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On the continuum limit of fermionic topological charge in lattice gauge theory

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It is proved that the fermionic topological charge of SU(N) lattice gauge fields on the four-torus, given in terms of a spectral flow of the Hermitian Wilson–Dirac operator or, equivalently, as the index of the overlap Dirac operator, reduces to the continuum topological charge in the classical continuum limit when the parameter m_0 is in the physical region $0<m_0<2$. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1415087]

I. INTRODUCTION

Let T^4 denote the Euclidean four-torus with fixed edge length L and fundamental domain $[0,L]^4 \subset \mathbf{R}^4$. A gauge potential on an SU(N) bundle over T^4 can be viewed as an su(N)-valued gauge field $A_\mu(x)$ on $\mathbf{R}^4$ satisfying

$$A_\mu(x+Le_\nu)=\Omega(x,\nu)A_\mu(x)\Omega(x,\nu)^{-1}+\Omega(x,\nu)\partial_\mu\Omega(x,\nu)^{-1}, \tag{1.1}$$

where e_ν is the unit vector in the positive ν-direction and $\Omega(x,\nu)$, $\nu=1,2,3,4$, are the SU(N)-valued monodromy fields which specify the principal SU(N) bundle over T^4. These also satisfy a cocycle condition which ensures that $A_\mu(x+Le_\nu+Le_\rho)$ is unambiguous and that Eq. (2.4) in this work is consistent. It is always possible to make a gauge transformation so that $\Omega(x,\nu)=1$ for $\nu=1,2,3$ and $\Omega(x,4)$ is periodic in x_1,x_2,x_3. Then for fixed x_4 $\Omega(x,4)$ determines a map $T^3 \to \mathrm{SU(N)}$. The degree of this map [which is independent of x_4 since $\Omega(x,4)$ depends smoothly on x_4] equals the Pontryargin number of the SU(N) bundle over T^4. The Pontryargin number of the bundle is encoded in the gauge field as its topological charge:

$$Q=\frac{-1}{8\pi^2}\int_{T^4}\mathrm{tr}(F\wedge F)=\frac{-1}{32\pi^2}\int d^4x\,\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}(x)F_{\rho\sigma}(x)). \tag{1.2}$$

The sections $\psi(x)$ in the standard spinor bundle over T^4 twisted by the SU(N) bundle can be viewed as spinor fields on $\mathbf{R}^4$ satisfying

$$\psi(x+Le_\nu)=\Omega(x,\nu)\psi(x). \tag{1.3}$$

The Dirac operator $\partial\!\!\!/^A=\gamma^\mu(\partial_\mu+A_\mu)$ acts on these, and the Index Theorem[1] gives

$$Q=\mathrm{index}\,\partial\!\!\!/^A. \tag{1.4}$$

The index$\partial\!\!\!/^A$ is equal to the spectral flow of the Hermitian operator $-\gamma_5(i\partial\!\!\!/^A-m)$ as m increases from any negative to any positive value [note that eigenvalues can only cross the origin at $m=0$ since $(\gamma_5(i\partial\!\!\!/-m))^2=\partial\!\!\!/^2+m^2$.]

The spectral flow description of Q motivates a fermionic definition of topological charge Q_{lat} in lattice gauge theory,[2–4] which has been extensively studied numerically in its various guises;

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see, e.g., Refs. 2 and 4–11. The purpose of this article is to analytically prove that Q_{lat} reduces to Q in the classical continuum limit. (This result was announced in Ref. 12 although the argument we give here is simpler and more direct than the one sketched there.)

II. STATEMENT OF THE MAIN RESULT

Put a hyper-cubic lattice on $\mathbf{R}^4$ with sites $a\mathbf{Z}^4$. We consider only the lattice spacings a for which L/a is a whole number. Furthermore, we restrict to lattice spacings with the property $a_1\mathbf{Z}^4 \subset a_2\mathbf{Z}^4$ for $a_2 < a_1$. This implies that if $x \in \mathbf{R}^4$ is a lattice site in the lattice with spacing a, then it is also a lattice site in all the other lattices with spacing $a' < a$. In the following, in statements concerning $a \to 0$ limits (in particular Proposition 2 to follow) the variable x always denotes such a point in $\mathbf{R}^4$; it is fixed in $\mathbf{R}^4$ and does not change as we go from one lattice to another.

The lattice transcript of A,

$$U_\mu(x) = T\exp\left(\int_0^1 aA_\mu(x + tae_\mu)\,dt\right) \tag{2.1}$$

$(T = t-\text{ordering})$, satisfies

$$U_\mu(x + Le_\nu) = \Omega(x,\nu)U_\mu(x)\Omega(x + ae_\mu,\nu)^{-1}. \tag{2.2}$$

Given such a lattice, let $\mathcal{C}$ denote the infinite-dimensional complex vectorspace of lattice spinor fields $\psi(x)$ (i.e., functions on the lattice sites taking values in $\mathbf{C}^4 \otimes \mathbf{C}^N$) and define the inner product

$$\langle \psi_1, \psi_2 \rangle = a^4 \sum_{x \in a\mathbf{Z}^4} \psi_1(x)^* \psi_2(x), \tag{2.3}$$

where a contraction of spinor and color indices is implied. Let $\mathcal{H} \subset \mathcal{C}$ denote the Hilbert space of spinor fields with $\|\psi\| < \infty$ and let $\mathcal{C}_L \subset \mathcal{C}$ denote the finite-dimensional subspace of spinor fields satisfying the lattice version of (1.3):

$$\psi(x + Le_\nu) = \Omega(x,\nu)\psi(x), \quad \forall x \in a\mathbf{Z}^4. \tag{2.4}$$

The fields $\psi \in \mathcal{C}_L$ are determined by their restriction to $\mathcal{F}_L :=$ the set of lattice sites contained in $[0,L)^4 \subset \mathbf{R}^4$. We define an inner product in $\mathcal{C}_L$ by

$$\langle \psi_1, \psi_2 \rangle_L = a^4 \sum_{x \in \mathcal{F}_L} \psi_1(x)^* \psi_2(x). \tag{2.5}$$

The covariant forward (backward) finite difference operators $(1/a)\,\nabla_\mu^+((1/a)\,\nabla_\mu^-)$ are defined on $\mathcal{C}$ by

$$\nabla_\mu^+ \psi(x) = U_\mu(x)\psi(x + ae_\mu) - \psi(x), \tag{2.6}$$

$$\nabla_\mu^- \psi(x) = \psi(x) - U_\mu(x - ae_\mu)^{-1}\psi(x - ae_\mu). \tag{2.7}$$

These are bounded $(\|\nabla_\mu^\pm\| \leqslant 2)$ and therefore map $\mathcal{H}$ to $\mathcal{H}$. They also preserve (2.4) and therefore map $\mathcal{C}_L$ to $\mathcal{C}_L$. Note that

$$(\nabla_\mu^\pm)^* = -\nabla_\mu^\mp \tag{2.8}$$

on $\mathcal{H}$ and $\mathcal{C}_L$. The lattice version of $i\partial\!\!\!/^A$ is the Wilson–Dirac operator:

$$D_w = i\frac{1}{a}\nabla\!\!\!/ + \frac{r}{2}a\left(\frac{1}{a^2}\Delta\right), \quad r>0, \tag{2.9}$$

where $(1/a)\,\nabla\!\!\!/ = \Sigma_\mu \gamma^\mu \frac{1}{2}(\nabla_\mu^+ + \nabla_\mu^-)$ is the naive lattice Dirac operator and $(1/a^2)\,\Delta = (1/a^2)\,\Sigma_\mu(\nabla_\mu^- + \nabla_\mu^+) = (1/a^2)\,\Sigma_\mu(\nabla_\mu^+)^*\nabla_\mu^+ = (1/a^2)\,\Sigma_\mu(\nabla_\mu^-)^*\nabla_\mu^-$ is the lattice Laplace operator. We are following the mathmatical convention where the γ^μ's are anti-Hermitian [this explains the factor i in $i(1/a)\,\nabla\!\!\!/$ in (2.9) which is not usually present in the physics literature where the γ^μ's are Hermitian]. Then $\nabla\!\!\!/$ is Hermitian due to (2.8) and Δ is Hermitian and positive. [The Wilson term, i.e., the second term in (2.9), which formally vanishes in the $a\to 0$ limit, is included to avoid the fermion doubling problem: a degeneracy of the nullspace of $\nabla\!\!\!/$ which is a lattice artifact unrelated to the continuum theory.[13,14]] The lattice version of $\gamma_5(i\partial\!\!\!/ - m)$ is the Hermitian operator $(1/a)\,H_m$:

$$\frac{1}{a}H_m = \gamma_5\left(D_w - \frac{rm}{a}\right), \tag{2.10}$$

$$H_m = \gamma_5(i\nabla\!\!\!/ + r(\tfrac{1}{2}\Delta - m)). \tag{2.11}$$

It can be shown that the spectrum of H_m is symmetric and without zero for all $m<0$. Hence the spectral flow of $-H_m$ as m increases from any negative value to some positive value m_0 is equal to half the spectral asymmetry of $-H_{m_0}$.[3,4] This suggests the following fermionic definition of the topological charge of the lattice gauge field $U_\mu(x)$:

$$Q_{\mathrm{lat}} = Q_{m_0} := -\frac{1}{2}\mathrm{Tr}\left(\frac{H_{m_0}}{|H_{m_0}|}\right), \tag{2.12}$$

where H_{m_0} is acting on $\mathcal{C}_L$. The spectral flow of H_m was first studied numerically in Ref. 2. The definition (2.12) arose in the overlap formulation of chiral gauge theory on the lattice.[3,4] Q_{m_0} also arises as an index: $Q_{m_0} = \mathrm{index}(D_{m_0}) := \mathrm{Tr}(\gamma_5|_{\ker D_{m_0}})$ where $D = (1/a)\,(1+\gamma_5\,(H/|H|))$ is the overlap Dirac operator.[15]

Unlike in the continuum case, the spectral flow of $-H_m$ depends on the final value $m_0>0$ of m. Numerical studies have shown that for reasonably smooth lattice gauge fields, e.g., when $U_\mu(x)$ is the lattice transcript of a smooth continuum gauge field and the lattice is reasonably fine, the eigenvalue crossings of $-H_m$ are localized around $m=0,2,4,6,8$.[2,8] Furthermore, when the lattice gauge field is the lattice transcript of a continuum field the spectral flow due to crossings close to $m=0$ was found to reproduce the continuum topological charge Q. In this article we complement the previous numerical studies with the following analytical result:

Theorem: In the above setting, where $U_\mu(x)$ is the lattice transcript (2.1) and $m_0 \notin \{0,2,4,6,8\}$, there exists an $a_0>0$ [depending on $A_\mu(x)$ and m_0] such that

$$Q_{m_0} = I(m_0)Q \quad \text{for all lattice spacings } a<a_0, \tag{2.13}$$

where

	$0<m_0<2$	$2<m_0<4$	$4<m_0<6$	$6<m_0<8$	$m_0 \notin [0,8]$
$I(m_0) =$	1	-3	3	-1	0

(2.14)

Remarks: (i) The dependence on m_0 in (2.13) and (2.14) coincides with that found in the above-mentioned numerical studies with smooth lattice gauge fields. (ii) The definition (2.12) of

Q_{m_0} is only meaningful when H_{m_0} does not have zero-modes. In the present case this is guaranteed when $m_0 \notin \{0,2,4,6,8\}$ and a is sufficiently small. Indeed, it is known that when $\|1-U(p)\|<\epsilon$ for all lattice plaquettes p, where $U(p)$ is the product of the link variables $U_\mu(x)$ around p, then there is a lower bound $H_{m_0}^2>b$, depending only on ϵ and m_0, such that for fixed $m_0 \notin \{0,2,4,6,8\} b>0$ when ϵ is sufficiently small. This bound was established in Ref. 16 (and improved in Ref. 17) for the case where $0<m_0<2$ and can be generalized to arbitrary $m_0 \notin \{0,2,4,6,8\}$.[18] In the present case, where $U_\mu(x)$ is the lattice transcript (2.1), we have

$$1-U(p_{x,\mu\nu})=a^2F_{\mu\nu}(x)+O(a^3)(x) \tag{2.15}$$

leading to

$$\|1-U(p)\|\sim O(a^2). \tag{2.16}$$

Hence the above-mentioned lower bound $H_{m_0}^2>b>0$ holds for all sufficiently small a. Here and in the following $O(a^p)(x)$ denotes a function on the lattice sites $x\in\mathcal{F}_L$ such that the operator norm of $O(a^p)(x)$, considered as a multiplication operator on $\mathcal{C}$, satisfies $\|O(a^p)(x)\|\leqslant a^pK$ for all $x\in\mathcal{F}_L$ where K is a constant independent of a and x. [In (2.15) $O(a^p)(x)$ takes values in the space of linear maps on $\mathbf{C}^N$; sometimes $O(a^p)(x)$ will just be a $\mathbf{C}$-valued function of x, in which case we have $|O(a^p)(x)|\leqslant a^pK$.] We discuss the derivation of (2.15) and (2.16), and other bounds used in the following, in the appendix. In general, to conclude (2.16) from (2.15) we need the $O(a^3)(x)$ term to satisfy $\|O(a^3)(x)\|\leqslant a^3K$ for all $x\in a\mathbf{Z}^4$. For general gauge field $A_\mu(x)$ on $\mathbf{R}^4$ this holds when $\|A_\mu(x)\|$ and $\|\partial_\mu A_\nu(x)\|$ are bounded on $\mathbf{R}^4$ (cf. the Appendix). In the present case the condition (1.1) generally results in divergence of $A_\mu(x)$ at infinity (for topologically nontrivial field). Nevertheless we still have (2.16) in this case: it is a consequence of (2.2) [note that $\|U_\mu(x)\|=1$ since $U_\mu(x)$ is unitary] and the fact that the $O(a^3)(x)$ term satisfies $\|O(a^3)(x)\|\leqslant a^3K$ when x is restricted to be in the fundamental domain $\mathcal{F}_L$.

III. PROOF OF THE THEOREM

The strategy for proving the theorem is to express Q_{m_0} as the sum of a density,

$$Q_{m_0}=a^4\sum_{x\in\mathcal{F}_L} q_L(x), \tag{3.1}$$

and show that

$$q_L(x)=I(m_0)q^A(x)+O(a)(x) \quad (x\in\mathcal{F}_L), \tag{3.2}$$

where

$$q^A(x)=\frac{-1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}F_{\mu\nu}(x)F_{\rho\sigma}(x). \tag{3.3}$$

Then $\lim_{a\to 0}Q_{m_0}=I(m_0)Q$, and since Q_{m_0} is integer it follows that Q_{m_0} must coincide with $I(m_0)Q$ for small nonzero a as stated in the theorem.

To specify the density $q_L(x)$ in (3.1) we introduce the following definitions. We decompose $\mathcal{C}=\mathcal{C}^{sc}\otimes(\mathbf{C}^4\otimes\mathbf{C}^N)$, $\mathcal{H}=\mathcal{H}^{sc}\otimes(\mathbf{C}^4\otimes\mathbf{C}^N)$ where $\mathcal{C}^{sc}$, $\mathcal{H}^{sc}$ denote the corresponding spaces of scalar lattice fields. $\mathcal{H}^{sc}$ has the orthonormal basis $\{\delta_x/a^2\}_{x\in a\mathbf{Z}^4}$ where $\delta_x(y)=\delta_{xy}$. For linear operator $\mathcal{O}_\mathcal{H}$ on $\mathcal{H}$ we define $\mathcal{O}_\mathcal{H}(x,y)=(1/a^4)\langle(\delta_x/a^2),\mathcal{O}_\mathcal{H}(\delta_y/a^2)\rangle$; this is a linear operator on $\mathbf{C}^4\otimes\mathbf{C}^N$ satisfying

$$\mathcal{O}_\mathcal{H}\psi(x)=a^4\sum_{y\in a\mathbf{Z}^4}\mathcal{O}_\mathcal{H}(x,y)\psi(y) \quad \forall\psi\in\mathcal{H}. \tag{3.4}$$

There is also an obvious decomposition $\mathcal{C}_L = \mathcal{C}_L^{sc} \otimes (\mathbf{C}^4 \otimes \mathbf{C}^N)$ with $\mathcal{C}_L^{sc}$ having the basis $\{\phi_x\}_{x \in \mathcal{F}_L}$ where $\phi_x(y) = (1/a^2)\,\delta_{xy}$ for $y \in \mathcal{F}_L$ and is extended to $a\mathbf{Z}^4$ in accordance with (2.4):

$$\phi_x(y+Ln) = \frac{1}{a^2}\Omega^{(n)}(x)\,\delta_{xy}\,, \quad \Omega^{(n)}(x) = \prod_\nu \Omega(x,\nu)^{n_\nu}, \quad n \in \mathbf{Z}^4. \tag{3.5}$$

For linear operator $\mathcal{O}_L$ on $\mathcal{C}_L$ we define $\mathcal{O}_L(x,y) = (1/a^4)\,\langle \phi_x, \mathcal{O}_L \phi_y \rangle_L$ for $x,y \in \mathcal{F}_L$; this is a linear operator on $\mathbf{C}^4 \otimes \mathbf{C}^N$ satisfying

$$\mathcal{O}_L \psi(x) = a^4 \sum_{y \in \mathcal{F}_L} \mathcal{O}_L(x,y)\,\psi(y) \quad \forall\ \psi \in \mathcal{C}_L\,, \quad x \in \mathcal{F}_L\,. \tag{3.6}$$

The Cauchy–Schwarz inequality gives $\|\mathcal{O}_{\mathcal{H}}(x,y)\| \leqslant (1/a^4)\,\|\mathcal{O}_{\mathcal{H}}\|$ and $\|\mathcal{O}_L(x,y)\| \leqslant (1/a^4)\,\|\mathcal{O}_L\|_L$.

The definition (2.12) of Q_{m_0} can now be rewritten as (3.1) with

$$q_L(x) = -\frac{1}{2}\mathrm{tr}\left(\frac{H}{\sqrt{H^2}}\right)_L (x,x), \tag{3.7}$$

where $H = H_{m_0}$ and the trace is over spinor and color indices (i.e., over $\mathbf{C}^4 \otimes \mathbf{C}^N$). The strategy for deriving (3.2) and (3.3) is now to relate $q_L(x)$ to $q_{\mathcal{H}}(x)$, defined by replacing $(H/\sqrt{H^2})_L$ by $(H/\sqrt{H^2})_{\mathcal{H}}$ in (3.7). (The latter is defined via the spectral theory for bounded operators on Hilbert space.) This approach was suggested to me by Martin Lüscher.[19] The point is that (3.2) and (3.3) are relatively easy to derive for $q_{\mathcal{H}}(x)$; in fact, this has essentially already been done in previous works.[20–23] One potentially problematic aspect with regards to these previous calculations is that in the present case $A_\mu(x)$ can diverge for $|x| \to \infty$. However, we will get around this by exploiting the locality property of $(H/\sqrt{H^2})_{\mathcal{H}}$,[16] which will allow us to replace $A_\mu(x)$ by a gauge field which vanishes outside a bounded region of $\mathbf{R}^4$.

The relation between $q_L(x)$ and $q_{\mathcal{H}}(x)$ is as follows:

Proposition 1:

$$\left(\frac{H}{\sqrt{H^2}}\right)_L (x,y) = \sum_{n \in \mathbf{Z}^4} \left(\frac{H}{\sqrt{H^2}}\right)_{\mathcal{H}} (x,y+Ln)\,\Omega^{(n)}(y) \quad (x,y \in \mathcal{F}_L), \tag{3.8}$$

where $\Omega^{(n)}(x)$ is defined in (3.5). In particular, setting $y = x$ and substituting in (3.7) we get

$$q_L(x) = q_{\mathcal{H}}(x) - \frac{1}{2} \sum_{n \in \mathbf{Z}^4 - \{0\}} \mathrm{tr}\left(\frac{H}{\sqrt{H^2}}\right)_{\mathcal{H}} (x,x+Ln)\,\Omega^{(n)}(x). \tag{3.9}$$

Proof: We begin by deriving a relation between $\mathcal{O}_L(x,y)$ and $\mathcal{O}_{\mathcal{H}}(x,y)$ for bounded operators $\mathcal{O}$ on $\mathcal{C}$ which leave $\mathcal{C}_L$ invariant. The proposition will then follow by exploiting the fact that $(H/\sqrt{H^2})_L$ and $(H/\sqrt{H^2})_{\mathcal{H}}$ can be simultaneously approximated by such operators. The approximation part is necessary since $H/\sqrt{H^2}$ is not a well-defined operator on the whole of $\mathcal{C}$; the technicalities are related to the fact that $\mathcal{C}_L \not\subset \mathcal{H}$, i.e., elements in $\mathcal{C}_L$ can have infinite norm.

Let $\mathcal{O}$ be a bounded operator on $\mathcal{C}$ which maps $\mathcal{C}_L$ to itself. Then it follows from the above definitions and (3.5) that, for $x,y \in \mathcal{F}_L$,

$$\begin{aligned}
\mathcal{O}_L(x,y) &= \frac{1}{a^4}\langle \phi_x, \mathcal{O}\phi_y\rangle_L \\
&= \sum_{z\in\mathcal{F}_L} \phi_x(z)(\mathcal{O}\phi_y)(z) \\
&= \frac{1}{a^2}(\mathcal{O}\phi_y)(x) = a^2 \sum_{z\in a\mathbf{Z}^4} \mathcal{O}_{\mathcal{H}}(x,z)\,\phi_y(z) \\
&= \sum_{n\in\mathbf{Z}^4} \mathcal{O}_{\mathcal{H}}(x,y+Ln)\,\Omega^{(n)}(y).
\end{aligned} \tag{3.10}$$

We now exploit the fact[16] that $1/\sqrt{H^2}$ has a power series expansion $\kappa\Sigma_{k=0}^{\infty} t^k P_k(H^2)$ norm-convergent to $(1/\sqrt{H^2})_L$ and $(1/\sqrt{H^2})_{\mathcal{H}}$ on $\mathcal{C}_L$ and $\mathcal{H}$, respectively. $P_k(\cdot)$ is a Legendre polynomial of order k; $\|P_k(H^2)\|\leqslant 1$; $t=e^{-\theta}$; and the constants $\kappa,\theta>0$ depend only on the (strictly positive) lower and upper bounds on H^2.[16] [We are assuming that a is sufficiently small so that H^2 has a lower bound $b>0$ cf. remark (ii) above]. Set

$$P^{(N)} := H\left(\kappa \sum_{k=0}^{N} t^k P_k(H^2)\right).$$

For arbitrary finite N this is a bounded operator on $\mathcal{C}$ which maps $\mathcal{C}_L$ to itself. In light of (3.10), to prove the proposition it suffices to show that $(H/\sqrt{H^2})_L(x,y) - P_L^{(N)}(x,y)$ and $\Sigma_{n\in\mathbf{Z}^4}[(H/\sqrt{H^2})_{\mathcal{H}}(x,y+Ln) - P_{\mathcal{H}}^{(N)}(x,y+Ln)]\Omega^{(n)}(y)$ both vanish in the $N\rightarrow\infty$ limit. The former is obvious. To show the latter it suffices to show that $\Sigma_{n\in\mathbf{Z}^4}\Sigma_{k=N+1}^{\infty} t^k \|P_k(x,y+Ln)\|$ vanishes in the $N\rightarrow 0$ limit. (We have set $P_k(x,z)=[P_k(H^2)](x,z)$.) For simplicity we show this for $y=x$ [the relevant case for (3.9)]; the argument in the general case is a straightforward generalization. Since $P_k(H^2)$ is of order k in H^2, and H couples only nearest neighbor sites, we have $P_k(x,x+Ln)=0$ when $(L/a)\,\Sigma_\mu |n_\mu|>2k$. Since $\|P_k(x,z)\|\leqslant (1/a^4)\,\|P_k(H^2)\|\leqslant a^4$ it follows that

$$\begin{aligned}
\sum_{n\in\mathbf{Z}^4}\sum_{k=N+1}^{\infty} t^k\|P_k(x,x+Ln)\| &\leqslant \frac{1}{a^4} t^N \left(\sum_{n\in\mathbf{Z}^4, L/2a\,\Sigma_\mu|n_\mu|\leqslant N}\ \sum_{k=1}^{\infty} t^k\right) \\
&\quad + \frac{1}{a^4}\left(\sum_{n\in\mathbf{Z}^4, 1/2a\,\Sigma_\mu|n_\mu|>N} t^{((L/2a)\,\Sigma_\mu|n_\mu|)} \sum_{k=1}^{\infty} t^k\right).
\end{aligned} \tag{3.11}$$

The first sum over n vanishes as $N^4 t^N$ for $N\rightarrow\infty$, while the second clearly vanishes for $N\rightarrow\infty$ since it is convergent for finite N. This completes the proof of the proposition.

We now derive a small a bound on the second term in (3.9). The facts that $P_k(x,x+Ln)=0$ for $(L/a)\,\Sigma_\mu|n_\mu|>2k$ and $\|P_k(H^2)\|\leqslant 1$ imply the following locality property of $(1/\sqrt{H^2})_{\mathcal{H}}$:[16]

$$\begin{aligned}
\left\|\left(\frac{1}{\sqrt{H^2}}\right)_{\mathcal{H}}(x,x+Ln)\right\| &\leqslant \left\|\kappa \sum_{k\geqslant (L/2a)\Sigma_\mu|n_\mu|} t^k P_k(x,y)\right\| \\
&\leqslant \kappa\, t^{((L/2a)\Sigma_\mu|n_\mu|)} \sum_{k=0}^{\infty} t^k \frac{1}{a^4} = \tilde{\kappa}\frac{1}{a^4}\exp\left(-\theta\frac{L}{2a}\sum_\mu |n_\mu|\right),
\end{aligned} \tag{3.12}$$

where $\tilde{\kappa} := \kappa/(1-e^{-\theta})$. For sufficiently small a this gives

$$\left\| \sum_{n\in\mathbf{Z}^4-\{0\}} \left(\frac{1}{\sqrt{H^2}}\right)_{\mathcal{H}}(x,x+Ln) \right\| \leqslant \sum_{n\in\mathbf{Z}^4-\{0\}} \frac{\tilde{\kappa}}{a^4}\prod_{\mu} \exp\left(-\theta\frac{L}{2a}|n_\mu|\right)$$

$$\leqslant \frac{\tilde{\kappa}}{a^4}\prod_{\mu}\left[2\int_{1/2}^{\infty}\exp\left(-\theta\frac{L}{2a}t_\mu\right)dt_\mu\right] = \tilde{\kappa}\left(\frac{4}{\theta L}\right)^4 \exp\left(-\frac{\theta L}{a}\right). \tag{3.13}$$

The second inequality follows from the fact that $\int_{1/2}^{\infty}\exp(-(\theta L/2a)\,t)\,dt \geqslant \exp(-(\theta L/2a))$ for sufficiently small a. It now follows from (3.9) that $q_L(x)=q_{\mathcal{H}}(x)+O(e^{-\rho/a})$ for sufficiently small a. (This had already been noted by M. Lüscher in the Abelian case in Ref. 24 although the derivation was not provided there.)

To prove the theorem it now suffices to show (3.2) and (3.3) for $q_{\mathcal{H}}(x)$ instead of $q_L(x)$, i.e., to show

$$q_{\mathcal{H}}(x)=I(m_0)q^A(x)+O(a)(x) \quad \text{for } x\in\mathcal{F}_L. \tag{3.14}$$

To simplify the derivation we exploit the fact that $q_{\mathcal{H}}(x)$ is local in the gauge field.[16] Because of this it suffices to show (3.14) in the case where $A_\mu(x)$ is replaced by another SU(N) gauge field $\tilde{A}_\mu(x)$ on $\mathbf{R}^4$ with $\tilde{A}_\mu(x)=A_\mu(x)$ in a neighborhood of $[0,L]^4$ and $\tilde{A}_\mu(x)=0$ outside a bounded region of $\mathbf{R}^4$. Specifically, we can take $\tilde{A}_\mu(x)=\lambda(x)A_\mu(x)$ where $\lambda(x)$ is a smooth function on $\mathbf{R}^4$ equal to 1 on $[-d,L+d]^4$ $(d>0)$ and vanishing outside a bounded region. To see this, let H and $\tilde{H}$ denote the operators defined by (2.11) with lattice gauge fields U and $\tilde{U}$ being the lattice transcripts [defined by (2.1)] of A and $\tilde{A}$, respectively. Then, for small a, just as for H^2 we have $\tilde{H}^2>b>0$ and an expansion $(1/\sqrt{\tilde{H}^2})_{\mathcal{H}}=\kappa\sum_{k=0}^{\infty}t^k\tilde{P}_k$ where $\tilde{P}_k=P_k(\tilde{H}^2)$. Since H and $\tilde{H}$ only couple nearest neighbor sites, $P_k(H^2)$ and $P_k(\tilde{H}^2)$ can only couple a lattice site in $[0,L]^4$ to another lattice site in $[0,L]^4$ via a site outside of $[-d,L+d]^4$ if $k\geqslant 2(d/2a)$. Therefore, $P_k(x,y)=\tilde{P}_k(x,y)$ for $x,y\in\mathcal{F}_L$ when $k<d/a$, and we find by an analogous argument to the one leading to (3.12) that, for $x,y\in\mathcal{F}_L$,

$$\left\| \left(\frac{1}{\sqrt{H^2}}\right)_{\mathcal{H}}(x,y)-\left(\frac{1}{\sqrt{\tilde{H}^2}}\right)_{\mathcal{H}}(x,y) \right\| \leqslant \kappa\sum_{k\geqslant d/a}^{\infty} t^k\|P_k(x,y)-\tilde{P}_k(x,y)\| \leqslant \frac{2\tilde{\kappa}}{a^4}e^{-\theta d/a}. \tag{3.15}$$

This together with the ultra-locality of H and $\tilde{H}$ implies

$$q_{\mathcal{H}}(x)=\tilde{q}_{\mathcal{H}}(x)+O\left(\frac{1}{a^4}e^{-\rho/a}\right)(x) \quad \text{for } x\in\mathcal{F}_L \tag{3.16}$$

In light of this, the theorem now follows from (a special case of) the following:

Proposition 2: Let $A_\mu(x)$ be a general smooth SU(N) gauge field on $\mathbf{R}^4$ with the property that $\|A_\mu(x)\|$, $\|\partial_\nu A_\mu(x)\|$, and $\|\partial_\sigma\partial_\nu A_\mu(x)\|$ are all bounded. Let $H=H_{m_0}$ be defined as in (2.11) with the lattice gauge field being the lattice transcript (2.1) of $A_\mu(x)$. Then $q_{\mathcal{H}}(x)=-\frac{1}{2}\operatorname{tr}(H/\sqrt{H^2})_{\mathcal{H}}(x,x)$ satisfies $q_{\mathcal{H}}(x)=I(m_0)q^A(x)+O(a)(x)$ for all $x\in a\mathbf{Z}^4$, where $\|O(a)(x)\|\leqslant aK$ for some constant K independent of x and small a.

Clearly the gauge field $\tilde{A}_\mu(x)$ introduced above satisfies the conditions of the proposition (since it vanishes outside a bounded region). Combining the proposition with (3.16) then gives (3.14), proving the theorem.

To prove Proposition 2 we use an integral representation to expand $1\sqrt{H^2}$ as a power series following Refs. 12 and 21. (This gives a more explicit power series expansion than the expansion in Legendre polynomials[16] discussed earlier.) Henceforth all operators are assumed to be acting on

$\mathcal{H}$ and we drop the subscript "$\mathcal{H}$" in the notation. Also, from now on $O(a^p)(x)$ denotes a term with $\|O(a^p)(x)\|\leqslant a^p K$ for all $x\in a\mathbf{Z}^4$ (not just for $x\in\mathcal{F}_L$). We first decompose

$$H^2=L-V, \tag{3.17}$$

where

$$L=-\nabla^2+r^2(\tfrac{1}{2}\Delta-m_0)^2, \tag{3.18}$$

$$V==ir\,\tfrac{1}{2}\gamma_\mu V_\mu-\tfrac{1}{4}[\gamma_\mu,\gamma_\nu]V_{\mu\nu}, \tag{3.19}$$

with

$$V_\mu=\frac{1}{2}\left[(\nabla_\mu^+ +\nabla_\mu^-),\sum_\nu\,(\nabla_\nu^- -\nabla_\nu^+)\right], \tag{3.20}$$

$$V_{\mu\nu}=\tfrac{1}{4}[(\nabla_\mu^+ +\nabla_\mu^-),(\nabla_\nu^+ +\nabla_\nu^-)]. \tag{3.21}$$

As pointed out in Ref. 16, the norms of the commutators of the $\nabla_\mu^\pm$'s are bounded by $\max_p\|1-U(p)\|$. The bound (2.16) on $\|1-U(p)\|$ is valid when the conditions of Proposition 2 are satisfied (cf. the Appendix), hence

$$\|V\|\sim O(a^2). \tag{3.22}$$

It follows that for small a we have $\|V\|<b/2$ where b is the lower bound on H^2 mentioned earlier in remark (ii). This in turn implies the lower bound $L>b/2>0$ for the positive operator L in (3.18). Thus for sufficiently small a the operator L is invertible, $\|L^{-1}\|\cdot\|V\|<1$, and we can make the expansion

$$\frac{H}{\sqrt{H^2}}=H\int_{-\infty}^{\infty}\frac{d\sigma}{\pi}\,\frac{1}{H^2+\sigma^2}=H\int_{-\infty}^{\infty}\frac{d\sigma}{\pi}\left(\frac{1}{1-(L+\sigma^2)^{-1}V}\right)\left(\frac{1}{L+\sigma^2}\right)=\int_{-\infty}^{\infty}\frac{d\sigma}{\pi}\sum_{k=0}^{\infty}H(G_\sigma V)^k G_\sigma, \tag{3.23}$$

where $G_\sigma:=(L+\sigma^2)^{-1}$. Note that the γ-matrices in (3.17) are all contained in V. Since the trace of γ_5 times a product of less than four γ-matrices vanishes, the $k=0$ and $k=1$ terms in (3.23) give vanishing contribution to $q_{\mathcal{H}}(x)$. On the other hand, the terms with $k\geqslant 3$ satisfy the following bound:

$$\begin{aligned}\left\|\int_{-\infty}^{\infty}\frac{d\sigma}{\pi}\sum_{k=3}^{\infty}[H(G_\sigma V)^k G_\sigma](x,x)\right\| &\leqslant \frac{1}{a^4}\|H\|\int_{-\infty}^{\infty}\frac{d\sigma}{\pi}\sum_{k=3}^{\infty}\|G_\sigma\|^{k+1}\|V\|^k\\ &\leqslant a^2K^3\|H\|\left[\int_{-\infty}^{\infty}\frac{d\sigma}{\pi}\,\frac{1}{(b/2+\sigma^2)^4}\right]\sum_{k=0}^{\infty}\left(\frac{2}{b}a^2K\right)^k,\end{aligned} \tag{3.24}$$

where we have used (3.22) and the bounds $G_\sigma<(b/2+\sigma^2)^{-1}\leqslant 2/b$. This is $O(a^2)$ since the integral and sum are finite and remain so in the $a\to 0$ limit. Hence only the $k=2$ term in (3.23) contributes in the $a\to 0$ limit:

$$q_{\mathcal{H}}(x)=q_{\mathcal{H}}^{(2)}(x)+O(a^2)(x), \tag{3.25}$$

where

$$q_{\mathcal{H}}^{(2)}(x) = -\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\sigma}{\pi}\operatorname{tr}[HG_\sigma VG_\sigma VG_\sigma](x,x). \tag{3.26}$$

For lattice operators $\mathcal{O}$ which are polynomials in $\nabla_\mu^\pm$ we denote by $\mathcal{O}^{(0)}$ the operator obtained by setting $U=1$ in (2.6) and (2.7). Standard arguments give (cf. the Appendix) $\|H-H^{(0)}\| \sim O(a)$ and $\|L-L^{(0)}\|\sim O(a)$. The latter implies $\|G_\sigma - G_\sigma^{(0)}\|\sim O(a)$; this follows from $G_\sigma - G_\sigma^{(0)} = G_\sigma^{(0)}(L^{(0)}-L)G_\sigma$ since G_σ and $G_\sigma^{(0)}$ are bounded from above by $2/b$ when a is sufficiently small. This allows us to replace H and G_σ by $H^{(0)}$ and $G_\sigma^{(0)}$ in (3.26) at the expense of an $O(a)(x)$ term. Furthermore, we have $\|[L^{(0)},V]\|\sim O(a^3)$ (cf. the Appendix). This leads to $\|[G_\sigma^{(0)},V]\|\sim O(a^3)$ as follows: The bound $\|\nabla_\mu^\pm\|\le 2$ and triangle inequalities lead to an a-independent upper bound $L<c$ which allows us to expand

$$G_\sigma = \left(\frac{1}{c+\sigma^2}\right)\left(\frac{1}{1-(c-L)/(c+\sigma)}\right) = \frac{1}{c+\sigma^2}\sum_{m=0}^{\infty}\left(\frac{c-L}{c+\sigma^2}\right)^m.$$

Now, since

$$\|[(c-L^{(0)})^m,V]\|\le m\|[L^0,V]\|\cdot\|c-L\|^{m-1}\le m(a^3K)(c-b/2)^{m-1},$$

we get

$$\|[G_\sigma^{(0)},V]\|\le \frac{a^3K}{c^2}\sum_{m=0}^{\infty}(m+1)\left(\frac{c-b/2}{c}\right)^m,$$

and this is $\sim O(a^3)$ since the sum converges (since $0<b/2<c$). Taking this into account in (3.26), it follows from (3.25) that

$$\begin{aligned} q_{\mathcal{H}}(x) &= -\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\sigma}{\pi}\operatorname{tr}[H^{(0)}V^2(G_\sigma^{(0)})^3](x,x) + O(a)(x) \\ &= -\frac{1}{2}\operatorname{tr}\left[H^{(0)}V^2\int_{-\infty}^{\infty}\frac{d\sigma}{\pi}\frac{1}{(L^{(0)}+\sigma^2)^3}\right](x,x)+O(a)(x) \\ &= \frac{-3}{16}\operatorname{tr}[H^{(0)}V^2(L^{(0)})^{-5/2}](x,x)+O(a)(x). \end{aligned} \tag{3.27}$$

Evaluating the trace over spinor indices we find [with $\nabla_\mu = \frac{1}{2}(\nabla_\mu^+ + \nabla_\mu^-)$]

$$\begin{aligned} q_{\mathcal{H}}(x) = \frac{-3r}{16}\epsilon_{\mu\nu\rho\sigma}\operatorname{tr}\left[\left(-\nabla_\mu^{(0)}(V_\nu V_{\rho\sigma}+V_{\nu\rho}V_\sigma)+\left(\frac{1}{2}\Delta^{(0)}-m_0\right)V_{\mu\nu}V_{\rho\sigma}\right)(L^{(0)})^{-5/2}\right](x,x) \\ +O(a)(x), \end{aligned} \tag{3.28}$$

where V_μ and $V_{\mu\nu}$ are given by (3.20) and (3.21). Calculations give (cf. the Appendix)

$$[\nabla_\mu^\pm,\nabla_\nu^\pm]\psi(x) = (a^2F_{\mu\nu}(x)+O(a^3)(x))\psi(x\pm ae_\mu \pm ae_\nu), \tag{3.29}$$

$$[\nabla_\mu^\pm,\nabla_\nu^\mp]\psi(x) = (a^2F_{\mu\nu}(x)+O(a^3)(x))\psi(x\pm ae_\mu \mp ae_\nu). \tag{3.30}$$

These determine the relevant contributions of V_μ and $V_{\mu\nu}$ in (3.28).

We now exploit the fact that there is a Fourier transformation on $\mathcal{H}^{sc}$ (=the space of scalar lattice fields with $\|\phi\|^2=\Sigma_{x\in a\mathbf{Z}^4}|\phi(x)|^2<\infty$); in particular, δ_x has the Fourier expansion

$$\delta_x = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} e^{-ikx/a} \phi_k , \tag{3.31}$$

where $\phi_k(y) := e^{iky/a}$. For a general operator $\mathcal{O}$ this leads to

$$\begin{aligned} \mathcal{O}_{\mathcal{H}}(x,x) &= \frac{1}{a^4} \left\langle \frac{\delta_x}{a^2}, \mathcal{O} \frac{\delta_x}{a^2} \right\rangle = \frac{1}{a^4} \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} e^{-ikx/a} \frac{1}{a^4} \langle \delta_x , \mathcal{O}\phi_k \rangle \\ &= \frac{1}{a^4} \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} e^{-ikx/a} (\mathcal{O}\phi_k)(x). \end{aligned} \tag{3.32}$$

In the case where

$$\mathcal{O} = \epsilon_{\mu\nu\rho\sigma}(-\nabla_\mu^{(0)}(V_\nu V_{\rho\sigma} + V_{\nu\rho} V_\sigma) + (\tfrac{1}{2}\Delta^{(0)} - m_0) V_{\mu\nu} V_{\rho\sigma})(L^{(0)})^{-5/2}, \tag{3.33}$$

a calculation using (3.20) and (3.21) with (3.29) and (3.30) gives

$$(\mathcal{O}\phi_k)(x) = 32\pi^2 a^4 \lambda(k;r,m_0)(q^A(x) + O(a)(x)) \phi_k(x), \tag{3.34}$$

where

$$\lambda(k;r,m_0) = \frac{\Pi_\nu \cos k_\nu (-m_0 + \Sigma_\mu (1-\cos k_\mu) - \Sigma_\mu (\sin^2 k_\mu / \cos k_\mu))}{[\Sigma_\mu \sin^2 k_\mu + r^2(-m_0 + \Sigma_\mu(1-\cos k_\mu))^2]^{5/2}}. \tag{3.35}$$

It follows from (3.28) and (3.32) that

$$q_{\mathcal{H}}(x) = I(r,m_0) q^A(x) + O(a)(x), \tag{3.36}$$

where

$$I(r,m_0) = \frac{-3r}{8\pi^2} \int_{-\pi}^{\pi} d^4k \, \lambda(k;r,m_0). \tag{3.37}$$

This integral was evaluated earlier in Refs. 21 and 23. It was found to be independent of $r > 0$ and a locally constant function of m_0 with values given by (2.14). This completes the proof of Proposition 2.

Remark: It is straightforward to generalize the results of this paper to SU(N) gauge fields on the $2n$-torus for arbitrary $n \geqslant 2$ and to U(1) gauge fields on the two-torus.

Finally, following the suggestion of a referee, we emphasize that a key point in this work is that it is the topological charge (i.e., the integrated Chern character) rather than the topological density that is shown to have the correct continuum limit. In this respect the treatment differs from all earlier treatments which are essentially limited to small (hence topologically trivial) fields.

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APPENDIX

In this Appendix we recall, for completeness, certain standard facts concerning the lattice transcript of a smooth continuum gauge field on $\mathbf{R}^4$ which lead to the bounds used in this article. The lattice transcript (2.1) can be written as

$$U_\mu(x)=\sum_{n=0}^{\infty} a^n \int_{0\le t_1\le\cdots\le t_n\le 1} dt_n\cdots dt_1\, A_\mu(x,t_n)\cdots A_\mu(x,t_1), \tag{A1}$$

where $A_\mu(x,t)=A_\mu(x+(1-t)ae_\mu)$. When A is bounded, i.e., $\|A_\mu(x)\|\le K$ for all x,μ, we have

$$\|\sum_{n=p}^{\infty} a^n \int_{0\le t_1\le\cdots\le t_n\le 1} dt_n\cdots dt_1\, A_\mu(x,t_n)\cdots A_\mu(x,t_1)\,\|\le\sum_{n=p}^{\infty} a^n \frac{1}{n!}K^n\le a^pK^pe^{aK}\sim O(a^p). \tag{A2}$$

Therefore, to derive the $O(a^p)$ and $O(a^p)(x)$ bounds used in the text it suffices to consider only a finite number of terms in the expansion (A1) (typically just the first few terms). An immediate consequence of (A2) with $p=1$ is the following: If A is bounded, then for any operator $P=P(\nabla_\mu^\pm)$ which is a polynomial in the covariant finite difference operators (2.6) and (2.7) we have

$$\|P-P^{(0)}\|\sim O(a).$$

The bounds $\|H-H^{(0)}\|\sim O(a)$ and $\|L-L^{(0)}\|\sim O(a)$ are particular examples of this. If we furthermore assume that the first order partial derivatives of A are bounded, i.e., $\|\partial_\mu A_\nu(x)\|\le K$ for all x,μ,ν, we have

$$\|[\nabla_\mu^{\pm(0)},U_\nu]\|\sim O(a). \tag{A3}$$

To see this, note that

$$\begin{aligned}[\nabla_\mu^{+(0)},U_\nu]\psi(x)&=(U_\nu(x+ae_\mu)-U_\nu(x))\psi(x+ae_\mu)\\ &=\left(a\int_{0\le t\le 1}dt(A_\nu(x+ae_\mu,t)-A_\nu(x,t))+O(a^2)\right)\psi(x+ae_\mu).\end{aligned} \tag{A4}$$

By the middle-value theorem,

$$A_\nu(x+ae_\mu,t)-A_\nu(x,t)=\partial_\mu A_\nu(x+sae_\mu,t)$$

for some $s\in[0,1]$. Since $\|\partial_\mu A_\nu\|$ is bounded (A3) now follows from (A4). The bound (A3) has the following easy generalization: Let $P=P(\nabla_\mu^\pm)$ be a polynomial of degree k in the $\nabla_\mu^\pm$'s; then, if all the partial derivatives of A of order $\le k$ are bounded, we have

$$\|[P^{(0)},U_\nu]\|\sim O(a). \tag{A5}$$

Moreover, with the same boundedness assumptions on $A_\mu(x)$ and $\partial_\mu A_\nu(x)$, straightforward calculations using the middle-value theorem give

$$1-U(p_{x,\mu,\nu})=a^2F_{\mu\nu}(x)+O(a^3)(x). \tag{A6}$$

Noting that[16]

$$[\nabla_\mu^+,\nabla_\nu^+]\psi(x)=(1-U(p_{x,\mu\nu}))U_\mu(x)U_\nu(x+ae_\mu)\psi(x+ae_\mu+ae_\nu) \tag{A7}$$

and similar formulas for the other commutators, a straightforward refinement of the arguments leading to (A5) and (A6) shows

$$\|[P^{(0)},[\nabla_\mu^\pm,\nabla_\nu^\pm]]\|\sim O(a^3),\quad \|[P^{(0)},[\nabla_\mu^\pm,\nabla_\nu^\mp]]\|\sim O(a^3). \tag{A8}$$

The requirement for this is that A and all its partial derivatives up to order r be bounded, where $r=\min\{k,2\}$. Since V is a linear combination of commutators of the $\nabla_\mu^\pm$'s we have, in particular,

$\|[L^{(0)},V]\| \sim O(a^3)$ when A and its partial derivatives up to order 2 are bounded. Finally, we remark that (3.29) and (3.30) follow from combining (A7) and the corresponding formulas for the other commutators with (A6).

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