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Characters of the discrete Heisenberg group and of its completion

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Abstract

In this paper we describe the relationship between characters of finitely generated torsion-free nilpotent groups of class 2 and their completions. In particular we give a detailed description of this connection for the discrete Heisenberg group.

1. *Introduction*

We consider a discrete group G which is nilpotent, finitely generated and torsion-free. If $\widetilde{G}$ is the Mal'cev completion of G, the theory of characters of $\widetilde{G}$ is well understood [**6**]. How then do we relate the character theory of G to that of $\widetilde{G}$ with the aim of finding a simple description of the characters of G? In fact, the situation is not simple even in the case when G has nilpotence class 2 since not all characters of G extend to those of $\widetilde{G}$ nor do all characters of $\widetilde{G}$ restrict to those of G. To be specific this paper is concerned with comparing the characters of the discrete Heisenberg group

$$G := \left\{ \begin{bmatrix} 1 & m & p \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} : m, n, p \in \mathbb{Z} \right\}$$

with those of its completion

$$\widetilde{G} := \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{Q} \right\}.$$

In Section 2, we cover all basic concepts, notation and related matters. In Section 3, we give some important facts related to the so-called AIC groups of [**3**]. Next in Section 4 we introduce the concept of an induced trace of a discrete countable

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nilpotent group of class 2. In Section 5, we develop the main theory in which we deal with a discrete group G which is of class 2, finitely generated and torsion-free. Finally, in Section 6, using the main theorem in Section 5, we describe some relations between characters of the discrete Heisenberg group and those of its completion.

All groups in this paper are assumed to be discrete and countable.

2. *Notation, terminology and related matters*

For subsets A and B of a group G we write $[A, B]$ to denote the group generated by elements of the form $[a, b] := a^{-1}b^{-1}ab$, $a \in A$, $b \in B$. We write $Z(G)$ and 1_G respectively to be the centre and the identity element of G. If G/B is a quotient group, we write $\overline{Z}(G/B)$ for the centre of G/B, and $Z(G/B) := \{x \in G : [x, G] \subseteq B\}$, where $[x, G] := \{[x, g] : g \in G\}$. We shall write $\psi|_A$ to denote the restriction of ψ to a set A, and $B(G)$ for the set of all bounded complex functions on G. If G is an abelian group, $G^\frown$ is the dual of G, that is, the set of all complex homomorphisms from G to the unit circle.

We recall that a group G is *nilpotent* if its upper central series defined by $Z^1(G) := Z(G)$, and $Z^{k+1}(G) := \{x \in G : [x, G] \subseteq Z^k(G)\}$ for $k \in \{1, 2, \ldots\}$, terminates after a finite number of steps in G.

A group G is called *complete* if, for every integer $n \neq 0$ and every $a \in G$, the equation $x^n = a$ has a unique solution. It is known that every torsion-free nilpotent group can be embedded in a complete torsion-free nilpotent group ([**4**, page 18]). A minimal complete torsion-free nilpotent group containing a given torsion-free nilpotent group G is termed a *Mal'cev completion* of G (it is unique up to isomorphism), or shortly *a completion* of G, and denoted by $\widetilde{G}$. In fact, any element of $\widetilde{G}$ is an element some power of which lies in G ([**11**, page 256]).

Assume now that the group G is discrete. A positive definite function $\varphi: G \to \mathbb{C}$ satisfying $\varphi(g_1g_2) = \varphi(g_2g_1)$ for all $g_1, g_2 \in G$ (φ is *central*) and $\varphi(1_G) = 1$ is called a *trace* of G. The set $\mathrm{Tr}(G)$ of all traces of G equipped with the pointwise topology is a compact convex set. The extreme points of this set (i.e. those f which cannot be written as a convex combination of two different traces) are called *characters* of G. We denote the set of all characters of G by $\mathrm{Ch}(G)$. A character $\varphi \in G$ is said to be *faithful* if $k(\varphi) := \{x \in G : \varphi(x) = 1\} = (1_G)$.

If φ is a faithful character of G then, for all $g \in G, h \in Z(G), \varphi(gh) = \varphi(g)\varphi(h)$. Notice that, if φ is a character of G, then $\overline{\varphi}$ with $\overline{\varphi}(\overline{g}) := \varphi(g)$, where $\overline{g} := gk(\varphi)$, defines a character of $G/k(\varphi)$. If $g \in G$ and $h \in Z(G/k(\varphi))$, then $\overline{\varphi}(\overline{g}\overline{h}) = \overline{\varphi}(\overline{g})\overline{\varphi}(\overline{h})$, and therefore $\varphi(gh) = \varphi(g)\varphi(h)$.

For a group G, let G_f be the normal subgroup consisting of all elements with finite conjugacy classes, that is the set of all x such that $x^G := \{g^{-1}xg : g \in G\}$ is finite. Then, G is said to be *flat* if, for every $x \in G_f$, x^G is a coset of some subgroup of G. The group is termed a group with *absolutely idempotent characters* (AIC) if, for every character φ of G we have $|\varphi|^2 \equiv |\varphi|$, that is $|\varphi(g)| = 1$ or $|\varphi(g)| = 0$ for all g ([**3**, page 182]). (This is equivalent to saying that for every character φ of G, $\varphi \equiv 0$ off $Z(G/k(\varphi))$.) It is straightforward to see that every nilpotent group of class 2 is AIC, and it is well known that every complete nilpotent group is AIC ([**5**, theorem 4.2]).

Let H be a normal subgroup of a group G. Suppose that G acts on $\mathrm{Tr}(H)$ by conjugation, that is, if $g \in G$ and $\varphi \in \mathrm{Tr}(H)$, $g \cdot \varphi := \varphi^g$ where $\varphi^g(h) := \varphi(g^{-1}hg)$, $h \in H$.

A trace $\varphi \in \mathrm{Tr}(H)$ is called *G-invariant* if $\varphi^g = \varphi$ for all $g \in G$. It is well known that, if φ is an G-invariant trace of H, then $\overline{\varphi}$ on G with $\overline{\varphi}|_H = \varphi$ and $\overline{\varphi}$ is 0 off H, is a trace of G (see [**16**]), and therefore it is easy to show that φ extends to some character of G. If H is not necessarily normal and $\mathrm{N}_G(H)$ is its normalizer in G, for a character φ of H, we write $\mathrm{Stab}_\varphi(G)$ for the *stabilizer* of φ in G, that is $\mathrm{Stab}_\varphi(G) := \{g \in \mathrm{N}_G(H) : \varphi^g = \varphi\}$.

3. *Some facts about AIC nilpotent groups*

The AIC groups appear to be the most tractable non-abelian nilpotent groups from the point of view of character theory. Here we give some facts about these groups. We shall use them in the sequel.

FACT 3·1. *Let G be a nilpotent group. Then G is AIC if and only if, for all* $\varphi \in \mathrm{Tr}(G)$, *if* $|\varphi| \equiv 1$ *on* $Z(G/k(\varphi))$, *then* $\varphi \equiv 0$ *off* $Z(G/k(\varphi))$.

Proof. The part ($\Leftarrow$) is obvious. We show the part ($\Rightarrow$). Let $\varphi \in \mathrm{Tr}(G)$. Without loss of generality we can assume that φ is faithful. Note that $\mathrm{Tr}(G)$ is metrizable since G is countable. By Choquet's theorem ([**13**, page 19]),

$$\varphi = \int_{\mathrm{Ch}(G)} \psi \, d\mu(\psi) \tag{3·1}$$

where μ is a probability measure on $\mathrm{Ch}(G)$. Suppose that $|\varphi| \equiv 1$ on $Z(G)$. From (3·1) we note that

$$\varphi|_{Z(G)} = \int_{\mathrm{Ch}(G)} \psi|_{Z(G)} \, d\mu(\psi).$$

Since φ is extremal on $Z(G)$, it follows from [**13**, proposition 1·4] that

$$\mathrm{supp}(\mu) \subseteq \{\psi \in \mathrm{Ch}(G) : \psi|_{Z(G)} = \varphi|_{Z(G)}\}.$$

Now let $\psi \in \mathrm{Ch}(G)$ such that $\psi|_{Z(G)} = \varphi|_{Z(G)}$. As φ is faithful, we have $k(\psi) \cap Z(G) = (1_G)$. Since G is nilpotent, it follows that $k(\psi) = (1_G)$, and hence $\psi \equiv 0$ off $Z(G)$ because G is AIC. Hence $\varphi \equiv 0$ off $Z(G)$.

PROPOSITION 3·2. *Let G be a group and* $\varphi \in \mathrm{Tr}(G)$. *Consider the following statements*:

(i) φ *is a character of G*;

(ii) $|\varphi| \equiv 1$ *on* $Z(G/k(\varphi))$;

(iii) φ *is a homomorphism on* $Z(G/k(\varphi))$.

Then (ii) *and* (iii) *are equivalent, and* (i) *implies* (ii). *If G is an AIC nilpotent group, then all statements are equivalent.*

Proof. The equivalence between (ii) and (iii) is obvious. The fact that (i) implies (ii) is also obvious. Suppose that φ satisfies condition (ii). By Fact 3·1, $\varphi \equiv 0$ off $Z(G/k(\varphi))$. But then the argument in the proof of Fact 3·1 also shows that φ is a character of G since it follows from $\varphi = \int_{\mathrm{Ch}(G)} \psi \, d\mu(\psi)$ that $\varphi = \psi$ for some $\psi \in \mathrm{Ch}(G)$. Thus (i) holds.

The following proposition characterizes the elements of $\mathrm{Ch}(G)$ when G is AIC and nilpotent.

PROPOSITION 3·3. *Let G be an AIC nilpotent group. Then a complex function φ is a character of G if and only if there exists a normal subgroup $B \lhd G$ and faithful character (injective homomorphism) χ on $\overline{Z}(G/B)$ such that*

$$\varphi(g) = \begin{cases} \chi(\overline{g}) & \text{if } \overline{g} \in \overline{Z}(G/B) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The part ($\Rightarrow$) is obvious. For the part ($\Leftarrow$), since φ is a trace of G, then it is a character by Proposition 3·2.

PROPOSITION 3·4. *Let H be a nilpotent group and N be a subgroup. Let ψ be a character of H, and $\varphi := \psi|_N$. Suppose that N is AIC. Then φ is a character of N if and only if $Z(N/k(\varphi)) \subseteq Z(H/k(\psi))$.*

Proof. The part ($\Leftarrow$) follows from Proposition 3·2 since φ is a homomorphism on $Z(N/k(\varphi))$. For the part ($\Rightarrow$), suppose on the contrary that there exists an $x \in Z(N/k(\varphi)) \setminus Z(H/k(\psi))$. By Proposition 3·2, $|\varphi(x)| = |\psi(x)| = 1$. This contradicts the fact that $|\psi(x)| < 1$ ([**5**, proposition 2.6]).

COROLLARY 3·5. *Let G be a torsion-free nilpotent group. Then, every character of G extends to a character of $G\widetilde{Z(G)}$. If G is AIC, every character of $G\widetilde{Z(G)}$ restricts to a character of G.*

Proof. The first statement is straightforward, while the second one follows from Proposition 3·4.

4. *The induced trace of a nilpotent group of class 2*

Let N be a normal subgroup of a nilpotent group H of class 2. We discuss here the concept of an induced trace of H by a trace of N. This clearly has connections with induced representations. However, for simplicity, we ignore the connections here.

PROPOSITION 4·1. *Let H be a group and N be a subgroup (which is not necessarily normal in H). Let φ be a trace of N. Suppose that, for all $x \in H$, we define ψ^x on H by*

$$\psi^x(h) := \begin{cases} \varphi(h^{-1}xh) & \text{if } h^{-1}xh \in N \\ 0 & \text{otherwise.} \end{cases}$$

If M is an invariant mean on $B(G)$, then ϕ with

$$\phi(x) := M(\psi^x), \quad x \in H$$

is a trace.

Proof. We shall show first that ϕ is positive definite. Since for $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$, $h_1, h_2, \ldots, h_n \in H$ we have

$$\sum_{i,j=1}^{n} \alpha_i \overline{\alpha}_j M\left(\psi^{h_i h_j^{-1}}\right) = M\left(\sum_{i,j=1}^{n} \alpha_i \overline{\alpha}_j \psi^{h_i h_j^{-1}}\right),$$

it suffices to show that $\sum_{i,j=1}^{n} \alpha_i \overline{\alpha}_j \psi^{h_i h_j^{-1}} \geqslant 0$. Let $x \in H$, and $g_i := x^{-1}h_i$ for every $i \in \{1, 2, \ldots, n\}$. Let $Nx_1, Nx_2, \ldots, Nx_{n_0}$ be different right cosets of N such that

$\bigcup_{i=1}^{n} Ng_i = \bigcup_{i=1}^{n_0} Nx_i$. Let for every $k \in \{1, 2, \ldots, n_0\}$, $S_k := \{g_1, g_2, \ldots, g_n\} \cap Nx_k$, and write $S_k = \{g_{k,i} : i = 1, 2, \ldots, |S_k|\}$. Since $\psi^{h_i h_j^{-1}}(x) = 0$ if $g_i g_j^{-1} \notin N$, we have

$$\sum_{i,j=1}^{n} \alpha_i \overline{\alpha}_j \psi^{h_i h_j^{-1}}(x) = \sum_{k=1}^{n_0} \sum_{g_{k,i}, g_{k,j} \in S_k} \alpha_{k,i} \overline{\alpha}_{k,j} \varphi\big(g_{k,i} g_{k,j}^{-1}\big).$$

But for every $k \in \{1, 2, \ldots, n_0\}$ and $i \in \{1, 2, \ldots, |S_k|\}$, $g_{k,i} = w_{k,i} x_k$ for some $w_{k,i} \in N$, so that $g_{k,i} g_{k,j}^{-1} = w_{k,i} w_{k,j}^{-1}$. Therefore

$$\sum_{i,j=1}^{n} \alpha_i \overline{\alpha}_j \psi^{h_i h_j^{-1}}(x) = \sum_{k=1}^{n_0} \sum_{i,j \in \{1,2,\ldots,|S_k|\}} \alpha_{k,i} \overline{\alpha}_{k,j} \varphi\big(w_{k,i} w_{k,j}^{-1}\big) \geqslant 0$$

as φ is a positive definite function on N.

We shall next show that ϕ is central. Fix $x, y \in G$. Let ${}_y\psi^x$ be the left translation of ψ^x by y. Let h be an arbitrary element of H. If $h^{-1}(yxy^{-1})h \in N$, then

$$\begin{aligned} {}_y\psi^x(h) &= \psi^x(y^{-1}h) = \varphi((y^{-1}h)^{-1} x y^{-1} h) \\ &= \varphi(h^{-1}(yxy^{-1})h) = \psi^{yxy^{-1}}(h). \end{aligned}$$

If $h^{-1}(yxy^{-1})h \notin N$ then

$${}_y\psi^x(h) = \psi^x(y^{-1}h) = 0 = \psi^{yxy^{-1}}(h).$$

Therefore,

$$\phi(yxy^{-1}) = M\left(\psi^{yxy^{-1}}\right) = M({}_y\psi^x) = M(\psi^x) = \phi(x)$$

for all $x, y \in H$.

We need the following two facts.

FACT 4·2. ([**8**, page 38]). *Let G be a topological group and f be a continuous function in $B(G)$. If f is weakly almost periodic, then all invariant means on $B(G)$ agree at f.*

FACT 4·3. ([**14**, theorem 3.1]). *Let G be a topological group. Then a continuous function $f \in B(G)$ is weakly almost periodic if and only if* $\lim_i \lim_j f(x_i y_j)$ *and* $\lim_j \lim_i f(x_i y_j)$ *are equal whenever they both exist.*

We also recall that every nilpotent group G has an invariant mean on $B(G)$ ([**7**, page 517]).

PROPOSITION 4·4. *Let H be a nilpotent group of class 2, $N \lhd H$ and $\varphi \in \mathrm{Tr}(N)$. Suppose that for all $x \in H$, ψ^x is as in Proposition 4·1. Then all invariant means on $B(H)$ agree on ψ^x.*

Proof. In view of Fact 4·2, we shall show that, for all $x \in H$, ψ^x is weakly almost periodic, and for this we will employ Fact 4·3. Suppose now that there exists sequences (h_i) and (g_j) in H such that $\lim_i \lim_j \psi^x_{h_i}(g_j)$ and $\lim_j \lim_i \psi^x_{h_i}(g_j)$ both exist. We will show that they are the same. It is enough to consider this for $x \in N$. Then we have

$$\begin{aligned} \psi^x(h_i g_j) &= \varphi\big(g_j^{-1} h_i^{-1} x h_i g_j\big) \\ &= \varphi\big(g_j^{-1} x [x, h_i] g_j\big) \\ &= \varphi\big(g_j^{-1} x g_j [x, h_i]\big) \\ &= \varphi\big(g_j^{-1} x g_j\big) \psi([x, h_i]), \end{aligned}$$

so that, since $\lim_j \psi^x(h_i g_j)$ exists, we have

$$\begin{aligned}\lim_i \lim_j \psi^x(h_i g_j) &= \lim_i \lim_j \varphi\big(g_j^{-1} x g_j\big)\varphi([x,h_i]) \\ &= \lim_i \Big(\varphi([x,h_i]) \lim_j \varphi\big(g_j^{-1} x g_j\big)\Big) \\ &= \Big(\lim_i \varphi([x,h_i])\Big)\Big(\lim_j \varphi\big(g_j^{-1} x g_j\big)\Big).\end{aligned}$$

We also have the same limit for $\lim_j \lim_i \psi^x(h_i g_j)$.

We now have the following definition.

DEFINITION 4·5. Let N, H and φ be as in Proposition 4·4. An *induced trace* $\varphi \uparrow_N^H$ of H by φ is defined by

$$\varphi \uparrow_N^H (x) := M(\psi^x), \quad x \in H$$

where M is an invariant mean on $B(H)$.

We next consider the following lemma whose proof is a simple modification of that of [**9**, theorem 18.10].

LEMMA 4·6. *Let G be a countable group. Suppose that S is a subspace of $B(G)$ on which all left invariant means of $B(G)$ agree. If M is a left invariant mean of $B(G)$, then there exists an increasing sequence (U_n) of finite subsets of G with $\bigcup_{n=1}^{\infty} U_n = G$ and*

$$M(f) = \lim_{n\to\infty} \frac{1}{|U_n|} \sum_{g\in U_n} f(g)$$

for all $f \in S$.

THEOREM 4·7. *Let H be a discrete, countable, nilpotent group of class 2. Let $N \lhd H$ and $\varphi \in \mathrm{Tr}(N)$. Then*

$$\varphi \uparrow_N^H (x) = \begin{cases} \varphi(x) & \textit{if } x \in A(\varphi; N, H) \\ 0 & \textit{otherwise} \end{cases}$$

where $A(\varphi; N, H) := \{x \in N : \varphi([x,h]) = 1, \forall h \in H\}$.

Proof. Since $N \lhd H$, $\psi^x = 0$ if $x \notin N$. We now consider the value of $\varphi \uparrow_N^H$ on N. By Lemma 4·6, there exists an increasing sequence (F_n) of finite subsets of H with $\bigcup_{n=1}^{\infty} F_n = H$ such that for all $x \in H$,

$$\varphi \uparrow_N^H (x) = \lim_{n\to\infty} \frac{1}{|F_n|} \sum_{h\in F_n} \psi^x(h).$$

If $h \in H$, then φ_h with $\varphi_h(x) := \varphi([x,h])$, $x \in N$, is a complex homomorphism on the abelian group $N/[N,G]$. Since

$$\psi^x(h) = \varphi(h^{-1}xh) = \varphi(x[x,h]) = \varphi(x)\varphi([x,h]),$$

we have

$$\varphi \uparrow_N^H (x) = \varphi(x) \lim_{n\to\infty} \frac{1}{|F_n|} \sum_{h\in F_n} \varphi_h(x).$$

As H is discrete, $\Gamma := \mathrm{Cl}\{\varphi_h : h \in H\}$ is a compact group, and

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{h\in F_n} \varphi_h(x) = \int_\Gamma \gamma(x)\, d_{\mu_\Gamma}(\gamma)$$

where μ_Γ is the normalized Haar measure on Γ. Now let

$$A(\varphi; N, H) := \{x \in N : \gamma(x) = 1, \forall \gamma \in \Gamma\}.$$

If $x \in A(\varphi; N, H)$, then clearly $\int_\Gamma \gamma(x) d_{\mu_\Gamma}(\gamma) = 1$. If $x \notin A(\psi; N, H)$, then $\gamma_0(x) \neq 1$ for some $\gamma_0 \in \Gamma$, and

$$\int_\Gamma \gamma(x) d_{\mu_\Gamma}(\gamma) = \gamma_0(x) \int_\Gamma \gamma_0^{-1}(x)\gamma(x)\, d_{\mu_\Gamma}(\gamma)$$

so that $\int_\Gamma \gamma(x)\, d_{\mu_\Gamma}(\gamma) = 0$. Therefore we find that, for $x \in N$,

$$\varphi \uparrow_N^H (x) = \begin{cases} \varphi(x) & \text{if } x \in A(\varphi; N, H) \\ 0 & \text{otherwise} \end{cases}$$

where $A(\varphi; N, H)$ is given by

$$A(\varphi; N, H) = \{x \in N : \varphi([x,h]) = 1, \forall h \in H\}.$$

5. *The main results*

We recall that all groups we are considering here are assumed to be discrete and countable.

THEOREM 5·1. *Let H be a nilpotent group of class 2, $N \lhd H$ and $\varphi \in \mathrm{Ch}(N)$. Then, $\varphi \uparrow_N^H$ is a character of H if and only if, for all $x \notin N$, there exists some $h \in H$ such that*

$$[x,h] \in N \quad \text{and} \quad \varphi([x,h]) \neq 1. \tag{5·2}$$

Proof. Let $\varphi \uparrow_N^H$ be a character of H. By Proposition 3·3, there exists a subgroup $B \lhd H$ such that $Z(H/B) \subseteq A(\varphi; N, H) \subseteq N$, and for $x \in Z(H/B)$, $\varphi(x) = 1$ if and only if $x \in B$. Now let $x \notin N$. Then there exists an $h_0 \in H$ such that $[x, h_0] \notin B$. Since $[x, h_o] \in Z(H/B)$, $[x, h_0] \in N$ and $\varphi([x, h_0]) \neq 1$, so that the condition (5·2) holds. Suppose, conversely, that the condition (5·2) is satisfied. Set $B := k(\varphi) \cap A(\varphi; N, H)$. If $h \in H$, $b \in B$, then

$$\varphi(h^{-1}bh) = \varphi(b[b,h]) = \varphi(b)\varphi([b,h]) = 1.$$

As $A(\varphi; N, H) \lhd H$, it follows that $B \lhd H$. If $x \in A(\varphi; N, H)$, then

$$[x, H] \subseteq k(\varphi) \cap A(\varphi; N, H) = B,$$

so that $A(\varphi; N, H) \subseteq Z(H/B)$. Let $x \in Z(H/B)$, that is $\varphi([x,h]) = 1$ for all $h \in H$. Then, it follows from condition (5.2) that $x \in N$, and hence $x \in A(\varphi; N, H)$. Therefore $A(\varphi; N, H) = Z(H/B)$. Now we notice that $A(\varphi; N, H) \subseteq Z(N/k(\varphi))$, so that φ is a complex homomorphism on $Z(H/B)$ satisfying, for $x \in Z(H/B)$, $\varphi(x) = 1$ if and only if $x \in B$. Hence $\varphi \uparrow_N^H$ is a character.

LEMMA 5·2. *Let S be a nilpotent group and $A \lhd S$ such that $[S,S] \subseteq A$. Let φ_1, $\varphi_2 \in \mathrm{Ch}(S)$ with φ_1 and φ_2 agreeing on A. Then there exists $\gamma \in (S/A)^\frown$ such that $\varphi_1 = \gamma\varphi_2$.*

Proof. This is an immediate consequence of [**5**, lemma 2.2].

LEMMA 5·3. *Let H be a nilpotent group of class 2 and $N \lhd H$. Let $\varphi \in \mathrm{Ch}(N)$, and write S_φ for* $\mathrm{Stab}_\varphi(H)$. *Then, for every $\psi \in \mathrm{Ch}(S_\varphi)$ such that $\psi|_N = \varphi$, $\psi \uparrow_{S_\varphi}^H$ is a character of H.*

Proof. If $x \notin S_\varphi$, then there exists $n \in N$ such that $\varphi(x^{-1}nx) \neq \varphi(n)$, that is $\varphi([x,n]) \neq 1$ or $\psi([x,n]) \neq 1$. Thus, according to Theorem 5·1, $\varphi \uparrow_{S_\varphi}^H$ is a character.

In view of Lemma 5·3, we have a map $s\colon \varphi \mapsto s(\varphi)$, where

$$s(\varphi) := \{\phi \in \mathrm{Ch}(H) : \phi = \psi \uparrow_{S_\varphi}^H \text{ for some } \psi \in \mathrm{Ch}(S_\varphi) \text{ with } \psi|_N = \varphi\}.$$

Then we have the following lemma.

LEMMA 5·4. *For all $\phi \in \mathrm{Ch}(H)$, $\phi \in s(\varphi)$ for some $\varphi \in \mathrm{Ch}(N)$.*

Proof. Let $\phi \in \mathrm{Ch}(H)$. We will set $\varphi \in \mathrm{Ch}(N)$ and $\psi \in \mathrm{Ch}(S_\varphi)$ such that $\psi|_N = \varphi$ and $\phi = \psi \uparrow_{S_\varphi}^H$. Let N_1 be the subgroup generated by $Z(H/k(\phi))$ and N. Let φ_1 be a character of N_1 which extends $\phi_1 := \phi|_{Z(H/k(\phi))}$. We shall show that $\varphi := \varphi_1|_N$ is a character of N. By Proposition 3·4, it suffices to show that $Z(N/k(\varphi)) \subseteq Z(N_1/k(\varphi_1))$. Let $x \in Z(N/k(\varphi))$. Let $g := \prod \ell_i n_i$, where $\prod$ is a finite product, $\ell_i \in Z(H/k(\phi))$ and $n_i \in N$. Then we have

$$\begin{aligned} \varphi_1([x,g]) &= \varphi_1\Big([x, \prod \ell_i n_i]\Big) \\ &= \prod \varphi_1([x,\ell_i])\varphi_1([x,n_i]) \\ &= \prod \phi([x,\ell_i])\varphi([x,n_i]) \\ &= 1. \end{aligned}$$

Thus $x \in Z(N_1/k(\varphi_1))$, and we have proved that φ is a character of N. Now let $g \in Z(H/k(\phi))$ and $x \in N$. Then $[g,x] \in N \cap k(\varphi_1) \subseteq k(\varphi)$, so that

$$\varphi(g^{-1}xg) = \varphi(x[x,g]) = \varphi(x)\varphi([x,g]) = \varphi(x).$$

Thus $Z(H/k(\phi)) \subseteq S_\varphi$, and hence $N_1 \subseteq S_\varphi$. Observe that $N_1 \lhd S_\varphi$. We shall now show that φ_1 is S_φ-invariant. Let $g \in S_\varphi$ and $x := \prod \ell_i n_i \in N_1$, where $\prod$ is a finite product, $\ell_i \in Z(H/k(\phi))$ and $n_i \in N$. Notice that $Z(H/k(\phi)) \subseteq Z(N_1/k(\varphi_1))$ and $[N_1, H] \subseteq Z(N_1)$. Thus we note that

$$\varphi_1(x) = \prod \varphi_1(\ell_i)\varphi_1\Big(\prod n_i\Big)$$

and

$$\varphi_1([x,g]) = \prod \varphi_1([\ell_i,g]) \prod \varphi_1([n_i,g]) = \prod \varphi_1([n_i,g]).$$

Hence we have

$$\begin{aligned}\varphi_1(g^{-1}xg) &= \varphi_1(x)\varphi_1([x,g]) \\ &= \prod \varphi_1(\ell_i)\varphi_1\Big(\prod n_i\Big)\prod \varphi_1([n_i,g]) \\ &= \prod \varphi_1(\ell_i)\varphi_1\Big(\prod n_i[n_i,g]\Big) \\ &= \prod \varphi_1(\ell_i)\varphi_1\Big(g^{-1}\Big(\prod n_i\Big)g\Big) \\ &= \prod \varphi_1(\ell_i)\varphi_1\Big(\prod n_i\Big) \\ &= \varphi_1(x),\end{aligned}$$

which shows that φ_1 is S_φ-invariant. Let ψ be a character of S_φ which extends φ_1. Since φ and ϕ agree on $Z(H/k(\phi))$ and $[H,H] \subseteq Z(H/k(\phi))$, we observe that

$$A(\psi; S_\varphi, H) = \{x \in S_\varphi : \psi([x,H]) = (1)\} = Z(H/k(\phi)).$$

Therefore $\phi = \psi \uparrow_{S_\varphi}^H \in s(\varphi)$.

Let H be a nilpotent group of class 2, and $N \triangleleft H$ such that $[H,H] \subseteq N$. Let s be the map defined earlier, and for $\psi \in \mathrm{Ch}(N)$, let

$$\overline{s}((N/[H,H])^\frown\psi) := \bigcup_{\gamma\in(N/[H,H])^\frown} s(\gamma\psi).$$

Then $\overline{s}$ defines a map from $(N/[H,H])^\frown$–orbits of $\mathrm{Ch}(N)$ onto

$$\mathfrak{D} := \{\overline{s}((N/[H,H])^\frown\psi) : \psi \in \mathrm{Ch}(N)\}.$$

THEOREM 5·5. *For such N and H, we have the following facts.*

(i) *The map $\overline{s}$ is a* 1-1 *correspondence, and $\mathfrak{D}$ is a decomposition of* $\mathrm{Ch}(H)$.

(ii) *For every $\varphi \in \mathrm{Ch}(N)$ such that $\{x \in H : \varphi([x,H]) = (1)\} = [H,H]$, $s(\varphi)$ is a singleton, that is $s(\varphi) = \{\varphi\uparrow_N^H\}$. Moreover, if $\varphi_1 = \gamma\varphi_2$ for some $\gamma \in (N/[H,H])^\frown$, then $s(\varphi_1) = s(\varphi_2)$.*

(In case $Z(H) \subseteq N$, the theorem also works if we replace the group $[H,H]$ with $Z(H)$.)

Proof. We shall show (i). Let φ_1 and φ_2 be characters of N such that

$$\overline{s}((N/[H,H])^\frown\varphi_1) \cap \overline{s}((N/[H,H])^\frown\varphi_2) \neq \varnothing.$$

Then $s(\gamma_1\varphi_1) \cap s(\gamma_2\varphi_2) \neq \varnothing$ for some $\gamma_1, \gamma_2 \in (N/[H,H])^\frown$. Suppose that $\psi_1 \uparrow_{S_{\gamma_1\varphi_1}}^H = \psi_2 \uparrow_{S_{\gamma_2\varphi_2}}^H$ for some $\psi_1 \in \mathrm{Ch}(S_{\varphi_1})$ and $\psi_2 \in \mathrm{Ch}(S_{\varphi_2})$ with $\psi_1|_N = \gamma_1\varphi_1$ and $\psi_2|_N = \gamma_2\varphi_2$. Since $[H,H] \subseteq N$, ψ_1 and ψ_2 agree on $[H,H]$, so that $\gamma_1\varphi_1$ and $\gamma_2\varphi_2$ agree on $[H,H]$. Hence, by Lemma 5·2, there exists $\gamma \in (N/[H,H])^\frown$ such that $\gamma_1\varphi_1 = \gamma\gamma_1\varphi_2$. Thus $(N/[H,H])^\frown\varphi_1 = (N/[H,H])^\frown\varphi_2$. We conclude that all elements in $\mathfrak{D}$ are mutually disjoint sets and $\overline{s}$ is a 1-1 correspondence. By Lemma 5·4, $\mathfrak{D}$ is a decomposition of $\mathrm{Ch}(H)$.

We shall now show (ii). Let $\varphi \in \mathrm{Ch}(N)$. Since all ψ's, where $\psi \in \mathrm{Ch}(S_\varphi)$ and $\psi|_N = \varphi$, agree on $[H,H] \subseteq N$, it follows that all $A(\psi; S_\varphi, H)'$s are the same, that is $\{x \in S_\varphi : \varphi([x,H]) = (1)\}$. According to the given condition, this common set is contained in N, and hence equal to $\{x \in N : \varphi([x,H]) = (1)\} = A(\varphi; N, H)$. Thus $s(\varphi)$ has exactly one element, that is $\varphi \uparrow_N^H$. If $\varphi_1 = \gamma\varphi_2$ for some such γ, since φ_1 and φ_2 agree on $[H,H]$, it is clear that $s(\varphi_1) = s(\varphi_2)$.

Let G be a finitely generated, torsion-free, nilpotent group of class 2, and $\widetilde{G}$ be its completion. It is important to note that $Z(\widetilde{G}) = \widetilde{Z(G)}$ ([**12**, theorem 1]). We abbreviate for $Z(G)$ and $Z(\widetilde{G})$ respectively as Z and $\widetilde{Z}$. We then have the following notation and facts for the next theorem:

$$\begin{aligned}
\mathrm{Ch}_1(G) &:= \{\varphi \in \mathrm{Ch}(G) : \varphi(Z) = (1)\},\\
\mathrm{Ch}_2(G) &:= \{\varphi \in \mathrm{Ch}(G) : \varphi(Z) \neq (1)\},\\
\mathrm{Ch}_1(G\widetilde{Z}) &:= \{\psi \in \mathrm{Ch}(G\widetilde{Z}) : \psi(\widetilde{Z}) = (1)\},\\
\mathrm{Ch}_2(G\widetilde{Z}) &:= \{\psi \in \mathrm{Ch}(G\widetilde{Z}) : \psi(Z) \neq (1)\},\\
\mathrm{Ch}'_2(G\widetilde{Z}) &:= \{\psi \in \mathrm{Ch}(G\widetilde{Z}) : \psi(\widetilde{Z}) \neq (1), \psi(Z) = (1)\},\\
\mathrm{Ch}_1(\widetilde{G}) &:= \{\phi \in \mathrm{Ch}(\widetilde{G}) : \phi(\widetilde{Z}) = (1)\},\\
\mathrm{Ch}_2(\widetilde{G}) &:= \{\phi \in \mathrm{Ch}(\widetilde{G}) : \phi(Z) \neq (1)\},\\
\mathrm{Ch}'_2(\widetilde{G}) &:= \{\phi \in \mathrm{Ch}(\widetilde{G}) : \phi(\widetilde{Z}) \neq (1), \phi(Z) = (1)\}.
\end{aligned}$$

FACT 5·6. *Let G be a torsion-free, nilpotent group of class 2.*

(i) *Every character in* $\mathrm{Ch}_1(G)$ *extends uniquely to a character in* $\mathrm{Ch}_1(G\widetilde{Z})$*, and every character in* $\mathrm{Ch}_1(G\widetilde{Z})$ *extends to a character in* $\mathrm{Ch}_1(\widetilde{G})$.

(ii) *Every character in* $\mathrm{Ch}_2(G)$ *extends to a character in* $\mathrm{Ch}_2(G\widetilde{Z})$.

(iii) *Every character in* $\mathrm{Ch}_1(G\widetilde{Z})$ (*resp.* $\mathrm{Ch}_2(G\widetilde{Z})$) *restricts to a character in* $\mathrm{Ch}_1(G)$ (*resp.* $\mathrm{Ch}_2(G)$)*, and every character in* $\mathrm{Ch}_1(\widetilde{G})$ *restricts to a character in* $\mathrm{Ch}_1(G\widetilde{Z})$.

Proof. For the first part of (i), let $\varphi \in \mathrm{Ch}_1(G)$. Define $\overline{\varphi}$ with $\overline{\varphi}(gz) := \varphi(g)$, $g \in G$, $z \in \widetilde{Z}$. If $g_1, g_2 \in G$, $z_1, z_2 \in \widetilde{Z}$ such that $g_1 z_1 = g_2 z_2$, then $g_2^{-1} g_1 = z_2 z_1^{-1} \in \widetilde{Z} \cap G = Z$, so that $\varphi(g_2^{-1} g_1) = \varphi(z_2 z_1^{-1}) = 1$, and hence $\overline{\varphi}(g_1 z_1) = \overline{\varphi}(g_2 z_2)$ as φ is a homomorphism. We see therefore that $\overline{\varphi}$ is a well defined homomorphism on $Z(G\widetilde{Z}/k(\overline{\varphi})) = G\widetilde{Z}$, so that $\overline{\varphi}$ is a character in $\mathrm{Ch}_1(G\widetilde{Z})$ which extends φ. This $\overline{\varphi}$ is unique, for if $\psi \in \mathrm{Ch}_1(G\widetilde{Z})$ such that $\psi|_G = \varphi$, then $\psi = \gamma\overline{\varphi}$ for some $\gamma \in (G\widetilde{Z}/G)^\wedge$ by Lemma 5·2, and hence $\gamma|_{\widetilde{Z}} \equiv 1$, which implies $\gamma \equiv 1$. For the second part of (i), let $\psi \in \mathrm{Ch}_1(G\widetilde{Z})$. As $\psi(\widetilde{Z}) = (1)$, ψ is $\widetilde{G}$-invariant, so that by Fact 3·2, this extends to some character of $\widetilde{G}$. Part (ii) follows immediately from the first part of Corollary 3·5. The first part of (iii) follows from the second part of Corollary 3·5, while the second part is obvious according to Proposition 3·4, for if $\phi \in \mathrm{Ch}_1(\widetilde{G})$ and $\psi := \phi|_{G\widetilde{Z}}$, then $Z(G\widetilde{Z}/k(\psi)) \subseteq \widetilde{G} = Z(\widetilde{G}/k(\phi))$.

Let s_1, $\overline{s}_1$ and $\mathfrak{D}_1$ be the maps s, $\overline{s}$ and decomposition $\mathfrak{D}$ defined earlier, where, at this point N and H are G and $G\widetilde{Z}$ respectively, and instead of the $(G/[G,G])^\wedge$-orbits we rather consider the $(G/Z)^\wedge$-orbits (see the comment following Theorem 5·5). Also, let s_2, $\overline{s}_2$ and $\mathfrak{D}_2$ be the same maps and the decomposition, with $N = G\widetilde{Z}$ and $H = \widetilde{G}$, where we consider the $(G\widetilde{Z}/\widetilde{Z})^\wedge$-orbits instead of the $(G\widetilde{Z}/[G,G])^\wedge$-orbits. We now define a $(G\widetilde{Z}/G)^\wedge *-$action on $\mathfrak{D}_2$ as follows. If $\delta \in (G\widetilde{Z}/G)^\wedge$ and $\psi \in \mathrm{Ch}(G\widetilde{Z})$,

$$\delta * \overline{s}_2((G\widetilde{Z}/\widetilde{Z})^\wedge\psi) := \overline{s}_2((G\widetilde{Z}/\widetilde{Z})^\wedge\delta\psi).$$

Consider $\mathrm{Ch}(G)$ decomposed into $\mathrm{Ch}_1(G)$ and $\mathrm{Ch}_2(G)$; $\mathrm{Ch}(G\widetilde{Z})$ into $\mathrm{Ch}_1(G\widetilde{Z})$, $\mathrm{Ch}_2(G\widetilde{Z})$ and $\mathrm{Ch}'_2(G\widetilde{Z})$; $\mathrm{Ch}(\widetilde{G})$ by $\mathrm{Ch}_1(\widetilde{G})$, $\mathrm{Ch}_2(\widetilde{G})$ and $\mathrm{Ch}'_2(\widetilde{G})$. Since every $\psi \in \mathrm{Ch}(G\widetilde{Z})$ and $\phi \in s(\psi)$ agree on $\widetilde{Z}$, we have $\psi \in \mathrm{Ch}_1(G\widetilde{Z})$ if and only if $s(\psi) \subseteq \mathrm{Ch}_1(\widetilde{G})$, and $\psi \in \mathrm{Ch}_2(G\widetilde{Z})$ (resp. $\mathrm{Ch}'_2(G\widetilde{Z})$) if and only if $s(\psi) \subseteq \mathrm{Ch}_2(\widetilde{G})$ (resp. $\mathrm{Ch}'_2(\widetilde{G})$). Let $\mathfrak{D}_{1,1}$ and $\mathfrak{D}_{1,2}$ denote the images of $\overline{s}_1$ restricted to $(G/Z)^\wedge-$orbits of $\mathrm{Ch}_1(G)$ and $\mathrm{Ch}_2(G)$ respectively. Let $\mathfrak{D}_{2,1}$ and $\mathfrak{D}_{2,2}$ denote the images of $\overline{s}_2$ restricted to $(G\widetilde{Z}/\widetilde{Z})^\wedge-$orbits of $\mathrm{Ch}_1(G\widetilde{Z}) \cup \mathrm{Ch}_2'(G\widetilde{Z})$ and $\mathrm{Ch}_2(G\widetilde{Z})$ respectively. Notice that $\{\mathfrak{D}_{1,1}, \mathfrak{D}_{1,2}\}$ and $\{\mathfrak{D}_{2,1}, \mathfrak{D}_{2,2}\}$ decompose $\mathfrak{D}_1$ and $\mathfrak{D}_2$ respectively.

For each $\varphi \in \mathrm{Ch}_1(G)$ (resp. $\mathrm{Ch}_2(G)$) we shall fix $\overline{\varphi}$ as a representative of those members of $\mathrm{Ch}_1(G\widetilde{Z})$ (resp. $\mathrm{Ch}_2(G\widetilde{Z})$) which extend φ. Also, for each $\psi \in \mathrm{Ch}_1(G\widetilde{Z})$ we shall fix $\overline{\psi}$ as a representative of those members of $\mathrm{Ch}_1(\widetilde{G})$ which extend ψ.

THEOREM 5·7. *Let G be a finitely generated, torsion-free, nilpotent group of class 2, and let $\widetilde{G}$ be its completion. Then we have the following correspondences.*

(i) *There is a 1-1 correspondence between $\mathrm{Ch}_1(G)$ and the $(\widetilde{G}/G\widetilde{Z})^\frown$–orbit space of $\mathrm{Ch}_1(\widetilde{G})$ according to the map*

$$\varphi \longmapsto (\widetilde{G}/G\widetilde{Z})^\frown \overline{\overline{\varphi}}$$

where $\overline{\overline{\varphi}}$ is a representative of those members of $\mathrm{Ch}_1(\widetilde{G})$ which extend $\overline{\varphi}$.

(ii) *Suppose that G satisfies $[x, \widetilde{G}] = \widetilde{Z}$ for all $x \notin \widetilde{Z}$. Then there is a 1-1 correspondence between the $(G\widetilde{Z}/\widetilde{Z})^\frown$–orbit space of $\mathrm{Ch}(G)$ and the $(G\widetilde{Z}/G)^\frown *$–orbit space of $\{\mathrm{Ch}_1(\widetilde{G})\} \cup \mathrm{Ch}_2'(\widetilde{G}) \cup \mathrm{Ch}_2(\widetilde{G})$. Specifically:*

(a) *the $(G\widetilde{Z}/\widetilde{Z})^\frown$–orbit space of $\mathrm{Ch}_1(G)$, which is the singleton $\{\mathrm{Ch}_1(G)\}$, corresponds with the $(G\widetilde{Z}/G)^\frown *$–orbit space of $\{\mathrm{Ch}_1(\widetilde{G})\} \cup \mathrm{Ch}_2'(\widetilde{G})$, which is the singleton $\{\{\mathrm{Ch}_1(\widetilde{G})\} \cup \mathrm{Ch}_2'(\widetilde{G})\}$; and*

(b) *the $(G\widetilde{Z}/\widetilde{Z})^\frown$–orbit space of $\mathrm{Ch}_2(G)$ corresponds in a 1-1 way with the $(G\widetilde{Z}/G)^\frown *$–orbit space of $\mathrm{Ch}_2(\widetilde{G})$ according to the map*

$$(G\widetilde{Z}/\widetilde{Z})^\frown \varphi \longmapsto (G\widetilde{Z}/G)^\frown * s_2(\overline{\varphi}).$$

Proof. We shall first show (i). By the first part of each Fact 5·6(i) and Fact 5·6(iii), the map $\varphi \mapsto \overline{\varphi}$ is a 1-1 correspondence between $\mathrm{Ch}_1(G)$ and $\mathrm{Ch}_1(G\widetilde{Z})$. Noting the second part of Fact 5·6(i), we have a map $r : \psi \mapsto r(\psi)$, $\psi \in \mathrm{Ch}_1(G\widetilde{Z})$, with

$$r(\psi) := \{\phi \in \mathrm{Ch}_1(\widetilde{G}) : \phi|_{G\widetilde{Z}} = \psi\},$$

that is, by Lemma 5·2, $r(\psi) = (\widetilde{G}/G\widetilde{Z})^\frown \overline{\psi}$. Clearly, this map is injective. It follows from the second part of Fact 5·6(iii) that this map is a surjection from $\mathrm{Ch}_1(G\widetilde{Z})$ onto $(\widetilde{G}/G\widetilde{Z})^\frown$-orbits of $\mathrm{Ch}_1(\widetilde{G})$, and hence (i) follows.

We shall now show (ii). Suppose that G satisfies $[x, \widetilde{G}] = \widetilde{Z}$ for all $x \notin \widetilde{Z}$. Then we notice that for all $\psi \in \mathrm{Ch}_2(G\widetilde{Z}) \cup \mathrm{Ch}_2'(G\widetilde{Z})$,

$$\{x \in \widetilde{G} : \psi([x, \widetilde{G}]) = (1)\} = [\widetilde{G}, \widetilde{G}] = \widetilde{Z},$$

and by Theorem 5·5(ii), $s_2(\psi)$ is a singleton, so that we can identify it as an element of $\mathrm{Ch}_2(\widetilde{G}) \cup \mathrm{Ch}_2'(\widetilde{G})$. For convenience, we first show part (b). Consider the dual $(G\widetilde{Z}/\widetilde{Z})^\frown$ of $G\widetilde{Z}/\widetilde{Z} = G/(G \cap \widetilde{Z}) = G/Z$ which acts both on $\mathrm{Ch}_2(G)$ and $\mathrm{Ch}_2(G\widetilde{Z})$. First note that we have a 1-1 map from $(G\widetilde{Z}/\widetilde{Z})^\frown$-orbits of $\mathrm{Ch}_2(G)$ onto $\mathfrak{D}_{1,2}$, with

$$(G\widetilde{Z}/\widetilde{Z})^\frown \varphi \longmapsto \overline{s}_1((G\widetilde{Z}/\widetilde{Z})^\frown \varphi) = (G\widetilde{Z}/G)^\frown (G\widetilde{Z}/\widetilde{Z})^\frown \overline{\varphi}.$$

(The equality follows immediately from Fact 5·6(ii), the definition of $\overline{s}$ and Lemma 5·2.) Let $\{\varphi_\alpha\}_{\alpha \in \Lambda}$ be a set of all representatives of $(G\widetilde{Z}/\widetilde{Z})^\frown$-orbits of $\mathrm{Ch}_2(G)$. For every $\alpha \in \Lambda$, let $\overline{\varphi}_\alpha$ denote a representative of the characters in $\mathrm{Ch}_2(G\widetilde{Z})$ which extend φ_α. Then we notice that

$$\mathfrak{D}_{1,2} = \{(G\widetilde{Z}/G)^\frown (G\widetilde{Z}/\widetilde{Z})^\frown \overline{\varphi}_\alpha : \alpha \in \Lambda\}$$

which is a decomposition of $\mathrm{Ch}_2(G\widetilde{Z})$. Next, we also have a 1-1 map from $(G\widetilde{Z}/\widetilde{Z})\hat{}$-orbits of $\mathrm{Ch}_2(G\widetilde{Z})$ onto $\mathfrak{D}_{2,2} = \mathrm{Ch}_2(\widetilde{G})$, with

$$(G\widetilde{Z}/\widetilde{Z})\hat{}\psi \longmapsto \overline{s}_2((G\widetilde{Z}/\widetilde{Z})\hat{}\psi) = s_2(\psi).$$

(The equality follows from Theorem 5·5(ii).) Then the relation

$$(G\widetilde{Z}/G)\hat{}(G\widetilde{Z}/\widetilde{Z})\hat{}\,\overline{\varphi} = (G\widetilde{Z}/\widetilde{Z})\hat{}(G\widetilde{Z}/G)\hat{}\,\overline{\varphi} \longmapsto \overline{s}_2((G\widetilde{Z}/G)\hat{}\,\overline{\varphi}) = (G\widetilde{Z}/G)\hat{}* s_2(\overline{\varphi})$$

defines a map σ from $\mathfrak{D}_{1,2}$ onto $(G\widetilde{Z}/G)\hat{}*$-orbits $\mathrm{Ch}_2(\widetilde{G})$. Hence the map $\sigma \circ \overline{s}_1$ is the desired 1-1 correspondence for part (b).

For part (a), let us fix $\varphi_0 \in \mathrm{Ch}_1(G)$. Since $G\widetilde{Z}/\widetilde{Z} = G/Z$, it follows that $(G\widetilde{Z}/\widetilde{Z})\hat{}\varphi_0 = \mathrm{Ch}_1(G)$. Then we have the following association:

$$\mathrm{Ch}_1(G) = (G\widetilde{Z}/\widetilde{Z})\hat{}\varphi_0 \longmapsto \overline{s}_1((G\widetilde{Z}/\widetilde{Z})\hat{}\varphi_0) = (G\widetilde{Z}/G)\hat{}(G\widetilde{Z}/\widetilde{Z})\hat{}\,\overline{\varphi}_0.$$

(The equality follows immediately from Fact 5·6(i), the definition of $\overline{s}$ and Lemma 5·2.) Here, $\{(G\widetilde{Z}/G)\hat{}(G\widetilde{Z}/\widetilde{Z})\hat{}\,\overline{\varphi}_0\} = \mathfrak{D}_{1,1} = \{\mathrm{Ch}_1(G\widetilde{Z}) \cup \mathrm{Ch}_2'(G\widetilde{Z})\}$. We also have a map from $(G\widetilde{Z}/\widetilde{Z})\hat{}$-orbits of $\mathrm{Ch}_1(G\widetilde{Z}) \cup \mathrm{Ch}_2'(G\widetilde{Z})$ onto $\mathfrak{D}_{2,1} = \{\mathrm{Ch}_1(\widetilde{G})\} \cup \mathrm{Ch}_2'(\widetilde{G})$, with

$$(G\widetilde{Z}/\widetilde{Z})\hat{}\psi \longmapsto \overline{s}_2((G\widetilde{Z}/\widetilde{Z})\hat{}\psi),$$

where $\overline{s}_2((G\widetilde{Z}/\widetilde{Z})\hat{}\psi)$, by Fact 5·6(i) and Lemma 5·2, is equal to $\mathrm{Ch}_1(\widetilde{G})$ if $\psi \in \mathrm{Ch}_1(G\widetilde{Z})$ (since $(G\widetilde{Z}/\widetilde{Z})\hat{}\psi = \mathrm{Ch}_1(G\widetilde{Z})$), and equal to $s_2(\psi)$ (which is a singleton in $\mathrm{Ch}_2'(\widetilde{G})$) if $\psi \in \mathrm{Ch}_2'(G\widetilde{Z})$. Then we have a map τ from $\mathfrak{D}_{1,1}$ onto $(G\widetilde{Z}/G)\hat{}*$-orbits of $\{\mathrm{Ch}_1(\widetilde{G})\} \cup \mathrm{Ch}_2'(\widetilde{G})$, with

$$\begin{aligned}(G\widetilde{Z}/G)\hat{}(G\widetilde{Z}/\widetilde{Z})\hat{}\,\overline{\varphi}_0 \longmapsto \{\mathrm{Ch}_1(\widetilde{G})\} \cup \left(\bigcup_{\delta \in (G\widetilde{Z}/G)\hat{} \setminus \{1_{(G\widetilde{Z}/G)\hat{}}\}} s_2(\delta\overline{\varphi}_0) \right) \\ = \{\mathrm{Ch}_1(\widetilde{G})\} \cup \mathrm{Ch}_2'(\widetilde{G}) \\ = (G\widetilde{Z}/G)\hat{}* \{\mathrm{Ch}_1(\widetilde{G})\}.\end{aligned}$$

Thus, the map $\tau \circ \overline{s}_2$ is the desired map for part (a).

6. *Characters of the discrete Heisenberg group and of its completion*

Consider the discrete Heisenberg group G, that is

$$G = \left\{ \begin{bmatrix} 1 & m & p \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} : m, n, p \in \mathbb{Z} \right\}.$$

This is finitely generated, torsion-free and nilpotent of class 2. Then the completion $\widetilde{G}$ of G is

$$\widetilde{G} = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{Q} \right\}.$$

That G arises (implicitly or explicitly) in various situations in analysis. It effectively gives rise to the irrational rotation C^*-algebras ([**15**]). In fact these are quotient algebras of the C^*-group algebra of G. Its completion $\widetilde{G}$ is the *rational Heisenberg* *group*. Whereas the character theory of G is complicated by subgroups of finite index (in $\mathbb{Z}$ but then also in G), no such subgroups exist in $\widetilde{G}$ and so the character theory of the rational Heisenberg group is in principal much simpler. Of course the dual of $\mathbb{Q}$ is involved in the description of $\mathrm{Ch}(\widetilde{G})$ and this complicates it but this object is just another abelian compact group. Thus to understand the relationship between the "simpler" character theory of the rational Heisenberg group and the more complicated theory of the integer Heisenberg group helps elucidate the latter.

For that G, we shall look at the characters in both $\mathrm{Ch}(G)$ and $\mathrm{Ch}(\widetilde{G})$ and the relation between them. Since $\mathrm{Ch}(G) = \mathrm{Prim}(C^*(G))$ and $\mathrm{Ch}(\widetilde{G}) = \mathrm{Prim}(C^*(\widetilde{G}))$ (see [**3**, **5**]), we also have the same relationship between primitive ideals of $C^*(G)$ and $C^*(\widetilde{G})$ as between the characters of these two groups.

We first consider the following fact.

Fact 6·1. *Let G be a torsion-free nilpotent group with centre of rank* 1.

(i) *If φ is a character of G, then $k(\varphi) = (1_G)$ if and only if $Z(G/k(\varphi)) = Z(G)$.*

(ii) *If $\widetilde{\varphi}$ is a character of $\widetilde{G}$, then either $Z(\widetilde{G}/k(\widetilde{\varphi})) = Z(\widetilde{G})$ or $k(\widetilde{\varphi}) \supseteq Z(\widetilde{G})$.*

Proof. For (i), the part ($\Rightarrow$) is obvious. For the part ($\Leftarrow$), let $k(\varphi) \neq (1_G)$. Choose some $x_0 \in Z^2(G) \setminus Z(G)$. Since $k(\varphi) \cap Z(G)$ is of finite index in $Z(G)$, there exists a positive integer n such that

$$\left[x_0^n, g\right] = [x_0, g]^n \in k(\varphi) \cap Z(G)$$

for all $g \in G$. This implies $x_0^n \in Z(G/k(\varphi))$ for some n. Since G is torsion-free, so is $Z^2(G)/Z(G)$ (see Warfield [**17**, 2.2]), so that $x_0^n \notin Z(G)$. Hence $Z(G/k(\varphi)) \supset Z(G)$. For (ii), suppose that $Z(\widetilde{G}) \subset Z(\widetilde{G}/k(\widetilde{\varphi}))$. As $\widetilde{G}/k(\widetilde{\varphi})$ is nilpotent, we have

$$Z(\widetilde{G}/k(\widetilde{\varphi})) \cap Z^2(\widetilde{G}) \supset Z(\widetilde{G}).$$

Now choose $y_0 \in (Z(\widetilde{G}/k(\widetilde{\varphi})) \cap Z^2(\widetilde{G})) \setminus Z(\widetilde{G})$. Then there exists $g \in \widetilde{G}$ such that $[y_0, g] \neq 1_G$. As $Z(G)$ is of rank 1, let $Z(G) = (z_0)$. Since $[y_0, g] \in Z(\widetilde{G})$, it follows that $[y_0, g]^n \in Z(G)$ for some n, so that $[y_0, g]^n = z_0^k$ for some $k \neq 0$. Writing $g = y^k$ for some $y \in \widetilde{G}$, we have

$$[y_0, y]^{nk} = [y_0, y^k]^n = z_0^k.$$

Since G is torsion-free, $[y_0, y]^n = z_0$. Now let $z \in Z(\widetilde{G})$. Then $z^\ell = z_0^m$ for some ℓ and $m \neq 0$. Writing $y^m = u^\ell$ for some $u \in \widetilde{G}$, we have

$$z^\ell = [y_0, y]^{nm} = [y_0, y^m]^n = [y_0, u^\ell]^n = [y_0, u]^{\ell n}.$$

Thus, $z = [y_0, u]^n = [y_0, u^n]$ as G is torsion-free. This shows that $Z(\widetilde{G}) \subseteq [y_0, \widetilde{G}]$. Since $y_0 \in Z(\widetilde{G}/k(\widetilde{\varphi})$, we have $Z(\widetilde{G}) \subseteq k(\widetilde{\varphi})$.

Fact 6·1 is helpful in reducing the computation of characters of torsion-free nilpotent groups with centre of rank 1. The characters of the discrete Heisenberg group G are well understood ([**2**, **10**]), so are the characters of $\widetilde{G}$ (see [**1**, example 2·4]). With the aid of Fact 6·1 and Proposition 3·3, we can also find the characters of both G and $\widetilde{G}$.

A faithful character φ of G is of the form

$$\varphi(m, n, p) = \begin{cases} e^{2\pi i \gamma_0 p} & \text{if } m = n = 0 \\ 0 & \text{otherwise} \end{cases}$$

where γ_0 is an irrational number mod $\mathbb{Z}$; while a non-faithful character φ of G is of the form

$$\varphi(m,n,p) = \begin{cases} e^{2\pi i\left(\alpha\frac{m}{q}+\beta\frac{n}{q}+\gamma_0 p\right)} & \text{if } m,n\in q\mathbb{Z},\ p\in\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

where α,β,γ_0 are real numbers mod $\mathbb{Z}$, $\gamma_0 = s/q$ for some $s,q\in\mathbb{Z}$ with $(s,q)=1$.

Let $\mathbb{Q}^\frown$ be the dual of the rationals $\mathbb{Q}$. (See [**9**, 25.4], for the dual of $\mathbb{Q}$.) Then a character $\widetilde{\varphi}$ of $\widetilde{G}$ is of the form

$$\widetilde{\varphi}(a,b,c) = \begin{cases} \delta(c) & \text{if } a=b=0 \\ 0 & \text{otherwise} \end{cases}$$

for some $\delta\in\mathbb{Q}^\frown$; or

$$\widetilde{\varphi}(a,b,c) = \delta_1(a)\delta_2(b)$$

for some $\delta_1,\ \delta_2\in\mathbb{Q}^\frown$.

Using Theorem 5·7, we shall now see how characters of G correspond to those of $\widetilde{G}$. We write the following for convenience:

$$\mathrm{Ch}_1(G) = \{(\alpha,\beta,0)_{\mathrm{n}} : \alpha,\beta\in\mathbb{R}/\mathbb{Z}\}$$

and

$$\mathrm{Ch}_2(G) = \{(\alpha,\beta,\gamma)_{\mathrm{n}} : \alpha,\beta\in\mathbb{R}/\mathbb{Z}, \gamma\in\mathbb{Q}/\mathbb{Z}, \gamma\notin\mathbb{Z}\}$$
$$\cup\ \{(0,0,\gamma)_{\mathrm{f}} : \gamma\in(\mathbb{R}/\mathbb{Z})\setminus(\mathbb{Q}/\mathbb{Z})\}.$$

A triple with a subscript 'n' indicates that this is non-faithful, and that with a subscript 'f' indicates that this is faithful. Now let $(\mathbb{Q}^\frown)_1$ be the subset of $\mathbb{Q}^\frown$ consisting the faithful elements. Let $(\mathbb{Q}^\frown)_2$ be the subset of $\mathbb{Q}^\frown$ containing every non-faithful element such that the restriction to $\mathbb{Z}$ is not equal to 1, and $(\mathbb{Q}^\frown)_3$ be the subset of $\mathbb{Q}^\frown$ containing every non-faithful element such that the restriction to $\mathbb{Z}$ is equal to 1. Then we note that

$$\mathrm{Ch}_1(\widetilde{G}) = \mathbb{Q}^\frown\times\mathbb{Q}^\frown\times(\theta),$$
$$\mathrm{Ch}_2(\widetilde{G}) = ((\theta)\times(\theta)\times(\mathbb{Q}^\frown)_1)\cup((\theta)\times(\theta)\times(\mathbb{Q}^\frown)_2),$$

and

$$\mathrm{Ch}_2'(\widetilde{G}) = (\theta)\times(\theta)\times((\mathbb{Q}^\frown)_3\setminus(\theta))$$

where $\mathrm{Ch}_1(\widetilde{G})$, $\mathrm{Ch}_2(\widetilde{G})$, $\mathrm{Ch}_2'(\widetilde{G})$ are notation defined earlier, and θ is the trivial homomorphism in $\mathbb{Q}^\frown$. By Theorem 5·7(i), all characters in $\mathrm{Ch}_1(G)$ correspond in a 1-1 way to all $((\mathbb{Q}/\mathbb{Z})^\frown\times(\mathbb{Q}/\mathbb{Z})^\frown\times(\theta))$-orbits of $\mathrm{Ch}_1(\widetilde{G})$ according to the map

$$(\alpha,\beta,0)_{\mathrm{n}} \longmapsto ((\mathbb{Q}/\mathbb{Z})^\frown\times(\mathbb{Q}/\mathbb{Z})^\frown\times(\theta))\overline{\overline{(\alpha,\beta,0)}}.$$

By Theorem 5·7(ii)(a), $(\mathbb{Z}^\frown\times\mathbb{Z}^\frown\times(\theta))$-orbits of $\mathrm{Ch}_1(G)$ (which is the single orbit $\{\mathrm{Ch}_1(G)\}$) corresponds to $((\theta)\times(\theta)\times(\mathbb{Q}/\mathbb{Z})^\frown)$-orbits of $\{\mathrm{Ch}_1(\widetilde{G})\}\cup\mathrm{Ch}_2'(\widetilde{G})$ (which is the single orbit $\{\{\mathrm{Ch}_1(\widetilde{G})\}\cup\mathrm{Ch}_2'(\widetilde{G})\}$) according to the map

$$(\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}\times(0))_{\mathrm{n}} \longmapsto \{\mathbb{Q}^\frown\times\mathbb{Q}^\frown\times(\theta)\}\cup(\theta)\times(\theta)\times((\mathbb{Q}/\mathbb{Z})^\frown\setminus\{\theta\}).$$

By Theorem 5·7(ii)(b), all $(\mathbb{Z}\hat{}\times\mathbb{Z}\hat{}\times(\theta))$-orbits of $\mathrm{Ch}_2(G)$ correspond in a 1-1 way to all $((\theta)\times(\theta)\times(\mathbb{Q}/\mathbb{Z})\hat{})$-orbits of $\mathrm{Ch}_2(\widetilde{G})$ according to the map

$$(\mathbb{Z}\hat{}\times\mathbb{Z}\hat{}\times(\theta))(\alpha,\beta,\gamma)_{\mathrm{n}} = (\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}\times\{\gamma\})_{\mathrm{n}} \longmapsto (\theta)\times(\theta)\times(\mathbb{Q}/\mathbb{Z})\hat{}\ast s(\overline{(\alpha,\beta,\gamma)})$$

and

$$(\mathbb{Z}\hat{}\times\mathbb{Z}\hat{}\times(\theta))(0,0,\gamma)_{\mathrm{f}} = ((\theta)\times(\theta)\times\{\gamma\})_{\mathrm{f}} \longmapsto (\theta)\times(\theta)\times(\mathbb{Q}/\mathbb{Z})\hat{}\ast s(\overline{(0,0,\gamma)}).$$

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REFERENCES

[**1**] R. J. Archbold and E. Kaniuth. On the primitive ideal space of a finitely generated nilpotent group. *Forum Math.* **9** (1997), 193–210.
[**2**] L. W. Baggett and J. Packer. The primitive ideal space of two-step nilpotent group C^*-algebras. *J. Func. Anal.* **124** (1994), 389–426.
[**3**] L. W. Baggett, E. Kaniuth and W. Moran. Primitive ideal spaces, characters, and Kirillov theory for discrete nilpotent groups. *J. Func. Anal.* **150** (1997), 175–203.
[**4**] G. Baumslag. *Lecture notes on nilpotent groups.* CBSM Regional Conference Series in Math., No. 2 (Conf. Boardmath Sci., Washington, DC, 1971).
[**5**] A. L. Carey and W. Moran. Characters of nilpotent groups. *Math. Proc. Camb. Phil. Soc.* **96** (1984), 123–137.
[**6**] A. L. Carey, W. Moran and C. E. M. Pearce. A Kirillov theory for divisible nilpotent groups. *Math. Ann.* **301** (1995), 119–133.
[**7**] M. M. Day. Amenable semi-groups. *Illinois J. Math.* **1** (1957), 509–544.
[**8**] F. P. Greenleaf. *Invariant Means on Topological Groups and Their Applications* (Van Nostrand, Reinhold Company, 1969).
[**9**] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis. I* (Springer-Verlag, 1963).
[**10**] R. E. Howe. On representations of discrete, finitely generated, torsion-free, nilpotent groups. *Pacific J. Math.* **73** (1977), 281–305.
[**11**] A. G. Kurosh. *The Theory of Groups. II* (Chelsea, 1956).
[**12**] A. I. Mal'cev. Nilpotent torsion-free groups. *Izv. Akad. Nauk SSSR Ser. Mat.* **13** (1949), 201–212 (in Russian).
[**13**] R. R. Phelps. *Lectures on Choquet's theorem* (D. Van Nostrand Company, Inc., 1966).
[**14**] J. S. Pym. The convolution of functionals on spaces of bounded functions. *Proc. London Math. Soc.* (3) **15** (1965), 84–104.
[**15**] M. A. Rieffel. C^*-algebras associated with rational rotations. *Pacific J. Math.* **93** (1981), no. 2, 415–429.
[**16**] E. Thoma. Über unitäre Darstellungen abzählbarer, diskreter Gruppen. *Math. Ann.* (3) **153** (1964), 111–138.
[**17**] R. B. Warfield. Nilpotent groups. Lecture Notes in Mathematics, vol. 513 (Springer-Verlag, 1976).