LATTICE QCD, GAUGE FIXING AND THE TRANSITION TO THE PERTURBATIVE REGIME

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Perturbative QCD uses the Faddeev-Popov gauge-fixing procedure, which leads to ghosts and the local BRST invariance of the gauge-fixed perturbative QCD action. In the asymptotic regime, where perturbative QCD is relevant, Gribov copies can be neglected. In the nonperturbative regime, one must adopt either a nonlocal Gribov-copy free gauge (e.g., Laplacian gauge) or attempt to maintain local BRST invariance at the expense of admitting Gribov copies. These issues are explored and discussed. In addition, the relationship between recent Dyson-Schwinger based calculations of the infrared behavior of QCD Green's functions and the lattice calculation of these quantities is examined.

1. Introduction

Perturbative quantum chromodynamics (QCD) is formulated using the Faddeev-Popov gauge-fixing procedure, which introduces ghost fields and leads to the local BRST invariance of the gauge-fixed perturbative QCD action. These perturbative gauge fixing schemes include, e.g., the standard choices of covariant, Coulomb and axial gauge fixing. These are entirely adequate for the purpose of studying perturbative QCD, however, they fail in the nonperturbative regime due to the presence of Gribov copies. Perturbative QCD works because in doing a weak-field expansion around the $A_\mu = 0$ configuration these Gribov copies are not encountered [1].

One could define nonperturbative QCD by imposing a non-local Gribov-copy free gauge fixing (such as Laplacian gauge) or, alternatively, one could attempt to maintain local BRST invariance at the cost of admitting Gribov copies. One of the well-known difficulties for the latter option is the problem of pairs of Gribov copies with opposite sign giving a vanishing path integral[2,3,4,5]. Whether or not a local BRST invariance for QCD can be maintained in the nonperturbative regime remains an open problem.

The standard lattice definition of QCD is equivalent to the choice of a

Gribov copy free gauge-fixing. There is a negligible chance of selecting two gauge-equivalent configurations (strictly zero except for numerical round-off error). Calculations of *physical observables* are unaffected by arbitrary gauge transformations on the configurations in the ideal gauge-fixed ensemble. A lattice QCD calculation using an ideal gauge-fixed ensemble will give a result for a gauge-invariant (i.e., physical) quantity which is identical to doing no gauge fixing at all, i.e., equivalent to the standard lattice calculation of physical quantities.

We begin by reviewing the standard arguments for constructing QCD perturbation theory, which use the Faddeev-Popov gauge fixing procedure to construct the perturbative QCD gauge-fixed Lagrangian density. The naive Lagrangian density of QCD is $\mathcal{L}_{\rm QCD} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \sum_f \bar{q}_f(i\not{D} - m_f)q_f$, where the index f corresponds to the quark flavours. The naive Lagrangian is neither gauge-fixed nor renormalized, however it is invariant under local $SU(3)_c$ gauge transformations $g(x)$. For arbitrary, small $\omega^a(x)$ we have $g(x) \equiv \exp\{-ig_s\,(\lambda^a/2)\,\omega^a(x)\} \in SU(3)$, where the $\lambda^a/2 \equiv t^a$ are the generators of the gauge group $SU(3)$ and the index a runs over the eight generator labels $a = 1, 2, ..., 8$.

Consider some gauge-invariant Green's function (for the time being we shall concern ourselves only with gluons) $\langle\Omega|\ T(\hat{O}[A])\ |\Omega\rangle = \int \mathcal{D}A\ O[A]\ e^{iS[A]} / \int \mathcal{D}A\ e^{iS[A]}$, where $O[A]$ is some gauge-independent quantity depending on the gauge field, $A_\mu(x)$. We see that the gauge-independence of $O[A]$ and $S[A]$ gives rise to an infinite quantity in both the numerator and denominator, which must be eliminated by gauge-fixing. The Minkowski-space Green's functions are defined as the Wick-rotated versions of the Euclidean ones.

The gauge orbit for some configuration A_μ is defined to be the set of all of its gauge-equivalent configurations. Each point A^g_μ on the gauge orbit is obtained by acting upon A_μ with the gauge transformation g. By definition the action, $S[A]$, is gauge invariant and so all configurations on the gauge orbit have the same action, e.g., see the illustration in Fig. 1.

2. Gribov Copies and the Faddeev-Popov Determinant

Any gauge-fixing procedure defines a surface in gauge-field configuration space. Fig. 2 is a depiction of these surfaces represented as dashed lines intersecting the gauge orbits within this configuration space. Of course, in general, the gauge orbits are hypersurfaces as are the gauge-fixing surfaces. Any gauge-fixing surface must, by definition, only intersect the gauge orbits

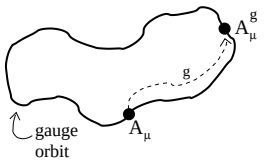

Figure 1. Illustration of the gauge orbit containing A_μ and indicating the effect of acting on A_μ with the gauge transformation g. The action $S[A]$ is constant around the orbit.

at distinct isolated points in field configuration space. For this reason, it is sufficient to use lines for the simple illustration of the concepts here. An ideal (or complete) gauge-fixing condition, $F[A] = 0$, defines a surface called the Fundamental Modular Region (FMR) that intersects each gauge orbit once and only once and typically where possible contains the trivial configuration $A_\mu = 0$. A non-ideal gauge-fixing condition, $F'[A] = 0$, defines a surface or surfaces which intersect the gauge orbit more than once. These multiple intersections of the non-ideal gauge fixing surface(s) with the gauge orbit are referred to as Gribov copies [2,3,4,5] Lorentz gauge ($\partial_\mu A^\mu(x) = 0$) for example, has many Gribov copies per gauge orbit. By definition an ideal gauge fixing is free from Gribov copies. The ideal gauge-fixing surface $F[A] = 0$ specifies the FMR for that gauge choice. Typically the gauge fixing condition depends on a space-time coordinate, (e.g., Lorentz gauge, axial gauge, *etc.*), and so we write the gauge fixing condition more generally as $F([A]; x) = 0$.

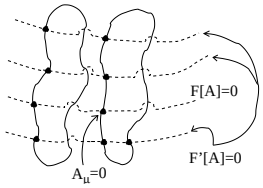

Figure 2. Ideal, $F[A]$, and non-ideal, $F'[A]$, gauge-fixing.

Let us denote one arbitrary gauge configuration per gauge orbit, A^0_μ, as the origin for that gauge orbit, i.e., corresponding to $g = 0$ on that orbit. Then each gauge orbit can be labelled by A^0_μ and the set of all such A^0_μ is equivalent to one particular, complete specification of the gauge. Under a gauge transformation, g, we move from the origin of the gauge orbit to the configuration, A^g_μ, where by definition $A^0_\mu \xrightarrow{g} A^g_\mu = gA^0_\mu g^\dagger - (i/g_s)(\partial_\mu g)g^\dagger$.

Let us denote for each gauge orbit the gauge transformation, $\tilde{g} \equiv \tilde{g}[A^0]$, as the transformation which takes us from the origin of that orbit, A^0_μ, to the corresponding configuration on the FMR, $A^{\rm FMR}_\mu \equiv A^{\tilde{g}}_\mu$, which is specified by the ideal gauge fixing condition $F([A^g];x) = 0$. In other words, an ideal gauge fixing has a unique $\tilde{g}$ which satisfies $F([A^g];x)|_{\tilde{g}} = 0$ and hence specifies the FMR as $A^{\tilde{g}} \equiv A^{\rm FMR}_\mu \in$ FMR. Note then that we have $\int \mathcal{D}A = \int \mathcal{D}A^0 \int \mathcal{D}g = \int \mathcal{D}A^{\rm FMR} \int \mathcal{D}(g-\tilde{g})$.

The *inverse Faddeev-Popov determinant* is defined as the integral over the gauge group of the gauge-fixing condition, i.e.,

$$\Delta_F^{-1}[A^{\rm FMR}] \equiv \int \mathcal{D}g\ \delta[F[A]] = \int \mathcal{D}g\ \delta(g-\tilde{g}) \left|\det\left(\frac{\delta F([A];x)}{\delta g(y)}\right)\right|^{-1} \quad (1)$$

Let us define the matrix $M_F[A]$ as $M_F([A];x,y)^{ab} \equiv \delta F^a([A];x)/\delta g^b(y)$. Then the *Faddeev-Popov determinant* for an arbitrary configuration A_μ can be defined as $\Delta_F[A] \equiv |\det M_F[A]|$. (The reason for the name is now clear). Note that we have consistency, since $\Delta_F^{-1}[A^{\rm FMR}] \equiv \Delta_F^{-1}[A^{\tilde{g}}] = \int \mathcal{D}g\ \delta(g-\tilde{g})\Delta_F^{-1}[A]$.

We have $1 = \int \mathcal{D}g\ \Delta_F[A]\ \delta[F[A]] = \int \mathcal{D}(g-\tilde{g})\ \Delta_F[A]\ \delta[F[A]]$ by definition and hence

$$\int \mathcal{D}A^{\rm FMR} \equiv \int \mathcal{D}A^{\rm FMR} \int \mathcal{D}(g-\tilde{g})\ \Delta_F[A]\delta[F[A]] = \int \mathcal{D}A\ \Delta_F[A]\delta[F[A]] \quad (2)$$

Since for an ideal gauge-fixing there is one and only one $\tilde{g}$ per gauge orbit, such that $F([A];x)|_{\tilde{g}} = 0$, then $|\det M_F[A]|$ is non-zero on the FMR. It follows that since there is at least one smooth path between any two configurations in the FMR and since the determinant cannot be zero on the FMR, then it cannot change sign on the FMR. The *first Gribov horizon* is defined to be those configurations with $\det M_F[A] = 0$ which lie closest to the FMR. By definition the determinant can change sign on or outside this horizon. Clearly, the FMR is contained within the first Gribov horizon and for an ideal gauge fixing, since the sign of the determinant cannot change, we can replace $|\det M_F|$ with $\det M_F$, [i.e., the overall sign of the functional integral is normalized away in the ratio of functional integrals].

These results are generalizations of results from ordinary calculus, where $|\det(\partial f_i/\partial x_j)|^{-1}_{\vec{f}=0} = \int dx_1 \cdots dx_n\, \delta^{(n)}(\vec{f}(\vec{x}))$ and if there is one and only one $\vec{x}$ which is a solution of $\vec{f}(\vec{x}) = 0$ then the matrix $M_{ij} \equiv \partial f_i/\partial x_j$ is invertible (i.e., non-singular) on the hypersurface $\vec{f}(\vec{x}) = 0$ and hence $\det M \neq 0$.

3. Generalized Faddeev-Popov Technique

Let us now assume that we have a family of *ideal* gauge fixings $F([A];x) = f([A];x) - c(x)$ for any Lorentz scalar $c(x)$ and for $f([A];x)$ being some Lorentz scalar function, (e.g., $\partial^\mu A_\mu(x)$ or $n^\mu A_\mu(x)$ or similar or any non-local generalizations of these). Therefore, using the fact that we remain in the FMR and can drop the modulus on the determinant, we have $\int \mathcal{D}A^{\rm FMR} = \int \mathcal{D}A \ \det M_F[A] \ \delta[f[A]-c]$. Since $c(x)$ is an arbitrary function, we can define a new "gauge" as the Gaussian weighted average over $c(x)$, i.e.,

$$\begin{aligned}\int \mathcal{D}A^{\rm FMR} &\propto \int \mathcal{D}c \, \exp\left\{-\frac{i}{2\xi}\int d^4x c(x)^2\right\} \int \mathcal{D}A \ {\rm det}M_F[A] \ \delta[f[A]-c] \\ &\propto \int \mathcal{D}A \ {\rm det}M_F[A] {\rm exp}\left\{-\frac{i}{2\xi}\int d^4x f([A];x)^2\right\} \\ &\propto \int \mathcal{D}A\mathcal{D}\chi\mathcal{D}\bar{\chi} \ \ \exp\left\{-i\int d^4x d^4y \ \bar{\chi}(x) M_F([A];x,y)\chi(y)\right\} \\ &\quad \times \exp\left\{-\frac{i}{2\xi}\int d^4x f([A];x)^2\right\} \, , \end{aligned} \tag{3}$$

where we have introduced the anti-commuting ghost fields χ and $\bar{\chi}$. Note that this kind of ideal gauge fixing does not choose just one gauge configuration on the gauge orbit, but rather is some Gaussian weighted average over gauge fields on the gauge orbit. We then obtain

$$\langle\Omega|\ T(\hat{O}[...])\ |\Omega\rangle = \frac{\int \mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}A\mathcal{D}\chi\mathcal{D}\bar{\chi}\ O[...]\ e^{iS_\xi[...]}}{\int \mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}A\mathcal{D}\chi\mathcal{D}\bar{\chi}\ e^{iS_\xi[...]}} \, , \tag{4}$$

where

$$\begin{aligned} S_\xi[q,\bar{q},A,\chi,\bar{\chi}] = &\int d^4x \left[-\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} - \frac{1}{2\xi}\left(f([A];x)\right)^2 + \sum_f \bar{q}_f(i\not{D} - m_f)q_f\right] \\ &+ \int d^4x d^4y \ \bar{\chi}(x) M_F([A];x,y)\chi(y) \, . \end{aligned} \tag{5}$$

4. Standard Gauge Fixing

We can now recover standard gauge fixing schemes as special cases of this generalized form. First consider standard covariant gauge, which we obtain by taking $f([A];x) = \partial_\mu A^\mu(x)$ and by *neglecting* the fact that this leads to Gribov copies. We need to evaluate $M_F[A]$ in the vicinity of the gauge-

fixing surface (specified by $\tilde{g}$):

$$M_F([A];x,y)^{ab} = \frac{\delta F^a([A];x)}{\delta g^b(y)} = \frac{\delta[\partial_\mu A^{a\mu}(x) - c(x)]}{\delta g^b(y)} = \partial^x_\mu \frac{\delta A^{a\mu}(x)}{\delta g^b(y)]}\,. \quad (6)$$

Under an infinitesimal gauge transformation about the FMR, $\delta g \equiv g - \tilde{g}$, we have $(A^{\tilde{g}})_\mu \to (A^{\tilde{g}+\delta g})_\mu$, where

$$(A^{\tilde{g}+\delta g})^a_\mu(x) = (A^{\tilde{g}})^a_\mu(x) + g_s f^{abc}\omega^b(x)A^c_\mu(x) - \partial_\mu\omega^a(x) + \mathcal{O}(\omega^2) \quad (7)$$

and hence in the neighbourhood of the gauge fixing surface (i.e., for small fluctations along the gauge orbit around $A^{\rm FMR}_\mu$), we have

$$\begin{aligned} M_F([A];x,y)^{ab} &= \partial^x_\mu \frac{\delta A^{a\mu}(x)}{\delta(\delta\omega^b(y)]}\bigg|_{\omega=0} \quad (8)\\ &= \partial^x_\mu \Big([-\partial^{x\mu}\delta^{ab} + g_s f^{abc}A^{c\mu}(x)]\delta^{(4)}(x-y)\Big)\,. \end{aligned}$$

We then recover the standard covariant gauge-fixed form of the QCD action

$$\begin{aligned} S_\xi[q,\bar{q},A,\chi,\bar{\chi}] = \int d^4x \left[-\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 + \sum_f \bar{q}_f(i\not{D} - m_f)q_f \right] \\ +(\partial_\mu\bar{\chi}_a)(\partial^\mu\delta^{ab} - g f_{abc}A^\mu_c)\chi_b\,. \quad (9) \end{aligned}$$

However, this gauge fixing has not removed the Gribov copies and so the formal manipulations which lead to this action are not valid. This Lorentz covariant set of naive gauges corresponds to a Gaussian weighted average over generalized Lorentz gauges, where the gauge parameter ξ is the width of the Gaussian distribution over the configurations on the gauge orbit. Setting $\xi = 0$ we see that the width vanishes and we obtain Landau gauge (equivalent to Lorentz gauge, $\partial^\mu A_\mu(x) = 0$). Choosing $\xi = 1$ is referred to as "Feynman gauge" and so on. We can similarly derive the QCD action for axial gauge.

5. Discussion and Conclusions

There is no known Gribov-copy-free gauge fixing which is a *local* function of $A_\mu(x)$. In other words, such a gauge fixing cannot be expressed as a function of $A_\mu(x)$ and a finite number of its derivatives, i.e., $F([A];x) \neq F(\partial_\mu, A_\mu(x))$ for all x. Hence, the ideal gauge-fixed action, $S_\xi[\cdots]$, in Eq. (5) becomes non-local and gives rise to a nonlocal quantum field theory. Since this action serves as the basis for the proof of the renormalizability of QCD, the proof of asymptotic freedom, local BRST invariance, and the

Schwinger-Dyson equations[6,7] (to name but a few) the nonlocality of the action leaves us without a first-principles proof of these features of QCD in the nonperturbative context.

The lattice implementation of Landau gauge finds local minima of the gauge fixing functional, which correspond to configurations lying inside the first Gribov horizon. The remaining Gribov copies after this partial gauge fixing then necessarily all have the same sign (positive) for the Faddeev-Popov determinant and hence add coherently in the functional integral. This ensures that the ghost propagator is positive definite[7,8]. The derivation of the Dyson-Schwinger equations is based on the fact that the integral of a total derivative vanishes[6] provided that the surface integral of the integrand vanishes when integrated over the boundary of the region. Since the Faddeev-Popov determinant vanishes on the first Gribov horizon, then we can still derive the standard Dyson-Schwinger equations from the Landau gauge fixed QCD action even if we restrict the gauge fields to lie within the first Gribov horizon. This is equivalent to requiring that the ghost propagator be positive definite. Thus it is valid to compare lattice Landau-gauge calculations with Dyson-Schwinger based calculations (with a positive definite ghost propagator), since these both consist of considering configurations within the first Gribov horizon. An extensive body of lattice calculations exist for the Landau gauge gluon[9] and quark[10,12] propagators and most recently for the quark-gluon vertex[13]. Similarly, calculations in Laplacian gauge (an ideal gauge) fixing have also recently become available[14,15].

It is well-established that QCD is asymptotically free, i.e., it is weak-coupling at large momenta. In the weak coupling limit the functional integral is dominated by small action configurations. As a consequence, momentum-space Green's functions at large momenta will receive their dominant contributions in the path integral from configurations near the trivial gauge orbit, i.e., the orbit containing $A_\mu = 0$, since this orbit minimizes the action. If we use standard lattice gauge fixing, which neglects the fact that Gribov copies are present, then at large momenta $\int \mathcal{D}A$ will be dominated by configurations lying on the gauge-fixed surfaces in the neighbourhood of *each* of the Gribov copies on the trivial orbit. Since for small field fluctuations the Gribov copies cannot be aware of each other, we merely overcount the contribution by a factor equal to the number of copies on the trivial orbit. This overcounting is normalized away in the ratio of functional integrals. Thus it is possible to understand why Gribov copies can be neglected at large momenta and why it is sufficient to use standard

gauge fixing schemes as the basis for calculations in perturbative QCD.

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