D5-branes (with orientation given in Table 2.1) is a BPS configuration.

In §2.14 the action of a D5-brane in a background spacetime $AdS_5 \times S^5$ (in IIB string theory) generated by D3-branes will be investigated. The motivation for this is that such an orientation of branes describes the much researched Brane Baryon Vertex (see [23, 29, 56, 68, 99, 135] and others). In the near horizon limit of N parallel D3-branes, where the background becomes $AdS_5 \times S^5$, it was discussed in [99, 135] that the configuration of a D5-brane wrapped on S^5 and connected to each D3-brane by an open string, is equivalent in the $\mathcal{N} = 4,\ D = 4$ SYM theory (which is equivalent to the string theory via the Maldacena conjecture) to the $SU(N)$ baryon vertex, ie a bound state of N quarks. This configuration is also a BPS configuration. The D5-brane action originally found in §2.14 is for a D5-brane free to be orientated in any directions in the background space. The constraints that need to be applied to the D-brane, as well as the projectors that need to be applied to the spinors to break the supersymmetry to reflect the presence of D3-branes and strings, are outlined at the end of the section.

In §2.15 the action of a D5-brane probe free to move in the entire $AdS_3 \times S^3 \times M^4$ space is searched for. This was because this action is more appropriate to finding an effective action equivalent to the IR components of the 2dim SCFT then the D5-brane action restricted to $AdS_3 \times S^3$ found in §2.13. The compact space M^4 was chosen to be T^4.

In Appendix E the Killing gauge is applied to the actions found in this thesis. The Killing gauge breaks the κ symmetry, reducing the number of nonzero terms in the fermionic spinors by half.

Before the analysis into determining D-brane actions is done, the Cartan forms, Maurer-Cartan equations and the field strength F are investigated in §2.11 for the spacetime backgrounds being considered. This was first done for $AdS_5 \times S^5$ in [90, 91] (see §2.7) and adapted to $AdS_3 \times S^3$ and $AdS_3 \times S^3 \times T^4$ here.

The rest of the work in this chapter is original, except when material has been cited.

2.11 Cartan 1 Forms and the Supercoset Method

2.11.1 Cartan Forms and the Action

The notations and superalgebra used when finding actions in this Chapter, in the three different backgrounds of interest: $AdS_3 \times S^3$, $AdS_5 \times S^5$ and $AdS_3 \times S^3 \times T^4$, can be found in Appendix A. They have been formulated in such a way that the resulting Maurer-Cartan Equations, the Cartan form variations and various other useful relations can be written out in the same way for the three different backgrounds. The notation used is based upon that in [91,97].

As in [91,97], for a particular coset superspace the left invariant Cartan 1 forms are:

$$L^A = dX^M L_M^A,$$

where $X^M = (x, \theta)$ are the background super coordinates and

$$G^{-1}dG = L^{\hat{a}} P_{\hat{a}} + \tfrac{1}{2} L^{\hat{a}\hat{b}} J_{\hat{a}\hat{b}} + L^{\hat{\alpha}} Q_{\hat{\alpha}}, \tag{2.11.1}$$

where $G = G(x, \theta)$ is a coset representative of $\frac{SU(1,1|2)^2}{SO(2,1)\times SO(3)}$ in $AdS_3 \times S^3$, $\frac{SU(2,2|4)}{SO(4,1)\times SO(5)}$ in $AdS_5 \times S^5$ and $\frac{SU(1,1|2)^2}{SO(2,1)\times SO(3)} \times U(1)^4$ in $AdS_3 \times S^3 \times T^4$ (see §2.3).

As described in §2.4, the Cartan forms L^A are inherently supersymmetric. They are also defined such that there is no mixing between the AdS_m, S^n and $\mathcal{M}^4$ dimensions of space (ie: $L^{aa'} = L^{aa''} = L^{a'a''} = 0$). In the rest of this thesis, the spinor index $\hat{\alpha}$ in $L^{\hat{\alpha}}$ is not explicitly written.

As in [91], the gauge choice $G(x, \theta) = g(x)e^{\theta Q}$ is chosen, needed to explicitly find the supersymmetric field strength F and the R-R fields in S_{WZ}. $g(x)$ is a coset representative of $\frac{SO(2,2)\times SO(4)}{SO(2,1)\times SO(3)} \cong \frac{SO(2,1)^2\times SO(3)^2}{SO(2,1)\times SO(3)}$ [108] in $AdS_3 \times S^3$, $\frac{SO(4,2)\times SO(6)}{SO(4,1)\times SO(5)}$ in $AdS_5 \times S^5$ and $\frac{SO(2,1)^2\times SO(3)^2}{SO(2,1)\times SO(3)} \times U(1)^4$ in $AdS_3 \times S^3 \times T^4$. $g(x)$ satisfies:

$$g(x)^{-1}dg(x) = e^{\hat{a}} P_{\hat{a}} + \frac{1}{2}\omega^{\hat{a}\hat{b}} J_{\hat{a}\hat{b}}. \tag{2.11.2}$$

The method used is to find the supersymmetric DBI-action and to then use κ-symmetry to find the WZ-action. In §2.6 and §2.7 it was seen that the total effective action of a Dp-brane can be written as:

$$S = S_{DBI} + S_{WZ}, \tag{2.11.3}$$

$$S_{DBI} = -\int_{M_{p+1}} d^{p+1}\sigma \sqrt{-\det(G_{ij} + F_{ij})}, \tag{2.11.4}$$

$$G_{ij} = L_i^{\hat{a}} L_j^{\hat{b}} \eta_{\hat{a}\hat{b}} = \partial_i X^M L_M^{\hat{a}} \partial_j X^N L_N^{\hat{b}} \eta_{\hat{a}\hat{b}}, \qquad L^{\hat{a}} = d\sigma^i L_i^{\hat{a}}, \tag{2.11.5}$$

$$S_{WZ} = \int_{M_{p+1}} e^{-F} \wedge C, \qquad C = \bigoplus_{n=0}^{p+1} C_n, \tag{2.11.6}$$

where C_n are the R-R super-n-forms and the brane tension has been set to 1 [3,10,14, 25–27]. The notation of eqn (2.11.6) is explained by eqn (2.6.14). As explained in §2.6, the Dirac-Born-Infeld action is inherently supersymmetric because it is constructed from supersymmetric Cartan forms. The construction of G_{ij} is shown above, and the construction of F_{ij} is outlined in §2.11.2.

An important observation [90] in the construction of the Cartan forms is that the action should be invariant in terms of the isotropy group of the $AdS_m \times S^n$ space, ie $SO(m-1,1) \times SO(n)$. Thus $L^{\hat{\alpha}}$ should be a $m+n$ dimensional spinor, decomposed into two spinors of AdS_m and S^n. $L^{\hat{a}}$, a $m+n$ dimensional vector, is decomposed into AdS_m and S^n vectors, and the tensor $L^{\hat{a}\hat{b}}$ is decomposed into AdS_m and S^n tensors. In Appendix A, the notation used for these Cartan forms, and how the AdS_m and S^n Cartan forms are merged to give Cartan forms for the whole space which have indices running over both the AdS_m and S^n indices is outlined. As mentioned earlier, this is a notation based on [90,91] and allows the supersymmetry algebra (Appendix A) and Maurer-Cartan equations (Appendix B) to be expressed with the same formulas for the different backgrounds considered (the difference being the range the indices can run over).

The general form of $d(e^{-F} \wedge C)$ can be deduced from [27] (which is explained in §2.6) which endeavours to find the D-brane actions on a general background. However as will be seen extra terms are needed for the WZ-actions here to make them fully κ-symmetric (as for the $H_{5,t=0}$ term of the D3-brane action in §2.7).

The κ-transformation in $AdS_m \times S^n$ can be derived from eqns (2.6.4) to (2.6.8):

$$\delta_\kappa x^{\hat{a}} = 0, \qquad \delta_\kappa x^{\hat{a}\hat{b}} = 0, \qquad \delta_\kappa \theta = \kappa, \qquad \frac{1}{2}(1+\Gamma)\kappa = \kappa. \tag{2.11.7}$$

This is applied to the DBI-action and then the WZ-action is defined such that it cancels the κ variation of the DBI-action. From section 2.6 Γ must have ± 1 eigenvalues and $\Gamma^2 = 1$. Thus $\frac{1}{2}(1+\Gamma)$ projects out half of the fermionic degrees of freedom, as needed for supersymmetry. Also, $d^{p+1}\sigma\ \Gamma$ must be a $p+1$ form in order to relate $\delta_\kappa S_{DBI}$ and $\delta_\kappa S_{WZ}$.

The Γ for a Dp-brane is from eqn (2.6.10) [10,13,14,25–27,91]:

$$d^{p+1}\sigma\ \Gamma = (-1)^n \frac{e^{-F}}{-\mathcal{L}_{DBI}} \wedge \bigoplus_n \Gamma_{(2n)} \mathcal{K}^n \mathcal{E}|_{vol}, \qquad \Gamma_{(2n)} = L^{\hat{a}_1}\Gamma_{\hat{a}_1} \cdots L^{\hat{a}_{2n}}\Gamma_{\hat{a}_{2n}}, \tag{2.11.8}$$

where $\mathcal{K}$ and $\mathcal{E}$ are defined in (A.1.4).

2.11.2 Defining F

As in [91], F is defined using the gauge choice $G(x,\theta) = g(x)e^{\theta Q}$:

$$F = dA + 2i\int_0^1 dt\ \bar{\theta}\widehat{L}_t \wedge \mathcal{K}L_t, \qquad F = \frac{1}{2}d\sigma^i \wedge d\sigma^j F_{ij}, \qquad \widehat{L} = L^{\hat{a}}\Gamma_{\hat{a}}, \tag{2.11.9}$$

where $L_t^{\hat{a}}(x,\theta) = L^{\hat{a}}(x,t\theta)$, $L_t(x,\theta) = L(x,t\theta)$. This definition holds in all the three backgrounds used in this thesis, due to the choice of the notation used (Appendix A).

It is not obvious that F is supersymmetric, as it is not defined entirely by Cartan forms. It, however, can be shown that:

$$dF = i\bar{L} \wedge \widehat{L} \wedge \mathcal{K}L, \tag{2.11.10}$$

which is clearly supersymmetric due to its expression in terms of supersymmetric Cartan forms [91]. This is shown using equations (B.1), (B.2), (B.21), (B.22) and the Fierz identity [97]:

$$(\bar{A}\Gamma_{\hat{a}}B)(\bar{C}\Gamma^{\hat{a}}D) = -\frac{1}{2}(\bar{A}\Gamma_{\hat{a}}e_lD)(\bar{B}\Gamma^{\hat{a}}e_lC) + \frac{1}{2}(\bar{A}\Gamma_{\hat{a}}e_lC)(\bar{B}\Gamma^{\hat{a}}e_lD), \tag{2.11.11}$$

where A, B, C, D are fermionic 0-forms, and $e_l = \{\mathbf{1}, \mathcal{E}, \mathcal{J}, \mathcal{K}\}$.

Begin by finding dF in terms of the t integration in eqn (2.11.9).

$$dF = 2i\int_0^1 dt\ (d\bar{\theta}d\bar{\theta}\widehat{L}_t \wedge \mathcal{K}L_t + \bar{\theta}d\widehat{L}_t \wedge \mathcal{K}L_t - \bar{\theta}\widehat{L}_t \wedge \mathcal{K}dL_t) \tag{2.11.12}$$

$$\begin{aligned} &= 2i\int_0^1 dt\ (d\bar{\theta}\widehat{L}_t \wedge \mathcal{K}L_t + \frac{i}{2}\bar{\theta}\mathcal{E}\widehat{L}_t\sigma_+ \wedge \widehat{L}_t \wedge \mathcal{K}L_t + \frac{1}{4}\bar{\theta}\widehat{L}_t \wedge L_t^{\hat{a}\hat{b}}\Gamma_{\hat{a}\hat{b}} \wedge \mathcal{K}L_t \\ &\quad -\bar{\theta}\Gamma_{\hat{a}}L_t^{\hat{a}\hat{b}} \wedge L_{t\hat{b}} \wedge \mathcal{K}L_t - i(\bar{\theta}\Gamma^{\hat{a}}\mathcal{K}L_t)(\bar{L}_t\Gamma^{\hat{a}} \wedge L_t)), \end{aligned} \tag{2.11.13}$$

where eqns (B.1) and (B.2) were used and:

$$\Gamma_{\hat{a}_1\ldots\hat{a}_n} = \frac{1}{n!}\Gamma_{[\hat{a}_1}\ldots\Gamma_{\hat{a}_n]}. \tag{2.11.14}$$

The relation:

$$\widehat{L}_t \wedge L_t^{\hat{a}\hat{b}}\Gamma_{\hat{a}\hat{b}} = -L_t^{\hat{a}\hat{b}}\Gamma_{\hat{a}\hat{b}} \wedge \widehat{L}_t + 4\Gamma_{\hat{a}}L_t^{\hat{a}\hat{b}} \wedge L_{t\hat{b}}, \tag{2.11.15}$$

derived using the anticommutation relations of the Γ matrices, can be used to simplify dF.

$$\begin{aligned} dF &= 2i\int_0^1 dt\ (d\bar{\theta}\widehat{L}_t \wedge \mathcal{K}L_t + \frac{i}{2}\bar{\theta}\mathcal{E}\widehat{L}_t\sigma_+ \wedge \widehat{L}_t \wedge \mathcal{K}L_t - \frac{1}{4}\bar{\theta}L_t^{\hat{a}\hat{b}}\Gamma_{\hat{a}\hat{b}} \wedge \widehat{L}_t \wedge \mathcal{K}L_t \\ &\quad -i(\bar{\theta}\Gamma^{\hat{a}}\mathcal{K}L_t)(\bar{L}_t\Gamma^{\hat{a}} \wedge L_t)). \end{aligned} \tag{2.11.16}$$

With an aim to expressing the integrand as a derivative with respect to t, eqn (B.22) can then be used to express dF as:

$$dF = i\int_0^1 dt\ (\partial_t\bar{L}_t \wedge \widehat{L}_t \wedge \mathcal{K}L_t + \bar{L}_t \wedge \widehat{L}_t \wedge \mathcal{K}\partial_t L_t - 2i(\bar{\theta}\Gamma^{\hat{a}}\mathcal{K}L_t)(\bar{L}_t\Gamma^{\hat{a}} \wedge L_t)). \tag{2.11.17}$$

The last term of this can be rearranged using eqn (2.11.11):

$$(\bar{\theta}\Gamma^{\hat{a}}\mathcal{K}L_t)(\bar{L}_t\Gamma^{\hat{a}}\wedge L_t) = -(\bar{\theta}\Gamma^{\hat{a}}L_t)(\bar{L}_t\Gamma^{\hat{a}}\wedge\mathcal{K}L_t), \tag{2.11.18}$$

Using this and eqn (B.21) it is easy to see:

$$dF = i\int_o^1 dt\ \partial_t(\bar{L}_t\wedge\widehat{L}_t\wedge\mathcal{K}L_t)\ =\ i\bar{L}\wedge\widehat{L}\wedge\mathcal{K}L \tag{2.11.19}$$

Thus, F is supersymmetric if the variation of A is defined such that it cancels the variation of the second term in δF [3,26,27].

2.12 D1-brane in $AdS_3\times S^3$

The specialisation of eqn (2.11.4) to the D1-brane gives the D1-brane DBI-action:

$$S_{DBI} = -\int_{M_2} d^2\sigma\ \sqrt{-\det(G_{ij}+F_{ij})}, \tag{2.12.1}$$

where $i,j\ \in\ \{0,1\}$ are brane coordinates. This is supersymmetric but not κ-invariant.

From (2.11.8) the operator Γ which defines the κ variation for this brane is:

$$\Gamma = \frac{\varepsilon^{i_1i_2}(\Gamma_{i_1i_2}\mathcal{J}+F_{i_1i_2}\mathcal{E})}{2\mathcal{L}_{DBI}}. \tag{2.12.2}$$

The next step is to find $\delta_\kappa S_{DBI}$. This is done using:

$$\delta\det(G+F) = \det(G+F)\mathrm{Tr}((G+F)^{-1}(\delta G+\delta F)), \tag{2.12.3}$$

Thus

$$\begin{aligned}\delta_\kappa S_{DBI} &= \delta_\kappa\left(-\int_{M_2} d^2\sigma\ \sqrt{-\det(G_{ij}+F_{ij})}\right)\\ &= \int_{M_2} d^2\sigma\frac{-\det(G+F)}{2\sqrt{-\det(G+F)}}\mathrm{Tr}((G+F)^{-1}(\delta_\kappa G+\delta_\kappa F)).\end{aligned} \tag{2.12.4}$$

The κ variations of G and F can be shown to be:

$$\delta_\kappa G_{ij} = -4i\bar{\kappa}\widehat{L}_{(i}L_{j)}, \qquad \delta_\kappa F_{ij} = 4i\bar{\kappa}\widehat{L}_{[i}\mathcal{K}L_{j]}. \tag{2.12.5}$$

using $Tr(M_{ij}) = M_{ii}$, $\delta_\kappa S_{DBI}$ shown to be:

$$\begin{aligned}\delta_\kappa S_{DBI} &= \int_{M_2} d^2\sigma - 2\sqrt{-\det(G+F)}(G+F)^{-1\ ij}(i\bar{\kappa}\widehat{L}_{[j}\mathcal{K}L_{i]} - i\bar{\kappa}\widehat{L}_{(j}L_{i)})\\ &= \int_{M_2} d^2\sigma - 2i\sqrt{-\det(G+F)}(G+F)^{-1\ ij}(\bar{L}_{[j}\widehat{L}_{i]}\mathcal{K}\kappa + \bar{L}_{(j}\widehat{L}_{i)}\kappa)\end{aligned}$$

Insert $\kappa = \Gamma\kappa$

$$= \int_{M_2} d^2\sigma - 2i\sqrt{-\det(G+F)}(G+F)^{-1\ ij}(\bar{L}_{[j}\widehat{L}_{i]}\Gamma\mathcal{K}\kappa + \bar{L}_{(j}\widehat{L}_{i)}\Gamma\kappa),$$

where Γ is given by eqn (2.12.2). The next step is to rearrange the equations such that the $\mathcal{K}$ is absorbed into $(G+F)^{-1}$. The purpose of this will later be seen to be to cancel contributions arising from $\Gamma\kappa$.

$$\begin{aligned}
\delta_\kappa S_{DBI} &= \int_{M_2} d^2\sigma \; -i\sqrt{-\det(G+F)}(G+F)^{-1\;ij}((\bar{L}_j\widehat{L}_i - \bar{L}_i\widehat{L}_j)\Gamma\mathcal{K}\kappa \\
&\quad +(\bar{L}_j\widehat{L}_i + \bar{L}_i\widehat{L}_j)\Gamma\kappa) &(2.12.6)\\
&= \int_{M_2} d^2\sigma - i\sqrt{-\det(G+F)}((\bar{L}_j(G+F)^{-1\;ij}\mathcal{K}\widehat{L}_i - \bar{L}_j(G-F)^{-1\;ij}\mathcal{K}\widehat{L}_i)\Gamma\kappa \\
&\quad +(\bar{L}_j(G+F)^{-1\;ij}\widehat{L}_i + \bar{L}_j(G-F)^{-1\;ij}\widehat{L}_i)\Gamma\kappa) &(2.12.7)\\
&= \int_{M_2} d^2\sigma - 2i\sqrt{-\det(G+F)}(\bar{L}_j(G+\mathcal{K}F)^{-1\;ij}\widehat{L}_i\Gamma\kappa) &(2.12.8)\\
&= i\int_{M_2} d^2\sigma\bar{L}_j(G+\mathcal{K}F)^{-1\;ij}\widehat{L}_i\varepsilon^{kl}(\Gamma_{kl}\mathcal{J} + F_{kl}\mathcal{E})\kappa, &(2.12.9)
\end{aligned}$$

where the symmetry of G and antisymmetry of F were used in (2.12.7) and (2.12.8) and the Fierz identity (2.12.2) was used in (2.12.9).

Using:

$$\varepsilon^{kl}\widehat{L}_i\Gamma_{kl} = \varepsilon^{kl}(\Gamma_{ikl} + 2G_{ik}\Gamma_l) = 2\varepsilon^{kl}G_{ik}\Gamma_l, \tag{2.12.10}$$

($\Gamma_{ikl} = 0$ as i, j & k can only equal 0 or 1) and:

$$\varepsilon^{kl}\Gamma_{[i}F_{kl]} = 0 = \varepsilon^{kl}(2\Gamma_i F_{kl} - 4\Gamma_k F_{il}), \tag{2.12.11}$$

$$\varepsilon^{kl}\Gamma_i F_{kl} = -2\Gamma_l F_{ik}\varepsilon^{kl}. \tag{2.12.12}$$

$\delta_\kappa S_{DBI}$ is equivalent to:

$$\begin{aligned}
\delta_\kappa S_{DBI} &= \int_{M_2} d^2\sigma \; 2i\varepsilon^{kl}\bar{L}_j((G+\mathcal{K}F)^{-1\;ij}(G+F\mathcal{K})_{ki}\mathcal{J}\Gamma_l\kappa) &(2.12.13)\\
&= \int_{M_2} 2i\bar{L}\wedge\widehat{L}\mathcal{J}\kappa. &(2.12.14)
\end{aligned}$$

From $(i\bar{L}\wedge\widehat{L}\wedge\mathcal{J}\kappa)^\dagger = \bar{L}\wedge\widehat{L}\wedge\mathcal{J}\kappa$ (derived by imposing the reality of the action) it is therefore apparent that the κ variation of S_{DBI} is:

$$\delta_\kappa S_{DBI} = 2i\int_{M_2}\bar{\kappa}\widehat{L}\wedge\mathcal{J}L. \tag{2.12.15}$$

Next, one shows that the κ-variation of the following supersymmetric WZ-action:

$$\begin{aligned}
S_{WZ} &= -2i\int_{M_2}\int_0^1 dt\;\bar{\theta}\widehat{L}_t\wedge\mathcal{J}L_t &(2.12.16)\\
&= -i\int_{M_3}\bar{L}\wedge\widehat{L}\wedge\mathcal{J}L, &(2.12.17)
\end{aligned}$$

cancels this variation. Equivalence of the two different forms of S_{WZ} can be seen via:

$$
\begin{aligned}
S_{WZ} &= -i\int_{M_3} \bar{L}\wedge\widehat{L}\wedge\mathcal{J}L \\
&= \int_{M_3}\int_0^1 dt\ \partial_t(-i\bar{L}_t\wedge\widehat{L}_t\wedge\mathcal{J}L_t) \quad \text{note } L_t=0 \\
&= \int_{M_3}\int_0^1 dt\ d(-2i\bar{\theta}\widehat{L}_t\wedge\mathcal{J}L_t) \qquad (2.12.18)\\
&= \int_{M_2}\int_0^1 dt\ -2i\bar{\theta}\widehat{L}_t\wedge\mathcal{J}L_t,
\end{aligned}
$$

where (B.1), (B.2), (B.21), (B.22), (B.19) and (2.11.11) were used.

S_{WZ} is invariant under supersymmetry transformations as eqn (2.12.17) is composed entirely of Cartan forms which are inherently supersymmetric.

To show that the κ variation of S_{WZ} cancels the κ variation of S_{DBI} apply the Lie derivative $\delta_\kappa = \{d, \imath_\kappa\}$ to eqn (2.12.16). (See eqn (2.6.9) for details on $\imath_\kappa$.)

$$
\begin{aligned}
\delta_\kappa S_{WZ} &= -2i\int_{M_2}\delta_\kappa\int_0^1 dt\ \bar{\theta}\widehat{L}_t\wedge\mathcal{J}L_t \qquad (2.12.19)\\
&= -2i\int_{M_2}(d\imath_\kappa+\imath_\kappa d)\int_0^1 dt\ \bar{\theta}\widehat{L}_t\wedge\mathcal{J}L_t \qquad (2.12.20)\\
&= -2i\int_{M_2}\imath_\kappa\int_0^1 dt\ d(\bar{\theta}\widehat{L}_t\wedge\mathcal{J}L_t) \qquad (2.12.21)\\
&= -i\int_{M_2}\imath_\kappa\int_0^1 dt\ \partial_t(\bar{L}_t\wedge\widehat{L}_t\wedge\mathcal{J}L_t) \qquad (2.12.22)\\
&= -i\int_{M_2}\imath_\kappa\,(\bar{L}\wedge\widehat{L}\wedge\mathcal{J}L) \qquad (2.12.23)\\
&= -2i\int_{M_2}(\bar{\kappa}\widehat{L}\wedge\mathcal{J}L), \qquad (2.12.24)
\end{aligned}
$$

where in (2.12.20) the first term which is a total derivative is set to zero. Thus $\delta_\kappa S_{DBI}+\delta_\kappa S_{WZ}=0$.

The full supersymmetric, κ-invariant D1-brane action in $AdS_3\times S^3$ is thus given by

$$
S^{D1} = -\int_{M_2} d^2\sigma\,\sqrt{-\det(G_{ij}+F_{ij})} - 2i\int_{M_2}\int_0^1 dt\ \bar{\theta}\widehat{L}_t\wedge\mathcal{J}L_t. \qquad (2.12.25)
$$

This can be expressed in terms of the spacetime supercoordinates x and θ via the expressions of the Cartan forms in terms of these coordinates, given by eqn (B.29) and eqn (B.30). However substituting in thcsc formulas greatly increases the complexity of the action, so it is left in this brief form here.

The results of applying the Killing gauge of [76] to this action are contained in Appendix E (see §2.8 for details on the Killing gauge). This gauge was adapted to the fundamental string in $AdS_3\times S^3$ in [108]. This gauge effectively takes action eqn

(2.12.25), and after expansion via substitution of eqn (B.29) and eqn (B.30) gauge fixes the κ symmetry to simplify the action. Since the results do not simplify the D1 and D5 brane actions as greatly as for the fundamental string the results are not contained in the main body of the thesis.

2.13 D5-brane in $AdS_3 \times S^3$

As mentioned earlier, the action described here is for the D5-brane lying entirely in $AdS_3 \times S^3$, which can be interpreted as a D5-brane in $AdS_3 \times S^3 \times \mathcal{M}^4$ with constraints imposed on the fields such that the solutions of the action must lie in $AdS_3 \times S^3$, with the $\mathcal{M}^4$ compactified away. Such constraints on a D5-brane solution in $AdS_3 \times S^3 \times \mathcal{M}^4$ describe branes in a BPS configuration [130] (as described in §2.10). This action does not have solutions that describe the branes whose gravity is warping spacetime to an $AdS_3 \times S^3 \times \mathcal{M}^4$ background. Such a D5-brane, with 4 directions compactified in $\mathcal{M}^4$, would have an action like the one of the D1-brane in $AdS_3 \times S^3$ just found (2.12.25). See section 2.9.4 for more details on the orientation of the background D1 and D5-branes.

This case of a D5-brane lying entirely in $AdS_3 \times S^3$, with $\mathcal{M}^4$ being compactified away, has similarities to the case studied in [4] where a D9-brane filling ten-dimensional space is investigated.

The specialisation of eqn (2.11.4) to the D5-brane shows the D5-brane has a DBI-action of:

$$S_{DBI} = -\int_{M_6} d^6\sigma \sqrt{-\det(G_{ij} + F_{ij})}, \qquad i,j \in \{0,\ldots,5\}. \tag{2.13.1}$$

This time the operator Γ that defines the κ-invariance is:

$$\Gamma = \frac{\varepsilon^{i_1\ldots i_6}}{\mathcal{L}_{DBI}} \left(\frac{\Gamma_{i_1\ldots i_6}}{6!}\mathcal{J} + \frac{\Gamma_{i_1\ldots i_4}F_{i_5 i_6}}{2.4!}\mathcal{E} + \frac{\Gamma_{i_1 i_2}F_{i_3 i_4}F_{i_5 i_6}}{2^4}\mathcal{J} + \frac{F_{i_1 i_2}F_{i_3 i_4}F_{i_5 i_6}}{3!.2^3}\mathcal{E} \right). \tag{2.13.2}$$

Using this the same procedure as for the D1-brane is followed.

Through a similar but more complicated calculation to finding the κ variation of the D1-brane's DBI action, the variation $\delta_\kappa S_{DBI}$ turns out to be given by[6]:

$$\delta_\kappa S_{DBI} = 2i \int_{M_6} \left(\frac{(\bar{\kappa}(\widehat{L})^5 \wedge \mathcal{J}L)}{5!} + \frac{(\bar{\kappa}(\widehat{L})^3 \wedge \mathcal{E}L) \wedge F}{3!} + \frac{(\bar{\kappa}\widehat{L} \wedge \mathcal{J}L) \wedge F \wedge F}{2} \right). \tag{2.13.3}$$

The WZ term whose κ-variation cancels this is:

$$S_{WZ} = -2i \int_{M_6}\int_0^1 dt \left(\frac{(\bar{\theta}(\widehat{L})^5 \wedge \mathcal{J}L)}{5!} + \frac{(\bar{\theta}(\widehat{L})^3 \wedge \mathcal{E}L) \wedge F}{3!} + \frac{(\bar{\theta}\widehat{L} \wedge \mathcal{J}L) \wedge F \wedge F}{2} \right). \tag{2.13.4}$$

[6] Alternatively the results in §2.6 can be used by applying eqn (2.6.17) to eqn (2.6.15).

How this was arrived at can be found in Appendix D.

The κ-variation of this is:

$$\delta_\kappa S_{WZ} = -2i\int_{M_6}\left(\frac{(\bar\kappa(\widehat{L})^5\wedge\mathcal{J}L)}{5!}+\frac{(\bar\kappa(\widehat{L})^3\wedge\mathcal{E}L)\wedge F}{3!}+\frac{(\bar\kappa\widehat{L}\wedge\mathcal{J}L)\wedge F\wedge F}{2}\right)+Y, \tag{2.13.5}$$

where:

$$\begin{aligned} Y &= \int_{M_6}\imath_\kappa\int_0^1 dt\ \left(-\frac{8}{5!}(\bar\theta\sigma_-((L_t^a\Gamma_a)^5\wedge L_t^{b'}\Gamma_{b'}-L_t^a\Gamma_a\wedge(L_t^{b'}\Gamma_{b'})^5)\wedge\mathcal{K}L_t)\right)\\ &+\int_{M_6}\imath_\kappa\int_0^1 dt\ \left(\frac{1}{3}(\bar\theta\sigma_-((L_t^a\Gamma_a)^4-(L_t^{a'}\Gamma_{a'})^4)\wedge L_t)\wedge F_t\right), \end{aligned} \tag{2.13.6}$$

and where $\imath_\kappa$ is from $\delta_\kappa=\{d,\imath_\kappa\}$ and $\imath_\kappa L=\kappa$, $\imath_\kappa L^{\hat a}=0$, $\imath_\kappa F=0$ [10].

Due to the indices $a\in\{0,1,2\}$ and $a'\in\{3,4,5\}$ having only 3 possibilities each, $Y=0$.

The total effective action of a D5-brane in $AdS_3\times S^3$ is thus given by:

$$\begin{aligned} S^{D5} &= -\int_{M_6} d^6\sigma\sqrt{-\det(G_{ij}+F_{ij})}\\ &-2i\int_{M_6}\int_0^1 dt\ \left(\frac{(\bar\theta(\widehat{L}_t)^5\wedge\mathcal{J}L_t)}{5!}+\frac{(\bar\theta(\widehat{L}_t)^3\wedge\mathcal{E}L_t)\wedge F_t}{3!}+\frac{(\bar\theta\widehat{L}_t\wedge\mathcal{J}L_t)\wedge F_t\wedge F_t}{2}\right). \end{aligned} \tag{2.13.7}$$

Eqn's (B.29) and (B.30) can be substituted into this action to express it in terms of x and θ, the $AdS_5\times S^5$ bosonic and fermionic coordinates respectively.

When calculating this action the Fierz identity eqn (2.11.11) was required as well as identities [26,27,91,114]

$$(\Gamma^{\hat a\hat b\hat c}\mathcal{E})_{(\hat\alpha\hat\beta}(\Gamma_{\hat c})_{\hat\gamma\hat\delta)}-2(\Gamma^{[\hat a}\mathcal{E})_{(\hat\alpha\hat\beta}(\Gamma^{\hat b]}\mathcal{K})_{\hat\gamma\hat\delta)}=0, \tag{2.13.8}$$

$$(\Gamma^{\hat a\hat b\hat c\hat d\hat e}\mathcal{J})_{(\hat\alpha\hat\beta}(\Gamma_{\hat e})_{\hat\gamma\hat\delta)}-4(\Gamma^{[\hat a\hat b\hat c}\mathcal{E})_{(\hat\alpha\hat\beta}(\Gamma^{\hat d]}\mathcal{K})_{\hat\gamma\hat\delta)}=0. \tag{2.13.9}$$

The proof of Fierz identity (2.13.9) can be found in Appendix C. This Fierz identity was derived by the author as the author could not find it in the literature, however it appeared in [114] shortly after.

The Killing gauge fixed D5-brane action in $AdS_3\times S^3$ is given in Appendix E.

2.14 D5-brane in $AdS_5\times S^5$

The mathematics in this case is almost identical to the previous case (this is due to the intentional similarities in the notation between the $AdS_3\times S^3$ Cartan forms and the $AdS_5 \times S^5$ Cartan forms), however in this case Y (eqn (2.13.6)) is not zero, but we can use:

$$\epsilon^{a_1 \ldots a_5} = -i\sigma_1 \Gamma^{a_1 \ldots a_5}, \qquad \epsilon^{a'_1 \ldots a'_5} = \sigma_2 \Gamma^{a'_1 \ldots a'_5}, \tag{2.14.1}$$

to show that:

$$\begin{aligned} S^{D5} &= -\int_{M_6} d^6\sigma \sqrt{-\det(G_{ij}+F_{ij})} \\ &\quad -i \int_{M_7} \left(\frac{(\bar{L} \wedge (\widehat{L})^5 \wedge \mathcal{J}L)}{5!} + \frac{(\bar{L} \wedge (\widehat{L})^3 \wedge \mathcal{E}L) \wedge F}{3!} + \frac{(\bar{L} \wedge \widehat{L} \wedge \mathcal{J}L) \wedge F \wedge F}{2} \right) \\ &\quad + \int_{M_7} \left(\frac{\epsilon_{a_1 \ldots a_5}}{30} L^{a_1} \wedge \ldots \wedge L^{a_5} \wedge F + \frac{\epsilon_{a'_1 \ldots a'_5}}{30} L^{a'_1} \wedge \ldots \wedge L^{a'_5} \wedge F \right), \end{aligned} \tag{2.14.2}$$

is the full κ-invariant, supersymmetric action.

Eqn (2.14.2) can be rewritten as:

$$\begin{aligned} S^{D5} = S_{DBI} - 2i \int_{M_6} \int_0^1 dt \quad & \left(\frac{(\bar{\theta}(\widehat{L}_t)^5 \wedge \mathcal{J}L_t)}{5!} + \frac{(\bar{\theta}(\widehat{L}_t)^3 \wedge \mathcal{E}L_t) \wedge F_t}{3!} + \right. \\ & \left. \frac{(\bar{\theta}\widehat{L}_t \wedge \mathcal{J}L_t) \wedge F_t \wedge F_t}{2} \right) + \int_{M_7} \mathcal{L}_{WZ}^{BOSE}, \end{aligned} \tag{2.14.3}$$

where

$$\mathcal{L}_{WZ}^{BOSE} = \mathcal{L}_{WZ}|_{\theta=0} = \frac{\epsilon_{a_1 \ldots a_5}}{30} e^{a_1} \wedge \ldots \wedge e^{a_5} \wedge dA + \frac{\epsilon_{a'_1 \ldots a'_5}}{30} e^{a'_1} \wedge \ldots \wedge e^{a'_5} \wedge dA. \tag{2.14.4}$$

Eqn's (B.29) and (B.30) can be substituted into this action to express it in terms of x and θ, the $AdS_5 \times S^5$ bosonic and fermionic coordinates respectively.

The Killing gauge fixed D5-brane action in $AdS_5 \times S^5$ is contained in Appendix E.

If this action was to be restricted such that it describes the baryon vertex of [23,29, 56,68,99,135], the following constraints need to be put in place. Firstly, the D5-brane must be wrapped on the S^5, thus: $\sigma^i = x^i$ for $i \in \{5,6,7,8,9\}$.

Secondly the restrictions on the spinors that break the supersymmetry according to the presence of the D5-brane and the background D3-branes and open strings are [68]:

$$\begin{aligned} \Gamma^{0123456}\theta &= \theta, \\ \Gamma^{0123}\theta &= \theta, \\ \Gamma^{4}\theta &= \theta. \end{aligned} \tag{2.14.5}$$

2.15 D5-brane in $AdS_3 \times S^3 \times T^4$

As mentioned in the introduction, an attempt was made to find the action of a general D5-brane free to move in all the dimensions of $AdS_3 \times S^3 \times T^4$. The same method as used to find the other actions in this thesis is used, however as will be shown, this method does not succeed in producing a fully κ-invariant action in this case.

The case of extending the D1-brane action to $AdS_3 \times S^3 \times T^4$ is trivial, and in the notation of Appendix A, appears unchanged to (2.12.25).

As mentioned in Appendix A, the coset representation for T^4 that is used here describes only translations in T^4 (ie $U(1)^4$).

The D5-brane DBI-action is

$$\begin{aligned} S_{DBI} &= -\int_{M_6} d^6\sigma\sqrt{-\det(G_{ij}+F_{ij})}, \qquad\qquad i,j \in \{0,\dots,5\}, \\ G_{ij} &= L_i^a L_{ja} + L_i^{a'} L_{ja'} + L_i^{a''} L_{ja''}, \qquad\qquad (2.15.1) \\ F &= dA + 2i\int_0^1 dt\ \bar{\theta}(L_t^a\Gamma_a + L_t^{a'}\Gamma_{a'} + L_t^{''}\Gamma_{a''})\mathcal{K}\wedge L_t, \end{aligned}$$

where Γ in given by (2.13.2).

The κ-variation of the DBI-action appears the same as (2.13.3) in the $AdS_3\times S^3\times T^4$ notation of Appendix A.

The WZ-action required to cancel this should be of the form:

$$\begin{aligned} S_{WZ}^{D5} &= -2i\int_{M_6}\int_0^1 dt\left(\frac{(\bar{\theta}(\widehat{L}_t)^5\wedge\mathcal{J}L_t)}{5!} + \frac{(\bar{\theta}(\widehat{L}_t)^3\wedge\mathcal{E}L_t)\wedge F_t}{3!}\right. \\ &\qquad\left. + \frac{(\bar{\theta}\widehat{L}_t\wedge\mathcal{J}L_t)\wedge F_t\wedge F_t}{2}\right) + \int_{M_7} E_7^{BOSE} \qquad (2.15.2) \\ &= -i\int_{M_7}\left(\frac{(\bar{L}\wedge(\widehat{L})^5\wedge\mathcal{J}L)}{5!} + \frac{(\bar{L}\wedge(\widehat{L})^3\wedge\mathcal{E}L)\wedge F}{3!}\right. \\ &\qquad\left. + \frac{(\bar{L}\wedge\widehat{L}\wedge\mathcal{J}L)\wedge F\wedge F}{2}\right) + \int_{M_7} E_7, \qquad (2.15.3) \end{aligned}$$

where E_7 should have the following properties to make the action κ-invariant

- E_7 should be supersymmetric.
- $\imath_\kappa E_7 = 0$.
-

$$\begin{aligned} \delta_\kappa E_7 &= \frac{8}{5!}(\bar{\kappa}\sigma_-((L^a\Gamma_a)^5\wedge L^{\tilde{b}}\Gamma_{\tilde{b}} - L^a\Gamma_a\wedge(L^{\tilde{b}}\Gamma_{\tilde{b}})^5)\wedge\mathcal{K}L) \\ &\quad -\frac{1}{3}(\bar{\kappa}\sigma_-((L^a\Gamma_a)^4 - (L^{\tilde{a}}\Gamma_{\tilde{a}})^4)\wedge L)\wedge F \qquad (2.15.4) \end{aligned}$$

where $\tilde{a} \in \{a', a''\}$.

No supersymmetric 7 form can be constructed out of the Cartan forms and F that satisfies these conditions.

Conclusions

The discussion of the results of this chapter, and future research directions arising from it, can be found in the Conclusion (Chapter 4).

Chapter 3

D-brane charges on a Group Manifold

This chapter discusses research into the charge groups of D-branes on a group manifold. This research can be read independently of the research that appears in Chapter 2, however there is some linkage between the topics.

In Chapter 2 the research focused on finding supersymmetric and κ symmetric GS actions of branes in $AdS_3 \times S^3$ and $AdS_5 \times S^5$ backgrounds, which respectively have even coset spaces: $\frac{SO(2,1)^2 \times SO(3)^2}{SO(2,1) \times SO(3)}$ and $\frac{SO(4,2) \times SO(6)}{SO(4,1) \times SO(5)}$ (the supersymmetry has been ignored here). Such coset spaces are obviously cosets of group manifolds and string theory in such spaces can be reexpressed as conformal field theory on (non-compact) group manifolds (WZW models for non-compact group manifolds).

D-branes in WZW models of such coset spaces have been investigated for the $AdS_3 \times S^3$ case (see [62, 109] for a list of references). The WZW model equivalent to IIB string theory with D-branes in AdS_3 space is $SL(2,\mathbf{R})_{\mathbf{k+2}}$ open string WZW theory, and the S^3 sector is equivalent to open string WZW on $SU(2)_{k-2}$ [62]. The D-brane content of such models has been investigated through the study of conjugacy classes and Cardy states on the manifold [62, 109].

Other research has been done to show that the spectrum of D-brane solutions for a D-brane DBI action on a group manifold agrees with results derived from D-branes in WZW models on the same group manifold, up to errors introduced due to the perturbative method used. See [17] for a list of references. This is equivalent to comparing results of the effective action of the D-brane on the group manifold (DBI action) and the Dirichlet conditions defining the D-brane in the string action on the manifold (WZW model).

As mentioned, the research in this chapter does not further investigate this correspondence between GS actions of D-branes in $AdS \times S$ space and D-branes in the WZW model, but rather investigates the charge groups of the D-branes in the WZW model. It was previously known [40] that the charges of D-branes should correspond to the dimension of the highest weight representation they correspond to, modulo some positive integer x, and that the charges satisfy an algebra determined by the fusion rules of the affine Lie algebra of the WZW model (see eqn (3.5.16)). Therefore the charge group for the D-brane is $\mathbb{Z}_x = \mathbb{Z}/x\mathbb{Z}$.

The values of x have important implications. Twisted K-theory also predicts that the D-brane charge group on group manifolds is of the form $\mathbb{Z}_x$. Thus if the predicted values of x from twisted K-theory can be compared to the actual charge groups of the D-branes on the group manifold (determined via D-brane condensation in WZW theory), then this would be a very strong test for the important conjecture that the D-branes' charge group is a twisted K-group.

Before commencing a review of our research, some background information shall be reviewed for the topics of D-brane condensation and D-brane charge in WZW models.

§3.1 is a brief review of the relevant parts of WZW theory without boundaries (ie, for closed strings). §3.2 reviews fusion rules and fusion ideals in WZW theory, which shall become important for determining x. In §3.3 a brief discussion of D-brane condensation shall be given, as well as a brief description as to why it is believed that the D-brane charge algebra is given by a K-theory. In the context of WZW models, D-branes exist in open string WZW theory, and in §3.4 a discussion on the relation of the closed WZW theory of §3.1 and open WZW theory is given, as well as how D-branes arise from the boundary conditions and their relationship to conjugacy classes, boundary states and the representations of the affine Lie algebra of the WZW model. Section 3.5 gives a brief review of the reasoning of [40] in deriving the brane charge algebra eqn (3.5.16). Section 3.5.1 briefly summarises the analysis of [40] in determining the charge groups of the $\hat{A}_{N,k}$ algebras and their comparison of this analysis with the predictions of K-theory. In §3.6 an expanded discussion of the motivation and aims of the research in this chapter is given before outlining the research itself.

§3.7 outlines the initial investigations into finding the charge group $\mathbb{Z}_x$ for the untwisted affine Lie algebra. The lists of exact results for x found are in Appendix H. §3.8 discusses the exact low level and rank results for x and discusses a conjectured formula for predicting x for all simple, compact, connected, simply connected Lie groups G (eqn (3.8.4)). This predicted formula does not apply for low level k in some affine Lie algebras. A method of using the fusion ideals as an analytical way of proving the conjectured x formula of §3.8 is discussed in §3.9. In this analysis a precise but unwieldy formula for calculating x for general x is found (eqn (3.9.5)). However it can be used to prove the more concise eqn (3.8.4) for some algebras, and to be numerically equivalent to it for the rest.

The reason (eqn (3.8.4)) does not apply for some low values of k, and the resolution to this are discussed in §3.10 and §3.11.

The conjectured x formula is proved analytically for the $\hat{A}_{N,k}$ algebras in §3.12 and for $\hat{C}_{N,k}$ in §3.14. A slightly different method utilising fusion potentials is used to prove the same results in §3.13 and §3.15.

In §3.16 it was proven that for $\hat{G}_{2,k}$ the conjectured formula for x is a divisor of the a true formula of x, and is probably indeed the correct formula for x. In §3.17 generators of the ideal are constructed for all G, which numerically agree with the conjectured formula up to very high level and rank (ie, for all levels and ranks tested).

§3.18 the symmetries of the D-brane charge lattice determined by the values of x are analysed. A very interesting and hitherto unsuspected symmetry was discovered. The constraints provided by this symmetry alter the predicted values of x for the $\hat{C}_{N,k}$ algebras, which results in a conjectured formula for x that is much more mathematically elegant than the conjectured formula of §3.8, in that it can be written in a form that is invariant for any affine Lie algebra of G.

Appendix F outlines the notation used in this chapter, and Appendix G summarises the knowledge of Weyl and affine Weyl groups needed in this chapter. Besides this summary of Weyl groups, knowledge of Lie, affine Lie algebra and conformal field theory is assumed.

The research in this chapter is published in [20].

3.1 An introduction to WZW Models

WZW (Wess-Zumino-Witten) models are two dimensional conformal field theories on group manifolds and are a unique branch of CFT in that they can be described by an action [133]. This allows the powerful analytical techniques of both actions and CFT to be used on any system described by such models.

WZW models are very important in string theory. String theory is a two dimensional conformal field theory on a 26 dimensional target space for bosonic string theory. and 10 dimensions for supersymmetric string theory. In a realistic string theory, only four of these dimensions can be either flat or nearly so (including the timelike dimension) and thus the others should be compactified away in small volumes. This allows the possibility of a realistic string theory picture of the universe existing on many different manifolds. When the string theory exists on a Lie group manifold, the theory in these dimensions is described by a WZW model.

In general, analysing string theory on curved space-times is very difficult, however WZW models can be analysed quite thoroughly due to simplifications introduced by the symmetry of the group manifold. This serves to increase the importance of such models, as ideas from compactification on group manifolds can yield clues as to how string theory on other curved spacetimes should behave.

This section shall focus on introducing WZW models for closed string theory. The relationship between WZW models for open string field theory and D-branes will be introduced in §3.4.

3.1.1 The WZW action

The action of WZW models are nonlinear sigma models with a Wess-Zumino term added to extend the sigma model's symmetry [133]. As such it is wise to start a discussion of WZW models with a discussion of nonlinear sigma models on a group manifold.

Nonlinear sigma models are actions describing the dynamics of a CFT on a manifold. On a group manifold the model contains a bosonic field $g(\tau, \sigma)$ which is an element of the finite dimensional group G. τ and σ are coordinates on the two dimensional world-sheet.

The action can be written as [54,133]:

$$S_0 = \frac{1}{16\pi} \int_{\partial B} d^2\sigma \ \mathrm{Tr}(\partial^\mu g^{-1} \partial_\mu g), \tag{3.1.1}$$

where ∂B is the two dimensional world-sheet, $\partial_\mu g^{-1} \partial^\mu g = \partial_\tau g^{-1} \partial^\tau g + \partial_\sigma g^{-1} \partial^\sigma g$ and the trace is normalised by:

$$\mathrm{Tr}(\tau^a \tau^b) = -\frac{1}{2} \delta_{a,b}, \tag{3.1.2}$$

Here τ^a are generators of the a representation of an algebra of the group G, and Tr is the trace over these generators. The trace is normalised such that the action is invariant when evaluated for any representation.[1]

It is a condition of the nonlinear sigma model being real that the field $g(\tau, \sigma)$ be valued in a unitary representation of the Lie Group (which implies the norm of all states in the representation are positive). When applied to closed string theory, the boundary condition $g(\tau, \sigma) = g(\tau, \sigma + 2\pi)$ must be imposed.

The bosonic field $g(\tau, \sigma)$ has a global $G_L \times G_R$ symmetry, transforming as:

$$g(\tau, \sigma) \to \Omega g(\tau, \sigma) \bar{\Omega}^{-1},$$

where in Ω and $\bar{\Omega}$ are elements of G.

The nonlinear sigma model seems like an appropriate candidate for an action describing two dimensional CFT on a group manifold. However while it has a Lie group symmetry, it does not have an affine Lie algebra symmetry. This lack of an affine Lie algebra can be demonstrated by the nonlinear sigma model's lack of independent holomorphic and antiholomorphic currents (because affine Lie algebra in a CFT is manifested through the current algebra, as will be seen in the next subsection). See [54,133] for more detail.

[1]This normalised trace in the action is: $Tr = \frac{2 \dim g}{\dim |\lambda| (\lambda, \lambda + 2\rho)} Tr'$, where Tr' is the trace on the generators of representation λ, dim g is the dimension of the group and dim $|\lambda|$ is the dimension of the representation.

The Wess Zumino term that needs to be added to the nonlinear sigma model is [54,133]:

$$\Gamma = \frac{1}{24\pi}\int_B d^3y\ \epsilon_{\alpha\beta\gamma}\mathrm{Tr}(\tilde{g}^{-1}\partial^\alpha\tilde{g}\tilde{g}^{-1}\partial^\beta\tilde{g}\tilde{g}^{-1}\partial^\gamma\tilde{g}), \tag{3.1.3}$$

where B is a three dimensional manifold with the two dimensional manifold ∂B of eqn (3.1.1) as its boundary, and $\tilde{g}(y_0, y_1, y_2)$ is the three dimensional extension of $g(\tau,\sigma)$ to B, and again takes values in the group G.

There is an ambiguity inherent in the choice of manifold B. A compact two dimensional manifold ∂B can be the boundary of two different three dimensional spaces (for eg, for S^2 embedded in a locally flat three dimensional space, $\partial B = S^2$ can be the boundary of the volume inside S^2 and the volume outside S^2). The integration over the difference between these two boundaries must be a multiple of $2\pi i$ in order for the contribution of this ambiguity to the action to cancel in a path integral [133].

If G is semi-simple, compact, connected and simply connected this is done by considering that the addition of the two possible B manifolds, after compensating for their opposite orientation, gives the three dimensional manifold S^3. To integrate eqn (3.1.3) over S^3, first decompose G into its $SU(2)$ subgroups and select the world-volume S^3 to lie along an $SU(2)$ submanifold. Integrating over these coordinates gives eqn (3.1.3) equal to 2π for all such group manifolds [133]. Similar analysis is possible for non-semisimple groups.

Thus to cancel out the ambiguity over the choice of the manifold B, eqn (3.1.3) must have integer coefficient in the WZW action. Therefore:

$$S_{WZW} = kS_0 + k\Gamma \tag{3.1.4}$$

with k an integer (k is even for $SO(3)$).[2] This is the first appearance of what will later be identified as the level of the affine Lie algebra.

To show Γ is an appropriate choice, it is sufficient to show that its variation gives equations of motion which give the correct current invariance.

The equation of motion of S_{WZW} is:

$$\partial_z(g^{-1}\partial_{\bar{z}}g) = 0. \tag{3.1.5}$$

where it has been expressed in terms of of complex variables $z = e^{i(\tau+\sigma)}$, $\bar{z} = e^{i(\tau-\sigma)}$.

$\partial_z(g^{-1}\partial_{\bar{z}}g) = 0$ implies $\partial_{\bar{z}}(\partial_z g g^{-1}) = 0$ due to:

$$\partial_z(g^{-1}\partial_{\bar{z}}g) = g^{-1}\partial_{\bar{z}}(\partial_z g g^{-1})g. \tag{3.1.6}$$

Thus $J(z) = -\frac{1}{2}k\partial_z g g^{-1}$ and $\bar{J}(\bar{z}) = -\frac{1}{2}kg^{-1}\partial_{\bar{z}}g$. [3]

[2] The sigma term is also multiplied by k to consistently cancel extra terms from the sigma term with extra terms from the WZ term that arise while transforming by eqn (3.1.8).

[3] The $\frac{1}{2}k$ factors are included so that the current algebra found later shall be in a more convenient form for expressing the current algebra in the standard notation of an affine Lie algebra.

The solution for $g(z,\bar{z})$ of the equations of motion is:

$$g(z,\bar{z}) = f(z)\bar{f}(\bar{z}), \tag{3.1.7}$$

which shows it factorizes as expected for a CFT.

These currents generate the variation:

$$g(z,\bar{z}) \to \Omega(z)g(z,\bar{z})\bar{\Omega}^{-1}(\bar{z}), \tag{3.1.8}$$

where $\Omega(z)$ and $\bar{\Omega}^{-1}(\bar{z})$ are elements of G.

This indicates that the global $G_L \times G_R$ invariance of the non-linear sigma model has been extended to a local $G(z)_L \times G(\bar{z})_R$ invariance (where $G(z)_L$ and $G(\bar{z})_R$ denote left and right acting G symmetry on $g(z,\bar{z})$, where these left and right transformations can vary with respect to z and $\bar{z}$ respectively). This is reflected by the algebra of the currents generating the local $G(z)_L \times G(\bar{z})_R$ invariance being an affine Lie algebra, as shall be discussed in the next subsection.

At this stage it is useful to delve into the mode expansions of the theory, whereupon it will be seen that the currents of the WZW model generate an affine Lie algebra.

3.1.2 Mode Expansions and the Affine Lie Algebra

The transformation (3.1.8) applies not only to g but to any primary field ϕ of the theory [54,135].

$$\phi(z,\bar{z}) \to \Omega(z)\phi(z,\bar{z})\bar{\Omega}^{-1}(\bar{z}) \tag{3.1.9}$$

where in this more general case $\Omega(z)$ and $\bar{\Omega}^{-1}(\bar{z})$ can be functions of generators of two separate representations of the algebra $\mathfrak{g}$ of G.

For such are case where $\Omega(z)$ and $\bar{\Omega}^{-1}(\bar{z})$ are matrices in two different representations with generators t^a and $\bar{t}^a$ respectively, the infinitesimal transformations are:

$$\Omega(z) = I + w^a t^a = I + w, \tag{3.1.10}$$

$$\bar{\Omega}^{-1}(\bar{z}) = I + \bar{w}^a \bar{t}^a = I + \bar{w}. \tag{3.1.11}$$

The generators t^a and $\bar{t}^a$ are antihermitian obey algebra: $[t^a, t^b] = f^{abc}t^c$. It follows from this that the infinitesimal transformations of ϕ are:

$$\delta_w \phi = w^a t^a \phi, \quad \delta_{\bar{w}}\phi = -\phi \bar{w}^a \bar{t}^a. \tag{3.1.12}$$

They are the left and right isospin rotations and are generated by $J(z)$ and $\bar{J}(\bar{z})$ respectively.

The current $J(z) = J^a t^a$ transforms a field A by:

$$\delta_w A(z,\bar{z}) = \oint_C d\varsigma\, w^a(\varsigma) J^a(\varsigma) A(z,\bar{z}) \tag{3.1.13}$$

and acts on a correlator of primary fields as [54]:

$$\langle J^a(z)\phi_1(z_1,\bar{z}_1)\ldots\phi_n(z_n,\bar{z}_n)\rangle = \sum_{j=1}^{n}\frac{t_j^a}{z-z_j}\langle\phi_1(z_1,\bar{z}_1)\ldots\phi_n(z_n,\bar{z}_n)\rangle. \tag{3.1.14}$$

t_j^a acts on the j^{th} field. Similarly for $\bar{J}(\bar{z})$.

$J(z)$ itself is not a primary field with respect to its own transformation and transforms as [54]:

$$\delta_w J(z) = [w(z), J(z)] + \frac{1}{2}k\partial w(z). \tag{3.1.15}$$

where $w(z) = w^a(z)t'^a$ and here t'^a are the generators of the adjoint representations. Ie, the currents are not Lie algebra primary fields.

With respect to the conformal transformation:

$$\delta_\varepsilon A(z,\bar{z}) = \oint_C d\varsigma\varepsilon(\varsigma)T(\varsigma)A(z,\bar{z}),$$

where $T(\varsigma)$ is the stress energy tensor. $J(z)$ is a primary field with conformal dimension 1:

$$\delta_\varepsilon J^a(z) = \varepsilon(z)\partial J^a(z) + \partial\varepsilon(z)J^a(z). \tag{3.1.16}$$

$J(z)$ is a Virasoro primary field.

The OPE's arising from these transformations are [54]:

$$T(z)J^a(z') = \frac{1}{(z-z')^2}J^a(z') + \frac{1}{z-z'}\partial' J^a(z') + \ldots \tag{3.1.17}$$

$$J^a(z)J^b(z') = \frac{k\delta^{ab}}{2(z-z')^2} + \sum_c \frac{f^{abc}}{z-z'}J^c(z') + \ldots \tag{3.1.18}$$

$$\bar{J}^a(\bar{z})\bar{J}^b(\bar{z}') = \frac{k\delta^{ab}}{2(\bar{z}-\bar{z}')^2} + \sum_c \frac{f^{abc}}{z-z'}\bar{J}^c(\bar{z}') + \ldots \tag{3.1.19}$$

These are found using the Ward identities.

Taking the Laurent series of $J^a(z) = \sum_n J_n^a z^{-n-1}$, the Kac Moody algebra is arrived at through eqn (3.1.18) and eqn (3.1.19) [54].

$$[J_n^a, J_m^b] = f^{abc}J_{n+m}^c + \frac{1}{2}kn\delta^{ab}\delta_{n+m}, \tag{3.1.20}$$

$$[\bar{J}_n^a, \bar{J}_m^b] = f^{abc}\bar{J}_{n+m}^c + \frac{1}{2}kn\delta^{ab}\delta_{n+m}, \tag{3.1.21}$$

$$[J_n^a, \bar{J}_m^b] = 0. \tag{3.1.22}$$

Thus the current algebra forms two copies of the Kac-Moody, or affine Lie algebra of the group G, the level of which is given by k.

The next step in identifying the WZW model with a CFT is to identify the stress energy tensor and the Virasoro algebra that arises from it.

3.1.3 Stress Energy Tensor and the Virasoro Algebra

On a group manifold, J_n^a is equivalent to the creation and annihilation operators α_m^i in flat space. For example in flat space closed string theory, the bosonic field X^i is a sum of excitations:

$$X^i(\tau,\sigma) = x^i + 2p^i\tau + i\sum_{n\neq 0}\frac{1}{n}(\alpha_n^i e^{-in(\tau+\sigma)} + \bar{\alpha}_n^i e^{-in(\tau-\sigma)}). \tag{3.1.23}$$

The currents in this case are:

$$\partial_z X^i = -i\sum_n \alpha_n^i z^{-n-1}, \quad \partial_{\bar{z}} X^i = -i\sum_n \bar{\alpha}_n^i z^{-n-1}. \tag{3.1.24}$$

In the flat space case $T(z) = \sum_n L_n z^{-n-1}$ and $L_n = \frac{1}{2}\sum_m : \alpha_{n-m}\alpha_m :$. In analogy, the stress energy tensor for a WZW model action can readily be shown to be [54]:

$$T(z) = \frac{1}{2\kappa}\sum_a : J^a(z)J^a(z) := \frac{1}{2\kappa}\sum_a : J_{n-m}^a J_m^a : z^{-n-2} \tag{3.1.25}$$

where $\kappa = (k + g^\vee)/2$ and $g^\vee = \frac{1}{2}\sum_{b,c} f_{abc}f_{abc}$. The normal ordering is such that the creation operators J_n^a with $n < 0$ are pushed to the left of the annihilation operators with $n > 0$.

The Virasoro generators can thus be constructed from the Kac-Moody currents [54]:

$$L_n = \frac{1}{k+g^\vee}\sum_m : J_m^a J_{n-m}^a :, \tag{3.1.26}$$

The Virasoro algebra is [54]:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}c(n^3-n)\delta_{n+m}, \tag{3.1.27}$$

$$[L_n, J_m^a] = -mJ_{n+m}^a, \tag{3.1.28}$$

$$c = \frac{kD}{k+g^\vee}, \tag{3.1.29}$$

where D is the dimension of the group G. The operators $\bar{L}_n$ follow from $\bar{T}(\bar{z})$ in the same fashion.

Any string theory that is to be a realistic model of the universe should have 3+1 flat dimensions, and the rest compactified away, possibly on a group manifold or multiple group manifolds (a semi simple group manifold).

The WZW model corresponding to a semi-simple Lie algebra $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ has an energy momentum tensor corresponding to a summation of the respective energy momentum tensors, and as these energy momentum tensors commute, the resultant central charge gained from the OPE of the total energy momentum tensor with itself is the sum of the separate central charges. Incorporating the flat space is merely a matter of adding the flat space stress energy tensor as well [54].

Thus for flat space and a group manifold [54]:

$$T(z) = \frac{1}{2}\sum_n L_n z^{-n-1}, \tag{3.1.30}$$

$$L_n = \frac{1}{2}\sum_{i=1}^{d} : \alpha^i_m \alpha^i_{n-m} : + \frac{1}{k+g^\vee}\sum_{a=i}^{D} : J^a_m J^a_{n-m} : . \tag{3.1.31}$$

As is well known for bosonic string theory, the central charge is 26 and the critical dimension of such a string theory with some dimensions flat and some in a group manifold is thus give by:

$$d + \frac{kD}{k+g^\vee} = 26, \tag{3.1.32}$$

where the central charge of flat space is equal the number of its dimensions d.

The next step in analysing WZW is to study its spectrum of states.

3.1.4 Primary Fields and the Spectrum of States

Before studying the spectrum of states, first return to the primary fields.

A WZW primary field is defined as a field that is transformed covariantly by a $G_L(z) \times G_R(\bar{z})$ transformation [54].

From eqn (3.1.14) isospin current transformations of the primary fields can be represented by the OPE's:

$$J^a(z)\phi_{\lambda,\mu}(w,\bar{w}) \sim \frac{t^a_\lambda \phi_{\lambda,\mu}(w,\bar{w})}{z-w}, \tag{3.1.33}$$

$$\bar{J}^a(\bar{z})\phi_{\lambda,\mu}(w,\bar{w}) \sim \frac{\phi_{\lambda,\mu}(w,\bar{w})t^a_\mu}{\bar{z}-\bar{w}}, \tag{3.1.34}$$

where the primary field is transformed by a representation λ on the left and a representation μ on the right. t^a_λ and t^a_μ are matrices in these representations. Particles with spin are produced by having the left and right representations different.

Next we will consider only the left transformation on the primary fields $\phi_{\lambda,\mu}$ and thus drop the μ.

By taking the OPE eqn (3.1.33) and the OPE of the stress energy tensor $T(z)$ with ϕ_λ:

$$T(w)\phi_\lambda(z,\bar{z}) = \frac{\Delta_\lambda}{(w-z)^2}\phi_\lambda(z,\bar{z}) + \frac{1}{w-z}\partial_w\phi_\lambda(z,\bar{z}) + \dots , \tag{3.1.35}$$

the constraints on applying the Kac-Moody and Virasoro generators of the system can be found to be:

$$L_n\phi_\lambda = J^a_n\phi_\lambda(z,\bar{z}) = 0, \quad n > 0, \tag{3.1.36}$$

$$L_0\phi_\lambda = \Delta_\lambda\phi_\lambda, \tag{3.1.37}$$

$$J^a_0\phi_\lambda = t^a_\lambda\phi_\lambda. \tag{3.1.38}$$

where Δ_λ is the conformal dimension of ϕ_λ.

The states:

$$\prod_{n>0}(J^a_{-n})^{m_n}\phi_\lambda|0\rangle, \tag{3.1.39}$$

form a basis of the representation with highest integrable weight λ.

As mentioned in the previous section, due to eqn (3.1.26) all states in the WZW model can be written in terms of:

$$\prod_{n>0}(J_{-n})^{p_n}(\bar{J}_{-n})^{q_n}\phi_{\lambda,\mu}|0\rangle \otimes \overline{|0\rangle}, \tag{3.1.40}$$

where $\overline{|0\rangle}$ is the ground state of the antiholomorphic sector of the theory.

The physical state conditions from combining the holomorphic and antiholomorphic sectors are:

$$(L_0 + \bar{L}_0 - 2)|phys\rangle = 0, \tag{3.1.41}$$

$$(L_0 - \bar{L}_0)|phys\rangle = 0. \tag{3.1.42}$$

Applying these two conditions to eqn (3.1.40) gives the on shell conditions:

$$m^2 = -p^2 = -2 + \frac{c_\lambda + c_\mu}{k + g^\vee} + N + \bar{N}, \tag{3.1.43}$$

$$N + \frac{c_\lambda}{k + g^\vee} = \bar{N} + \frac{c_\mu}{k + g^\vee}, \tag{3.1.44}$$

where N and $\bar{N}$ are the number operators for J and $\bar{J}$, and c_γ is the quadratic Casimir of representation γ: $c_\gamma\delta_{ij} = -(t^a_\gamma t^a_\gamma)_{ij}$.

3.1.5 Integrable Highest Weight Representations in WZW Theory

As you would expect because the representations in the WZW model are Kac-Moody representations, for finite level k not all representations of the finite dimensional Lie algebra $\mathfrak{g}$ are present in the theory.[4].

To see this, consider the following $SU(2)$ sub-algebra of the current algebra, for a WZW model on a semi-simple Lie group [54]:

$$[J^{-\theta}_{+1}, J^{\theta}_{-1}] = m - J^{\theta}_0,$$

$$[m - J^{\theta}_0, J^{-\theta}_1] = 2J^{-\theta}_{+1},$$

[4]Strictly speaking, all the representations are still present in the theory, but they are not integrable, and thus decouple from the WZW model and any primary fields corresponding to such representations appearing in a correlation function of the theory makes the correlation function vanish [54]

$$[m - J_0^\theta, J_{-1}^\theta] = -2J_{-1}^\theta, \tag{3.1.45}$$

where $h_\theta = [\tau^\theta, \tau^{-\theta}]$, $m = 2k/(\theta, \theta) = k$, $J_{+1}^{-\theta}$ is the annihilation current corresponding to the generator τ^θ of the Lie algebra (similarly for the other J's), and the connection to $SU(2)$ is made via $P^+ = J_{+1}^{-\theta}$, $P^- = J_{-1}^\theta$ and $P^3 = m - J_0^{h_\theta}$. This $SU(2)$ and all the other $SU(2)$ subalgebras which the semi-simple Lie group decomposes with respect to contribute an $SU(2)$ pseudo spin.

For a representation in the $SU(2)$ subgroup we are concentrating on with highest weight λ, the representation can be written as:

$$\{\phi_\lambda, P^-\phi_\lambda, (P^-)^2\phi_\lambda, \dots\}, \tag{3.1.46}$$

for ϕ_λ being the primary field corresponding to the highest weight.

Using the normalisation $(\theta, \theta) = 2$, the pseudo spin of the representation is:

$$P^3\phi_\lambda = (k - (\lambda, \theta))\phi_\lambda. \tag{3.1.47}$$

For a finite dimensional Lie algebra, a representation with weight λ is finite dimensional if the eigenvalue of P^3 is a positive integer. For the affine Lie case, the representation is said to be integrable if the eigenvalue is:

$$M = k - (\lambda, \theta) \geq 0, \qquad M \in \mathbb{Z}_{\geq 0}. \tag{3.1.48}$$

This excludes many representations from the fundamental chamber, as seen in the discussion of the fundamental chamber and Weyl symmetry in Appendix G.2. Therefore, only representations with level $0 \leq (\lambda, \theta) \leq k$ are integrable.

Reviews of the topic of closed string WZW models and their affine Lie algebras can be found in [32,49,54].

3.2 Fusion Rules of Untwisted Affine Lie Algebra and their Application to WZW Models

3.2.1 Introduction: What are Fusion Rules?

Fusion rules in an affine Lie algebra are closely related to the operator product algebra. They are the analogy of the tensor product rules in a finite dimensional Lie Algebra, and are indeed, as you would expect, closely related to the tensor product and can be arrived at through careful truncation of the tensor product algebra.

Fusion rules are very important in WZW models, as they predict how states interact in the theory, by constraining its operator product expansions. Fusion rules are valid for the WZW model for string world-sheets with and without boundaries. A WZW model with world-sheets with boundaries describes string theory on a group manifold with open as well as closed strings, and thus it is these WZW models that can have D-branes, making it the relevant case for study in this thesis.

However the results of WZW models with string world-sheets without boundaries are valid for open string WZW models and thus in this section the fusion rule discussion is for WZW models of world-sheets without boundaries. In §3.4 the complications introduced by boundaries are discussed. This summary shall also focus on the fusion rules for untwisted affine Lie algebras, as it was for such WZW models that the D-brane charge analysis was carried out.

The reason fusion rules are important in this research is because the fusion rules can be shown to give constraints on the possible D-brane charge group and thus can be used to determine the charge group.

This summary shall be structured as follows: after a brief introduction to the concept of fusion rules, the Verlinde formula [24, 131] shall be discussed in §3.2.2. In §3.2.3 the fusion rules and their symmetries, along with the origins of their symmetries, shall be discussed [32]. In section 3.2.4 examples of $\widehat{SU(N)}_k$ fusion rules shall be derived with level truncation and Young Tableaux methods. §3.2.5 discusses the Kac Walton formula from which the fusion rules for all the affine Lie algebras can be derived. An algorithm based upon it shall be discussed and an example shall be quickly worked through. Finally, §3.2.6 discusses fusion rings, fusion potentials and fusion ideals [52], which shall become important concepts in this research.

3.2.2 The Verlinde Formula

It was shown in [131] that the fusion rules, which are the selection rules of the OPEs of the primary fields of the WZW model, are for a general affine Lie algebra:

$$\phi_\lambda \otimes_f \phi_\mu = \sum_{\nu \in P_+^{(k)}} \mathcal{N}_{\lambda\mu}^{(k)\nu} \phi_\nu, \tag{3.2.49}$$

$$P_+^{(k)} = \{\lambda \in P_+ | (\lambda, \theta) \leq k\}, \tag{3.2.50}$$

where $\otimes_f$ denotes the fusion of two primary fields (the f differentiates the fusion symbol from the tensor product symbol), P_+ is the set of highest weights of the irreducible representations in the fundamental chamber of the finite dimensional Lie algebra (ie, the 'dominant' integral weights). The fusion rules can be rewritten in terms of the highest weights of the representations:

$$\lambda \otimes_f \mu = \bigoplus_{\nu \in P_+^{(k)}} \mathcal{N}_{\lambda\mu}^{(k)\nu} \nu. \tag{3.2.51}$$

The fusion coefficients $\mathcal{N}$ are given by the Verlinde formula which is [24, 131]:

$$\tilde{\mathcal{N}}_{\hat{\lambda}\hat{\mu}}^{(k)\hat{\nu}} = \sum_{\hat{\sigma} \in P_+^{\prime(k)}} \frac{S_{\hat{\lambda}\hat{\sigma}} S_{\hat{\mu}\hat{\sigma}} \bar{S}_{\hat{\nu}\hat{\sigma}}}{S_{\hat{0}\hat{\sigma}}} = \mathcal{N}_{\lambda\mu}^{(k)\nu}, \tag{3.2.52}$$

where $\hat{0}$ stands for the vacuum representation with weight $k\hat{\Lambda}_0$, $\mathcal{S}$ is the modular matrix, $\bar{\mathcal{S}}_{\hat{\nu}\hat{\sigma}} = \mathcal{S}_{\hat{\nu}^*\hat{\sigma}}$ (where $\hat{\nu}^*$ is the conjugate representation to $\hat{\nu}$, defined by $\hat{\nu}^* = -w_0 \cdot \hat{\nu}$ where w_0 is the longest element of the finite Weyl group, which is discussed in Appendix G), $P'^{(k)}_+$ is defined separately to $P^{(k)}_+$ and is the set of affine highest weights in the affine fundamental chamber ($P^{(k)}_+$ is obtained from $P'^{(k)}_+$ by projecting out the imaginary roots of the weights in $P'^{(k)}_+$) and $\mathcal{N}^{(k)\nu}_{\lambda\mu}$ are the sought after fusion coefficients. The $\mathcal{N}^{(k)\nu}_{\lambda\mu}$ and $\tilde{\mathcal{N}}^{(k)\hat{\nu}}_{\hat{\lambda}\hat{\mu}}$ are equal to each other, however $\tilde{\mathcal{N}}^{(k)\hat{\nu}}_{\hat{\lambda}\hat{\mu}}$ is expressed in terms of the highest weights of the affine Lie algebra fundamental chamber ($P'^{(k)}_+$) (which it is derived with respect to) and $\mathcal{N}^{(k)\nu}_{\lambda\mu}$ is expressed in terms of the finite dimensional truncation of these weights ($P^{(k)}_+$). Both $\mathcal{N}^{(k)\nu}_{\lambda\mu}$ and $\tilde{\mathcal{N}}^{(k)\hat{\nu}}_{\hat{\lambda}\hat{\mu}}$ are used, as in some formulas the coefficients of the fusion ideal are contracted with indices that take values in $P'^{(k)}_+$ (for eg. eqn (3.2.70)) and in other formulas they are contracted with indices which take values in $P^{(k)}_+$ (for eg. eqn (3.2.49)). More information on the distinction between the affine weights and the finite dimensional weights is discussed in Appendices F and G.2.

The fusion coefficients $\mathcal{N}^{(k)\nu}_{\lambda\mu}$ that relate primary fields of finite dimensional representations in $P^{(k)}_+$ via eqn (3.2.49) is derived in terms of the modular $\mathcal{S}_{\hat{\mu}\hat{\sigma}}$ matrices, which relate affine characters parametrised by $P'^{(k)}_+$, and has indices which take values in the affine weights.[5]

The modular $\mathcal{S}$ matrix relates the characters of representations that are evaluated at weights that differ by a modular transformation $(\mu;\tau;t) \to (\mu/\tau; -1/\tau; t + |\mu|^2/2\tau)$:

$$\chi_{\hat{\lambda}}(\mu/\tau; -1/\tau; t + |\mu|^2/2\tau) = \sum_{\hat{\nu} \in P^{(k)}_+} \mathcal{S}^{\hat{\nu}}_{\hat{\lambda}}\, \chi_{\hat{\nu}}(\mu;\tau;t), \tag{3.2.53}$$

for the affine weight notation discussed in Appendix G.2.

The formula for $\mathcal{S}$ is:

$$\mathcal{S}_{\hat{\lambda}\hat{\mu}} = i^{|\Delta_+|}|P/Q^\vee|^{-\frac{1}{2}} \frac{1}{(k+g^\vee)^{\frac{N}{2}}} \sum_{w\in W} \epsilon(w) \exp\left(-2\pi i \frac{(w(\lambda+\rho), \mu+\rho)}{k+g^\vee}\right), \tag{3.2.54}$$

where the scalar product acts on the finite component of the affine weights[6], and W is the finite dimensional Lie algebra's Weyl group. $|\Delta_+|$ is the number of positive roots in the algebra, and $P/Q^\vee$ is the weight lattice modulo the coroot lattice (ie: the set of weights in an elementary cell of the coroot lattice).

An outline of the proof of the Verlinde formula in the presence of a boundary shall be given in §3.4.

3.2.3 Symmetries of the Fusion Rules

An aspect of fusion rules that has vital implications for the research in §3.17 and §3.18 is the fact that fusion rules have various automorphisms which simplify their structure.

[5] For the rest of this thesis the index (k) of the fusion coefficients $\mathcal{N}$ shall be dropped.

[6] This means that the scalar product is that defined by eqn (F.1) and acts on the finite weights λ and μ obtained by projecting out the imaginary roots from the affine weights $\hat{\lambda}$ and $\hat{\mu}$ in eqn (3.2.54).

In this research the fusion rules are used to find the set $\mathbb{Z}_x$ in which D-brane charges take values. The fusion rules act as constraints on what values D-brane charges can take, and the greater the number of symmetries that the fusion rules possess, the simpler the generating set of constraints is.

The symmetries involved can be divided up into two types. Firstly there are symmetries which come about from the action of the outer automorphisms of the finite dimensional Lie algebra ($\mathrm{Out}(\mathfrak{g})$) on the representations of the affine Lie Algebra, and secondly there are further symmetries on the fusion coefficients from the automorphisms of the affine Lie algebra (symmetries of the affine Dynkin diagram modulo the symmetries of the finite dimensional Lie algebra Dynkin diagram: $\mathrm{Out}(\hat{\mathfrak{g}})/\mathrm{Out}(\mathfrak{g})$) [15,98].

Of these symmetries, it is the second that is important in our investigations. Symmetries of the first type are not important in the work in this thesis and shall not be covered here.

So where does this outer automorphism symmetry come from, and what form does it take in the fusion rules? Its origins come the Verlinde formula (3.2.52) and the modular $\mathcal{S}$ matrix eqn (3.2.54).

Consider $A \in \mathrm{Out}(\hat{\mathfrak{g}})/\mathrm{Out}(\mathfrak{g}))$. The action of A on a weight is to permute some of its elements, and its action on $\mathcal{S}$ is [32]:

$$\begin{aligned} A\mathcal{S}_{\hat{\lambda}\hat{\mu}} &= \mathcal{S}_{A\hat{\lambda}\hat{\mu}} \\ &= i^{|\Delta_+|}|P/Q^\vee|^{-\frac{1}{2}}\frac{1}{(k+g^\vee)^{\frac{N}{2}}}\sum_{w\in W}\epsilon(w)e^{\left(-2\pi i\frac{(w(A\hat{\lambda}+\rho),\mu+\rho)}{k+g^\vee}\right)} \\ &= i^{|\Delta_+|}|P/Q^\vee|^{-\frac{1}{2}}\frac{1}{(k+g^\vee)^{\frac{N}{2}}}\sum_{w\in W}\epsilon(w)e^{\left(-2\pi i\frac{((k+g^\vee)wA\Lambda_0,\mu+\rho)+(ww_A(\lambda+\rho),\mu+\rho)}{k+g^\vee}\right)}. \end{aligned}$$

Using $(wA\Lambda_0, \mu+\rho) = (A\Lambda_0, \mu+\rho) \mod \mathbb{Z}$ the term $\exp(-2\pi i(A\Lambda_0, \mu+\rho))$ can be taken out the front of the summation, and a change of variables for the Weyl reflection $w' = ww_A$ and inserting $\epsilon(w') = \epsilon(w)\epsilon(w_A)$ yields:

$$A\mathcal{S}_{\hat{\lambda}\hat{\mu}} = \epsilon(w_A)\exp\left(-2\pi i(A\Lambda_0, \mu+\rho)\right)\mathcal{S}_{\hat{\lambda}\hat{\mu}}. \tag{3.2.55}$$

Since $\epsilon(w_A) = \exp(2\pi i(A\Lambda_0, \rho))$ this becomes:

$$A\mathcal{S}_{\hat{\lambda}\hat{\mu}} = \mathcal{S}_{\hat{\lambda}\hat{\mu}}e^{-2\pi i(A\Lambda_0,\mu)}. \tag{3.2.56}$$

See Appendix G.2.2 for more detail on the actions of the outer automorphisms on weights.

Inserting this into the Verlinde formula yields the fusion coefficient symmetries ($A, A' \in \mathrm{Aut}(\hat{\mathfrak{g}})$):

$$\begin{aligned} \tilde{\mathcal{N}}_{A(\hat{\lambda})A'(\hat{\mu})}^{AA'(\hat{\nu})} &= \tilde{\mathcal{N}}_{\hat{\lambda}\hat{\mu}}^{\hat{\nu}}, \\ \tilde{\mathcal{N}}_{A(\hat{\lambda})\hat{\mu}}^{\hat{\nu}} &= \tilde{\mathcal{N}}_{\hat{\lambda}A(\hat{\mu})}^{\hat{\nu}}. \end{aligned} \tag{3.2.57}$$

From this symmetry a condition on the fusion coefficients can be found. By replacing $\hat{\sigma}$ by $A(\hat{\sigma})$ in eqn (3.2.52), the following is found [32]:

$$\tilde{\mathcal{N}}_{\hat{\lambda}\mu}^{\hat{\nu}} = \tilde{\mathcal{N}}_{\hat{\lambda}\hat{\mu}}^{\hat{\nu}} e^{-2\pi i(A\Lambda_0,\lambda+\mu-\nu)} = \tilde{\mathcal{N}}_{\hat{\lambda}\hat{\mu}}^{\hat{\nu}} e^{-2\pi i((A\Lambda_0,\lambda)+(A\Lambda_0,\mu)-(A\Lambda_0,\nu))}. \tag{3.2.58}$$

The first line is allowed as if $\hat{\sigma} \in P_+^{(k)}$, then $A\hat{\sigma} \in P_+^{(k)}$. For $\tilde{\mathcal{N}}$ to be nonzero, the coefficient must be equal to 1. This requires:

$$(A\Lambda_0, \lambda + \mu - \nu) \in \mathbb{Z}, \tag{3.2.59}$$

which in turn puts a constraint on what representations can be produced from fusing λ and μ:

$$\lambda + \mu - \nu \in Q, \tag{3.2.60}$$

where Q is the root lattice.

3.2.4 $\widehat{SU(N)}$ Fusion Rules using Young Tableaux

$SU(N)$ Tensor Products using the Pieri Formula

In this section, to give some simple examples of fusion rules, the Pieri formula shall be introduced and then modified to deal with fusion rules rather than tensor products, and using this some $\widehat{SU(N)}$ fusion rules shall be derived.

The Pieri formula is the formula that describes how Young Tableaux are multiplied and thus determines the $SU(N)$ tensor products. When adjusted for the affine Lie algebra the Pieri formula directly gives the fusion rules for such an algebra, at least for the fundamental representations, and all others can be derived from it.

Recall that the Young Tableau for any weight $\lambda = (\lambda_1, \ldots, \lambda_{N-1})$ in $SU(N)$ is given by:

$$\lambda = \{l_1; l_2; \ldots; l_{N-1}\} = \{\sum_{i=1}^{N-1} \lambda_i; \sum_{i=2}^{N-1} \lambda_i; \ldots; \lambda_{N-1}\}, \tag{3.2.61}$$

where l_i is the length of the i^{th} row of the tableau.

The Pieri formula describes the multiplication of a fundamental representation's tableau by another tableau (which does not need to be fundamental). A fundamental representation of weight $\Lambda_j = (0, \ldots, 1, 0, \ldots, 0)$ with the one appearing in the j^{th} position, is given by the tableau:

$$x_j = \{1; 1; \ldots; 1\}, \tag{3.2.62}$$

which describes a tableau which is a vertical column of height j, where the $l_i = 0$ terms are omitted (ie, there are j 1's in eqn (3.2.62)).

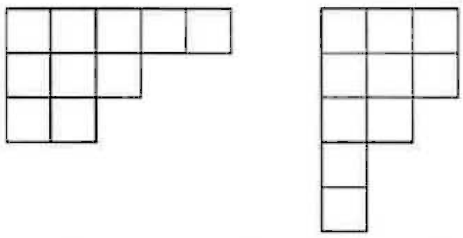
The Pieri formula is:

$$x_j \otimes \{l_1; \dots ; l_{N-1}\} = \bigoplus_{\substack{l_i \leq p_i \leq l_i + 1 \leq p_{i-1} \\ \sum_i p_i = j + \sum_i l_i}} \{p_1; \dots ; p_N\}. \tag{3.2.63}$$

The Pieri formula can thus be used to derive the tensor product rule of two representations, assuming that one of the representations can be decomposed in terms of fundamental representations. The Giambelli formula can be used to do this.

The Giambelli formula expresses a representation as a polynomial of fundamental representations. It is given by finding a determinant whose entries are fundamental weights, determined by the transpose of the Young Tableau of the original representation.

To illustrate the point regarding a transpose of a Young Tableau, a tableau $\{5; 3; 2\}$ has a transpose $\{3; 3; 2; 1; 1\}$.



{5; 3; 2} {3; 3; 2; 1; 1}

Figure 3.1: *Young tableau* $\{5; 3; 2\}$ *and it's transpose* $\{3; 3; 2; 1; 1\}$.

Here the transpose of a tableau $\{l_1; \dots ; l_{N-1}\}$ shall be denoted by $\{\tilde{l}_1; \dots ; \tilde{l}_s\}$ for some value of s (s denotes the index of the last non zero $\tilde{l}_i$).

The Giambelli formula is:

$$\{l_1; \dots ; l_{N-1}\} = \det x_{\tilde{l}_i + j - i} = \det \begin{pmatrix} x_{\tilde{l}_1} & x_{\tilde{l}_1 + 1} & \cdots & x_{\tilde{l}_1 + s - 1} \\ x_{\tilde{l}_2 - 1} & x_{\tilde{l}_2} & \cdots & x_{\tilde{l}_2 + s - 2} \\ \vdots & & & \\ x_{\tilde{l}_s - s + 1} & x_{\tilde{l}_s - s + 2} & \cdots & x_{\tilde{l}_s} \end{pmatrix}, \tag{3.2.64}$$

where in this formula $x_0 = x_N = 1$ and $x_i = 0 \;\; \forall\, i < 0,\; i > N$.

The validity of the Giambelli formula can be shown through inductive use of the Pieri formula.

So let's consider an example of using these two formulas to find the tensor product of two representations. Consider the product of the representations of weight $(1, 2, 0)$ and $(3, 0, 1)$ in $SU(4)$.

The Young Tableaux of these two representations are respectively $\{3; 2\}$ and $\{4; 1; 1\}$. The decomposition of $\{3; 2\}$ is:

$$\{3; 2\} = \det x_{\tilde{l}_i + j - i}\,, \tag{3.2.65}$$

where the transpose of the Tableau $\{3;2\}$ is $\{2;2;1\}$. The decomposition thus becomes:

$$\begin{aligned}
\{3;2\} &= \det\begin{pmatrix} x_2 & x_3 & x_4 \\ x_1 & x_2 & x_3 \\ x_{-1} & x_0 & x_1 \end{pmatrix} \\
&= \det\begin{pmatrix} x_2 & x_3 & 1 \\ x_1 & x_2 & x_3 \\ 0 & 1 & x_1 \end{pmatrix} \\
&= x_1 \oplus x_1 x_2^2 \ominus x_1^2 x_3 \ominus x_2 x_3.
\end{aligned}$$

Using the Pieri formula, it can be seen that:

$$\begin{aligned}
x_1 \otimes \{4;1;1\} &= \{5;1;1\} \oplus \{4;2;1\} \oplus \{4;1;1;1\} \\
&= \{5;1;1\} \oplus \{4;2;1\} \oplus \{3\} \\
x_2 \otimes \{4;1;1\} &= \{5;2;1\} \oplus \{5;1;1;1\} \oplus \{4;2;2\} \oplus \{4;2;1;1\} \\
&= \{5;2;1\} \oplus \{4\} \oplus \{4;2;2\} \oplus \{3;1\},
\end{aligned}$$

where the tableaux containing columns of 4 elements have been reduced.

The final tensor product rule between the tableaus is:

$$\begin{aligned}
\{3;2\} \otimes \{4;1;1\} = \quad & \{4;4;3\} \oplus \{5;1;1\} \oplus \{3;3;1\} \oplus \{6;1\} \oplus \{3;2;2\} \oplus 2\{5;2\} \\
& \oplus\{5;3;3\} \oplus \{4;3\} \oplus \{7;3;1\} \oplus 2\{4;2;1\} \oplus \{5;4;2\} \oplus \{6;4;1\} \\
& \oplus\{6;3;2\},
\end{aligned}$$

which in terms of weights is:

$$\begin{aligned}
&(1,2,0) \otimes (3,0,1) = (0,1,3) \oplus (4,0,1) \oplus (0,2,1) \oplus (5,1,0) \oplus (1,0,2) \oplus 2(3,2,0) \\
&\oplus(2,0,3) \oplus (1,3,0) \oplus (4,2,1) \oplus 2(2,1,1) \oplus (1,2,2) \oplus (2,3,1) \oplus (3,1,2). \quad (3.2.66)
\end{aligned}$$

See [32] for a review of Young Tableaus and the Pieri and Giambelli formulas.

Level Truncation to find the $\widehat{SU(N)}_k$ Fusion Rules

In $\widehat{SU(N)}_k$, while the affine weights do not have Young Tableaux themselves, there is a procedure for multiplying the Young Tableaux of the finite weights they correspond to. This procedure is the truncated Pieri formula [52].

The truncated Pieri formula contains an extra summation condition $p_1 - p_N \leq k$ which reflects that the product of a fundamental representation with another representation in the fundamental chamber cannot contain fusion products of level greater than k.

The Giambelli formula (eqn (3.2.64)) also picks up the condition that the transposed tableaus can have no more than k rows, ie, $s \leq k$ [52].

For example, when the condition $p_1 - p_N \leq k$ is applied to the example earlier (3.2.66), for level $k = 4$, the result is (in terms of weights of representations):

$$(1,2,0) \otimes_f (3,0,1) = (0,2,1) \oplus (1,0,2) \oplus (0,1,3) \oplus (2,1,1). \qquad (3.2.67)$$

0

3.2.5 Kac-Walton Formula and the Algorithm for Finding Fusion Rules

In the previous section the easily understood and intuitive approach for finding fusion rules in $\widehat{SU(N)}_k$ using Young Tableaux was discussed. Obviously this approach does not generalise well to the other Lie algebras which do not have such a nice Young Tableau description. In this section the Kac-Walton formula shall be briefly discussed, and how it can be used to generate the fusion rules for all the other Lie algebras.

The Kac-Walton formula describes the relationship between the tensor product coefficients for a finite dimensional Lie algebra, and the fusion coefficients for an affine Lie algebra. They are related by the Weyl reflections of the affine Lie algebra (Appendix G.2). The following is a very brief outline of how to derive the Kac-Walton formula. More detail can be found in [73,132].

The characters (eqn (G.1.12)) of the irreducible finite dimensional representations of the underlying finite dimensional Lie algebra are given by ratios of the modular matrix $\mathcal{S}$ values when the character is evaluated at a special point:

$$\begin{aligned} \chi_\mu(\xi_\sigma) &= \frac{\mathcal{S}_{\hat{\sigma}\hat{\mu}}}{\mathcal{S}_{\hat{\sigma}0}} = \gamma_{\hat{\mu}}^{(\hat{\sigma})}, \\ \xi_\sigma &= -\frac{2\pi i(\sigma+\rho)}{k+g^\vee}. \end{aligned} \qquad (3.2.68)$$

That $\chi_\mu(\xi_\sigma) = \frac{\mathcal{S}_{\hat{\sigma}\hat{\mu}}}{\mathcal{S}_{\hat{\sigma}0}}$ can be seen immediately from the form of $\mathcal{S}$ eqn (3.2.54), and it is quickly seen that substituting ξ_σ into the character eqn (G.1.12) completes this proof.

The $\gamma_{\hat{\mu}}^{(\hat{\sigma})}$'s satisfy the fusion rules of the affine Lie algebra (eqn (3.2.49)). To show this use the unitarity of $\mathcal{S}$ to rewrite the Verlinde formula into the form:

$$\sum_{\hat{\mu}\in P_+^{\prime(k)}} \mathcal{S}_{\hat{\mu}\hat{\nu}} \tilde{\mathcal{N}}_{\hat{\delta}\hat{\lambda}}^{\hat{\mu}} = \frac{\mathcal{S}_{\hat{\delta}\hat{\nu}}\mathcal{S}_{\hat{\lambda}\hat{\nu}}}{\mathcal{S}_{\hat{0}\hat{\nu}}}, \qquad (3.2.69)$$

and replace the $\mathcal{S}$'s by $\gamma_{\hat{\mu}}^{(\hat{\sigma})}$'s to get:

$$\gamma_{\hat{\delta}}^{\hat{\sigma}} \gamma_{\hat{\lambda}}^{\hat{\sigma}} = \sum_{\hat{\mu}\in P_+^{(k)}} \tilde{\mathcal{N}}_{\hat{\delta}\hat{\lambda}}^{\hat{\mu}} \gamma_{\hat{\mu}}^{\hat{\sigma}}, \qquad (3.2.70)$$

which is the fusion rules in terms of $\gamma_{\hat{\mu}}^{(\hat{\sigma})}$'s and therefore also $\chi_\mu(\xi_\sigma)$'s.

Resulting from the fact that the characters evaluated at any general point obey the tensor product rules of the finite dimensional Lie algebra, at the point ξ_σ a relation between the coefficients of the fusion rules and the tensor products can be found:

$$\sum_{\nu\in P_+^{(k)}} \mathcal{N}_{\lambda\mu}^{\nu}\chi_\nu(\xi_\sigma) = \sum_{\nu\in P_+} N_{\lambda\mu}^{\nu}\chi_\nu(\xi_\sigma). \tag{3.2.71}$$

where the notation in eqn (3.2.52) has been used.

To relate these two summations it is necessary to alter them so that they sum over the same range. The left sums over the affine fundamental chamber $P_+^{(k)}$, for a particular value of k, whereas the right sums over the infinite number of representations in the finite dimensional algebra's fundamental chamber P_+. The two summations can thus be related by a shifted affine Weyl reflection (see Appendix G.2 for details on such reflections). Any weight in P_+ can be reflected and shifted into $P_+^{(k)}$ with respect to the shifted action of some element $\hat{w} \in \widehat{W}$, the affine Weyl group. Weights lying on the boundary of $P_+^{(k)}$ for which the all the weights obey $\sum_{i=1}^{N} a^i\lambda_i = k+1$ are reflected into themselves via the reflection s_{α_0} with a reflection signature $\epsilon(s_{\alpha_0}) = -1$ and thus their character at ξ_σ is zero and such representations can be ignored. This reflection can be stated as:

$$\exists\ \hat{w} \in \widehat{W} \ \text{ s.t. } \ \hat{w}\cdot\hat{\nu}' = (\nu;\ k_\nu;\ 0), \tag{3.2.72}$$

where $\nu \in P_+$ and $\hat{\nu}' \in P_+^{\prime(k)}$ This allows eqn (3.2.71) to be rewritten as:

$$\sum_{\nu\in P_+^{(k)}} \mathcal{N}_{\lambda\mu}^{\nu}\chi_\nu(\xi_\sigma) = \sum_{\hat{\nu}\in P_+^{\prime(k)}} \sum_{\substack{w\,\cdot\,\nu'\,\in\,P_+ \\ \hat{w}\,\in\,\widehat{W}}} N_{\lambda\mu}^{w\cdot\nu'}\chi_{w\cdot\nu'}(\xi_\sigma), \tag{3.2.73}$$

where $w\cdot\nu'$ denotes the finite part of the affine weight $\hat{w}\cdot\hat{\nu}'$ obtained by projecting out the imaginary roots.

Using the fact that the Weyl transformation acts on the character as:

$$\chi_{w\cdot\mu}(\epsilon_\sigma) = \epsilon(\hat{w})\chi_\mu(\epsilon_\sigma), \tag{3.2.74}$$

(where again $w\cdot\mu$ is the finite part of $\hat{w}\cdot\hat{\mu}$) the relation between the fusion coefficients can be derived:

$$\mathcal{N}_{\lambda\mu}^{(k)\nu} = \sum_{\substack{w\,\cdot\,\nu\,\in\,P_+ \\ \hat{w}\,\in\,\widehat{W}}} N_{\lambda\mu}^{w.\nu}\epsilon(w). \tag{3.2.75}$$

This is the Kac-Walton formula, which has the advantage of clearly showing that, as $k\to\infty$, the fusion coefficients approach the tensor product coefficients. This will be made clearer with the example to follow.

The basic algorithm for using the Kac-Walton formula is to first derive the tensor product rules for all the representations whose affine extension appear in the affine fundamental chamber.

Then, for each tensor product relation, consider the representations that fall outside the fundamental chamber. The representations which either lie on the boundary of the fundamental chamber, or can be mapped to the boundary of the fundamental chamber through a shift in weight by integer values of $\frac{k+2}{a_i^\vee}$ (or through such shifts and a shifted Weyl reflection - Appendix G) can all be ignored, as such representations have vanishing character when evaluated at ξ_σ.

The representations that lie outside the affine fundamental chamber, must then be Weyl reflected (using a shifted Weyl reflection) into the affine fundamental chamber. A weight λ that has been reflected to λ' in the affine fundamental chamber thus has its contribution in the fusion product altered from $\oplus\lambda$ to $\oplus\epsilon(w)\lambda'$.

Let us apply this to the example used earlier (eqn (3.2.67)). This is the example of finding the fusion $(1,2,0) \otimes_f (3,0,1)$ in $\widehat{SU(4)}_{k=4}$. Start by using the Littlewood-Richardson rule for finding the tensor product eqn (3.2.66). Identify all representations that lie outside the fundamental chamber. These are:

$$(4,0,1)\ \ 2(3,2,0)\ \ (2,0,3)\ \ (1,2,2)\ \ (5,1,0)\ \ (2,3,1)\ \ (3,1,2)\ \ (4,2,1) \qquad (3.2.76)$$

Next identify the representations that lie on the boundary of the fundamental chamber (ie, those with $\sum_{i=1}^{N-1} \lambda_i = k+1 \equiv \lambda_0 = -1$) and eliminate these representations. These are:

$$(4,0,1)\ \ 2(3,2,0)\ \ (2,0,3)\ \ (1,2,2) \qquad (3.2.77)$$

Applying the affine shifted Weyl reflection s_{α_0} to the final representations should reveal which representations lie inside the fundamental chamber. The shifted affine Weyl reflection acts on a weight as [32]:

$$s_{\alpha_0}.\hat{\lambda} = \hat{\lambda} + \hat{\rho} - (\lambda_0 + 1)\alpha_0 - \hat{\rho}, \qquad (3.2.78)$$

where the affine root α_0 in terms of weights is $\alpha_0 = (2,-1,0,-1)$ where the first index is λ_0. Elsewhere when affine weights have been written in this thesis, λ_0 has been dropped unless explicitly mentioned.

Thus the s_{α_0} shifted weights are (including λ_0 in weights):

$$\begin{aligned}
s_0.(-2,5,1,0) &= (-2,5,1,0) + (2,-1,0,-1) = (0,4,1,-1),\\
s_0.(-3,4,2,1) &= (-3,4,2,1) + 2(2,-1,0,-1) = (1,2,2,-1),\\
s_0.(-2,2,3,1) &= (-2,2,3,1) + (2,-1,0,-1) = (0,1,3,0),\\
s_0.(-2,3,1,2) &= (-2,3,1,2) + (2,-1,0,-1) = (0,2,1,1).
\end{aligned}$$

Clearly both $(5,1,0)$ and $(4,2,1)$ can be mapped to a boundary of the fundamental chamber (ie $(5,1,0)$ and $(4,2,1)$ lie on the boundary of a different chamber) and thus can be eliminated. The other two representations can be mapped to the fundamental chamber interior, with a reflection coefficient of $\epsilon(s_0) = -1$, and thus cancel out other representations.

The final result is:

$$\begin{aligned}(1,2,0) \otimes_f (3,0,1) &= (0,2,1) \oplus (1,0,2) \oplus (0,1,3) \oplus (1,3,0) \oplus 2(2,1,1) \\ &\quad \ominus (1,3,0) \ominus (2,1,1) \\ &= (0,2,1) \oplus (1,0,2) \oplus (0,1,3) \oplus (2,1,1). \end{aligned} \tag{3.2.79}$$

In the next section, the results are for a reinterpreted in terms of quantum dimensions.

The Kac-Walton procedure is obviously time consuming, and in the bulk of our research, the software *Kac*, written by Bert Schellekens [118] was used to generate fusion rules.

3.2.6 Fusion Rings and Fusion Potentials

Consider a general two dimensional rational conformal field theory with a finite set of irreducible representations, labelling each representation by $i \in I$ (I is the set of representations). The RCFT will have a 'fusion ring,' which in terms of the preferred basis of the primary fields associated with the irreducible representations $\{\phi_i \mid i \in I\}$ is:

$$\phi_i \otimes_f \phi_j = \sum_{k \in I} \mathcal{N}_{ij}^k \phi_k. \tag{3.2.80}$$

The fusion ring is a commutative, associative, unital ring, which we shall call $\mathcal{F}$ [43,52].

The coefficients $\mathcal{N}_{ij}^k$ are the fusion coefficients, and take values in $\mathbb{Z}_{\geq 0}$. They are given by the Verlinde formula (3.2.52).

For a simple, connected, simply-connected, Lie groups G, the fusion coefficient matrices $(\mathcal{N}_i)_j^k$ have eigenvalues given by $\gamma_{\hat{\mu}}^{(\hat{\sigma})}$ (eqn (3.2.68)). For a general RCFT these eigenvalues are:

$$\lambda_i^{(l)} = \frac{\mathcal{S}_{li}}{\mathcal{S}_{l1}}, \tag{3.2.81}$$

where the 1 index is such that ϕ_1 is the unit element of $\mathcal{F}$.

In [52] it was argued that the fusion ring $\mathcal{F}$ is isomorphic to the abstract ring:

$$\mathcal{F} \cong \mathbb{C}[x_1, x_2, \dots, x_M]/\mathcal{I}. \tag{3.2.82}$$

When the generators $x_1, \dots, x_M$ correspond to the primary fields (or a subset thereof), the ideal is characterized as th polynomials $P \in \mathbb{C}[x_1, \dots, x_M]$ which vanish when the $(x_1, \dots, x_M)$'s take values in the set of vectors:

$$\{(\frac{\mathcal{S}_{li_1}}{\mathcal{S}_{l1}}, \dots, \frac{\mathcal{S}_{li_M}}{\mathcal{S}_{l1}}) \in \mathbb{C}^N | l \in I\} \tag{3.2.83}$$

This ideal of a RCFT, usually referred to as the fusion ideal, has the unusual property of corresponding to a Jacobian variety [33, 43, 47, 52]. This means that the generating polynomials of the ideal can be integrated into a 'fusion potential' V, thus the fusion ideal can be written as:

$$\mathcal{I} = \langle \frac{\partial V}{\partial \phi_1}, \dots, \frac{\partial V}{\partial \phi_M} \rangle. \tag{3.2.84}$$

This will be explored further in §'s 3.13 and 3.15.

The analysis of this thesis has been done for WZW models on compact, simple, connected, simply-connected, Lie groups G, rather than general RCFT. Thus the representations of the fields correspond to the integrable highest weight modules of the particular affine Lie algebra $\hat{\mathfrak{g}}$ (only untwisted algebras are considered here, the twisted case is left for future analysis).

As discussed in §3.2.5, for WZW models of G, the eigenvalues of the $(\mathcal{N}_i)_j^k$ can be expressed in terms of a limit of the characters of the finite dimensional representations of weight λ, eqns (3.2.68). The characters are given by eqn (G.1.12).

As for eqn (3.2.70), from this replacement of the fields ϕ_i with the characters χ_μ the fusion rules for the WZW model become:

$$\chi_\lambda(\xi_\sigma)\chi_\mu(\xi_\sigma) = \sum_{\nu \in P_+^{(k)}} \mathcal{N}_{\lambda\mu}^{\nu} \chi_\nu(\xi_\sigma), \tag{3.2.85}$$

where $P_+^{(k)}$ is the set of representations in the fundamental chamber.

These fusion rules are a truncation of the tensor products of the Lie algebra, which form the polynomials of characters with integer coefficients. The constraints imposed by the Kac-Walton formula to truncate the tensor product polynomials to the fusion polynomials can be summed up by the constraints of the fusion ideal. Thus for WZW models on a Lie group G, with an untwisted affine Lie algebra of level k, the fusion ring becomes:

$$\mathcal{F}_k = \mathbb{Z}[\chi_1, \dots, \chi_N]/\mathcal{I}_k, \tag{3.2.86}$$

where χ_i represent the N fundamental representations of a rank N finite dimensional Lie algebra. The fusion ring, and thus the fusion ideal, only needs to be expressed in terms of these representations, as all other representations can be expressed as polynomials of these representations. The character polynomials comprising the generators of the fusion ring ideal $\mathcal{I}_k$ must vanish when evaluated at a point ξ_σ (because $\chi_\mu(\xi_\sigma) = \lambda_\mu^{(\sigma)}$ and the ideal generators vanish when evaluated at $\lambda_\mu^{(\sigma)}$).

Thus the question of how to find a generating set for $\mathcal{I}_k$ arises.

The relation (3.2.74) describing the action of the affine Weyl group $\widehat{W}$ on the finite dimensional representation characters (evaluated at the special point ξ_σ) can be used to identify polynomials of characters which can be used to build up a generating set of $\mathcal{I}_k$. To reiterate the relation, for an element of the affine Weyl group $w \in \widehat{W}$, the reflection of the character is:

$$\chi_{w\cdot\lambda}(\xi_\sigma) = \epsilon(\hat{w})\chi_\lambda(\xi_\sigma), \tag{3.2.87}$$

where $w \cdot \lambda$ is the finite part of $\hat{w} \cdot \hat{\lambda}$ obtained by projecting out the imaginary roots.

Good candidates for generators of the ideal are thus the representations on the reflection boundaries of the fundamental chamber, such that $\hat{w} \cdot \hat{\lambda} = \hat{\lambda}$. For the simple reflections $\hat{w} = \{s_{\alpha_0}, \ldots, s_{\alpha_N}\}$ of the fundamental chamber into the adjacent chambers, the reflections all have signature $\epsilon(\hat{w}) = -1$ and thus eqn (3.2.87) becomes:

$$\chi_\lambda = 0 \tag{3.2.88}$$

From eqn (G.2.23) and eqn (G.2.27), the these boundaries occur at weights such that: $(\hat{\lambda} + \hat{\rho}, \alpha_i) = 0$ (a different boundary for each simple root).

It was conjectured in [52] that the characters on the boundary due to the reflection s_{α_0} provide a complete set of generators for the ideal (ie, the set of characters with weights such that $(\lambda, \theta) = k + 1$). This is generally believed to be true (and has been proved to be true for $\hat{A}_{N,k}$ [52, 57]), and certainly no contradiction was found in this research except for low k, for details of which see §'s 3.10 and 3.11. Ie, Gepner's conjectured general generators of the ideal are:

$$\begin{aligned} \mathcal{I}_k &= \langle \chi_\mu : \ \mu \in \ P_+, \ (\mu|\theta) = k + 1\rangle \\ &= \langle P_\mu : \ \mu \in \ P_+, \ (\mu|\theta) = k + 1\rangle, \end{aligned} \tag{3.2.89}$$

where P_μ is a polynomial expressing the character χ_μ in terms of the characters of the fundamental representations.

3.3 Tachyon Condensation of D-branes and K-theory

Pioneering work in [12,64,118–120,136] has lead to a new understanding of the spectrum of D-branes in string theory. The BPS branes reviewed and studied in Chapter 2 have been well studied for some time, but they are not the total spectrum of branes found in string theory. There are also many non-BPS D-branes.

For example, in *IIB* string theory the BPS D-branes are D($2p-1$)-branes for $p \in \{0, 1, 2, 3, 4, 5\}$. But there are non-BPS, meta stable branes of dimension $2p$ possible as well. These meta stable branes have a non-zero tachyon field, which causes their decay, and also do not have R-R charge.

Similarly *IIA* string theory has non-BPS D($2p-1$)-branes for $p \in \{0, 1, 2, 3, 4, 5\}$, and there are many ways to formulate non-BPS branes on orbifolds, and type *I* string theory contains non-BPS D(-1), D0, D7 and D8-branes [39, 118, 120, 136]. While the non-BPS D-branes in type IIA and IIB string theory are unstable for most background spacetimes, they are often stable when the background spacetime is an orbifold, as the orbifold symmetry may project out the tachyon state. Such configurations may still have no R-R charge, and yet this does not prevent their stability.

We shall briefly study the decay of unstable non-BPS D-branes, as well as decay of unstable configurations of BPS branes (needed to form non-BPS branes).

Let us study IIA and IIB non-BPS branes. Each Dp-brane has a R-R charge gained from integrating a $p+1$ form gauge field (see §2.1.3). The D-branes have an orientation. If the orientation of a Dp-brane with some R-R charge is reversed, then the charge of the Dp-brane is negative the charge it originally had. In comparison to a D-barne with the same original orientation, the reversed brane is a $\bar{\mathrm{D}}p$-brane, an anti-brane.

Consider a system of a D-brane and its anti-brane lying parallel and coincident to each other. Individually these two branes both preserve half of the supersymmetry of the system, but their presence together breaks all of the supersymmetry. As well as this, for this system the the negative energy open string NS ground state survives the GSO projection, giving tachyon states of mass $m^2 = -\frac{1}{2}$ [59]. The presence of the tachyonic field means the D-$\bar{\mathrm{D}}$-brane system is unstable.

The tachyonic potential $V(T)$ of this system is pictured in Figure 3.2, where the

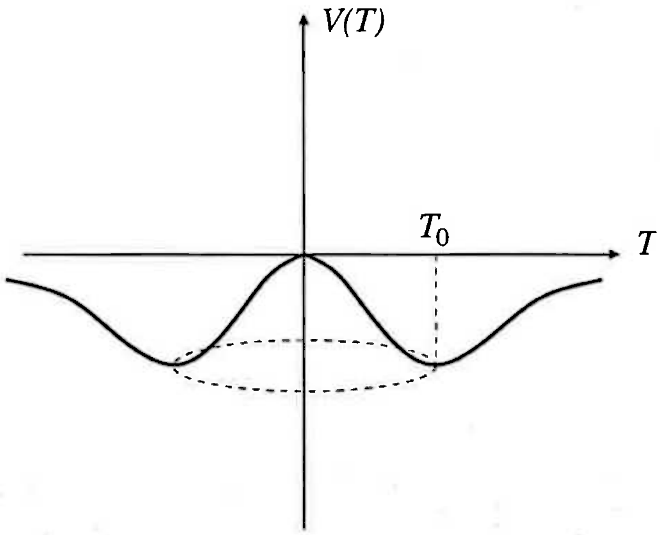


Figure 3.2: *Tachyonic potential of a D-brane and anti-D-brane system [121].*

tachyonic field T is complex (note the tachyonic field can be shown to be complex from the same arguments that prove the GSO projection does not eliminate it). This has a maximum for the tachyon field $T = 0$. Due to the $U(1) \times U(1)$ gauge field of the two brane system, the tachyon potential $V(T)$ picks up a phase invariance, and thus the depends only on $|T|$: $V(|T|)$.

At the minimum of the potential $|T| = T_0$ the tension energy T_p of the two branes cancels the negative potential energy [118, 120]:

$$2T_p + V(T_0) = 0. \tag{3.3.1}$$

Thus at the tachyon potential minimum, the D-$\bar{\text{D}}$-brane system has vanishing energy, and no R-R charge (as the D-brane and $\bar{\text{D}}$-brane charges cancel). Therefore the tachyonic ground state is in reality the vacuum.

An excited tachyonic kink solution (see Figure 3.3) also exists. This solution is real

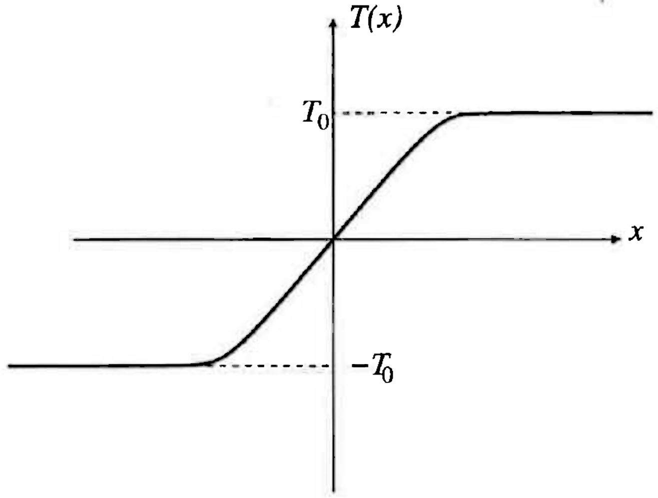


Figure 3.3: *Tachyonic kink solution of the Dp-$\bar{\text{D}}$p-brane system, equivalent to condensation to a non-BPS D$(p-1)$-brane [121].*

(Im(T)=0), time independent and spatially independent in $p-1$ of Dp-branes spatial dimensions. Figure 3.3 describes the spatial dependence in the last dimension (x) along the brane and predicts:

$$T(x) \to T_0, \quad x \to \infty,$$
$$T(x) \to -T_0, \quad x \to -\infty.$$

Thus at $x \to \pm\infty$ the kink solution is the ground state vacuum in Figure 3.2. The energy of this tachyonic solution is concentrated around $x = 0$, thus this kink solution seems to be equivalent to a D$(p-1)$-brane lying in the space of the parallel and coincident Dp-$\bar{\text{D}}p$-brane system. Indeed it has been shown that this does indeed imply that Dp-$\bar{\text{D}}p$-brane system in IIA/IIB string theory decays into a non-BPS D$(p-1)$-brane [118, 120]. This was shown by study of a marginal deformation of the bulk and boundary operators of the Dp-$\bar{\text{D}}p$-brane system into the operators of the non-BPS D$(p-1)$-brane which interpolates between $T = 0$ of the Dp-$\bar{\text{D}}p$-brane system to the kink solution. This interpolation deforms the CFT of the D-$\bar{\text{D}}$ to the CFT of the non-BPS D-brane.

A less rigorous way of looking at it is that the tachyon ground state solution becomes the vacuum $|T| = T_0$. And the kink solution is also equivalent to the vacuum $|T| = T_0$

away from $x = 0$, and thus open strings cannot end away from $x = 0$. However at $x = 0$, $T = 0$ and the open strings can end here, forming a D-brane.

That this D-brane is an unstable non-BPS D-brane is confirmed by looking at the topography of the kink solution. The tachyon potential minimum is a circle which has $\pi_0(S^1) = 0$. On the other hand the kink solution can only be stable on some manifold M for which $\pi_0(M) \neq 0$ and thus the kink solution and hence the D-brane is not stable.

There is also a tachyonic mode corresponding to a freedom to redefine the kink solution such that:

$$\begin{aligned} T(x) &\to T_0 e^{i\theta}, \qquad x \to \infty, \\ T(x) &\to -T_0, \qquad x \to -\infty. \end{aligned}$$

This reflects the $U(1) \times U(1)$ phase invariance. For $\theta = \pi$, $T = -T_0$ for $x \to \pm\infty$ and thus the kink solution becomes the vacuum solution.

Not only can the Dp-$\bar{\text{D}}p$-brane system decay into a non-BPS D$(p-1)$-brane, this non-BPS brane can in turn decay into a BPS D$(p-2)$-brane.

As discussed, the non-BPS D$(p-1)$-brane has a real tachyon field T' living on it. The potential on this brane can be shown to be the $\mathbb{Z}_2$ symmetric potential in Figure 3.4 [118,120].

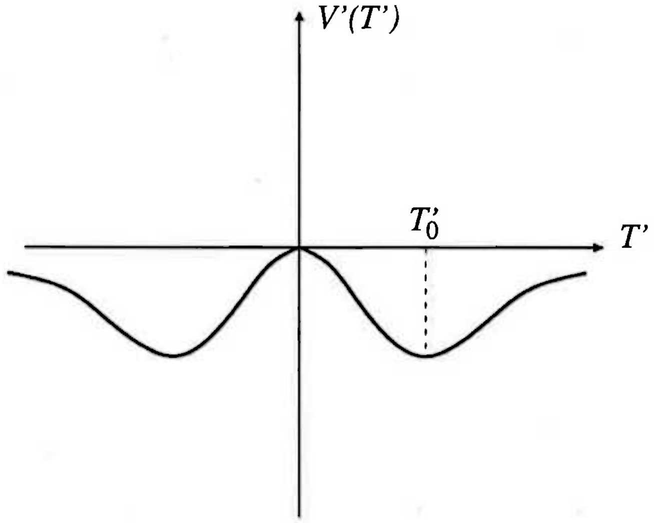


Figure 3.4: *Tachyonic potential of a non-BPS D-brane [121].*

The potential energy at the minimums of of the potential can again be shown to cancel the brane tension:

$$V'(T'_0) + T'_{p-1} = 0. \tag{3.3.2}$$

Thus again the tachyon ground state is equivalent to the vacuum.

Again there is a kink solution present on the D$(p-1)$-brane world-volume, which is independent of $p-2$ spatial dimensions and the timelike dimension, and is dependent

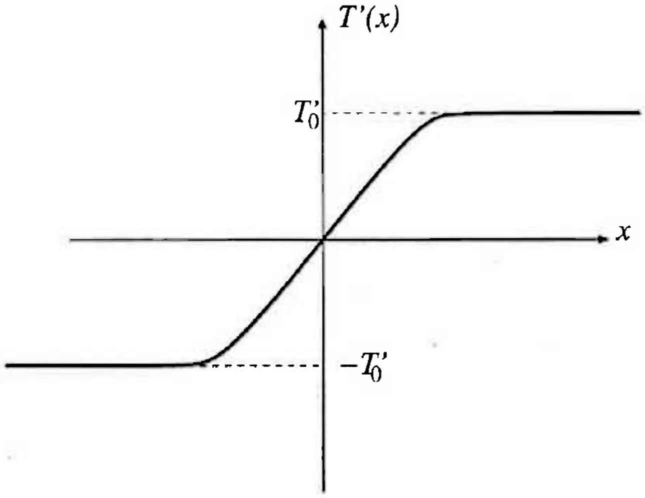


Figure 3.5: *Tachyonic kink solution of the non-BPS D$(p-1)$-brane potential, equivalent to condensation to a BPS D$(p-2)$-brane [121].*

on the last coordinate y. The solution is shown in Figure 3.5. Again, through the same argument used previously, this implies the non-BPS D$(p-1)$-brane condenses into a D$(p-2)$-brane. However, this time the brane it condenses into is stable, as the ground state of the potential in Figure 3.5 has a manifold M of two unconnected points and thus $\pi_0(M) \neq 0$. As mentioned before, this is the condition required for the kink solution to be stable. The D$(p-2)$-brane also picks up a R-R charge [120] and thus this brane is a BPS solution.

This situation describes the 'descent relations' of the D-branes in IIA/IIB string theory. Two BPS Dp-branes, the D-$\bar{\text{D}}$ pair, are only metastable and condense to a metastable non-BPS D$(p-1)$-brane, which in turn condenses to a BPS D$(p-2)$-brane.

Bosonic D-branes have similar descent relations.

This behaviour, combined with the fact that D-branes with trivial de Rham charge can be stable lead to the idea that D-brane charge groups are not a de Rham cohomology as previously thought. To illustrate this, any non-BPS D-brane which has zero R-R charge with respect to the de Rham cohomology/charge group should be unstable. However, some such branes are stable, as seen above. The answer must be that they have non-trivial charge, but not with respect to a de Rham cohomology. What then is the charge group?

It is easy to see that any BPS Dp-brane in a IIA (IIB) string theory could be the end product of the decay of a number of D8 (D9) branes and an equal number of $\bar{\text{D}}$8 ($\bar{\text{D}}$9) antibranes. Consider the IIB example. The space filling D9-branes are determined solely by the two $U(N)$ gauge fields E and F living on the branes and antibranes (where N is the number of D9 branes). If extra D9-$\bar{\text{D}}$9-brane pairs are added, with a $U(M)$ gauge bundle H on them, the tachyon field of open strings between these branes is a section of a trivial bundle and these branes condense to the vacuum

[121, 136]. The D-brane charges are classified by their $U(N)$ vector bundles (E, F and H) subject to the pair of vector bundles (E, F) being equivalent to $(E + F, F + H)$. This also defines a K-group, and thus the D-brane charge group should indeed be a K-group [79, 136]. (Similar analysis is possible for the other branches of string theory besides IIB.)

In particular this analysis [136] showed that in type IIA and IIB string theory on a background manifold X with a vanishing NS-NS three form H, or a 3-form $[H] \in H^3(X, \mathbb{Z})$ (where $[H]$ is the cohomology class of H) such that an integer multiple of $[H]$ vanishes, then the charge group is a $K^*(X)$ group or a twisted $K^*(X, [H])$ group respectively. It was shown in [21] that for the more general case of $[H] \in H^3(X, \mathbb{Z})$ and the H field is not torsion class (ie, some integer multiple of it does not vanish) then the charge group should also be a twisted K-group, $K^*(X, [H])$. However, in this case the twisted K-theory is more complicated than that considered in [136].

Indeed the idea that the D-brane charge group is a K-group explains how non-BPS D-branes can be stable. The Chern map demands an isomorphism between the non-torsion components of a K-group on a background and the corresponding de Rham cohomology. However in general the torsion components of the group are different. Thus for most D-brane interactions the two charge groups predict identical results, but for stable non-BPS branes, the branes could have no de Rham charge, but have non-trivial charge in the torsion components of the K-group, making them stable [128, 136]. [7]

See [121] for a useful review of this topic.

3.4 Open String WZW Theory and D-branes on Group Manifolds

3.4.1 3-form Quantisation in WZW theory

In section 3.1.1 the quantisation of the WZ term of the WZW action was discussed:

$$H = \frac{k}{24\pi}\mathrm{Tr}[(g^{-1}dg)^3], \qquad (3.4.1)$$
$$\int_Z H = 2\pi k,$$

where the manifold Z is a 3-cycle in the group G (for a WZW model in group manifold G) and represents the difference between the two choices of three dimensional manifolds that have as a boundary the two dimensional world-sheet of the sigma model term.[8]

[7]The torsion components of a finitely generated Abelian group C [40] ($C = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \oplus \mathbb{Z}_{x_1} \oplus \ldots \oplus \mathbb{Z}_{x_n}$) are $\mathrm{Tor}(C) := \mathbb{Z}_{x_1} \oplus \ldots \oplus \mathbb{Z}_{x_n}$. $\mathrm{Tor}(C)$ is the 'torsion' of C.

[8]Note that locally $H = dB$ where B is the NS-NS 2-form.

From this it can be seen that $[H]$ is an element of an integer cohomology [70,136]:

$$\frac{[H]}{2\pi} \in H^3(G, \mathbb{Z}). \tag{3.4.2}$$

From the definition in the previous section, for a group manifold H which is not vanishing, the charge group should be given by a twisted K-group $K^*(G, [H])$, instead of a normal $K^*(X)$ group (X is spacetime manifold) [21,136].

3.4.2 Open WZW Theory

When extending closed string WZW theory to the open string case, complications come from both the boundary conditions of the open string world sheets and the $U(1)$ gauge fields on the string ends that need to be included into the theory.

In §2.2 it was revealed that $\mathcal{F} = dA + B$ is a gauge invariant quantity where A is the $U(1)$ gauge field and B is the NS-NS 2-form.

The WZW model is extended to open string theory by:

$$S_{\text{open}} = \frac{k}{16\pi} \int_{\Sigma} d^2\sigma \; \text{Tr}(\partial^\mu g^{-1} \partial_\mu g) + \int_M H - \int_D \mathcal{F}, \tag{3.4.3}$$

where Σ is the open string world-sheet and D is a two dimensional manifold not yet specified. The manifold M is a three dimensional compact manifold with a boundary that should be given by the combination of $g(\Sigma)$ and D ($g(\Sigma)$ is the projection of Σ onto the group manifold). Due to the facts that $D + g(\Sigma)$ needs to be closed in order for it to be a boundary of the compact manifold M, and that the open strings will end on the D-branes of the system, the obvious choice for D is for it to be disk of a D-brane in the group manifold, such as in Figure 3.6 [70]. The world-sheet in

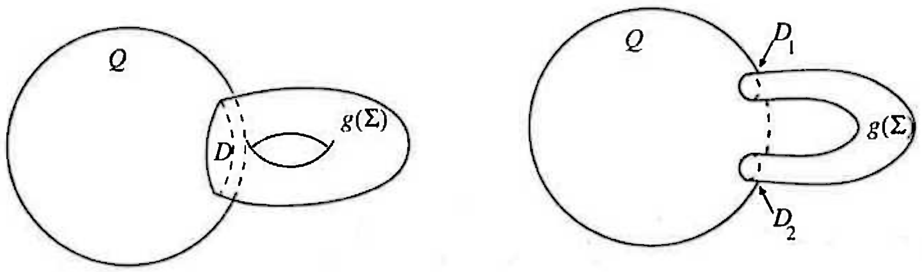


Figure 3.6: *Map of string world sheet Σ into the group manifold, ending on the D-brane wrapped on conjugacy class Q (subspace of the group manifold). The disk D is the subspace of Q sharing the same boundary as $g(\Sigma)$. The three dimensional manifold M is the space enclosed by $g(\Sigma)$ and D. The manifolds on the left are equivalent (via a conformal transformation) to the second set of manifolds shown, displaying a closed string being emitted and absorbed by the D-brane [70].*

this diagram is equivalent to the world-sheet of an open string starting and ending on the same D-brane. Let $g(\Sigma)$ be the map of the open string world-sheet to the group manifold and Q be the subspace of the group manifold swept out by the D-brane.

In general, to describe strings between branes D may be replaced by several disks in the integration of $\mathcal{F}$.

Just as for the closed string case, there is some ambiguity in the choice of the three dimensional manifold M, which extends in this case to D. Consider two choices of the manifolds M and D defined modulo the brane world-sheet Q: (M_1, D_1) and (M_2, D_2), such that the boundary of both D_1 and D_2 is $\partial g(\Sigma)$ and the boundary of M_1 is $D_1 + g(\Sigma)$ and similarly for M_2.

Then letting $S = D_1 - D_2$ and $Z = M_1 - M_2$ as in Figure 3.7, the quantity:

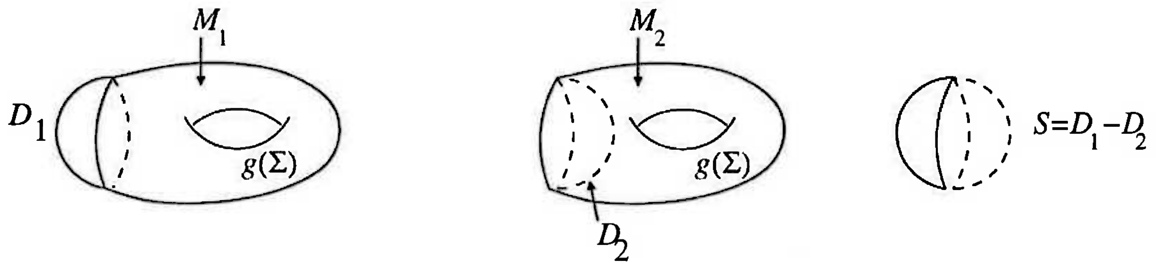


Figure 3.7: *S and Z with respect to the two choices of manifolds,* (M_1, D_1) *and* (M_2, D_2) *[70].*

$$\int_Z H - \int_S \mathcal{F} \equiv C, \tag{3.4.4}$$

must be an integer multiple of 2π for the the effect of this ambiguity to cancel in the path integration of the action.

This quantisation is outlined in [70], the main results being:

$$\begin{aligned} C = - \int_S dA \quad \mathrm{mod}\ 2\pi k, \\ \frac{[dA]}{2\pi} \in H^2(Q, \mathbb{Z}). \end{aligned} \tag{3.4.5}$$

C is interpreted as the D$(p-2)$-brane charge on the Dp-brane.

3.4.3 Current Boundary Conditions, D-branes and Conjugacy Classes

As discussed in §3.1.2 the Kac-Moody currents $J(z)$ and $\bar{J}(\bar{z})$ generate the left and right isospin transformations on g:

$$g'(z, \bar{z}) = \Omega(z) g(z, \bar{z}) \bar{\Omega}^{-1}(\bar{z}).$$

At the boundary of the open string world sheet, some boundary condition must be employed. Such a boundary condition should preserve half of the chiral $G_L(z) \times G_R(\bar{z})$

symmetry and thus should map J into $\bar{J}$. The boundary condition should ensure that energy is conserved at the boundary as well, so $T(z) = T(\bar{z})$ at the boundary.

The allowed boundary conditions are [7,37,40,80,124]:

$$J(z) = A'\bar{J}(\bar{z}), \tag{3.4.6}$$

where $A' \in \mathrm{Aut}(\mathfrak{g})$, the automorphism group of the finite dimensional Lie algebra. A' are always of the form:

$$A' = A \circ \mathrm{Ad}_g. \tag{3.4.7}$$

where $A \in \mathrm{Out}(\mathfrak{g})$ and $\mathrm{Ad}_g \in \mathrm{Int}(\mathfrak{g})$. Translations of type Ad_g merely translate the conjugacy classes and branes on the group manifold and leave the D-brane charge invariant up to a possible change of sign, and are thus are not of interest [86,127]. The automorphisms of interest are: $A \in \mathrm{Out}(\mathfrak{g})$ where $\mathrm{Out}(\mathfrak{g})$ are the outer automorphisms of the current algebra.

In general the research of this chapter will involve untwisted affine Lie algebras, given by $A' = 1$. Thus the review here shall concentrate on this case.

$A' = 1$ implies the isospin transformations:

$$g'(z,\bar{z}) = a(\Omega)g(z,\bar{z})\bar{\Omega}^{-1}, \tag{3.4.8}$$

where $\Omega = a(\Omega)$. The transformation $f_a : G \to G$ is the action of A' on the group G. For $A' = 1$, $a(\bar{\Omega}) = \bar{\Omega}$.

This transformation translates the open string ends over a subspace of the group manifold defined by the conjugacy class:

$$\mathcal{C}_a(h) = \{a(g)hg^{-1} |\; g \in G\}, \quad \mathcal{C}_1(h) = \{ghg^{-1} |\; g \in G\}. \tag{3.4.9}$$

This is interpreted as the world-volume in the group manifold of a D-brane. All points h in the same conjugacy class are on the same D-brane. This formulation of D-branes via conjugacy classes predicts an infinite number of D-branes, one for each possible conjugacy class, entirely filling the group manifold. However it will be shown in the next section that flux quantisation will impose a condition on where D-branes can lie, and thus in truth there is only a finite number of D-branes.

The untwisted boundary condition ($A' = 1$) can be rewritten as:

$$(1 + \mathrm{Ad}_g)\partial_\tau g = (1 - \mathrm{Ad}_g)\partial_\sigma g, \tag{3.4.10}$$

for a boundary at $\sigma = 0$ and $\mathrm{Ad}_g(v) = gvg^{-1}$ (v is a tangent vector v on some point $g \in G$ on the group manifold).

Denoting the tangent space of the group manifold G at point g as T_gG, the tangent space can be decomposed into parallel and orthogonal directions to the action of Ad_g:

$$T_gG = T_g^{\perp}G \oplus T_g^{\parallel}G. \tag{3.4.11}$$

In the perpendicular directions, eqn (3.4.10) becomes:

$$(\partial_\tau g)^\perp = 0, \tag{3.4.12}$$

because $\mathrm{Ad}_g = 1$ in $T_g^\perp G$. Applying this the boundary condition (3.4.10) becomes:

$$(\partial_\sigma g)^{||} = (1 - \mathrm{Ad}_g)^{-1}(1 + \mathrm{Ad}_g)(\partial_\tau g)^{||}, \tag{3.4.13}$$

in the $T_g^{||}G$ space.

Inserting local coordinates X^a of the manifold G, and the tangential coordinates ϵ^a at point $g \in G$, these boundary conditions become:

$$\begin{aligned} \text{Dirichlet}: &\quad (\partial_\tau X)^\perp|_{\partial\Sigma} = 0, && (3.4.14)\\ \text{Neumann}: &\quad \partial_\sigma X_{||}^a - \mathcal{F}^a{}_b \partial_\tau X_{||}^b|_{\partial\Sigma} = 0, && (3.4.15)\\ &\quad \mathcal{F}^b{}_a e_b = (1 - \mathrm{Ad}_g)^{-1}(1 + \mathrm{Ad}_g)e_a. && (3.4.16) \end{aligned}$$

Thus we have the Dirichlet and Neumann conditions for the D-branes defined by the conjugacy classes.

The same gauge invariant 2-form $\mathcal{F} = dA + B$ of the previous section can be rewritten as:

$$\mathcal{F} = \frac{k}{32\pi}\mathrm{Tr}\left[(dgg^{-1})^{||}(1 - \mathrm{Ad}_g)^{-1}(1 + \mathrm{Ad}_g)(dgg^{-1})^{||}\right]. \tag{3.4.17}$$

A method of labelling D-branes on the group manifold that summarises their properties is needed. Since it is inconvenient to classify D-branes by just any point on their conjugacy class, it would be preferable to have a subset of the group manifold for which every conjugacy class touches at a single point. This unique point could then be used to label the conjugacy class. Fortunately there is always such a submanifold present, the 'Maximal Torus' T. The maximal torus is defined by the exponentiation of the Abelian Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The symmetry of the maximal torus is $\mathcal{Z}(h) = \{g \in G | ghg^{-1} = h\}$. This obviously implies that each conjugacy class can only contain a single point h in T.

$\mathcal{Z}(h)$ is the 'centraliser' of h. The D-brane passing through h has a world-volume given by the space:

$$\mathcal{C}(h) = G/\mathcal{Z}(h). \tag{3.4.18}$$

This also implies the dimension of a D-brane defined by $\mathcal{C}(h)$ is equal to: $\dim G - \dim \mathcal{Z}(h)$. Generally $\dim Z(h)$= rank of G, however at some points the symmetry is enhanced and the dimension of $\dim Z(h)$ is greater, reducing the dimension of the D-brane. See [70] for the details of brane dimensions in various group manifolds.

3.4.4 D-brane Flux, Stability and Highest Weights

In this section we shall review the flux quantisation of D-branes on a group manifold, and its implications for D-brane stability.

In [9] the flux quantisation was studied for $G = SU(2)$. This system has D2-branes as S^2 spheres embedded in the S^3 manifold, as well as two D0-branes at the poles. Geometrically there is nothing stopping the D2-branes deforming to the poles and condensing to a point. The question is what prevents it?

It was found in [9] that the D2-brane action is minimised when the D0-brane flux dA (ie the D(2-2)-brane flux of eqn (3.4.5)) is parametrised by a subspace of G such that coordinate g in this subspace corresponds to coordinate $2\pi k\psi$ on the maximal torus $T = U(1)$. The D0-brane flux being quantised fixes ψ:

$$\psi = n/k, \quad n \in \mathbb{Z}_k \, .$$

This fixes the D2-brane to a conjugacy class passing through $\psi = n/k$ and thus the D2-brane cannot collapse to a D0-brane at the poles. It is this flux that prevents the D2-brane collapsing to a D0-brane.

Note also that the finite number of values of n fixes the number of possible D-branes to $k-1$ D2-branes and 2 D0-branes (at $n = 0$ and $n = k$).

Generalising this to the other compact Lie groups is simple. The same stabilisation technique can be applied individually to each $SU(2)$ subgroup of the Lie group.

Decompose G into its $SU(2)$ subgroups, labelling each by $SU(2)_\alpha$ where α is the corresponding root for each subgroup. Now the manifold S in eqn (3.4.4) can be decomposed into two cycles $S_\alpha = SU(2)_\alpha/U(1)_\alpha$ such that:

$$\int_S dA = \sum_\alpha c_\alpha \int_{S_\alpha} dA, \tag{3.4.19}$$

$$S \cong \sum_\alpha c_\alpha S_\alpha, \tag{3.4.20}$$

where c_α are constants and represents the winding number of S over S_α.

Choosing $(c_1, \ldots, c_r)$ (r=rank G) as a unitary vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ for each α and applying the eqn (3.4.5) gives:

$$\int_{S_\alpha} dA = 2\pi n_\alpha \mod 2\pi k, \quad \forall \alpha \; n \in \mathbb{Z}. \tag{3.4.21}$$

Comparing this with:

$$\int_{S_\alpha} dA = \int_{S_\alpha} \mathcal{F} - \int_{Z_\alpha} H = 2\pi k \vec{\alpha} \cdot \vec{\psi}, \tag{3.4.22}$$

where $\vec{\psi}$ is the vector coordinates on the maximal torus $T = U(1)^r$, and this formula is derived using the explicit forms of H and $\mathcal{F}$ (eqn (3.4.1) and eqn (3.4.17)) [70].

This directly implies $k\vec{\psi}$ is a highest weight of an integrable representation of the affine Lie algebra.

This thus implies that there is only a discrete number of branes rather than a continuous spectrum of them and that each D-brane corresponds to the highest weight of a representation of the affine Lie algebra of the group G.

As will be seen in the next section the D-branes can be repressed in terms of boundary states and boundary conditions on the open string world-sheet boundary corresponding to the irreducible representations of the algebra.

Before continuing, it should be mentioned that the level k used so far in describing WZW models must be shifted to $k + g^\vee$ when considering fermions, but as the effects of these fermions decouple from the rest of the WZW action other than to shift k, the rest of the theory does not need to be adjusted [40].

3.4.5 Boundary Operators

Let us consider the fusion of primary fields on the boundary of the world sheet [24,32, 44,69]. First consider a primary field $\phi(z)$, $z \in \mathbb{C}$ on the upper half plane, near the boundary/real axis. By mapping the upper half plane to the entire plane (by joining it with its mirror image lower half plane), the field $\phi(z)$ interacts with its mirror image $\phi(z^*)$, and thus has an OPE:

$$\phi(z)\phi(z^*) \approx \sum_i (z - z^*)^{h_i - 2h} \phi_B^i(\frac{z + z^*}{2}), \tag{3.4.23}$$

where h and h_i are the conformal dimensions of $\phi(z)$ and ϕ_B^i respectively and ϕ_B^i are fields which belongs to the same operator algebra as $\phi(z)$ but lie on the boundary.

Now consider the world sheet to be an annulus, with the coordinate τ in the periodic direction, with period T, and the coordinate σ running from 0 to L. Let there be a boundary condition labelled by α at $\sigma = 0$, and a boundary condition β at $\sigma = L$. For τ a timelike direction, the Hamiltonian is dependent on the boundary conditions for any time τ. Label the Hamiltonian by $H_{\alpha\beta}$.

Transforming the annulus coordinate $w = \tau + i\sigma$ by $z = e^{\frac{\pi}{L}w}$ transforms the annulus into the upper plane, with boundary condition α on the negative real line, and β on the positive real line. The interpretation of the boundary operators of eqn (3.4.23) is that a boundary operator $\phi_{B\alpha\beta}(0)$ placed at $x = 0$, the junction between these two boundary conditions, transforms one condition into another.

3.4.6 Verlinde Formula on the Boundary

We will now outline the standard proof of the Verlinde formula in the presence of a boundary. This will be done for a purely Virasoro algebra. Even though the results are believed to be true for Lie algebras as well, showing this is still an open problem.

Consider the partition function on the annulus, with the time like direction running along the periodic dimension:

$$\begin{aligned} Z_{\alpha\beta} &= \text{Tr } q^{H_{\alpha\beta}}, \quad q = e^{2\pi i\tau}, \quad \tau = iT/2L, & (3.4.24)\\ &= \sum_i \mathcal{N}^i_{\alpha\beta}\chi_i(q), \quad \chi_i(q) = q^{-c/24}\text{Tr}_i q^{L_0}, & (3.4.25) \end{aligned}$$

where the partition function can be re-expressed in terms of characters because the eigenfunctions of the Hamiltonian form irreducible representations of the Virasoro algebra. The trace Tr_i takes the trace over the states in the Verma module of the irreducible representation i. $\mathcal{N}^i_{\alpha\beta}$ labels the number of copies of each representation i, and shall be shown to be the fusion coefficients.

Under the modular transformation $\tau \to -1/\tau$ the characters are transformed:

$$\chi_i(q) = \sum_j \mathcal{S}_{ij}\chi_j(q'), \quad q' = e^{-2\pi i/\tau}, \tag{3.4.26}$$

where $\mathcal{S}_{ij}$ is the Virasoro modular matrix, and the partition function becomes:

$$Z_{\alpha\beta}(q) = \sum_i \mathcal{N}^i_{\alpha\beta}\mathcal{S}_{ij}\chi_j(q'), \tag{3.4.27}$$

In $q' = e^{-4\pi L/T}$ the roles of T and L are opposite to in q and this can be considered as a swapping of the time and space directions such that now time moves along the direction σ from 0 to L, and thus the Hamiltonian now moves from a state $|\alpha\rangle$ at $\sigma = 0$ to $|\beta\rangle$ at $\sigma = L$. To see this transform the annulus to the flat ring via:

$$\varsigma = e^{\frac{2\pi i u}{T}}.$$

The world-sheet is now a ring, with the timelike direction σ moving from the inner circle at $\sigma = 0$ to the outer ring at $\sigma = L$. The Hamiltonian for motion in the radial timelike direction is thus:

$$H' = \frac{2\pi}{T}(L^\varsigma_0 + \bar{L}^\varsigma_0 - \frac{c}{12}), \tag{3.4.28}$$

where L^ς_n is a Virasoro generator in the ς coordinates.

The partition function becomes:

$$Z_{\alpha\beta}(q) = \langle\beta|e^{LH'}|\alpha\rangle. \tag{3.4.29}$$

Thus the partition function on the annulus with the Hamiltonian dependent on the boundary conditions has been transformed into the partition function of the ring with the Hamiltonian independent of the boundary conditions, but evolving the state $|\alpha\rangle$ to $|\beta\rangle$ (the boundary state $|\alpha\rangle$ is equivalent to the boundary condition α). The challenge now is to find possible boundary states.

The first step in doing this is deriving constraints the states must obey. Firstly we must restrict momentum and energy flowing off the ends of the annulus. This is equivalent to placing constraints of $T(\tau,\sigma)$, the stress energy tensor on the annulus:

$$T(w)=\bar{T}(w^*), \quad \text{if} \quad \frac{w-w^*}{2}=0 \text{ or } iL. \tag{3.4.30}$$

In the ς coordinates, this is equivalent to the condition:

$$(L_n^\varsigma-\bar{L}_{-n}^\varsigma)|\alpha\rangle=0. \tag{3.4.31}$$

In [69] it is shown that the states that satisfy the constraint (3.4.30) are given by:

$$|j\rangle=\sum_m |j;m\rangle\otimes U\overline{|j;m\rangle}, \tag{3.4.32}$$

where $|j;m\rangle$ is state m in representation j and U is an anti unitary operator.[9] $U\overline{|j;m\rangle}=\overline{|j;m\rangle}^*$ where $\overline{|j;m\rangle}^*$ is the complex conjugate of $\overline{|j;m\rangle}$.

Inserting $\sum_i |i\rangle\langle i|=1$ into eqn (3.4.29) gives:

$$\begin{aligned}
Z_{\alpha\beta}(q) &= \sum_{ij}\langle\alpha|i\rangle\langle i|q'^{\frac{1}{2}L_0^\varsigma+\frac{1}{2}\bar{L}_0^\varsigma-\frac{c}{24}}|j\rangle\langle j|\beta\rangle \\
\langle i|q'^{\frac{1}{2}L_0^\varsigma+\frac{1}{2}\bar{L}_0^\varsigma-\frac{c}{24}}|j\rangle &= \sum_{m,n}\langle i;m|\otimes\overline{\langle i;m|}U^\dagger q'^{\frac{1}{2}L_0^\varsigma+\frac{1}{2}\bar{L}_0^\varsigma-\frac{c}{24}}|j;n\rangle\otimes U\overline{|j;n\rangle} \\
&= \delta_{i,j}q'^{-c/24}\mathrm{Tr}_i q^{L_0} \\
Z_{\alpha\beta}(q) &= \sum_i\langle\alpha|j\rangle\langle j|\beta\rangle\chi_j(q'),
\end{aligned} \tag{3.4.33}$$

for an appropriate choice in the bases of $|i\rangle$ and $|j\rangle$. Comparing this with eqn (3.4.27) gives:

$$\sum_i \mathcal{S}_{ij}\mathcal{N}^i_{\alpha\beta}=\langle\alpha|j\rangle\langle j|\beta\rangle. \tag{3.4.34}$$

It was shown in [69] that the following states are solutions to this equation:

$$|\tilde{j}\rangle=\sum_i\frac{\mathcal{S}_{ji}}{\sqrt{\mathcal{S}_{0j}}}|j\rangle. \tag{3.4.35}$$

These are the Cardy states, and means there is a boundary state corresponding to each irreducible representation. While these states satisfy the conditions for a boundary state, they are not a complete set of all such states. These are the states that correspond to Neumann conditions.

Substituting the Cardy states into eqn (3.4.34) yields:

$$\sum_i \mathcal{S}_{ij}\mathcal{N}_{kl}=\frac{\mathcal{S}_{ki}\mathcal{S}_{lj}}{\mathcal{S}_{0j}}. \tag{3.4.36}$$

[9] An antiunitary operator U acting on a vector space V acts as $U(cv)=c^*U(v)$ where $v\in V$ and $c\in\mathbb{C}$.

This argument is based on the discussion in [32], which summarises arguments of [24,69].

The subject of applying knowledge of Cardy states to D-branes on a group manifold G is developed in [7,37,49,109,124] amongst others. For open string WZW theory the Dirichlet boundary condition:

$$J^a(z) - \bar{J}^a(\bar{z})\big|_{z=\bar{z}} = 0 \tag{3.4.37}$$

is applied to the boundary state $|\tilde{\lambda}\rangle$ (where the meaning of the λ shall be clear momentarily and the boundary condition is $J^a(z) - \bar{J}^a(\bar{z})\big|_{z=\bar{z}} |\tilde{\lambda}\rangle = 0)$. The resulting boundary state is (for untwisted affine Lie algebra) [37]:

$$\left|\tilde{\lambda}\right\rangle = \sum_{\hat{\mu}\in P_+^{\prime(k)}} \frac{\mathcal{S}_{\hat{\lambda}\hat{\mu}}}{\mathcal{S}_{\Lambda_0\hat{\lambda}}} \left|\mu\right\rangle \tag{3.4.38}$$

$$= \sum_{\mu\in P_+^{(k)}} \chi_\mu(\xi_\lambda) \left|\mu\right\rangle \tag{3.4.39}$$

$$|\mu\rangle = \sum_{m\in\mathcal{H}_{\hat{\mu}}} |\mu; m\rangle \otimes U\overline{|j; m\rangle}, \tag{3.4.40}$$

where eqn (3.4.40) gives the Ishibashi state corresponding to the affine Lie algebra integrable representation with highest weight $\hat{\mu}$. $\mathcal{H}_{\hat{\mu}}$ is the highest weight module of integrable representation with highest weight $\hat{\mu}$. Eqn (3.4.38) follows from eqn (3.4.39) via eqn (3.2.68). χ_λ is the finite dimensional Lie algebra character of representation with highest weight λ (the weight obtained by projecting the imaginary roots from $\hat{\lambda}$) [10]. $\left|\tilde{\lambda}\right\rangle$ is the Cardy boundary state corresponding to a D-brane. λ is a highest weight in $P_+^{(k)}$.

Comparing this to the conjugacy class description of D-branes, it is easy to see that a D-brane defined by boundary state eqn (3.4.38) corresponds to a conjugacy class $C(h_\lambda)$, $h_\lambda = e^{-\xi_\lambda}$ (ξ_μ given by eqn (3.2.68)) where h_λ is an element of the maximal torus of the Lie group G.

3.4.7 States on the Infinite Strip

Let us look at states flowing along a flat strip in Figure 3.8. When the boundary conditions both correspond to the vacuum representation there is just the null state on the strip. At $\tau = p_1$ there is a boundary operator ϕ_{0i} which swaps the boundary condition to i and means the states flowing here belong to the affine Lie algebra integrable representations given by $\bigoplus_l \tilde{\mathcal{N}}_{0i}^l l = \bigoplus_l \delta_i^l l = i$. At $\tau = p_2$ there is boundary operator ϕ_{0j} after which the states on the strip belong to $\bigoplus_k \tilde{\mathcal{N}}_{ji}^k k$.

[10]The difference between the forms of eqn (3.4.36) and eqn (3.4.39) (ie: a factor of $1/\sqrt{\mathcal{S}}$) is merely a difference in normalisation.

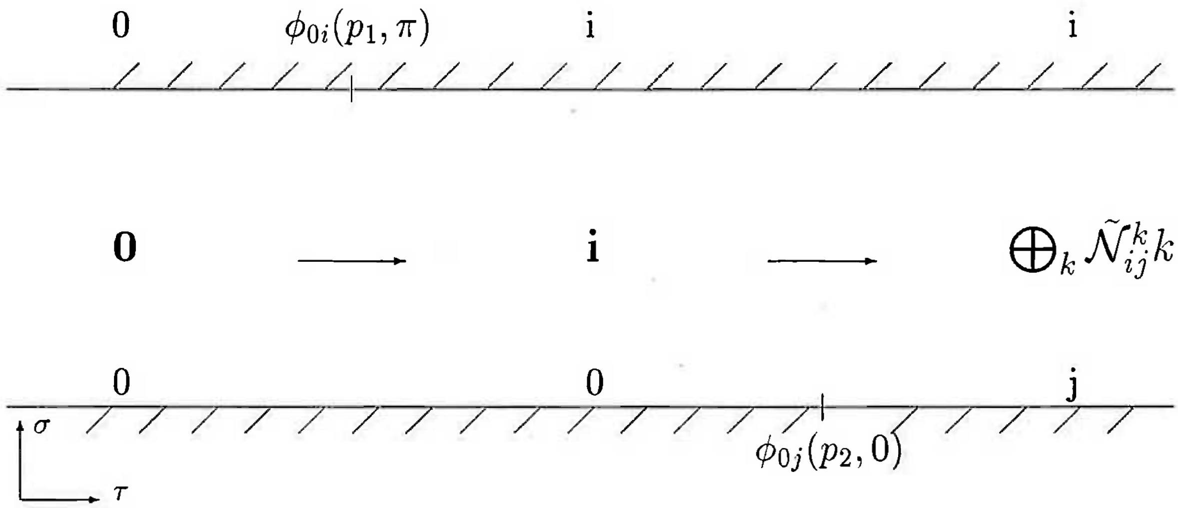


Figure 3.8: *States on an Infinite Strip with Boundary Operators [24].*

Conversely, in the picture when time flows from side to side of the strip, the states exist on the boundaries, characterised by the Cardy states, for which there is a single state for each representation.

In the discussions of the previous few subsections the arguments suggest that the D-branes are sharply defined subspaces of the the group manifolds. In reality, non-commutative effects slightly smear out the conjugacy classes around which the D-branes are wrapped, as shown in [5,37]. However the details of this do not affect the research in this thesis and thus are not reviewed here.

3.5 Condensation on Group Manifolds

This section summarises the arguments of [40] which investigates how to determine the R-R charge group of D-branes on a group manifold and comparing the charge group to the predictions of twisted K-theory. See §2.1.3 for a discussion of R-R charges of D-branes.

The authors study the condensation of a stack of identical D-branes to a configuration of other D-branes. Each brane has a specific R-R charge determined by the conjugacy class which defines the D-brane (see §3.4 for details of conjugacy classes of D-branes). Therefore the charges of the stack of D-branes must be conserved by the process of condensation, such that the sum of charges of the branes resulting from condensation equals the sum of charges of the original branes.

The charge group in which this charge is conserved is a discrete Abelian group by construction, but the details are not certain. As mentioned in §3.3 while it was for some time believed that the charge algebra is a de Rham cohomology, recent work [21,128,136] suggests that in reality the charge algebra is rather a more complicated K-group.

As mentioned in §3.3, for string theory on a particular background the non-torsion components of the corresponding K-group and de-Rham cohomology will be isomorphic. The differences in the charge groups come from there torsion components. As such, the authors of [40] chose to study brane condensation on a group manifold, because such backgrounds have a nontrivial, non-torsion class NS-NS three form, the WZW three form H (see §3.4.1) for which $[H] = (k + g^\vee)[H_0]$, and the charge group should be a twisted K-group $K^\star(G, (k + g^\vee)[H_0])$ ($[H_0]$ is the generator of $H^3(G, \mathbb{Z}) \cong \mathbb{Z}$). In such a group manifold background the K-group and de Rham cohomology torsion components are different, providing a perfect test bed for studying non-BPS D-branes and their charges, providing an important direct test of whether or not the charge group is a twisted K-group.

The case of D-branes in a $SU(2)$ group manifold was studied previously in [5,7–9,96,103,129] where it was shown that the D-branes (conjugacy classes) of $SU(2)$ are D0-branes and D2-branes. The D2-branes are S^2 spheres wrapped on the S^3 manifold. The two D0-branes which lay at opposite poles of the S^3 (ie at $\pm e$, where e is the group unit). It was shown that a stack of D0-branes at $+e$ condenses into a D2-brane, and as more and more D0-branes are condensed the resultant D2-brane is moved further along the S^3 until after a critical number of D0-branes are condensed at $+e$, they form a D0-brane at $-e$. This implies that the charge group is $\mathbb{Z}_x$ for some integer x (this is explained in more detail in §3.5.1).

The key ingredient to the analysis in [40] was the realisation that R-R charges of D-branes are RG-invariants. This is because in the CFT description of D-branes as boundary conditions on the world-sheet, brane condensation is due to boundary operators. The IR limit of the RG trajectories associated with these boundary operators determines the decay products. This implies conserved R-R charges are RG-invariants.

The analysis in [40] was done for all the twisted as well as untwisted affine Lie algebras $\hat{\mathfrak{g}}_k^w$ of the group G. In the summary of WZW models given in §3.1 twisted algebras were not reviewed as are not covered in the research of this chapter. As such the topic of twisted affine Lie algebras shall receive only the minimum attention here needed to explain the results of [40]. The twisting labelled by w is an outer automorphism of the finite dimensional Lie algebra. This is a symmetry of the finite dimensional Lie algebra Dynkin diagram. The D-branes are defined by the conjugacy classes with respect to these outer automorphisms, as in eqn (3.4.9) (where w is the a in eqn (3.4.9)).

The possible D-branes can be labelled by $\Xi = (\alpha, w)$ where w is the twisted conjugacy class and α specifies the boundary condition. α takes values in some set $\mathcal{J}_k^w$, and for untwisted affine Lie algebra ($w = I$, the identity outer automorphism), $\mathcal{J}_k^w = P_+^{(k)}$ where $P_+^{(k)}$ and is the finite dimensional Lie algebra highest weight obtained by projecting out the imaginary root of it's corresponding affine weight.

The partition function of an open string stretched between two D-branes defined by the boundary conditions (α, w) and (β, w) is:

$$Z^w_{\alpha\beta}(q) = \sum_{\gamma \in P_k^{(+)}} \mathcal{N}^{w;\beta}_{\gamma\alpha} \chi_\gamma(q), \tag{3.5.1}$$

where $\chi_\gamma(q)$ are the characters of the untwisted affine Lie algebra $\hat{\mathfrak{g}}_k$. $\mathcal{N}^{w;\beta}_{\gamma\alpha}$ are the twisted fusion coefficients for which $\gamma \in P_+^{(k)}$ and $\alpha,\ \beta \in P_+^{w(k)}$. They are given by the twisted Verlinde formula:

$$\mathcal{N}^{w;\beta}_{\gamma\alpha} = \sum_{\lambda \in P_k^{w(+)}} \frac{\bar{\mathcal{S}}^w_{\lambda\beta} \mathcal{S}^w_{\lambda\alpha} \mathcal{S}_{\lambda\gamma}}{\mathcal{S}_{\lambda 0}}, \tag{3.5.2}$$

where $\mathcal{S}^w$ is the twisted modular matrix.

Just as for the untwisted fusion coefficients, the twisted fusion rules form a representation of the untwisted fusion rules:

$$\sum_{\beta \in P_k^{w(k)}} \mathcal{N}^{w;\beta}_{\mu\alpha} \mathcal{N}^{w;\gamma}_{\nu\beta} = \sum_{\beta \in P_k^{(k)}} \mathcal{N}^{\beta}_{\mu\nu} \mathcal{N}^{w;\gamma}_{\beta\alpha}. \tag{3.5.3}$$

The group manifold has an effective "volume" dictated by the level of the affine Lie algebra. For example, the larger the level, the larger the number of representations in the untwisted affine Lie algebra, and the larger the number of corresponding D-branes allowed in the group manifold.

In the $k \to \infty$ limit the affine Lie algebra approaches the finite dimensional Lie algebra, and the number of representations in the theory goes to infinity.

Analysis in the large volume limit is much simpler than for the small volume limit and renormalization group methods are understood in the large volume limit. However it is in the small volume/low k limit that better describes string theory (in a realistic string theory the group manifold is compactified away). The method used was thus to find solutions for D-brane condensation in the large volume limit and then to deform these solutions into the small volume realm. This method could potentially miss RG fixed points that appear only in the small volume limit, however it does provide solutions in the tricky small volume limit.

The case of a stack of M identical D-branes was studied, each of type (α, w). Because this stack of branes are parallel and coincident, they preserve the full chiral current algebra $\hat{\mathfrak{g}}_k$.

When studying the large volume limit it was realised that only the massless modes need to be considered, as the mass of the massive modes will blow up in the small volume limit and decouple from the effective theory. For $k \neq \infty$ the only massless fields are the Kac-Moody currents, thus only perturbations with action:

$$S_{\text{pert}} = \int_{\partial\Sigma} dx A_a J^a(x), \tag{3.5.4}$$

were considered. x is the coordinate on $\partial\Sigma$ (the boundary of the string world-sheet Σ)) and A_a where $a \in \{1, \ldots, \dim G\}$ are $M \times M$ Chan Paton matrices.

The effective action for the Chan-Paton fields A^a was found to be:

$$S_{M\Xi}(A) = \mathrm{Tr}\left(-\frac{1}{4}[A_a, A_b][A^a, A^b] + \frac{i}{3k}f^{abc}A_a[A_b, A_c] + \text{constant}\right). \qquad (3.5.5)$$

The trace is over the space of $M \times M$ Chan Paton matrices. The equation of motion of this effective action is:

$$[A^a, [A_a, A_b] - \frac{i}{3k}f^{abc}A^c] = 0. \qquad (3.5.6)$$

In [5], for branes in the untwisted algebra ($w = I$) it was found through RG methods that for some representation with highest weight σ of $\mathfrak{g}$, where $\dim \sigma = M$, and the $M \times M$ matrices A^a take values in the matrix space of this representation, that this equation of motion implies that the stack of M branes of type (α, I) condenses to a meta stable configuration of branes determined by:

$$M(\alpha, I) \to \sum_{\gamma \in P_k} N^{\gamma}_{\sigma\alpha}(\gamma, I), \qquad (3.5.7)$$

where N are the tensor product coefficients (in the $k \to \infty$ limit the fusion coefficients become the tensor product coefficients). The new configuration of branes is only metastable, and can in turn condense to a lower energy configuration [5].

In the low volume limit (low k) the techniques used to derive eqn (3.5.7) no longer work. However the mathematical problem becomes identical to the Kondo model. The Kondo problem was originally analysed in terms of magnetic impurities in low temperature conductors. The analogy between the brane condensation and system of magnetic impurities in low temperature conductors is that the electrons in the conductor will have an integer number (k) of conductance bands. The electrons fermionic fields have spin current which gives rise to a Kac-Moody current algebra $\hat{\mathfrak{g}}_k$. The analysis for the Kondo effect can be found in [1] and its interpretation for D-brane condensation in [40], and again relies on RG methods. The essential result is that the characters of M D-branes are transformed by RG flow into:

$$M\chi_\mu(q) \to \sum_{\gamma} \mathcal{N}^{\gamma}_{\sigma\mu}\chi_\gamma(q), \qquad (3.5.8)$$

where again σ is the highest weight of a representation with dimension M, the same as for the large volume case.

Now consider the partition function of an open string stretched between a stack of M D-branes of type (α, w) and a D-brane of type (β, w).

$$Z^w_{M\alpha\beta}(q) := MZ^w_{\alpha\beta}(q). \qquad (3.5.9)$$

From eqn (3.5.8) this becomes:

$$\begin{aligned} MZ^w_{\alpha\beta}(q) &\to \sum_{\mu \in P_k^{(+)}} \sum_{\gamma \in P_k^{(+)}} \mathcal{N}^{w;\beta}_{\mu\alpha}\mathcal{N}^{\beta}_{\mu\alpha}\chi_l(q) \\ &= \sum_{\gamma \in P_k^{w(+)}} \mathcal{N}^{w;\gamma}_{\sigma\alpha} Z^w_{\gamma\beta}(q). \end{aligned} \qquad (3.5.10)$$

The second line is derived using eqn (3.5.3). Thus the partition function of an open string stretched between M (α, w) D-branes and a (β, w) D-brane condenses into a sum of partition functions of various open strings stretched between these D-branes (γ, w) and the brane (β, w).

As pictured in Figure 3.9, the D-brane (β, w) is merely a spectator in this process describing the condensation of M D-branes into its decay products, described by:

$$M(\alpha, w) \to \sum_{\gamma \in P_k^{w(+)}} \mathcal{N}_{\sigma\alpha}^{w;\gamma}(\gamma, w). \tag{3.5.11}$$

For $w = I$ and the limit $k \to \infty$ for which $\mathcal{N}_{\sigma\alpha}^{\gamma} \to N_{\sigma\alpha}^{\gamma}$, this formula approaches the large volume condensation formula (3.5.7).

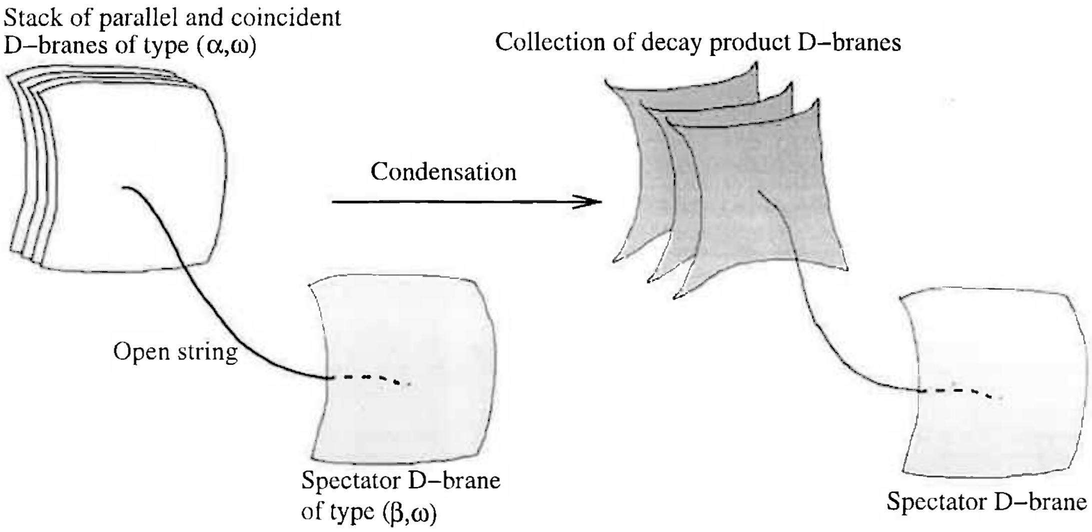


Figure 3.9: *A stack of M D-branes of type (α, w) decay in the presence of a spectator D-brane into a collection of D-branes determined by eqn (3.5.11).*

Charge conservation implies from eqn (3.5.11) that:

$$\dim \sigma \cdot q_{(\alpha,w)} = \sum_{\gamma \in P_k^{w(+)}} \mathcal{N}_{\sigma\alpha}^{w;\gamma} q_{(\gamma,w)}, \tag{3.5.12}$$

where $q_{(\alpha,w)}$ is the R-R charge of the D-brane defined by the boundary condition (α, w).

For the untwisted case, evaluating this formula for $\alpha = 0$ (the fundamental representative) gives:

$$\begin{aligned} \mathcal{N}_{\sigma 0}^{\gamma} &= \delta_{\sigma}^{\gamma}, \\ q_{\beta} &= \dim \beta \cdot q_0, \end{aligned} \tag{3.5.13}$$

where $q_\beta = q_{(\beta,I)}$. Normalising $q_0 = 1$ gives the brane charge $q_\beta = \dim \beta$. Note that this implies $\dim \sigma = q_\sigma$.

This implies the brane charge relation:

$$\dim \sigma \cdot \dim \alpha = \sum_{\gamma \in P_k^{w(+)}} \mathcal{N}_{\sigma\alpha}^{w;\gamma} \dim \gamma. \tag{3.5.14}$$

However this cannot be precisely correct because the relation:

$$\dim \sigma \cdot \dim \alpha = \sum_{\gamma \in P_k} N_{\sigma\alpha}^{w;\gamma} \dim \gamma, \tag{3.5.15}$$

is one of the main properties of the tensor product of representations (where N are the tensor product coefficients), so in reality the charge relation should be of the form:

$$\dim \sigma \cdot \dim \alpha = \sum_{\gamma \in P_k^{w(+)}} \mathcal{N}_{\sigma\alpha}^{w;\gamma} \dim \gamma \mod x, \tag{3.5.16}$$

for some integer x. Thus the charges take values in $\mathbb{Z}_x$. This is the equation that needs to be solved for x to determine the charge group for comparison with a twisted K-group.

3.5.1 Condensation on A_N Manifold

Armed with the relation (3.5.16) the authors of [40] investigated the charge group of $\hat{A}_{N,k}$. The first case studied was that of $\hat{A}_{1,k}$.

From [5, 7–9, 96, 103, 129] the open string $\hat{A}_{1,k}$ WZW theory has $k+1$ D-branes, labelled by the affine Lie algebra highest weight $\hat{\alpha} = (k - ja_1^\vee)\Lambda_0 + j\Lambda_1$, $j \in \{0, \ldots, k\}$. For convenience this is usually replaced by the finite dimensional Lie algebra weight $\alpha = j\Lambda_1$, because the D-brane charge relation (3.5.16) is in terms of the dimensions of the finite dimensional Lie algebra representations of highest weights α. The branes with $\alpha = 0$ & $k\Lambda_1$ are D0-branes and rest are D2-branes, S^2 spheres embedded in the S^3 sphere. It was noted that in [5, 7–9, 96, 103, 129] that condensing m D0-branes of type $\alpha = 0$ gives a D2-brane. Considering that charge must be conserved, and that q_0 has been normalised to 1, the D2-brane is thus $q_{m\Lambda_1} = m+1$. Condensing k D0-brane of type $\alpha = 0$ gives a D0-brane of type $\alpha = k\Lambda_1$, which must have charge $q_{k\Lambda_1} = k+1$. This D0-brane should also have opposite charge to the D0-brane $\alpha = 0$, $q_{k\Lambda_1} = -1$. Therefore to identify charge -1 with $k+1$ the charges must be the integers modulo $k+2$. This is explained further in Figure 3.10.

This behaviour can be re-derived with eqn (3.5.16). In $\hat{A}_{1,k}$ the irreducible representation of highest weight $k\Lambda_1$ is a simple current, and weight Λ_1 is the fundamental representation.

Letting $\sigma = k\Lambda_1$ and $\alpha = \Lambda_1$ in eqn (3.5.16) gives:

$$(k+1) \cdot 2 = \dim k \cdot 2 = q_{k-1} = k \mod x, \tag{3.5.17}$$

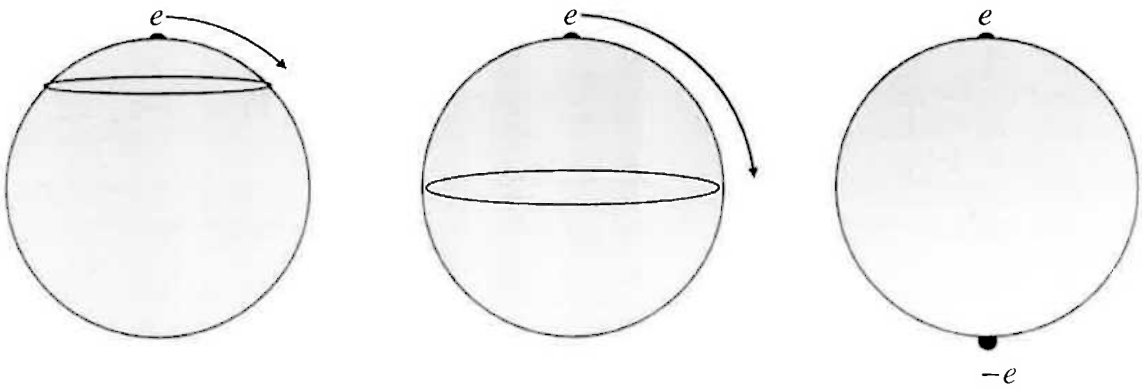


Figure 3.10: *M D0-branes at e condense into a D2-brane in the group manifold with charge $M \times q_0$. The greater the charge, the further the D2-brane moves along the group manifold, until at a critical charge, it becomes a D0-brane at $-e$, the anti D0-brane of a D0-branes at e [40].*

where the fusion rule can be calculated to be $\mathcal{N}^{\gamma}_{k\Lambda_1,\Lambda_1} = \delta_{\gamma,(k-1)\Lambda_1}$. The maximum value of x that this constraint is true for is $k+2$. This value of x can be shown to be consistent for all representations σ and α. Thus:

$$\dim\sigma \cdot \dim\alpha = \sum_{\beta \in P_k^{(+)}} \mathcal{N}^{\beta}_{\sigma\alpha} \dim\beta \mod (k+2). \tag{3.5.18}$$

For the general $\hat{A}_{N,k}$ the general set of constraints on x are given by:

$$\dim\sigma \cdot \dim\alpha = \sum_{\beta \in P_k^{(+)}} \mathcal{N}^{\beta}_{\sigma\alpha} \dim\beta \mod x, \tag{3.5.19}$$

for all $\sigma,\ \alpha \in P_+^{(k)}$. Using simple current arguments, it can be shown that this set of constraints can be reduced to the constraints produced by condensing a set of Λ_i D-branes of number M, where $M = \dim J$, (J is the simple current $J = k\Lambda_1$ of $\hat{A}_{N,k}$) and Λ_i $i \in \{1, \ldots, N\}$ are the highest weights of the fundamental representations [40]:

$$\dim J \cdot q_{w_i} = \sum_{\beta \in P_k^{(+)}} \mathcal{N}^{\beta}_{Jw_i} \mod x. \tag{3.5.20}$$

Thus x is given by the greatest common divisor (gcd) of the values a_i:

$$\begin{aligned} a_i &= \dim J \cdot \dim\Lambda_i - \sum_{\beta \in P_k^{(+)}} \mathcal{N}^{\beta}_{J,\Lambda_i} \dim\beta \\ &= \frac{(k+1)\ldots(k+i-1)(k+i+1)\ldots(k+N+1)}{(i-1)!(N-1+1)!}. \end{aligned} \tag{3.5.21}$$

From this, x was found to be:

$$x = \frac{k+N+1}{\gcd(k+N+1, \mathrm{lcm}(1, \ldots, N))}. \tag{3.5.22}$$

As will be discussed in the research in the next section, it is only in the simplicity of the cases of the $\hat{A}_{N,k}$ and $\hat{C}_{N,k}$ algebras that we were able to solve the constraints provided by eqn (3.5.16) analytically for x away from low k, in a concise form such as in eqn (3.5.22), (however a more complicated solution is possible, see eqn (3.9.5)). As will be discussed, this is due to the simplicity of the fusion ideals in these cases.

The analysis of [40] goes on to find some constraints on the more complicated case of the twisted $\hat{A}_{N,k}$ algebra. The extra complications meant that x could not be fully determined.

The total charge group for adding D-brane charges needs to take into account that the D-branes may belong to twisted or untwisted D-branes, which have differing values of x. Also, for twisted algebras, there are in general multiple valid options for assigning charge to a D-brane, other than the option $q_{(\alpha,w)} = \dim(\alpha, w)$. Thus the charge group must also consider these possibilities, for each of which there will be a separate unknown charge group $\mathbb{Z}_x$.

The total charge group of the charges on group manifold G, with affine level k is thus of the form [40]:

$$C(\hat{G}_k) = \mathbb{Z}_x \oplus \bigoplus_{\substack{\nu \in \{1,\dots,s\} \\ w \in \mathrm{Out}(\mathfrak{g})}} \mathbb{Z}_{x^w_\nu}\ , \tag{3.5.23}$$

where $\nu \in \{1,\dots,s\}$ runs over each possible charge assigning system of each twisting w and $\mathbb{Z}_x$ is the untwisted charge group. This is the torsion components of the K-group.

Thus for $\hat{A}_{N,k}$ the charge group is:

$$C(\hat{A}_{N,k}) = \mathbb{Z}_x \oplus \bigoplus_{\nu \in \{1,\dots,s\}} \mathbb{Z}_{x^w_\nu}\ , \tag{3.5.24}$$

where x is given by eqn (3.5.22) and $w \in \mathrm{Out}(\mathfrak{g})$:

$$w(\lambda_1,\dots,\lambda_N) = (\lambda_N,\dots,\lambda_1). \tag{3.5.25}$$

Some calculations have been done to find the charge group predicted by twisted K-theory for $\hat{A}_{1,k}$ and $\hat{A}_{2,k}$ [111]. The results are that:

$$K^*(A_1,(k+2)[H_0]) = \mathbb{Z}_{(k+2)}, \tag{3.5.26}$$

which agrees with the results of eqn (3.5.22) (there is no twisting for $\hat{A}_{1,k}$) and:

$$K^*(A_2,(k+3)[H_0]) = \mathbb{Z}_r \oplus \mathbb{Z}_r, \tag{3.5.27}$$

where r is known to be a divisor of $k+3$, which again agrees with the results of [40], although is not as precise as these results.

It is believed that the techniques used in [111] to analyse the charge groups predicted by twisted K-theory for $\hat{A}_{1,k}$ and $\hat{A}_{2,k}$ should generalise to other groups. If this is done, comparison with the results of [40], and with the results of the research of this chapter, can be done, providing a strong test to the conjecture that the D-brane charge group is a twisted K-group on a group manifold.

3.6 Statement of the Problem, Finding D-Brane Charge Groups

It is well known that D-branes contain a charge, which is conserved over D-brane interactions [105]. It was originally believed that the D-brane charge group was of the most simple form possible to fit the model, that of a de Rham cohomology.

More recently it has come to be understood that the D-brane charge conservation relations are more complicated than originally thought, owing to the discovery of unstable D-branes, stable non-BPS branes and D-brane condensation [118] (see §3.3).

Such behaviour is not consistent with the D-brane charge group being a de Rham cohomology §3.3. However the charge group being a K-group is consistent with all knowledge of D-brane behaviour up to date.

Nevertheless, the evidence mentioned in §3.3 is not conclusive, thus further checks of this are necessary.

In order to conduct just such a check, Fredenhagen and Schomerus [40] (§3.5 & §3.5.1) analysed the charge group structure for WZW D-branes on a group manifold. CFT on the group manifold has the advantages of both being tractable due to the great amount of symmetry in the theory, and also in these cases K-theory and de Rham cohomology give different predictions of the charge group due to the presence of torsion components of these groups on a group manifold [21, 40].

Fredenhagen and Schomerus did this by studying the condensation of D-branes on the manifold and deriving a relation that relates the dimensions of the representations of a stack of D-branes prior to condensation to the dimensions of the resultant D-branes after condensation (discussed in §3.5). This relation only holds true modulo some positive integer x, and thus the dimension of the representation of a D-brane is interpreted to be the brane charge, and charges are said to take values in $\mathbb{Z}_x$.

When $\mathbb{Z}_x$ for D-branes in the twisted and untwisted affine Lie algebra sectors are known, the charge group of the group manifold can be written as (see §3.5.1):

$$C(G, k + g^\vee) = \mathbb{Z}_x \oplus \bigoplus_{\nu=1}^{s} \mathbb{Z}_{x_\nu} \,, \tag{3.6.1}$$

where G is the Lie group manifold, k is the level of the affine Lie algebra, $g^\vee$ is the dual Coxeter number, $\mathbb{Z}_x$ is the charge group for charge belonging to the untwisted affine Lie algebra sector and $\bigoplus_{\nu=1}^{s} \mathbb{Z}_{x_\nu}$ sums over the twisted sectors.

It is this charge group that can be compared to the predicted charge group of K-theory. However at this stage little is known about the values of x K-theory demands, so a direct comparison is not yet possible.

The relation that Fredenhagen and Schomerus found relating the dimensions of the representation of D-branes before and after a stack of M D-branes condense is eqn (3.5.16). Using this formula, Fredenhagen and Schomerus found x for untwisted $\widehat{A}_N$ and put some constraints on the form of x for twisted $\widehat{A}_N$ (see §3.5.1).

They found that the charge group for untwisted $\widehat{A}_N$ is determined by:

$$x = \frac{k+N+1}{\gcd(k+N+1, \mathrm{lcm}(1, \dots, N))}, \tag{3.6.2}$$

for level k and rank N.

The defining problem investigated in this research was to find the values of x defining the charge groups for all the untwisted affine Lie algebras of compact, simple, connected, simply connected Lie groups, either through solving equation (3.5.16) or through any other means possible.

As will be shown, in the process the case of $\widehat{A}_N$ was solved again using alternative methods. It was first done by expounding and clarifying the proof in [40], then proved again by using the knowledge from [52, 53] that the greatest common divisor (gcd) of the ideal of the fusion ring gives x, and lastly (and in a closely related way) by using fusion potentials. This was then generalised to $\widehat{C}_N$.

For the other affine Lie algebras, their non unit comarks prevented such methods from easily generalising. Instead, these were first solved on a case by case basis for particular ranks and levels of each affine Lie algebra, and this data was used to conjecture the correct form of the formulas for x for each affine Lie algebra for general level and rank (eqn (3.8.4)).

Using fusion ideal arguments a precise though unwieldy formula for x is derived for all the untwisted affine Lie algebras of Lie groups G (see eqn (3.9.5) and eqn (3.9.6)). This is shown numerically to be equivalent to the more concise conjectured formula given by eqn (3.8.4).

The symmetries of the charge lattice are then studied, and using this, a more elegant possible formula for x is derived.

3.7 Initial Investigations into Using Fusion Rules to Find x

Our initial approach to finding values of x for various affine Lie algebras was to explicitly list the fusion rules of the Lie algebra:

$$\phi_\lambda \otimes_f \phi_\gamma = \sum_{\delta \in P_+^{(k)}} \mathcal{N}_{\lambda\gamma}^{\delta} \phi_\delta, \tag{3.7.1}$$

(where ϕ_λ is the primary field corresponding to the representation with weight λ) and then to replace the fields in the fusion rules with the finite dimensions of the representations, as according to eqn (3.5.16). This thus provides a set of constraints that can be used to find x. Due to its level by level and rank by rank nature, this method is obviously not a method that can be used to analytically find x for all levels and ranks for the affine Lie algebra, but it is a good starting point.

The program *Kac* [115], was the main source of fusion rules. It generates fusion rules using the Kac-Walton formula, see §3.2.5.

As an example, let us study the first two levels of $\hat{C}_{N,k}$. For $\hat{C}_{2,1}$, the fusion rules produced by the program *Kac* are:

$$\begin{aligned} (0,0) \otimes_f (0,0) &= (0,0), \\ (0,0) \otimes_f (0,1) &= (0,1), \\ (0,0) \otimes_f (1,0) &= (1,0), \\ (0,1) \otimes_f (0,1) &= (0,0), \\ (0,1) \otimes_f (1,0) &= (1,0), \\ (1,0) \otimes_f (1,0) &= (0,0) \oplus (0,1), \end{aligned} \tag{3.7.2}$$

where representations are represented by their weights, in terms of Dynkin labels (λ_1, λ_2) and the affine Dynkin label is not shown.

The dimensions of the representations $(0,0)$, $(0,1)$ and $(1,0)$ are respectively: **1**, **5** and **4** (in this subsection most dimensions were found using the software *Kac*, but they can be found easily by hand, see eqn (G.1.14)). Inserting the fusion coefficients determined by (3.7.2) and these dimensions into eqn (3.5.16) yields the constraints:

$$\begin{aligned} 1 &= 1 \bmod x, \\ 1 \times 5 &= 5 \bmod x, \\ 1 \times 4 &= 4 \bmod x, \\ 5 \times 5 &= 1 \bmod x, \\ 5 \times 4 &= 4 \bmod x, \\ 4 \times 4 &= 1 + 6 \bmod x. \end{aligned} \tag{3.7.3}$$

From these constraints it is easy to see that for $\hat{C}_{2,1}$, x must equal 1 or 2. It is also easy to see that the first three constraints of (3.7.3), which arise from fusions involving the vacuum representation $(0,0)$ in the product, are trivial and can be ignored.

It is an important question to decide if the charge group is indeed defined $\mathbb{Z}_1$ or $\mathbb{Z}_2$. In general, the constraints (3.5.16) will be satisfied by a maximum x, and also all divisors of x. In this analysis, the maximum x is always found, in order to not lose any generality (its divisors can always be determined later at will) and it may eventuate that when data for x values are known for K-theory (such analysis is yet to be done) that divisors of the x's found here may be of more crucial importance than the maximum x's.

$\hat{C}_{2,2}$ has fusion rules:

$$\begin{aligned}
(0,2) \otimes_f (0,2) &= (0,0), \\
(0,2) \otimes_f (0,1) &= (0,1), \\
(0,2) \otimes_f (1,0) &= (1,1), \\
(0,2) \otimes_f (1,1) &= (1,0), \\
(0,2) \otimes_f (2,0) &= (2,0), \\
(0,1) \otimes_f (0,1) &= (0,0) \oplus (0,2) \oplus (2,0), \\
(0,1) \otimes_f (1,0) &= (1,0) \oplus (1,1), \\
(0,1) \otimes_f (1,1) &= (1,0) \oplus (1,1) \\
(0,1) \otimes_f (2,0) &= (0,1) \oplus (2,0), \\
(1,0) \otimes_f (1,0) &= (0,0) \oplus (0,1) \oplus (2,0), \\
(1,0) \otimes_f (1,1) &= (0,2) \oplus (0,1) \oplus (2,0), \\
(1,0) \otimes_f (2,0) &= (1,0) \oplus (1,1), \\
(1,1) \otimes_f (1,1) &= (0,0) \oplus (0,1) \oplus (2,0), \\
(1,1) \otimes_f (2,0) &= (1,0) \oplus (1,1), \\
(2,0) \otimes_f (2,0) &= (0,0) \oplus (0,2) \oplus (0,1),
\end{aligned} \tag{3.7.4}$$

where the trivial fusions involving the vacuum in the product have been dropped. The dimensions of the representations $(0,0)$, $(0,2)$, $(0,1)$, $(1,0)$, $(1,1)$ and $(2,0)$ are respectively: **1**, **14**, **5**, **4**, **16** and **10**. Inserting these dimensions into (3.5.16) yields the constraints:

$$\begin{aligned}
14 \times 14 &= 1 \bmod x, \\
14 \times 5 &= 5 \bmod x, \\
14 \times 4 &= 16 \bmod x, \\
14 \times 16 &= 4 \bmod x, \\
14 \times 10 &= 10 \bmod x, \\
5 \times 5 &= 1 + 14 + 10 \bmod x, \\
5 \times 4 &= 4 + 16 \bmod x, \\
5 \times 16 &= 4 + 16 \bmod x, \\
5 \times 10 &= 5 + 10 \bmod x, \\
4 \times 4 &= 1 + 5 + 10 \bmod x, \\
4 \times 16 &= 14 + 5 + 10 \bmod x, \\
4 \times 10 &= 4 + 16 \bmod x, \\
16 \times 16 &= 1 + 5 + 10 \bmod x, \\
16 \times 10 &= 4 + 16 \bmod x, \\
10 \times 10 &= 1 + 14 + 5 \bmod x.
\end{aligned} \tag{3.7.5}$$

This yields $x_{\text{maximum}} = 5$.

The first few levels for each affine Lie algebra were analysed by hand, however the next step was to analyse many levels and ranks for each affine Lie algebra, in an attempt to see if any easily discerned patterns in x could be found, from which it may be possible to conjecture formulas for x for the various affine Lie algebras and use these conjectured formulas as hints towards finding analytical formulas for all ranks and levels.

The approach described above is effective, but quite slow by hand, and very easily automated, so a program was written (in the language Python) to take the fusion rule and representation dimension output of the program *Kac* and analyse it by inserting the dimensions into the fusion rules and finding a series of constraints of the form: $0 = a_{\alpha\beta} \bmod x$ ($a_{\alpha\beta} = \dim \alpha \cdot \dim \beta - \sum_{\gamma \in P_+^{(k)}} \mathcal{N}_{\alpha\beta}^{\gamma} \dim \gamma$) and find x via:

$$x = gcd\{a_{\alpha\beta} \mid \alpha \ \& \ \beta \in P_+^{(k)}\}. \tag{3.7.6}$$

The results from of this search for x for the various affine Lie algebras can be found in Appendix H.

While calculating these values of x, the fact that the finite dimensions and fusion rules are symmetric under the action of the automorphisms of the finite dimensional Lie algebra Dynkin diagrams $\mathrm{Out}(\mathfrak{g})$, can be used to simplify the fusion rules, and to reduce considerably the number of constraints from fusion rules that need to be calculated (as the constraints of form (3.5.16) are symmetric under the action of $\mathrm{Out}(\mathfrak{g})$).

3.8 Conjectured x Formulas

The next step was to analyse the data from Appendix H in an attempt to see some logic in the pattern of x, which would then give important clues as to how to analytically find x for all rank and level.

The necessity of finding such a clue first is that for any case except the $\hat{A}_{N,k}$ affine Lie algebras, the analysis involved in finding a general formula for x is made quite difficult due to the more complicated form of the fusion rules and dimension formulas.

It was very enlightening to compare the data to the formula for determining x for $\hat{A}_{N,k}$ [40]. This formula is:

$$x = \frac{k+N+1}{\gcd(k+N+1, \mathrm{lcm}(1, \dots, N))}. \tag{3.8.1}$$

We noticed that $k+N+1$ in this formula can be replaced by $k+g^\vee$, where $g^\vee = N+1$ is the dual Coxeter number of the Lie algebra A_N, and is determined by the sum of the comarks: $g^\vee = \sum_{i=1}^{r} a_i^\vee + 1$.

When comparing this to the data in Appendix H, it was noted that except for some cases at low k, all the x's were divisors of $k+g^\vee$ for their respective affine Lie algebras, and thus it seemed very likely that the general formula for x looked very similar to eqn (3.8.1).

Through studies into finding what it was appropriate to divide $k+g^\vee$ by in the general x formulas, it was realised that the numbers that appear in lcm in eqn (3.8.1) can be rewritten in two ways:

$$\begin{aligned} \mathrm{lcm}(1, \dots, N) &= \mathrm{lcm}(1, \dots, g^\vee - 1) && (3.8.2) \\ &= \mathrm{lcm}(\text{exponents of } A_N) && (3.8.3) \end{aligned}$$

where the exponents of A_N are: $\{1, 2, \dots, N\}$.

It was thus realised that the x data found in Appendix H can be given by the following formulas:

$$x = \frac{k+g^\vee}{\gcd(k+g^\vee, y_N)}, \tag{3.8.4}$$

with y_N given in Table 3.1. The A_N result was derived in [40]. The form of x as $k+g^\vee$ divided by a divisor of $k+g^\vee$ ensures that x is an integer.

The y_N's can be re-expressed in terms of the dual Coxeter numbers and the exponents, as seen in Table 3.2.

The formula (3.8.4) along with the values of y_N given in table 3.2 is a conjecture. Before the analysis of the next few sections, it is merely the best guess of a formula that reproduces the data in Appendix H, using eqn (3.8.1) as a basis of the educated guess, and the properties of the dual Coxeter number and the exponents of algebra to put the formula for x in a somewhat algebra independent form. There are other formulas that would reproduce the finite amount of data in Appendix H.

$\mathfrak{g}$	y_N
A_N	$\mathrm{lcm}(1,2,\ldots,N)$
B_N	$\mathrm{lcm}(1,2,\ldots,2N-1)$
C_N	$\mathrm{lcm}(1,2,\ldots,N,1,3,5,\ldots,2N-1)$
D_N	$\mathrm{lcm}(1,2,\ldots,2N-3)$
E_6	$\mathrm{lcm}(1,2,\ldots,11)$
E_7	$\mathrm{lcm}(1,2,\ldots,17)$
E_8	$\mathrm{lcm}(1,2,\ldots,29)$
F_4	$\mathrm{lcm}(1,2,\ldots,11)$
G_2	$\mathrm{lcm}(1,2,\ldots,5)$

Table 3.1: *Values of y_N for Various Algebras*

$\mathfrak{g}$	$g^\vee$	exponents	y_N
A_N	$N+1$	{1,2, ... ,N}	$\mathrm{lcm}(1,2,\ldots,g^\vee-1,exponents)$
B_N	$2N-1$	{1,3, ... ,2N-1}	$\mathrm{lcm}(1,2,\ldots,g^\vee-1,exponents)$
C_N	$N+1$	{1,3, ... ,2N-1}	$\mathrm{lcm}(1,2,\ldots,g^\vee-1,exponents)$
D_N	$2N-2$	{1,3, ... ,2N-3,N-1}	$\mathrm{lcm}(1,2,\ldots,g^\vee-1,exponents)$
E_6	12	{1,4,5,7,8,11}	$\mathrm{lcm}(1,2,\ldots,g^\vee-1,exponents)$
E_7	18	{1,5,7,9,11,13,17}	$\mathrm{lcm}(1,2,\ldots,g^\vee-1,exponents)$
E_8	30	{1,7,11,13,17,19,23,29}	$\mathrm{lcm}(1,2,\ldots,g^\vee-1,exponents)$
F_4	9	{1,5,7,11}	$\mathrm{lcm}(1,2,\ldots,g^\vee,exponents)$
G_2	4	{1,5}	$\mathrm{lcm}(1,2,\ldots,g^\vee,exponents)$

Table 3.2: *y_N in Terms of Exponents*

The actual constraints put on y_N by the data in Appendix H are contained in Table 3.3. When constructing y_N values from the x data of Appendix H it was realised that the constraints imposed on y_N for some values of x for low levels of k, usually $k=1$, contradicted the constraints imposed by higher level k. Either that of the x value was larger than $k+g^\vee$ and thus not allowed by eqn (3.8.4). Thus when constructing the constraints on y_N, if a low level constraint contradicted the high level constraints, then the lower level constraints were ignored. Obviously the resulting conjectured formula for x (eqn (3.8.4)) along with the corresponding values of y_N would not apply to these skipped levels.

By studying the exceptional levels which do not have x values that can be given by eqn (3.8.4) along with the corresponding values of y_N, it can be seen that all the exceptions occur for low k, and indeed only occur when k is less than the highest comark of the algebra. Thus $\hat{A}_{N,k}$ and $\hat{C}_{N,k}$ which both have all comarks equal to 1

$\hat{\mathfrak{g}}_N$	Levels Cut	y_N Constraints
$\hat{B}_2$	-	$2^1 3^1 5^0 7^0 11^0 13^0 17^0 \ldots$
$\hat{B}_3$	1	$2^2 3^1 5^{1+} 7^0 11^0 13^0 \ldots$
$\hat{B}_4$	-	$2^2 3^1 5^{1+} 7^{1+} 11^0 13^0 \ldots$
$\hat{B}_5$	1	$2^2 3^{2+} 5^{1+} 7^{1+} 11^0 13^0 \ldots$
$\hat{B}_6$	1	$2^3 3^{2+} 5^{1+} 7^{1+} \ldots$
$\hat{B}_7$	-	$2^3 \ldots$
$\hat{C}_2$	-	$2^1 3^1 5^0 7^0 11^0 13^0 17^0 \ldots$
$\hat{C}_3$	-	$2^1 3^1 5^{1+} 7^0 11^0 \ldots$
$\hat{C}_4$	-	$2^2 3^1 5^{1+} 7^{1+} 11^0 \ldots$
$\hat{C}_5$	-	$2^2 3^2 5^{1+} 7^{1+} 11^0 \ldots$
$\hat{C}_6$	-	$2^2 3^2 5^{1+} 7^{0+} 11^{1+} \ldots$
$\hat{D}_4$	-	$2^2 3^1 5^{1+} 7^{0+} 11^0 13^0 \ldots$
$\hat{D}_5$	-	$2^{2+} 3^1 5^{1+} 7^{1+} 11^0 13^0 \ldots$
$\hat{D}_6$	-	$2^{2+} 3^{1+} 5^{1+} 7^{1+} 11^0 13^0 \ldots$
$\hat{D}_7$	-	$2^3 3^{2+} 5^{1+} 7^{1+} 11^{0+} 13^0 17^0 \ldots$
$\hat{D}_8$	1	$2^3 3^{2+} 5^{0+} 7^{0+} 11^{0+} 13^{0+} 17^0 \ldots$
$\hat{E}_6$	1	$2^3 3^{2+} 5^{1+} 7^{1+} 11^{0+} 13^{0+} 17^0 \ldots$
$\hat{E}_7$	1	$2^{3+} 3^{2+} 5^{1+} 7^{1+} 11^{1+} 13^{0+} 17^{0+} 19^{0+} 23^0 \ldots$
$\hat{E}_8$	1, $2^\dagger$	$2^{2+} 3^{2+} 5^{1+} 7^{1+} 11^{1+} 13^{0+} 17^{1+} \ldots$
$\hat{F}_4$	1	$2^3 3^{2+} 5^{1+} 7^{1+} 11^{1+} 13^0 \ldots$
$\hat{G}_2$	1	$2^2 3^1 5^{1+} 7^0 11^0 13^0 \ldots$

Table 3.3: *Table of y_N constraints. The "Levels Cut" column lists which levels of the affine Lie algebra did not have their x value considered as a constraint on y_N, due to contradictions with higher level constraints, or because the x value was larger than $k + g^\vee$, and thus could not come from eqn (3.8.4). The values of y_N are expressed as products of powers of prime factors. A factor of p^{n+} for p a prime and n an integer should be interpreted as a constraint on the power of the factor p^m in y_N such that $m \geq n$. Above the highest prime factor that the data in Appendix H gives constraints for, the product series is denoted by dots. † Level 2 is omitted to allow y_N to follow the pattern in Table 3.2 rather than because it contradicts higher level constraints. It is believed though that if x for $\hat{E}_{8,18}$ could be calculated (a huge numerical task), the condition would contradict that of $\hat{E}_{8,2}$.*

have no exceptions.

Thus it can be speculated at this stage that while eqn (3.8.4) can predict x very well, it for some reason does not apply when k is less than the highest comark of the algebra. In §3.10 this will indeed be shown to be the case, from analysing the method of finding x through using generators of the fusion ideal, studied in §3.9.

Exact formulas for x for cases where $k < \max\{a_1^\vee, \ldots, a_N^\vee\}$ shall be derived analytically in §3.11.

Before moving to the next section, it is worth noting that the regularity of the formulas for x given by eqn (3.8.4) and table 3.2 is very suggestive of a unified derivation to find x for all the untwisted affine Lie algebras of Lie groups G, except perhaps $\hat{F}_4$ and $\hat{G}_2$. At first it is slightly perplexing why the formulas for F_4 and G_2 follow a different pattern, however as will be seen in §3.18.2 the formulas for y_N can all be rewritten into a unified form, with a slight adjustment made to y_N for the $\hat{C}_{N,k}$ case, justified by charge group symmetry arguments in §3.18.2.

3.9 x from Fusion Ideals

In this section deriving x from the fusion ring shall be discussed.

In §3.2.6 fusion rings were discussed for WZW models on a Lie group G, with an untwisted affine Lie algebra of level k. It was concluded that the fusion ring for such a WZW model is eqn (3.2.86), and that one generating set of the ideal is the characters at level $(\lambda, \theta) = k + 1$.

Now it will be discussed how this data is used to generate x.

The main set of constraints we need to solve to determine x is eqn (3.5.16). To reiterate, for the untwisted algebra at level k:

$$\dim\lambda \dim\mu = \sum_{\gamma \in P_+^{(k)}} \mathcal{N}_{\lambda\mu}^{\gamma} \dim\gamma \mod x. \tag{3.9.1}$$

From eqn (3.9.1), applying the dimension operation $\dim : \chi \to \mathbb{Z}$ to the characters in the fusion rules, which is equivalent to applying the dimension operation to the characters of the fusion ring (3.2.86) would yield:

$$\dim\lambda \dim\mu - \sum_{\gamma \in P_+^{(k)}} \mathcal{N}_{\lambda\mu}^{\gamma} \dim\gamma \in \mathbb{Z}_x \,. \tag{3.9.2}$$

If this same operation was applied to the tensor product polynomials, the result would be:

$$\dim\lambda \dim\mu - \sum_{\gamma} N_{\lambda\mu}^{\gamma} \dim\gamma = 0, \tag{3.9.3}$$

which is satisfied without any need to mod out by x. Indeed this relation is strongly linked to how the tensor product coefficients are derived. Thus the dimension operation applied to the characters of the tensor product polynomial ring yields:

$$\dim : \mathbb{Z}[\chi_1, \ldots, \chi_N] \to \mathbb{Z}.$$

Therefore it is easy to see that a necessary and sufficient condition of:

$$\dim : \mathcal{F}_k = \mathbb{Z}[\chi_1, \dots , \chi_N]/\mathcal{I}_k \to \mathbb{Z}_x = \mathbb{Z}/x\mathbb{Z}\ , \tag{3.9.4}$$

is that the dimension operation applied to all the character polynomials generating the ideal $\mathcal{I}_k$ gives a set of integers that are elements of $x\mathbb{Z}$, and not only that, but that the gcd of these numbers gives the same x as the maximum allowed x of eqn (3.9.1).

Thus a method of generating x is:

$$x = \gcd\{\dim : p(\chi_i) \mid p(\chi_i) \in \mathcal{I}_k\}, \tag{3.9.5}$$

where $p(\chi_i)$ are the character polynomials generating the ideal $\mathcal{I}_k$.

Using the generators of eqn (3.2.89) gives the formula for x:

$$x = \gcd\{\dim\lambda \mid (\lambda, \theta) = k + 1\}\,. \tag{3.9.6}$$

These formulas for x grow very complicated as the level and rank of the algebra is increased, as the number of characters in the ideal, and the complexity of the dimensions of their corresponding representations increases greatly with level and rank. As such it is advantageous to show eqn (3.9.5)/eqn (3.9.6) are equivalent to eqn (3.8.4).

Eqns (3.9.6) and (3.9.5) are used in §3.12 and §3.14 to derive eqn (3.8.4) for $\hat{A}_{N,k}$ and $\hat{C}_{N,k}$.

In §3.16 it is shown that eqn (3.8.4) is a divisor of eqn (3.9.6).

For the other affine Lie algebras of Lie groups G, an extensive computer analysis was carried out that showed that for each algebra, level k and rank N, eqn (3.9.6) agrees with eqn (3.8.4).

Before continuing into proving the conjectured formula (3.8.4) for some algebras an intuitive (though incomplete) argument can be used to find reasons as to why eqn (3.9.6) should agree with eqn (3.8.4). Firstly, from eqn (3.9.6) x is determined by a gcd of dimensions of finite dimensional highest weight representations on the maximum level boundary of the affine Lie algebra fundamental chamber (defined by eqn (3.2.89)).

For a representation with highest weight λ, the dimension is given by:

$$\dim\lambda = \prod_{\alpha\in\Delta_+} \frac{(\lambda + \rho|\alpha)}{(\rho|\alpha)}. \tag{3.9.7}$$

When λ is a boundary weight of the affine Lie algebra's fundamental chamber, $(\lambda + \rho|\theta) = k + g^\vee$, and thus each term in the gcd of eqn (3.9.6) will have a $k + g^\vee$ factor, the numerator of the conjectured x formula (3.8.4).

The denominator $(\rho|\alpha)$ of dimension formula also contains the ingredients of the denominator of eqn (3.8.4). For simply laced Lie algebras the factors $(\rho|\alpha)$ run from 1 up to $(\rho|\theta) = g^\vee - 1$ (with some repetitions). These are the numbers whose least common multiple (lcm) gives y_N, the number in the denominator of eqn (3.8.4) whose gcd with $k + g^\vee$ is divided into the numerator $(k + g^\vee)$.

For non-simply laced algebras, the factors $(\rho|\alpha)$ need not be integers. It is easily verified that setting

$$y_\alpha = \begin{cases} (\rho|\alpha) & \text{if } (\rho|\alpha) \in \mathbb{Z}, \\ (\rho|\alpha^\vee) & \text{if } (\rho|\alpha) \notin \mathbb{Z}, \end{cases} \tag{3.9.8}$$

and taking $y = \text{lcm}\{y_\alpha|\alpha \in \Delta_+\}$, reproduces the results for y_N given in Table 3.2.

Thus all the essential ingredients for the conjectured x formula (3.8.4) exist in eqn (3.9.6).

3.10 Why the Conjecture does not Apply for Low k

As mentioned in §3.8 the conjectured formula (3.8.4) does not apply for the level of the algebra lower than the highest comark of the algebra. In this section it shall be explained why.

From the previous section it is understood that x can be found through using the constraints of the ideal $\mathcal{I}_k$. However let us consider again how the x formula is generated. It takes advantage of the constraints imposed on the fusion rules provided by the ideal of the fusion ring $\mathcal{I}_k$. Just as in transforming the fusion rules (3.7.1) into the constraints (3.5.16), the idea is to take the polynomials of representations that form the generators of the ideal, and substitute the finite dimensions of these representations, ie $\chi_\lambda \to \dim\lambda$ in the polynomials. Being generators of an ideal, these polynomials are set to zero, so when the representations have been replaced by their dimensions, they are set to zero modulo x.

Thus x is given by the gcd of these constraints, which are zero modulo x.

It is reasonable to expect that a formula for x for general level k must use the same generating set (level adjusted) for each level. From §3.9 the most obvious and general is generating set (3.2.89).

The ideal is expressed in terms of P_μ, polynomials expressing representations of highest weight μ in terms of the fundamental representations. Each polynomial must vanish individually. Any other generating set for general k will similarly be able to be expressed in terms of the fundamental weights.

However the question arises, what happens when $k < \max\{a_i^\vee|i \in \{0, \dots, N\}\}$ for rank N, when not all the fundamental representations are integrable, ie when some fundamental representations will not lie in the fundamental chamber? Obviously at such levels k, any generating set that expresses representations at level $k+1$ as polynomials of all the fundamental representations should not apply.

For example, consider $\hat{G}_{2,1}$. The integrable representations are Λ_2 with weight $(0, 1)$ and Λ_0, the vacuum. The representations which generate the ideal are Λ_1 with weight $(1, 0)$ and $2\Lambda_2$ with weight $(0, 2)$.

The only non trivial fusion rule present in $\hat{G}_{2,1}$ is:

$$\chi_{\Lambda_2} \times \chi_{\Lambda_2} = \chi_{\Lambda_0} + \chi_{\Lambda_2}, \tag{3.10.1}$$

whereas the corresponding tensor product is:

$$\chi_{\Lambda_2} \times \chi_{\Lambda_2} = \chi_{\Lambda_0} + \chi_{\Lambda_2} + \chi_{2\Lambda_2} + \chi_{\Lambda_1}, \tag{3.10.2}$$

so $\chi_{2\Lambda_2} = \chi_{\Lambda_2}^2 - \chi_{\Lambda_2} - \chi_{\Lambda_0} - \chi_{\Lambda_1}$. Thus the ideal (3.2.89) is:

$$\langle \chi_{\Lambda_1}, \chi_{2\Lambda_2} \rangle = \langle \chi_{\Lambda_1}, \chi_{\Lambda_2}^2 - \chi_{\Lambda_2} - \chi_{\Lambda_0} - \chi_{\Lambda_1} \rangle.$$

A fusion ring with the ideal (3.2.89) would be of form:

$$F = \frac{\mathbb{Z}[\chi_{\Lambda_1}, \chi_{\Lambda_2}]}{\langle \chi_{\Lambda_1}, \chi_{\Lambda_2}^2 - \chi_{\Lambda_2} - \chi_{\Lambda_0} - \chi_{\Lambda_1} \rangle}. \tag{3.10.3}$$

The dimensions $\dim\{2\Lambda_2\} = 27$ and $\dim\{\Lambda_1\} = 14$ can be use to find the predicted value of x.

$$x = \gcd(\dim(2\Lambda_2), \dim(\Lambda_1)) = \gcd(27, 14) = 1.$$

As expected, this differs from the value of x found from direct calculation with the fusion rules, of $x = 41$.

As mentioned earlier, this difference is due to the fact that one of the fundamental representations is non-integrable. Thus when constraining the tensor product rule (3.10.2) to the fusion rule (3.10.1) χ_{Λ_1} cannot be set to zero alone, but only in combination with $\chi_{2\Lambda_2}$. The constraint is thus $\chi_{2\Lambda_2} + \chi_{\Lambda_1} \to 0$ and the fusion ring is:

$$F = \frac{\mathbb{Z}[\chi_{\Lambda_2}]}{\langle \chi_{\Lambda_2}^2 - \chi_{\Lambda_2} - \chi_{\Lambda_0} - \chi_{\Lambda_1} + \chi_{\Lambda_1} \rangle} = \frac{\mathbb{Z}[\chi_{\Lambda_2}]}{\langle \chi_{\Lambda_2}^2 - \chi_{\Lambda_2} - \chi_{\Lambda_0} \rangle}. \tag{3.10.4}$$

From this ideal $x = \gcd(\dim(2\Lambda_2) + \dim(\Lambda_1)) = \gcd(27 + 14) = 41$, the correct answer.

To summarise, choices of generating sets of fusion ideals, such as that use in eqn (3.9.6), which apply for $k \geq \max\{a_1^\vee, \ldots, a_N^\vee\}$ will in general not apply for $k < \max\{a_1^\vee, \ldots, a_N^\vee\}$. Thus a new generating set of the ideal must be applied to eqn (3.9.5) to find x for these cases (if you choose to use this method for such low k - in the next section it is not used for finding x for low k due to the ease of other methods in this case).

It should be obvious from the analysis of this section that while the eqn (3.9.6) using the fixed set of generators for the ideal (eqn (3.2.89)) can not apply for low k, if the greatest comark of the algebra is greater than one, the more general formula eqn (3.9.5) for finding x should always apply, as long as a suitable ideal is chosen for low k.

3.11 x for Low k

In this section x shall be found for $k < \max\{a_1^\vee, \ldots a_N^\vee\}$ for the affine Lie algebras of the Lie groups G.

The comarks of the simple Lie algebras can be found in Appendix (F).

For $\hat{A}_{N,k}$ and $\hat{C}_{N,k}$ all the comarks are equal to 1 and thus no such exceptions to eqn (3.8.4) occur.

For the $\hat{B}_{N,k}$ series of algebras, the comarks are:

$$(a_0^\vee, a_1^\vee, a_2^\vee, \ldots, a_{N-1}^\vee, a_N^\vee) = (1, 1, 2, \ldots, 2, 1). \tag{3.11.1}$$

Thus x may differ from eqn (3.8.4) for $k = 1$.

The representations present for $\hat{B}_{N,1}$ have weights:

$$\Lambda_0 = (0, \ldots, 0), \quad \Lambda_1 = (1, 0, \ldots, 0), \quad \Lambda_N = (0, \ldots, 0, 1), \tag{3.11.2}$$

where the affine Dynkin label has been dropped.

The fusion rules for $\hat{B}_{N,1}$ are (excluding fusions with the vacuum in the product):

$$\begin{aligned} \Lambda_1 \otimes_f \Lambda_1 &= \Lambda_0, \\ \Lambda_1 \otimes_f \Lambda_N &= \Lambda_N, \\ \Lambda_N \otimes_f \Lambda_N &= \Lambda_0 \oplus \Lambda_1. \end{aligned} \tag{3.11.3}$$

The finite component dimensions of the representations, obtained by finding the dimension of the finite dimensional representations obtained by projecting out the imaginary roots from the affine representations, can be found using the dimensional formula (G.1.14).

The dimensions are calculated to be:

$$\mathrm{Dim}(\Lambda_1) = 2N + 1, \tag{3.11.4}$$

$$\mathrm{Dim}(\Lambda_N) = 2^N. \tag{3.11.5}$$

When this information is introduced to eqn (3.5.16), the following constraints are arrived at:

$$0 = 4N^2 + 4N \bmod x, \tag{3.11.6}$$

$$0 = 2^{N+1} N \bmod x, \tag{3.11.7}$$

$$0 = 2(2^{2N-1} - N - 1) \bmod x. \tag{3.11.8}$$

These constraints yield x for $\hat{B}_{N,1}$:

$$x = 2 \gcd(2N(N+1), 2^N N, 2^{2N-1} - N - 1). \tag{3.11.9}$$

An identical analysis yields x for $\hat{D}_{N,1}$:

$$x = \gcd(2N - 1, 2^{2N-3} - N, 2^{2N-2} - 1). \tag{3.11.10}$$

For $\hat{E}_{N,k}$, $\hat{F}_{4,k}$ and $\hat{G}_{2,k}$, there is only a finite number of algebras of level $k < \max\{a_1^\vee, \ldots, a_N^\vee\}$, and all of the x values of all of these cases were found in §3.7 and Appendix H and thus deriving general formulas describing x for these values would be repeating the work of these sections.

The x values for the levels $k < \max\{a_1^\vee, \ldots, a_N^\vee\}$ of $\hat{E}_{N,k}$, $\hat{F}_{4,k}$ and $\hat{G}_{2,k}$ (listed in Appendix H) agreed with eqn (3.8.4) in most cases, and thus only the values of x which differ from eqn (3.8.4) are listed here in Table 3.4. This $x = \infty$ for $\hat{E}_{8,1}$ reflects the charge value is unconstrained and can take any value in $Z_{\geq 0}$. Considering that the only D-brane allowed in this algebra is the one corresponding to the vacuum representation - the conjugacy class passing through the 'origin' of the maximal torus - which has charge equal to 1, having $x = \infty$ does not in this case mean that there are an infinite number of D-brane charges possible. In fact it should be remembered in all cases that the number of different types of D-branes allowed in the WZW model is always restricted to the number of integrable representations in the affine Lie algebra, and that x merely determines a limit on what values the charges of these D-branes can take.

$\mathfrak{g}$	k	x
B_N	1	$2\gcd(2N(N+1), N2^N, 2^{2N-1} - N - 1)$
D_N	1	$\gcd(2N-1, 2^{2N-3} - N, 2^{2N-2} - 1)$
E_6	1	26
E_7	1	3135
E_8	1	∞
	2	4
F_4	1	649
G_2	1	41

Table 3.4: *Exceptions to the Conjectured Formulas for x.*

It should be noted that for $\hat{F}_{4,2}$, $E_{6,2}$, $E_{7,k=2,3}$ and $E_{8,k=3,4,5}$ there is no reason why x should agree with eqns (3.8.4). It is merely coincidence that the less restrictive conditions of the lower levels agree with eqns (3.8.4) for these affine Lie algebras.

3.12 Proving the x Conjecture Using Fusion Ideals for $\hat{A}_{N,k}$

3.12.1 Fredenhagen and Schomerus Ideal

In this section the formula for x (3.8.1) is analytically proven for $\hat{A}_{N,k}$. This has already been done in [40,86] (see section 3.5.1), however as a lead up to further results, it was redone in terms of fusion ideals here.

The analysis in [40] was essentially equivalent to using the following fusion ideal:

$$\{k\Lambda_1 + \Lambda_i | i \in \{1, \ldots, N\}\}. \tag{3.12.1}$$

This ideal was proven using induction, involving the simple currents of $A_{N,k}$.

Therefore, from eqn (3.9.5):

$$x = \gcd\{\dim(k\Lambda_1 + \Lambda_i) | i \in \{1, \ldots, N\}\}. \tag{3.12.2}$$

To show this is equivalent to eqn (3.8.1), the result proved in [86] that:

$$\gcd\left(\begin{pmatrix} X \\ 1 \end{pmatrix}, \begin{pmatrix} X \\ 2 \end{pmatrix}, \ldots, \begin{pmatrix} X \\ N \end{pmatrix}\right) = \frac{X}{\gcd(X, \mathrm{lcm}(1, 2, \ldots, N))}, \tag{3.12.3}$$

shall be required. This can be used to rewrite eqn (3.8.1) as:

$$x = \gcd\left\{\binom{k+N+1}{i} | i \in \{1, \ldots, N\}\right\}, \tag{3.12.4}$$

in which form it is easier to show equivalence with eqn (3.12.2).

Begin by calculating the $\dim(k\Lambda_1 + \Lambda_i)$ using eqn (G.1.14):

$$\begin{aligned} \dim(k\Lambda_1 + \Lambda_i) &= \frac{i}{k+i}\begin{pmatrix} N+1 \\ i \end{pmatrix}\begin{pmatrix} k+N+1 \\ k \end{pmatrix} \\ &= \begin{pmatrix} k+i-1 \\ k \end{pmatrix}\begin{pmatrix} k+N+1 \\ N+1-i \end{pmatrix}. \end{aligned} \tag{3.12.5}$$

From the second form of the dimension, it is obvious from the fact that the dimensions are equal to $\begin{pmatrix} k+N+1 \\ N+1-i \end{pmatrix}$ multiplied by a positive integer that:

$$\begin{aligned} x &= \gcd\{\dim(k\Lambda_1 + \Lambda_i) | i \in \{1, \ldots, N\}\} \\ &\geq \gcd\{\begin{pmatrix} k+N+1 \\ N+1-i \end{pmatrix} | i \in \{1, \ldots, N\}\} \\ &= \gcd\{\begin{pmatrix} k+N+1 \\ i \end{pmatrix} | i \in \{1, \ldots, N\}\}. \end{aligned} \tag{3.12.6}$$

Next it will be shown that:

$$\begin{aligned} x &= \gcd\{\dim(k\Lambda_1 + \Lambda_i) | i \in \{1, \ldots, N\}\} \\ &\leq \gcd\{\begin{pmatrix} k+N+1 \\ i \end{pmatrix} | i \in \{1, \ldots, N\}\}. \end{aligned} \tag{3.12.7}$$

which will prove that x is given by eqn (3.8.1).

The second inequality is true if $\binom{k+N+1}{i}$ can be expressed in terms of a sum of the dimensions where the dimensions have integer coefficients. It indeed can be, and is given by:

$$\begin{pmatrix} k+N+1 \\ j \end{pmatrix} = \sum_{i=1}^{N+1-j} (-1)^{i-1} \begin{pmatrix} N+1-i \\ j \end{pmatrix} \dim(k\Lambda_1 + \Lambda_i). \tag{3.12.8}$$

Below this relation shall be proven. Begin by substituting the first equation in eqn (3.12.5) into eqn (3.12.8) and dividing both sides by the binomial on the LHS.

$$\begin{pmatrix} k+M \\ k \end{pmatrix} \sum_{i=0}^{M} (-1)^{i-1} \frac{i}{k+i} \begin{pmatrix} M \\ i \end{pmatrix} = 1, \tag{3.12.9}$$

where $M = N+1-j$. This has the advantage of concentrating the variables N and j into a single variable, which allows the result to be induced over M instead of N and j separately.

This is re-expressed using $\frac{i}{k+i} = 1 - \frac{i}{k+i}$ and the binomial relation:

$$\sum_{i=0}^{M} (-1)^i \binom{M}{i} = 0, \tag{3.12.10}$$

to give:

$$k\binom{k+M}{k} \sum_{i=0}^{M} (-1)^i \frac{1}{k+i} \binom{M}{i} = 1. \tag{3.12.11}$$

This relation is true for $M = 0$ for $k \in \mathbb{Z}_+$, and thus can be used as the basis of an induction over M, where eqn (3.12.11) is the induction assumption.

Investigate the induction conclusion:

$$k\binom{k+M+1}{k} \sum_{i=0}^{M+1} (-1)^i \frac{1}{k+i} \binom{M+1}{i} = 1. \tag{3.12.12}$$

Use the binomial relation:

$$\binom{M+1}{i} = \binom{M}{i} + \binom{M}{i-1}, \tag{3.12.13}$$

to express eqn (3.12.12) as:

$$\begin{aligned} & k\binom{k+M+1}{k} \left[\sum_{i=0}^{M} (-1)^i \frac{1}{k+i} \left(\binom{M}{i} + \binom{M}{i-1} \right) + \frac{(-1)^{M+1}}{k+M+1} \right] \\ &= k\binom{k+M+1}{k} \left[\sum_{i=0}^{M} \frac{(-1)^i}{k+i} \binom{M}{i} + \sum_{i=0}^{M-1} \frac{(-1)^{i+1}}{k+i+1} \binom{M}{i} + \frac{(-1)^{M+1}}{k+M+1} \right] \\ &= k\binom{k+M+1}{k} \left[\sum_{i=0}^{M} (-1)^i \left(\frac{1}{k+i} - \frac{1}{k+i+1} \right) \binom{M}{i} \right] = 1. \end{aligned} \tag{3.12.14}$$

The induction assumption can be used to simplify the first term, and changing variables for $k' = k+1$ gives:

$$\begin{aligned}
&\frac{k'+M}{M+1} + (k'-1)\binom{k'+M}{k'-1}\sum_{i=0}^{M}\frac{(-1)^{i+1}}{k'+i}\binom{M}{i} \\
&= \frac{k'+M}{M+1} - \frac{(k'-1)k'}{M+1}\binom{k'+M}{k'-1}\sum_{i=0}^{M}\frac{(-1)^{i}}{k'+i}\binom{M}{i} \\
&= \frac{k'+M}{M+1} - \frac{k'-1}{M+1} = 1.
\end{aligned} \tag{3.12.15}$$

Where in the second step the induction assumption was used a second time.

Thus using the ideal (3.12.1) proves x is given by eqn (3.8.1).

3.12.2 Gepner Ideal

The basis used in the previous subsection was a complicated basis for the ideal to use, due to the complicated dimension formula for these representations. A new result is to use the generators of the ideal for $\hat{A}_{N,k}$ derived in [52, 95]. This ideal basis is:

$$\mathcal{I}_k = \{(k+i)\Lambda_1 | i \in \{1, \dots, N\}\}, \tag{3.12.16}$$

for which:

$$\dim((k+i)\Lambda_1) = \binom{k+i+N}{N}. \tag{3.12.17}$$

From eqn (3.9.5) x is given by:

$$x = \gcd\left(\binom{k+N+1}{N}, \binom{k+N+2}{N}, \dots, \binom{N+2N}{N}\right). \tag{3.12.18}$$

The aim is to now show this is equivalent to eqn (3.12.4). This can be done through repeated use of eqn (3.12.13). Begin by replacing the first term by:

$$\binom{k+1+N}{N} = \binom{k+N+2}{N} - \binom{k+N+1}{N-1}.$$

By using the gcd relation $\gcd(a, a \pm b) = \gcd(a, b)$, inserting this into the gcd yields:

$$\begin{aligned}
x &= \gcd\left(\binom{k+N+1}{N}, \binom{k+N+2}{N}, \dots, \binom{k+2N}{N}\right) \\
&= \gcd\left(\binom{k+N+2}{N} - \binom{k+N+1}{N-1}, \binom{k+N+2}{N}, \dots, \binom{k+2N}{N}\right) \\
&= \gcd\left(\binom{k+N+1}{N-1}, \binom{k+N+2}{N}, \dots, \binom{k+2N}{N}\right).
\end{aligned}$$

This can iteratively be used to lower the integers in the bottom of the binomials until the following gcd is reached:

$$x = \gcd\left(\binom{k+N+1}{1}, \binom{k+N+2}{2}, \dots, \binom{k+2N}{N}\right). \tag{3.12.19}$$

To show this is equivalent to eqn (3.12.4), eqn (3.12.13) will again need to be used iteratively, this time to lower the upper arguments in the binomials. For example, substitute:

$$\binom{k+2N}{N} = \binom{k+2N-1}{N} + \binom{k+2N-1}{N-1}$$

into the gcd.

$$\begin{aligned} x &= \gcd\left(\binom{k+N+1}{1}, \ldots, \binom{k+2N-1}{N-1}, \binom{k+2N}{N}\right) \\ &= \gcd\left(\binom{k+N+1}{1}, \ldots, \binom{k+2N-1}{N-1}, \binom{k+2N-1}{N-1} + \binom{k+2N-1}{N}\right) \\ &= \gcd\left(\binom{k+N+1}{1}, \ldots, \binom{k+2N-1}{N-1}, \binom{k+2N-1}{N}\right). \end{aligned}$$

Applying this procedure to all the entries of the gcd (except the first) will, after many iterations, yield:

$$x = \gcd\left(\binom{k+N+1}{1}, \ldots, \binom{k+N+1}{N-1}\right), \tag{3.12.20}$$

which is equal to eqn (3.12.4), and thus also equal to the main result for $\hat{A}_{N,k}$, eqn (3.8.1).

3.13 The Fusion Potential for $\hat{A}_{N,k}$

Fusion potentials were briefly mentioned in section 3.2.6. Here a more in-depth account shall be given of the fusion potential, to show how the fusion potential also determines x, in that it encapsulates the fusion ideal. While this section does not reveal any new knowledge about x, it is useful to investigate as a method that could potentially be generalised to the other affine Lie algebras, providing analytical proof of the x formulas for these algebras (this analysis is done for $\hat{C}_{N,k}$ in §3.15).

The potential, constructed in [52], is expressed in terms of variables q_i. These variables are constructed from the over-complete basis of the weight space of $\hat{A}_{N,k}$, $\{\epsilon_i\}_{i=1}^{N+1}$ ($\sum_{i=1}^{N+1} \epsilon_i = 0$, $\epsilon_i \cdot \epsilon_j = \delta_{ij} - \frac{1}{N+1}$) by $q_i = e^{\epsilon_i}$.

These q_i's can be used to re-express the finite dimensional Lie algebra character formula G.1.10 for the fundamental irreducible representations with highest weight Λ_m, $m \in \{1, \ldots, N\}$ of A_N as:

$$\chi_m = \sum_{1 \le i_1 < \ldots < i_m \le N+1} q_{i_1} \cdots q_{i_m}. \tag{3.13.1}$$

The fusion potential in terms of q_i's is:

$$V(\chi_1, \ldots, \chi_N) = \frac{1}{k+N+1} \sum_{i=1}^{N+1} q_i^{k+N+1}. \tag{3.13.2}$$

In this form, how to derive the fusion constraints by taking the derivative with respect to the fundamental representation characters is not obvious. To see how this is done, first generalise the fusion potential to:

$$V_m = \frac{1}{m} \sum_{i=1}^{N+1} q_i^m. \tag{3.13.3}$$

In this notation, the fusion potential is V_{N+k+1}.

A generating function $V(t)$ can be constructed from the V_m's, which takes an expression in terms of q_i's and constructs one in terms of χ_m's.

$$V(t) = \sum_{m=1}^{\infty} (-1)^{m-1} V_m t^m = \log\left(\sum_{i=0}^{N+1} \chi_i t^i\right). \tag{3.13.4}$$

In this formulation, χ_0 and χ_{N+1}, which are not characters of fundamental representations, are set equal to 1.

An expression for V_{N+k+1} in terms of χ_n's is arrived at by Taylor expanding the right hand formula of eqn (3.13.4) around $t = 0$ and taking the coefficient of t^{N+k+1}.

$$V_{N+k+1} = \frac{(-1)^{N+k+1}}{(N+k+1)!} \frac{d^{N+k+1}}{dt^{N+k+1}} \left(\log(\sum_{i=0}^{N+1} \chi_i t^i) \right) \Bigg|_{t=0}.$$

Considering that the fusion potential is equivalent to the integration of all the constraints in the fusion ideal (where the integration is done with respect to the characters in the constraints), the constraints are given by the derivatives $\frac{\partial V_{N+k+1}}{\partial \chi_j}$. These constraints are turned into constraints on x by inserting the dimensions (equivalent to evaluating the characters at $q_i = 1$), such that:

$$\chi_j = \dim(\Lambda_j) = \binom{N+1}{j}, \tag{3.13.5}$$

$$0 = \frac{\partial V_{N+k+1}}{\partial \chi_j}\Bigg|_{\chi_i=\dim(\Lambda_i)} \mod x. \tag{3.13.6}$$

The values of $\frac{\partial V_{N+k+1}}{\partial \chi_j}\Big|_{\chi_i=\dim(\Lambda_i)}$ are:

$$\begin{aligned}
\frac{\partial V_{N+k+1}}{\partial \chi_j}\Bigg|_{\chi_i=\dim(\Lambda_i)} &= \frac{(-1)^{N+k+1}}{(N+k+1)!} \frac{d^{N+k+1}}{dt^{N+k+1}} \left(\frac{t^j}{\sum_{i=0}^{N+1} \chi_i t^i} \right) \Bigg|_{\substack{\chi_i = \dim(\Lambda_i) \\ t=0}} \\
&= \frac{(-1)^{N+k+1}}{(N+k+1)!} \frac{d^{N+k+1}}{dt^{N+k+1}} \left(\frac{t^j}{(1+t)^{N+1}} \right) \Bigg|_{\substack{\chi_i = \dim(\Lambda_i) \\ t=0}} \\
&= \frac{(-1)^{N+k+1}}{(N+k+1)!} \frac{d^{N+k+1}}{dt^{N+k+1}} \left(\frac{t^j}{N!} \frac{d^N}{dt^N} (1+t)^{-1} (-1)^N \right) \Bigg|_{\substack{\chi_i = \dim(\Lambda_i) \\ t=0}} \\
&= \frac{(-1)^{N+k+1}}{(N+k+1)!N!} \frac{d^{N+k+1}}{dt^{N+k+1}} \left(t^j \frac{d^N}{dt^N} \sum_{r=0}^{\infty} (-t)^r \right) \Bigg|_{\substack{\chi_i = \dim(\Lambda_i) \\ t=0}} \\
&= \binom{k+2N+1-j}{N}.
\end{aligned} \tag{3.13.7}$$

x is given by the gcd of these values of $\left.\frac{\partial V_{N+k+1}}{\partial \chi_j}\right|_{\chi_i=\dim(\Lambda_i)}$:

$$x = \gcd\left\{ \left. \binom{k+2N+1-j}{N} \right| j \in \{1,\dots,N\} \right\} = \gcd\left\{ \left. \binom{k+N+j}{N} \right| j \in \{1,\dots,N\} \right\} \tag{3.13.8}$$

which is equivalent to eqn (3.12.18), thus showing the fusion potential can also be used to derive x.

3.14 Proving the x Conjecture Using Fusion Ideals for $\hat{C}_{N,k}$

The fusion ideal generating set used here to derive the x formula for $\hat{C}_{N,k}$ is that of [18, 53, 95]. The authors used an educated guess to construct this set, and then in [18] found the fusion potential this ideal generating set corresponds to, and showed that this fusion potential has the correct properties expected.

The generating set of the ideal comes from identifying the polynomials in the following set with zero:

$$\{\chi_{(k+1)\Lambda_1}, \chi_{(k+2)\Lambda_1} + \chi_{k\Lambda_1}, \chi_{(k+3)\Lambda_1} + \chi_{(k-1)\Lambda_1}, \dots, \chi_{(k+N)\Lambda_1} + \chi_{(k+2-N)\Lambda_1}\}. \tag{3.14.1}$$

As an indication as to why this set may be good, consider the set $\{\chi_{(k+i)\Lambda_1} \mid I \in \{1,\dots,N\}\}$ under the action of the shifted affine Weyl reflection s_{α_0}. The elements of this set do not lie on the boundary of an affine Lie algebra chamber and thus by themselves should not be identified with zero when evaluated at ξ_σ (see eqn (3.2.68)) to form generators of the fusion ideal. However as explained in §'s 3.2.6, 3.9 and 3.2.5, when evaluated at ξ_σ, the representation and its shifted s_{α_0} reflection partner's (see eqn (3.2.74)) characters together always are identified with zero $\chi_{s_{\alpha_0}\cdot\lambda} - \epsilon(s_{\alpha_0})\chi_\lambda \in \mathcal{I}_k$ where $\epsilon(s_{\alpha_0}) = -1$ is the Weyl reflection coefficient. See Appendix G.2 for more details on affine reflections.

The reflections in the affine direction are:

$$s_{\alpha_0} \cdot (k+i)\Lambda_1 = (k+2-i)\Lambda_1, \tag{3.14.2}$$

thus from this the character summations in eqn (3.14.1) are valid generators of the fusion ideal. This in itself is not a proof that these generators are sufficient to generate the whole ideal. However, as mentioned, it was shown in [18] that this set of constraints is sufficient to form a basis of the ideal.

Using this ideal, the task is thus to prove the x formula for $\hat{C}_{N,k}$:

$$x = \frac{k+N+1}{\gcd(k+N+1, \operatorname{lcm}(1,2,\dots,N,1,3,5,\dots,2N-1))}. \tag{3.14.3}$$

The dimension of an irreducible representation with highest weight $l\Lambda_1$ is:

$$\dim(l\Lambda_1) = \binom{l+2N-1}{2N-1}, \qquad l \in \mathbb{Z}_+ . \tag{3.14.4}$$

This can be used in conjunction with the ideal to yield the x formula:

$$\begin{aligned} x &= \gcd\left(\binom{k+2N}{2N-1}, \binom{k+2N+1}{2N-1} + \binom{k+2N-1}{2N-1}, \right. \\ &\left. \binom{k+2N+2}{2N-1} + \binom{k+2N-2}{2N-1}, \dots, \binom{k+3N-1}{2N-1} + \binom{k+N+1}{2N-1}\right). \end{aligned} \tag{3.14.5}$$

Similarly to the $\hat{A}_{N,k}$ case, repeated use of the binomial identity (3.12.13) can transform this formula into:

$$x = \gcd\left(\binom{k+N+1}{1}, \binom{k+N+2}{3}, \dots, \binom{k+2N-1}{2N-3}, \binom{k+2N}{2N-1}\right). \tag{3.14.6}$$

Returning to eqn (3.12.3), the proof for this, outlined in Appendix C of [86], can be generalised to apply to the above formula, even though the numbers appearing in the lower portion of the binomials are not a sequential series of integers. This can thus be used to show that eqn (3.14.6) is equivalent to:

$$x = \frac{k + g^\vee}{\gcd(k + g^\vee, \mathrm{lcm}(1, 2, \dots, N, 1, 3, 5, \dots, 2N-1))}. \tag{3.14.7}$$

and thus we have proved the conjectured formula (3.8.4) for $\hat{C}_{N,k}$.

The following ideal: $\{k\Lambda_1 + \Lambda_i | i \in \{1, \dots, N\}\}$ can also be used to find the identical result again.

3.15 The Fusion Potential for $\hat{C}_{N,k}$

The same fusion potential method as for $\hat{A}_{N,k}$ can be used for $\hat{C}_{N,k}$, with regard to the fusion potential for $\hat{C}_{N,k}$ of [18, 53], given by:

$$V(\chi_1, \chi_2, \dots, \chi_N) = \frac{1}{k+N+1} \sum_{i=1}^{N} (q_i^{k+N+1} + q_i^{-(k+N+1)}). \tag{3.15.1}$$

As in §3.13, $q = e^{\epsilon_i}$ for the orthonormal basis $\{\epsilon_i\}_{i=1}^{N}$ of the weight space of $\hat{C}_{N,k}$.

The relation of the q_i's to the characters of the fundamental representations Λ_m is given by:

$$\chi_j = E_j - E_{j-2}\ , \quad (E_j = 0 \text{ when } j < 0), \tag{3.15.2}$$

$$\sum_{j=0}^{\infty} E_j t^j = \prod_{i=1}^{N} (1 + q_i t)(1 + q_i^{-1} t). \tag{3.15.3}$$

As a consequence of eqn (3.15.3), $E_j = 0$ if $j > 2N$ as the RHS of eqn (3.15.3) is a polynomial of t of order $2N$. Another consequence of eqn (3.15.3) is that $E_j = E_{2n-j}$.[11] This combined with eqn (3.15.2) means $\chi_j + \chi_{2N+2-j} = 0$. Considering that there are $2N+1$ E_j's, which would imply $2N+3$ independent fundamental representations χ_N, when there should be only N, this relationship reduces the number to $N+1$, which is further reduced to N characters when it is considered that $\chi_0 = E_0 = 1$.

As in §3.13, the fusion potential is generalised, in this case to:

$$V_m = \frac{1}{m}\sum_{i=1}^{N}(q_i^m + q_i^{-m}). \tag{3.15.4}$$

The equivalent of eqn (3.13.4) for $\hat{C}_{N,k}$ is:

$$V(t) = \sum_{m=1}^{\infty}(-1)^{m-1}V_m t^m = \log\left(\sum_{j=0}^{2N} E_j t^j\right), \tag{3.15.5}$$

expressed in terms of E_j's instead of characters.

And again, in analogy to §3.13, the constraints of the fusion ideals are found by finding the coefficients of t^{N+k+1} in a Taylor expansion of the generating function, taking the derivatives of this coefficient V_{k+N+1} with respect to the fundamental characters.

$$\frac{\partial V_{k+N+1}}{\partial \chi_i} = \frac{1}{(k+N+1)!}\frac{\partial^{k+N+1}}{\partial t^{k+N+1}}\left(\frac{t^i + t^{i+2} + \ldots + t^{2N-2-i} + t^{2N-i}}{\sum_{j=0}^{2N} E_j t^j}\right)\Bigg|_{t=0}. \tag{3.15.6}$$

When these constraints are evaluated at $q_i = 1$, the points where the character is replaced by the finite dimensions $\dim(\Lambda_i) = \binom{2N}{i} - \binom{2N}{i-2}$, they become constraints on x.

$$\frac{\partial V_{k+N+1}}{\partial \chi_i}\bigg|_{\chi_j = \dim \Lambda_j} = \sum_{j=0}^{N-i}\binom{k+3N-i-2j}{2N-1}. \tag{3.15.7}$$

Taking the gcd of these constraints gives:

$$\begin{aligned}
x &= \gcd\left\{\sum_{j=0}^{N-i}\binom{k+3N-i-2j}{2N-1} \mid i \in \{1,\ldots,N\}\right\}\\
&= \gcd\left\{\sum_{j=0}^{i-1}\binom{k+2N+1-i+2j}{2N-1} \mid i \in \{1,\ldots,N\}\right\}\\
&= \gcd\left(\binom{k+2N}{2N-1}, \binom{k+2N+1}{2N-1} + \binom{k+2N-1}{2N-1},\right.\\
&\quad \binom{k+2N+2}{2N-1} + \binom{k+2N+1}{2N-1} + \binom{k+2N-1}{2N-1} + \binom{k+2N-2}{2N-1}, \ldots,\\
&\quad \left.\binom{k+3N-1}{2N-1} + \ldots + \binom{k+N+1}{2N-1}\right).
\end{aligned}$$

[11] This is seen as $(1+q_i t)(1+q_i^{-1}t) = (1 + (q_i + q_i^{-1})t + t^2)$, for which the coefficients of t^m are equal to the coefficients of t^{2-m}. As eqn (3.15.3) is constructed from a product of N of these terms, the E_j coefficients should equal E_{2N-j}.

Thus

$$x = \gcd\left(\binom{k+2N}{2N-1}, \binom{k+2N+1}{2N-1} + \binom{k+2N-1}{2N-1}, \right. \tag{3.15.8}$$
$$\left. \binom{k+2N+2}{2N-1} + \binom{k+2N-2}{2N-1}, \dots, \binom{k+3N-1}{2N-1} + \binom{k+N+1}{2N-1}\right).$$

Which is eqn (3.14.5) and thus equivalent to eqn (3.14.3).

3.16 Showing Conjectured x Formula is a Lower Bound for x in $\hat{G}_2$

In [86] it was realised that the relation:

$$\frac{k}{\gcd(k, \mathrm{lcm}(1,2,\dots,N))} \equiv \gcd\left(\binom{k}{1}, \binom{k}{2}, \dots, \binom{k}{N}\right), \tag{3.16.1}$$

proved in Appendix C of [86], can be used to describe the x formula for $\hat{A}_{N,k}$ for suitable values of k and N.

Using this same identity, our conjectured formula for x takes the form (see eqn (3.8.4)):

$$\begin{aligned} x &= \frac{k+g^\vee}{\gcd\left(k+g^\vee, \mathrm{lcm}(a_1,\dots,a_m)\right)} \\ &= \gcd\left(\binom{k+g^\vee}{a_1}, \binom{k+g^\vee}{a_2}, \dots, \binom{k+g^\vee}{a_m}\right), \end{aligned} \tag{3.16.2}$$

where the numbers in the lowest common multiple are given in Table 3.2. This identity works for all the affine Lie algebras except $\hat{C}_{N,k}$, because for $\hat{C}_{N,k}$ y_N is not a lcm of a sequential set of integers, but is the lowest common multiple of $\{1, 2, \dots, N, 1, 3, 5, \dots, 2N-1\}$. This would seem to be a problem, but it can be shown that the proof in [86] can be adjusted for $\hat{C}_{N,k}$ to:

$$\begin{aligned} x &= \frac{k+g^\vee}{\gcd(k+g^\vee, \mathrm{lcm}(1,2,\dots,N,1,3,5,\dots,2N-1))} \\ &= \gcd\left(\binom{k+g^\vee}{1}, \binom{k+g^\vee}{2}, \dots, \binom{k+g^\vee}{N}, \right. \\ &\quad \left. \binom{k+g^\vee}{1}, \binom{k+g^\vee}{3}, \binom{k+g^\vee}{5}, \dots, \binom{k+g^\vee}{2N-1}.\right) \end{aligned} \tag{3.16.3}$$

Next, using the ideal (3.2.89), the exact formula for x can be found by taking the greatest common divisor of the dimensions of all the representations at level $k+1$ to give eqn (3.9.6):

$$x = \gcd\{\dim \lambda | (\lambda|\theta) = k+1\}. \tag{3.16.4}$$

However, this formula for x can be unwieldy.

Considering the similar forms for eqn (3.16.4) and (3.16.2) it was thought that it may be possible to show that the gcd of the dimensions of level $(k+1)$ (3.16.4) (these representations usually form an over complete generating set for the fusion ideal and thus the gcd of the dimensions of these reps gives x [52]) can be shown to be equal to eqn (3.16.2), and thus to the original conjectured formula for x.

The example $\hat{G}_2$ was studied first.

In this case it was shown using eqn (G.1.14) that the dimension of a representation of level $(k+1)$ with weight $(\lambda, k+1-2\lambda)$ is:

$$
\begin{aligned}
D_\lambda &= \sum_{i=1}^{5} a_i \begin{pmatrix} k+4 \\ i \end{pmatrix}, && (3.16.5)\\
a_1 &= -\frac{1}{5}\begin{pmatrix} \lambda+2 \\ 4 \end{pmatrix}, && (3.16.6)\\
a_2 &= \frac{1}{2}\begin{pmatrix} \lambda+1 \\ 3 \end{pmatrix}, && (3.16.7)\\
a_3 &= -\frac{1}{2}(\lambda+1)(3\lambda-2), && (3.16.8)\\
a_4 &= -(\lambda+1)(\lambda-3), && (3.16.9)\\
a_5 &= (2\lambda+2). && (3.16.10)
\end{aligned}
$$

It was realised that any x of the form (3.16.2) which is a gcd of the binomials in (3.16.5) must obviously be a divisor of the whole dimensional formula, and thus must also be a divisor of eqn (3.16.4) (as the coefficients a_i are integers). Thus we have shown that that the conjectured x (eqn (3.16.2)) is at the least a lower bound to the true x.

If the formula relating the binomials and the dimensions could be inverted, and if the coefficients in the formulas expressing the binomials as a sum of dimensions had integer coefficients for the dimensions, it was realised that (3.16.4) must then be a divisor of x of form (3.16.2). This would show that x of form (3.16.2) is equal to the true x of form (3.16.4).

To investigate this, we expressed the transformation from binomials to dimensions as a matrix equation:

$$\begin{pmatrix} D_0 \\ D_1 \\ \cdot \\ \cdot \\ \cdot \\ D_{\lfloor \frac{k+1}{2} \rfloor} \end{pmatrix} = \begin{pmatrix} a_1(\lambda = 0) & \ldots & a_5(\lambda = 0) \\ a_1(\lambda = 1) & \ldots & a_5(\lambda = 1) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_1(\lambda = \lfloor \frac{k+1}{2} \rfloor) & \ldots & a_5(\lambda = \lfloor \frac{k+1}{2} \rfloor) \end{pmatrix} \begin{pmatrix} \binom{k+4}{1} \\ \binom{k+4}{2} \\ \binom{k+4}{3} \\ \binom{k+4}{4} \\ \binom{k+4}{5} \end{pmatrix} . \quad (3.16.11)$$

The matrix is square for $k = 7$ & 8, so the inverse of this matrix was investigated for these cases first. However it was found that in these cases there are no inverses as the determinate of the matrix in (3.16.11) is equal zero.

In the general k case, where there are many left inverse options, we used the inverse

$$L = (A^T A)^{-1} A^T,$$

where A is the transformation matrix from (3.16.11). For no level k checked did an inverse exist.

While we could not confirm our conjectured x formula is correct for $\hat{G}_2$, we did show it is a divisor of the true answer (and of course also a lower bound). We also demonstrated this method of proving (3.16.2) is equivalent to (3.16.4) does not work for $\hat{G}_2$ and is unlikely to work for any affine Lie algebra (except perhaps $\hat{A}_n$ and $\hat{C}_n$).

This analysis has not been done for any case except $\hat{G}_2$ as it is not certain it will generalise well. In the $\hat{G}_2$ case things worked well as the binomial which is the highest order polynomial has order 5 in k, the same order as the level $(k+1)$ representation dimension equations, so the decomposition of the dimensional formula in terms of the binomials produced nice k independent coefficients. However in most cases the dimensional formula will be of much higher order than the binomials and so the coefficients will be k dependent.

3.17 Ideals for the Other Affine Lie Algebras

In §'s 3.12 to 3.14 exact generating sets for the ideals of the fusion rings of $\hat{A}_{N,k}$ and $\hat{C}_{N,k}$ were discussed and from them the x formulas were derived. The generating sets of these ideals are very succinct, summarising the effects of the ideal in as few constraints as possible. This brevity is what allows the formula for x (eqn (3.9.5)) to be shown to be equivalent to the very succinct formula (3.8.4).

In contrast, calculating x from the general generators of the ideal of eqn (3.9.6). which is sufficient to define x for all $k \geq \max\{a_i^\vee | i \in \{0, \dots, N\}\}$, is quite complicated. not only due to the complexity of the dimension formulas for all representations at level $k+1$, but also due to the fact that the number of generators in this ideal grows quickly as a function of k.

Therefore in order to be able to construct x from fusion ideals, simpler generating sets for the algebras $\hat{B}_{N,k}$, $\hat{D}_{N,k}$, $\hat{E}_{N,k}$, $\hat{F}_{4,k}$ and $\hat{G}_{2,k}$ were needed.

While fusion potentials have been formulated for these affine algebras [30], these potentials are not in a formulation that can be expressed in terms of the representations of the affine Lie algebra and thus could not be used to generate constraints for the fusion ideal (in terms of representations).

As a result, new generating sets of the ideals for these algebras were searched for. The method used was to construct hypothetical sets of constraints and to use these constraints to calculate x. Considering that x from any correct generating set should agree with the generating set (3.2.89), the test of a set was to see if it predicted the same x as eqn (3.2.89) for as many levels and ranks as could be reasonably numerically calculated.

When constructing these sets, inspiration was gained through study of the generating sets of the ideals for $\hat{A}_{N,k}$ (3.17.1) and $\hat{C}_{N,k}$ (3.17.3) [53]. These generating sets consist of a subset of the generating set (3.2.89), of all the representations nearest to a corner of the plane defined by $(\lambda|\theta) = k+1$ (for representations with weight λ) in the weight lattice/fusion diagram.

For example, from eqn (3.17.1) consider $\hat{A}_{2,3}$. The generating set is:

$$\{4\Lambda_1, 3\Lambda_1 + \Lambda_2\}.$$

which as can be seen from Figure 3.11 lay in one corner of the boundary to the fundamental chamber.

Often when a generating set was found in this way, further experimentation discovered that some of the constraints were not necessary. In contrast, it was also discovered that for low k sometimes extra conditions were required. Ie for $\hat{E}_{7,2}$ and $\hat{E}_{7,3}$ the extra constraints of setting the dimensions of $(0,0,0,1,0,0,0)$ and $(0,0,0,0,0,0,2)$ to zero mod x are respectively required, and similarly for $\hat{E}_{8,4}$ and $\hat{E}_{8,12}$, adding $(0,1,0,0,0,0,1,0)$ and $(0,2,1,0,0,0,0,1)$ respectively. For $\hat{E}_{8,2}$ setting all the dimensions of representations of level $(\theta,\lambda) = k+1$ to 0 mod x is still insufficient and weights outside the fundamental chamber are required.

When finding these generating sets, a guideline that was always followed was that a constraint in the generating set is equivalent to another constraint in the same set if related by an outer automorphism of the finite dimensional Lie algebra Dynkin diagram.

The results of this search are contained below.

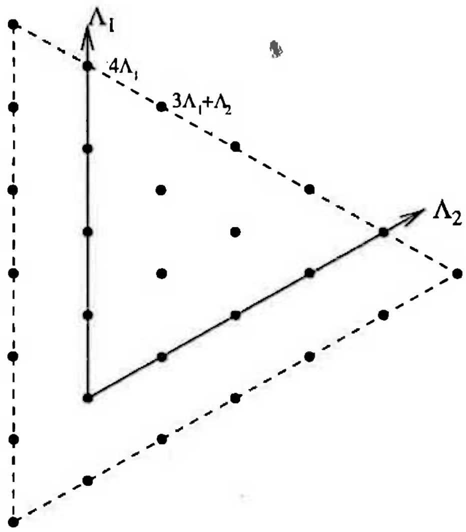


Figure 3.11: *The characters which form generators of the ideal 3.17.1 have highest weights clustered around* $(k+1)\Lambda_1$*, along the* $(\lambda|\theta) = k+1$ *line. The dashed line is the boundary of the affine fundamental chamber.*

- $\hat{\boldsymbol{A}}_{N,k}$

$$\{k\Lambda_1 + \Lambda_i : i = 1, \dots, N\}. \tag{3.17.1}$$

- $\hat{\boldsymbol{B}}_{N,k}$

$$\begin{aligned}&\{(k+1)\Lambda_1, (k-1)\,\Lambda_1 + 2\Lambda_N\}\\ &\cup \{(k-1)\,\Lambda_1 + \Lambda_i, (k-2)\,\Lambda_1 + \Lambda_i + \Lambda_N : i = 2, \dots, N-1\}.\end{aligned} \tag{3.17.2}$$

- $\hat{\boldsymbol{C}}_{N,k}$

$$\{k\Lambda_1 + \Lambda_i : i = 1, \dots, N\}. \tag{3.17.3}$$

- $\hat{\boldsymbol{D}}_{N,k}$

$$\begin{aligned}&\{k\Lambda_1 + \Lambda_N, (k-1)\,\Lambda_1 + 2\Lambda_N, (k-1)\,\Lambda_1 + \Lambda_{N-1} + \Lambda_N,\\ &\qquad (k-3)\,\Lambda_1 + \Lambda_{N-1} + 3\Lambda_N\}\\ &\cup \{(k-1)\,\Lambda_1 + \Lambda_i, (k-3)\,\Lambda_1 + \Lambda_i + 2\Lambda_N : i = 2, \dots, N-2\}\\ &\cup \{(k-3)\,\Lambda_1 + \Lambda_i + \Lambda_j : 2 \leqslant i < j \leqslant N-2\}.\end{aligned} \tag{3.17.4}$$

- $\hat{\boldsymbol{E}}_{6,k}$ For k odd,

$$\begin{aligned}&\{\left(0,0,0,0,0,\tfrac{k+1}{2}\right), \left(0,0,0,0,2,\tfrac{k-1}{2}\right), \left(1,0,0,0,1,\tfrac{k-1}{2}\right), \left(0,1,0,1,0,\tfrac{k-3}{2}\right),\\ &\left(0,0,0,0,6,\tfrac{k-5}{2}\right), \left(0,0,0,3,0,\tfrac{k-5}{2}\right), \left(3,0,0,0,3,\tfrac{k-5}{2}\right), \left(1,0,1,0,4,\tfrac{k-7}{2}\right)\}.\end{aligned} \tag{3.17.5}$$

For k even,

$$\{(0,0,0,0,1,\tfrac{k}{2}),(0,0,0,0,3,\tfrac{k-2}{2}),(0,0,0,1,1,\tfrac{k-2}{2}),\\ (0,0,1,0,0,\tfrac{k-2}{2}),(0,2,0,0,1,\tfrac{k-4}{2}),(1,0,1,0,1,\tfrac{k-4}{2}),\\ (0,1,0,0,5,\tfrac{k-6}{2}),(0,1,0,2,1,\tfrac{k-6}{2}),(2,0,0,1,3,\tfrac{k-6}{2})\}. \tag{3.17.6}$$

- $\hat{E}_{7,k}$ For k odd,

$$\{(\tfrac{k+1}{2},0,0,0,0,0,0),(\tfrac{k-1}{2},0,0,0,1,0,0),\\ (\tfrac{k-3}{2},0,1,0,0,0,0),(\tfrac{k-5}{2},0,0,2,0,0,0),(\tfrac{k-7}{2},0,0,0,0,8,0),\\ (\tfrac{k-7}{2},0,0,0,4,0,0),(\tfrac{k-7}{2},0,1,0,2,0,0),(\tfrac{k-9}{2},0,0,2,0,4,0),\\ (\tfrac{k-9}{2},0,1,0,0,6,0),(\tfrac{k-9}{2},0,1,0,2,2,0)\}. \tag{3.17.7}$$

For k even,

$$\{(\tfrac{k}{2},0,0,0,0,1,0),(\tfrac{k-2}{2},1,0,0,0,0,0),\\ (\tfrac{k-2}{2},0,0,0,0,1,1),(\tfrac{k-4}{2},1,0,0,0,2,0),(\tfrac{k-4}{2},1,0,0,1,0,0),\\ (\tfrac{k-4}{2},0,0,1,0,0,1),(\tfrac{k-6}{2},1,0,1,0,1,0),(\tfrac{k-6}{2},0,0,1,1,0,1),\\ (\tfrac{k-8}{2},1,0,0,3,0,0),(\tfrac{k-8}{2},0,0,0,0,7,1),(\tfrac{k-8}{2},0,0,0,3,1,1),\\ (\tfrac{k-10}{2},1,0,1,0,5,0),(\tfrac{k-10}{2},1,0,1,1,3,0)\}. \tag{3.17.8}$$

- $\hat{E}_{8,k}$ For k odd,

$$\{(\tfrac{k+1}{2},0,0,0,0,0,0,0),(\tfrac{k-1}{2},0,0,0,0,0,1,0),(\tfrac{k-3}{2},0,1,0,0,0,0,0),\\ (\tfrac{k-5}{2},0,0,0,1,0,0,0),(\tfrac{k-5}{2},0,1,0,0,0,1,0),(\tfrac{k-5}{2},0,0,0,0,2,0,0),\\ (\tfrac{k-7}{2},0,0,0,0,2,0,0),(\tfrac{k-7}{2},0,1,0,0,0,2,0),(\tfrac{k-9}{2},0,0,0,0,1,0,2),\\ (\tfrac{k-9}{2},0,1,0,1,0,0,0),(\tfrac{k-9}{2},1,0,0,0,1,0,1),(\tfrac{k-11}{2},0,0,0,0,0,6,0),\\ (\tfrac{k-11}{2},0,0,0,2,0,0,0),(\tfrac{k-13}{2},0,0,0,1,0,4,0),(\tfrac{k-15}{2},0,0,0,0,4,0,0),\\ (\tfrac{k-15}{2},0,0,2,0,0,3,0),(\tfrac{k-17}{2},0,0,0,0,0,0,6),(\tfrac{k-17}{2},0,0,0,3,0,0,0),\\ (\tfrac{k-17}{2},0,1,0,1,2,0,0),(\tfrac{k-19}{2},0,1,2,0,0,0,2),(\tfrac{k-19}{2},0,2,0,2,0,0,0)\} \tag{3.17.9}$$

For k even,

$$\{(\tfrac{k-2}{2},0,0,0,0,0,0,1),(\tfrac{k-2}{2},1,0,0,0,0,0,0),(\tfrac{k-4}{2},0,0,1,0,0,0,0),\\ (\tfrac{k-6}{2},0,0,0,0,1,0,1),(\tfrac{k-6}{2},1,0,0,0,0,2,0),(\tfrac{k-8}{2},0,0,0,0,0,0,3),\\ (\tfrac{k-8}{2},0,0,0,1,0,0,1),(\tfrac{k-8}{2},0,0,1,0,0,2,0),(\tfrac{k-8}{2},3,0,0,0,0,0,0),\\ (\tfrac{k-10}{2},0,1,0,0,1,0,1),(\tfrac{k-10}{2},0,0,0,1,0,1,1),(\tfrac{k-10}{2},1,0,1,0,0,0,1),\\ (\tfrac{k-12}{2},0,0,0,0,0,5,1),(\tfrac{k-12}{2},0,0,1,1,0,1,0),(\tfrac{k-14}{2},0,0,1,0,1,3,0),\\ (\tfrac{k-16}{2},0,0,1,0,3,0,0),(\tfrac{k-16}{2},0,1,1,0,1,2,0),(\tfrac{k-18}{2},0,0,2,0,0,0,3),\\ (\tfrac{k-18}{2},0,1,0,0,0,0,5),(\tfrac{k-18}{2},0,1,0,2,0,0,1),(\tfrac{k-18}{2},0,2,0,0,2,0,1)\} \tag{3.17.10}$$

- $\hat{F}_{4,k}$ For k odd,

$$\begin{aligned}\{&(\tfrac{k+1}{2},0,0,0)\,,(\tfrac{k-1}{2},0,0,2)\,,(\tfrac{k-1}{2},0,1,0)\,,(\tfrac{k-3}{2},1,0,1)\,,\\&(\tfrac{k-3}{2},0,2,0)\,,(\tfrac{k-5}{2},0,1,4)\,,(\tfrac{k-5}{2},1,0,3)\}\,.\end{aligned} \tag{3.17.11}$$

For k even,

$$\begin{aligned}\{&(\tfrac{k}{2},0,0,1)\,,(\tfrac{k-2}{2},0,1,1)\,,(\tfrac{k-2}{2},1,0,0)\,,(\tfrac{k-4}{2},0,0,5)\,,\\&(\tfrac{k-4}{2},1,0,2)\,,(\tfrac{k-4}{2},0,2,1)\,,(\tfrac{k-4}{2},1,1,0)\,,(\tfrac{k-6}{2},1,0,4)\}\,.\end{aligned} \tag{3.17.12}$$

- $\hat{G}_{2,k}$ For k odd,

$$\{(\tfrac{k+1}{2},0)\,,(\tfrac{k-1}{2},2)\,,(\tfrac{k-3}{2},4)\}\,. \tag{3.17.13}$$

For k even,

$$\{(\tfrac{k}{2},1)\,,(\tfrac{k-2}{2},3)\,,(\tfrac{k-4}{2},5)\}\,. \tag{3.17.14}$$

As previously mentioned, the x values determined by identifying the above representations with zero were exhaustively tested numerically. For example, the $\hat{B}_{N,k}$ case was tested for levels $k \in \{2,\ldots,5000\}$ for ranks $N \in \{3,\ldots,9\}$, and for levels $k \in \{2,\ldots,500\}$ for ranks $N \in \{10,\ldots,17\}$.

The $\hat{D}_{N,k}$ case was studied for levels $k \in \{3,\ldots,5000\}$ for ranks $N \in \{4,\ldots,8\}$, and for levels $k \in \{3,\ldots,500\}$ for ranks $N \in \{9,\ldots,18\}$ and for $k = 2$ for $n \in \{19,\ldots,57\}$.

It should be remarked that this is not a proof that the above generating sets fully define the ideal but it is strong numerical evidence that they do.

3.18 Outer Automorphisms and Charge Lattice Symmetries

In this section results pertaining to the symmetries of the D-brane charge lattice in the fundamental chamber shall be discussed. These include the symmetries of $\mathrm{Out}(\hat{\mathfrak{g}})/\mathrm{Out}(\mathfrak{g})$ on the D-brane charge, as well as an unexpected symmetry resulting from translations of the weight lattice.

3.18.1 Outer Automorphisms and Charge

In Appendix G the action of the outer automorphisms $\mathrm{Out}(\hat{\mathfrak{g}})/\mathrm{Out}(\mathfrak{g})$ on the weights of representations is discussed. As mentioned there, it maps weights in $P_+^{(k)}$ to other weights in $P_+^{(k)}$.

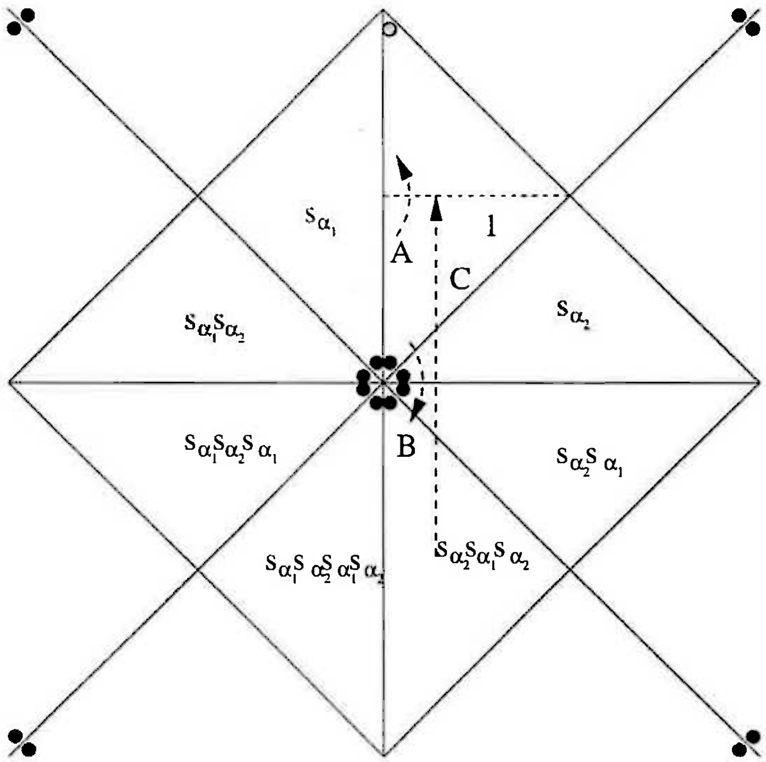


Figure 3.12: *This diagram shows the affine fundamental chamber of $\hat{C}_{2,k}$ for a shifted action of $\widehat{W}$ (marked by the corresponding Weyl group element 1), as well as the action of the outer automorphism on the weights of the fundamental chamber (reflection A around the dashed line). The black circles represent the highest weights that the vacuum weight is transformed into in each chamber by the action of $\widehat{W}$. The grey circle is the highest weight the outer automorphism maps the vacuum weight to (the action of the outer automorphism is represented by the flip A around the dashed line). It is also shown that the action of A is equivalent to the shifted Weyl reflection B (reflecting by $s_{\alpha_2}s_{\alpha_1}s_{\alpha_2} \in W$) and the translation C (the translation is $(k+g^\vee)\Lambda_2$ which is not an element of $\widehat{W}$).*

Such a transformation cannot be an element of the affine Weyl group $\widehat{W} = T(Q^\vee) \rtimes W$, which maps the affine fundamental chamber into all other affine chambers, but with a particular orientation for each chamber. The outer automorphisms $A \in \mathrm{Out}(\hat{\mathfrak{g}})/\mathrm{Out}(\mathfrak{g})$ can flip or rotate this orientation, such as in Figure 3.12 of the $\hat{C}_{2,4}$ case where the outer automorphism acting on the fundamental chamber is equivalent to: $A(\lambda_0, \lambda_1, \dots, \lambda_N) = (\lambda_n, \lambda_{n-1}, \dots, 0)$ which is a flip around a line in the middle of the chamber.

This corresponds to a Weyl reflection by the Weyl group element $w = s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}$ and a translation t equivalent to a weight shift of $(k+g^\vee)\Lambda_2$ (see Appendix G.2). For $\hat{C}_{2,4}$, the fundamental weights are expressed in terms of roots as: $\Lambda_1 = \alpha_1 + \frac{1}{2}\alpha_2, \;\; \Lambda_2 = \alpha_1 + \alpha_2$. This in turn implies $t = (k+g^\vee)\Lambda_2 = 7\alpha_1 + 7\alpha_2$ which is not an element of the coroot lattice $Q^\vee$ (the coroot lattice contains weights of form $\lambda = 2m\Lambda_1 + 2n\Lambda_2, \;\; m, n \in \mathbb{Z}$). It has been shown [73] that for general affine Lie algebras, the outer automorphism group $\mathrm{Out}(\hat{\mathfrak{g}})/\mathrm{Out}(\mathfrak{g})$ and the affine Weyl group $\widehat{W}$ combine into a larger group $T(Q^*) \rtimes W$. Q^* is the dual lattice to the roots lattice $Q = \mathbb{Z}\alpha_1 + \ldots + \mathbb{Z}\alpha_N$, defined by:

$$Q^* = \{\lambda \in P \mid (\lambda, \mu) \in \mathbb{Z}\ \forall \mu \in Q\}. \tag{3.18.1}$$

This allows translations such as $t = (k + g^\vee)\Lambda_2$ for $\hat{C}_{2,4}$.

As a consequence of the group symmetry $T(Q^*) \rtimes W$, the following symmetry was noticed on the D-brane charge lattice, where $q_\lambda = \dim \lambda \mod x$ is the D-brane charge:

$$q_{A\lambda} = (\epsilon(w_A) q_\lambda) \mod x = (e^{2\pi i (A\Lambda_0, \rho)} q_\lambda) \mod x\ . \tag{3.18.2}$$

where $A\lambda$ is the finite component of representation $A\hat{\lambda}$ and the Weyl reflection w_A is the reflection corresponding to the reflection component of A in $T(Q^*) \rtimes W$, and is given by Appendix G.

Another way of thinking of this symmetry is as a consequence of the symmetry of the fusion coefficients (3.2.57) under the action of $A \in \mathrm{Out}(\hat{\mathfrak{g}})/\mathrm{Out}(\mathfrak{g})$. Take for example the fusion relation:

$$\mathcal{N}_{A\lambda\mu}^{A\nu} = \mathcal{N}_{\lambda\mu}^{\nu}\ . \tag{3.18.3}$$

Applying this to eqn (3.18.2) and eqn (3.5.16) gives:

$$\begin{aligned} \dim(A\lambda) \dim \mu &= \sum_{\nu \in P_+^{(k)}} \mathcal{N}_{A\lambda\mu}^{A\nu} \dim(A\nu) \mod x \\ q_{A\lambda} q_\mu &= \sum_{\nu \in P_+^{(k)}} \mathcal{N}_{\lambda\mu}^{\nu} q_{A\nu} \mod x \\ \epsilon(w_A) q_\lambda q_\mu \mod x &= \epsilon(w_A) \sum_{\nu \in P_+^{(k)}} \mathcal{N}_{\lambda\mu}^{\nu} q_\nu \mod x \\ \therefore \dim \lambda \dim \mu &= \sum_{\nu \in P_+^{(k)}} \mathcal{N}_{\lambda\mu}^{\nu} \dim \nu \mod x. \end{aligned} \tag{3.18.4}$$

This would indeed work for any constant factor n_A in the relation: $q_{A\lambda} = n_A q_\lambda \mod x$, so this is not a proof that $n_A = \epsilon(w_A)$, it is consistent with eqn (3.18.2).

Combining the D-brane charge relation eqn (3.18.2) with $q_{\hat{w}\cdot\lambda} = (\epsilon(\hat{w}) q_\lambda) \mod x$, for $\hat{w} \in \widehat{W}$,[12] gives the total charge relation for the extended Weyl group $T(Q^*) \rtimes W$.

This symmetry (3.18.2) of the D-brane charges has been noticed before for $\hat{A}_{2,k}$ [126, 127], and for $\hat{A}_{N,k}$ in [86] however it has not been proven elsewhere. In these papers it was noticed that acting with an element of the center (which in Appendix G, is discussed as being isomorphic to the outer automorphism group $\mathrm{Aut}(\hat{\mathfrak{g}})$ of the affine Dynkin diagram symmetries) corresponds to rotating the conjugacy class on the

[12]This can be seen from realising $\dim \lambda$ (eqn (G.1.14)) is a specialisation of $\chi_\lambda(\xi_\mu)$ (eqn (3.2.68)) for $\mu \to -\rho$, whereas $\chi_{\hat{w}\cdot\lambda} = \epsilon(\hat{w})\chi_\lambda$ from (eqn (3.2.74)). The relation $q_{\hat{w}\cdot\lambda} = \epsilon(\hat{w}) q_\lambda \mod x$ follows.

manifold around which the D-brane is wrapped, and thus should only have an effect on the position of the D-brane on the manifold, but leave unchanged the D-brane charge, up to a possible sign change. In $\hat{A}_{N,k}$ it was shown that the outer automorphisms generate multiplets of like charge [127].

In general, the size of such multiplets related by eqn (3.18.2) will be determined by the order of $\mathrm{Out}(\hat{\mathfrak{g}})$.

The symmetry of the automorphisms can thus be used as a constraint on x.

$$q_{A\lambda} - \epsilon(w_A)q_\lambda \mod x = 0 \;\equiv\; \dim(A\lambda) - \epsilon(w_A)\dim(\lambda) \mod x = 0. \quad (3.18.5)$$

These constraints could potentially be used to determine x, and at the very least to put constraints on x.[13]

The symmetry eqn (3.18.2) can of course be augmented by the symmetry of the finite dimensional Lie algebra Dynkin diagram outer automorphisms on the weights, but such automorphisms map representations to other representations with the same finite dimension and as such provide no new information on the possible values of x.

A program was written (in Python language) to generate the constraints (3.18.5) for particular affine Lie algebras and then to use these constraints to determine a maximum value of x. This numerical analysis proved that usually the constraints (3.18.5) do fully determine x (by which is meant the results from this analysis agree with the data in Appendix H). However, there were some cases in which the data from these constraints did not correctly determine x. These were mostly for low k which from analogy to section 3.10 was to expected. However there still some examples for $k \geq \max\{\text{comarks}\}$, suggesting for some algebras the conditions provided by eqn (3.18.2) are not quite restrictive enough, resulting in values of x that are too large.

Of course for the cases when the center of the group is zero, this method could not be applied because the center of the group is isomorphic to $\mathrm{Out}(\hat{\mathfrak{g}})/\mathrm{Out}(\mathfrak{g})$

A small technical point is that the charges lying on boundaries of reflections, with constraints where $\epsilon(w_A) = -1$ such that: $q_{A\hat{\lambda}} + q_{\hat{\lambda}} \mod x = 0$, can have two possible values, $q = 0$ or $q = \frac{x}{2}$. The program considered all such cases as $q = 0$, however some cases for $q = \frac{x}{2}$ were done by hand. Neither approach ensured that the resulting x would agree with our conjectured formulas.

The results of this analysis, along with a summary of the outer automorphisms of each algebra and the sign of $\epsilon(w_A)$ for these outer automorphisms is contained in Table 3.5.

3.18.2 Weight Lattice Translation Symmetry

While studying charge lattice diagrams an extra, unexpected symmetry was found. Further analysis revealed that this symmetry is a manifestation of $T(P) \rtimes W$ where

[13] These constraints really are the constraints provided by one element $\hat{w} \in \widehat{W}$, and by the translations $T(Q^*)$, the latter not affecting the brane charge.

Algebra	Out($\hat{\mathfrak{g}}$)/Out($\mathfrak{g}$) generators	Sign	Exceptions to Aut($\hat{\mathfrak{g}}$) determining x: (x_{aut}, x)
$\hat{A}_{N,k}$ $N \geq 1$	$\mathbb{Z}_{N+1} : (\lambda_0, \lambda_1, \ldots, \lambda_N)$ $\rightarrow (\lambda_N, \lambda_0, \ldots, \lambda_{N-1})$	$(-1)^N$	none
$\hat{B}_{N,k}$ $N \geq 3$	$\mathbb{Z}_2 : (\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_N)$ $\rightarrow (\lambda_1, \lambda_0, \lambda_2, \ldots, \ldots_N)$	-1	$\hat{B}_{N,2}$: $(2x, x)$ N odd $\hat{B}_{14,1}$: $(2, 14)$
$\hat{C}_{N,k}$ $N \geq 2$	$\mathbb{Z}_2 : (\lambda_0, \lambda_1, \ldots, \lambda_N)$ $\rightarrow (\lambda_N, \lambda_{N-1}, \ldots, \lambda_0)$	$(-1)^{\lceil \frac{N}{2} \rceil}$	$\hat{C}_{5,1}$: $(7,1)$, $\hat{C}_{5,2}$: $(8,2)$ $\hat{C}_{6,2}$: $(3,1)$, $\hat{C}_{7,3}$: $(11,1)$ $\hat{C}_{8,4}$: $(13,1)$, $\hat{C}_{9,1}$: $(11,1)$ $\hat{C}_{9,6}$: $(4,2)$
$\hat{D}_{N,k}$ $N \geq 4$ N even	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 : (\lambda_0, \lambda_1, \ldots, \lambda_N) \rightarrow$ $(\lambda_1, \lambda_0, \lambda_2, \ldots, \lambda_{N-2}, \lambda_N, \lambda_{N-1})$ $(\lambda_0, \lambda_1, \ldots, \ldots_N) \rightarrow (\lambda_N, \ldots, \lambda_0)$	$+1$ $(-1)^{\frac{N}{2}}$	$\hat{D}_{8,1} : (1,3)$ $\hat{D}_{20,1} : (1,3)$
$\hat{D}_{N,k}$ $N \geq 5$ N odd	$\mathbb{Z}_4 : (\lambda_0, \lambda_1, \ldots, \lambda_N) \rightarrow$ $(\lambda_{N-1}, \lambda_N, \lambda_{N-2}, \ldots, \lambda_2, \lambda_1, \lambda_0)$	$(-1)^{\frac{N-1}{2}}$	$\hat{D}_{11,1} : (1,3)$ $\hat{D}_{23,1} : (5,15)$
$\hat{E}_{6,k}$	$\mathbb{Z}_3 : (\lambda_0, \lambda_1, \ldots, \lambda_6)$ $\rightarrow (\lambda_1, \lambda_5, \lambda_4, \lambda_3, \lambda_6, \lambda_0, \lambda_2))$	$+1$	$\hat{E}_{6,2} : (7,1)$
$\hat{E}_{7,k}$	$\mathbb{Z}_2 : (\lambda_0, \lambda_1, \ldots, \lambda_7)$ $\rightarrow (\lambda_6, \lambda_5, \ldots, \lambda_0, \lambda_7)$	-1	$\hat{E}_{7,1} : (57, 3135)$, $\hat{E}_{7,2} : (8,1)$, $\hat{E}_{7,4} : (11,1)$
$\hat{E}_{8,k}$,	$\{0\}$	-	-
$\hat{F}_{4,k}$,	$\{0\}$	-	-
$\hat{G}_{2,k}$	$\{0\}$	-	-

Table 3.5: *Generators of Out($\hat{\mathfrak{g}}$)/Out($\mathfrak{g}$), sign of reflection of charge, and exceptions to Out($\hat{\mathfrak{g}}$)/Out($\mathfrak{g}$) determining x. See [15, 98] for more on the outer automorphisms.*

$P = \mathbb{Z}\Lambda_1 + \ldots \mathbb{Z}\Lambda_N$ is the weight lattice. Thus we have gone from the affine Weyl group, $T(Q^\vee) \rtimes W$, to the combined group of the affine Weyl group and Out($\hat{\mathfrak{g}}$)/Out($\mathfrak{g}$), $T(Q^*) \rtimes W$, to the most general transformations $T(P) \rtimes W$. It can be seen that this has been getting gradually more general, as $P \supset Q^* \supset Q^\vee$. Let us label the charge lattice symmetry $T(P) \rtimes W$ as the extended affine Weyl symmetry $\widehat{W}'$.

Let us study the case of $\hat{C}_{2,4}$, which is displayed in Figures 3.13 & 3.14. The finite dimensions of the weights are displayed, along with their corresponding charges for $x = 7$. As can be seen, there are three lines of symmetry through the fundamental chamber, around which the charges are flipped ($q \rightarrow -q \mod x$).

Reflection around the middle horizontal line is the outer automorphism symmetry discussed earlier. Reflection around the top line is shown in Figures 3.13 and 3.14.

The most immediate thing to observe is that this symmetry does not reflect the fundamental chamber into another chamber. It can be seen from these figures to be equivalent to a shifted reflection of the fundamental chamber by Weyl element s_{α_2} followed by a translation $t = (k + g^\vee)(\Lambda_2 - \Lambda_1)$, where $t \notin Q^*$, but $t \in P$. Similarly, reflection around the lower line is equivalent to a shifted reflection of the fundamental chamber by Weyl element $s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$ followed by a translation $t = -(k+g^\vee)(\Lambda_1)$, where again $t \notin Q^*$, but $t \in P$.

To make transparent what translations are allowed by the various translation groups, for the example of $\hat{C}_{2,4}$, it is easy to show that the shifts by the weight lattice $T(P)$ allow (see eqn (G.2.32)):

$$\Delta\lambda = (k + g^\vee)(\mathbb{Z}\Lambda_1 + \mathbb{Z}\Lambda_2). \tag{3.18.6}$$

Shifts for the lattice Q^* allow:

$$\Delta\lambda = \begin{cases} (k + g^\vee)(2\mathbb{Z}\Lambda_1 + \mathbb{Z}\Lambda_2), & \text{if } k + g^\vee \in 2\mathbb{Z} + 1, \\ (k + g^\vee)(\mathbb{Z}\Lambda_1 + \mathbb{Z}\Lambda_2), & \text{if } k + g^\vee \in 2\mathbb{Z}. \end{cases} \tag{3.18.7}$$

Shifts for the coroot lattice $Q^\vee$ allow:

$$\Delta\lambda = \begin{cases} (k + g^\vee)(2\mathbb{Z}\Lambda_1 + 2\mathbb{Z}\Lambda_2), & \text{if } k + g^\vee \in 2\mathbb{Z} + 1, \\ (k + g^\vee)(\mathbb{Z}\Lambda_1 + \mathbb{Z}\Lambda_2), & \text{if } k + g^\vee \in 2\mathbb{Z}. \end{cases} \tag{3.18.8}$$

These symmetries were observed to appear for any non-simply laced Lie algebra, for which the weights are not integer sums of the coroots. Figure 3.15 displays this symmetry for the other rank two case, $\hat{G}_{2,k}$.

Using this symmetry, a constraint on x can be constructed, that as a subset includes the constraints from the $\mathrm{Out}(\hat{\mathfrak{g}})/\mathrm{Out}(\mathfrak{g})$ of the previous section. Consider that the constraints of the previous section originate from a Weyl reflection, and the translations of Q^*. If these conditions are combined with the Weyl group conditions (the ideals $\mathcal{I}_k$ of previous sections), then to generate the extra information from the constraints (3.18.2) it is merely necessary to shift each element in the fundamental chamber by the minimum allowed shifts in Q^* and demand that the charges of these two weights in different chambers are equal.

The new conditions, coming from $T(P) \rtimes W$ then become:

$$q_\lambda - q_{\lambda+t} = 0, \tag{3.18.9}$$

$$t \in (k + g^\vee)(\mathbb{Z}\Lambda_1 + \ldots + \mathbb{Z}\Lambda_N), \tag{3.18.10}$$

combined with the generators of the ideal in eqn (3.2.89). More generally stated then, the constraints that determine x can be expressed as:

$$q_{w\cdot\lambda} - \epsilon(w) q_\lambda \mod x = 0, \qquad w \in \widehat{W}' \tag{3.18.11}$$

A program (written in the language Python) analysed x for these conditions, and found exact agreement with the earlier conjectured formula for x eqn (3.8.4) (with the normal exceptions for low k) except for $\hat{C}_{N,k}$ when N is not a power of two, for some levels k (n.b. that the case of $\hat{C}_{2,4}$ studied in Figures 3.13 and 3.14 obeys eqn (3.18.11)).

Thus it would appear that either this symmetry is broken for $\hat{C}_{N,k}$ when N is a power of two or x is different to what was thought for these cases.

Studying the exceptions, if this new symmetry is accepted as relevant, it predicts that for $\hat{C}_{N,k}$:

$$x = \frac{k + g^\vee}{\gcd(k + g^\vee, \operatorname{lcm}(1, \dots, g - 1))} \tag{3.18.12}$$

where g is the Coxeter number, not the dual Coxeter number (see Appendix F for the Coxeter numbers).

Study of eqn (3.8.4) made it apparent that all other formulas for x can be rewritten in an identical way. Thus, if this new symmetry of translations on the weight lattice is accepted, it means that eqn (3.18.12) is a general unifying expression for x!

It is unclear if it is valid to use these weight lattice translation symmetries to determine the charge group $\mathbb{Z}_x$. While it would seem natural to apply such a symmetry to determine the charge group, this on its own is not a justification to use it to enhance the constraints provided by the fusion rules. Applying this symmetry implies that the analysis of [40] used to derive the brane condensation charge relation (3.5.16) does not take into account all the physical restrictions on the brane charge algebra. It would be intriguing to determine x for the untwisted $\hat{C}_{N,k}$ algebras via K-theory, so that a direct comparison between predictions eqn (3.8.4) and eqn (3.18.12) would be possible.

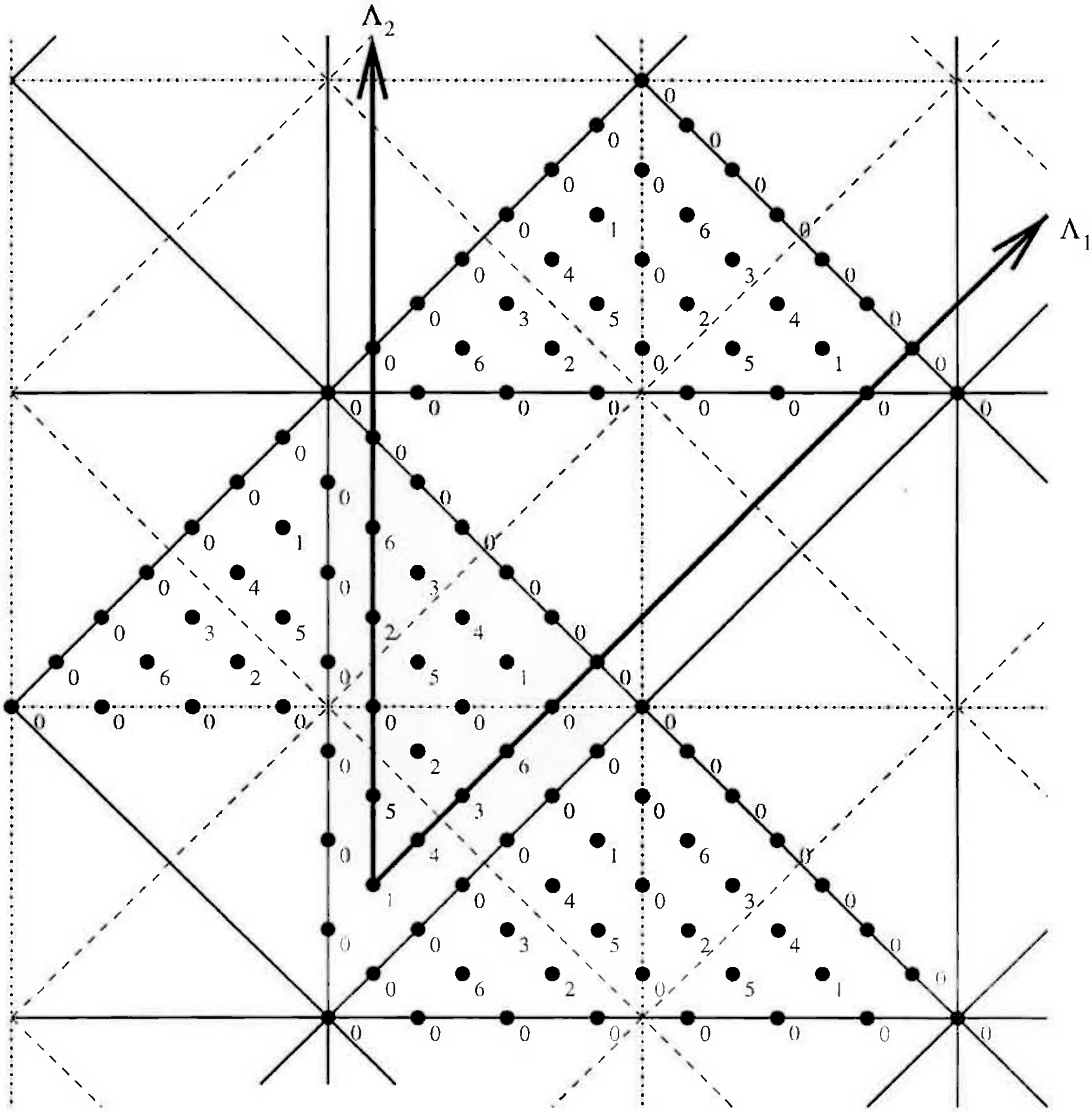


Figure 3.13: *This is the weight lattice for $\hat{C}_{2,4}$. Weights of interest here have been marked by their corresponding D-brane charges. The shaded triangle is the affine fundamental chamber, for a shifted action of $\widehat{W}$. The solid lines are the chamber boundaries (corresponding to shifted reflections and translations of $T(Q^\vee) \rtimes W$). The dotted lines are the extra reflections present due to the $T(Q^*) \rtimes W$ symmetry. The dashed lines are the extra reflections present due to the $T(P) \rtimes W$ symmetry. Notice the charges are antisymmetric ($q_\lambda \to -q_{\lambda'} \mod x$) under reflections about all of these lines.*

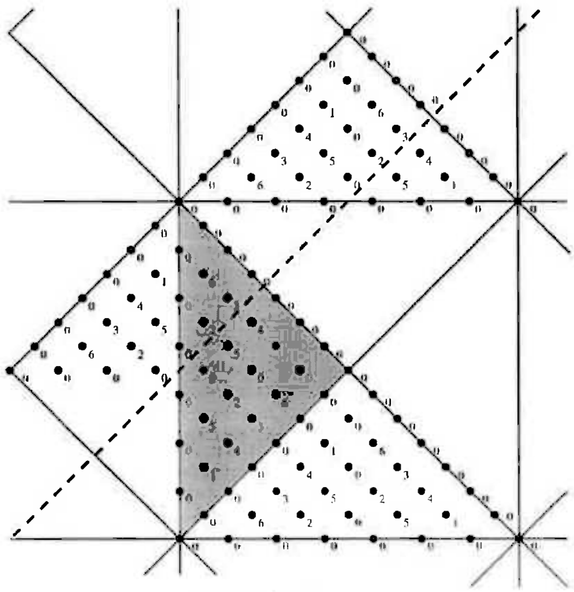


Fig 3.14.i

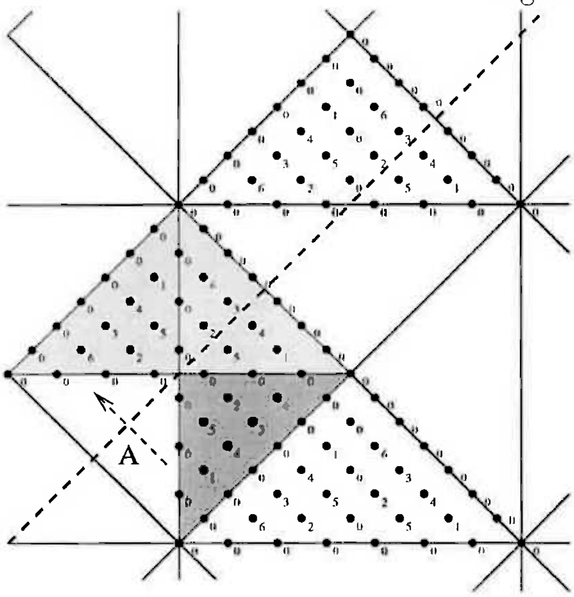


Fig 3.14.ii

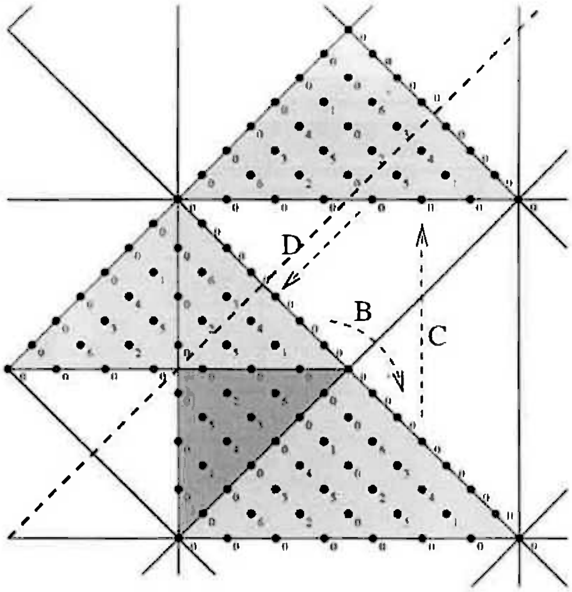


Fig 3.14.iii

Figure 3.14: *Fig. 3.14.i highlights the fundamental chamber of Fig. 3.13. The dashed line corresponds to a reflection of the weight lattice by* $w \in T(P) \rtimes W$ *(for* w *not an element of* $T(Q^*) \rtimes W$*). In Fig. 3.14.ii transformation A demonstrates the reflection of the fundamental chamber with respect to* w*. In Fig. 3.14.iii, transformations B, C and D demonstrate that the reflection A of Fig. 3.14.ii is equivalent to a reflection* $w \in W$ *(refl. B), followed by two translations of the weight lattice, C and D, equivalent to a trans:* $(k + g^\vee)(\Lambda_2 - \Lambda_1)$ *(this shift is not an action of an element of* $T(Q^*)$*, but of* $T(P)$*).*

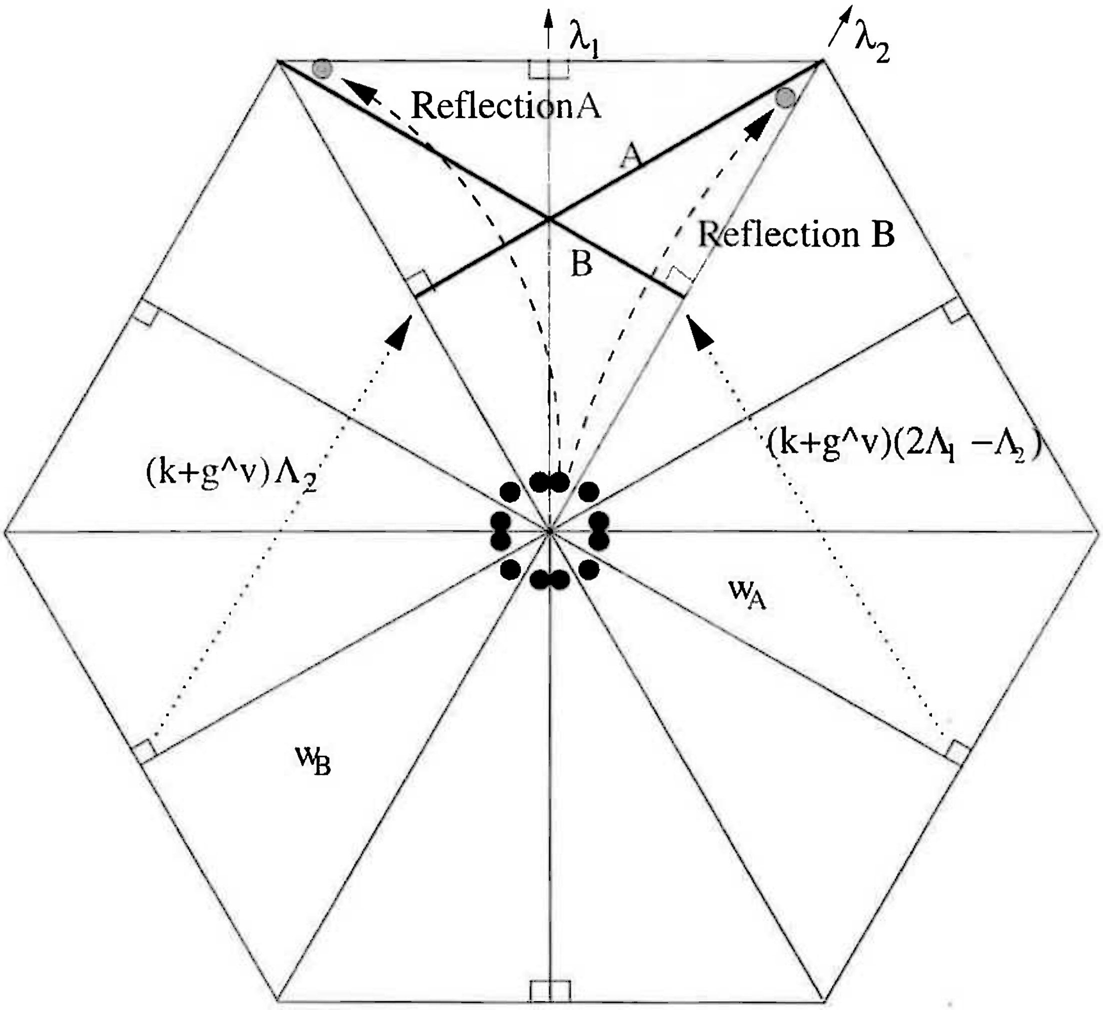


Figure 3.15: *Weight lattice transformation symmetries of $\hat{G}_{2,k}$. When the highest weights λ in this diagram are replaced by the D-brane dimensions of the corresponding integrable representations (q_λ), the charges in the lattice are antisymmetric with respect to reflections about the lines A and B. Reflection around line A is equivalent to the mapping of the fundamental chamber (shaded) to another by $w_A \in W$, followed by a translation $(k + g^\vee)(2\Lambda_1 - \Lambda_2)$ (which is not an element of Q^*). Reflection around line B is equivalent to the mapping of the fundamental chamber to another by $w_B \in W$, followed by a translation $(k + g^\vee)(\Lambda_2)$ (which is not an element of Q^*). The chambers are those defined by the shifted action of $\widehat{W}$.*

Chapter 4

Conclusions

4.1 Discussion of Results for Effective Brane Actions in $AdS_m \times S^n$ and Future Directions

In this thesis the fully κ-invariant and supersymmetric actions are found for the D1 and D5-branes in $AdS_3 \times S^3$ and the D5 action in $AdS_5 \times S^5$.

The D1-brane action found (eqn (2.12.25)) acts as a probe action for the D1 and D5-branes lying parallel and coincident that warp spacetime in such a way as to generate the IIB $AdS_3 \times S^3$ supersymmetric spacetime background. This action is expressed in terms of the supersymmetric background coordinates using eqns (B.29) and (B.30). The killing gauge is fixed to simplify the action in Appendix E.1. It is discussed in §2.15 that the extension of this action to $AdS_3 \times S^3 \times T^4$ is trivial and in the notation of Appendix A.1 the D1-brane action in $AdS_3 \times S^3 \times T^4$ appears unchanged to eqn (2.12.25).

The D5-brane action in an $AdS_3 \times S^3$ is found (eqn (2.13.7)). Again it can be expressed in terms of the supersymmetric background coordinates using eqns (B.29) and (B.30). As mentioned earlier, the action described here is for the D5-brane lying entirely in $AdS_3 \times S^3$, which can be interpreted as a D5-brane in $AdS_3 \times S^3 \times \mathcal{M}^4$ with constraints imposed on the fields such that the solutions of the action must lie in $AdS_3 \times S^3$, with the $\mathcal{M}^4$ compactified away. Such constraints on a D5-brane solution in $AdS_3 \times S^3 \times \mathcal{M}^4$ describe branes in a BPS configuration. This is because the D5-D5 brane configuration, sharing one common dimension is S-dual to the NS5-NS5 brane configuration sharing one common dimension, which was shown in [130] to be a BPS configuration.

The D5-brane action eqn (2.13.7) does not have solutions that describe the branes whose gravity is warping spacetime to an $AdS_3 \times S^3 \times \mathcal{M}^4$ background. Such a D5-brane, with 4 directions compactified in $\mathcal{M}^4$, has an action like the D1-branes in $AdS_3 \times S^3$. As such, this brane action is not as useful as the D1-brane action (eqn (2.12.25)) for probing the branes generating the warped spacetime metric, or as useful for finding an effective action of the 2dim SCFT that IIB string theory in $AdS_3 \times S^3$ corresponds to via the Maldacena conjecture.

The D5-brane action in $AdS_5 \times S^5$ found (eqn (2.14.3)) is of interest as it can be adjusted to describe the D5, D3, F1 configuration of [23, 29, 56, 68, 135] which exhibits a baryon vertex, by placing some restrictions on the degrees of freedom. After also applying the eqns (B.29) and (B.30) to express the action in terms of background supersymmetric coordinates, the constraints described at the end of §2.14 make this action the full supersymmetric action equivalent to an effective action of the baryon vertex in $\mathcal{N}=4$ $D=4$ SYM theory.

A fully κ-symmetric D5 action in $AdS_3 \times S^3 \times T^4$ failed to be found in this case. However, this does not mean that it does not exist. The WZ-action in this case may take a different form to more standard D-brane actions. It should be said though that most of the interesting dynamics occurs in the $AdS_3 \times S^3$, not in T^4, so the action found in (2.12.25) is sufficient for most purposes.

In Appendix E, the Killing gauge is applied to the D5-brane actions in $AdS_3 \times S^3$ and $AdS_5 \times S^5$. However, even though the D5-brane action in $AdS_3 \times S^3$ is reduced to being eighth order in fermions, it is still somewhat complicated. The D5-brane in $AdS_5 \times S^5$ remains very complicated even in this gauge. It remains to be seen if some method, such as the T-duality transformation of [78] can be used to simplify the actions.

There are several directions that could be investigated in future. It would still be advantageous to continue the analysis for finding a D5-brane action in the full $AdS_3 \times S^3 \times T^4$ background. This would allow a better comparison between the brane probe action and the IR components of the two dimensional SCFT that the IIB string theory is equivalent to via the Maldacena conjecture. Carrying out such a comparison between the D1 and D5 brane actions and the 2D SCFT could potentially yield new insights into the Maldacena conjecture.

A first step to doing such a comparison would be to attempt to simplify the actions as much as possible. A beginning is made towards this, in fixing the κ and reparametrization symmetries in Appendix E. However, these actions still involve quite complicated WZ terms (except for the D-string action, which becomes quite simple). Perhaps further simplification could be made along the lines of [78], in which the 2dim scalar-scalar duality is used to simplify the GS action of the IIB in $AdS_5 \times S^5$.

The actions found in this thesis are of course all Abelian actions. However, considering the presence of multiple branes, it would be advantageous to be able to write the actions as non-Abelian actions, where the fields (x, θ, A) are replaced by $N \times N$ matrices. Thus the gauge fields are enhanced to $U(N)$ gauge fields, and the spacetime coordinates are described via non-commutative geometry. Considering that little is known about how to find a non-Abelian DBI action on curved space, this is a difficult problem.

Since the publishing of [31] based on the research in Chapter 2, it has been discovered that IIB string theory is maximally symmetric in the background of plane-waves with five form R-R flux [16]. This has lead to a huge research effort into the field of string theory in the background of R-R flux (see for eg [88, 89, 92, 112]). While it is believed that string theory in such a background is not as physically relevant as string theory in an $AdS_m \times S^n$ background, it is similar to the $AdS_m \times S^n$ background generated by the gravity of D-branes, and is also significantly simpler. Thus analysis of string theory in this background is leading to new information about how to tackle string theory in $AdS_m \times S^n$. One particular research problem of interest that remains open is to re-derive the D1-D5 actions in this background. Similar analysis was recently done for the D3-brane in [89].

Finally the D1/D5-brane system should be dual to a F1/NS5-brane system under an $SL(2;\mathbb{Z})$ duality (S-duality) of the IIB theory in $AdS_3 \times S^3 \times T^4$. Along similar lines of [97] (which studied the self-S-duality of the D3-branes [91]) it would be a useful check of the D1 and D5 actions found here obey this duality.

4.2 Discussion of Research into D-brane Charges on Group Manifolds and Future Directions

The research in this chapter has several primary results. The first of these is the derivation of exact values of x which determine the D-brane charge group $\mathbb{Z}_x$ for untwisted affine Lie algebras of compact, simple, simply connected group manifolds G.

Initially x is determined by solving eqn (3.5.16) for specific values or k and N. The results of this are listed in the tables of Appendix H and in Table 3.4.

The second main result, from §3.9, is that for a particular affine Lie algebra of a Lie group G, x can be determined in great generality by determining a set of generators for the fusion ideal and replacing the characters in the generators by the dimension of the irreducible representations the characters correspond to, creating a list of polynomials of representation dimensions. Then x is taken to be the greatest common divisor of all these resulting polynomials. The constraints on x provided by this procedure are shown to be equivalent to the constraints imposed by eqn (3.5.16) which can also be used to define x. A bonus of this method is that it can be more easily done for all ranks and levels in general, rather than for one pair of rank and level at a time. This yields the very general formula (3.9.5) for x. However this formula is reliant on choosing a correct set of generators for the fusion ideal for the levels being investigated.

Eqn (3.9.5) can be applied to the specific set of generators of the fusion ideal (3.2.89), to give the formula for x:

$$x = \gcd\{\dim \lambda | (\lambda, \theta) = k + 1, \forall \lambda \in P_+\}. \tag{4.2.1}$$

This result is very general, for all untwisted affine Lie algebras of Lie groups G. However, it has a drawback. In general the number of highest weights on the affine fundamental chamber boundary defined by $(\lambda, \theta) = k+1$ grows quickly with rank and level, and each formula for $\dim \lambda$, given by eqn (G.1.14), can be very complicated. Due to this, if a precise formula for the charge group predicted by twisted K-theory is derived, it would be probably be difficult to analytically prove that the K-theory results for x are given by the above result. Another drawback of this formula is that the generators of the ideal used to find eqn (4.2.1) are not valid for k less than the greatest comark of the algebra.

It is of course desirable to be able to calculate x for general rank and level. This is done for $\hat{A}_{N,k}$ and $\hat{C}_{N,k}$ using fusion ideals in sections §3.12 and §3.14. The result for $\hat{A}_{N,k}$ had been derived previously in [40], using a method that we show is equivalent to using the generating set (3.12.1) for the ideal. We then rederive the result using the generators of the ideal (3.12.16), a generating set for the ideal which was derived in [52,95].

The formulas for x are found to be:

$$\hat{A}_{N,k}: \quad x = \frac{k+N+1}{\gcd(k+N+1, \mathrm{lcm}(1,\ldots,N))}, \tag{4.2.2}$$

$$\hat{C}_{N,k}: \quad x = \frac{k+N+1}{\gcd(k+N+1, \mathrm{lcm}(1,\ldots,N,1,3,\ldots,2N-1))}. \tag{4.2.3}$$

In §3.13 and §3.15, these results are rederived, this time using the method of fusion potentials. It was believed that the method of fusion potentials may generalise to the other affine Lie algebras, but in practice this does not seem to be practical. The fusion potentials for all untwisted affine Lie algebras (of compact, simple, simply connected group manifolds G) have been derived in [30] but they are of a form that does not seem to be able to be transformed into something that can produce fusion ideals as polynomials of characters.

Using the x data in Appendix H a formula for all values of x for untwisted affine Lie algebras of Lie groups G is conjectured:

$$\hat{\mathfrak{g}}_{N,k}: \quad x = \frac{k+g^\vee}{\gcd(k+g^\vee, y_N)}, \tag{4.2.4}$$

$$y_N = \begin{cases} \mathrm{lcm}(1,2,\ldots,g^\vee-1, exponents), & \text{for } \mathfrak{g} \neq F_4 \text{ or } G_2 \\ \mathrm{lcm}(1,2,\ldots,g^\vee, exponents), & \text{else} \end{cases}$$

It is shown that eqn (4.2.5) is equivalent to eqn (4.2.1) for general level and rank for $\hat{A}_{N,k}$ and $\hat{C}_{N,k}$. Due to the complicated form of eqn (4.2.1) no method is presented here to show eqn (4.2.5) is equivalent to eqn (4.2.1) for the other untwisted affine Lie algebras of Lie groups G. However numerically they agree up to extremely high level and rank, so it is believed that they are the same. Likewise, both formulas do not apply for k less than the greatest comark of the algebra.

The reasons the conjectured formulas above do not apply when $k < \max_{i\in\{1,\ldots,N\}} a_i^\vee$

are explained in §3.10 and the x values for which it does not apply are found in §3.11. Thus x is found for all ranks and levels for the untwisted affine Lie algebras studied.

In §3.16 it is shown analytically that for $\hat{G}_{2,k}$ the conjectured x formula (4.2.5) for $\hat{G}_{2,k}$ is a divisor and thus a lower bound for the x given by eqn (4.2.1).

Considering the complication of the generating sets for fusion ideals given by eqn (3.2.89), much simpler generating sets for the untwisted $\hat{\mathfrak{g}}$ are found in §3.17.

Finally in §3.18 the symmetries of the D-brane charge lattice are studied. As expected the charges had a symmetry under the shifted affine Weyl group:

$$q_{\hat{w}\cdot\lambda} - \epsilon(\hat{w})q_\lambda \mod x = 0, \quad \hat{w} \in \widehat{W} = T(Q^\vee) \rtimes W, \tag{4.2.5}$$

where $q_\lambda = \dim\lambda \mod x$ is the brane charge of an irreducible representation with highest weight λ.

Another expected symmetry is a symmetry under the action of $\mathrm{Out}(\hat{\mathfrak{g}})/\mathrm{Out}(\mathfrak{g})$. It is found to combine with the shifted affine Weyl transformations to give a symmetry:

$$q_{\hat{w}\cdot\lambda} - \epsilon(\hat{w})q_\lambda = 0, \quad \hat{w} \in \widehat{W} = T(Q^*) \rtimes W, \tag{4.2.6}$$

where $\epsilon(\hat{w}) = \epsilon(w)$. w is such that $\hat{w} = t\cdot w$ where $w \in W$ and $t \in Q^*$. Q^* is the lattice dual to the root lattice ($Q^* = \{\lambda \in P \mid (\lambda,\mu) \in \mathbb{Z}\ \forall\mu \in Q)\}$).

An unexpected symmetry of the charge lattice is then discovered in §3.18.2. This symmetry is a symmetry under the transformations of the weight lattice:

$$q_\lambda - q_{\lambda+t} = 0, \tag{4.2.7}$$

$$t \in (k+g^\vee)P, \tag{4.2.8}$$

where P is the weight lattice. This expanded the total charge lattice symmetry to:

$$q_{\hat{w}\cdot\lambda} - \epsilon(\hat{w})q_\lambda \mod x = 0, \quad \hat{w} \in \widehat{W} = T(P) \rtimes W, \tag{4.2.9}$$

where $\epsilon(\hat{w}) = \epsilon(w)$. w is such that $\hat{w} = t\cdot w$ where $w \in W$ and $t \in P$. P is the highest weight lattice. This symmetry is only distinguishable from the symmetry (4.2.6) when P is not equivalent to Q^* (ie, for algebras with short roots).

This latter symmetry held true for all affine Lie algebras studied for the $\mathbb{Z}_x$'s determined earlier, except for $\hat{C}_{N,k}$ when N is not a power of two. For such cases the charges satisfy eqn (4.2.9) only if a divisor to the value of x predicted by eqn (4.2.3) is substituted into $q_\lambda = \dim\lambda \mod x$. In these cases if the above symmetry is employed as a constraint on x, as well as the constraints that come from the generators of the ideal, then this divisor of x is determined. If this new value is taken to be the correct value of x a formula can be determined which expresses these new values of x for all rank and level for $\hat{C}_{N,k}$ (eqn (4.2.10)). It was realised that this new form of x is true for all the affine Lie algebras studied (under appropriate replacement of the values of $g^\vee$ and g), giving:

$$x = \frac{k+g^\vee}{\gcd(k+g^\vee, \mathrm{lcm}(1,2,\dots,g-1))}, \tag{4.2.10}$$

where g is the Coxeter number of the algebra.

This more restrictive[1] value of x for $\hat{C}_{N,k}$ does not contradict eqn (4.2.3) because eqn (4.2.3) was derived to find the maximum possible value of x allowed by eqn (3.5.16), but the main constraints for determining x (eqn (3.5.16)) will be satisfied by any divisor of this maximal number, and eqn (4.2.10) is obviously a divisor of eqn (4.2.3).

It is not certain that this symmetry should really be imposed for $\hat{C}_{N,k}$. Comparison with twisted K-group data should provide a good test to discriminate between eqn (4.2.3) and eqn (4.2.10).

Now it is time to compare these results to twisted K-theory of a group manifold. From [111], it is known that the values of x predicted by K-theory must be divisors of $k + g^\vee$, which agrees with our results for the level k greater than or equal to the greatest comark of the Lie algebra. However for k less than the greatest comark, the values of x are not determined by eqn (4.2.5), (as determined in §3.10 and §3.11) and in general the values for x for these low values of k are not divisors of $k + g^\vee$ and thus disagree with the twisted K-theory analysis of [111] for these values of k.

At this stage little else is known about what constraints twisted K-theory places on the values of x, except for the cases of $\hat{A}_{1,k}$ and $\hat{A}_{2,k}$ [111] (see §3.5.1). The twisted K-theory constraints on x were shown to agree with the x data for these cases in [40].

There are many future directions of this work. Firstly it would be good to be able to prove eqn (4.2.5) analytically for the B, D, E, F and G algebras. However the evidence presented in support of eqn (4.2.5) for these algebras in Chapter 3 is very strong and it is not pressing to have more evidence.

An obvious direction that needs to be taken is the calculation of the $K^*(G, (k + g^\vee)[H_0])$ charge group for the Lie groups G, using methods along the lines of those in [40]. Until this is done there can be no further comparison of the charge groups $\mathbb{Z}_x$ for untwisted (symmetry preserving) D-branes found in this work and the results of K-theory. Once such K-theory results have been found, if they agree with the charge groups found here it will provide strong evidence for both the idea that brane charges are given by twisted K-groups in string theory, and that eqn (3.5.11) correctly describes D-brane condensation on a group manifold. As mentioned earlier it should also allow comparison of eqn (4.2.3) and eqn (4.2.10) and thus provide a test as to whether the weight lattice translation symmetry of the D-brane charge lattice should be respected.

At the moment there is no physical argument for including or excluding the charge symmetry eqn (4.2.9) as a constraint for determining x for $\hat{C}_{N,k}$ (N not a power of 2). It would be very advantageous to have such an argument to give a second way of discriminating between eqn (4.2.5) and eqn (4.2.10) for x. Such an argument could also explain why this symmetry is automatically satisfied for all the other untwisted affine Lie algebras of Lie groups G, as a result of applying the constraints (3.5.16), giving a weight lattice translation symmetry to the charge lattice.

[1] Restrictive in the sense that it is determined using more constraints, resulting in smaller value.

Another obvious direction to be pursued is the derivation of formulas for x for the twisted D-branes, ie the non-symmetry preserving branes corresponding to the twisted Kac-Moody current boundary conditions (see eqn (3.4.6)). This is needed to completely specify the D-brane charge group, along the lines of eqn (3.5.23). This has been partially done for twisted $\hat{A}_N$ in [40] (strong constraints have been placed on the possible values of x for $N \leq 5$, and very weak constraints placed on x for $N > 5$). This subject is complicated by uncertainty and contradictions in the literature towards how to calculate twisted fusion rules.

As mentioned earlier, in K-theory it is understood why x is a divisor of $k + g^\vee$ (through the Atiyah-Hirzebruch spectral sequence [111]), but it is not understood why it must be so through our approach. It would be interesting and useful to see why it occurs.

It has been shown [42] that the fusion algebra $\mathcal{F}_k$ for a level k WZW model can be identified with a twisted equivariant K-theory $K_G(G, (k + g^\vee)[H_0])$ where $[H_0]$ is the generator of $H^3(G, \mathbb{Z}) \cong \mathbb{Z}$. Considering that the K-group describing the charge group is non-equivariant, it would be interesting to establish the relationship between the equivariant and non-equivariant K-theory of G on one hand, and the relation between the fusion algebra $\mathcal{F}_k$ and the charge group $\mathbb{Z}_x$ on the other.

Recently the bundle gerbe for $G = SU(N)$ has been constructed in [51,87]. Twisted K-theory can be described in terms of the Grothendieck K-theory in terms of isomorphism classes of bundle gerbe modules [19]. These bundle gerbes show up naturally in the discussion of the WZW model [51, 94]. Thus it would be interesting to explicitly construct the bundle gerbe modules corresponding to a particular D-brane charge.

Finally it would be interesting to study brane condensation on a $SL(2; R) \times SU(2)$ manifold using WZW methods, which is equivalent to D-brane condensation in $AdS_3 \times S^3$ space [62], and see if it has any implications for the Maldacena conjecture.

Appendix A

Notation and Algebra for Various $AdS \times S$ Spaces

The algebra follows along the lines of [91, 100, 104, 108].

A.1 $AdS_3 \times S^3 \times T^4$

The coset representation for T^4 that is used here describes only translations. That is, $J_{a''b''}$ are absent from the algebra, done for simplicity.

The notation and algebra used (with the radii of the spaces AdS_3, S^3 and T^4 set to 1) is as follows:

- $a, b, c \in \{0, 1, 2\}$ are indices of $SO(2,1)$ (AdS_3).
- $a', b', c' \in \{3, 4, 5\}$ are indices of $SO(3)$ (S^3).
- $a'', b'', c'' \in \{6, 7, 8, 9\}$ are indices of $U(1)^4$ (T^4).
- $\hat{a}, \hat{b}, \hat{c} \in \{0, \ldots, 9\}$ are a combination of (a, b, c), (a', b', c') and (a'', b'', c'').
- $\alpha, \beta, \gamma \in \{1, 2\}$ are $SO(2,1)$ spinor indices.
- $\alpha', \beta', \gamma' \in \{1, 2\}$ are $SO(3)$ spinor indices.
- $\alpha'', \beta'', \gamma'' \in \{1, \ldots, 4\}$ are $SO(4)$ spinor indices.
- $\hat{\alpha}, \hat{\beta}, \hat{\gamma} \in \{1, \ldots, 32\}$ are $D = 10$ Majorana-Weyl spinor indices.
- $I, J, K \in \{1, 2\}$ are $\mathcal{N} = 2$ supersymmetry labels.
- $P_a, P_{a'} \& P_{a''}$ are the AdS_3, S^3 and T^4 translations respectively. J_{ab} and $J_{a'b'}$) are the $SO(2,1)$ and $SO(3)$ rotations corresponding to the AdS_3 and S^3 spaces respectively. They are defined to be antihermitian.

- 32×32 Γ matrices of $AdS_3 \times S^3 \times T^4$ are:

$$\begin{aligned}
&\Gamma^a = \gamma^a \otimes 1_2 \otimes 1_4 \otimes \sigma^1, \quad \gamma^a: \ \gamma^0 = i\sigma^3, \ \gamma^1 = \sigma^1, \ \gamma^2 = \sigma^2, \\
&\Gamma^{a'} = 1_2 \otimes \gamma^{a'} \otimes \bar{\gamma}_5 \otimes \sigma^2, \quad \gamma^{a'}: \ \gamma^{3'} = \sigma^1, \ \gamma^{4'} = \sigma^2, \ \gamma^{5'} = \sigma^3, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \bar{\gamma}_5 = \gamma_6\gamma_7\gamma_8\gamma_9, \\
&\Gamma^{a''} = 1_2 \otimes 1_2 \otimes \gamma^{a''} \otimes \sigma^2.
\end{aligned} \tag{A.1.1}$$

- $$\Gamma^{\hat{a}_1\hat{a}_2\ldots\hat{a}_n} = \Gamma^{[\hat{a}_1} \ldots \Gamma^{\hat{a}_n]} \tag{A.1.2}$$
$$\Gamma_{(n)} = \frac{1}{n!} L^{\hat{a}_1} \ldots L^{\hat{a}_n} \Gamma_{\hat{a}_1\ldots\hat{a}_n} \tag{A.1.3}$$

- $\pi_+ = 1_2 \otimes 1_2 \otimes \ 1_4 \otimes \frac{1}{2}(1_2 + \sigma_3)$, $\ \sigma_+ = 1_2 \otimes 1_2 \otimes \ 1_4 \otimes \frac{1}{2}(\sigma_1 + i\sigma_2)$
$\sigma_- = 1_2 \otimes 1_2 \otimes \ 1_4 \otimes \frac{1}{2}(\sigma_1 - i\sigma_2)$

- $\hat{C} = C \otimes C' \otimes \ C'' \otimes i\sigma^2$ is the charge conjugation matrix, where C, C' and C'' are the charge conjugation matrices of $SO(2,1)$, $SO(3)$ and $SO(4)$.

- $\{\Gamma^{\hat{a}}, \Gamma^{\hat{b}}\} = 2\eta^{\hat{a}\hat{b}}$ where $\eta^{\hat{a}\hat{b}} = (-, +, \ldots, +)$.

- Supersymmetry generators $Q^{I\hat{\alpha}} = \begin{pmatrix} 0 \\ -Q^{I\alpha\alpha'\alpha''} \end{pmatrix}$. In this notation $\theta_{I\hat{\alpha}}$ and $L_{I\hat{\alpha}}$ have the opposite chirality: $L^{I\hat{\alpha}} = \begin{pmatrix} L^{I\alpha\alpha'\alpha''} \\ 0 \end{pmatrix}$.

- $Q^{1\hat{\alpha}}$ and $Q^{2\hat{\alpha}}$ can be combined as: $Q = \begin{pmatrix} Q^1 \\ Q^2 \end{pmatrix}$.

- $$\mathcal{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathcal{K} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ act on } Q = \begin{pmatrix} Q^1 \\ Q^2 \end{pmatrix}. \tag{A.1.4}$$

The algebra:

$$\begin{aligned}
&[P_a, P_b] = J_{ab}, \qquad [P_{a'}, P_{b'}] = -J_{a'b'}, \qquad [P_{a''}, P_{b''}] = 0, \\
&[P_a, J_{bc}] = \eta_{ab}P_c - \eta_{ac}P_b, \qquad [P_{a'}, J_{b'c'}] = \eta_{a'b'}P_{c'} - \eta_{a'c'}P_{b'}, \\
&[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}, \\
&[J_{a'b'}, J_{c'd'}] = \eta_{b'c'}J_{a'd'} + \eta_{a'd'}J_{b'c'} - \eta_{a'c'}J_{b'd'} - \eta_{b'd'}J_{a'c'},
\end{aligned} \tag{A.1.5}$$

$$\begin{aligned}
&[Q, P_{\hat{a}}] = \frac{i}{2} Q\mathcal{E}\sigma_+\Gamma_{\hat{a}}, \qquad [Q, J_{\hat{a}\hat{b}}] = -\frac{1}{2} Q\Gamma_{\hat{a}\hat{b}} \text{ if } \hat{a}, \hat{b} \leq 5, \\
&\{Q_{\hat{\alpha}}, Q_{\hat{\beta}}\} = -2i(\hat{C}\Gamma^{\hat{a}}\pi_+)_{\hat{\alpha}\hat{\beta}}P_{\hat{a}} + \mathcal{E}[(\hat{C}\Gamma^{ab}\sigma_-)_{\hat{\alpha}\hat{\beta}}J_{ab} - (\hat{C}\Gamma^{a'b'}\sigma_-)_{\hat{\alpha}\hat{\beta}}J_{a'b'}]
\end{aligned} \tag{A.1.6}$$

A.2 $AdS_3 \times S^3$

The notation and algebra for just $AdS_3 \times S^3$ space (ie, assuming the T^4 is compactified to zero volume limit and ignoring it) is almost identical to the case of the more general $AdS_3 \times S^3 \times T^4$, except any terms with double primed indices are absent, thus any difficulties in treating T^4 are removed. The notable changes besides this are:

- $\hat{\alpha}, \hat{\beta}, \hat{\gamma} \in \{1, \ldots, 8\}$ are $D = 6$ complex chiral spinor indices.
- 8×8 Γ matrices of $AdS_3 \times S^3$ are:

$$\begin{aligned} \Gamma^a &= \gamma^a \otimes 1 \otimes \sigma^1, \quad \gamma^a: \quad \gamma^0 = i\sigma^3, \quad \gamma^1 = \sigma^1, \quad \gamma^2 = \sigma^2, \\ \Gamma^{a'} &= 1 \otimes \gamma^{a'} \otimes \sigma^2, \quad \gamma^{a'}: \quad \gamma^{3'} = \sigma^1, \quad \gamma^{4'} = \sigma^2, \quad \gamma^{5'} = \sigma^3. \end{aligned} \tag{A.2.1}$$

A.3 $AdS_5 \times S^5$

The third set of notation and algebra we use is for $AdS_5 \times S^5$. This is exactly the same notation as used in [91].

- $a, b, c \in \{0, \ldots, 4\}$ are indices of $SO(4,1)$.
- $a', b', c' \in \{5, \ldots, 9\}$ are indices of $SO(5)$.
- $\hat{a}, \hat{b}, \hat{c} \in \{0, \ldots, 9\}$ are a combination of (a, b, c) and (a', b', c').
- $\alpha, \beta, \gamma \in \{1, \ldots, 4\}$ are $SO(4.1)$ spinor indices.
- $\alpha', \beta', \gamma' \in \{1, \ldots, 4\}$ are $SO(5)$ spinor indices.
- $\hat{\alpha}, \hat{\beta}, \hat{\gamma} \in \{1, \ldots, 32\}$ are $D = 10$ Majorana-Weyl spinor indices.
- $I, J, K \in \{1, 2\}$, $\mathcal{N} = 2$ supersymmetry labels.
- P_a and $P_{a'}$ are the AdS_5 and S^5 translation operators. J_{ab} and $J_{a'b'}$ and are the $SO(4,1)$ and $SO(5)$ rotations respectively. They are defined to be antihermitian.
- 32×32 Γ matrices of $AdS_5 \times S^5$ are:

$$\begin{aligned} \Gamma^a &= \gamma^a \otimes 1_4 \otimes \sigma^1, \quad \gamma^a: \quad \gamma^0 = i\sigma^3 \otimes 1_2, \quad \gamma^{1,\ldots,3} = \sigma^2 \otimes \sigma^{1,\ldots,3}, \\ & \qquad\qquad \gamma^4 = \sigma^1 \otimes 1_2, \\ \Gamma^{a'} &= 1_4 \otimes \gamma^{a'} \otimes \sigma^2, \quad \gamma^{a'}: \quad \gamma^{5,\ldots,7} = \sigma^2 \otimes \sigma^{1,\ldots,3}, \quad \gamma^8 = \sigma^3 \otimes 1_2, \\ & \qquad\qquad \gamma^9 = \sigma^1 \otimes 1_2. \end{aligned} \tag{A.3.1}$$

- $\hat{C} = C \otimes C' \otimes i\sigma^2$ is the charge conjugation matrix, where C and C' are the charge conjugation matrices of $SO(4,1)$ and $SO(5)$.

- $\{\Gamma^{\hat{a}}, \Gamma^{\hat{b}}\} = 2\eta^{\hat{a}\hat{b}}$ where $\eta^{\hat{a}\hat{b}} = (-, +, \ldots, +)$.
- Supersymmetry generators $Q^{I\hat{\alpha}} = \begin{pmatrix} 0 \\ -Q^{I\alpha\alpha'} \end{pmatrix}$. In this notation $\theta_{I\hat{\alpha}}$ and $L_{I\hat{\alpha}}$ have the opposite chirality: $L^{I\hat{\alpha}} = \begin{pmatrix} L^{I\alpha\alpha'} \\ 0 \end{pmatrix}$.
- $Q^{1\hat{\alpha}}$ and $Q^{2\hat{\alpha}}$ can be combined as $Q = \begin{pmatrix} Q^1 \\ Q^2 \end{pmatrix}$.

Appendix B

Maurer-Cartan Equations and other Cartan Form Relations

The Maurer-Cartan Equations, the variations for the Cartan forms and their $\partial_t L$ equations (which will be explained below) for $AdS_3 \times S^3 \times T^4$, $AdS_3 \times S^3$ and $AdS_5 \times S^5$ can all be written out in the same form, as long as it is understood that the indices run over different ranges for each space, and that $L^{a''b''} = 0$ for $AdS_3 \times S^3 \times T^4$. The equations of course appear exactly the same as the corresponding equations in [91].

$$dL^{\hat{a}} = -L^{\hat{a}\hat{b}} \wedge L_{\hat{b}} - i\bar{L}\Gamma^{\hat{a}} \wedge L \tag{B.1}$$

$$dL = \frac{i}{2}\sigma_+\widehat{L} \wedge \mathcal{E}L - \frac{1}{4}L^{\hat{a}\hat{b}}\Gamma_{\hat{a}\hat{b}} \wedge L \tag{B.2}$$

$$dL^{ab} = -L^a \wedge L^b - L^{ac} \wedge L_c{}^b + \bar{L}\Gamma^{ab}\sigma_- \wedge \mathcal{E}L \tag{B.3}$$

$$dL^{a'b'} = L^{a'} \wedge L^{b'} - L^{a'c'} \wedge L_{c'}{}^{b'} - \bar{L}\Gamma^{a'b'}\sigma_- \wedge \mathcal{E}L \tag{B.4}$$

Even though the proof of these equations appears elsewhere [77, 90], it is still instructional to derive them. It shall be done here for $AdS_3 \times S^3$, but the extension to $AdS_3 \times S^3 \times T^4$ is simple.

Firstly the differential operator of the theory must be found [77]:

$$D = d + G^{-1}dG = d + L^a P_a + \frac{1}{2}L^{ab}J_{ab} + L^{a'}P_{a'} + \frac{1}{2}L^{a'b'}J_{a'b'} + L^{\alpha\alpha' I}Q_{\alpha\alpha' I} \,. \tag{B.5}$$

This differential operator has the property that $D^2 = 0$. Expanding this in terms of eqn (B.5) and using the supersymmetry algebra (A.1.6) gives:

$$\begin{aligned} D^2 = 0 \;\; = \;\; & dL^a P_a + dL^{a'}P_{a'} + \frac{1}{2}dL^{ab}J_{ab} + \frac{1}{2}dL^{a'b'}J_{a'b'} + dL^{\hat{\alpha}}Q_{\hat{\alpha}} \\ & + \frac{1}{2}L^a \wedge L^b[P_a, P_b] + \frac{1}{2}L^{a'} \wedge L^{b'}[P_{a'}, P_{b'}] + \frac{1}{8}L^{ab} \wedge L^{cd}[J_{ab}, J_{cd}] \\ & + \frac{1}{8}L^{a'b'} \wedge L^{c'd'}[J_{a'b'}, J_{c'd'}] - \frac{1}{2}L^{\hat{\alpha}} \wedge L^{\hat{\beta}}\{Q_{\hat{\alpha}}, Q_{\hat{\beta}}\} + \frac{1}{2}L^a \wedge L^{bc}[P_a, J_{bc}] \\ & + \frac{1}{2}L^{a'} \wedge L^{b'c'}[P_{a'}, J_{b'c'}] + L^{\hat{\alpha}} \wedge L^a[Q_{\hat{\alpha}}, P_a] + L^{\hat{\alpha}} \wedge L^{a'}[Q_{\hat{\alpha}}, P_{a'}] \\ & + \frac{1}{2}L^{\hat{\alpha}} \wedge L^{ab}[Q_{\hat{\alpha}}, J_{ab}] + \frac{1}{2}L^{\hat{\alpha}} \wedge L^{a'b'}[Q_{\hat{\alpha}}, J_{a'b'}] \,. \end{aligned} \tag{B.6}$$

When evaluating the momentum term of $L^{\hat\alpha} \wedge L^{\hat\beta}\{Q_{\hat\alpha}, Q_{\hat\beta}\}$, where the anticommutation is given in eqns (A.1.6):

$$-2iL^{\hat\alpha} \wedge L^{\hat\beta}(\hat{C}\Gamma^{\hat a}\pi_+)_{\hat\alpha\hat\beta}P_{\hat a} \ . \tag{B.7}$$

It must be remembered that for $AdS_3 \times S^3 \times T^4$ and $AdS_5 \times S^5$ the spinor L is Majorana-Weyl and for $AdS_3 \times S^3$ it is complex chiral, and thus in both cases:

$$L^{I\hat\alpha} = \begin{pmatrix} L^{I\alpha\alpha'} \\ 0 \end{pmatrix},$$

and thus $\pi_+ L^{\hat\beta} = L^{\hat\beta}$. And of course the charge conjugation $\hat{C}$ acts on L as $L^{\hat\alpha}\hat{C} = \bar{L}^{\hat\alpha}$. The rotational generator terms are evaluated similarly. Therefore D^2 is:

$$\begin{aligned} 0 = D^2 \ &= \ (dL^a + L^{ab} \wedge L_b + i\bar{L}\Gamma^a \wedge L)P_a + (dL^{a'} + L^{a'b'} \wedge L_{b'} + i\bar{L}\Gamma^{a'} \wedge L)P_{a'} \\ &\quad +(dL^{ab} + L^a \wedge L^b + L^{ac} \wedge L_c{}^b - \bar{L}\Gamma^{ab}\sigma_- \wedge \mathcal{E}L)J_{ab} \\ &\quad +(dL^{a'b'} - L^{a'} \wedge L^{b'} + L^{a'c'} \wedge L_{c'}{}^{b'} + \bar{L}\Gamma^{a'b'}\sigma_- \wedge \mathcal{E}L)J_{a'b'} \\ &\quad +\bar{Q}(\frac{i}{2}\sigma_+\widehat{L} \wedge \mathcal{E}L - \frac{1}{4}L^{\hat a\hat b}\Gamma_{\hat a\hat b} \wedge L). \end{aligned} \tag{B.8}$$

The Maurer-Cartan equations follow directly from this.

The variations of the Cartan forms are [77,90]:

$$\delta L^{\hat a} = d\delta x^{\hat a} + L^{\hat a\hat b}\delta x_{\hat b} - \delta x^{\hat a\hat b}L_{\hat b} + 2i\bar{L}\Gamma^{\hat a}\delta\theta, \tag{B.9}$$

$$\delta L = d\delta\theta - \frac{i}{2}\sigma_+\widehat{L}\mathcal{E}\delta\theta + \frac{1}{4}L^{\hat a\hat b}\Gamma_{\hat a\hat b}\delta\theta + \frac{i}{2}\sigma_+\delta x^{\hat a}\Gamma_{\hat a}\mathcal{E}L - \frac{1}{4}\delta x^{\hat a\hat b}\Gamma_{\hat a\hat b}L, \tag{B.10}$$

$$\delta x^{\hat a} \equiv \delta X^M L^{\hat a}_M, \quad \delta x^{\hat a\hat b} \equiv \delta X^M L^{\hat a\hat b}_M, \quad \delta\theta \equiv \delta X^M L^{\hat a\hat b}_M. \tag{B.11}$$

As an example of how these variations of the Cartan forms are derived, let us look at eqn (B.9).

$$\begin{aligned} \delta L^{\hat a} \ &= \ \delta(dX^M L^{\hat a}_M) && \text{(B.12)} \\ &= \ \delta(dX^M)L^{\hat a}_M + dX^M\delta X^N\frac{\delta L^{\hat a}_M}{\delta X^N} && \text{(B.13)} \\ &= \ d(\delta X^M L^{\hat a}_M) + dX^M\delta X^N\frac{\delta L^{\hat a}_M}{\delta X^N} && \text{(B.14)} \\ &= \ d\delta x^{\hat a} + dX^M\delta X^N\frac{\delta L^{\hat a}_M}{\delta X^N}, && \text{(B.15)} \end{aligned}$$

where we have used $dL^{\hat a}_M = 0$.

Eqn (B.1) can be re-expressed as:

$$dL^{\hat a} = dX^M \wedge dX^N\frac{\partial L^{\hat a}_N}{\partial X^M} = dX^M \wedge dX^N(-L^{\hat a\hat b}_M L_{N\hat b} - i\bar{L}_M\Gamma^{\hat a}L_N). \tag{B.16}$$

Using this, $dX^N\delta X^M\frac{\delta L^{\hat a}_N}{\delta X^M}$ is found to be:

$$\begin{aligned} dX^N\delta X^M\frac{\delta L^{\hat a}_N}{\delta X^M} \ &= \ -\delta X^M dX^N L^{\hat a\hat b}_M L_{N\hat b} + dX^N\delta X^M L^{\hat a\hat b}_N L_{M\hat b} - i\delta X^M dX^N \bar{L}_M\Gamma^{\hat a}L_N \\ &\quad +idX^N\delta X^M\bar{L}_N\Gamma^{\hat a}L_M \\ &= \ -\delta x^{\hat a\hat b}L_{\hat b} + L^{\hat a\hat b}\delta x_{\hat b} + 2i\bar{L}\Gamma^{\hat a}\delta\theta \ . \end{aligned} \tag{B.17}$$

Thus eqn (B.1) is arrived at.

Rescale $\theta \to t\theta$ to get the L_t Cartan forms.

$$L_t^{\hat{a}}(x,\theta) \equiv L^{\hat{a}}(x,t\theta), \quad L_t^{\hat{a}\hat{b}}(x,\theta) \equiv L^{\hat{a}\hat{b}}(x,t\theta), \quad L_t(x,\theta) \equiv L(x,t\theta). \tag{B.18}$$

The initial conditions of these are:

$$L_{t=0}^{\hat{a}} = e^{\hat{a}}, \qquad L_{t=0}^{\hat{a}\hat{b}} = \omega^{\hat{a}\hat{b}}, \qquad L_{t=0} = 0. \tag{B.19}$$

It can be seen from eqn (2.11.9):

$$\partial_t F_t = 2i\bar{\theta}\widehat{L}_t \wedge \mathcal{K} L_t. \tag{B.20}$$

The defining equations for the Cartan forms can be shown to be [90]:

$$\partial_t L_t^{\hat{a}} = -2i\bar{\theta}\Gamma^{\hat{a}} L_t, \tag{B.21}$$
$$\partial_t L_t = d\theta - \frac{i}{2}\sigma_+ \widehat{L}\mathcal{E}\theta + \frac{1}{4}\Gamma_{\hat{a}\hat{b}} L_t^{\hat{a}\hat{b}}\theta, \tag{B.22}$$
$$\partial_t L_t^{ab} = 2\bar{\theta}\mathcal{E}\Gamma^{ab}\sigma_- L_t, \qquad \partial_t L_t^{a'b'} = -2\bar{\theta}\mathcal{E}\Gamma^{a'b'}\sigma_- L_t. \tag{B.23}$$

where $L_t^A(x,\theta) = L^A(x,t\theta)$. The method for proving these is outlined in [77], and is to scale the group element G by t, ie $G_t = g(x)e^{t\theta^{\hat{\alpha}}Q_{\hat{\alpha}}}$ and to take the t derivative of the Cartan 1 forms:

$$\partial_t(G_t^{-1}dG_t) = \partial_t L^{\hat{a}} P_{\hat{a}} + \partial_t L^{\hat{a}\hat{b}} J_{\hat{a}\hat{b}} + \partial_t L^{\hat{\alpha}} Q_{\hat{\alpha}} \ . \tag{B.24}$$

By separating the Cartan forms into $L_t^A = L_0^A + \tilde{L}_0^A$ where:

$$\begin{aligned} g^{-1}dg &= L_0^{\hat{a}} P_{\hat{a}} + L_0^{\hat{a}\hat{b}} J_{\hat{a}\hat{b}} \ , && \text{(B.25)}\\ e^{-t\bar{\theta}.Q} g^{-1} dg e^{t\bar{\theta}.Q} &= g^{-1}dg + L_0^{\hat{\alpha}} Q_{\hat{\alpha}} \ , && \text{(B.26)}\\ (e^{-t\bar{\theta}.Q} de^{-t\bar{\theta}.Q}) &= \tilde{L}_t^{\hat{a}} P_{\hat{a}} + \tilde{L}_t^{\hat{a}\hat{b}} J_{\hat{a}\hat{b}} + \tilde{L}_t^{\hat{\alpha}} Q_{\hat{\alpha}} \ , && \text{(B.27)} \end{aligned}$$

$\partial_t(G_t^{-1}dG_t)$ becomes:

$$\begin{aligned} \partial_t(G_t^{-1}dG_t) &= \partial_t(e^{-t\bar{\theta}.Q}(L_0^{\hat{a}} P_{\hat{a}} + L_0^{\hat{a}\hat{b}} J_{\hat{a}\hat{b}})e^{-t\bar{\theta}.Q}) + \partial_t(e^{-t\bar{\theta}.Q} de^{-t\bar{\theta}.Q}) \\ &= [e^{-t\bar{\theta}.Q}(L_0^{\hat{a}} P_{\hat{a}} + L_0^{\hat{a}\hat{b}} J_{\hat{a}\hat{b}})e^{-t\bar{\theta}.Q}, \bar{\theta}.Q] + d\bar{\theta}.Q \\ &\quad + [\tilde{L}_t^{\hat{a}} P_{\hat{a}} + \tilde{L}_t^{\hat{a}\hat{b}} J_{\hat{a}\hat{b}} + \tilde{L}_t^{\hat{\alpha}} Q_{\hat{\alpha}}, \bar{\theta}.Q] \\ &= [L_0^{\hat{a}} P_{\hat{a}} + L_0^{\hat{a}\hat{b}} J_{\hat{a}\hat{b}}, \bar{\theta}.Q] + d\bar{\theta}.Q + [\tilde{L}_t^{\hat{a}} P_{\hat{a}} + \tilde{L}_t^{\hat{a}\hat{b}} J_{\hat{a}\hat{b}} + \tilde{L}_t^{\hat{\alpha}} Q_{\hat{\alpha}}, \bar{\theta}.Q] \\ &= d\bar{\theta}.Q - 2i\bar{\theta}\Gamma^{\hat{a}} L_t P_{\hat{a}} - \frac{i}{2}\bar{Q}\sigma_+ \widehat{L}\mathcal{E}\theta + \frac{1}{4}\bar{Q}\Gamma_{\hat{a}\hat{b}} L_t^{\hat{a}\hat{b}}\theta \\ &\quad + 2\bar{\theta}\mathcal{E}\Gamma^{ab}\sigma_- L_t J_{ab} - 2\bar{\theta}\mathcal{E}\Gamma^{a'b'}\sigma_- L_t J_{a'b'} \ . \end{aligned} \tag{B.28}$$

from which the defining equations for the Cartan forms can be extracted.

The solutions to equations (B.21), (B.22) and (B.23) are [77,90]:

$$
\begin{aligned}
L &= V(\theta)D\theta, && \text{(B.29)}\\
L^{\hat{a}} &= e^{\hat{a}} - 2i\bar{\theta}\Gamma^{\hat{a}}W(\theta)D\theta, && \text{(B.30)}\\
L^{ab} &= w^{ab} + 2\bar{\theta}\mathcal{E}\Gamma^{ab}\sigma_- W(\theta)D\theta, && \text{(B.31)}\\
L^{a'b'} &= w^{a'b'} - 2\bar{\theta}\mathcal{E}\Gamma^{a'b'}\sigma_- W(\theta)D\theta, && \text{(B.32)}\\
V(m(\theta)) &= \frac{\sinh\sqrt{m}}{\sqrt{m}}, &&\\
W(m(\theta)) &= \frac{\cosh\sqrt{m}-1}{m}, &&\\
m(\theta) &= -\theta_+\Gamma^{\hat{a}}\mathcal{E}\theta\bar{\theta}\Gamma^{\hat{a}} + \frac{1}{2}\Gamma^{ab}\theta\bar{\theta}\mathcal{E}\Gamma^{ab}\sigma_- - \frac{1}{2}\Gamma^{a'b'}\theta\bar{\theta}\mathcal{E}\Gamma^{a'b'}\sigma_- \ , &&
\end{aligned}
$$

and for $AdS_3 \times S^3 \times U(1)^4$, $L^{a''b''} = 0$. These solutions express the Cartan forms, and thus the actions found in this thesis, in terms of the background supercoordinate θ and the bosonic Cartan forms $e^{\hat{a}}$ and $w^{\hat{a}\hat{b}}$, which are the Cartan forms for the theory when the fermionic coordinates are set to zero, and are purely functions of $x^{\hat{a}}$, the background bosonic coordinates.

Using the Killing gauge the Cartan forms are simplified in [76,102].

Appendix C

Derivation of the Fierz Identities

In this Appendix Fierz identity (2.13.9) is proven.

This is done using the Fierz identity (2.13.8):

$$(\Gamma^{\hat{a}\hat{b}\hat{c}}\mathcal{E})_{(\hat{\alpha}\hat{\beta}}(\Gamma_{\hat{c}})_{\hat{\gamma}\hat{\delta})} - 2(\Gamma^{[\hat{a}}\mathcal{E})_{(\hat{\alpha}\hat{\beta}}(\Gamma^{\hat{b}]}\mathcal{K})_{\hat{\gamma}\hat{\delta})} = 0, \tag{C.1}$$

and applying this Fierz identity to the situation:

$$\begin{aligned} 0 = &((\Gamma^{\hat{a}\hat{b}\hat{c}}\mathcal{E})_{(\hat{\alpha}\hat{\beta}}(\Gamma_{\hat{c}})_{\hat{\gamma}\hat{\delta})} - 2(\Gamma^{[\hat{a}}\mathcal{E})_{(\hat{\alpha}\hat{\beta}}(\Gamma^{\hat{b}]}\mathcal{K})_{\hat{\gamma}\hat{\delta})})\times \\ &((-A^\alpha(\mathcal{K}\Gamma^{de}B)^\beta C^\gamma D^\delta + B^\alpha(\mathcal{K}\Gamma^{de}A)^\beta C^\gamma D^\delta - B^\alpha(\mathcal{K}\Gamma^{de}C)^\beta A^\gamma D^\delta \\ &-B^\alpha(\mathcal{K}\Gamma^{de}D)^\beta C^\gamma A^\delta) \\ &+((\mathcal{K}\Gamma^{de}A)^\alpha B^\beta C^\gamma D^\delta - (\mathcal{K}\Gamma^{de}B)^\alpha A^\beta C^\gamma D^\delta - (\mathcal{K}\Gamma^{de}C)^\alpha B^\beta A^\gamma D^\delta \\ &-(\mathcal{K}\Gamma^{de}D)^\alpha B^\beta C^\gamma A^\delta)), \end{aligned} \tag{C.2}$$

where A, B, C and D are spinors and thus are Grassmannian, and of course a spinor appearing on the left of (2.13.8), for example A in:

$$(\bar{A}\mathcal{K}\Gamma^{\hat{a}\hat{b}\hat{c}}\mathcal{E})_{(\hat{\alpha}\hat{\beta}}(\Gamma_{\hat{c}})_{\hat{\gamma}\hat{\delta})}B,$$

is really $\bar{A} = A^\dagger\Gamma^0$.

Consider the terms:

$$-((\Gamma^{\hat{a}\hat{b}\hat{c}}\mathcal{E})_{(\hat{\alpha}\hat{\beta}}(\Gamma_{\hat{c}})_{\hat{\gamma}\hat{\delta})} - 2(\Gamma^{[\hat{a}}\mathcal{E})_{(\hat{\alpha}\hat{\beta}}(\Gamma^{\hat{b}]}\mathcal{K})_{\hat{\gamma}\hat{\delta})})A^\alpha(\mathcal{K}\Gamma^{de}B)^\beta C^\gamma D^\delta = 0. \tag{C.3}$$

The other terms are arrived at through a relabelling of the spinors.

This equation can be shown to be equal to:

$$\begin{aligned} 0 \;=\; &-\varepsilon^{abde}[(\bar{A}\Gamma^{abc}\Gamma^{de}\mathcal{E}\mathcal{K}B)(\bar{C}\Gamma_c D) + (\bar{C}\Gamma^{abc}\Gamma^{de}\mathcal{E}\mathcal{K}B)(\bar{D}\Gamma_c A) \\ &+(\bar{D}\Gamma^{abc}\Gamma^{de}\mathcal{E}\mathcal{K}B)(\bar{A}\Gamma_c C) + (\bar{D}\Gamma_c\Gamma^{de}\mathcal{K}B)(\bar{B}\Gamma^{abc}\mathcal{E}C) \\ &+(\bar{C}\Gamma_c\Gamma^{de}\mathcal{K}B)(\bar{D}\Gamma^{abc}\mathcal{E}B) + (\bar{A}\Gamma_c\Gamma^{de}\mathcal{K}B)(\bar{C}\Gamma^{abc}\mathcal{E}D) \\ &-2[(\bar{A}\Gamma^{[a}\Gamma^{|de|}\mathcal{J}\mathcal{K}B)(\bar{C}\Gamma^{b]}\mathcal{K}D) + (\bar{C}\Gamma^{[a}\Gamma^{|de|}\mathcal{J}\mathcal{K}B)(\bar{D}\Gamma^{b]}\mathcal{K}A) \\ &+(\bar{D}\Gamma^{[a}\Gamma^{|de|}\mathcal{J}\mathcal{K}B)(\bar{A}\Gamma^{b]}\mathcal{K}C)] \\ &-2[(\bar{A}\Gamma^{[a}\mathcal{J}C)(\bar{D}\Gamma^{b]}\Gamma^{de}\mathcal{K}^2B) + (\bar{D}\Gamma^{[a}\mathcal{J}A)(\bar{C}\Gamma^{b]}\Gamma^{de}\mathcal{K}^2B) \\ &+(\bar{C}\Gamma^{[a}\mathcal{J}D)(\bar{A}\Gamma^{b]}\Gamma^{de}\mathcal{K}^2B)]]. \end{aligned} \tag{C.4}$$

When deriving this it is necessary to remember that $\mathcal{E}^\dagger = -\mathcal{E}$, $\mathcal{K}^\dagger = -\mathcal{K}$, $\mathcal{J}^\dagger = \mathcal{J}$.

Using this eqn (C.3) becomes:

$$\begin{aligned}
\varepsilon^{abde}[\quad & (\Gamma^{abc}\Gamma^{de}\mathcal{J})_{(\alpha\beta}(\Gamma_c)_{\gamma\delta)} + (\Gamma^{de}\Gamma^{abc}\mathcal{J})_{(\alpha\beta}(\Gamma_c)_{\gamma\delta)} \\
- \quad & (\Gamma_c\Gamma^{de}\mathcal{K})_{(\alpha\beta}(\Gamma^{abc}\mathcal{E})_{\gamma\delta)} + (\Gamma^{de}\Gamma_c\mathcal{K})_{(\alpha\beta}(\Gamma^{abc}\mathcal{E})_{\gamma\delta)} \\
- \quad & 2(\Gamma^{[a}\Gamma^{|de|}\mathcal{E})_{(\alpha\beta}(\Gamma^{b]}\mathcal{K})_{\gamma\delta)} - 2(\Gamma^{de}\Gamma^{[a}\mathcal{E})_{(\alpha\beta}(\Gamma^{b]}\mathcal{K})_{\gamma\delta)} \\
+ \quad & 2(\Gamma^{[a}\mathcal{J})_{(\alpha\beta}(\Gamma^{b]}\Gamma^{de})_{\gamma\delta)} - 2(\Gamma^{[a}\mathcal{J})_{(\alpha\beta}(\Gamma^{|de|}\Gamma^{b]})_{\gamma\delta)}]A^\alpha B^\beta C^\gamma D^\delta = 0. \quad \text{(C.5)}
\end{aligned}$$

Next, merge the Γ^{de}'s into antisymmetric combinations with the other Γ's present (made easier by the antisymmetry imposed by ε^{abde}), and after some simplification the Fierz identity is arrived at:

$$\varepsilon^{abde}((\Gamma^{abcde}\mathcal{J})_{\alpha\beta}(\Gamma_c)_{\gamma\delta} - 4(\Gamma^{[abd}\mathcal{E})_{(\alpha\beta}(\Gamma^{e]}\mathcal{K})_{\gamma\delta)})A^\alpha B^\beta C^\gamma D^\delta = 0. \qquad \text{(C.6)}$$

Appendix D

Calculations for D-5 Brane in $AdS_3 \times S^3$

This appendix gives a brief summary of the extensive calculations required to find the WZ term of the D5-brane action in $AdS_3 \times S^3$.

From [27] and §2.6 it can be seen that the S_{WZ} for a D6-brane in $AdS_3 \times S^3$ is of the form:

$$\delta_\kappa S_{WZ} = \int_{M_6} \imath_\kappa (e^F \wedge R). \tag{D.1}$$

The integration is over a 6 form, such that $\imath_\kappa$ must be acting on a 7 form. This notation thus really implies:

$$\delta_\kappa S_{WZ} = \int_{M_6} \imath_\kappa (\sum_{j=0}^{3} \frac{1}{j!} F^j \wedge R_{(7-2j)}), \tag{D.2}$$

where $R_{(n)}$ is an n form (eqn (2.6.17)):

$$\begin{aligned} R_{(n)} &= -\frac{d\sigma^\alpha \wedge \sigma^\beta \wedge d\sigma^{i_{n-2}} \wedge \ldots \wedge d\sigma^{i_1}}{2(n-2)!} R_{[i_1 \ldots i_{n-2}\alpha\beta]} && \text{(D.3)} \\ &= \frac{d\sigma^\alpha}{2} \wedge (R_{[i_1 \ldots i_{n-2}\alpha\beta]} \frac{d\sigma^{i_{n-2}} \wedge \ldots \wedge d\sigma^{i_1}}{2(n-2)!} d\sigma^\beta (-1)^n && \text{(D.4)} \\ &= (-1)^{\frac{(n-1)(n-2)}{2}} i\bar{L} \wedge (\frac{\widehat{L}^{n-2}}{(n-2)!} \mathcal{K}^{\frac{n-1}{2}} \mathcal{E}) \wedge L). && \text{(D.5)} \end{aligned}$$

Thus $\delta_\kappa S_{WZ}$ is:

$$\begin{aligned} \delta_\kappa S_{WZ} &= -i \int_{M_6} \imath_\kappa \left(\frac{(\bar{L} \wedge (\widehat{L})^5 \wedge \mathcal{J} L)}{5!} + \frac{(\bar{L} \wedge (\widehat{L})^3 \wedge \mathcal{E} L) \wedge F}{3!} \right. && \text{(D.6)} \\ & \left. + \frac{(\bar{L} \wedge \widehat{L} \wedge \mathcal{J} L) \wedge F \wedge F}{2} \right) + Y, \end{aligned}$$

for some as yet unknown Y.

Now consider that S_{WZ} for a Dp-brane can be expressed as an integral over either a $p+1$ or a $p+2$ dimensional manifold, for example eqns (2.12.16) and (2.12.17) for the D1-brane case. For the D5-brane the $p+1$ dimensional manifold action is [27] (and see §2.6):

$$S_{WZ} = \int_{M_6} (e^F \wedge C). \tag{D.7}$$

For the second form, the formula is [27,91]:

$$S_{WZ} = \int_{M_7} d(e^F \wedge C) = \int_{M_7} e^F \wedge R + \dots \tag{D.8}$$

The extra terms on the right must have zero κ invariance to obey eqn (D.1).

For a D5-brane in $AdS_3 \times S^3$, the second form of S_{WZ} is not possible due to there being only 6 spacetime degrees of freedom, such that a seven form in the the background spacetime coordinates must be zero due to the 7 forms anticommuting indices (which can only take 6 values). This can be seen by studying a 7-form Z that has been pulled back to the bosonic components of spacetime in $AdS_3 \times S^3$ in which the manifold M_7 is embedded.

$$Z = dX^{\hat{a}_1} \wedge \dots \wedge dX^{\hat{a}_7} Z_{[\hat{a}_1 \dots \hat{a}_7]} = 0 \quad \text{for } \hat{a}_i \in \{1, \dots, 6\} \tag{D.9}$$

S_{WZ} must be found in terms of eqn (D.7). This can be done using the same techniques as for the D1-brane.

Let X_7 be the 7 form:

$$\begin{aligned} X_7 &= e^F \wedge R \\ &= -i\left(\frac{(\bar{L} \wedge (\widehat{L})^5 \wedge \mathcal{J} L)}{5!} + \frac{(\bar{L} \wedge (\widehat{L})^3 \wedge \mathcal{E} L) \wedge F}{3!} + \frac{(\bar{L} \wedge \widehat{L} \wedge \mathcal{J} L) \wedge F \wedge F}{2}\right). \end{aligned} \tag{D.10}$$

($X_{7\ t=0} = 0$) which is equivalent to:

$$X_7 = \int_0^1 dt\ \partial_t X_{7\ t} + X_{7\ t=0}\ . \tag{D.11}$$

The objective is to get $\partial_t X_{7\ t}$ in terms of a total exterior derivative which can thus be used to reduce the integration from an M_7 manifold to an M_6 manifold via Stokes' Theorem.

Carrying ∂_t into $\partial_t X_{7\ t}$ and simplifying gives :

$$\begin{aligned} \partial_t X_{7\ t} &= i(\frac{2}{5!}(\partial_t \bar{L}_t) \wedge \widehat{L}_t^5 \wedge \mathcal{J} L_t + \frac{1}{4!}\bar{L}_t \wedge (\partial_t \widehat{L}_t) \wedge \widehat{L}_t^4 \wedge \mathcal{J} L_t \\ &\quad -\frac{1}{3!}\bar{L}_t \wedge (\partial_t L_t^{\hat{a}}) \wedge L_{t\ \hat{a}} \wedge \widehat{L}_t^3 \wedge \mathcal{J} L_t + \frac{2}{3!}\partial_t \bar{L}_t \wedge \widehat{L}_t^3 \mathcal{E} \wedge L_t \wedge F_t \\ &\quad +\frac{1}{2}\bar{L}_t \wedge (\partial_t \widehat{L}_t) \wedge \widehat{L}_t^2 \wedge \mathcal{E} L_t \wedge F_t - \bar{L}_t \wedge \partial_t L_t^{\hat{a}} \wedge L_{t\hat{a}} \wedge \widehat{L}_t \wedge \mathcal{E} L_t \wedge F_t \\ &\quad +\frac{1}{3!}\bar{L}_t \wedge \widehat{L}_t^3 \mathcal{E} \wedge L_t \wedge \partial_t F_t + (\partial_t \bar{L}_t) \wedge \widehat{L}_t \wedge \mathcal{J} L_t \wedge F_t^2 \\ &\quad +\frac{1}{2}\bar{L}_t \wedge (\partial_t \widehat{L}_t) \wedge \mathcal{J} L_t \wedge F_t^2 + \bar{L}_t \wedge \widehat{L}_t \wedge \mathcal{J} L_t \wedge F_t \wedge \partial_t F_t). \end{aligned} \tag{D.12}$$

From analogy with the calculations for the D1-brane in $AdS_3 \times S^3$ (see eqns (2.12.18)) some terms can be simplified:

$$\begin{aligned}
\frac{i}{2}(2(\partial_t \bar{L}_t) \wedge \widehat{L}_t \wedge \mathcal{J} L_t + \bar{L}_t \wedge (\partial_t \widehat{L}_t) \wedge \mathcal{J} L_t) \wedge F_t^2 &= \frac{i}{2}\partial_t(\bar{L}_t \wedge \widehat{L}_t \wedge \mathcal{J} L_t) \wedge F_t^2 \\
&= id(\bar{\theta} \wedge \widehat{L}_t \wedge \mathcal{J} L_t) \wedge F_t^2 .
\end{aligned}$$

The analysis in [91] for the D3-brane in $AdS_5 \times S^5$ is very similar to the analysis done here, and some of the analysis done in [91] for the S_{WZ} of the D3-brane can be borrowed directly. In particular, using the $AdS_5 \times S^5$ versions of eqns (2.11.11), (2.13.8), (B.21), (B.22), (B.1) and (B.2), the authors of [91] found that:

$$\begin{aligned}
i\partial_t(\frac{\bar{L}_t \wedge \widehat{L}_t^3 \wedge \mathcal{E} L_t}{3!} + \bar{L}_t \wedge \widehat{L}_t \wedge \mathcal{J} L_t \wedge F_t) &= 2i\ d(\frac{\bar{\theta}\widehat{L}_t^3 \wedge \mathcal{E} L_t}{3!} + \bar{\theta}\widehat{L}_t \wedge \mathcal{J} L_t \wedge F_t) \\
&-\frac{1}{3}\bar{\theta}\mathcal{E}\widehat{L}_t\sigma_+ \wedge (L_t^a \wedge L_t^b\Gamma_{ab} + L_t^{a'} \wedge L_t^{b'}\Gamma_{a'b'}) \wedge \widehat{L}_t \wedge \mathcal{E} L_t
\end{aligned} \tag{D.13}$$

Substituting this into $\partial_t X_{7\ t}$, and expanding using eqns (B.21), (B.22) and (B.20) yields:

$$\begin{aligned}
\partial_t X_{7\ t} = & -\frac{2i}{5!}(d\bar{\theta} \wedge \widehat{L}_t^5 \wedge \mathcal{J} L_t) + \frac{1}{5!}(\bar{\theta}\mathcal{E}\widehat{L}_t\sigma_+ \wedge \widehat{L}_t^5 \wedge \mathcal{J} L_t) \\
& + \frac{i}{2\ 5!}(\bar{\theta} L_t^{\hat{a}\hat{b}}\Gamma_{\hat{a}\hat{b}} \wedge \widehat{L}_t^5 \wedge \mathcal{J} L_t) + \frac{2}{4!}(\bar{\theta}\Gamma^{\hat{a}} L_t) \wedge (\bar{L}_t\Gamma_{\hat{a}} \wedge \widehat{L}_t^4 \wedge \mathcal{J} L_t) \\
& - \frac{2}{3!}(\bar{\theta}\widehat{L}_t \wedge L_t) \wedge (\bar{L}_t \wedge \widehat{L}_t^3 \wedge \mathcal{J} L_t) + \frac{2}{3!}(\bar{\theta} \wedge \widehat{L}_t \wedge \mathcal{K} L_t) \wedge (\bar{L}_t \wedge \widehat{L}_t^3 \wedge \mathcal{E} L_t) \\
& - \frac{2i}{3!}d(\bar{\theta}\widehat{L}_t^3 \wedge \mathcal{E} L_t) \wedge F_t - id((\bar{\theta}\widehat{L}_t \wedge \mathcal{J} L_t) \wedge F_t^2) \\
& + \frac{1}{3}(\bar{\theta}\mathcal{E}\widehat{L}_t\sigma_+ \wedge (L_t^a \wedge L_t^b\Gamma_{ab} + L_t^{a'} \wedge L_t^{b'}\Gamma_{a'b'}) \wedge \widehat{L}_t \wedge \mathcal{E} L_t) \wedge F_t .
\end{aligned} \tag{D.14}$$

To simplify this further, the following relations are needed. Firstly:

$$\sigma_+ \widehat{L}_t^4 = \widehat{L}_t^4 \sigma_+ + 8\sigma_+(L_t^a\Gamma_a \wedge L_t^{b'}\Gamma_{b'} \wedge L_t^{c'}\Gamma_{c'} \wedge L_t^{d'}\Gamma_{d'} + L_t^a\Gamma_a \wedge L_t^b\Gamma_b \wedge L_t^c\Gamma_c \wedge L_t^{d'}\Gamma_{d'}), \tag{D.15}$$

which is found using (A.2.1), secondly:

$$\begin{aligned}
& \frac{i}{4!}(\bar{\theta}\Gamma^{\hat{a}} L_t) \wedge (\bar{L}_t\Gamma_{\hat{a}} \wedge \widehat{L}_t^4) + \frac{2i}{3}(\bar{\theta}\widehat{L}_t \wedge \mathcal{K} L_t) \wedge (\bar{L}_t \wedge \widehat{L}_t^3 \wedge \mathcal{E} L_t) \\
& - \frac{i}{3!}(\bar{\theta}\widehat{L}_t \wedge L_t) \wedge (\bar{L}_t \wedge \widehat{L}_t^3 \wedge \mathcal{J} L_t) + \frac{i}{3!}(\bar{\theta}\widehat{L}_t^3 \wedge \mathcal{E} L_t) \wedge (\bar{L}_t \wedge \widehat{L}_t \wedge \mathcal{K} L_t) = 0,
\end{aligned} \tag{D.16}$$

which is arrived at through the Fierz identity (2.13.9), and thirdly eqns (B.1) and (B.2) are needed to express $\partial_t X_{7\ t}$ in terms of exterior derivatives. Using this information,

$\partial_t X_{7\ t}$ is found to be:

$$\begin{aligned}
\partial_t X_{7\ t} &= \frac{2i}{5!}(-d\bar{\theta}\widehat{L}_t^5 \wedge \mathcal{J} L_t - \bar{\theta} d(\widehat{L}_t^5) \wedge \mathcal{J} L_t + \bar{\theta}\widehat{L}_t^5 \wedge \mathcal{J} dL_t) - \frac{2i}{3!}(d(\bar{\theta}\widehat{L}_t^3 \wedge \mathcal{E} L_t) \wedge F_t, \\
&\quad +(\bar{\theta}\widehat{L}_t^3 \wedge \mathcal{E} L_t) \wedge dF_t) - 2id(\frac{1}{2}(\bar{\theta}\widehat{L}_t \wedge \mathcal{J} L_t) \wedge F_t^2) + Y \qquad (D.17) \\
&= -2id(\frac{\bar{\theta}\widehat{L}_t^5 \wedge \mathcal{J} L_t}{5!} + \frac{(\bar{\theta}\widehat{L}_t^3 \wedge \mathcal{E} L_t) \wedge F_t}{3!} + \frac{(\bar{\theta}\widehat{L}_t \wedge \mathcal{J} L_t) \wedge F_t^2}{2}) + Y' \qquad (D.18) \\
Y' &= \frac{8}{5!}(\bar{\theta}\mathcal{E}\widehat{L}_t \sigma_+ \wedge (L_t^a \Gamma_a \wedge L_t^{b'} \Gamma_{b'} \wedge L_t^{c'} \Gamma_{c'} \wedge L_t^{d'} \Gamma_{d'} \\
&\quad + L_t^{a'} \Gamma_{a'} \wedge L_t^b \Gamma_b \wedge L_t^c \Gamma_c \wedge L_t^d \Gamma_d) \wedge \widehat{L}_t \wedge \mathcal{J} L_t) \\
&\quad + \frac{1}{3}(\bar{\theta}\mathcal{E}\widehat{L}_t \sigma_+ (L_t^a \wedge L_t^b \Gamma_{ab} + L_t^{a'} \wedge L_t^{b'} \Gamma_{a'b'}) \wedge \widehat{L}_t \wedge \mathcal{E} L_t) \wedge F_t\ . \qquad (D.19)
\end{aligned}$$

Now that $\partial_t X_{7\ t}$ has been found in this form, it should be obvious the Stokes' Theorem and equations (D.11) and (D.8) can be used to put $\delta_\kappa S_{WZ}$ into the form of eqns (2.13.5) and (2.13.6). As previously explained in the main text, Y in eqn (2.13.6) is equal to zero due to the antisymmetry of its indices.

Thus from eqn (D.7), S_{WZ} is equivalent to:

$$S_{WZ} = -2i \int_{M_6} \int_0^1 dt \left(\frac{(\bar{\theta}(\widehat{L})^5 \wedge \mathcal{J} L)}{5!} + \frac{(\bar{\theta}(\widehat{L})^3 \wedge \mathcal{E} L) \wedge F}{3!} + \frac{(\bar{\theta}\widehat{L} \wedge \mathcal{J} L) \wedge F \wedge F}{2} \right). \qquad (D.20)$$

It is possible to check that the κ variation of this does indeed give eqn (2.13.4) for $Y = 0$.

Appendix E

Killing Gauge Fixed Actions

In this appendix the Killing gauge of [74,76] is applied to the D1-brane in $AdS_3 \times S^3$, and the D5-brane actions in both $AdS_3 \times S^3$ and $AdS_5 \times S^5$. See §2.8 for more details.

E.1 D1-brane in $AdS_3 \times S^3$

The Killing gauge [76,108] is defined by

$$\mathcal{P}_{\pm}^{IJ} = \frac{1}{2}(\delta^{IJ} \pm \Gamma_* \epsilon^{IJ}), \tag{E.1.1}$$

$$\mathcal{P}_{-}\theta = 0, \tag{E.1.2}$$

where $\Gamma_* = i\Gamma_{01}$ in $AdS_3 \times S^3$. This gauge consistently breaks the κ symmetry of the D-brane probe actions in this background.

The most appropriate Killing gauge will alter for different backgrounds. It must be remembered that the curved spacetimes studied here are being generated by other D-branes lying in the coordinates $\{0, \dots, p\}$.

From §2.8, Γ_* for other $AdS \times S$ spaces, is given by $\Gamma_* = i^n \Gamma_{0\dots p}$. It is known (§2.8) that Γ_* must be antihermitian and anticommute with Γ_0 (where p is the number of spatial dimensions of the branes whose gravity generates the spacetime background).

From:

$$\Gamma_{0\dots p}^{\dagger} = -\Gamma_{p\dots 0} = (-1)^{1+\lfloor \frac{p+1}{2} \rfloor} \Gamma_{0\dots p} \; . \tag{E.1.3}$$

It is seen that for Γ_* to be antihermitian, $n = \lfloor \frac{p+1}{2} \rfloor \mod 2$.

Finally, we are applying this to IIB string theory, which only contains Dp-branes for odd p, and it is easy to show: $\{\Gamma_0, \Gamma_{0\dots p}\} = 0$ for odd p.

Therefore:

$$\Gamma_* = i^{(\lfloor \frac{p+1}{2} \rfloor \mod 2)} \Gamma_{0\dots p} \; . \tag{E.1.4}$$

The basis used to represent the spinors in the simplified action is:

$$\theta_{\pm} = \mathcal{P}_{\pm}\theta, \tag{E.1.5}$$
$$\theta^1_{\pm} = \frac{1}{2}(\theta^1 \pm \Gamma_* \theta^2), \tag{E.1.6}$$
$$\theta^2_{\pm} = \frac{1}{2}(\theta^2 \mp \Gamma_* \theta^1) = \mp \Gamma_* \theta^1_{\pm}. \tag{E.1.7}$$

Letting $p \in \{0, 1\}$ be the coordinates along the background D1-branes and $q \in \{2, \ldots, 5\}$ be the transverse coordinates, θ and the Cartan forms can be shown to simplify to [76, 108]:

$$\theta^I_+ = \sqrt{|y|}\eta^I_+, \tag{E.1.8}$$
$$L^I_{t\,+} = t\sqrt{|y|}d\eta^I_+, \tag{E.1.9}$$
$$L^I_- = 0, \tag{E.1.10}$$
$$L^p_t = |y|(dx^p - it^2\bar{\eta}^I_+\Gamma^p d\eta^I_+), \tag{E.1.11}$$
$$L^q_t = \frac{1}{|y|}dy^q. \tag{E.1.12}$$

This is done using the solutions for the Cartan forms eqns (B.29)-(B.30) Using this, the DBI-action is:

$$S_{DBI} = -\int_{M_2} d^2\sigma \sqrt{-\det(G_{ij} + F_{ij})}, \tag{E.1.13}$$
$$G_{ij} = |y|^2(\partial_i x_p - 2i(\bar{\eta}^1_+\Gamma_p\partial_i\eta^1_+))(\partial_j x^p - 2i(\bar{\eta}^1_+\Gamma^p\partial_j\eta^1_+)) + \frac{1}{|y|^2}\partial_i y^q \partial_j y^q,$$
$$\epsilon^{ij}F_{ij} = \epsilon^{ij}(2\partial_i A_j + 4i(\bar{\eta}^1_+\Gamma_q\partial_i y^q\partial_j\eta^1_+)),$$

while the WZ-action is found to be

$$S_{WZ} = -2\int_{M_2}(\bar{\eta}^1_+(dy^q\Gamma_q)\Gamma_{01} \wedge d\eta^1_+). \tag{E.1.14}$$

E.2 D5-brane in $AdS_3 \times S^3$

The DBI-action of the D5-brane looks identical to that of the D1-brane, except of course that $i, j \in \{0, \ldots, 5\}$ instead of $\{0, 1\}$.

The WZ-action is quite complicated in this gauge, however some terms disappear as the Killing gauge leaves only 8 fermionic degrees of freedom, so no term can be higher than eighth order in η^1_+.

$$\begin{aligned}
S_{WZ} \;=\; & -2i \int_{M_6} \Big[-\frac{2}{5!}\bar{\eta}^1_+ \{ \frac{1}{2|y|^4}(dy^q\Gamma_q)^5 + 10(dy^q\Gamma_q)^3 \wedge [\frac{1}{2}(dx^p\Gamma_p)^2 \\
& -i(dx^p\Gamma_p) \wedge ((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p) - \frac{2}{3}((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)^2] \\
& +5|y|^5(dy^q\Gamma_q) \wedge [\frac{1}{2}(dx^p\Gamma_p)^4 - 2i(dx^p\Gamma_p)^3 \wedge ((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p) \\
& -4(dx^p\Gamma_p)^2 \wedge ((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)^2 + 4i(dx^p\Gamma_p) \wedge ((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)^3]\} \wedge \Gamma_* d\eta^1_+ \\
& +\frac{1}{3!}\bar{\eta}^1_+\{-2|y|^4[\frac{1}{2}(dx^p\Gamma_p)^3 - \frac{3i}{2}(dx^p\Gamma_p)^2 \wedge ((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p) \\
& -2(dx^p\Gamma_p) \wedge ((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)^2 + i((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)^3] \\
& -3(dx^p\Gamma_p) \wedge (dy^q\Gamma_q)^2 + 3i(dy^q\Gamma_q)^2 \wedge ((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)\} \wedge \Gamma_* d\eta^1_+ \wedge dA \\
& +\frac{i}{3}\bar{\eta}^1_+\{-2|y|^4[\frac{1}{4}(dx^p\Gamma_p)^3 - i(dx^p\Gamma_p)^2 \wedge ((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p) \\
& -\frac{3}{2}(dx^p\Gamma_p) \wedge ((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)^2 + \frac{4i}{5}((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)^3] - \frac{3}{2}(dx^p\Gamma_p) \wedge (dy^q\Gamma_q)^2 \\
& +2i((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p) \wedge (dy^q\Gamma_q)^2\} \wedge \Gamma_* d\bar{\eta}^1_+ \wedge (\bar{\eta}^1_+(dy^q\Gamma_q) \wedge d\eta^1_+) \\
& -(\bar{\eta}^1_+(dy^q\Gamma_q) \wedge \Gamma_* d\eta^1_+) \wedge \{\frac{1}{2}dA \wedge dA + idA \wedge (\bar{\eta}^1_+(dy^q\Gamma_q) \wedge d\eta^1_+) \\
& -\frac{2}{3}(\bar{\eta}^1_+(dy^q\Gamma_q) \wedge d\eta^1_+)^2\}\Big] .
\end{aligned} \tag{E.2.15}$$

E.3 Killing Gauge Fixed D5-brane Action in $AdS_5 \times S^5$

Again the DBI-action is written as eqn (E.1.13) with $i,j \in \{0,\dots,5\}$, and the WZ-action is found to be very similar to the $AdS_3 \times S^3$ case, except that now $\Gamma_* = \Gamma_{0123}$ and the WZ-action contains some extra terms due to the greater number of fermionic degrees of freedom, and also due to eqn (2.14.4).

$$\begin{aligned}
S_{WZ} \;=\; & -2i\int_{M_6}\Big[-\frac{2}{5!}\bar{\eta}^1_+\{\frac{1}{2|y|^4}(dy^q\Gamma_q)^5 + 10(dy^q\Gamma_q)^3 \wedge [\frac{1}{2}(dx^p\Gamma_p)^2 \\
& -i(dx^p\Gamma_p)\wedge((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p) - \frac{2}{3}((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)^2] \\
& +5|y|^5(dy^q\Gamma_q)\wedge[\frac{1}{2}(dx^p\Gamma_p)^4 \\
& -2i(dx^p\Gamma_p)^3\wedge((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p) - 4(dx^p\Gamma_p)^2\wedge((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)^2 \\
& +4i(dx^p\Gamma_p)\wedge\ ((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)^3 + \frac{8}{5}((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)^4]\}\wedge\Gamma_* d\eta^1_+ \\
& +\frac{1}{3!}\bar{\eta}^1_+\{-2|y|^4[\frac{1}{2}(dx^p\Gamma_p)^3 - \frac{3i}{2}(dx^p\Gamma_p)^2\wedge((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p) \\
& -2(dx^p\Gamma_p)\wedge((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)^2 + i((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)^3] \\
& -3(dx^p\Gamma_p)\wedge(dy^q\Gamma_q)^2 + 3i(dy^q\Gamma_q)^2\wedge((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)\}\wedge\Gamma_* d\eta^1_+\wedge dA \\
& +\frac{i}{3}\bar{\eta}^1_+\{-2|y|^4[\frac{1}{4}(dx^p\Gamma_p)^3 - i(dx^p\Gamma_p)^2\wedge((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p) \\
& -\frac{3}{2}(dx^p\Gamma_p)\wedge((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)^2 + \frac{4i}{5}((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)^3] - \frac{3}{2}(dx^p\Gamma_p)\wedge(dy^q\Gamma_q)^2 \\
& +2i((\bar{\eta}^1_+\Gamma^p d\eta^1_+)\Gamma_p)\wedge(dy^q\Gamma_q)^2\}\wedge\Gamma_* d\eta^1_+\wedge\ (\bar{\eta}^1_+(dy^q\Gamma_q)\wedge d\eta^1_+) \\
& -(\bar{\eta}^1_+(dy^q\Gamma_q)\wedge\Gamma_* d\eta^1_+)\wedge\{\frac{1}{2}dA\wedge dA + idA\wedge(\bar{\eta}^1_+(dy^q\Gamma_q)\wedge d\eta^1_+) \\
& -\frac{2}{3}(\bar{\eta}^1_+(dy^q\Gamma_q)\wedge d\eta^1_+)^2\}\Big] + \int_{M_7}\mathcal{L}^{BOSE}_{WZ},
\end{aligned} \tag{E.3.16}$$

$$\mathcal{L}^{BOSE}_{WZ} = 4|y|^3 dx^0\wedge\ldots\wedge dx^3\wedge dy^4\wedge dA + \frac{4}{|y|^5}dy^5\wedge\ldots\wedge dy^9\wedge dA. \tag{E.3.17}$$

Appendix F

Lie Algebra and Affine Lie Algebra Notation

Throughout this thesis finite dimensional simple, connected, simply-connected, Lie groups are denoted by G, and their finite dimensional Lie algebras by $\mathfrak{g}$. The untwisted affine extension of this algebra is: $\hat{\mathfrak{g}} = (\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}k$. The Cartan subalgebra of of $\mathfrak{g}$ is denoted by $\mathfrak{h}$. Its dual, the algebra of roots α, is $\mathfrak{h}^*$. The positive roots are denoted by Δ_+. The root lattice is labelled Q, the weight lattice as P and the coroot lattice as $Q^\vee$.

Often in this thesis the algebra A_N is written as $SU(N+1)$, which usually denotes the group, instead of $su(N+1)$, the normal algebra notation. This is an abuse of notation common in physics, but it is hoped the reader understands what is meant. The rank of a Lie algebra is usually denoted by N in this work.

The fundamental weights Λ_i of a Lie algebra $\mathfrak{g}$ of rank N are defined by:

$$(\Lambda_i, \alpha_j^\vee) = \delta_{ij}, \tag{F.1}$$

where α_j ($j \in \{1, \ldots, N\}$) is a simple root and $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$.

The non-degenerate positive bilinear form has been normalised such that $(\theta, \theta) = 2$, where $\theta = \sum_{i=1}^N a_i^\vee \alpha_i^\vee$ is the highest root ($a_i^\vee$ is the comark corresponding to the simple root $\alpha_i^\vee$).

Highest weights $\lambda = \sum_{i=1}^N \lambda_i \Lambda_i$ are often denoted by the vector $\{\lambda_1, \ldots, \lambda_N\}$. The Weyl vector ρ is defined by:

$$\rho \;=\; \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha = \sum_{i=1}^N \Lambda_i = (1, 1, \ldots, 1). \tag{F.2}$$

The affine highest weights are:

$$\begin{aligned} \hat{\lambda} &= \lambda + \lambda_0 \Lambda_0 && \text{(F.3)} \\ &= \sum_{i=0}^N \lambda_i \Lambda_i && \text{(F.4)} \\ &= (\lambda_0, \lambda_1, \ldots, \lambda_N), && \text{(F.5)} \end{aligned}$$

for rank N (the affine weights $\Lambda_i, i \in \{0, \ldots, N\}$ are explained in Appendix G.2). Often when writing an affine weight as a vector of the Dynkin labels λ_i, the λ_0 is dropped. It is usually easily understood when this is done in the text. An alternative way of representing an affine weight is eqn (G.2.17). The affine Weyl vector is $\hat{\rho} = \sum_{i=0}^{N} \Lambda_i$

The set of all highest weights of irreducible representations in the finite fundamental chamber is P_+. The set of all highest weights in the affine fundamental chamber is $P_+^{(k)}$. The set of finite dimensional Lie algebra highest weights obtain by projecting out the imaginary roots from the affine weights in $P_+^{\prime(k)}$ is $P_+^{(k)}$:

$$P_+^{(k)} = \{\lambda \in P_+ | (\lambda, \theta) \leq k\}. \tag{F.6}$$

The group of automorphisms of a finite dimensional Lie algebra $\mathfrak{g}$ is labelled $\mathrm{Aut}(\mathfrak{g})$. The group of inner automorphisms of the same algebra is labelled $\mathrm{Int}(\mathfrak{g})$. The group of outer automorphisms of $\mathfrak{g}$ is:

$$\mathrm{Out}(\mathfrak{g}) := \mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g}) \tag{F.7}$$

$\mathrm{Out}(\mathfrak{g})$ is isomorphic to the set of symmetries of the finite dimensional Lie algebra Dynkin diagram.

Replacing $\mathfrak{g}$ by $\hat{\mathfrak{g}}$ in the above paragraph, the outer automorphisms of the affine Lie algebra is:

$$\mathrm{Out}(\hat{\mathfrak{g}}) := \mathrm{Aut}(\hat{\mathfrak{g}})/\mathrm{Int}(\hat{\mathfrak{g}}) \tag{F.8}$$

$\mathrm{Out}(\hat{\mathfrak{g}})$ is isomorphic to the group of symmetries of the untwisted affine Dynkin diagram. The center of the covering group of G, $Z(G)$ is isomorphic to:

$$Z(G) \simeq \mathrm{Out}(\hat{\mathfrak{g}})/\mathrm{Out}(\mathfrak{g}) \tag{F.9}$$

The following is some more data on the Lie algebras [32], summarising the Cartan matrices, the algebra dimension $\dim \mathfrak{g}$, the dual Coxeter number $g^\vee = 1 + \sum_{i=1}^{N} a_i^\vee$, the Coxeter number g, the order of the Weyl group $|W|$, the highest root θ.[1] Also included are the Dynkin diagrams (using notation of [32]), which gives the node labelling conventions used in this thesis, as well a the marks and comarks of each node. The vectors beside each node in the Dynkin diagram are: (Dynkin label, mark a_i, comark $a_i^\vee$). The dark nodes correspond to the short simple roots.

The rank of an algebra is usually denoted by N throughout this work.

[1]The conjugacy classes of the algebra, isomorphic to the outer automorphism group of the untwisted affine Lie algebras Dynkin diagram, are listed in Table 3.5. The exponents of each algebra appear in Table 3.2.

A_N

$\dim \mathfrak{g} = N^2 + 2N$

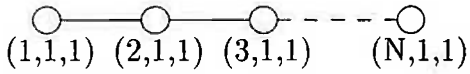
$g^\vee = N + 1$

$g = N + 1$

$|W| = (N+1)!$

$\theta = (1, 0, \ldots, 0, 1)$

$$A = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 2 \end{pmatrix}$$



$B_{N\geq 3}$

$\dim \mathfrak{g} = 2N^2 + N$

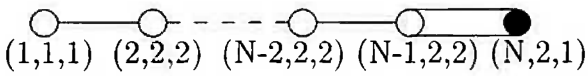
$g^\vee = 2N - 1$

$g = 2N$

$|W| = 2^N N!$

$\theta = (0, 1, 0, \ldots, 0)$

$$A = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -2 \\ 0 & 0 & 0 & \ldots & -1 & 2 \end{pmatrix}$$



$C_{N\geq 2}$

$\dim \mathfrak{g} = 2N^2 + N$

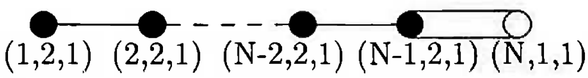
$g^\vee = N + 1$

$g = 2N$

$|W| = 2^N N!$

$\theta = (2, 0, \ldots, 0)$

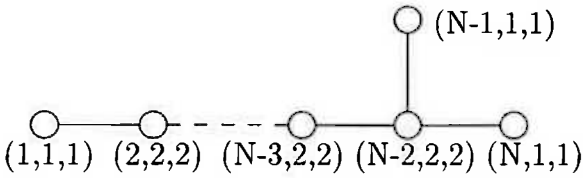
$$A = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & 0 & \ldots & -2 & 2 \end{pmatrix}$$



$D_{N\geq 4}$

$\dim \mathfrak{g} = 2N^2 - N$
$g^\vee = 2N - 2$
$g = 2N - 2$
$|W| = 2^{N-1}N!$
$\theta = (0, 1, 0, \ldots, 0)$

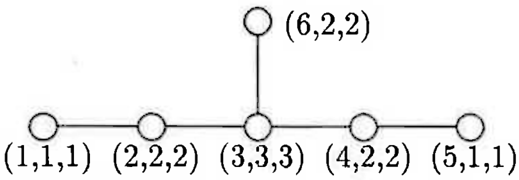
$$A = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -1 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 2 & 0 \\ 0 & 0 & 0 & \ldots & -1 & 0 & 2 \end{pmatrix}$$



E_6

$\dim \mathfrak{g} = 78$
$g^\vee = 12$
$g = 12$
$|W| = 51840$
$\theta = (0, \ldots, 0, 1)$

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$



E_7

$\dim \mathfrak{g} = 133$

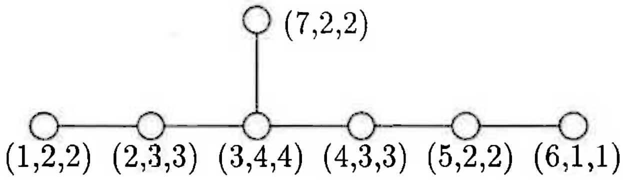
$g^\vee = 18$

$g = 18$

$|W| = 2903040$

$\theta = (1, 0, \ldots, 0)$

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$



E_8

$\dim \mathfrak{g} = 248$

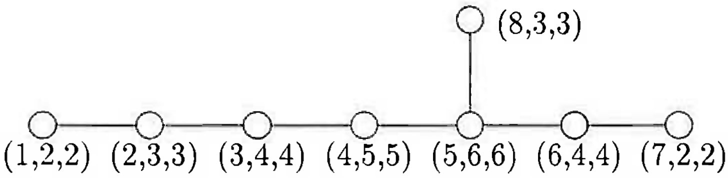
$g^\vee = 30$

$g = 30$

$|W| = 696729600$

$\theta = (1, 0, \ldots, 0)$

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$



F_4

$\dim \mathfrak{g} = 52$

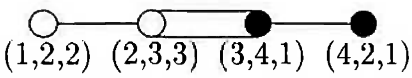
$g^\vee = 9$

$g = 12$

$|W| = 1152$

$\theta = (1, 0, 0, 0)$

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$



G_2

$\dim \mathfrak{g} = 14$

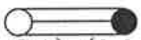
$g^\vee = 4$

$g = 6$

$|W| = 12$

$\theta = (1,0)$

$$A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$



(1,2,2) (2,3,1)

Appendix G

Weyl Reflections and Characters

It is assumed the reader has a basic knowledge of conformal field theory, Lie groups, Lie algebras and affine Lie algebras, so such things shall not be discussed here, although the notation used in this paper, as well as a few critical relations from Lie algebra and affine Lie algebra are contained in Appendix F. Reviews of these large topics can be found in [32, 73].

In contrast, a knowledge of the Weyl group and the affine Weyl group is vital for some of the analysis in this chapter, and as such a brief review of this topic is presented here.

G.1 Weyl Group

The Weyl group W consists of elements s_{α_i} which transform the roots, such that a root is reflected into another root, through a hyperplane perpendicular to the simple root α_i. A root β therefore transforms as:

$$s_{\alpha_i}\beta = \beta - (\alpha_i^\vee, \beta)\alpha_i. \tag{G.1.1}$$

When acted upon the simple roots, this is merely:

$$s_{\alpha_i}\alpha_j = \alpha_j - A_{ji}\alpha_i. \tag{G.1.2}$$

The basic properties of the reflections s_i are that:

$$(s_{\alpha_i}s_{\alpha_j})^{m_{ij}} = 1, \tag{G.1.3}$$

$$m_{ij} = \begin{cases} 1 & \text{if } i = j \\ \frac{\pi}{\pi-\theta_{ij}} & \text{if } i \neq j \end{cases}, \tag{G.1.4}$$

where θ_{ij} is the angle between the simple roots α_i and α_j.

The Weyl group transformations are built up through combinations:

$$w = \prod_i s_{\alpha_i}\ . \tag{G.1.5}$$

The Weyl reflections also act on the weights λ as:

$$s_{\alpha_i}\lambda = \lambda - (\alpha_i^\vee, \lambda)\alpha_i \ , \tag{G.1.6}$$

$$(w\lambda, \lambda') = (\lambda, w^{-1}\lambda') \ . \tag{G.1.7}$$

The action of the Weyl reflection w on the weight space is to divide it up into chambers C_w, where each chamber is defined by:

$$C_w = \{\lambda | \ (\alpha_i, w\lambda) \geq 0, \ \forall \, i \in \{1, \dots, N\}\}. \tag{G.1.8}$$

The chamber defined with respect to the identity element $w = I$, $w \in W$, is called the fundamental chamber. Weights in any chamber can be reflected into the other chambers via the action of W. To illustrate this, consider the Figure G.1 of the root system of A_2 with the weight space mapped on top of the root system.

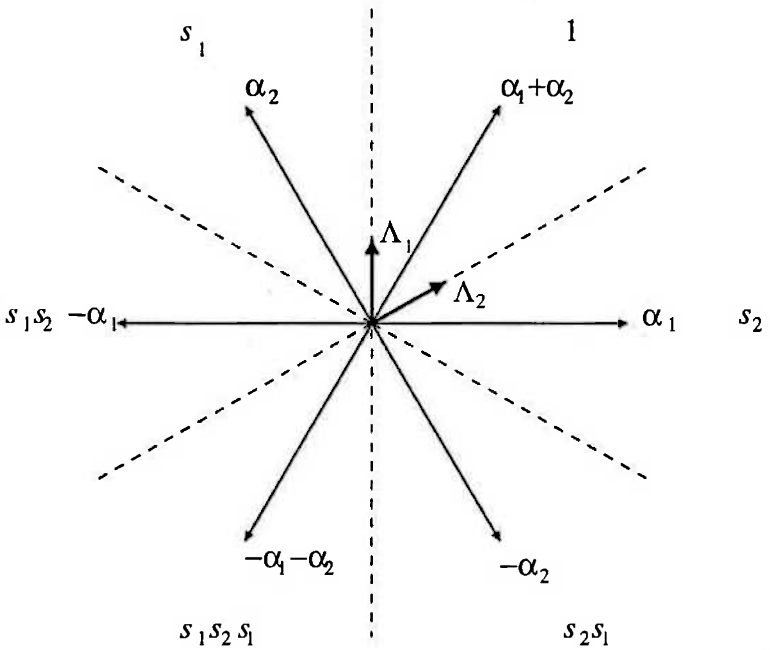


Figure G.1: *Weyl Chambers and Roots of A_2 [32]. The length of the vectors represent the lengths of the roots and weights. The angles between the vectors represent the angles determined by taking the scalar product of the vectors. The Weyl Chambers are denoted as the spaces between the dashed lines. Each is labelled by the element of the Weyl group needed to reflect a weight from the fundamental chamber (the chamber denoted by the identity element of the Weyl group) to the other chamber.*

As can be seen from studying this figure, the fundamental chamber is reflected into the neighbouring chambers by the two simple reflections s_{α_1} and s_{α_2}. The rest of the chambers correspond to the other independent Weyl reflections, with the fundamental chamber corresponding to the identity reflection.

When considering characters of representations, the characters are not symmetric under the action of the Weyl group, but rather under the action of the shifted Weyl group. The shifted Weyl reflections $w \in W$ on the weights λ are:

$$w \cdot \lambda \equiv w(\lambda + \rho) - \rho, \tag{G.1.9}$$

where ρ is the Weyl vector with all Dynkin labels equal one: $\rho = (1, \dots, 1)$.

The shifted Weyl reflections have the effect of extending the boundaries of the fundamental chamber by one in the weight space (ie, for example for $\hat{A}_{2,3}$, the shifted s_{α_i} reflects a weight λ through the boundary at $\lambda_i = -1$, as opposed to through the boundary at $\lambda_i = 0$ for the unshifted reflection), such as in Figure G.2

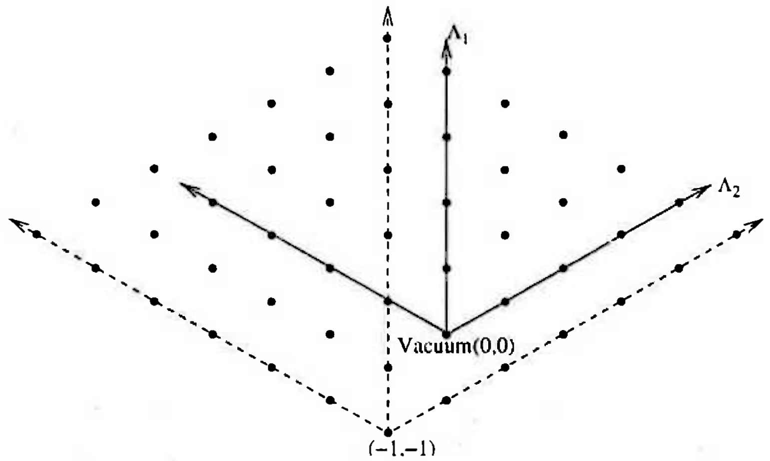


Figure G.2: *The finite dimensional Lie algebra fundamental chamber is the set of weights in the infinite triangle between the Λ_1 and Λ_2 axii. The weights in this chamber are reflected around the Λ_1 axis by s_{α_1}. However the action of the shifted reflection with respect to s_{α_1} reflects weights around the dashed vertical axis, corresponding to $\lambda_2 = -1$.*

The formal character of a irreducible representation with highest weight λ is given by:

$$\chi_\lambda = \frac{\sum_{w \in W} \epsilon(w) e^{w(\lambda+\rho)}}{\sum_{w \in W} \epsilon(w) e^{w\rho}}, \tag{G.1.10}$$

where $\epsilon(w)$ is the signature of w and is defined by:

$$\epsilon(w) = (-1)^{l(w)}. \tag{G.1.11}$$

$l(w)$ is the length of w, equal to the minimum number of simple Weyl reflections s_{α_i} that can be used to construct w.

The formal character eqn (G.1.10) evaluated at a particular value is:

$$\chi_\lambda(\xi) = \frac{\sum_{w \in W} \epsilon(w) e^{(w(\lambda+\rho),\xi)}}{\sum_{w \in W} \epsilon(w) e^{(w\rho,\xi)}}, \tag{G.1.12}$$

where $\xi \in \mathfrak{h}^*$, the dual Cartan subalgebra (ie, the set of roots).

Let us study the character of a representation with highest weight $w' \cdot \lambda$.

$$\begin{aligned}
\chi_{w'\cdot\lambda} &= \frac{\sum_{w\in W}\epsilon(w)e^{w(w'\cdot\lambda+\rho)}}{\sum_{w\in W}\epsilon(w)e^{w\rho}} \\
&= \frac{\sum_{w\in W}\epsilon(w)e^{w(w'(\lambda+\rho)-\rho+\rho)}}{\sum_{w\in W}\epsilon(w)e^{w\rho}} \\
&= \frac{\sum_{w\in W}\epsilon(w)e^{ww'(\lambda+\rho)}}{\sum_{w\in W}\epsilon(w)e^{w\rho}} \\
&= \frac{\sum_{w''\in W}\epsilon(w'')\epsilon(w')e^{w''(\lambda+\rho))}}{\sum_{w\in W}\epsilon(w)e^{w\rho}} \\
&= \epsilon(w')\chi_\lambda\ .
\end{aligned} \tag{G.1.13}$$

Due to the fact that the dimension of a representation can be determined by evaluating the character in the limit:

$$\dim\lambda = \lim_{t\to 0}\chi_\lambda(t\rho) = \prod_{\alpha\in\Delta_+}\frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)} \tag{G.1.14}$$

this transfers into a symmetry of the dimensions as well.

See [32,45,46] for reviews on this topic.

G.2 Affine Weyl Group

The following is a review of the relevant points of the affine Weyl group $\widehat{W}$ [73].

The extension of the Weyl group W to $\widehat{W}$ for the untwisted affine Lie algebras is straightforward. The main extensions are that the simple Weyl reflections are augmented by an extra reflection corresponding to the affine root α_0, and that the roots and weights being reflected by the Weyl group now contain an extra label corresponding to this simple root.

The weights and roots in the affine Lie algebra can have contributions from the imaginary root of the algebra δ. Therefore a finite weight λ becomes:

$$\hat{\lambda} = \lambda + n\delta, \quad n\in\mathbb{Z}, \tag{G.2.15}$$

and a finite root α becomes:

$$\hat{\alpha} = \alpha + n\delta, \quad n\in\mathbb{Z}. \tag{G.2.16}$$

The notation used to describe such weights and roots is:

$$\hat{\lambda} = (\lambda; k_\lambda; n_\lambda), \tag{G.2.17}$$

where λ is a finite weight, n_λ denotes the level of δ contributions and k_λ is the affine level of the weight ($\lambda_0 = k_\lambda - \sum_{i=1}^N \lambda_i a_i^\vee$).

In this notation the simple roots of the affine Lie algebra are:

$$\hat{\alpha}_i = (\alpha_i; 0; 0), \quad i \neq 0,$$
$$\hat{\alpha}_0 = (-\theta; 0; 1) \tag{G.2.18}$$

and the fundamental weights are:

$$\hat{\Lambda}_i = (\Lambda_i; a_i^\vee; 0), \ i \neq 0,$$
$$\hat{\Lambda}_0 = (0; 1; 0). \tag{G.2.19}$$

The comark of the root $\hat{\alpha}_0$ is $a_0^\vee = 1$. Usually the fundamental weights $\hat{\Lambda}_i$, and affine simple roots $\hat{\alpha}_i$ are denoted by Λ_i and α_i respectively, they are equivalent to the fundamental weights of the finite dimensional Lie algebra (except of course for $\hat{\Lambda}_0$ and $\hat{\alpha}_0$ which have no corresponding weights or roots in $\mathfrak{g}$, but for this reason denoting them by Λ_0 and α_0 shall not cause any confusion).

The affine scalar product is:

$$(\hat{\lambda}, \hat{\mu}) \;=\; (\lambda, \mu) + k_\lambda n_\mu + k_\mu n_\lambda \ , \tag{G.2.20}$$
$$\hat{\lambda} \;=\; (\lambda; k_\lambda; n_\lambda), \quad \hat{\mu} = (\mu; k_\mu; n_\mu). \tag{G.2.21}$$

The length of the simple affine roots and fundamental weights is unchanged from the finite case:

$$(\hat{\alpha}_i, \hat{\alpha}_i) = (\alpha_i, \alpha_i),$$
$$(\hat{\Lambda}_i, \hat{\Lambda}_i) = (\Lambda_i, \Lambda_i). \tag{G.2.22}$$

where here Λ_i are the finite dimensional Lie algebra fundamental weights.

In an affine Lie algebra, the simple Weyl reflections $s_{\hat{\alpha}_i}$, $i \neq 0$ on an affine weight $\hat{\beta}$ become:

$$s_{\hat{\alpha}_i}\hat{\beta} \;=\; \hat{\beta} - (\hat{\beta}, \hat{\alpha}_i^\vee)\hat{\alpha}_i = \hat{\beta} - (\beta, \alpha_i^\vee)\hat{\alpha}_i \ , \tag{G.2.23}$$
$$= \; (\beta - (\beta, \alpha_i^\vee)\alpha_i; \ 0 \ ; \ n_\beta) \ ,$$

These Weyl reflections $s_{\hat{\alpha}_i}$ $i \neq 0$ act on finite weight λ of affine weight $\hat{\lambda}$ in the same fashion as before. The new behaviour comes from:

$$s_{\hat{\alpha}_0}\hat{\lambda} \;=\; \hat{\lambda} - (\hat{\lambda}, \hat{\alpha}_0^\vee)\hat{\alpha}_0 = \hat{\lambda} - \lambda_0\hat{\alpha}_0 \ , \tag{G.2.24}$$
$$s_{\hat{\alpha}_0} \cdot \hat{\lambda} \;=\; s_{\hat{\alpha}_0}(\hat{\lambda} + \hat{\rho}) - \hat{\rho} = \hat{\lambda} - (\lambda_0 + 1)\hat{\alpha}_0 \ , \tag{G.2.25}$$

where the affine root α_0 in terms of weights is:

$$\hat{\alpha}_0 = (\sum_{i=1}^{N} \theta_i, -\theta_1, \ldots, -\theta_N), \tag{G.2.26}$$

for the highest root θ of the finite dimensional Lie algebra.

The elements $\hat{w} \in \widehat{W}$ are combinations of reflections $\hat{w} = s_{\hat{\alpha}_i} \dots s_{\hat{\alpha}_j}$.

The shifted Weyl reflections are:

$$\hat{w} \cdot \hat{\beta} = \hat{w}(\hat{\beta} + \hat{\rho}) - \hat{\rho}, \tag{G.2.27}$$

$\hat{\rho} = (1, \dots, 1)$, including the affine index equal to 1[1].

Due to $\lambda_0 = k - (\lambda, \theta) = k - \sum_{i=1}^{N} \lambda_i$, the shifted reflections of the affine root $\hat{\alpha}_0$ (G.2.25) are dependent on the level k. Indeed the shifted reflection $s_{\hat{\alpha}_0}$ really corresponds to a reflection through the line at level $\sum_{i=1}^{N} \lambda_i = k+1$. This encapsulates why the fundamental chamber for a Lie algebra (which contains an infinite number of weights) is finite in size in the affine Lie algebra (although each representation now is infinite dimensional due to the affine roots[2]).

Now let us reanalyse the affine Weyl group in terms of a semi direct product of the Weyl group and translations by the coroot lattice. First let us quickly review the semi direct product.

G.2.1 The Semi-Direct Product

The rigorous mathematical definition of a semi direct product can be found in [116]. Here the concept shall be illustrated with an example.

Consider the Poincaré transformations. They combine the rotations R and translations T of a vector. For $A \in R$ and $\vec{b} \in T$ the transformation $(A, \vec{b})$ of a vector $\vec{y}$ is:

$$\vec{y}' = A \cdot \vec{y} + \vec{b} \equiv (A, \vec{b})\vec{y}. \tag{G.2.28}$$

A second transformation $(A', \vec{b}')$ of this vector transforms to:

$$\vec{y}'' = A' \cdot A \cdot \vec{y} + A' \cdot \vec{b} + \vec{b}' = (A' \cdot A, A' \cdot \vec{b} + \vec{b}')(\vec{y}). \tag{G.2.29}$$

The Poincaré transformation, built up of two simpler transformations, one of which interferes with the other, is an example of a Semi-Direct Product between two different transformations. In the notation of semi-direct products, the Poincaré group is: $R \ltimes T$.

The affine Weyl-transformations are another example of semi-direct product of groups: $\widehat{W} = T(Q^\vee) \rtimes W$, where W is the finite dimensional Lie algebra Weyl group and $T(Q^\vee)$ are transformations on the on the weight space by elements of the coroot lattice $Q^\vee = \mathbb{Z}\alpha_1^\vee + \dots + \mathbb{Z}\alpha_N^\vee$ [73].

When acted upon a weight, the shifted translation t_γ for $\gamma \in Q^\vee$ corresponds to a shift in the weight of $(k + g^\vee)\gamma$. To see this reexamine the reflection of the fundamental chamber of $\hat{C}_{2,k}$ by $s_{\hat{\alpha}_0}$. As can be seen by Figure G.3, this reflection is equivalent to a

[1]The convention used when writing affine weights $\hat{\lambda}$ is that the affine Dynkin label λ_0 is first, ie: $\hat{\lambda} = (\lambda_0, \lambda_1, \dots, \lambda_N)$.

[2]More correctly, the finite dimensional Lie algebra fundamental chamber is obtained from the affine Lie algebra fundamental chamber of infinite level k via projecting out the imaginary roots.

reflection of the fundamental chamber by $w \in W$ into chamber C_w, and a translation by an element of the coroot lattice t_γ (see (G.2.32)).

The translations along the simple coroots can be represented by reflections of the simple roots:

$$t_{\hat{\alpha}_i^\vee} = s_{\hat{\alpha}_i} s_{\hat{\alpha}_i+\delta} , \tag{G.2.30}$$

$$t_{\hat{\alpha}_i^\vee}\hat{\lambda} = (\lambda + k\alpha_i^\vee;\ k;\ n - \lambda_i - \frac{2}{|\alpha_i|^2}k), \tag{G.2.31}$$

which is a shift by $k\alpha^\vee$. If this operator is applied as a shifted Weyl reflection it becomes:

$$t_{\hat{\alpha}_i^\vee} \cdot \hat{\lambda} = (\lambda + (k + g^\vee)\alpha_i^\vee;\ k;\ n - \lambda_i - \frac{2}{|\alpha_i|^2}(k + g^\vee)). \tag{G.2.32}$$

This is a shift by $(k + g^\vee)\alpha_i^\vee$ and thus the affine Weyl group $\widehat{W} = T(Q^\vee) \rtimes W$ can be used to transform the affine fundamental chamber into all of the infinite number of other chambers. Indeed the semi-direct product nature of this is confirmed by studying the action of $t_{\hat{\beta}'^\vee} s_{\hat{\alpha}'} t_{\hat{\beta}^\vee} s_{\hat{\alpha}}$ on the finite dimensional λ component of affine weight $\hat{\lambda}$:

$$t_{\hat{\beta}'^\vee} s_{\hat{\alpha}'} t_{\hat{\beta}^\vee} s_{\hat{\alpha}} \hat{\lambda} = (\lambda + k(\beta'^\vee + \beta^\vee - (\alpha'^\vee, \{\lambda + k\beta^\vee - (\alpha^\vee, \lambda)\alpha)\}); \ldots ; \ldots) \tag{G.2.33}$$

where $\beta^\vee$, $\beta'^\vee$, $\alpha'^\vee$ and $\alpha^\vee$ are the finite dimensional roots obtained by projecting the imaginary roots from the affine roots.

For an element $\hat{w} \in \widehat{W}$ where $\hat{w} = \hat{v} \cdot t_{\hat{\alpha}^\vee}$ and $\hat{v}$ is a simple reflection (ie, $\hat{w}$ comprises a translation and a reflection) the signature of $\hat{w}$ is equal the signature of $\hat{v}$ as the length $l(t)$ of a translation is always even and thus does not affect the signature.

G.2.2 Outer Automorphism Group

The outer automorphism group of affine Lie algebra $\hat{\mathfrak{g}}$, modulo the finite algebra outer automorphisms $\mathrm{Out}(\hat{\mathfrak{g}})/\mathrm{Out}(\mathfrak{g})$, is isomorphic to the center of the Lie group G [15, 73, 98].

The action of $A \in \mathrm{Out}(\hat{\mathfrak{g}})/\mathrm{Out}(\mathfrak{g})$ on the weights can be expressed in terms of the the action of $\widehat{W}$:

$$A\hat{\lambda} = k(A - 1)\hat{\Lambda}_0 + w_A\hat{\lambda}, \tag{G.2.34}$$

where $w_A = w_{\hat{\alpha}_i} w_0$, for w_0 being the finite Weyl transformation with the largest value of $l(w)$, (ie the 'longest' reflection), and s_{α_i} is such that $A\hat{\Lambda}_0 = \hat{\Lambda}_i$. Thus the first term in eqn (G.2.34) is a translation, and the second terms a reflection of the the weight.

The signature of w_A is:

$$\epsilon(w_A) = e^{2\pi i (A\hat{\Lambda}_o, \rho)}. \tag{G.2.35}$$

The scalar product $(A\hat{\Lambda}_o, \rho)$ is a finite scalar product, between the finite Weyl vector ρ and the finite dimensional Lie algebra weight obtained by projecting out the imaginary root δ from $A\hat{\Lambda}_0$.

G.2.3 Affine Lie Algebra Characters and the Weyl Group

The formal character of a representation with integrable highest weight $\hat{\lambda}$ for an affine Lie algebras is:

$$\chi_{\hat{\lambda}} = e^{-m_{\hat{\lambda}}\delta} \frac{\sum_{\hat{w}\in\widehat{W}} \epsilon(\hat{w}) e^{\hat{w}(\hat{\lambda}+\hat{\rho})}}{\sum_{\hat{w}\in\widehat{W}} \epsilon(\hat{w}) e^{\hat{w}\hat{\rho}}}, \tag{G.2.36}$$

$$m_{\hat{\lambda}} = \frac{(\hat{\lambda}+\hat{\rho},\hat{\lambda}+\hat{\rho})}{2(k+g^\vee)} - \frac{(\hat{\rho},\hat{\rho})}{2g^\vee} = \frac{(\lambda+\rho,\lambda+\rho)}{2(k+g^\vee)} - \frac{(\rho,\rho)}{2g^\vee}. \tag{G.2.37}$$

The imaginary root δ of the affine Lie algebra can be represented in terms of the coroots by: $\delta = \sum_{i=0}^{N} a_i^\vee \hat{\alpha}_i^\vee$.

The shifted Weyl reflections $w \in W$ act on this character in the same way the shifted finite Weyl reflections act on the finite character [73].

$$\chi_{w.\hat{\lambda}} = \epsilon(w)\chi_{\hat{\lambda}} \; . \tag{G.2.38}$$

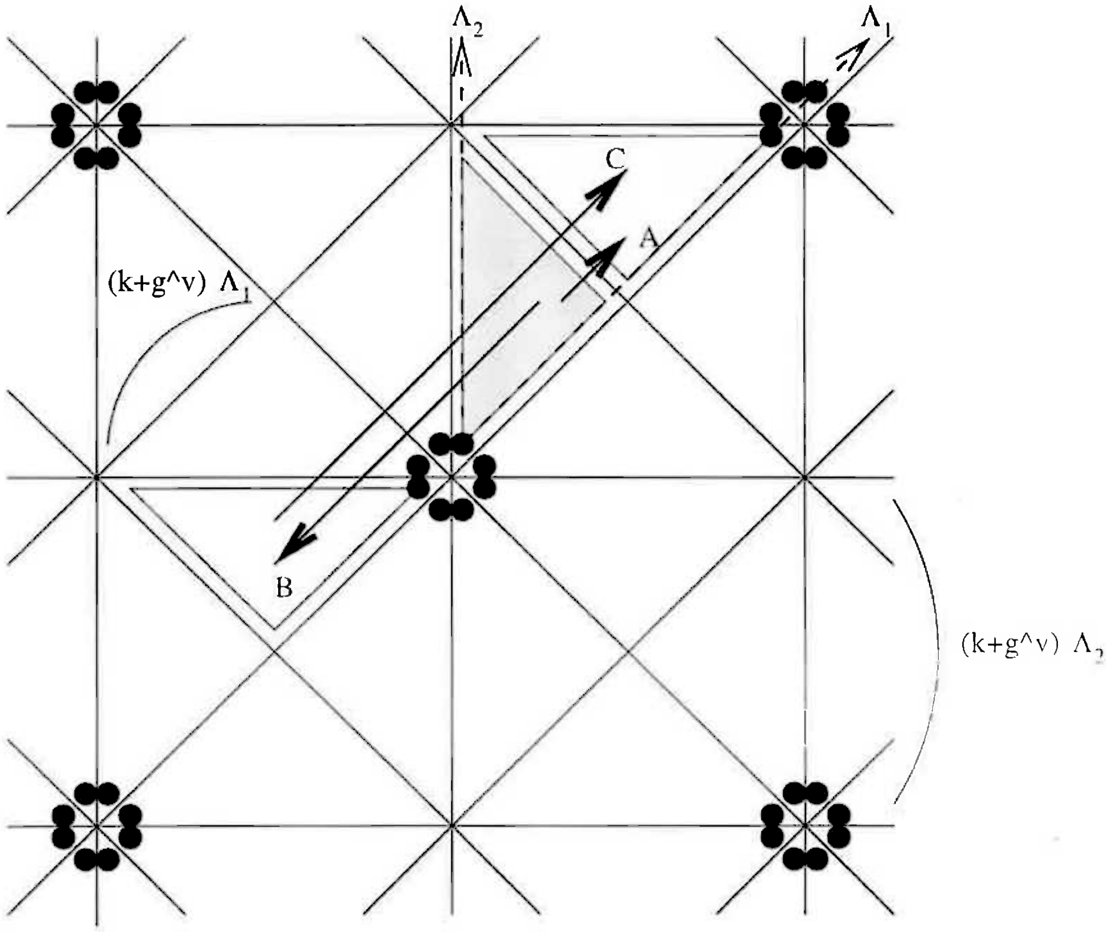


Figure G.3: *For $\hat{C}_{2,k}$, the shifted affine Weyl transformation $s_{\hat{\alpha}_0}$ (denoted by transformation A) transforms the affine fundamental chamber defined with respect to a shifted action of $\widehat{W}$ (dark shaded) to the chamber above it. This shifted reflection is equivalent to reflecting the fundamental chamber by the shifted action of $s_{\hat{\alpha}_1} s_{\hat{\alpha}_2} s_{\hat{\alpha}_1} \in W$ (denoted by the transformation B), followed by a shift C, equivalent to a shift by $2(k+g^\vee)\Lambda_1$ given by the action of $t^2_{\hat{\alpha}_1^\vee}$. The dark dots represent the vacuum representations and the other representations it is mapped into by $\widehat{W}$.*

Appendix H

x Values for Affine Lie Algebras

This appendix contains the results for calculating the x values that determine the charge group $\mathbb{Z}_x$ of an affine Lie algebra. These results were found through obtaining the fusion rules and the dimensions of the finite components of all representations in the affine Lie algebra using the program *Kac* [115]. Using a different especially written computer program, this data was then applied to eqn (3.5.16), which derived all the constraints determining x, from which the maximum x could then be found using eqn (3.7.6).

This analysis was not done for the $\hat{A}_{N,k}$ affine Lie algebras, because the analysis has been done analytically for all ranks and levels elsewhere [40].

The following tables contain x data for all the untwisted affine Lie algebras of compact, connected, simple, simply connected Lie groups. The blank parts of the data correspond to x's of various k levels that were skipped, as they were deemed unnecessary to corroborate the model of x that was being built up at the time. Less levels were studied for higher ranks due to the fact that for higher ranks and for higher levels, the matrix S which the program *Kac* uses to calculate the fusion rules gets too big for the program to easily handle (when it is above 256×256 the program does not work well).

For $\hat{E}_{8,1}$, $x = \infty$ as the only representation present is the vacuum with finite dimension 1, so eqn (3.5.16) yields only the constraint $0 = 0 \bmod x$, which is satisfied for all x (and we are listing the largest possible x in the tables).

k	$\hat{B}_{2,k}$	$\hat{B}_{3,k}$	$\hat{B}_{4,k}$	$\hat{B}_{5,k}$	$\hat{B}_{6,k}$	$\hat{B}_{7,k}$
1	2	8	2	4	2	-
2	5	7	3	11	13	-
3	1	2	1	1	1	2
4	7	3	11	13	1	
5	4	1	1	1	2	
6	3	11	13	1	17	
7	5	1	1	2	1	
8	11	13	-	-		
9	2		4	1		
10	13					
11	7					
12	5					
13	8					
14	17					
15	3					

Table H.1: x *Data for* $\hat{B}_{N,k}$

k	$\hat{C}_{2,k}$	$\hat{C}_{3,k}$	$\hat{C}_{4,k}$	$\hat{C}_{5,k}$	$\hat{C}_{6,k}$
1	2	1	1	1	2
2	5	1	1	2	1
3	1	7	2	1	1
4	7	4	3	1	1
5	4	3	1	11	
6	3	1	11		
7	5	11			
8	11				
9	2				
10	13				
11	7				
12	5				
13	8				
14	17				
15	3				

Table H.2: x *Data for* $\hat{C}_{N,k}$

k	$\hat{D}_{4,k}$	$\hat{D}_{5,k}$	$\hat{D}_{6,k}$	$\hat{D}_{7,k}$	$\hat{D}_{8,k}$
1	7	3	11	13	3
2	2	1	1	1	2
3	3	11	13	1	17
4	1	1	1	2	1
5	11	13	1	17	
6	1	1		1	
7	13				

Table H.3: x *Data for* $\hat{D}_{N,k}$

k	$\hat{E}_{6,k}$	$\hat{E}_{7,k}$	$\hat{E}_{8,k}$
1	26	3135	∞
2	1	1	4
3	1	1	1
4	2	1	1
5	17	23	1
6	1	1	1

Table H.4: x *Data for* $\hat{E}_{N,k}$

k	$\hat{F}_{4,k}$
1	649
2	1
3	1
4	13
5	1
6	1
7	2
8	-
9	1

Table H.5: x *Data for* $\hat{F}_{4,k}$

k	$\hat{G}_{2,k}$
1	41
2	1
3	7
4	2
5	3
6	1
7	11
8	1
9	13

Table H.6: x *Data for* $\hat{G}_{2,k}$

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