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A note on monopole moduli spaces

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We discuss the structure of the framed moduli space of Bogomolny monopoles for arbitrary symmetry breaking and extend the definition of its stratification to the case of arbitrary compact Lie groups. We show that each stratum is a union of submanifolds for which we conjecture that the natural L^2 metric is hyper-Kähler. The dimensions of the strata and of these submanifolds are calculated, and it is found that for the latter, the dimension is always a multiple of four. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1590056]

I. INTRODUCTION

Recently there has been much interest in monopoles with nonmaximal symmetry breaking at infinity. In particular questions have been raised as to when they are manifolds and when they have hyper-Kähler metrics. This note gathers together some mathematical results concerning the structure of the moduli spaces and their L^2 metrics. These range from theorems which have been proved in full generality through partially proved theorems to outright conjectures.

Recall that we generally expect that moduli spaces of solutions of the self-duality equations and their reductions such as the Bogomolny equations and Nahm's equations should be hyper-Kähler manifolds. One reason for this is that formally such moduli spaces arise as hyper-Kähler quotients. To recall this, fix a compact, connected Lie group G, with Lie algebra $\mathfrak{g}$, and consider the space $\mathcal{A}$ of G-connections (vector potentials) on the trivial G-bundle over flat $\mathbb{R}^4$. By identifying

$$A_0 dx_0 + A_1 dx_1 + A_2 dx_2 + A_3 dx_3$$

with the $\mathfrak{g}\otimes\mathbb{H}$-valued function

$$A_0 + iA_1 + jA_2 + kA_3,$$

where i, j, and k are unit quaternions, $\mathcal{A}$ becomes a quaternionic vector space. Formally, $\mathcal{A}$ can be equipped with the L^2 metric, making it a flat hyper-Kähler manifold. Because $\mathbb{R}^4$ is not compact, the convergence of this metric will depend upon subjecting our connections to suitable asymptotic conditions, and these will be considered in detail below. Setting this aside for the moment, it is a straightforward exercise to check that the hyper-Kähler moment map for the action of the gauge group $\mathcal{G}$ on $\mathcal{A}$ is given by

$$A \mapsto F_A^+ \in \Omega^2(X,\mathfrak{g}) \otimes \mathrm{Im}\,\mathbb{H}.$$

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Hence the hyper-Kähler quotient $\mathcal{A}///\mathcal{G}$ should be the same as the space of anti-self-dual connections divided by the action of the gauge group, and the L^2-metric will descend to define a hyper-Kähler metric on the moduli space.

A monopole on $\mathbb{R}^3$ is a pair $c=(A,\Phi)$, where A is a connection on the trivial G-bundle $E\to\mathbb{R}^3$, and Φ is a section of the adjoint bundle $E\times_G\mathfrak{g}$. The monopole c satisfies the Bogomolny equations

$$\mathrm{d}_A\Phi=*F_A \tag{1.1}$$

if and only if the connection Φdx_0+A is anti-self-dual on $\mathbb{R}\times\mathbb{R}^3$. In particular, from this four-dimensional point of view, Φ cannot vanish at infinity, because it is independent of x_0. Thus the convergence of the L^2 metric and the nondegeneracy of the hyper-Kähler symplectic forms are important issues in this case.

These issues were fully resolved when $G=\mathrm{SU}(2)$ by Atiyah and Hitchin:[2] they showed that the moduli space of (framed) monopoles of charge k is, indeed, a complete hyper-Kähler manifold. Its dimension is $4k$ where the charge of the monopole is k.

For a general compact Lie group of rank r it is expected that the moduli space of monopoles with *maximal symmetry* breaking is a hyper-Kähler manifold although this has not been proved in generality. Except for very simple low charge cases, there are mostly partial results which compute the metric asymptotically near the edge of the moduli space; see, for example, Refs. 4, 5, and 17, and references therein.

The real complications, however, arise when there is *nonmaximal symmetry breaking* which is our primary interest below. The case of SU(3) monopoles with minimal symmetry breaking was treated in detail in Ref. 7, but beyond this little seems to be known.

We shall present here a summary of the results discussed in the article: the reader will have to refer forward for precise definitions.

The full moduli space of (framed) monopoles of mass μ and charge m is denoted by $\mathcal{M}(u,\mu,[\phi]=m)$. Here $0\neq\mu\in\mathfrak{g}$ is arbitrary (*maximal* symmetry breaking is precisely the condition that μ should be regular) and u is a unit vector in $\mathbb{R}^3$. m is a homotopy class, essentially a string of integers. The boundary conditions imposed guarantee that for some $k\in\mathfrak{g}$,

$$\Phi(tu)=\mu-\frac{k}{2t}+o(t^{-1})\quad\text{for}\quad t\gg 0. \tag{1.2}$$

There is therefore a map $e:\mathcal{M}(u,\mu,[\phi]=m)\to\mathfrak{g}$ which assigns k to (A,Φ). The image $\mathcal{K}$ of e in $\mathfrak{g}$ is not the whole of $\mathfrak{g}$, but rather a disjoint union of $C(\mu)$-orbits

$$\mathcal{K}=C(\mu)k_1\cup C(\mu)k_2\cup\cdots\cup C(\mu)k_n. \tag{1.3}$$

It turns out that the k_j are *integral* elements of $\mathfrak{g}$. The set of all monopoles (A,Φ) with $e(A,\Phi)\in C(\mu)k_j$ is the jth stratum $\mathcal{M}_j$, say, of the moduli space. This was defined in a different way for $G=\mathrm{SU}(r+1)$ in Ref. 19. In general, $\mathcal{M}_j$ does not have dimension divisible by 4, so it cannot be hyper-Kähler. However, if we define, for $k\in\mathcal{K}$,

$$\mathcal{M}(u,\mu,k)=\{(A,\Phi)\in\mathcal{M}(u,\mu,[\phi]=m):e(A,\Phi)=k\} \tag{1.4}$$

[the moduli space of framed monopoles of *type* (μ,k)], then we shall see that $\mathcal{M}(u,\mu,k)$ has dimension divisible by 4 and the natural conjecture is that the L^2 metric makes $\mathcal{M}(u,\mu,k)$ into a hyper-Kähler manifold.

At least one of the strata, $\mathcal{M}_1$, say, must be open, hence of the same dimension as $\mathcal{M}(u,\mu,[\phi]=m)$, but this stratum need not be hyper-Kähler. If, however, $C(\mu)k_1=k_1$, then $\mathcal{M}_1=\mathcal{M}(u,\mu,k_1)$ and then this stratum *is* a candidate to be hyper-Kähler. Notice more generally that if k and k' lie in $C(\mu)k_j$, an element $g\in C(\mu)$ with $\mathrm{ad}(g)k=k'$ can be regarded as a constant gauge transformation which maps $\mathcal{M}(u,\mu,k)$ diffeomorphically to $\mathcal{M}(u,\mu,k')$.

In Sec. IV, magnetic charges $m_1,\dots,m_s$ and holomorphic charges $h_1,\dots,h_{r-s}$ are defined for monopoles in $\mathcal{M}(u,\mu,k)$. The information in the magnetic charges is topological and is equivalent to the homotopy class m. In particular, the magnetic charges do not vary from stratum to stratum. By contrast the holomorphic charges determine the stratum $\mathcal{M}_j$. (The number s of magnetic charges is completely determined by the mass μ.)

We shall show that if $\mathcal{M}(u,\mu,k)$ is nonempty, then the charges are all nonnegative, and that

$$\dim \mathcal{M}(u,\mu,k)=4(m_1+\cdots+m_s+h_1+\cdots+h_{r-s}). \tag{1.5}$$

Dimensions of the strata and full moduli space are also determined in Sec. VI.

II. THE MODULI SPACE AS A MANIFOLD

In this section we shall introduce various different monopole moduli spaces and explain carefully which of them are smooth manifolds, and which are likely to admit hyper-Kähler metrics. Throughout we shall be considering *Euclidean* monopoles, that is to say, monopoles on flat $\mathbb{R}^3$. Note that the metric enters the Bogomolny equation (1.1) through the Hodge star operator. Some work has also been done on hyperbolic monopoles, where $\mathbb{R}^3$ is replaced by hyperbolic three-space $\mathcal{H}^3$. It is expected that moduli spaces of hyperbolic monopoles will be diffeomorphic to the corresponding moduli spaces of Euclidean monopoles, but this has not been proved in general. On the other hand, the issue of natural metrics on moduli spaces of hyperbolic monopoles is completely open: all that is known for certain is that the L^2 metric is infinite in this case.

There are two reasons why there are so many different monopole moduli spaces. The first is that the monopoles must be *framed*, and this can be done either at a base-point in $\mathbb{R}^3$ or "at infinity." The second has to do with the specification of the asymptotics of the Higgs field Φ.

A. Notation

In order to discuss monopoles, we shall fix the following:

(i) G is a compact, connected, semi-simple Lie group of rank r. The complexification is denoted G^c and Lie algebra $\mathfrak{g}$.

(ii) If $a\in\mathfrak{g}$, $O_a\subset\mathfrak{g}$ is the orbit of a in $\mathfrak{g}$ under the adjoint action of G. $C(a)\subset G$ is the centralizer of a, with Lie algebra $\mathfrak{c}(a)$.

(iii) As a homogeneous space, $O_a=G/C(a)=G^c/P_a$, where P_a is the appropriate parabolic subgroup. The latter description gives O_a the structure of a compact complex manifold.

(iv) μ and k are commuting elements of $\mathfrak{g}$, $[\mu,k]=0$.

(v) $E\to\mathbb{R}^3$ will denote the trivial principal G-bundle over $\mathbb{R}^3$.

B. Boundary conditions and moduli spaces

The physically natural condition to impose on a solution of the Bogomolny equations is the finite-energy condition

$$\int |F_A|^2=\int |\mathrm{d}_A\Phi|^2<\infty. \tag{2.1}$$

We shall impose apparently rather stronger asymptotic conditions. It follows from the work of Taubes if $G=\mathrm{SU}(2)$, that (2.1) together with (1.1) implies these stronger conditions, but for general groups this must remain a conjecture.

Following Jarvis, we assume the following.

(BC1) Along each straight line, there is a gauge in which

$$\Phi=\mu-\frac{k}{2r}+O\left(\frac{1}{r^{1+\delta}}\right)$$

for all sufficiently large r.

(BC2) In this same gauge,

$$\mathrm{d}_A\Phi = \frac{k}{2r^2}\,\mathrm{d}r + O\left(\frac{1}{r^{2+\delta}}\right)$$

for all sufficiently large r.

These conditions are closely related to the Bogomolny–Prasad–Sommerfield (BPS) boundary conditions of Ref. 12.

Define

$$\mathcal{C} = \{(A,\Phi): \mathrm{d}_A\Phi = *F_A,\ \ (A,\Phi)\text{satisfies BC1 and BC2}\}.$$

Notice that we do not yet fix μ and k: we merely assert that the boundary conditions are satisfied for *some* elements μ and k satisfying

$$\mu \neq 0, \quad [\mu,k]=0. \tag{2.2}$$

Denote by $\mathcal{G}$ the group of all automorphisms g of E that preserve the boundary conditions (i.e., g and ∇g have limits as r goes to infinity along any straight line, and the limiting values are continuously differentiable when viewed as functions on the sphere at infinity). Then $\mathcal{G}$ acts on $\mathcal{C}$ and we would like to define the monopole moduli space as the quotient $\mathcal{M}=\mathcal{C}/\mathcal{G}$. This will have singularities because $\mathcal{G}$ does not act freely. In addition it will contain components of arbitrarily high dimension. We shall now explain how these two problems are eliminated.

C. The degree of a monopole

The asymptotic value of Φ is a section ϕ, say, of $\mathrm{ad}(E_\infty)$, where E_∞ is the restriction of E to the two-sphere at infinity. Since $\mathrm{ad}(E_\infty)$ is a trivial bundle, we can view ϕ as a continuous map into $\mathfrak{g}$. By BC1, this takes values in the adjoint orbit O_μ. This orbit is preserved by the action of gauge transformations g on E_∞, but $g(\phi)=\mathrm{ad}(g)\phi$, so that this map is not gauge-invariant. However, its homotopy class $m=[\phi]$ *is* gauge invariant, because $\pi_2(G)=0$, so that any gauge transformation can be deformed to the identity. The homotopy class m is called the *degree* of the monopole. This discussion suggests the definition of spaces

$$\mathcal{C}(O_\mu,[\phi]=m),$$

where the adjoint orbit as well as the homotopy class of ϕ are fixed. This is referred to as the set of monopoles of mass μ and charge m. Note that $O_\mu = G/C(\mu)$.

D. Radial scattering and interior framing

Let $x\in\mathbb{R}^3$ be any point. The moduli space of monopoles framed at x, of mass μ and charge m, is the quotient

$$\mathcal{M}(x,O_\mu,[\phi]=m) = \mathcal{C}(O_\mu,[\phi]=m)/\mathcal{G}(x),$$

where

$$\mathcal{G}(x) = \{g\in\mathcal{G}: g(x)=1\}.$$

In Ref. 15 Jarvis proved the following:

Theorem 2.1: *There is a natural bijection*

$$r_x: \mathcal{M}(x,O_\mu,[\phi]=m) \to \mathcal{R}(O_\mu,m)$$

where the set on the RHS is the space of all holomorphic maps $v:S^2\to O_\mu$, *with* $[v]=m$.

In defining $\mathcal{R}(O_\mu, m)$ recall from Sec. II A that O_μ is in a natural way a complex manifold. It is known[6] that $\mathcal{R}(O_\mu, m)$ is a finite-dimensional smooth manifold, often referred to as a space of rational maps. It follows that our framed moduli space can be identified with a smooth manifold. It should be the case that r_x is naturally a diffeomorphism, but to prove that one would have to equip $\mathcal{M}(x, O_\mu, [\phi]=m)$ with a smooth structure. Although this should be possible, we are not aware of a detailed treatment of this issue.

E. Framing at infinity and parallel scattering

To frame monopoles "at infinity" we pick a point $u \in S^2$, viewed as the sphere at infinity in $\mathbb{R}^3$. Returning to BC1, we define

$$\mathcal{C}(u,\mu,[\phi]=m) = \{(A,\Phi) \in \mathcal{C}: \lim_{t\to\infty} \Phi(tu) = \mu, [\phi]=m\}$$

and

$$\mathcal{C}(u,\mu,k) = \{(A,\Phi) \in \mathcal{C}: \Phi(tu) = \mu - k/2t + o(t^{-1})\}$$

and introduce the corresponding gauge group

$$\mathcal{G}(u) = \{g \in \mathcal{G}: \lim_{t\to\infty} g(tu) = 1\}.$$

The corresponding moduli spaces are

$$\mathcal{M}(u,\mu,[\phi]=m) = \mathcal{C}(u,\mu,[\phi]=m)/\mathcal{G}(u) \quad \text{and} \quad \mathcal{M}(\mu,k) = \mathcal{C}(u,\mu,k)/\mathcal{G}(u).$$

The first of these is called the moduli space of (framed) monopoles with mass μ and degree m. The second is called the moduli space of (framed) monopoles of type (μ,k).

These can also be identified with spaces of rational maps:

Theorem 2.2: *(a) There is a natural bijection $r_u : \mathcal{M}(u,\mu,m) \to \tilde{\mathcal{R}}(O_\mu, m)$. (b) There is a natural bijection $\hat{r}_u : \mathcal{M}(u,\mu,k) \to \tilde{\mathcal{R}}(O_{\mu k}, m)$.*

Here $\tilde{\mathcal{R}}(O_\mu, m) \subset \mathcal{R}(O_\mu, m)$ is the set of *based* rational maps, that is, those which send $u \in S^2$ to μ. In part (b),

$$O_{\mu k} = G/H_{\mu k} = G^c/P_{\mu k}, \quad \text{where} \quad H_{\mu k} = C(\mu) \cap C(k), \tag{2.3}$$

and $P_{\mu k}$ is the corresponding parabolic subgroup.

Part (a) of this result was proved first by Donaldson[8] for $G=\mathrm{SU}(2)$, then by Hurtubise[11] for classical groups by a generalization of Donaldson's approach. Both parts were proved for general G by Jarvis[13,14] using parallel scattering to associate a rational map to a monopole, and nonlinear analysis to invert this procedure.

We note in passing that Jarvis shows that the restriction of r_u to $\mathcal{M}(u,\mu,k)$ is the composition of $\hat{r}_u$ with the projection $\tilde{\mathcal{R}}(O_{\mu k}, m) \to \tilde{\mathcal{R}}(G^c/P, m)$.

Once again, it is not clear that smooth structures have been defined on these framed moduli spaces. One conjectures that natural smooth structures should exist, such that these bijections are diffeomorphisms.

As we indicated in the Introduction, it is the moduli spaces $\mathcal{M}(u,\mu,k)$ that have dimensions divisible by 4 and which are therefore candidates to be hyper-Kähler spaces. In Proposition 6.2 the dimension of $\tilde{\mathcal{R}}(O_{\mu k}, m)$ will be explicitly computed.

F. Discussion

Let $x(t)=ut$, and consider the bijection $r_{x(t)}$, for t large, of Theorem 2.1. It is tempting to believe that this should approach the map r_u of Theorem 2.2. However, they cannot be compared directly since they have different targets. But we could divide both sides by the appropriate groups to get bijections

$$\tilde{r}_u:\mathcal{M}(O,[\phi]=m)\to\tilde{\mathcal{R}}(O_\mu,m)/C(\mu) \quad\text{and}\quad \tilde{r}_{x(t)}:\mathcal{M}(O_\mu,[\phi]=m)\to\mathcal{R}(O_\mu,m)/G$$

and then compare them via the natural isomorphism induced by the inclusion of based maps into unbased maps. A straightforward calculation shows that the limit of $\tilde{r}_{tu}(A,\Phi)$ typically does not exist because evaluated in coordinates it blows up. Some kind of renormalization or scaling must be required to find the relationship between the limit of $\tilde{r}_{x(t)}$ and $\tilde{r}_u$.

III. THE L^2 METRIC

Formally, a tangent vector to (A,Φ) in $\mathcal{C}$ is a pair $(\dot{A},\dot{\Phi})$ satisfying the linearization at (A,Φ) of the Bogomolny equations. The L^2 metric gives this vector length-squared equal to

$$\int_{\mathbb{R}^3}(|\dot{A}|^2+|\dot{\Phi}|^2)\mathrm{d}x; \tag{3.1}$$

due to the noncompactness of $\mathbb{R}^3$, this need not converge. Looking back at BC1 and BC2, it is clear that (3.1) cannot converge if the variation $\dot{\Phi}$ changes μ or k in BC1. It is natural, therefore, to focus on $\mathcal{M}(u,\mu,k)$ as the obvious candidate to carry a hyper-Kähler metric. Our first task is to show that if the Bogomolny equations hold asymptotically, then the pair (μ,k) determines the leading asymptotics of the monopole on the whole of the two-sphere at infinity.

We begin by noting that the boundary conditions imply that the connection A restricts to give a connection a on E_∞ and that BC1 gives

$$\Phi(tz)=\phi(z)-\frac{f(z)}{2t}+o(t^{-1}), \tag{3.2}$$

where ϕ and f are smooth functions of $z\in S^2$ and the framing condition is

$$\phi(u)=\mu, \qquad f(u)=k. \tag{3.3}$$

The Bogomolny equations reduce to

$$\nabla f=0,\ \ \nabla\phi=0,\ \ F_a=\frac{f}{2}\mathrm{dvol}, \tag{3.4}$$

where dvol denotes the standard area-form of the unit two-sphere. A pair (ϕ,f) satisfying (3.3) and (3.4) are called *monopole boundary data*.

We now prove that, up to gauge, the pair (ϕ,f) is completely determined by its value $(\mu,k/2)$ at the base-point u.

Proposition 3.1: Let (ϕ,f) and (ϕ',f') be boundary data for a monopole:

(i) If u and v are in S^2, then there is a $g\in G$ such that $\phi(u)=\mathrm{ad}(g)(\phi(v))$ and $f(u)=\mathrm{ad}(g)(f(v))$.

(ii) If there is an $h\in G$ such that $\phi(u)=\mathrm{ad}(h)(\phi'(u))$ and $f(u)=\mathrm{ad}(h)(f'(u))$, then there is a $g:S^2\to G$ such that $\phi^g=\phi'$ and $f^g=f'$.

Proof: If $\phi=0$, this is a trivial case of the results of Ref. 1 classifying equivalence classes or Yang–Mills connections over a Riemann surface. We follow the proof in Ref. 1. Recall that $E_\infty\to S^2$ is a principal G-bundle. Then ϕ and f can be viewed as equivariant maps $E_\infty\to\mathfrak{g}$. Fix a point $p_0\in E_\infty$ and let $\phi(p_0)=\mu$ and $f(p_0)=k$. Because ϕ and f are covariantly constant they are constant along any horizontal path. If $p\in P$, we can join p_0 to some point pg with a horizontal curve and then $\phi(p)=\mathrm{ad}(g)(\mu)$ and $f(p)=\mathrm{ad}(g)(k)$ as required.

From the discussion in the preceding paragraph it follows that we have a map

$$(\phi,f):E_\infty \to O_{\mu k}=G/H_{\mu k}, \quad H_{\mu k}=C(\mu)\cap C(k).$$

Here $O_{\mu k}$ is the orbit of (μ,k). The preimage of the coset $H_{\mu k}$, i.e., the set of all points p in E_∞ at which $\phi(p)=\mu$ and $f(p)=k$, is a reduction of E_∞ to $H_{\mu k}$ which we denote by $E_{\mu k}$. If $p \in E_{\mu k}$, then any horizontal curve is also in $E_{\mu k}$ because ϕ and k are constant along horizontal curves so the connection also reduces to $P_{\mu k}$.

Because S^2 is simply connected, standard results on reduction of bundles to their holonomy subgroups can be used.[16] It follows from the Ambrose–Singer theorem that the holonomy subgroup at p_0 is the subgroup $H\subset H_{\mu k}$ obtained by exponentiating k and that $E_{\mu k}$ reduces to a bundle E_0 with structure group H.

For the final point we need to know that k is an integral element of the Lie algebra. This is done in Ref. 9 and in a different fashion in Ref. 13. We proceed as follows. Because $[\mu,k]=0$ the closure of the subgroup generated by $\exp(t\mu+sk)$ for any t and s will be an Abelian subgroup of G so a torus and hence inside a maximal torus containing H. If λ is any weight of this maximal torus, we can form an associated line bundle which will have integer chern class $\lambda(k)$. It follows that k is an integer element of $\mathfrak{g}$ and that it exponentiates to define a circle subgroup and a homomorphism $\chi:U(1)\to G$.

We have now reduced our original bundle to a subbundle $Q\to S^2$ which is a circle bundle. It has a connection A and a curvature F with $*F=k/2$ a constant so that it is a circle bundle of degree 1. If A' is another connection with curvature $F'=F$, then $A-A'=a$ with $da=0$ so $a=d(\exp(g))$ for $g:S^2\to \mathrm{U}(1)$ and hence the connections A and A' are equal after a gauge transformation.

This gives us a method of constructing the original bundle, connection and Higgs field from the data μ and k. First take the standard U(1) bundle $Q\to S^2$ with its SU(2) invariant connection and fix $q_0\in Q$ in the fiber over the point u. Let $\chi:\mathrm{U}(1)\to G_{\mu k}\subset G$ be the homomorphism defined by exponentiating k. We can then form $Q\times_\chi G$, the associated bundle, using the action $(q,k)z=(qz,\chi(z)^{-1}k)$ for $z\in \mathrm{U}(1)$. This inherits a connection and the Higgs field is defined by $\hat{\phi}([q,k])=\mathrm{ad}(k)(\mu)$. $\square$

Let $\mathcal{C}^\infty$ denote the set of all monopole boundary data (ϕ,f) and let $\mathcal{G}^\infty$ be the space of all gauge transformations at infinity, that is maps $g:S^2\to G$. Define the *moduli space of boundary data* to be the quotient $\mathcal{M}^\infty=\mathcal{C}^\infty/\mathcal{G}^\infty$. We have the boundary map

$$\partial:\mathcal{M}\to\mathcal{M}^\infty, \tag{3.5}$$

which sends (A,Φ) to the value of the Higgs field and curvature at infinity. Our reason for introducing the boundary map is that we believe that the methods of Atiyah and Hitchin[2] can be adapted to show that

Conjecture 3.2: If $\partial(A,\Phi)=\partial(A',\Phi')$, then there is a gauge transformation g such that A^g-A' and $\Phi^g-\Phi'$ are L^2.

The idea here is that if the condition holds, then for some gauge transformation g, Φ^g and Φ' should agree up to order $1/r$, so that $\Phi^g-\Phi'$ will be square integrable. Similar considerations should apply to the difference between the connections.

Let $\mathcal{G}^\infty(u)$ be all gauge transformations which are the identity at u and let $\mathcal{C}^\infty(u,\mu,m)$ be all pairs (ϕ,f) with $\phi(u)=\mu$ and $[\phi]=m$. Denote $\mathcal{M}^\infty(u,\mu,m)=\mathcal{C}^\infty(u,\mu,m)/\mathcal{G}^\infty(u)$. We have the commuting diagram

$$\begin{array}{ccc}\mathcal{M}(u,\mu,m) & \to & \mathcal{M}^\infty(u,\mu,m)\\ \downarrow & & \downarrow\\ \mathcal{M}(O,m) & \to & \mathcal{M}^\infty(O,m)\end{array}$$

where both vertical maps are quotienting by the group $C(\mu)$.

Conjecture 3.2 would imply that the L^2 metric is finite on each of the moduli spaces $\mathcal{M}(u,\mu,k)$ of monopoles of type (μ,k). This suggests the following.

Conjecture 3.3: The spaces $\mathcal{M}(u,\mu,k)$ are hyper-Kähler manifolds.

A natural approach to these conjectures is the analysis of the linearization $\mathcal{D}$ at $(A,\Phi) \in \mathcal{M}(u,\mu,k)$ of (1.1). Combined with the Coulomb gauge-fixing condition, $\mathcal{D}$ becomes a coupled Dirac operator on $\mathbb{R}^3$,

$$\mathcal{D}:C^\infty(\mathbb{R}^3,\mathbb{H}\otimes\mathbb{H}\otimes \mathrm{ad}(E))\rightarrow C^\infty(\mathbb{R}^3,\mathbb{H}\otimes\mathbb{H}\otimes \mathrm{ad}(E)),$$

where $\mathbb{H}$ is regarded as the spin-bundle of $\mathbb{R}^3$. Unfortunately this operator is not ''invertible at infinity'' and so it is not automatically a Fredholm operator in L^2. Taubes analyzed it in detail when $G=\mathrm{SU}(2)$, but in general, rigorous results about this operator are not available. Nonetheless, it should be possible to find a suitable space of functions such that $\mathcal{D}$ becomes a Fredholm operator, with index calculable in terms of the type data (μ,k). Formally $\mathcal{D}$ is quaternionic, so its L^2 null space will automatically be a quaternionic vector space with compatible inner product. The reader is referred to Ref. 2, Chaps. 3 and 4, for a detailed discussion of the case $G=\mathrm{SU}(2)$.

A. Group actions

Consider $\mathcal{E}$, the group of Euclidean transformations of $\mathbb{R}^3$, which is the semi-direct product of SO(3), the group of rotations and $\mathbb{R}^3$ the group of translations. As the monopole bundle $E\rightarrow\mathbb{R}^3$ is trivial the group $\mathcal{E}$ acts on the connection and Higgs field, preserves the Bogomolny equations and commutes with gauge transformations so it acts on the full-unframed moduli space. In general this action disturbs the framings. If $x\in\mathbb{R}^3$, then the subgroup $\mathcal{E}_x$ of transformations preserving x, which is isomorphic to SO(3), acts naturally on the moduli space of monopoles framed at x. If $u\in S^2$, then the subgroup of $\mathcal{E}_u$ of transformations preserving the line through u, which is isomorphic to $\mathrm{SO}(2)\times\mathbb{R}^2$, will act naturally on the moduli space of monopoles framed at u.

As well as these straightforward actions the moduli space $\mathcal{M}(u,\mu,k)$ also carries an action of the full group of Euclidean transformations. For this we need a different description of this moduli space (cf. Ref. 2, pp. 15 and 16). Note that Proposition 3.1 shows that k defines a representation of the circle in G, hence an associated G-bundle over the two-sphere. This carries a natural SO(3)-action and has a unique SO(3)-equivariant connection a and Higgs field ϕ such that $\phi(u)=\mu$ and $f(u)=k$. The moduli space $\mathcal{M}(\mu,k)$ is now defined to consist of configurations (A,Φ,q) where (A,ϕ) is a monopole and q is an isomorphism between $\partial(A,\Phi)$ and (ϕ,f), modulo the group of gauge transformations that approach the identity at infinity. Then $\mathcal{M}(\mu,k)$ has a natural SO(3)-action and can be shown to be diffeomorphic to $\mathcal{M}(u,\mu,k)$. The subtlety is [as in the case $G=\mathrm{SU}(2)$] that the diffeomorphism between $\mathcal{M}(\mu,k)$ and $\mathcal{M}(u,\mu,k)$ is not equivariant with respect to the copy of $\mathrm{SO}(2)\subset\mathrm{SO}(3)$ which fixes the direction u.

B. Discussion

Assuming that the L^2 metric does define a genuine hyper-Kähler metric on $\mathcal{M}(u,\mu,k)$, there are many interesting open questions surrounding it. First of all, there is the issue of whether it is complete for all μ and k. Second, there are questions relating to variation of the parameters μ and k. It is natural to conjecture that the metrics will vary smoothly with μ as long as the corresponding orbit O_μ does not jump. An interesting conjecture of Lee, Weinberg, and Yi[18] suggests that these hyper-Kähler metrics should also behave well with respect to specialization of μ. To state the conjecture, call a path $\mu:[0,\delta]\rightarrow\mathfrak{g}$ a *regular* deformation of $\mu_0=\mu(0)$ if $\mu(t)$ is *regular* for all $t>0$. Let $\mathcal{M}_t=\mathcal{M}(u,\mu_t,k)$, and let g_t be the L^2 metric on $\mathcal{M}_t$.

Conjecture 3.4: Given any $0\neq\mu_0\in\mathfrak{g}$, there is a regular deformation μ_t, such that $(\mathcal{M}_t,g_t)$ tends to $(\mathcal{M}_0,g_0)$ as $t\rightarrow 0$.

Note that Jarvis[13] describes a ''filling-out procedure'' which associates to any holomorphic map $v:S^2\rightarrow O_{\mu,k}$ a new map $\tilde{v}:S^2\rightarrow G/T$ where T is a maximal torus. This would appear to be closely related to the idea of regular deformation of a general element μ, but it says nothing about the behavior of the metrics.

We have now filled in the details of our account in the Introduction up to Eq. (1.4), though we have not yet shown that $\mathcal{K}$ has the structure claimed in (1.3). We turn to that in the next section.

IV. MAGNETIC AND HOLOMORPHIC CHARGES

We will now show how to calculate explicitly the *magnetic charges* of a monopole which determine the homotopy class m and the *holomorphic charges* which determine the strata. We will also make some conjectures about the possible values these can take.

In this section, μ and k are as before. In addition, T is a maximal torus whose Lie algebra $\mathfrak{t}$ contains both μ and k. Recall that a choice of Weyl chamber C in $\mathfrak{t}$ gives rise to a set of simple roots $\alpha_1,\dots,\alpha_r$ and the corresponding fundamental weights $\lambda_1,\dots,\lambda_r$ defined by

$$2\frac{\langle\alpha_i,\lambda_j\rangle}{\langle\alpha_i,\alpha_i\rangle}=\delta_{ij}. \tag{4.1}$$

We can always choose a fundamental Weyl chamber C satisfying

$$\alpha_1(\mu)>0,\dots,\alpha_s(\mu)>0,\ \text{and}\ \ \alpha_{s+1}(\mu)=0,\dots,\alpha_r(\mu)=0, \tag{4.2}$$

because this is just the condition that μ is in the closure of C and a particular ordering of the simple roots.

We would like to apply the corresponding fundamental weights to k but this is not possible as we only know that k is in the Lie algebra of the centralizer of μ. We can conjugate k by $C(\mu)$ until it is inside the torus but then we find that $\mathfrak{t}\cap C(\mu)k$ is not a single point but an orbit under $\mathcal{W}_\mu$ the subgroup of the Weyl group stabilising μ. Our first result resolves this problem by showing that we can pick out a unique element $\tilde{k}$ of $\mathfrak{t}\cap C(\mu)k$.

Proposition 4.1: Suppose that the moduli space $\mathcal{M}(u,\mu,k)$ is nonempty and we have fixed a maximal torus containing μ, a fundamental Weyl chamber C with μ in its closure and have ordered the simple roots so they satisfy (4.2). Then there exists a uniquely determined $\tilde{k}\in\mathfrak{t}\cap C(\mu)k$, such that

$$\alpha_{s+1}(\tilde{k})\leqslant 0,\dots,\alpha_r(\tilde{k})\leqslant 0.$$

Moreover, we have $\lambda_j(\tilde{k})\geqslant 0$ for $j=1,\dots,r$.

We shall give the proof of this proposition in a moment. For now, we shall use it to define the *charges* of the monopole to be the non-negative integers

$$\lambda_1(\tilde{k}),\dots,\lambda_r(\tilde{k}).$$

They are naturally divided into *magnetic charges*

$$m_1=\lambda_1(\tilde{k}),\dots,m_s=\lambda_s(\tilde{k})$$

and the *holomorphic charges*:

$$h_1=\lambda_{s+1}(\tilde{k}),\dots,h_{r-s}=\lambda_r(\tilde{k}).$$

In some examples the simple roots have a natural ordering and it is convenient not to reorder them. In that case we just choose $\tilde{k}$ to be the unique $\tilde{k}\in\mathfrak{t}\cap C(\mu)k$ such that whenever $\alpha_i(\mu)=0$ we have $\alpha_i(\tilde{k})\leqslant 0$. We then say that $\lambda_i(\tilde{k})$ is a magnetic charge if $\alpha_i(\mu)>0$ and a holomorphic charge if $\alpha_i(\mu)=0$.

The most important point to be made here is that it is easy to show that $\pi_2(O_\mu)=\mathbb{Z}^s$ and the magnetic charges determine the homotopy class of ϕ the Higgs field at infinity (see, for example, Ref. 3). The magnetic charges therefore cannot change under continuous deformation of a monopole. By contrast, the holomorphic charges can jump under continuous deformation of the monopole.

Note that the strata in the moduli space are all those monopoles with the same $\tilde{k}$.

As well as being non-negative the holomorphic charges satisfy the additional constraint that $\alpha_i(\tilde{k}) \leqslant 0$ for all $i = s+1,\dots,r$. This is equivalent to

$$\sum_{l=1}^{r-s} \frac{2\langle \alpha_i, \alpha_{l+s}\rangle}{\langle \alpha_{l+s}, \alpha_{l+s}\rangle} h_l + \sum_{j=1}^{s} \frac{2\langle \alpha_i, \alpha_j\rangle}{\langle \alpha_j, \alpha_j\rangle} m_j \leqslant 0, \text{ for } i = s+1,\dots,r. \tag{4.3}$$

We conjecture the following.

Conjecture 4.2: For a given μ there are monopoles with any collection of non-negative magnetic charges $(m_1,\dots,m_s)$. Given a choice of magnetic charges there are monopoles with any collection of holomorphic charges $(h_1,\dots,h_{r-s})$ satisfying (4.3).

It should be possible to prove this result using rational maps but it has eluded us. We can prove, however, the following.

Proposition 4.3: For a given μ and choice of magnetic charges there are at most a finite number of possible holomorphic charges satisfying (4.3).

We defer the proof to the next section but note that this gives the following.

Corollary 4.4: There are only a finite number of strata and in particular there must be an open stratum.

Note that this approach gives a nice picture in terms of Dynkin diagrams. For maximal symmetry-breaking, all charges are magnetic (i.e., topological) and the heuristic is that there are m_i fundamental monopoles of type i for each i a node on the Dynkin diagram. For nonmaximal symmetry breaking mark each node i with $\alpha_i(\mu) = 0$. Now each Dynkin node still has associated to it the non-negative integer $\lambda_i(\tilde{k})$. This number is a magnetic charge m_i if i is unmarked, and again the heuristic is that there are m_i fundamental monopoles of type i. If i is a marked node, then $\lambda_i(\tilde{k})$ is a holomorphic charge. This labels the strata and can jump under continuous deformation of the monopole. The possible holomorphic charges are constrained by inequalities which can be deduced from the Dynkin diagram and (4.3).

A. Proof of Proposition 4.1

Let $\mathcal{W}_\mu$ be the subgroup of the Weyl group fixing μ and note that it acts transitively on the set of all fundamental Weyl chambers with μ in their closure.[10]

To prove first that a $\tilde{k}$ exists we follow Jarvis[13] and consider the condition $\alpha(\mu - tk') > 0$ for large enough t and any $k' \in \mathfrak{t} \cap C(\mu)k$. As there are only a finite number of roots we can find an $\epsilon > 0$ such that for all $t \in (0,\epsilon]$ we have that $\alpha(\mu - tk') = 0$ if and only if $\alpha(\mu) = 0$ and $\alpha(k') = 0$ and $\alpha(\mu - tk') > 0$ implies $\alpha(k') = 0$ and $\alpha(k') < 0$. For any such t choose a fundamental Weyl chamber with $\mu - tk'$ in its closure. As $t \to 0$ we see that this has μ in its closure as well. If this is not the fundamental Weyl chamber we first thought of we can move it by $\sigma \in \mathcal{W}_\mu$ until it is and then let $\tilde{k} = \sigma(k')$. Then $\mu - t\tilde{k}$ is in the closure of our fundamental Weyl chamber so that $\alpha_i(\mu) > 0$ for $i = 1,\dots,s$ and $\alpha_j(\mu) = 0$ and $\alpha_j(\tilde{k}) \leqslant 0$ for $j = s+1,\dots,r$.

We will see in a moment that $\tilde{k}$ is unique but for now we show that $\lambda_i(\tilde{k}) \geqslant 0$ for all $i = 1,\dots,r$.

Consideration of the twistor construction for monopoles shows that ϕ and f satisfy the following non-negativity constraint for any direction u. Choose any maximal torus T so that $\phi(u), f(u) \in \mathfrak{t}$. Choose a fundamental Weyl chamber whose closure contains $\phi(u)$ and let $\alpha_1,\dots,\alpha_r$ be the corresponding simple roots. Define the fundamental weights $\lambda_1,\dots,\lambda_r$ by (4.1). Then

$$\lambda_i(f(z)) \geqslant 0 \quad \text{for all } \; i = 1,\dots,r$$

independent of all the choices made. Note that $\tilde{k}$ is a conjugate of k under an element of $C(\mu)$ and hence corresponds to the k for some different monopole which also satisfies the positivity constraint. Hence we must have $\lambda_i(\tilde{k}) \geqslant 0$ for all $i=1,\ldots,r$.

Consider lastly the uniqueness of $\tilde{k}$. So assume we have $\tilde{k}$ and $\sigma(\tilde{k})$ for $\sigma \in \mathcal{W}_\phi$ and $\alpha_j(\tilde{k}) \leqslant 0$ and $\alpha_j(\sigma(\tilde{k})) \leqslant 0$ for every $i=s+1,\ldots,r$. Let V be the span of the roots $\alpha_{s+1},\ldots,\alpha_r$. This is a root system with Weyl group $\mathcal{W}_\mu$. Let C_{ij} be the inverse of the matrix $D_{ij}=\langle \alpha_i, \alpha_j \rangle$. Then both C and D are symmetric. Define

$$\chi : \mathfrak{t} \to V$$

by

$$\chi(h) = \sum_{j,k=s+1}^{r} \alpha_j(h) C_{jk} \alpha_k .$$

Let σ_l be a simple root reflection for $s+1 \leqslant l \leqslant r$. Then

$$\begin{aligned} \chi(\sigma_l(h)) &= \sum_{j,k=s+1}^{r} \sigma_l(\alpha_j)(h) C_{jk} \alpha_k \\ &= \chi(h) - \sum_{j,k=s+1}^{r} \frac{2\langle \alpha_j, \alpha_l \rangle}{\langle \alpha_l, \alpha_l \rangle} C_{jk} \alpha_k(h) \\ &= \chi(h) - \frac{2\alpha_l(h)}{\langle \alpha_l, \alpha_l \rangle} \alpha_l . \end{aligned}$$

Moreover,

$$\begin{aligned} \sigma_l(\chi(h)) &= \chi(h) - \frac{2\langle \chi(h), \alpha_l \rangle}{\langle \alpha_l, \alpha_l \rangle} \\ &= \chi(h) - \sum_{j,k=s+1}^{r} \alpha_j(h) C_{jk} \frac{2\langle \alpha_k, \alpha_l \rangle}{\langle \alpha_l, \alpha_l \rangle} \\ &= \chi(h) - \frac{2\alpha_l(h)}{\langle \alpha_l, \alpha_l \rangle} \alpha_l \\ &= \chi(\sigma_l(h)). \end{aligned}$$

It follows that if $\sigma \in \mathcal{W}_\mu$, then $\chi(\sigma(\tilde{k})) = \sigma(\chi(\tilde{k}))$. We also have $\langle \alpha_l, \chi(h) \rangle = \alpha_l(h)$ so that $\chi(\tilde{k})$ and $\chi(\sigma(\tilde{k}))$ are in the closure of the same Weyl chamber in V. Applying Humphreys' 10.3 Lemma B[10] we see that $\chi(\sigma(\tilde{k})) = \sigma(\chi(\tilde{k})) = \chi(\tilde{k})$ and hence $\alpha_i(\tilde{k} - \sigma(\tilde{k})) = 0$ for $i=s+1,\ldots,r$. We have previously seen that $\lambda_i(\tilde{k} - \sigma(\tilde{k})) = 0$ for $i=1,\ldots,s$. Moreover, the span of the $\lambda_1,\ldots,\lambda_s$ is orthogonal to the span of the $\alpha_{s+1},\ldots,\alpha_r$, so together they must span $\mathfrak{t}^*$ and hence $\tilde{k} = \sigma(\tilde{k})$.

B. Proof of Proposition 4.3

Let ϵ be the sum of all the positive roots which are in the span of the simple roots $\alpha_{s+1},\ldots,\alpha_r$. Notice that $\epsilon(\tilde{k}) \leqslant 0$. Recall[10] that a simple root reflection σ_i permutes all the positive roots except α_i which it sends to $-\alpha_i$. So if $s+1 \leqslant i \leqslant r$, we have $\sigma_i(\epsilon) = \epsilon - 2\alpha_i$ so that

$$2\frac{\langle \epsilon,\alpha_i\rangle}{\langle \alpha_i,\alpha_i\rangle}=2.$$

So we have

$$\epsilon=\sum_{j=1}^{r} 2\frac{\langle \epsilon,\alpha_i\rangle}{\langle \alpha_i,\alpha_i\rangle}\lambda_i=\sum_{j=1}^{s} 2\frac{\langle \epsilon,\alpha_i\rangle}{\langle \alpha_i,\alpha_i\rangle}\lambda_i+\sum_{i=s+1}^{r} 2\lambda_i=\sum_{j=1}^{s} -p_j\lambda_j+\sum_{i=s+1}^{r} 2\lambda_i,$$

where $p_j\geqslant 0$ because if $1\leqslant j\leqslant s$ we have $\langle \epsilon,\alpha_j\rangle\leqslant 0$. Applying ϵ to $\tilde{k}$ gives

$$0\leqslant\sum_{i=1}^{r-s} h_i\leqslant\sum_{j=1}^{s} p_j m_j$$

and, as each h_j is non-negative, this means there can only be a finite number of possibilities.

V. EXAMPLES

Let $G=\mathrm{SU}(N)$ and μ be a diagonal matrix with eigenvalues $i\mu_1,i\mu_2,\ldots,i\mu_q$ with multiplicities $n_1,\ldots,n_q$ and assume that $\mu_1>\mu_2>\cdots>\mu_q$. Choose the usual fundamental Weyl chamber. That is, if d is any diagonal matrix with entries $id_1,\ldots,id_N$, then it is in the fundamental Weyl chamber if $d_1>d_2>\cdots>d_N$. Clearly this has μ in its closure. Define $x_j(d)=d_j$. Then the simple roots are $\alpha_i=x_{i+1}-x_i$ for $i=1,\ldots,N-1$. The fundamental weights satisfy

$$\lambda_j(d)=d_1+\cdots+d_j$$

and a weight is magnetic if $j=n_1,n_2,\ldots,n_{q-1}$ and holomorphic otherwise.

Let $\mathbb{C}^N=\mathbb{C}^{n_1}\oplus\cdots\oplus\mathbb{C}^{n_q}$ be the corresponding eigenvalue decomposition of $\mathbb{C}^N$. Assume that on $\mathbb{C}^{n_j}$ the eigenvalues of k are

$$k_{n_1+\cdots+n_{j-1}+1}\leqslant k_{n_1+\cdots+n_{j-1}+2}\leqslant\cdots\leqslant k_{n_1+\cdots+n_j}.$$

Then $\tilde{k}$ is the diagonal matrix with entries $ik_1,\ldots,ik_N$.

Let $\mathcal{M}_j$ be the stratum containing $\mathcal{M}(u,\mu,k)$. It was shown in Ref. 19 that

$$\dim\ \mathcal{M}_j=4\sum_{i=1}^{N}(k_1+\cdots+k_i)+\dim\ C(\mu)k,$$

and hence from the definition of the strata in the Introduction,

$$\dim\ (\mathcal{M}(u,\mu,k))=4\sum_{i=1}^{N}(k_1+\cdots+k_i),$$

so the dimension is divisible by four as required for a hyper-Kähler manifold. In Proposition 6.2 we shall show that this result is always true.

Notice that we could find a deformation μ_t of μ by choosing μ_t to be diagonal with entries $i\mu_j(t)$ such that

$$\mu_1(t)>\mu_2(t)>\mu_3(t)>\cdots>\mu_N(t),$$

and, of course, with $\mu(0)=\mu$. It follows from known results on the moduli spaces[12,20] that $\dim\mathcal{M}(u,\mu_t,k)=\dim\mathcal{M}(u,\mu,k)$. In fact, the method used in Ref. 19 to calculate the dimension formula shows that $\mathcal{M}(u,\mu(t),k))$ and $\mathcal{M}(u,\mu,k)$ are diffeomorphic spaces of holomorphic maps. This result was generalized to arbitrary G by Jarvis.[13]

VI. DIMENSIONS

In this section we compute the dimension of the moduli space $\mathcal{M}(u,\mu,m)$ by computing the dimension of $\tilde{\mathcal{R}}(O_\mu,m)$. We shall also compute the dimensions of the strata and the moduli space $\mathcal{M}(u,\mu,k)$ of monopoles of type (μ,k), by computing the dimension of $\tilde{\mathcal{R}}(O_{\mu k},m)$.

Fix a maximal torus T, a fundamental Weyl chamber and a set of simple roots $\alpha_1,\ldots,\alpha_r$. For a root α let $\mathfrak{g}_\alpha$ be the α root space. Denote by B the standard Borel determined by this choice of simple roots. That is the Lie algebra of B contains the root space of every simple root. The parabolic P is determined by the fact that its Lie algebra $\mathfrak{p}$ contains the root spaces for the negative roots $\alpha_{s+1},\ldots,\alpha_r$.

If $f:S^2\to G^c/P$ is a holomorphic map, then we can use it to pull back the tangent bundle to G^c/P and the Riemann–Roch theorem tell us that

$$\dim(H^0(S^2,f^{-1}(TG^c/P)))-\dim(H^1(S^2,f^{-1}(TG^c/P)))=\dim(G^c/P)+c_1(\det(f^{-1}TG^c/P)),$$

where $\det(TG^c/P)$ is the determinant line bundle of $f^{-1}TG^c/P$ and c_1 denotes the first Chern class. Because the group G acts holomorphically on G^c/P every element of $\mathfrak{g}$ defines a holomorphic vector field on G^c/P so we have a surjection of holomorphic vector bundles over S^2

$$\mathfrak{g}\times S^2\to f^{-1}TG^c/P\to 0$$

and it follows from the short exact sequence in cohomology that

$$\dim(H^1(S^2,f^{-1}(TG^c/P)))=0.$$

The tangent space to $\tilde{\mathcal{R}}(G^c/P,m)$ at the function f is just the subset of sections in $H^0(S^2,f^{-1}(TG^c/P))$ which vanish at the base point, say $P\in G^c/P$. This has *real* dimension

$$\dim\mathcal{R}(G^c/P,m)=2(\dim(H^0(S^2,f^{-1}(TG^c/P)))-\dim(G^c/P))=2c_1(\det(f^{-1}TG^c/P)).$$

Each of the fundamental weights $\lambda_1,\ldots,\lambda_s$ extend to one-dimensional representations of P and hence define homogeneous line bundles $L(\lambda_i)$ over G^c/P. The magnetic charges of a holomorphic map f are $m_i=-c_1(f^{-1}(L(\lambda_i)))$. Choose $\tilde{k}$ so that $m_i=\lambda_i(\tilde{k})$. Then $c_1(f^{-1}(L(-\lambda)))=\lambda(\tilde{k})$ for any weight λ.

Let ϵ be the weight defined by the adjoint representation of P on $\mathfrak{p}$. Then the weight defined by the adjoint representation of P on $\mathfrak{g}/\mathfrak{p}$ is $-\epsilon$. The bundle $\det(T(G^c/P)$ is then a homogeneous bundle over G^c/P induced by the character $-\epsilon$ so that

$$c_1(f^{-1}(\det(TG^c/P)))=c_1(f^{-1}(L(-\epsilon)))=\epsilon(\tilde{k}).$$

Hence

$$\dim\tilde{\mathcal{R}}(G^c/P,m)=2\epsilon(\tilde{k}).$$

In the case of maximal symmetry breaking where the parabolic P is a Borel B

$$\epsilon=\sum_{\alpha>0}\alpha=2\sum_{i=1}^{r}\lambda_i,$$

so that

$$\dim\tilde{\mathcal{R}}(G/B,m)=4\sum_{i=1}^{r}m_i.$$

In the nonmaximal symmetry breaking case we can proceed further. Because ϵ is a weight we know that $\epsilon=\Sigma_{i=1}^{r}-n_i\lambda_i$ for some integers n_i. We also know that ϵ is a character of P so invariant under the simple root reflections σ_i for $i=s+1,\ldots,r$. But $\sigma_j(\epsilon)=\epsilon+n_j\alpha_j$ so that we must have

$$\epsilon=\sum_{i=1}^{s} -n_i\lambda_i$$

and hence

$$\dim\widetilde{\mathcal{R}}(G^c/P,m)=2\sum_{i=1}^{s} n_i m_i .$$

We can obtain some further information about the n_i. First we note that

$$n_i=-2\frac{\langle\epsilon,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}.$$

Also, if ρ is one-half the sum of the positive roots and $\rho_{\mathfrak{p}}$ is one-half the sum of the positive roots α for which $\mathfrak{g}_{-\alpha}\subset\mathfrak{p}$, then we have that $\epsilon=-2\rho+2\rho_{\mathfrak{p}}$ and hence

$$n_i=2\frac{\langle 2\rho-2\rho_{\mathfrak{p}},\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}=2\left(1-2\frac{\langle\rho_{\mathfrak{p}},\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}\right)\lambda_i$$

using the standard fact that $\rho=\Sigma_{i=1}^{r}\lambda_i$. Hence

$$\dim(\widetilde{\mathcal{R}}(G^c/P,m))=4\sum_{i=1}^{s}\left(1-2\frac{\langle\rho_{\mathfrak{p}},\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}\right)m_i ,$$

which agrees with the result in Ref. 20. So we have the following proposition.

Proposition 6.1: The dimension of the moduli space $\mathcal{M}(u,\mu,m)$ *is*

$$4\sum_{i=1}^{s}\left(1-2\frac{\langle\rho_{\mathfrak{p}},\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}\right)m_i .$$

Notice that while the Lie theory guarantees ρ is a weight, the same may not be true of $\rho_{\mathfrak{p}}$ and hence expressions such as

$$2\frac{\langle\rho_{\mathfrak{p}},\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}$$

may not be integers. This is consistent with the fact that for nonmaximal symmetry breaking the moduli space may not be hyper-Kähler for the simple reason that its dimension is not a multiple of four.

Next we calculate $\dim\widetilde{\mathcal{R}}(O_{\mu k},m)$ where $P_{\mu k}$ is the parabolic subgroup containing all the positive roots and the negative roots α where $\alpha(\mu)=\alpha(\tilde{k})=0$ and we let $O_{\mu k}=G^c/P_{\mu k}$. This is the parabolic subgroup occurring in (2.3).

Then ϵ is the sum of all these roots so that

$$\epsilon(\tilde{k})=\sum_{\alpha>0}\alpha(\tilde{k})$$

and we have

$$\dim\widetilde{\mathcal{R}}(G^c/P_{\mu k},k)=4\left(\sum_{i=1}^{s} m_i+\sum_{j=1}^{r-s} h_j\right).$$

Hence we deduce the following.

Proposition 6.2: The dimension of the moduli space $\mathcal{M}(u,\mu,k)$ *is*

$$4\left(\sum_{i=1}^{s} m_i+\sum_{j=1}^{r-s} h_j\right).$$

In particular it is divisible by four.

Similarly for the strata, we have the following Corollary.

Corollary 6.3: The dimension of the stratum $\mathcal{M}_j$ *containing* $\mathcal{M}(u,\mu,k)$ *is*

$$4\left(\sum_{i=1}^{s} m_i+\sum_{j=1}^{r-s} h_j\right)+\dim C(\mu)-\dim C(\mu)\cap C(k).$$

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[1] Atiyah, A. and Bott, R., ''The Yang-Mills equations over Riemann surfaces,'' Philos. Trans. R. Soc. London, Ser. A **308**, 523–615 (1982).
[2] Atiyah, M. and Hitchin, N., ''The geometry and dynamics of magnetic monopoles,'' *M. B. Porter Lectures* (Princeton University Press, Princeton, NJ, 1988).
[3] Bernstein, I. N., Gel'fand, I. M., and Gel'fand, S. I., ''Schubert cells and cohomology of the spaces G^c/P,'' Russ. Math. Surveys **28** (3), 1–26 (1973).
[4] Bielawski, R., ''Monopoles and the Gibbons-Manton metric,'' Commun. Math. Phys. **194**, 297–321 (1998).
[5] Bielawski, R., ''Asymptotic metrics for SU*(N)*-monopoles with maximal symmetry breaking,'' Commun. Math. Phys. **199**, 297–325 (1998).
[6] Boyer, C. P., Mann, B. M., Hurtubise, J. C., and Milgram, R. J., ''The topology of the space of rational maps into generalized flag manifolds,'' Acta Math. **173**, 61–101 (1994).
[7] Dancer, A. S., ''Nahm data and SU(3) monopoles,'' Nonlinearity **5**, 1355–1373 (1992).
[8] Donaldson, S. K., ''Nahm's equations and the Classification of Monopoles,'' Commun. Math. Phys. **96**, 387–407 (1984).
[9] Goddard, P., Nuyts, J., and Olive, D. I., ''Gauge theories and magnetic charges,'' Nucl. Phys. B **125**, 1–28 (1977).
[10] Humphreys, J. E., *Linear Algebraic Groups*, Graduate Texts in Mathematics (Springer-Verlag, New York, 1981).
[11] Hurtubise, J., ''The classification of monopoles for the classical groups,'' Commun. Math. Phys. **120**, 613–641 (1989).
[12] Hurtubise, J. and Murray, M. K., ''On the construction of monopoles for the classical groups,'' Commun. Math. Phys. **122**, 35–89 (1989).
[13] Jarvis, S., ''Euclidean monopoles and rational maps,'' Proc. London Math. Soc. **77**, 170–192 (1998).
[14] Jarvis, S., ''Construction of Euclidean monopoles,'' Proc. London Math. Soc. **77**, 193–214 (1998).
[15] Jarvis, S., ''A rational map for Euclidean monopoles via radial scattering,'' J. Reine Angew. Math. **524**, 17–41 (2000).
[16] Koboyashi, S. and Nomizu, K., *Foundations of Differential Geometry* (Wiley-Interscience, New York, 1963), Vol. 1.
[17] Lee, K., Weinberg, E. J., and Yi, P., ''Moduli space of many BPS monopoles for arbitrary gauge groups,'' Phys. Rev. D **54**, 1633–1643 (1996).
[18] Lee, K., Weinberg, E. J., and Yi, P., ''Massive and massless monopoles with non-abelian magnetic charges,'' Phys. Rev. D **54**, 6351–6371 (1996).
[19] Murray, M., ''Stratifying monopoles and rational maps,'' Commun. Math. Phys. **125**, 661–674 (1989).
[20] Weinberg, E. J., ''Fundamental monopoles in theories with arbitrary symmetry breaking,'' Nucl. Phys. B **203**, 445–471 (1982).