

PUBLISHED VERSION

Maltman, Kim; Wolfe, Carl E.

[Isospin Breaking in the Relation Between the \$\tau^- \rightarrow \nu \pi^- \pi^0\$ and \$e^+e^- \rightarrow \pi^+\pi^-\$ Versions of \$|F\(s\)|^2\$ and Implications for \$\(g - 2\)_\mu\$](#) Physical Review D, 2006; 73(1):013004

©2006 American Physical Society

<http://link.aps.org/doi/10.1103/PhysRevD.73.013004>

PERMISSIONS

<http://publish.aps.org/authors/transfer-of-copyright-agreement>

“The author(s), and in the case of a Work Made For Hire, as defined in the U.S. Copyright Act, 17 U.S.C.

§101, the employer named [below], shall have the following rights (the “Author Rights”):

[...]

3. The right to use all or part of the Article, including the APS-prepared version without revision or modification, on the author(s)' web home page or employer's website and to make copies of all or part of the Article, including the APS-prepared version without revision or modification, for the author(s)' and/or the employer's use for educational or research purposes.”

7th June 2013

<http://hdl.handle.net/2440/34433>

Isospin breaking in the relation between the $\tau^- \rightarrow \nu_\tau \pi^- \pi^0$ and $e^+ e^- \rightarrow \pi^+ \pi^-$ versions of $|F_\pi(s)|^2$ and implications for $(g - 2)_\mu$

Kim Maltman*

Department of Mathematics and Statistics, York University, 4700 Keele St., Toronto, ON Canada M3J 1P3[‡]Carl E. Wolfe[‡]

Department of Physics and Astronomy, York University, 4700 Keele St., Toronto, ON Canada M3J 1P3

(Received 3 November 2005; published 12 January 2006)

We investigate two points related to existing treatments of isospin-breaking corrections to the conserved vector current relation between $\sigma[e^+ e^- \rightarrow \pi^+ \pi^-]$ and $d\Gamma[\tau^- \rightarrow \nu_\tau \pi^- \pi^0]/ds$. Implications for the value of the hadronic contribution to $a_\mu = (g - 2)_\mu/2$ based on those analyses incorporating hadronic τ decay data are also considered. We conclude that the uncertainty on the isospin-breaking correction which must be applied to the τ decay data should be significantly increased, and that the central value of the ρ - ω mixing contribution to this correction may be significantly smaller than indicated by the present standard determination. Such a shift would contribute to reducing the discrepancy between the τ - and electroproduction-based determinations of the leading order hadronic contribution to a_μ .

DOI: 10.1103/PhysRevD.73.013004

PACS numbers: 13.40.Em, 13.35.Dx, 13.66.Bc, 14.60.Ef

I. INTRODUCTION

It is well known that, after the large, purely leptonic contribution, the largest of the remaining standard model (SM) contributions to the anomalous magnetic moment of the muon, $a_\mu \equiv (g - 2)_\mu/2$, is that due to the leading order hadronic vacuum polarization, $[a_\mu]_{\text{had}}^{LO}$. This contribution may be evaluated, in terms of experimental $e^+ e^- \rightarrow$ hadrons cross section data, using the dispersion integral representation [1]

$$[a_\mu]_{\text{had}}^{LO} = \frac{\alpha_{EM}^2(0)}{3\pi^2} \int_{4m_\pi^2}^{\infty} ds \frac{K(s)}{s} R(s), \quad (1)$$

where the form of $K(s)$ is well known [1] and $R(s)$ is the ratio of the “bare” $e^+ e^- \rightarrow$ hadrons cross section to that for $e^+ e^- \rightarrow \mu^+ \mu^-$ [2]. Since the isovector part of the electromagnetic (EM) spectral function is related by the conserved vector current (CVC) hypothesis to the charged current isovector vector spectral function, which can be obtained from the invariant mass distribution of states with zero net strangeness in the decay in $\tau^- \rightarrow \nu_\tau +$ hadrons, the high precision hadronic τ decay data of Refs. [4–6] can, in principle, be used to improve the determination of $[a_\mu]_{\text{had}}^{LO}$ [7–11].

The high accuracy achieved by the current experimental determination of a_μ [12] places a significant premium on reducing the error on $[a_\mu]_{\text{had}}^{LO}$, which currently dominates the uncertainty on the SM prediction for a_μ (see Ref. [13] for a recent review). At the desired level of precision, the τ decay data can be used only after taking into account the small isospin-breaking (IB) corrections to the CVC rela-

tion between the charged and neutral current isovector spectral functions. A detailed investigation of possible sources of such corrections, for the numerically dominant $\pi\pi$ contribution, has been made in Refs. [14,15], and the resulting s -dependent IB correction factor incorporated into the latest τ -based analyses [11,16] of $[a_\mu]_{\text{had}}^{LO}$. A comparison of the corrected, τ -based spectral data with that obtained from the recent high-precision CMD-2 experiment [17], however, shows significant residual disagreement in the $\pi\pi$ components of the two versions of the isovector spectral function [16]: the two are compatible below, and in the vicinity of, the ρ peak, but differ by $\sim 5\%$ - 10% for $m_{\pi\pi}$ between ~ 0.85 and ~ 1 GeV [16]. This discrepancy leads to incompatible determinations of $[a_\mu]_{\text{had}}^{LO}$, the τ -based determination lying $\sim 2\sigma$ higher [11,16,18], and producing a SM prediction for a_μ in agreement with the experimental result, while the EM-based determination yields a SM prediction which differs from experiment by $\sim 2.5\sigma$ (see Ref. [11] and references therein for more details).

The preliminary KLOE $e^+ e^- \rightarrow \pi^+ \pi^-$ radiative return data [19] supports the earlier EM-based determination, yielding a value of $[a_\mu]_{\text{had}}^{LO}$ compatible with that obtained using the CMD-2 $\pi\pi$ data [11]. However, as has been pointed out by many authors, the point-by-point agreement between the CMD-2 and KLOE cross sections is less than satisfactory [11], the KLOE data lying higher than CMD-2 below the ρ peak and lower than CMD-2 both on the peak and above it. The structure of the weight $K(s)/s$ is such that the effects of these discrepancies largely cancel in $[a_\mu]_{\text{had}}^{LO}$, but the situation nonetheless remains unsatisfactory.

Differences in the ρ^0 and ρ^\pm masses and widths, suggested as one possibility for resolving the $\pi\pi$ spectral function discrepancy [20], appear able to reduce locally, but not resolve fully the discrepancy [11].

*Electronic address: kmaltman@yorku.ca

[‡]CSSM, University of Adelaide, Adelaide, SA 5005, Australia[‡]Electronic address: wolfe@yorku.ca

Recent developments further complicate the picture. In Ref. [21], QCD sum rule constraints on the electroproduction and τ decay data were investigated. Sum rules of the form

$$\int_{s_{th}}^{s_0} ds w(s) \rho(s) = \frac{-1}{2\pi i} \int_{|s|=s_0} ds w(s) \Pi(s) \quad (2)$$

were employed, where $\Pi(s)$ is either the EM or charged isovector vector current correlator, $\rho(s)$ is the corresponding spectral function, s_{th} is the relevant threshold, and $w(s)$ is a function analytic inside and on the contour $|s| = s_0$. The OPE is employed on the right-hand side (RHS), providing the desired constraints. At the scales employed, the operator product expansion (OPE) for the vector current correlators is essentially entirely dominated by the dimension $D = 0$ perturbative contribution, and hence determined by the single input, α_s . This input may be taken from high-scale determinations of $\alpha_s(M_Z)$ which are independent of the EM and τ data being tested. It turns out that both the normalization and s_0 -dependence of the weighted spectral integrals generated from the hadronic τ decay data are in excellent agreement with OPE expectations [21]. In contrast, the weighted EM spectral integrals (obtained using CMD-2 data for the $\pi\pi$ spectral component) do not agree with OPE expectations, having (i) normalizations which are $\sim 2\sigma$ low, and (ii) slopes with respect to s_0 which are $\sim 2.5\sigma$ low [21]. These observations suggest either a problem with the EM data, or the presence of non-negligible non-one-photon physics contributions to the EM cross sections. In either case, the results favor determinations of $[a_\mu]_{had}^{LO}$ which incorporate hadronic τ decay data over those based on EM data alone, and a SM prediction for a_μ in agreement with the current Brookhaven (BNL) experimental result [12]. The recently released SND $e^+e^- \rightarrow \pi^+\pi^-$ cross section results [22] are compatible with the IB-corrected τ data, and support this conclusion.

In light of the above unsettled situation, we revisit the question of the reliability of the determination of the IB corrections which must be applied to the τ decay data, focussing on two aspects of the existing treatment. We denote the correction to $[a_\mu]_{had}^{LO}$ associated with these IB corrections by $[\delta a_\mu]_{had}^{LO}$.

The first point concerns the uncertainty on the estimate for the contribution to $[\delta a_\mu]_{had}^{LO}$ associated with “ ρ - ω mixing” [23] (present in the EM, but not the τ , spectral function). The most recent updates of the τ -based evaluation of $[a_\mu]_{had}^{LO}$ [11,16], employ the IB corrections of Ref. [15] (CEN). The CEN analysis is based on a version of the ChPT-constrained model for $F_\pi(s)$ developed by Guerrero and Pich [25] (GP). The original GP model, which involved only the isospin-conserving (IC) component of $F_\pi(s)$, was modified by CEN through the addition of an IB $\omega \rightarrow \pi\pi$ contribution having the nominal ρ - ω mixing form. We refer to the resulting model as the GP/

CEN model. Using the parameter values given by CEN, that part of the full IB correction associated with ρ - ω mixing becomes

$$[\delta a_\mu]_{had:mix}^{LO} = (3.5 \pm 0.8) \times 10^{-10}, \quad (3)$$

with the quoted uncertainty due essentially entirely to the 20% uncertainty on the parameter $\theta_{\rho\omega}$, which describes the overall strength of ρ - ω “mixing” in the model. The uncertainty in Eq. (3) represents only a minor component of the total $\pm 2.6 \times 10^{-10}$ uncertainty quoted by CEN for the full set of IB corrections [15].

The GP/CEN model, however, is not the only one available for $F_\pi(s)$. The Gounaris-Sakurai (GS) model [26], the Kuhn-Santamaria (KS) model [27], and the hidden local symmetry (HLS) model [28], for example, all predate the GP/CEN model and have been used extensively in the literature. The models differ in the form employed for the broad IC component of the $e^+e^- \rightarrow \pi^+\pi^-$ amplitude, which is given (or dominated) by the $e^+e^- \rightarrow \rho^0 \rightarrow \pi^+\pi^-$ contribution. Implicit in the CEN error estimate is the (given the narrowness of the ω , plausible) assumption that the value obtained for $[\delta a_\mu]_{had:mix}^{LO}$ will be largely insensitive to which of the models is employed in extracting the interference signal. The high level of cancellation in the $K(s)/s$ -weighted integrals of the interference components of the various model cross sections, however, makes $[\delta a_\mu]_{had:mix}^{LO}$ *much* more model-dependent than would be naturally anticipated. The resulting theoretical systematic error turns out to significantly exceed that associated with uncertainties on the fitted model parameters for any given model, including the GP/CEN model. This point is discussed in more detail in Section II below.

The second point concerns an IB correction not accounted for in the CEN analysis. In the limit that (as for the IC component) the IB component of the $e^+e^- \rightarrow \pi^+\pi^-$ amplitude is assumed dominated, away from threshold, by resonance contributions, three such contributions will, in principle, be present in the ρ , ω resonance region. These are shown in Fig. 1, where the open circles represent IC vertices and the crossed circles IB vertices. J_μ^3 and J_μ^8 are the isovector and isoscalar members of the vector current octet. The first two graphs represent the ρ - ω mixing and direct IB $\omega \rightarrow \pi\pi$ decay contributions to the amplitude. They are small away from the ω peak region, generate contributions to the flavor ‘38’ part of the EM spectral function, and combine to produce the prominent narrow interference shoulder in the experimental cross section. The remaining graph depicts the contribution associated with the IB (isoscalar) component of the ρ^0 EM decay constant. Such a component of the decay constant is unavoidable in the SM.

Because of the narrowness of the ρ - ω interference shoulder, the interference part of the cross section can, modulo the model dependence noted above, be determined experimentally. The corresponding contribution to

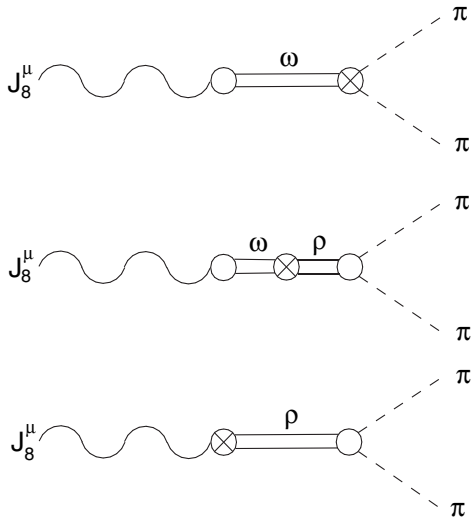


FIG. 1. Isospin-breaking resonance contributions to $e^+e^- \rightarrow \pi^+\pi^-$.

$[\delta a_\mu]_{\text{had}}^{LO}$ can thus also, with the same caveat, be determined experimentally. In contrast, the interference contribution associated with the isoscalar ρ^0 EM decay constant (which also belongs to the flavor '38' part of the cross section) is identical in shape to the dominant, broad, IC flavor '33' contribution and not (even in principle) extractable experimentally. The corresponding contribution to $[\delta a_\mu]_{\text{had}}^{LO}$, which is certainly present at some level in the SM, has not, to our knowledge, been investigated in the literature, and certainly is *not* included in the treatment of IB corrections employed by CEN, for the reason explained below. It is, in fact, analogous to the ρ - ω mixing contribution, which was also not present in the GP model approach [14], and hence had to be added by hand to the GP model expression by CEN [15].

The reason the three IB resonance contributions to $F_\pi(s)$ shown in Fig. 1 are not incorporated in the GP model framework is as follows. The GP model is constructed by implementing the constraints of unitarity, analyticity, and short-distance QCD, and requiring that the model expression for $F_\pi(s)$ match properly onto the known next-to-leading order (NLO) ChPT expression at low energy. This last constraint is realized using the resonance chiral effective theory approach, in which low-energy resonance effects appear through contributions proportional to the NLO low-energy constants (LEC's), L_k^r , of Gasser and Leutwyler [29]. It is, however, straightforward to demonstrate that, at NLO, $\langle \pi^+\pi^- | J_\mu^8 | 0 \rangle$ receives contributions only from loops, and not from the NLO LEC's [30]. As a result, none of the resonance-induced IB effects depicted in the figure are incorporated in the GP expression for $F_\pi(s)$. Although, numerically, the IB loop effects are tiny near threshold [30], the obvious experimental interference shoulder in the ρ - ω region shows that this does not remain the case at higher energies. In order to include ρ - ω inter-

ference, it was thus necessary for CEN to add a ρ - ω mixing contribution to the GP model by hand. The broad IB ρ contribution is similarly absent in the GP model approach, and would also have to be added by hand. In Section III we investigate sum rule constraints on this contribution, and, in addition, use the size of the analogous effect in the pseudoscalar sector, as evaluated at NLO in the chiral expansion, to obtain some guidance as to what the natural scale of the effect might be.

A brief summary, and our conclusions, is given in Section IV.

II. MODEL DEPENDENCE OF $[\delta a_\mu]_{\text{had};\text{mix}}^{LO}$

The pion form factor, in the GS model, is given by [26]

$$F_\pi^{(GS)}(s) = \frac{1}{(1 + \beta)} \left(\text{BW}_\rho^{(GS)}(s) \left[1 + \delta \frac{s}{m_\omega^2} P_\omega(s) \right] + \beta \text{BW}_{\rho'}^{(GS)}(s) \right), \quad (4)$$

where

$$P_\omega(s) = \frac{m_\omega^2}{(m_\omega^2 - s - im_\omega \Gamma_\omega)}, \quad (5)$$

$$\text{BW}_V^{(GS)}(s) = \frac{m_V^2 \left(1 + d(m_V) \frac{\Gamma_V}{m_V} \right)}{(m_V^2 - s + f(s, m_V, \Gamma_V) - im_V \Gamma_V(s, m_V, \Gamma_V))}$$

with

$$\begin{aligned} d(m_V) &= \frac{3m_\pi^2}{(\pi p_\pi^2(m_V^2))} \ell n \left(\frac{(m_V + 2p_\pi(m_V^2))}{2m_\pi} \right) \\ &\quad + \frac{m_V}{(2\pi p_\pi(m_V^2))} - \frac{m_\pi^2 m_V}{(\pi p_\pi^3(m_V^2))}, \\ f(s, m_V, \Gamma_V) &= \frac{\Gamma_V m_V^2}{p_\pi^2(m_V^2)} \left(p_\pi^2(s) [H(s) - H(m_V^2)] \right. \\ &\quad \left. + (m_V^2 - s) p_\pi^2(m_V^2) \frac{dH}{ds}(m_V^2) \right), \\ H(s) &= \frac{2p_\pi(s)}{\pi \sqrt{s}} \ell n \left(\frac{\sqrt{s} + 2p_\pi(s)}{2m_\pi} \right), \end{aligned} \quad (6)$$

where $p_\pi(s) = \sqrt{\frac{s}{4} - m_\pi^2}$ is the pion momentum in the center of mass system for squared invariant mass s , $\Gamma_V(s, m_V, \Gamma_V)$ is the standard s -dependent width for vector meson V implied by p -wave phase space, and $\Gamma_V = \Gamma_V(m_V^2, m_V, \Gamma_V)$.

Similarly, for the KS model, one has [27]

$$F_\pi^{(KS)}(s) = \left(\frac{P_\rho(s) \left(\frac{1 + \delta P_\omega(s)}{1 + \delta} \right) + \beta P_{\rho'}(s) + \gamma P_\rho''(s)}{1 + \beta + \gamma} \right) \quad (7)$$

with

$$P_V(s) = \frac{m_V^2}{m_V^2 - s - im_V \Gamma_V(s, m_V, \Gamma_V)}. \quad (8)$$

The HLS model [28], as implemented by CMD-2, has the form

$$F_\pi^{(HLS)}(s) = 1 - \frac{a_{HLS}}{2} + \frac{a_{HLS}}{2} \left(\frac{P_\rho(s)(1 + \delta P_\omega(s))}{1 + \delta} \right) \quad (9)$$

with a_{HLS} a constant. The model provides a good quality fit to the data below 1 GeV despite having no explicit ρ' contribution. It also turns out to reproduce the correct final state $\pi\pi$ phases after the model parameters have been fitted [31].

For all of the GS, KS, and HLS models, the constant δ , which parametrizes the strength of the narrow IB amplitude, is taken to be complex. A nonzero phase is, in general, unavoidable in the presence of an IB direct $\omega \rightarrow \pi\pi$ decay contribution [23].

The GP model for the pion form factor is given by [25]

$$F_\pi^{(GP)}(s) = P_\rho(s) \exp\left(\frac{-s}{96\pi^2 f_\pi^2} \left[\text{Re}L\left(\frac{m_\pi^2}{s}, \frac{m_\pi^2}{m_\rho^2}\right) + \frac{1}{2} \text{Re}L\left(\frac{m_K^2}{s}, \frac{m_K^2}{m_\rho^2}\right) \right]\right), \quad (10)$$

where

$$L\left(\frac{m^2}{s}, \frac{m^2}{m_\rho^2}\right) = \ell n\left(\frac{m^2}{m_\rho^2}\right) + \frac{8m^2}{s} - \frac{5}{3} + \beta(s)^3 \ell n\left[\frac{\beta(s) + 1}{\beta(s) - 1}\right] \quad (11)$$

with $\beta(s) = \sqrt{1 - 4m^2/s}$ and the s -dependent width, $\Gamma_\rho(s, m_\rho, \Gamma_\rho)$ appearing in $P_\rho(s)$ replaced by the resonance chiral effective theory expression

$$\Gamma_\rho(s) = \frac{m_\rho s}{96\pi f_\pi^2} \left(\theta(s - 4m_\pi^2) \beta_\pi(s)^3 + \frac{1}{2} \theta(s - 4m_K^2) \beta_K(s)^3 \right). \quad (12)$$

Some IB effects are incorporated into $F_\pi^{(GP)}(s)$ if one evaluates the phase space factors in the s -dependent width using the physical charged π and K masses. In Ref. [15], a small rescaling of the coefficient appearing on the RHS of Eq. (12) is allowed in order to account for the ~ 1.5 MeV contribution of $\pi\pi\gamma$ decays to the total width of the ρ [15,32]. The CEN modification of $F_\pi^{(GP)}(s)$, designed to incorporate the ρ - ω mixing contribution not included in the original GP model, then has the form

$$F_\pi^{(GP/CEN)}(s) = F_\pi^{(GP)}(s) - P_\rho(s) \left(\frac{\theta_{\rho\omega}}{3m_\rho^2} \right) \left(\frac{s}{m_\omega^2} \right) P_\omega(s). \quad (13)$$

The parameter $\theta_{\rho\omega}$ was assumed real by CEN.

The original version of the GP/CEN model, as parametrized by CEN, predates the most recent, corrected version of the CMD-2 data, and does not provide a good fit to it, producing a χ^2 of 80 for the 43 CMD-2 data points. The fit quality can be improved by allowing m_ρ , $\theta_{\rho\omega}$ (still assumed real) and the rescaling of the resonance chiral effective theory width to be fit to data, but the resulting optimized fit still has a χ^2 of 61 for the resulting 40 degrees of freedom. The corresponding mixing contribution, $[\delta a_\mu]_{\text{had;mix}}^{LO}$, is shifted only slightly from the original CEN value, from 3.5×10^{-10} to 3.7×10^{-10} . The fact that the ρ - ω mixing signal is actually a combination of mixing and direct $\omega \rightarrow \pi\pi$ effects, however, means that an effective representation for the combination of the form given by Eq. (13) is not generally possible without allowing $\theta_{\rho\omega}$ to have a nonzero phase [24]. If we extend the GP/CEN model in this way, treating the phase as a fourth parameter to be fit to the data, an acceptable fit, having $\chi^2 = 41$ for the remaining 39 degrees of freedom, becomes possible. We refer to this version of the GP/CEN model as GP/CEN* in the table below.

The values obtained for $[\delta a_\mu]_{\text{had;mix}}^{LO}$ in the various models are given in Table I. All results are generated using versions of the models optimized to the most recent versions of the CMD-2 [17] *bare* cross section data [33]. Only the GP/CEN* version of the GP/CEN model is included since the unmodified version does not produce an acceptable quality fit.

Two things are evident from the table. First, the sensitivity of $[\delta a_\mu]_{\text{had;mix}}^{LO}$ to the model employed is much larger than that associated with the uncertainties on the values of the fitted model parameters for a given model. Second, by comparing the GP/CEN* results to those for the GP/CEN version of the model (quoted above), we see that allowing a nonzero phase for the IB parameter $\theta_{\rho\omega}$ leads to a significant decrease in $[\delta a_\mu]_{\text{had;mix}}^{LO}$. (A similar effect is produced by the phase of δ in the other models.) The origin of these effects is easy to understand. In the interference region, the IC amplitude is $\simeq B_\rho(s)$, where $B_\rho(s)$ is the $\rho(770)$ Breit-Wigner-like form in the given model. Writing the IB amplitude in the “ ρ - ω mixing” form, generically $B_\rho(s)\delta P_\omega$, with $\delta = |\delta|e^{i\phi}$, the flavor ‘38’ component of the EM cross section is then given approximately by the expression

TABLE I. $[\delta a_\mu]_{\text{had;mix}}^{LO}$ for the various models discussed in the text fit to the most recent CMD-2 bare $e^+e^- \rightarrow \pi\pi$ cross sections.

| Model | χ^2/dof | $[\delta a_\mu]_{\text{had;mix}}^{LO} \times 10^{10}$ |
|---------|---------------------|---|
| GS | 36/38 | 2.0 ± 0.5 |
| HLS | 37/38 | 4.0 ± 0.6 |
| KS | 37/38 | 3.9 ± 0.6 |
| GP/CEN* | 41/39 | 2.0 ± 0.5 |

$$|B_\rho(s)|^2 \left[\frac{2|\delta|m_\omega^2}{[(m_\omega^2 - s)^2 + m_\omega^2 \Gamma_\omega^2]} (\cos(\phi)(m_\omega^2 - s) - m_\omega \Gamma_\omega \sin(\phi)) \right]. \quad (14)$$

Since the coefficient multiplying $\cos(\phi)$ in the square bracket of Eq. (14) is antisymmetric about $s = m_\omega^2$, the corresponding contribution to $[\delta a_\mu]_{\text{had:mix}}^{LO}$ vanishes in the limit that one neglects the variation of $|B_\rho(s)|^2 K(s)/s$ over the ω region. Since both $|B_\rho(s)|^2$ and $K(s)/s$ are decreasing functions in this region, a small residual positive contribution remains. The coefficient of $\sin(\phi)$, in contrast, is symmetric, so no analogous cancellation is present in the corresponding contribution to $[\delta a_\mu]_{\text{had:mix}}^{LO}$, i.e., the $\sin(\phi)$ integral is strongly enhanced relative to the $\cos(\phi)$ integral. It is the strong cancellation in the $\cos(\phi)$ integral, combined with small differences in the s -dependence of $B_\rho(s)$ in the different models, which accounts for the significant model dependence in the results for $[\delta a_\mu]_{\text{had:mix}}^{LO}$. Since fits to the data favor small positive ϕ for all the models considered here, the $\sin(\phi)$ contribution to $[\delta a_\mu]_{\text{had:mix}}^{LO}$ is negative. The relative enhancement of the $\sin(\phi)$ integral means that the cancellation against the $\cos(\phi)$ contribution, which is absent if one sets ϕ to zero from the outset, can be quite significant, even for relatively small ϕ .

Table II shows the impact of the choice of input data set on $[\delta a_\mu]_{\text{had:mix}}^{LO}$, giving the central values corresponding to optimized fits of each model to the bare CMD-2 and SND cross sections. In the case of the KLOE data, the optimized fits for all four models have $\chi^2/\text{dof} > 2$ [35]; the corresponding $[\delta a_\mu]_{\text{had:mix}}^{LO}$ results have therefore been omitted from the table. A $\chi^2/\text{dof} > 2$ is also obtained for the optimized fit of the GP/CEN* model to the SND data; the corresponding entry in the table, though included for completeness, has been enclosed in parentheses to remind the reader of this fact [36]. From the table we see that the variation in $[\delta a_\mu]_{\text{had:mix}}^{LO}$ values among the different models, for a fixed input data set, is significantly larger than the variation of the results for a given model over the different input data sets. It is thus the theoretical systematic error associated with choice of model used in separating the IC and IB components of the amplitude which dominates the uncertainty in the determination of $[\delta a_\mu]_{\text{had:mix}}^{LO}$.

TABLE II. Central values of $[\delta a_\mu]_{\text{had:mix}}^{LO}$, in units of 10^{-10} , for the optimized fits of the various models to the bare CMD-2 [17] and SND [22] $e^+e^- \rightarrow \pi\pi$ cross sections.

| Model | CMD-2 | SND |
|---------|-------|-------|
| GS | 2.0 | 2.2 |
| HLS | 4.0 | 4.5 |
| KS | 3.9 | 4.3 |
| GP/CEN* | 2.0 | (1.6) |

TABLE III. Central values for $[\delta a_\mu]_{\text{had:mix}}^{LO}$ for earlier model fits in the literature.

| Model | Reference | χ^2/dof | $[\delta a_\mu]_{\text{had:mix}}^{LO} \times 10^{10}$ |
|--------|-----------|---------------------|---|
| GS | ALEPH97 | 59/38 | 4.3 |
| KS | ALEPH97 | 75/38 | 6.3 |
| GS | DAVIER03 | 65/38 | 2.5 |
| GP/CEN | CEN02 | 80/42 | 3.5 |

For comparison, and to further illustrate the sensitivity of the mixing contribution to small changes in the data and the resulting model fits, results corresponding to some older fits from the literature are given in Table III. ‘‘CEN02’’ labels the original GP/CEN result [15] (with no phase for the parameter $\theta_{\rho\omega}$), ‘‘ALEPH97’’ the results corresponding to the ‘‘combined’’ (τ plus electroproduction) GS and KS fits of Ref. [4], and ‘‘DAVIER03’’ the results corresponding to the similarly combined GS fit (Table 3) of Ref. [34]. Details of the fit procedures, and data sets employed may be found in the original references. The total χ^2 of the fits *relative to the 2003 CMD-2 data* are also given. The quoted results correspond in each case to the central values of the fit parameters for the models. Typically, differences between the old and new values of $[\delta a_\mu]_{\text{had:mix}}^{LO}$ for a given model are much larger than one might have anticipated, given the relatively small changes in both the data and fitted parameter values. This sensitivity is again a reflection of the strong cancellation in the integral of the product of IC and IB amplitudes.

We conclude that the inability to separate the IC and IB components of the $e^+e^- \rightarrow \pi\pi$ amplitudes in a model independent manner leads to a significant uncertainty in the evaluation of $[\delta a_\mu]_{\text{had:mix}}^{LO}$. How one assesses this uncertainty depends on one’s attitude to the various models. One stance might be to argue that the GS and GP/CEN models, which explicitly incorporate the constraints of unitarity and analyticity, are to be favored in deciding on a central value. The variation of the results across the different models would then serve as a measure of the residual uncertainty. Alternately, since all of the models are, to greater or lesser extent, phenomenological, and, at least for the CMD-2 data, yield comparable quality fits, one could instead average the results to arrive at a central value, and assign an error large enough to incorporate the highest and lowest values allowed by the errors associated with those on the fitted parameters for the various models. The first stance would yield

$$[\delta a_\mu]_{\text{had:mix}}^{LO} = (2.0 \pm 2.9) \times 10^{-10}, \quad (15)$$

the second

$$[\delta a_\mu]_{\text{had:mix}}^{LO} = (3.1 \pm 1.8) \times 10^{-10}. \quad (16)$$

A smaller central value, and significantly larger uncertainty, would result if one ignored the poor quality of the

model fits and also took the KLOE-based results into account [37]. A variant of the first stance would be to average only those results obtained from the GS and GP/CEN* models, with an error sufficient to cover the range of values obtained from either model and from either of the SND or CMD-2 sets. This would yield a result

$$[\delta a_\mu]_{\text{had;mix}}^{LO} = (2.0 \pm 0.7) \times 10^{-10} \quad (17)$$

with significantly smaller errors (comparable to those of CEN). While it is true that the treatment of unitarity and analyticity in these models provides an argument in their favor, the explicit shapes of the GS and GP/CEN* $\pi\pi$ cross sections in the ρ region cannot be said to be a rigorous consequence of QCD. Since it is the small differences in the s -dependence of the cross sections in the interference region which are responsible for the model dependence of the $[\delta a_\mu]_{\text{had;mix}}^{LO}$ results above, and since the data at present is not sufficiently precise to distinguish (in terms of quality of fit) between the s -dependences predicted by the different models, we consider this latter stance, though defensible, potentially insufficiently conservative in its treatment of errors.

III. THE BROAD IB ρ CONTRIBUTION

Although the broad IB ρ contribution to the experimental $e^+e^- \rightarrow \pi^+\pi^-$ cross section has the same shape as the dominant IC contribution, and hence cannot be separated from it experimentally, it can, in principle, be determined theoretically through a QCD sum rule analysis of the IB vector current correlator, $\Pi^{38}(q^2)$, defined by

$$\begin{aligned} & i \int d^4x e^{iq \cdot x} \langle 0 | T(J_\mu^3(x) J_\nu^8(0)) | 0 \rangle \\ & \equiv (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi^{38}(q^2). \end{aligned} \quad (18)$$

A nonzero (IB) coupling of the ρ to J_μ^8 will produce a broad ρ contribution to the spectral function of Π^{38} , $\rho^{38}(s)$, whose strength, X_ρ , is proportional to the product of the IC isovector and IB isoscalar ρ^0 decay constants. This product, together with analogous IB products, X_V , for the other vector meson resonances, can, in principle, be determined by matching the appropriate weighted integrals of ρ^{38} to the corresponding OPE expressions.

In Ref. [38] such an analysis was performed using two different families of “pinch-weighted” finite energy sum rules (pFESR’s) [39]. The details of the analysis may be found in Ref. [38], and will not be repeated here. The following point is, however, worth noting. Because the numerically dominant term on the OPE side of the various sum rules is that with dimension $D = 4$, the contribution from the VEV’s of $D = 6$ four-quark operators, $c_6 O_6$, is not expected to be negligible. The fact that the VEV’s for such operators are typically not well known empirically

would normally present a problem for the sum rule analysis. It turns out that the dependence of the X_V on $c_6 O_6$ is different for the two different pFESR families, allowing, not only the X_V , but also $c_6 O_6$, to be determined from the combined analysis. The values of $c_6 O_6$ which make the different X_V consistent turn out to agree at the $\sim 1\%$ level, providing strong support for the reliability of the analysis.

Unfortunately, in the analysis of Ref. [38], the ρ - ω mixing contribution to $\rho^{38}(s)$ implied by the observed interference shoulder in the EM cross section was not input separately on the spectral side of the sum rules employed. As a result, the output X_ρ contains contributions from all three of the IB processes shown in Fig. 1. In order to separate out the experimentally inaccessible contribution associated with the third of those processes, we have redone the analysis of Ref. [38], this time inputting the “interference” component of ρ^{38} , as determined from the CMD-2 experimental data in the interference region. This input, as noted above, depends to some extent on the choice of model for the ρ contribution to the IC component of the amplitude. We have then used the various pFESR’s to solve for the residual broad ρ contribution, which yields directly the contribution to the flavor ‘38’ part of the EM spectral function associated with the third graph in Fig. 1. With current experimental errors, an accurate determination of this “direct” contribution, and hence of the associated contribution to $[\delta a_\mu]_{\text{had}}^{LO}$, $[\delta a_\mu]_{\text{had;direct}}^{LO}$, turns out to be impossible. We find central values of a natural scale (see below), but with errors, induced by the uncertainty in the integrated (model-dependent) ρ - ω interference term, much larger than these central values. Even more unfortunately, versions of the analysis using errors scaled down artificially by hand suggest that the reduction needed to allow even just a reliable determination of the sign of the direct effect are unlikely to be reachable in the foreseeable future.

In view of the weakness of the constraints arising from the sum rule analysis, we turn to the pseudoscalar sector, and study the size of analogous effects in the IB decay constant, f_π^8 , of the π^0 , defined by

$$\langle 0 | A_\mu^8 | \pi^0 \rangle = i f_\pi \epsilon_1 q_\mu \equiv i f_\pi^8 q_\mu. \quad (19)$$

At leading order in the chiral expansion, the IB parameter $\epsilon_1 = f_\pi^8/f_\pi$ is equal to $\theta_0 = \sqrt{3}(m_d - m_u)/4(m_s - \hat{m})$, where $\hat{m} = (m_d + m_u)/2$, and is due entirely to IB mixing on the external leg, induced by the nonzero value of $m_d - m_u$. At NLO, ϵ_1 receives contributions both from mixing and from IB in the low-energy representation of the axial current A_μ^8 . The full NLO expression for ϵ_1 is given in Ref. [29], while the ingredients necessary for separating the mixing and vertex contributions may be found in Ref. [40]. At NLO the separate mixing and vertex contributions are, as expected on physical grounds, scale dependent. The (scale-dependent) direct (vertex) contribution is

given by

$$\begin{aligned} \left[\frac{f_\pi^8}{f_\pi} \right]_{\text{direct}} &= \left(\frac{16m_K^2 + 4m_\pi^2}{f_\pi^2} \right) L_4^r(\mu) + 8 \frac{m_\pi^2}{f_\pi^2} L_5^r(\mu) \\ &\quad - \frac{m_K^2 \ell n(m_K^2/\mu^2)}{8\pi^2 f_\pi^2} + \left(\frac{m_K^2 - m_\pi^2}{12\pi^2 f_\pi^2} \right) \\ &\quad \times [1 + \ell n(m_K^2/\mu^2)], \end{aligned} \quad (20)$$

where μ is the ChPT renormalization scale, and $L_{4,5}^r(\mu)$ are the renormalized NLO LEC's, evaluated at scale μ [29].

Taking, for illustration, the central values for $L_{4,5}^r$ from the main fit of Ref. [41], one finds that the leading (NLO) contribution to f_π^8 varies from 0.58% to 0.46% of f_π as μ varies from m_η to 1 GeV. Using the slightly modified fit denoted ‘‘fit D’’ in Ref. [42], which produces good values for the threshold parameters for $\pi\pi$ and πK scattering, the ratio runs from 0.60% to 0.71% over the same range of μ . We thus conclude that the NLO direct contribution to the IB decay constant ratio is $\sim 1/2\%$ for typical hadronic scale choices. A similar value for the corresponding IB ratio of ρ^0 decay constants would produce a contribution $[\delta a_\mu]_{\text{had;direct}}^{LO} \simeq (2-3) \times 10^{-10}$. The central value obtained from the sum rule analysis turns out to be comparable to, or less than, this natural size for all four models considered here. The upper bound implied by the errors, however, is much larger, preventing us from using the sum rule constraints in a meaningful way.

Note that the presence of a small IB component in the ρ^0 EM decay constant would have an impact on the values of m_ρ and Γ_ρ obtained by fitting the various phenomenological models to data. At the natural scale (given above) for this effect, however, the (model-dependent) shift in m_ρ would be ~ 0.2 MeV or less, and that in Γ_ρ 0.4 MeV or less. The effect, while contributing to the apparent difference in charged and neutral ρ masses and widths, can thus account for at best a modest fraction of the differences obtained after fitting the models independently to the τ decay and electroproduction data.

IV. CONCLUSIONS

We have shown that the model dependence encountered in separating the IB from the IC component of the $e^+e^- \rightarrow \pi^+\pi^-$ cross section leads to a theoretical systematic uncertainty on the ρ - ω mixing contribution to $[\delta a_\mu]_{\text{had}}^{LO}$ which is several times that associated with the fit parameter uncertainties for any given model. There is also a potentially non-negligible IB contribution, associated with the direct IB coupling of the isoscalar part of the EM current to the ρ^0 , which cannot, even in principle, be determined experimentally. At present, we are able only to make a rough ‘‘natural size’’ estimate for the magnitude of this

contribution. Both of these effects would need to be taken into account when estimating the uncertainty in the IB correction required in order to incorporate the $\tau^- \rightarrow \nu_\tau \pi^- \pi^0$ spectral data into the evaluation of $[a_\mu]_{\text{had}}^{LO}$. The results make it clear that evaluating the mixing component of the IB correction using only a single model in the analysis of the cross section will lead to a significant underestimate of the uncertainties.

In view of the results of the tests involving independent high-scale OPE constraints reported in Ref. [21], and the compatibility of the SND and $\tau\pi\pi$ data sets, we base our final results on the SND data set. (The CMD-2 and KLOE values, quoted above, allow alternate choices to be made.) Choosing the GS model as the favored case would lead to a decrease of 1.3×10^{-10} in $[\delta a_\mu]_{\text{had;mix}}^{LO}$, relative to the standard CEN value. Such a shift would *lower* the τ -based prediction for $[\delta a_\mu]_{\text{had}}^{LO}$ in the SM, slightly increasing the difference between the SM prediction and the central experimental value, but leaving them compatible at the 1σ level. For a given model, the results for $[\delta a_\mu]_{\text{had;mix}}^{LO}$ obtained using either the CMD-2 or SND data are actually in good agreement. The decrease in central value relative to CEN would thus also reduce the discrepancy between the CMD-2 and τ -based determinations of the $\pi\pi$ contribution to $[\delta a_\mu]_{\text{had}}^{LO}$ by $\sim 10\%$. The HLS model, which produces an optimized fit to the SND data of comparable quality to that of the GS model, in contrast, shifts the mixing contribution *up* by 1.0×10^{-10} , relative to the CEN value. Since the GS model has the constraints of analyticity and unitarity explicitly built into it, while the HLS model does not, we have favored the GS model in our discussion of the central value above. It is, however, important to bear in mind that, in terms of quality of fit to the SND data, the models cannot be distinguished, and hence that a significant uncertainty, associated with the model dependence, must be attached to any particular chosen central value. The range of values allowed by the CMD-2 data is fully contained within that allowed by the SND data, but the latter range would need to be significantly extended, including to negative values of $[\delta a_\mu]_{\text{had;mix}}^{LO}$, if the lower-quality fits to the KLOE data were also taken into account.

Finally, taking into account the uncertainties already identified by CEN, and adding to these both the increased uncertainty on $[\delta a_\mu]_{\text{had;mix}}^{LO}$ and a possible direct IB isoscalar ρ^0 EM coupling contribution of the ‘‘natural size’’ discussed above, we arrive at a combined uncertainty for the IB correction one must apply in order to use the $\tau^- \rightarrow \nu_\tau \pi^- \pi^0$ data in computing $[\delta a_\mu]_{\text{had}}^{LO}$ which is $\sim 4 \times 10^{-10}$. The more optimistic assessment of the uncertainty in $[\delta a_\mu]_{\text{had;mix}}^{LO}$ represented by Eq. (17) (which we do not recommend at present) would decrease this only slightly, to $\sim 3.5 \times 10^{-10}$. It appears unlikely that this uncertainty can be significantly reduced. Should the new BNL experimental proposal [43] be approved, the uncertainty on the

IB corrections would thus exceed those on the experimental determination of a_μ , seriously limiting our ability to make use of hadronic τ decay data in determining the SM prediction for $[\delta a_\mu]_{\text{had}}^{LO}$.

ACKNOWLEDGMENTS

K. M. would like to acknowledge the hospitality of the CSSM, University of Adelaide, and the ongoing support of the Natural Sciences and Engineering Council of Canada.

-
- [1] M. Gourdin and E. de Rafael, Nucl. Phys. **B10**, 667 (1969).
- [2] See Ref. [3] for further details on the vacuum polarization corrections necessary to convert the cross-sections reported by various collaborations in the literature to the corresponding bare cross-section values.
- [3] M. Davier, S. Eidelman, A. Hocker, and Z. Zhang, Eur. Phys. J. C **27**, 497 (2003).
- [4] R. Barate *et al.* (ALEPH Collaboration), Z. Phys. C **76**, 15 (1997).
- [5] K. Anderson *et al.* (OPAL Collaboration), Eur. Phys. J. C **7**, 571 (1999).
- [6] S. Anderson *et al.* (CLEO Collaboration), Phys. Rev. D **61**, 112002 (2000).
- [7] R. Alemany, M. Davier, and A. Hocker, Eur. Phys. J. C **2**, 123 (1998).
- [8] M. Davier and A. Hocker, Phys. Lett. B **419**, 419 (1998); Phys. Lett. B **435**, 427 (1998).
- [9] G. Colangelo, Nucl. Phys. Proc. Suppl. **131**, 185 (2004).
- [10] J.F. de Troconiz and F.J. Yndurain, Phys. Rev. D **71**, 073008 (2005).
- [11] A. Hocker, hep-ph/0410081.
- [12] G.W. Bennett *et al.* (Muon $g - 2$ Collaboration), Phys. Rev. Lett. **92**, 161802 (2004).
- [13] M. Passera, J. Phys. G **31**, R75 (2005).
- [14] V. Cirigliano, G. Ecker, and H. Neufeld, Phys. Lett. B **513**, 361 (2001).
- [15] V. Cirigliano, G. Ecker, and H. Neufeld, J. High Energy Phys. **08** (2002) 002.
- [16] M. Davier, S. Eidelman, A. Hocker, and Z. Zhang, Eur. Phys. J. C **31**, 503 (2003).
- [17] R.R. Akhmetshin *et al.* (CMD-2 Collaboration), Phys. Lett. B **527**, 161 (2002); Phys. Lett. B **578**, 285 (2004).
- [18] K. Hagiwara, A.D. Martin, D. Nomura, and T. Teubner, Phys. Rev. D **69**, 093003 (2004).
- [19] A. Aloisio *et al.* (KLOE Collaboration), Phys. Lett. B **606**, 12 (2005).
- [20] S. Ghozzi and F. Jegerlehner, Phys. Lett. B **583**, 222 (2004).
- [21] K. Maltman, hep-ph/0504201.
- [22] M.N. Achasov *et al.* (SND Collaboration), hep-ex/0506076.
- [23] The IB $e^+e^- \rightarrow \omega^0 \rightarrow \pi^+\pi^-$ amplitude, whose interference with the leading isospin-conserving component, dominated by intermediate ρ^0 exchange, is responsible for the observed interference shoulder in the experimental cross section, is in fact generated by both ρ - ω mixing and direct IB $\omega \rightarrow \pi\pi$ coupling effects. For a given model of the ρ propagator, the experimentally extracted Orsay phase can be used, in principle, to separate these two contributions [24]. The recent CMD-2 data [17] shows clear evidence for a nonzero value of the direct IB $\omega \rightarrow \pi\pi$ coupling, as does the recent SND data [22]. At leading order in IB, the two effects both generate contributions to the flavor '38' part of the EM spectral function.
- [24] K. Maltman, H.B. O'Connell, and A.G. Williams, Phys. Lett. B **376**, 19 (1996).
- [25] F. Guerrero and A. Pich, Phys. Lett. B **412**, 382 (1997).
- [26] G.J. Gounaris and J.J. Sakurai, Phys. Rev. Lett. **21**, 244 (1968).
- [27] J.H. Kuhn and A. Santamaria, Z. Phys. C **48**, 445 (1990).
- [28] M. Bando *et al.*, Phys. Rev. Lett. **54**, 1215 (1985).
- [29] J. Gasser and H. Leutwyler, Nucl. Phys. **250**, 465 (1985).
- [30] H.B. O'Connell *et al.*, hep-ph/9707404.
- [31] M. Benayoun *et al.*, Eur. Phys. J. C **2**, 269 (1998).
- [32] We thank Vincenzo Cirigliano for clarifying this point.
- [33] Davier [34], working with the GS model, has noted that the m_ρ and Γ_ρ values obtained from fits to the bare and vacuum-polarization-dressed versions of the CMD-2 cross sections differ by ~ 1 MeV. Similar shifts also occur for the other models. These shifts produce corresponding shifts in $[\delta a_\mu]_{\text{had:mix}}^{LO}$. The versions associated with the fits to the bare cross sections, quoted in the text, lie higher than those associated with the fit to the vacuum-polarization-dressed values by $(0.4 \rightarrow 0.6) \times 10^{-10}$, depending on the model. The relevant versions for comparison to the τ data are, as explained by Davier, those obtained from the fit to the bare cross sections [34].
- [34] M. Davier, Nucl. Phys. Proc. Suppl. **131**, 123 (2004).
- [35] The fits to the KLOE data all require significantly larger values of the phase of δ (or $\theta_{\rho\omega}$) than do the corresponding fits to the CMD-2 and SND data sets. Such phase values lead to significantly lower $[\delta a_\mu]_{\text{had:mix}}^{LO}$, for the reasons explained in the text. In view of the poor quality of the corresponding fits, we have, however, excluded these results from consideration in arriving at a final assessment of the range of allowed values for $[\delta a_\mu]_{\text{had:mix}}^{LO}$. Were one to include them, the range would be considerably expanded, extending even to small negative values.
- [36] Adding an explicit ρ' contribution to the GP/CEN* model would, of course, allow one to obtain an improved fit to the SND data. Since the original CEN version did not include such a contribution, we do not tabulate results for such a further-modified version of the GP model here.
- [37] The optimized fits to the KLOE data yield $[\delta a_\mu]_{\text{had:mix}}^{LO} \times 10^{10} = -0.2 \pm 0.7, -0.6 \pm 0.8, 0.6 \pm 0.8,$ and -0.6 ± 0.7 for the GS, KS, HLS, and GP/CEN* models, respectively.
- [38] K. Maltman and C.E. Wolfe, Phys. Rev. D **59**, 096003

- (1999).
- [39] K. Maltman, Phys. Lett. B **440**, 367 (1998).
- [40] K. Maltman, Phys. Lett. B **351**, 56 (1995).
- [41] G. Amoros, J. Bijnens, and P. Talavera, Nucl. Phys. **B602**, 87 (2001).
- [42] J. Bijnens, AIP Conf. Proc. **768**, 153 (2005).
- [43] See e.g. D. W. Hertzog, Nucl. Phys. B, Proc. Suppl. **144**, 191 (2005).