Realizing 3-cocycles as obstructions

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The occurrence of a 3-cocycle in quantum mechanics or quantum field theory has been interpreted somewhat paradoxically as a breakdown of the Jacobi identity. The main result of this paper is that the 3-cocycle in chiral QCD arises as an obstruction which prevents the existence of a certain extension of one Lie algebra by another. This obstruction may be avoided by constructing a modified Lie algebra extension consisting of derivations on the algebra generated by the fields. However the 3-cocycle then appears when an attempt is made to implement these derivations by commutation with unbounded operators in the canonical equal-time formalism. Assuming the existence of these unbounded operators is what leads to the violation of the Jacobi identity. © *1995 American Institute of Physics.*

I. INTRODUCTION

In the physics literature the occurrence of 3-cocycles has been interpreted as the breakdown of the Jacobi identity for commutators of the field operators. However commutators of operators defined on a common dense invariant domain in Hilbert space must always satisfy the Jacobi identity. This forces the conclusion that when a 3-cocycle arises it must signal the absence of a representation by such operators of the algebra in question. Some time ago one of us[1] showed how one could understand this conclusion in terms of an "obstruction" to representing a symmetry group of a quantum system on a Hilbert space (this was an extension of the traditional mathematical approach to the existence of group extensions[2]). To apply this in an explicit way to the physical examples we found that further mathematical techniques were needed. A comprehensive investigation was undertaken in Ref. 3 where we adapted some results on cohomology theory of groups and Lie algebras, originally devised to study group actions on C^*-algebras, so that they might be used to explain the properties of certain 3-cocycles in quantum mechanics and quantum field theory. The fact which we wish to report here is that this framework does indeed show that in one of the principal examples found in the physics literature (chiral QCD_{3+1}) the 3-cocycle does arise as an obstruction.

For the earlier literature on anomalies in general including these 3-cocycles see Refs. 3–15 and references therein. The only explicit examples given in Ref. 3 were of a nonphysical nature. While the abstract theory of Ref. 3 is quite technical it turns out to be simple to apply and hence our aim in this account is to show how to describe two physically interesting cases. The first of these is inspired by the Dirac monopole problem and occupies Sec. II. The second is the problem of anomalous commutators in non-Abelian chiral gauge theories. We describe the results of the perturbation theory calculation for equal time commutators in Sec. III and our interpretation of the result in Sec. IV. We find that the 3-cocycle predicted by perturbation theory arises from an algebraic interpretation of the anomalous commutators followed by an application of Ref. 3 to this situation. It should be regarded as further evidence of the failure of the canonical equal time formalism, something which should not surprise the experts.[16] We preface our overall discussion with some mathematics in Section 1 which presents the theory behind these examples in a Lie algebra framework. The reader may omit Section 1 on the first reading, dipping into it for the motivation for the methods used in the examples.

Note that Ref. 3 deals, except for its last section, with groups acting on algebras of bounded operators (it has in mind the Haag–Kastler framework for quantum field theory) whereas here we are more concerned with the Wightman viewpoint (that is algebras of space–time smeared operator-valued distributions). Thus Section 1 is an amplification of the classical theory of Lie algebra extensions following Ref. 3 necessitated by this application to Wightman field theory.

Finally it is worth summarizing our interpretation of the results described here. In the axiomatic approach to quantum field theory it is assumed that the quantum fields are operator-valued distributions which are defined on a common dense invariant subspace of the Hilbert space of states of the theory. Moreover it is normally assumed that the test functions are, say, functions of fast decrease or of compact support on Minkowski space. These fields generate the so-called field algebra which is an algebra of unbounded operators. On the other hand the canonical approach to quantum field theory works with fields defined at sharp time and smeared only in the space variables. Now, the conclusion of the present paper is that the difficulty with the canonical equal-time formalism for chiral QCD suggested by perturbation theory methods has an independent mathematical formulation in terms of a version of extension theory for Lie algebras.

This raises the question of how one is to interpret the sharp time formalism. Calculations involving free field current algebras indicate that sharp time space-smeared electromagnetic currents exist only as derivations on the space–time smeared algebra of currents. In other words one may define sharp time currents by their commutation relations with the space–time smeared currents (which do exist). One cannot construct the sharp time currents as operator-valued distributions on the test functions defined on the space variables only: it is well known that the usual expressions for these currents which are written in the literature must be interpreted as defining quadratic forms (in other words only their matrix elements are well defined). The conclusion to be drawn from the present paper is that for chiral QCD_{3+1} one must also regard the equal-time formalism as being defined only by its algebraic relations with the full space–time smeared field algebra (which of course has not been shown to exist in any mathematically reasonable sense).

II. LIE ALGEBRA 3-COCYCLES

From a mathematical point of view the relationship between obstruction theory and 3-cocycles is well understood. The main task in applying this theory to the examples found in physics is to identify and interpret the Lie algebras from which the putative extensions are to be constructed and hence to give explicitly the 3-cocycles which act as obstructions to the existence of them. This task was carried out at the group level in Ref. 3 where the situation is in some ways more complex. As the physics literature deals mainly with Lie algebras it is worthwhile to present some of the key ideas *ab initio* at the infinitesimal level. The purpose of this section then is to explain the theory necessary for the ensuing discussion. It should be read in conjunction with Section 5 of Ref. 3.

We start from some (possibly infinite dimensional) Lie algebras Δ, Γ, where Δ is an ideal of Γ and $\Omega := \Gamma/\Delta$. We summarize this situation by saying there is a short exact sequence:

$$\Delta \to \Gamma \to \Omega .$$

In a quantum field theory Γ is given as derivations on the observables or algebra of quantum fields of the theory. Thus we assume the existence of a field algebra $\mathscr{A}$ on which Γ acts as derivations. One then seeks operators which implement these derivations and hence we assume that there is algebra $\mathcal{Y}$ consisting of operators preserving some common dense domain D of a Hilbert space $\mathcal{H}$ with Lie bracket given by the commutator. For $\mathcal{A}$ and $\mathcal{Y}$ to be compatible we need the field algebra $\mathcal{A}$ to also act as operators preserving D. Mathematically this is expressed by saying that $\Gamma \subset \mathrm{Der}\,\mathcal{A}$ (the derivations on $\mathcal{A}$) such that $\mathcal{Y}$ implements Γ, i.e., for each $g \in \Gamma$ there is a $y \in \mathcal{Y}$ such that $g(A) = \mathrm{ad}(y)(A) = [y, A]$ on D for all $A \in \mathcal{A}$, and $\mathrm{ad}|_{\mathcal{A}} \mathcal{Y} = \Gamma$. Let $\mathcal{X} := \{y \in \mathcal{Y} | [y, A] = 0$ for all $A \in \mathcal{A}\}$, then this situation may be summarized in the following diagram of exact sequences of Lie algebras:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{W} & \subseteq & \mathcal{X} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{V} & \subset & \mathcal{Y} & & & & \\
 & ad|_{\mathcal{A}} & \downarrow\uparrow\, v & ad|_{\mathcal{A}} & \downarrow\uparrow\, \hat{v} & & & & \\
0 & \longrightarrow & \Delta & \longrightarrow & \Gamma & \underset{\omega}{\rightleftarrows} & \Omega & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $\mathcal{V} := \{u \in \mathcal{Y} | \mathrm{ad}|_{\mathcal{A}} u \in \Delta\}$ and $\mathcal{W} = \mathcal{V} \cap \mathcal{X}$. As we shall see in the examples this diagram summarizes certain desiderata of quantum theory. Our aim will be to show how these desiderata may fail to exist in particular cases. This is analogous in a way to Wigner's theorem which forces us to consider representations up to a factor when *a priori* we might hope for ordinary representations. Before continuing we make one notational remark, in the following $Z(B)$ denotes the center of the algebra B.

For each of the exact sequences (both vertical and horizontal) in the preceding diagram we may choose linear sections v, $\hat{v}$, and ω, as indicated in the diagram, which then define consequent 2-cocycles μ, $\hat{\mu}$, σ via the relations

$$[v_d, v_k] = v_{[d,k]} + \mu(d,k), \quad \mu : \Delta^2 \to \mathcal{W},$$

$$[\hat{v}_g, \hat{v}_h] = \hat{v}_{[g,h]} + \hat{\mu}(g,h), \quad \hat{\mu} : \Gamma^2 \to \mathcal{X},$$

$$[w_g, w_h] = w_{[g,h]} + \sigma(g,h), \quad \sigma : \Omega^2 \to \Delta.$$

These equations simply summarize the fact that a Lie algebra extension is defined either by the short exact sequences above or explicitly via commutation relations using these 2-cocycles. In these circumstances we can prove some elementary facts.

Lemma: The algebra $\mathcal{V}$ is a Lie ideal of $\mathcal{Y}$.

Proof: Let $u \in \mathcal{V}$ and $y \in \mathcal{Y}$. Then we need to show $[u, y] \in \mathcal{V}$:

$$\mathrm{ad}|_{\mathcal{A}}([u, y]) = [\mathrm{ad}|_{\mathcal{A}} u, \mathrm{ad}|_{\mathcal{A}} y] \in \Delta$$

because $\mathrm{ad}|_{\mathcal{A}}$ is a Lie homomorphism, $\mathrm{ad}|_{\mathcal{A}} u \in \Delta$, $\mathrm{ad}|_{\mathcal{A}} y \in \Gamma$, and Δ is an ideal of Γ. Hence $[u, y] \in \mathcal{V}$. □

The key point is that we are given the action of Γ by Lie bracket on the ideal Δ. We implement the subalgebra Δ by operators in $\mathscr{V}$ and then ask: is it possible to define an action of Γ on these operators? The previous lemma shows that we can now define an action $\delta: \Gamma \to \mathrm{Der}\,\mathscr{V}$ by

$$\delta_g(v) := [\hat{v}_g, v] \quad \text{for all } v \in \mathscr{V}, g \in \Gamma.$$

Clearly δ is a linear map on Γ. For it to be an action, we need $\delta_{[g,h]} = [\delta_g, \delta_h]$. Let us check this relation:

$$\begin{aligned}[\delta_g, \delta_h](v) &= [[\hat{v}_g, \hat{v}_h], v] \quad \text{using the Jacobi identity}\\ &= [\hat{v}_{[g,h]}, v] + [\hat{\mu}(g,h), v] = \delta_{[g,h]}(v) + [\hat{\mu}(g,h), v].\end{aligned}$$

Thus δ is an action if and only if $[\hat{\mu}(g,h), v] = 0$, i.e.,

$$\hat{\mu}: \Gamma^2 \to Z(\mathscr{X}, \mathscr{V}) := \{y \in \mathscr{X} | [y, v] = 0 \quad \text{for all } v \in \mathscr{V}\}.$$

Henceforth assume this. Now we are in a position to apply the philosophy of Ref. 6, namely to measure the difference between v and $\hat{v}$ on Δ. Let

$$\varphi(d) := \hat{v}_d - v_d \quad \text{for all } d \in \Delta,$$

so $\varphi: \Delta \to \mathscr{W}$ is linear.

Theorem 2.1: Given v, $\hat{v}$, and δ as above, then $\delta: \Gamma \to \mathrm{Der}\,\mathscr{V}$ is an action such that
(i) $\delta_g(v_d) \in v_{[g,d]} + \mathscr{W}$ for all $g \in \Gamma$, $d \in \Delta$ and
(ii) $\delta_d(v) = [v_d, v]$ for all $d \in \Delta$, $v \in \mathscr{V}$,
if and only if $\varphi(d) \in Z(\mathscr{V}) \cap \mathscr{W}$ for all $d \in \Delta$.

In this case we may define a function λ (cf. Ref. 6) by

$$\lambda(g,d) := \delta_g(v_d) - v_{[g,d]} \in \mathscr{K} = Z(\mathscr{V}) \cap \mathscr{W}$$

and then

$$\lambda(g,d) = \hat{\mu}(g,d) - \delta_g(\varphi(d)) + \varphi([g,d]).$$

Proof: First we see that

$$\begin{aligned}(\mathrm{ad}|_{\mathscr{V}}\delta_g(v_d))(A) = [\delta_g(v_d), A] &= [[\hat{v}_g, v_d], A]\\ &= -[[A, \hat{v}_g], v_d] - [[v_d, A], \hat{v}_g]\\ &= [g(A), v_d] - [d(A), \hat{v}_g]\\ &= g(d(A)) - d(g(A))\\ &= [g,d](A) = [v_{[g,d]}, A].\end{aligned}$$

Thus $\delta_g(v_d) \in v_{[g,d]} + \mathscr{W}$. Next we have

$$\delta_d(v) = [\hat{v}_d, v] = [v_d, v] + [\varphi(d), v] = [v_d, v] \quad \text{for all } v \in \mathscr{V},\ d \in \Delta \quad \text{iff } [\varphi(d), v] = 0,$$

i.e.,

$$\varphi(d) \in Z(\mathscr{V}) \cap \mathscr{W} = \mathscr{K} \quad \text{for all } d \in \Delta.$$

For the last equation,

$$
\begin{aligned}
\lambda(g,d):=\delta_g(v_d)-v_{[g,\,d]}=[\hat{v}_g,\,v_d]-v_{[g,\,d]}
&=[\hat{v}_g,\hat{v}_d-\varphi(d)]-\hat{v}_{[g,\,d]}+\varphi([g,\,d])\\
&=\hat{v}_{[g,\,d]}+\hat{\mu}(g,d)-[\hat{v}_g,\,\varphi(d)]-\hat{v}_{[g,\,d]}+\varphi([g,\,d])\\
&=\hat{\mu}(g,d)-[\hat{v}_g,\,\varphi(d)]+\varphi([g,\,d])\\
&=\hat{\mu}(g,d)-\delta_g(\varphi(d))+\varphi([g,\,d])
\end{aligned}
$$

and clearly this is in $\mathscr{K}$ by prior assumptions. □

Remarks: (1) In Section 5 of Ref. 3 we showed that the map λ is the key ingredient for constructing 3-cocycles in this setting. The precise formula is given in Eq. (2.2) below from which we see that when $\hat{v}_d=v_d$ for all $d\in\Delta$ (i.e., $\varphi=0$) we have $\lambda(g,d)=\hat{\mu}(g,d)$. Hence in this special case if $\hat{\mu}=0$, the 3-cocycle is zero.

(2) On the other hand actions $\delta:\Gamma\to\mathrm{Der}\,\mathscr{V}$ satisfying the two conditions of Theorem 2.1 {but not necessarily of the form $\delta_g(v)=[\hat{v}_g,\,v]$} have also been studied in Ref. 3, and also determine a 3-cocycle $K:\Omega^3\to\mathscr{K}$ as in (2.2) below. The role of λ is explained by Ref. 3, Theorem 5.1, which says that, given an action $\delta:\Gamma\to\mathrm{Der}\,\mathscr{K}$ there will be an extension: $\delta:\Gamma\to\mathrm{Der}\,\mathscr{V}$ if and only if $\mu(d,k)\in\mathscr{K}$ for all $d,k\in\Delta$ and there exists a map $\lambda:\Gamma\times\Delta\to\mathscr{K}$ such that

$$\lambda(0,d)=0=\lambda(g,0), \tag{2.1a}$$

$$\delta_g(\mu(d,k))+\lambda(g,[d,\,k])=\mu([g,\,d],k)+\mu(d,[g,\,k]), \tag{2.1b}$$

$$\lambda([g,\,h],d)=\delta_g(\lambda(h,d))-\delta_h(\lambda(g,d))+\lambda(g,[h,\,d])-\lambda(h,[g,\,d]), \tag{2.1c}$$

$$\lambda(d,k)=\mu(d,k)\quad\text{for } d,k\in\Delta;g,h\in\Gamma. \tag{2.1d}$$

Note that usually $\mathscr{K}$ consists of scalars so that the starting point for this result, an action δ on $\mathscr{K}$, is not a problem. We emphasize that the existence of such a λ guarantees there is an extension of δ to the larger algebra $\mathscr{V}$ and conversely. In the examples we will encounter such actions (and consequent 3-cocycles K) in the case when the second vertical exact sequence in the previous diagram,

$$0\to\mathscr{X}\to\mathscr{Y}\to\Gamma\to 0,$$

is undefined. Now the existence of such a sequence is a desirable feature of any quantum field theory: it simply says that one may implement the symmetry defined by Γ by commutation with operators in $\mathscr{Y}$ on the Hilbert space of states. However we find that in the examples only the first sequence

$$0\to\mathscr{W}\to\mathscr{V}\to\Delta\to 0$$

can be assumed to exist and that the 3-cocycle is an obstruction which prevents us finding the other vertical sequence.

Definition: For each choice of λ satisfying conditions (2.1a)–(2.1d) above there is a 3-cocycle $K:\Omega^3\to\mathscr{K}$ given by

$$
\begin{aligned}
K(g,h,f)&=\lambda(\omega_g,\sigma(h,f))+\lambda(\omega_h,\sigma(f,g))+\lambda(\omega_f,\sigma(g,h))\\
&=\delta_{\omega_g}(v(h,f))+\delta_{\omega_h}(v(f,g))+\delta_{\omega_f}(v(g,h))-v([g,\,h],f)-v([f,\,g],h)-v([h,\,f],g)\\
&=(\partial v)(g,h,f),
\end{aligned}\tag{2.2}
$$

where $v(f,g):=v(\sigma(f,g))$. [We refer the reader to Ref. 3 for the verification that (2.2) satisfies the 3-cocycle identity.]

The main point of this section is to explain how this $[K]$ is to be interpreted as an obstruction. Hence let K be trivial, i.e., there is a linear antisymmetric 2-cochain $\rho:\Omega^2\rightarrow\mathscr{K}$ such that

$$K(g,h,f)=\delta_{\omega_g}(\rho(h,f))+\delta_{\omega_h}(\rho(f,g))+\delta_{\omega_f}(\rho(g,h))-\rho([g,\ h],f)-\rho([f,\ g],h)-\rho([h,\ f],g)$$
$$=(\partial\rho)(g,h,f).$$

Substituting this into $K=\partial v$ we obtain $\partial W=0$ where $W(g,h):=v(g,h)-\rho(g,h)$.

Theorem 2.2: Given the Lie algebras, sections, and action δ as above, with consequent 3-cocycle K, assume that K is trivial. Then

(i) $\mathscr{E}=\Omega\oplus\mathscr{V}$ with bracket

$$[g_1\oplus u_1,\ g_2\oplus u_2]:=[g_1,\ g_2]\oplus([u_1,\ u_2]+\delta_{\omega_{g_1}}(u_2)-\delta_{\omega_{g_2}}(u_1)+W(g_1,g_2))$$

defines a Lie algebra (W as above).

(ii) The map $\nu:\mathscr{E}\rightarrow\mathrm{Der}\,\mathscr{A}$ given by

$$\nu(g\oplus u)(A):=[u,\ A]+\omega_g(A),\quad A\in\mathscr{A},g\in\Omega,u\in\mathscr{V}$$

is an action.

Proof: (i) Clearly $\mathscr{E}$ is a linear space and the bracket is linear in each entry and antisymmetric. Check the Jacobi identity:

$$J=\mathrm{Cycl.}[g_1\oplus u_1,\ [g_2\oplus u_2,g_3\oplus u_3]]\quad\text{(sum of cyclic permutations over 1 2 3)}$$
$$=\mathrm{Cycl.}[g_1\oplus u_1,\ [g_2,\ g_3]\oplus([u_2,\ u_3]+\delta_{\omega_{g_2}}(u_3)-\delta_{\omega_{g_3}}(u_2)+W(g_2,g_3))]$$
$$=\mathrm{Cycl.}\{[g_1,[g_2,g_3]]\oplus([u_1,[u_2,u_3]]+[u_1,\delta_{\omega_{g_2}}(u_3)-\delta_{\omega_{g_3}}(u_2)+W(g_2,g_3)]$$
$$+\delta_{\omega_{g_1}}([u_2,u_3]+\delta_{\omega_{g_2}}(u_3)-\delta_{\omega_{g_3}}(u_2)+W(g_2,g_3))-\delta_{\omega_{[g_2,g_3]}}(u_1)+W(g_1,[g_2,\ g_3]))\}.$$

Now using

$$\mathrm{Cycl.}[g_1,\ [g_2,\ g_3]]=0=\mathrm{Cycl.}[u_1,\ [u_2,\ u_3]]$$

and

$$\mathrm{Cycl.}(\delta_{\omega_{g_1}}(W(g_2,g_3))+W(g_1,[g_2,\ g_3]))=(\partial W)(g_1,g_2,g_3)=0$$

and notation $0\oplus u=u$, we see

$$J=\mathrm{Cycl.}\{[u_1,\ \delta_{\omega_{g_2}}(u_3)-\delta_{\omega_{g_3}}(u_2)+W(g_2,g_3)]+(\delta_{\omega_{g_1}}\delta_{\omega_{g_2}}(u_3)$$
$$-\delta_{\omega_{g_1}}\delta_{\omega_{g_3}}(u_2)-\delta_{\omega_{[g_2,\ g_3]}}(u_1))+\delta_{\omega_{g_1}}([u_2,\ u_3])\}.$$

Now

$$\text{Cycl.}(\delta_{\omega_{g_1}}\delta_{\omega_{g_2}}(u_3)-\delta_{\omega_{g_1}}\delta_{\omega_{g_3}}(u_2)-\delta_{\omega_{[g_2,g_3]}}(u_1))$$

$$=([\delta_{\omega_{g_1}},\delta_{\omega_{g_2}}]-\delta_{\omega_{[g_1,g_2]}})(u_3)+([\delta_{\omega_{g_3}},\delta_{\omega_{g_1}}]-\delta_{\omega_{[g_3,g_1]}})(u_2)+([\delta_{\omega_{g_2}},\delta_{\omega_{g_3}}]-\delta_{\omega_{[g_2,g_3]}})(u_1)$$

$$=\delta_{[\omega_{g_1},\omega_{g_2}]-\omega_{[g_1,g_2]}}(u_3)+\cdots$$

$$=\delta_{\sigma(g_1,g_2)}(u_3)+\delta_{\sigma(g_3,g_1)}(u_2)+\delta_{\sigma(g_2,g_3)}(u_1)$$

$$=[v(g_1,g_2),\ u_3]+[v(g_3,g_1),\ u_2]+[v(g_2,g_3),\ u_1].$$

Observe $[u_1,\ W(g_2,g_3)]=-[v(g_2,g_3),\ u_1]$, so substituting into J and cancelling

$$J=\text{Cycl.}\{[u_1,\ \delta_{\omega_{g_2}}(u_3)-\delta_{\omega_{g_3}}(u_2)]+\delta_{\omega_{g_1}}([u_2,\ u_3])\}$$

$$=[u_1,\ \delta_{\omega_{g_2}}(u_3)-\delta_{\omega_{g_3}}(u_2)]+[u_3,\ \delta_{\omega_{g_1}}(u_2)-\delta_{\omega_{g_2}}(u_1)]+[u_2,\ \delta_{\omega_{g_3}}(u_1)-\delta_{\omega_{g_1}}(u_3)]$$

$$+\delta_{\omega_{g_1}}([u_2,\ u_3])+\delta_{\omega_{g_2}}([u_3,\ u_1])+\delta_{\omega_{g_3}}([u_1,\ u_2])=0$$

using $\delta_{\omega_{g_1}}([u_2,\ u_3])\ =\ [\delta_{\omega_{g_1}}(u_2),\ u_3]\ +\ [u_2,\ \delta_{\omega_{g_1}}(u_3)]$.

(ii) Clearly $\nu:\mathscr{E}\to\text{Der}\,\mathscr{A}$ is linear, so we need to show that

$$\nu([g_1\oplus u_1,\ g_2\oplus u_2])=[\nu(g_1\oplus u_1),\ \nu(g_2\oplus u_2)].$$

$$\text{rhs}=[\nu(g_1\oplus u_1),\ \nu(g_2\oplus u_2)](A)$$

$$=\nu(g_1\oplus u_1)\nu(g_2\oplus u_2)(A)-\nu(g_2\oplus u_2)\nu(g_1\oplus u_1)(A)$$

$$=[u_1,\ \nu(g_2\oplus u_2)(A)]+\omega_{g_1}(\nu(g_2\oplus u_2)(A))-[u_2,\ \nu(g_1\oplus u_1)(A)]-\omega_{g_2}(\nu(g_1\oplus u_2)(A))$$

$$=[u_1,\ [u_2,\ A]]+[u_1,\ \omega_{g_2}(A)]+\omega_{g_1}([u_2,\ A])+\omega_{g_1}(\omega_{g_2}(A))-[u_2,\ [u_1,\ A]]$$

$$-[u_2,\ \omega_{g_1}(A)]-\omega_{g_2}([u_1,\ A])-\omega_{g_2}(\omega_{g_1}(A))$$

$$=[[u_1,\ u_2],\ A]+[\delta_{\omega_{g_1}}(u_2),\ A]-[\delta_{\omega_{g_2}}(u_1),\ A]+[\omega_{g_1},\omega_{g_2}](A).$$

In this step we used the Jacobi identity and

$$\omega_g([u,A])=[\delta_{\omega_g}(u),\ A]+[u,\ \omega_g(A)]$$

which we prove as follows: recall that any $u\in\mathscr{V}$ can be written $u=v_d+\psi_d$ for some $d\in\Delta$, where $\psi_d\in\mathscr{W}$. Thus

$$\omega_g([u,\ A])=\omega_g([v_d,\ A])=(\omega_g\cdot d)(A)$$

$$=[\omega_g,\ d](A)+(d\cdot\omega_g)(A)$$

$$=[v_{[\omega_g,\ d]},\ A]+[v_d,\ \omega_g(A)]$$

$$=[\delta_{\omega_g}(v_d),\ A]+[u,\ \omega_g(A)]$$

$$=[\delta_{\omega_g}(u),\ A]+[u,\ \omega_g(A)].$$

So, returning to the main calculation:

$$\text{rhs}=[[u_1,\ u_2]+\delta_{\omega_{g_1}}(u_2)-\delta_{\omega_{g_2}}(u_1),\ A]+\sigma(g_1,g_2)(A)+\omega_{[g_1,g_2]}(A).$$

So since $\sigma(g_1,g_2)(A)=[v(g_1,g_2), A]=[W(g_1,g_2), A]$,

$$\text{rhs}=[[u_1,\ u_2]+\delta_{\omega_{g_1}}(u_2)-\delta_{\omega_{g_2}}(u_1)+W(g_1,g_2),\ A]+\omega_{[g_1,\ g_2]}(A)$$

$$=\nu([g_1\oplus u_1,g_2\oplus u_2])(A)=\text{lhs}.$$

□

Thus we interpret a nontrivial 3-cocycle K as an obstruction to the construction of this extension $\mathscr{E}$ and its action on $\mathscr{A}$ as above. (Observe that when K is nontrivial, we cannot define W hence the bracket of $\mathscr{E}$ is undefined.)

III. PARTICLE IN A MAGNETIC FIELD

One of the first examples of the appearance of a 3-cocycle in a physical context arose from the Dirac monopole. From a mathematical viewpoint this example is somewhat artificial: a precise analysis of a quantum mechanical particle in the field of a point monopole can be devised which avoids any discussion of 3-cocycles.[17,18]

However one special case of the monopole situation (a magnetic field $\mathbf{B}$ on $\mathbb{R}^3$ with $\nabla.\mathbf{B}\neq 0$) does fit into our framework. We include this example to explain the main ideas on 3-cocycles in a simple context.

So the situation we are considering is that of a particle moving in $\mathbb{R}^3$ in a magnetic field with nonvanishing divergence. The quantization of such a system leads immediately to difficulties. For example one would like to write for the velocity generators (setting Planck's constant equal to one),

$$v_i=\frac{1}{m}\left(\frac{1}{i}\frac{\partial}{\partial x_i}-eA_i\right), \tag{3.1}$$

where $\nabla\times\mathbf{A}=\mathbf{B}$ even though a smooth $\mathbf{A}$ does not exist unless the magnetic field is divergence free. Proceeding formally from (3.1) implies that one should expect anomalous commutators. Let $\{\alpha_j,\beta_j|j=1,2,3\}$ be real and define

$$\sigma(\Sigma_j\beta_jv_j,\Sigma_k\gamma_kv_k)=\xi e^{jkl}\beta_j\gamma_kB_l, \tag{3.2}$$

where B_l, $l=1,2,3$, are the components of $\mathbf{B}$ and $\xi=e/m^2$. The right-hand side of (3.2) is what one gets by formal calculation of the commutators $(-i)[\Sigma_j\beta_jv_j,\ \Sigma_k\gamma_kv_k]$ from (3.1). Now in order for the Jacobi identity for the commutators of the velocity generators to hold it is necessary for σ to satisfy a 2-cocycle identity [this is shown for example in Ref. 14 where a formal calculation from (3.1) yields the right-hand side of (3.8) below when one checks the Jacobi identity]. But with $\nabla.\mathbf{B}\neq 0$, σ is not a 2-cocycle and so (3.2) cannot be used to define a Lie bracket. However, this line of argument is suspect as we cannot assume that (3.1) with singular $\mathbf{A}$ defines an essentially self-adjoint operator on an invariant dense domain of the Hilbert space of states. Indeed one usually expects that one has to choose some self-adjoint extension by imposing boundary conditions (in Ref. 8 these additional conditions are found to lead to the Dirac quantization condition). Nevertheless the framework of Ref. 3 and the previous section may be used to give a consistent interpretation of (3.2).

To use this framework we construct some Lie algebras. Introduce

(i) the Abelian Lie algebra Δ of smooth real-valued functions of the generators of space translations modulo constants: that is, let Δ be all smooth functions from $\mathbb{R}^3$ to $\mathbb{R}$ where we identify d_1 and d_2 if d_1-d_2 is constant and we indicate the equivalence class of d by $\tilde{d}$;

(ii) the Abelian Lie algebra of velocity generators: $\Omega=\mathbb{R}^3$ with basis $\{v_1,v_2,v_3\}$.

Next one regards σ as a function from $\Omega\times\Omega$ to Δ [that is to take the right-hand side of (3.2) as a smooth function on $\mathbb{R}^3$ modulo constants]. To avoid confusion we denote this function by $\tilde{\sigma}$. Second we assume $\mathbf{B}$ satisfies

$$\nabla\cdot\mathbf{B}=\text{constant}=:c\neq 0.$$

There is an obvious way to construct a solution to this equation by requiring $\mathbf{B}$ to depend linearly on the $\mathbb{R}^3$ variables. All other solutions differ from this one by a solution of the homogeneous equation $\nabla\cdot\mathbf{B}=0$. Such fields do not give rise to a 3-cocycle and so we shall not consider them. Thus we make the global restriction that we consider only $\mathbf{B}$ which are linear functions. Henceforth fix such a $\mathbf{B}$ to make σ well defined.

Next we assume that Ω acts on $f\in\Delta$ in the usual way by

$$v_i(f)=\frac{\partial f}{\partial x_i}. \tag{3.3}$$

We now observe that $\tilde{\sigma}:\Omega\times\Omega\rightarrow\Delta$ satisfies the 2-cocycle identity. To see this rewrite the Ω action on Δ as

$$\delta_\beta(d)=\beta(d)=\sum_j \beta_j v_j(d)\quad\text{for } \beta=\sum_j \beta_j v_j\in\Omega,\quad d\in\Delta.$$

So the 2-cocycle relation is (where Cycl.$\{\cdot\}$ denotes sum over cyclic permutations of 1,2,3)

$$(\partial\tilde{\sigma})(\beta_1,\beta_2,\beta_3)=\text{Cycl.}\{\delta_{\beta_1}(\tilde{\sigma}(\beta_2,\beta_3))+\tilde{\sigma}(\beta_1,[\beta_2,\ \beta_3])\}=\text{Cycl.}\{\delta_{\beta_1}(\tilde{\sigma}(\beta_2,\beta_3))\}$$

and with notation $\beta_i=\Sigma_j\beta_j^{(i)}v_j$:

$$(\partial\tilde{\sigma})(\beta_1,\beta_2,\beta_3)=\text{Cycl.}\left\{\sum_j \beta_j^{(1)}v_j(\xi\epsilon^{kln}\beta_k^{(2)}\beta_l^{(3)}B_n)\right\}=\xi\ \text{Cycl.}\sum_{j,k,l,n}\beta_j^{(1)}\epsilon^{kln}\beta_k^{(2)}\beta_l^{(3)}v_j(B_n). \tag{3.4}$$

We want this to be proportional to $\nabla\cdot\mathbf{B}$ (which is zero under ~).

Now coefficients of the mixed terms are zero; for example, we check the coefficient of $v_1(B_2)$:

$$\beta_1^{(1)}(\beta_3^{(2)}\beta_1^{(3)}-\beta_1^{(2)}\beta_3^{(3)})+\beta_1^{(2)}(\beta_3^{(3)}\beta_1^{(1)}-\beta_1^{(3)}\beta_3^{(1)})+\beta_1^{(3)}(\beta_3^{(1)}\beta_1^{(2)}-\beta_1^{(1)}\beta_3^{(2)})=0.$$

Others are zero by symmetry. So

$$(\partial\tilde{\sigma})(\beta_1,\beta_2,\beta_3)=\xi\ \text{Cycl.}\sum_{k,l,n}\epsilon^{kln}\beta_k^{(2)}\beta_l^{(3)}\beta_n^{(1)}v_n(B_n).$$

By writing out each coefficient of $v_i(B_i)$, we find they are equal, and the common value is $\xi\epsilon^{kln}\beta_k^{(1)}\beta_l^{(2)}\beta_n^{(3)}=\xi\boldsymbol{\beta}_1\wedge\boldsymbol{\beta}_2\wedge\boldsymbol{\beta}_3$. So

$$(\partial\tilde{\sigma})(\beta_1,\beta_2,\beta_3)=\xi(\boldsymbol{\beta}_1\wedge\boldsymbol{\beta}_2\wedge\boldsymbol{\beta}_3)\cdot\sum_i v_i(B_i)=0 \tag{3.5}$$

when $\nabla\cdot\mathbf{B}=0$ and if $\nabla\cdot\mathbf{B}=c$ as assumed here, then under ~ (i.e., identifying the constants with zero) the right-hand side of (3.5) is zero in Δ. Having verified that $\tilde{\sigma}$ is a 2-cocycle on Ω with values in Δ one can form the corresponding extension of Lie algebras:

$$\Gamma=\left\{(\beta,\tilde{\delta})\middle|\beta\in\Omega,\beta=\sum_i \beta_i v_i,\quad \tilde{\delta}\in\Delta\right\}$$

with commutators

$$[(\beta,\tilde{\delta}),(\beta',\tilde{\delta}')]=(0,\beta(\tilde{\delta}')-\beta'(\tilde{\delta})+\tilde{\sigma}(\beta,\beta')),\tag{3.6}$$

where from (3.3) we have

$$\beta(\tilde{\delta}')=\sum_i \beta_i \frac{\tilde{\partial}\delta'}{\partial x_i},$$

and we have linearity:

$$(\beta,\tilde{\delta})+t(\beta',\tilde{\delta}')=(\beta+t\beta',\tilde{\delta}+t\tilde{\delta}').$$

The key observation is to note that, when we take a representation of the algebra of smooth functions on $\mathbb{R}^3$ by operators on a Hilbert space and then let them act on operators on this space by taking commutators, the scalars act trivially. Thus one really has an action of Δ. Thus it is possible to have an action of the Lie algebra Γ as derivations on the algebra $\mathscr{A}$ of observables (say by commutators). In particular we can define the action of $\tilde{\sigma}$ as a derivation by using the commutator with the function σ where the latter is regarded as an operator on the Hilbert space of states. This means that from the viewpoint of algebraic quantum theory we may use (3.2) to define the commutator between derivations. A problem arises only if one now wants to represent the velocities v_i's by operators on a Hilbert space rather than by derivations on an algebra.

To prepare this situation for an application of Sec. I, we restrict our attention to the subalgebra $\hat{\Gamma}\subset\Gamma$ generated by $\Omega\times\Delta_0$ where

$$\Delta_0:=\text{lin span}\{d_j|j=1,2,3\}\subset\Delta$$

and $d_j:\mathbb{R}^3\to\mathbb{R}$ are the functions $d_j(x)=x_j$, $j=1,2,3$, i.e., the position observables. So as a linear space $\hat{\Gamma}=\Omega\times\Delta_0$ and the bracket (3.6) restricts on $\hat{\Gamma}$ to

$$[(\beta,\tilde{\delta}),\ (\beta',\tilde{\delta}')]=(0,\tilde{\sigma}(\beta,\beta'))\tag{3.6$'$}$$

using the fact that $\beta(\tilde{\delta})=0$ since $\tilde{\delta}$ is linear and constants are factored out of Δ_0. Note that $\tilde{\sigma}$ takes its values in Δ_0 by assumption of linear B. This provides us with the short exact sequence

$$0\to\Delta_0\to\hat{\Gamma}\to\Omega\to 0,$$

which will play the role of the horizontal exact sequence in Sec. I. We also have a section $\omega:\Omega\to\hat{\Gamma}$ given by the canonical identification of Ω in $\hat{\Gamma}=\Omega\times\Delta_0$, with associated cocycle $\tilde{\sigma}$ given by (3.6′).

Next we need to construct the vertical exact sequence:

$$0\to\mathscr{W}\to\mathscr{V}\to\Delta_0\to 0.$$

Assume there is a representation v of Δ_0 as self-adjoint operators on a common dense invariant domain D, and let $\mathscr{V}$ be the set of operators: lin span $\{\mathbb{R}1,v(\tilde{\delta})|\tilde{\delta}\in\Delta_0\}$, so the commutators in $\mathscr{V}$

are all identically zero. Assume further that the field algebra $\mathscr{A}$ is irreducibly represented on D, so that $\mathrm{Ker}(\mathrm{ad}|_{\mathscr{A}})=\mathbb{C}\mathbf{1}$, hence $\mathrm{ad}|_{\mathscr{A}}(\mathscr{V})\cong\Delta_0$, and this identifies Δ_0 as derivations on $\mathscr{A}$. So with the map from $\mathscr{V}$ to Δ_0 in the exact sequence given by $\mathrm{ad}|_{\mathscr{A}}$, we find $\mathscr{W}=\mathbb{R}\mathbf{1}$. In the terminology of Sec. I the original representation v defines a section $v:\Delta_0\rightarrow\mathscr{V}$ with 2-cocycle $\mu=0$.

The data we need to apply Sec. I is completed by a map $\lambda:\hat{\Gamma}\times\Delta_0\rightarrow\mathbb{R}$ satisfying (2.1) and defined by

$$\lambda((\beta,\tilde{\gamma}),\tilde{\gamma}')=\frac{1}{m}\sum_k \beta_k\gamma_k' \tag{3.7}$$

(this choice of λ will be physically motivated below).

Conditions (2.1a), (2.2b) and (2.2d) are obviously satisfied—all expressions are zero. We check (2.1c) [omitting expressions $\delta_{(\beta,\gamma)}(\mu(\cdot))=0$]:

$$\lambda([(\beta,\tilde{\gamma}),(\beta',\tilde{\gamma}')],\tilde{\gamma}'')=\lambda((\beta,\tilde{\gamma}),[(\beta',\tilde{\gamma}'),(0,\tilde{\gamma}'')])+\lambda((\beta',\tilde{\gamma}'),[(\beta,\tilde{\gamma}),(0,\tilde{\gamma}'')]),$$

$$\mathrm{lhs}=\lambda((0,\tilde{\sigma}(\beta,\beta')),\tilde{\gamma}'')=0 \quad \text{by } (3.7),$$

$$\mathrm{rhs}=\lambda((\beta,\tilde{\gamma}),\tilde{\sigma}(\beta',0))+\lambda((\beta',\tilde{\gamma}'),\tilde{\sigma}(\beta,0))=0.$$

Hence λ defines a 3-cocycle K as in (2.2). Now using the identifications above:

$$K(\beta,\theta,\chi)=\lambda((\beta,0),\tilde{\sigma}(\theta,\chi))+\lambda((\theta,0),\tilde{\sigma}(\chi,\beta))+\lambda((\chi,0),\tilde{\sigma}(\beta,\theta))$$
$$=\frac{1}{m}\sum_k \{\beta_k(\tilde{\sigma}(\theta,\chi))_k+\theta_k(\tilde{\sigma}(\chi,\beta))_k+\chi_k(\tilde{\sigma}(\beta,\theta))_k\},$$

where $\tilde{\sigma}(\theta,\chi)=\Sigma_k(\tilde{\sigma}(\theta,\chi))_k d_k$, so $(\tilde{\sigma}(\theta,\chi))_k=(\partial/\partial x_k)\tilde{\sigma}(\theta,\chi)$ and using (3.2),

$$(\tilde{\sigma}(\theta,\chi))_k=\xi\epsilon^{jnl}\theta_j\chi_n\frac{\partial B_l}{\partial x_k},$$

hence

$$K(\beta,\theta,\chi)=\frac{\xi}{m}\sum_k \epsilon^{jnl}\{\theta_j\chi_n\beta_k+\chi_j\beta_n\theta_k+\beta_j\theta_n\chi_k\}\frac{\partial B_l}{\partial x_k}.$$

This is exactly of the same form as (3.4), hence

$$K(\beta,\theta,\chi)=\frac{\xi}{m}(\boldsymbol{\beta}\wedge\boldsymbol{\theta}\wedge\boldsymbol{\chi})(\nabla\cdot\mathbf{B}). \tag{3.8}$$

On setting $\beta=v_1$, $\theta=v_2$, $\chi=v_3$, this agrees with the 3-cocycle in Ref. 14.

This obstruction cannot be avoided once we fix (3.7) because the cohomology of Ω (an Abelian Lie algebra) with values in $\mathbb{R}$ is given by totally skew multilinear functionals on Ω, so it follows that (3.8) determines a nontrivial cohomology class.

We can read off from Theorem 2.2 that K is indeed an obstruction in the usual sense.

Finally, to justify the choice (3.7) for λ, recall that λ corresponds to an action $\delta:\Gamma\rightarrow\mathrm{Der}\,\mathscr{V}$ such that $\lambda(g,d)=\delta_g(v_d)-v_{[g,d]}$, $g\in\Gamma$, $d\in\Delta$. In this case $[(\beta,\tilde{\delta}),(0,\tilde{\delta}')]=(0,\tilde{\sigma}(\beta,0))=0$, so

$$\lambda((\beta,\tilde{\gamma}),\tilde{\gamma}')=\delta_{(\beta,\tilde{\gamma})}(v(\tilde{\gamma}'))=\delta_{(\beta,0)}(v(\tilde{\gamma}'))+\delta_{(0,\tilde{\gamma})}(v(\tilde{\gamma}')).$$

Now on identifying $d_j \in \Delta_0$ with the position variables and $v_j \in \Omega$ with generators of velocities, we expect to obtain the canonical commutation relations in any representation. This forces a choice of action $\delta:\Gamma \to \mathrm{Der}\,\mathscr{V}$ as follows. First, since a representation $v:\Delta_0 \to \mathrm{Op}(D)$ is already given, we will have

$$\delta_{(0,\tilde{\gamma})}(v(\tilde{\gamma}'))=(-i)[v(\tilde{\gamma}),\ v(\tilde{\gamma}')]=0.$$

[The $(-i)$ is due to the fact that we require the $v(\tilde{\gamma})$s to be self-adjoint.] Second, we assume we have self-adjoint operators $\{u(v_j)|j=1,2,3\}$ on $\mathscr{H}$ preserving D such that

$$[u(v_j),\ v(d_k)]=i\delta_{jk}/m, \tag{3.9}$$

the canonical commutation relations. Then for

$$\beta=\sum_j \beta_j v_j \in \Omega, \quad \tilde{\gamma}'=\sum_k \gamma_k' d_k \in \Delta_0$$

we find

$$\begin{aligned}\lambda((\beta,\tilde{\gamma}),\tilde{\gamma}')&=\delta_{(\beta,0)}(v(\tilde{\gamma}'))\\ &=\sum_{j,k} \beta_j\gamma_k'\delta_{(v_j,0)}(v(d_k)):\\ &=\sum_{j,k} \beta_j\gamma_k'(-i)[u(v_j),v(d_k)]=\frac{1}{m}\sum_k \beta_k\gamma_k'\end{aligned}$$

using (3.9), and this is exactly (3.7). This justification is necessarily loose because of the following lemma, in which we make precise the earlier comment that there is no (twisted) representation of Γ in which δ is implemented.

Lemma 3.1: Given the structures above, there is no second vertical exact sequence as in Sec. I.

$$0\to\mathbb{R}\to\mathscr{Y}\xrightarrow{\mathrm{ad}|_{\mathscr{V}}}\hat{\Gamma}\to 0$$

with section $\hat{v}:\hat{\Gamma}\to\mathscr{Y}$ coinciding with v on Δ_0 such that

$$\lambda((\beta,\tilde{\gamma}),\tilde{\gamma}')=(-i)[\hat{v}((\beta,\tilde{\gamma})),\ v(\tilde{\gamma}')].$$

Proof: Assume the contrary. Then by Remark (1) below Theorem 2.1 we have

$$\lambda((\beta,\tilde{\gamma}),\tilde{\gamma}')=\hat{\mu}((\beta,\tilde{\gamma}),(0,\tilde{\gamma}')), \tag{*}$$

where $\hat{\mu}$ is the 2-cocycle $\hat{\mu}:\Gamma^2\to\mathbb{R}$ associated with $\hat{v}$. Check the cocycle relation for $\hat{\mu}$:

$$\begin{aligned}(\partial\hat{\mu})((\beta,\tilde{\gamma}),(\beta',\tilde{\gamma}'),(\beta'',\tilde{\gamma}''))&=\text{Cycl.}\{\delta_{(\beta,\tilde{\gamma})}(\hat{\mu}((\beta',\tilde{\gamma}'),(\beta'',\tilde{\gamma}''))\\&\quad+\hat{\mu}((\beta,\tilde{\gamma}),[(\beta',\tilde{\gamma}'),(\beta'',\tilde{\gamma}'')]))\}\\&=\text{Cycl.}\{\hat{\mu}((\beta,\tilde{\gamma}),[(\beta',\tilde{\gamma}'),\,(\beta'',\tilde{\gamma}'')])\}\\&=\text{Cycl.}\{\hat{\mu}((\beta,\tilde{\gamma}),(0,\tilde{\sigma}(\beta',\beta'')))\}\\&=\text{Cycl.}\{\lambda((\beta,\tilde{\gamma}),\tilde{\sigma}(\beta',\beta''))\}\ \text{ by }\ (*)\\&=\text{Cycl.}\left\{\frac{1}{m}\sum_k\beta_k(\tilde{\sigma}(\beta',\beta''))_k\right\}\ \text{ by }\ (3.7)=K(\beta,\beta',\beta'')\end{aligned}$$

and from (3.8) we know this cannot be zero. Thus $\hat{\mu}$ is not a 2-cocycle, and $\hat{v}$ cannot be a section as assumed. □

The proof of this lemma is essentially the standard calculation in Ref. 14, and shows in what sense K is an obstruction to implementing the Lie algebra of derivations by operators.

Remark: In the analysis above it is essential that the algebras we are considering consist of unbounded operators. Proposition 2.6.4 of Ref. 19 shows that there can be no nontrivial 3-cocycle for Lie algebras of bounded derivations on algebras of bounded operators.

IV. SUMMARY OF JO'S CALCULATION

Probably the most interesting example of a 3-cocycle arises in the calculation of the Schwinger term for currents in QCD. We begin by reminding the reader of the standard calculation following Jo[20] (which, for our purposes, is the most useful treatment).

Jo considers chiral fermions in (3+1) dimensions coupled to a Yang–Mills gauge field. He defines an equal-time algebra starting with

$$A=\sum_{i=1}^{3}\sum_{a}A_i^a(x)T^a\,dx^i,$$

the Yang–Mills field "operator" at fixed time $t=0$. Here T^a are the generators of the Lie algebra, **g**, of the gauge group. Jo finds that the CCR become anomalous. Defining the equal-time commutators of the operators by the BJL method (see Ref. 14 for a discussion), Jo finds the following:

$$[A_i^a(x),\,A_j^b(y)]=0,$$

$$[E_i^a(x),\,A_j^b(y)]=-i\,\delta_{ij}\delta^{ab}\delta^3(x-y),$$

$$[E_i^a(x),\,E_j^b(y)]=i\alpha\epsilon^{ijk}\,\mathrm{tr}[(T^aT^b+T^bT^a)T^c]A_k^c(x)\delta^3(x-y)$$

(here ϵ^{ijk} is the antisymmetric tensor as usual, $\mathrm{tr}(\cdot)$ is the trace in **g**, α is a constant, $A_k\equiv\Sigma_aA_k^aT^a$ and **g** is n-dimensional and the E's are the fields which in the classical Lagrangian method are conjugate to the A's).

For our purposes it is convenient to rewrite this as follows.

Let $\mathscr{S}$ denote the smooth functions of fast decrease on $\mathbb{R}^3$ with values in $\mathbb{R}^3\times\mathbb{R}^n$, and denote the components of $f\in\mathscr{S}$ by $f_i^a(x)$. Then

$$A(f):=\sum_{i=1}^{3}\sum_{a=1}^{n}\int_{\mathbb{R}^3}A_i^a(x)f_i^a(x)d^3x \tag{4.1}$$

and similarly we smear $E(f)$ over $\mathscr{S}$. The commutation relations are

$$[A(f),\, A(g)]=0, \tag{4.2a}$$

$$[E(f),\, A(g)]=-i\sum_{j=1}^{3}\sum_{a=1}^{n}\int f_j^a(x)g_j^a(x)d^3(x)=:-i(f,g), \tag{4.2b}$$

$$[E(f),\, E(g)]=i\alpha A(f\circledast g), \tag{4.2c}$$

where

$$(f\circledast g)_k^c(x):=\sum_{a,b}^{n}\epsilon^{ijk}f_i^a(x)g_j^b(x)\mathrm{tr}[(T^aT^b+T^bT^a)T^c]. \tag{4.3}$$

The right-hand side of (4.2c) is not a 2-cocycle, only a 2-cochain with values in the Lie algebra generated by the $A(f)$. Comparing the equations (4.2) with (3.9) and (3.2) we note the similarity with the situation in Sec. III. One can now try to verify the Jacobi identity. Let

$$J(E(f),E(g),E(h))=\mathrm{Cycl.}[E(f),\,[E(g),\,E(h)]].$$

Then

$$\begin{aligned} J(E(f),E(g),E(h)) &= \mathrm{Cycl.}[E(f),i\alpha A(g\circledast h)] \\ &= \alpha\ \mathrm{Cycl.}(f,g\circledast h) \\ &= \alpha\ \mathrm{Cycl.}\sum_{a,b,c}\int e^{ijk}g_i^a(x)h_j^b(x)f_k^c(x)\mathrm{tr}[(T^aT^b+T^bT^a)T^c]d^3x \\ &= 3\alpha\sum_{a,b,c}\int \epsilon^{ijk}g_i^a(x)h_j^b(x)f_k^c(x)\mathrm{tr}[(T^aT^b+T^bT^a)T^c]d^3x, \end{aligned} \tag{4.4}$$

where Cycl. denotes summation over cyclic permutations of f,g,h.

At this point the view expounded in the literature is to regard the Jacobi identity as failing. So one has to conclude that the $E(f)$s are not operators on a common dense invariant domain. In fact, of course, it is not clear where the contradiction lies since one is effectively assuming that the $E(f)$s are such operators in order to define the anomalous commutators (4.2) and so perhaps the BJL formalism used to calculate them is the cause of the problem. We observe in Sec. V that the anomalous commutators (4.2) can be interpreted as specifying the commutation relations between derivations. This is consistent with the observation that a quadratic form may well define a derivation on the algebra of space–time smeared fields without necessarily defining an operator. Thus (4.2) has a meaning in field theory independent of perturbation theory as does the 3-cocycle (4.4).

V. THE CGRS FRAMEWORK

To use our framework we need to identify an exact sequence of Lie algebras

$$0\to\Delta\to\Gamma\to\Omega\to 0 \tag{5.1}$$

with Δ an Abelian Lie algebra and Γ an Abelian extension of Δ by Ω.

To obtain this sequence, we will think of the "operators" $E(f)$ and $A(g)$ as derivations on some field algebra $\mathscr{A}$, in which case scalars are factored out of the commutation relations (4.2).

Let Δ be the Abelian Lie algebra, identical with the test function space $\mathscr{S}$ as a linear space [its elements are thought of as the $A(f)$s]. Let Ω be another copy of $\mathscr{S}$ as an Abelian Lie algebra [but now its elements are thought of as the $E(f)$s without the right-hand side of (4.2c)].

The action of Ω on Δ is taken to be trivial, i.e., $\delta_\beta(d)=0$ for all $\beta\in\Omega$, $d\in\Delta$ [this is justified by (4.2b), factoring out constants]. Then the map $\sigma:\Omega^2\rightarrow\Delta$, which is defined by $\sigma(f,g):=\alpha f\circledast g$ (using the identification of Ω and Δ with $\mathscr{S}$), is a 2-cocycle because

$$(\partial\sigma)(f,g,h)=\text{Cycl.}\{\delta_f(\sigma(g,h))+\sigma(f,[g,\ h])\}=0$$

for all $f,g,h\in\Omega$. So we form the corresponding extension of Lie algebras:

$$\Gamma=\Omega\oplus\Delta \quad \text{with bracket } [f\oplus g,\ h\oplus k]=0\oplus\sigma(f,h) \tag{5.2}$$

as in (3.6). Now the horizontal exact sequence (5.1) is specified, and we identify the section $\omega:\Omega\rightarrow\Gamma$ by $\omega_f=f\oplus 0$, so

$$[\omega_f,\ \omega_h]=0\oplus\sigma(f,h),$$

and ω_f corresponds to $E(f)$, now satisfying (4.2c).

For the vertical exact sequence

$$0\rightarrow\mathscr{W}\rightarrow\mathscr{V}\xrightarrow{\text{ad}|_{\mathscr{A}}}\Delta\rightarrow 0$$

assume there is a representation v of Δ as self-adjoint operators on a common dense invariant domain D in a Hilbert space $\mathscr{H}$, on which the field algebra is also irreducibly represented and such that v implements Δ as derivations on $\mathscr{A}$. Let $\mathscr{V}$ be the linear space of operators spanned by $\{\mathbb{R}1,v_d|d\in\Delta\}$, which is an Abelian Lie algebra. Then $\mathscr{W}=\mathbb{R}1$, $\text{ad}|_{\mathscr{A}}(\mathscr{V})\cong\Delta$, and the original representation $v:\Delta\rightarrow\mathscr{V}$ is a section with 2-cocycle $\mu=0$.

To "guess" an appropriate map $\lambda:\Gamma\times\Delta\rightarrow\mathbb{R}$ satisfying (2.1), we calculate the action $\delta:\Gamma\rightarrow\text{Der }\mathscr{V}$ which would have been appropriate if (4.2b) were true. Identify $A(f)$ with v_f, $f\in\Delta=\mathscr{S}$. Suppose there are operators u_f, $f\in\Omega=\mathscr{S}$, preserving D [identified with $E(f)$] satisfying the equal-time commutation relations

$$[u_f,\ v_h]=i(f,h)$$

and providing the action $\delta:\Omega\rightarrow\text{Der }\mathscr{V}$ by $\delta_f(v_h)=(-i)[u_f,\ v_h]$. So $\delta:\Gamma\rightarrow\text{Der }\mathscr{V}$ will be

$$\delta_{f\oplus g}(v_h)=\delta_{f\oplus 0}(v_h)+\delta_{0\oplus g}(v_h)=\delta_{f\oplus 0}(v_h)=(f,h)$$

and thus $\lambda:\Gamma\times\Delta\rightarrow\mathbb{R}$ is given by

$$\lambda(f\oplus g,h)=\delta_{f\oplus g}(v_h)-v_{[f\oplus g,\ 0\oplus h]}=\delta_{f\oplus g}(v_h)=(f,h). \tag{5.3}$$

Since this λ was obtained by fallacious reasoning, we need to verify that it does indeed satisfy the conditions (2.1). Again, (2.1a), (2.1b), and (2.1d) are obviously true. As for (2.1c)

$$\lambda([f\oplus g,\ h\oplus k],m)=\lambda(f\oplus g,[h\oplus k,\ 0\oplus m])-\lambda(h\oplus k,[f\oplus g,\ 0\oplus m]),$$

$$\text{lhs}=\lambda(0\oplus\sigma(f,h),m)=0,$$

$$\text{rhs}=\lambda(f\oplus g,0)-\lambda(h\oplus k,0)=0.$$

Hence the λ given by (5.3) satisfies Eq. (2.1), hence defines a 3-cocycle $K:\Omega^3\rightarrow\mathbb{R}$:

$$K(f,g,h)=\lambda(f\oplus 0,\sigma(g,h))+\lambda(g\oplus 0,\sigma(h,f))+\lambda(h\oplus 0,\sigma(f,g))$$
$$=(f,\sigma(g,h))+(g,\sigma(h,f))+(h,\sigma(f,g))=\alpha\ \text{Cycl.}(f,g\circledast h) \tag{5.4}$$

which is exactly the same 3-cocycle as the one obtained in Sec. 3 from Jo's calculation. Observe firstly that K determines a nontrivial cohomology class as the cohomology group in degree n with real coefficients of an Abelian Lie algebra is given by the space of totally skew n-multilinear maps. Hence we can now proceed to prove the analogue of Lemma 3.1, which shows that the Lie algebra Γ cannot be represented in such a way that the action δ associated to λ is implemented. Moreover K will also be an obstruction to the existence of the extension of Theorem 2.2.

We conclude that the 3-cocycle is an obstruction to finding a representation of the Lie algebra Γ for which the canonical relations (3.1) hold. From the viewpoint of rigorous quantum field theory one should regard this result as further evidence that the canonical equal-time formalism is probably not appropriate in 3+1 dimensions: a fact which is not all that surprising.

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