Characterization of Matrix-exponential Distributions

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Signed Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

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Dedication

This thesis is dedicated to Associate Professor William (Bill) Henderson (1943–2001) who was a truly inspirational applied probabilist.

Abstract

A random variable that is defined as the absorption time of an evanescent finite-state continuous-time Markov chain is said to have a *phase-type* distribution. A phase-type distribution is said to have a *representation* $(\boldsymbol{\alpha}, \boldsymbol{T})$ where $\boldsymbol{\alpha}$ is the initial state probability distribution and $\boldsymbol{T}$ is the infinitesimal generator of the Markov chain. The distribution function of a phase-type distribution can be expressed in terms of this representation. The wider class of *matrix-exponential* distributions have distribution functions of the same form as phase-type distributions, but their representations do not need to have a simple probabilistic interpretation. This class can be equivalently defined as the class of all distributions that have rational Laplace-Stieltjes transform. There exists a one-to-one correspondence between the Laplace-Stieltjes transform of a matrix-exponential distribution and a representation $(\boldsymbol{\beta}, \boldsymbol{S})$ for it where $\boldsymbol{S}$ is a companion matrix.

In order to use matrix-exponential distributions to fit data or approximate probability distributions the following question needs to be answered:

> *"Given a rational Laplace-Stieltjes transform, or a pair* $(\boldsymbol{\beta}, \boldsymbol{S})$ *where* $\boldsymbol{S}$ *is a companion matrix, when do they correspond to a matrix-exponential distribution?"*

In this thesis we address this problem and demonstrate how its solution can be applied to the abovementioned fitting or approximation problem.