Departure processes from *MAP/PH/1* queues

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Signed Statement

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Dedication

This work is dedicated to Elizabeth Joseph, Margaret Edward and Carmel Anthony, three of my ten children who left this world before my wife and I could know them.

Requiem æternam dona eis Domine.

Et lux perpetua luceat eis.

Abstract

A *MAP/PH/1* queue is a queue having a Markov arrival process (*MAP*), and a single server with phase-type (*PH*-type) distributed service time. This thesis considers the departure process from these type of queues. We use matrix analytic methods, the Jordan canonical form of matrices, non-linear filtering and approximation techniques. The departure process of a queue is important in the analysis of networks of queues, as it may be the arrival process to another queue in the network. If a simple description were to exist for a departure process, the analysis of at least feed-forward networks of these queues would then be analytically tractable.

Chapter 1 is an introduction to some of the literature and ideas surrounding the departure process from *MAP/PH/1* queues.

Chapter 2 sets up the basic notation and establishes some results which are used throughout the thesis. It contains a preliminary consideration of *PH*-type distributions, *PH*-renewal processes, *MAP*s, *MAP/PH/1* queues, non-linear filtering and the Jordan canonical form.

Chapter 3 is an expansion of [7], where the question of whether a *MAP* description can exist for the departure process of a non-trivial *MAP/M/1* queue is considered. In a 1994 paper, Olivier and Walrand conjectured that the departure process of a *MAP/PH/1* queue is not a *MAP* unless the queue is a stationary *M/M/1* queue. This conjecture was prompted by their claim that the departure process of an *MMPP/M/1* queue is not *MAP* unless the queue is a stationary *M/M/1* queue. We show that their proof has an algebraic error, which leaves open the above question of whether the departure process of an *MMPP/PH/1* queue is a *MAP* or not.

In Chapter 4, the more fundamental problem of identifying stationary *M/M/1* queues in the class of *MAP/PH/1* queues is considered. It is essential to be able to determine from its generator when a stationary *MAP* is a Poisson process. This does not appear to have been discussed in the literature prior to the author's paper [5], where this deficiency was remedied using ideas from non-linear filtering theory, to give a characterisation as to when a stationary *MAP* is a Poisson process. Chapter 4 expands upon [5]. This investigation of higher order representations of the Poisson process is motivated by first considering when a higher order *PH*-type distribution is just negative exponential.

In Chapter 5, we consider the related question of minimal order representations for *PH*-type distributions, an issue which has attracted much interest in the literature. A discussion of other authors' ideas is given and these ideas are then inter-related to the work presented in Chapter 4 on the *PH*-type distributions.

The *MAP/M/1* queue is then considered in Chapter 6 from the perspective of whether having an exact level and phase independent stationary distribution of the geometric form

$$\boldsymbol{\Psi}_0 = [\boldsymbol{\pi}_0, \eta\boldsymbol{\pi}_0, \eta^2\boldsymbol{\pi}_0, \eta^3\boldsymbol{\pi}_0, \ldots],$$

where $0 < \eta < 1$ is a scalar, implies that the *MAP* is Poisson. The answer is in the affirmative for this question, but the converse is not strictly true. Apart from showing the ubiquitous asymptotic form of level and phase independence exhibited by all stable *MAP/M/1* queues, we prove that a very large class of stable queues, exhibits what we have termed *shift-one level and phase independence.* Stable *MAP/M/1* queues exhibiting shift-one level and phase independence, are characterised by a stationary distribution of the following form:

$$\boldsymbol{\Psi}_0 = [\boldsymbol{\pi}_0, \chi\boldsymbol{\xi}_0, \chi^2\boldsymbol{\xi}_0, \chi^3\boldsymbol{\xi}_0, \ldots],$$

where $0 < \chi < 1$ is a scalar and $\boldsymbol{\xi}_0$ is a positive row vector.

In Chapter 7, a family of approximations is proposed for the output process of a stationary *MAP/PH/1* queue. To check the viability of these approximations, they are used as input to another single server queue. Performance measures for the second server are obtained analytically in both the tandem and approximation cases, thus eliminating the need for simulation to compare results. Comparison of these approximations is also made against other approximation methods in the literature.

In Chapter 8, we show that our approximations from Chapter 7 have the property of exactly matching the inter-departure time distribution. Our k^{th} approximation also accurately captures the first $k-1$ lag-correlation coefficients of the stationary departure process. The proofs of this direct association between lag-correlation coefficients and the level of complexity k are given.

Chapter 1

Introduction

This thesis considers the departure processes from *MAP/PH/1* queues. Departure processes are important in the analysis of networks of such queues, as they may be the arrival process to another queue in the network. Descriptions of the *MAP* and *PH*-type distributions are given in detail in Chapter 2. An introductory description of the *MAP* (or Markov arrival process) is that it is a process which counts transitions of a finite Markov chain. A *PH*-type distribution (or phase-type distribution) is the distribution of the hitting time in a finite-state, continuous, absorbing Markov chain.

There have been many papers dealing with the output process of a single queue, for example [11], [12], [17] and [18]. One of the earlier notable results on the departure process from a single server queue was claimed in 1955 by Morse [29], where he stated that

> "A little thought will convince one that the efflux from a single-channel, exponential service channel, fed by Poisson arrivals, must be Poisson with the same rate as the arrivals."

The first proof of this was given by Burke [11] in 1956, and later by Reich [45] in 1957 by a different method. Essentially, Burke showed that under stationary conditions, the queue size at a departure epoch and the time elapsing until the next departure epoch are independent random variables. Reich on the other hand, showed that the Markov chain X, for any birth and death process (which includes all *M/M/s* systems) is reversible, and therefore that $X(t)$ and $X(-t)$, for $t \in \mathbb{R}$, have the same joint distribution. The deaths in X correspond to the births in the reversed process, thus implying that they must both be the epochs of a Poisson process. This result simplified much of the earlier analysis for tandems, such as that published in 1954 by R.R.P. Jackson [24]. It was noted in [24] that the stationary distribution of the number of customers in each queue of two negative exponential servers in series, fed with a Poisson process

> "...settles down in apparent independence."

Similarly, in 1954 O'Brien [36] stated that for two queues (gates) in series having Poisson arrivals and a negative exponential first server, that

> "The arrival of customers at gate 2 will be random and the average arrival rate will be the same as that for gate 1."

Therefore, some years before Burke's proof in 1956, the departure process from an *M/M/1* queue was observed to be Poisson.

The *M/M/1* queue is the simplest form of the *MAP/PH/1* queue, and the departure process has a simple finite description. A natural question which arises is whether a similar property holds for other queues. That is, for more general queues, does there exist a finite state Markov chain and a set of transitions for which the counting process of the transitions is identical to the departure process of the original queue. If a simple description were to exist for the departure process, the analysis of at least feed-forward networks of these queues would then be analytically tractable.

Olivier and Walrand [42] presented an argument to show that no such finite state chain exists for an *MMPP/M/1* queue (a special case of the *MAP/M/1* queue, detailed in Chapter 2) which is not an *M/M/1* queue and conjectured that this is also true for a *MAP/M/1* queue. Upon investigation, we found that their proof had an algebraic error (subsequently reported in [7]), leaving open the question of whether the departure process of an *MMPP/PH/1* queue can be a *MAP*.

There is also a more fundamental problem with Olivier and Walrand's proof. Since it is possible for the arrival process of a *MAP/M/1* queue to be Poisson, but with a possibly complicated description, and since we know that the output of such a queue *is* a *MAP* (as mentioned above, it is Poisson), it is essential to be able to tell from its generator when a *MAP* is, in fact, Poisson. This was not discussed in Olivier and Walrand [42], nor does it appear to have been discussed elsewhere in the literature. This deficiency is remedied here, and in [5], using ideas from non-linear filtering theory, to give a characterisation as to when a stationary *MAP* is a Poisson process. As a preliminary result, we will discuss how to recognise a higher order representation of a negative exponential distribution. This is closely allied to the non-trivial questions of determination of minimal order and non-uniqueness of representations for PH-type distributions. These two problems have been considered in many publications, such as [15], [16], [31], [37], [38], [39], [40], [41] and [47]. These papers are discussed, and some of our results are re-framed using ideas from these authors.

A stable *M/M/1* queue has a geometric stationary distribution $\boldsymbol{\pi}$ involving a scalar $0 < \rho = \left(\frac{\lambda}{\mu}\right) < 1$, where λ is the rate of the Poisson arrival process and μ is the negative exponential service rate. The stationary distribution is given by $\boldsymbol{\pi} = \pi_0(1, \rho, \rho^2, \rho^3, \ldots)$, where $0 < \pi_0 < 1$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$. For *MAP/M/1* queues, the stationary distribution $\boldsymbol{\Psi}$ is matrix geometric of the following form (see [31])

$$\boldsymbol{\Psi} = \boldsymbol{\pi}_0[I, R, R^2, R^3, \ldots],$$

for some non-negative matrix R with $sp(R) < 1$. We suppose that the level is exactly independent of phase, a property which has been considered in the more general setting of the quasi birth and death (*QBD*) process by Ramaswami and Taylor [44], and Latouche and Taylor [26]. Exact level and phase independence implies that the stationary distribution for the *MAP/M/1* queue may be written in the form

$$\boldsymbol{\Psi} = [\boldsymbol{\pi}_0, \eta\boldsymbol{\pi}_0, \eta^2\boldsymbol{\pi}_0, \eta^3\boldsymbol{\pi}_0, \ldots],$$

for a scalar $0 < \eta < 1$. We will show that exact level and phase independence in a *MAP/M/1* queue implies that the stationary *MAP* is Poisson. However, it is then shown that not all *MAP/M/1* queues, where the stationary *MAP* is in fact Poisson, have an exact level and phase independent stationary distribution. We further prove that a large class of *MAP/M/1* queues exhibits what we have termed *shift-one level and phase independence*. Shift-one level and phase independence has the following form,

$$\boldsymbol{\Psi} = [\boldsymbol{\pi}_0, \chi\boldsymbol{\xi}_0, \chi^2\boldsymbol{\xi}_0, \chi^3\boldsymbol{\xi}_0, \ldots],$$

for some scalar $0 < \chi < 1$ and positive row vector $\boldsymbol{\xi}_0$.

There exists an exact finite *MAP* description of the departure process of the *MAP/PH/1/k* queue. The *MAP* description of this queue has dimension mnk, where m and n are the respective dimensions of the matrix parameters for the arrival process and the service distribution. Although this description is finite, it can become unwieldy when used as a tool for the analysis of networks of such queues. Approximate techniques which reduce the size of these representations therefore become necessary. An extensive list of references for methods of analysis of various tandem queues with a finite intermediate buffer was given in [32].

In the infinite buffer case, no such exact finite description has been found. In fact, if Olivier and Walrand's claim turns out to be correct, there does not exist a finite *MAP* description for the output process of a stationary *MAP/PH/1* queue in which the *MAP* is not a Poisson process. We will develop a family of *MAP* approximations to the departure process of the *MAP/PH/1* queue. To check the viability of these approximations, they have been used as input to another single server queue, and the second queue length distributions have been compared with their "exact" counterparts, calculated using matrix-analytic techniques, thus avoiding simulation.

Other techniques to approximate point processes have been given in, for example, [1], [2], [9], [20], [49], [51] and [52]. Some comparison is made with previous work in the literature for further validation of the approximations. In particular, the published results of Whitt [52] are directly compared to results obtained for the same queues by our methods. The structure of the defining processes is exploited in our approximations, and as a direct result, all of our approximations yield the exact output process for the trivial situation of Poisson arrivals to a negative-exponential first server.

We will prove that the stationary inter-departure time distribution for the *MAP/PH/1* queue is identical to the stationary inter-event time distribution for our *MAP* approximations to the departure process of the *MAP/PH/1* queue. The family of approximations is indexed by a parameter $k \in \{1, 2, \ldots\}$, and for $k = 1$, the *MAP* approximation is a *PH*-renewal process. *PH*-renewal processes are non-correlated and hence this approximation contains no correlation information about the departures from the *MAP/PH/1* queue. For $k \geq 2$, the approximations are not *PH*-renewal processes, and we prove in Chapter 8 that the lag-correlation coefficients $c_1(k), \ldots, c_{k-1}(k)$ for the stationary inter-event times of the k^{th} approximation are identical to the lag-correlation coefficients $c_1, \ldots, c_{k-1}$ for the stationary inter-departure times of the *MAP/PH/1* queue.

Chapter 2

Preliminaries

In this chapter, notation is defined and some results are given for the common structures and ideas which are used throughout the thesis.

2.1 MAPs

A Markovian arrival process (or *(MAP)*), is a process which counts transitions of a *finite* state Markov chain. Consider a Markov chain, $X = \{x_t, t \geq 0\}$, having a *finite* state space $\boldsymbol{X}$, with conservative transition rate matrix D.

For $i \neq j \in \boldsymbol{X}$, let

$$[D_1]_{i,j} \leq [D]_{i,j}, \tag{2.1.1}$$

for $i \in \boldsymbol{X}$, let

$$0 \leq [D_1]_{i,i} < \infty, \tag{2.1.2}$$

and let

$$D_0 = D - D_1 \ . \tag{2.1.3}$$

If we assume that the Markov chain with transition rate matrix D is irreducible, then this Markov chain has a unique stationary distribution $\boldsymbol{\nu}$ such that $\boldsymbol{\nu} D = \mathbf{0}$.

The transitions with rates D_1 are *observed*, while those in D_0 are *hidden*. The process $J(t)$, which counts the *observed* transitions of this Markov chain, is a *MAP* of stationary arrival rate $\lambda = \boldsymbol{\nu} D_1 \boldsymbol{e}$, where $\boldsymbol{e}$ is a column of ones of the appropriate dimension. This definition for $\boldsymbol{e}$ will be used throughout this thesis, with subscripts used to denote specific column lengths only when ambiguities could arise. The simplest *MAP* is the Poisson process, with $D_1 = \lambda$ and $D_0 = -\lambda$, where $\lambda > 0$ is the rate of the Poisson process. The term *MAP* was originally coined in Lucantoni, Meier-Hellstern and Neuts [19]. For an excellent discussion on the *MAP* including examples, see Lucantoni [27].

It is possible for a process which counts transitions of an infinite state Markov chain to be statistically equivalent to a *MAP*. For example, let us consider a queue with Poisson arrivals of rate $\lambda > 0$ and a single server having negative exponentially distributed holding times with mean $\frac{1}{\mu} > 0$. This is known as an *M/M/1* queue, which we model by a Markov chain $X = \{x_t, t \geq 0\}$, on the state space $\mathbb{Z}^+$, where x_t represents the number of customers in the queue at time t. This Markov chain has the following infinite transition rate matrix

$$\begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & \dots \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \dots \\ 0 & \mu & -(\lambda+\mu) & \lambda & 0 & \dots \\ \vdots & & \ddots & \ddots & \ddots & \end{bmatrix}, \tag{2.1.4}$$

where the number of customers in the queue, increases with the row number of the matrix. In the situation where $\lambda < \mu$, the queue is positive recurrent, and under stationary conditions, the point process of occurrence of transitions $(n+1, n), n \geq 0$, is known (see Burke [11]) to be a Poisson process of rate λ, which is, of course, a trivial *MAP*.

2.2 *MAP/M/1* queues

The arrival process can be generalised for the transition rate matrix in (2.1.4) by relaxing the requirement that the inter-arrival times be negative exponentially distributed (see [27]). This is achieved by adding auxiliary states or phases to the arrival process, associating certain phase changes with each arrival and allowing phase changes to occur without arrivals. The Markovian simplicity is still preserved since the sojourn times within phases of the auxiliary process are still negative exponentially distributed. We define a two state Markov chain (x_t, y_t), where x_t represents the number of customers in the queue at time t and y_t represents the phase of the arrival process at time t. Let D_0 and D_1 be as in equations (2.1.1), (2.1.2) and (2.1.3), with dimension equal to the number of auxiliary states or phases m. Following the notation of Neuts [31], we get the following block matrix form for the conservative rate matrix of the *MAP/M/1* queue:

$$Q = \begin{bmatrix} D_0 & D_1 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix} . \tag{2.2.1}$$

When the queue is empty, the matrix D_0 governs the transitions of the arrival process which do not correspond to an arrival and the matrix D_1 governs those transitions which do. Let I_m be the identity matrix of dimension m. When the queue is occupied, the matrix $A_2 = \mu I_m$ governs departures, $A_0 = D_1$ governs arrivals and $A_1 = D_0 - A_2$ governs those transitions which do not correspond to an arrival or a departure. Note that

$$D = D_1 + D_0 = A_0 + A_1 + A_2 \tag{2.2.2}$$

is the conservative rate matrix of the process which governs changes in phase, so that

$$D\boldsymbol{e} = 0. \tag{2.2.3}$$

As in Neuts [31], we assume that Q defines an irreducible, regular Markov chain. Necessary conditions for this are that the matrices D_0 and A_1 are non-singular. Hence this Markov chain has at most one stationary distribution $\mathbf{\Psi}$ such that $\mathbf{\Psi}Q = 0$. This stationary distribution has a *matrix-geometric* form and is given by

$$\mathbf{\Psi} = [\boldsymbol{\pi}_0, \boldsymbol{\pi}_0 R, \boldsymbol{\pi}_0 R^2, \boldsymbol{\pi}_0 R^3, \ldots] , \tag{2.2.4}$$

where R is the minimal non-negative solution to the matrix quadratic equation

$$R^2 A_2 + R A_1 + A_0 = 0, \tag{2.2.5}$$

and $\boldsymbol{\pi}_0$ is the unique positive solution to the system of equations

$$\boldsymbol{\pi}_0(D_0 + RA_2) = \mathbf{0} \quad \text{and} \quad \boldsymbol{\pi}_0(I - R)^{-1}\boldsymbol{e} = 1 . \tag{2.2.6}$$

One special case of a *MAP* is the Markov-modulated Poisson process (*MMPP*), characterised by a matrix D_1 which is diagonal. This process is essentially a Poisson process in which the Poisson parameter is itself a random variable dependent on a Markov process. The Poisson parameters for a given *MMPP* are the diagonal entries of the matrix D_1.

2.3 *PH*-random variables and *PH*-renewal processes

Any distribution on $[0, \infty)$ which can be obtained as the distribution of time until absorption in a continuous-time, finite-space Markov chain which has a single absorbing state into which absorption is certain, is said to be of *PH*-type.

Consider a Markov chain with $n + 1$ states, initial probability vector $(\boldsymbol{\alpha}, \alpha_{n+1})$ and transition rate matrix

$$Q = \begin{bmatrix} S & \boldsymbol{S}^0 \\ \mathbf{0} & 0 \end{bmatrix},$$

where S is a non-singular $n \times n$ matrix with $S_{ii} < 0$, $S_{ij} \geq 0$ for all $i \neq j$, and $\boldsymbol{S}^0 \geq 0$ is an n-vector of rates such that $S\boldsymbol{e} + \boldsymbol{S}^0 = \mathbf{0}$. The conditional probabilities $r_j(t)$ that the process is in state j at time t, with initial conditions $(r_1(0), r_2(0), \ldots, r_n(0)) = \boldsymbol{r}(0) = (\boldsymbol{\alpha}, \alpha_{n+1})$, satisfy the differential equations

$$\frac{d\boldsymbol{r}(t)}{dt} = \boldsymbol{r}(t)Q,$$

which have solution

$$\boldsymbol{r}(t) = (\boldsymbol{\alpha}, \alpha_{n+1})e^{Qt}.$$

The conditional probability vector $\boldsymbol{v}(t)$ that the process is still in one of the states $1, \ldots, n$ at time t is given by

$$\boldsymbol{v}(t) = \boldsymbol{\alpha}e^{St}.$$

Thus

$$F(t) = 1 - \boldsymbol{\alpha}e^{St}\boldsymbol{e}$$

is the probability distribution of time until absorption into state $n + 1$. This is classified as a *PH*-type distribution with representation $(\boldsymbol{\alpha}, S)$.

If the $(n + 1)^{st}$ state of the *PH*-random variable is considered as an instantaneous state, in that the process is instantaneously restarted using the probability vector $\boldsymbol{\alpha}$, then the process consisting of absorption epochs is a *PH*-renewal process with representation $(\boldsymbol{\alpha}, S)$.

For a thorough treatment of *PH*-random variables and *PH*-renewal processes, see Chapter 2 of Neuts [31].

2.4 Filtering

One of the more common uses of filtering formulae is calculating the stochastic intensity of a point process, which in fact determines the law of the point process. (See Chapter 10 of Walrand [50].) The technique of filtering or extracting information from point processes also leads to methods whereby two point processes may be compared. These will now be outlined by first defining the filtration matrices Q_0 and Q_1. Consider a Markov chain, $X = \{x_t, t \geq 0\}$, having a countably *infinite* state space $\boldsymbol{X}$, with transition rate matrix Q.

For $i \neq j \in \boldsymbol{X}$, let

$$[Q_1]_{i,j} \leq [Q]_{i,j}, \tag{2.4.1}$$

for $i \in \boldsymbol{X}$ let

$$0 \leq [Q_1]_{i,i} < \infty, \tag{2.4.2}$$

and let

$$Q_0 = Q - Q_1. \tag{2.4.3}$$

The transitions with rates Q_1 are *observed*, while those in Q_0 are *hidden*.

Note that the specifications of the above filtration matrices are very similar to the specifications given for the matrices D_0 and D_1 of the *MAP*. The essential difference is that the matrices Q, Q_0 and Q_1 are not required to be finite. The matrices D_0 and D_1, however, may also be viewed as a filtration of the matrix D.

Let $J(t)$ count the number of observed transitions up to time t, and let

$$\Psi(t, k) = P\{x_t = k | J(t)\},$$

so that $\Psi(t, k)$ is the probability of being in state k at time t, conditioned by the observed process up to time t. Also let $\boldsymbol{\Psi}_t$ be the row vector

$$\boldsymbol{\Psi}_t = \{\Psi(t, k), k \in \boldsymbol{X}\}.$$

The process $J(t)$ is a *MAP* if and only if there exists another counting process $J(t)^*$, defined on a *finite* state Markov chain with rate matrix Q^* and corresponding filtration matrices ${Q_0}^*$ and ${Q_1}^*$ such that $Q^* = {Q_0}^* + {Q_1}^*$, with

$$\boldsymbol{\Psi}_t Q_1 \boldsymbol{e} = \boldsymbol{\Psi}_t^* {Q_1}^* \boldsymbol{e}, \text{ for all } t \in \mathbb{R}^+. \tag{2.4.4}$$

Let $\boldsymbol{\pi}_0$ be the initial distribution of the Markov chain X with transition matrix $Q = Q_0 + Q_1$. Let T_n denote the jump times of the process $J(t)$, and so by Theorem 10.2.2 of [50],

$$\boldsymbol{\Psi}(t) = \frac{\boldsymbol{v}_t}{\boldsymbol{v}_t \boldsymbol{e}},$$

where $\boldsymbol{v}_t$ is determined by

$$\begin{cases} \boldsymbol{v}_0 & = & \boldsymbol{\pi}_0, \\ \frac{d}{dt}\boldsymbol{v}_t & = & \boldsymbol{v}_t Q_0, t \neq T_n \text{ for } n \geq 1, \\ \boldsymbol{v}_t & = & \boldsymbol{v}_{t-} Q_1, t = T_n \text{ for some } n \geq 1. \end{cases}$$

Using this, and (2.4.4), Theorem 10.2.13 of [50] states that two processes, $J(t)$ and $J(t)^*$, have the same finite dimensional distributions if and only if, for any given initial distributions of states $\boldsymbol{\pi}_0$ and $\boldsymbol{\pi}_0^*$ respectively,

$$\frac{\boldsymbol{\pi}_0 e^{Q_0 t_1} Q_1 e^{Q_0 t_2} \ldots Q_1 e^{Q_0 t_k} Q_1 \boldsymbol{e}}{\boldsymbol{\pi}_0 e^{Q_0 t_1} Q_1 e^{Q_0 t_2} \ldots Q_1 e^{Q_0 t_k} \boldsymbol{e}} = \frac{\boldsymbol{\pi}_0^* e^{{Q_0}^* t_1} {Q_1}^* e^{{Q_0}^* t_2} \ldots {Q_1}^* e^{{Q_0}^* t_k} {Q_1}^* \boldsymbol{e}}{\boldsymbol{\pi}_0^* e^{{Q_0}^* t_1} {Q_1}^* e^{{Q_0}^* t_2} \ldots {Q_1}^* e^{{Q_0}^* t_k} \boldsymbol{e}},$$

$$\text{for all } k \geq 1, \text{ and } t_i \in [0, \infty) \text{ for } 1 \leq i \leq k.$$

Note that this last expression can be used to compare two point processes of any state space dimension, including those with a countably infinite state space.

2.5 Jordan canonical form

For any $m \times m$ matrix A, there exists a non-singular $m \times m$ matrix T which transforms A into an $m \times m$ matrix J, known as the Jordan-canonical form. See for example page 152 of Gantmacher [21], or Noble [35], from which the following text on the

Jordan-canonical form is adapted. The transformation is defined by

$$T^{-1}AT = J \quad = \begin{bmatrix} J_1 & 0 & \dots & \dots & 0 \\ 0 & J_2 & 0 & \dots & 0 \\ \vdots & \ddots & J_3 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & J_g \end{bmatrix}, \tag{2.5.1}$$

where each Jordan block J_i is given by

$$J_i \quad = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \lambda_i & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & \lambda_i \end{bmatrix}$$

and the λ_i are eigenvalues of A.

The same eigenvalue can occur in different Jordan blocks, with the number of Jordan blocks, g, being determined by the number of independent eigenvectors. For any given eigenvalue, λ_r, there is a collection of Jordan blocks containing λ_r, and the order of the largest of these Jordan blocks is given by the index of the nilpotent matrix $A - \lambda_r I$. The index of a nilpotent matrix, N, is the value p such that $N^p = 0$ and $N^{p-1} \neq 0$, which in the above context is always less than or equal to the algebraic multiplicity of the eigenvalue λ_r.

For example, if an 8×8 matrix A has 3 distinct eigenvalues, γ, β and δ, with respective algebraic multiplicities $1, 4$ and 3, with the respective number of corresponding independent eigenvectors (geometric multiplicity) being $1, 2$ and 2, and if the index of $A - \beta I$ is 3, then we may construct T using the following 5 Jordan blocks.

$$J_1 = [\gamma], J_2 = [\beta], J_3 = \begin{bmatrix} \beta & 1 & 0 \\ 0 & \beta & 1 \\ 0 & 0 & \beta \end{bmatrix}, J_4 = [\delta], J_5 = \begin{bmatrix} \delta & 1 \\ 0 & \delta \end{bmatrix},$$

to give

$$J = \begin{bmatrix} \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta \end{bmatrix}. \tag{2.5.2}$$

If we pre-multiply equation (2.5.1) by the transformation matrix T, we get $AT = TJ$. This relationship enables the calculation of the columns $\boldsymbol{\tau}_i$ of the transformation matrix T and hence makes possible the calculation of the rows $\bar{\boldsymbol{\tau}}_i$ of T^{-1}. From our example we can see, by using (2.5.2), that we get

$$\begin{aligned} A\boldsymbol{\tau}_1 &= \gamma\boldsymbol{\tau}_1 \\ A\boldsymbol{\tau}_2 &= \beta\boldsymbol{\tau}_2 \\ A\boldsymbol{\tau}_3 &= \beta\boldsymbol{\tau}_3 \\ A\boldsymbol{\tau}_4 &= \beta\boldsymbol{\tau}_4 + \boldsymbol{\tau}_3 \\ A\boldsymbol{\tau}_5 &= \beta\boldsymbol{\tau}_5 + \boldsymbol{\tau}_4 \\ A\boldsymbol{\tau}_6 &= \delta\boldsymbol{\tau}_6 \\ A\boldsymbol{\tau}_7 &= \delta\boldsymbol{\tau}_7 \\ A\boldsymbol{\tau}_8 &= \delta\boldsymbol{\tau}_8 + \boldsymbol{\tau}_7. \end{aligned} \tag{2.5.3}$$

We note that in our example, $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}_3, \boldsymbol{\tau}_6$ and $\boldsymbol{\tau}_7$ are right eigenvectors of the matrix A, with $\boldsymbol{\tau}_1$ corresponding to γ, $\boldsymbol{\tau}_2$ and $\boldsymbol{\tau}_3$ corresponding to β, and $\boldsymbol{\tau}_6$ and $\boldsymbol{\tau}_7$ corresponding to δ. The columns $\boldsymbol{\tau}_4, \boldsymbol{\tau}_5$ and $\boldsymbol{\tau}_8$ are known as generalised right eigenvectors which can be calculated from (2.5.3).

In our example, we have two right eigenvectors which we initially label $\boldsymbol{\tau}_{\beta_1}$ and $\boldsymbol{\tau}_{\beta_2}$, corresponding to the eigenvalue β. It is not clear which of these correspond to $\boldsymbol{\tau}_3$ in the Jordan block J_3. To ascertain this and to find the generalised right eigenvector $\boldsymbol{\tau}_4$, from (2.5.3), we look at

$$(A-\beta I)\boldsymbol{\tau}_4=\boldsymbol{\tau}_{\beta_1} \text{ and } (A-\beta I)\boldsymbol{\tau}_4=\boldsymbol{\tau}_{\beta_2},$$

only one of which will yield a solution for $\boldsymbol{\tau}_4$. Then to find $\boldsymbol{\tau}_5$, we can solve

$$(A-\beta I)\boldsymbol{\tau}_5=\boldsymbol{\tau}_4.$$

In a similar way for eigenvalue δ, we can find the appropriate right eigenvector which corresponds to $\boldsymbol{\tau}_7$ and the generalised right eigenvector $\boldsymbol{\tau}_8$. Note also that in this example, the rows $\bar{\boldsymbol{\tau}}_1, \bar{\boldsymbol{\tau}}_2, \bar{\boldsymbol{\tau}}_5, \bar{\boldsymbol{\tau}}_6$ and $\bar{\boldsymbol{\tau}}_8$ of T^{-1} are left eigenvectors of the matrix A, with $\bar{\boldsymbol{\tau}}_1$ corresponding to γ, $\bar{\boldsymbol{\tau}}_2$ and $\bar{\boldsymbol{\tau}}_5$ corresponding to β, and $\bar{\boldsymbol{\tau}}_6$ and $\bar{\boldsymbol{\tau}}_8$ corresponding to δ.

The Jordan canonical form for a matrix A can be written in the following way,

$$A=T\left(\sum_{j=1}^{g}\lambda_j E_j+N_j\right)T^{-1}, \tag{2.5.4}$$

where the E_j are idempotent matrices, and the N_j are nilpotent matrices which give the form for each Jordan block. Thus, in the previous example where $\lambda_3=\beta$, we have

$$E_3=\begin{bmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix} \text{ and } N_3=\begin{bmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}.$$

Let p_r be the order of the r^{th} Jordan block J_r of the matrix A, with eigenvalue λ_r. From Chapter 5 of Gantmacher [21], we see that the function $f(A)$ of a matrix A is defined if and only if the function $f(\lambda_r)$, and its first p_r-1 derivatives, are defined for all $r \in (1, 2, \ldots, g)$. Then, $f(A) = Tf(J)T^{-1}$, and the function $f(J)$ can be evaluated as

$$\begin{bmatrix} f(\lambda_1) & & & & & & & & & \\ & f(\lambda_2) & \frac{f'(\lambda_2)}{1!} & \ldots & \frac{f^{(p_2-1)}(\lambda_2)}{(p_2-1)!} & & & & & \\ & 0 & \ddots & \ddots & \vdots & & & & & \\ & \vdots & & f(\lambda_2) & \frac{f'(\lambda_2)}{1!} & & & & & \\ & 0 & \ldots & 0 & f(\lambda_2) & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & f(\lambda_g) & \frac{f'(\lambda_g)}{1!} & \ldots & \frac{f^{(p_g-1)}(\lambda_g)}{(p_g-1)!} \\ & & & & & & 0 & \ddots & \ddots & \vdots \\ & & & & & & \vdots & & f(\lambda_g) & \frac{f'(\lambda_g)}{1!} \\ & & & & & & 0 & \ldots & 0 & f(\lambda_g) \end{bmatrix},$$

where all elements in the non-diagonal blocks are zero. In other words,

$$f(A) = T\left(\sum_{j=1}^{g}\sum_{v=1}^{p_j} \frac{f^{(v-1)}(\lambda_j)}{(v-1)!} E_j N_j^{v-1} E_j\right) T^{-1}.$$

The exponential e^{At} can therefore be written as

$$\begin{aligned} e^{At} &= T\left(\sum_{j=1}^{g}\sum_{v=1}^{p_j} \left(\frac{t^{v-1}}{(v-1)!}\right) e^{\lambda_j t} E_j N_j^{v-1} E_j\right) T^{-1} \\ &= \sum_{j=1}^{g}\sum_{v=1}^{p_j} \left(\frac{t^{v-1}}{(v-1)!}\right) e^{\lambda_j t} A_{j,v} \quad , \end{aligned} \tag{2.5.5}$$

where[1] $A_{j,v} = TE_j N_j^{v-1} E_j T^{-1}$ and $N_j^0 \equiv$ the identity matrix. The inverse A^{-1}, if it exists, can also be written as

$$A^{-1} = T\left(\sum_{j=1}^{g}\sum_{v=1}^{p_j} \left(\frac{1}{\lambda_j}\right)\left(\frac{-1}{\lambda_j}\right)^{v-1} E_j N_j^{v-1} E_j\right) T^{-1}.$$

2.5.1 Special diagonalisable case

If the $m \times m$ matrix A has m independent eigenvectors then the Jordan canonical form reduces to a simple diagonal form $D = P^{-1}AP$, where D is a diagonal

[1] A nice analogy, since TENET in Latin means "he holds", synonymous with the matrix $A_{j,v}$ holding the information with respect to the matrix exponential.

matrix of the eigenvalues of A and the matrix P is constructed of the eigenvectors corresponding to each of the eigenvalues. Thus we can write

$$A = \sum_{j=1}^{m} \lambda_j \boldsymbol{r}_j \boldsymbol{l}_j,$$

where $\boldsymbol{r}_j$ and $\boldsymbol{l}_j$ are right and left eigenvectors respectively which correspond to each eigenvalue λ_j for $j \in [1, \ldots, m]$. The inverse A^{-1}, if it exists, and the exponential e^{At} also have simple forms written as

$$A^{-1} = \sum_{j=1}^{m} \frac{1}{\lambda_j} \boldsymbol{r}_j \boldsymbol{l}_j \quad \text{and} \quad e^{At} = \sum_{j=1}^{m} e^{\lambda_j t} \boldsymbol{r}_j \boldsymbol{l}_j.$$

Chapter 3

MAP output from a *MAP/M/1* queue?

3.1 Introduction

Recall that a Markovian arrival process or *MAP*, is a process which counts transitions of a *finite* state Markov chain. An example of an infinite state Markov chain which can generate a point process equivalent to a *MAP* , is the *M/M/1* queue, described in Chapter 2. If the rate of Poisson arrivals to this queue is $\lambda > 0$ and the average holding time is $\frac{1}{\mu} > 0$, then for $\lambda < \mu$, the queue is positive recurrent and, under stationary conditions, the point process of occurrence of transitions $(n+1, n)$, for $n \geq 0$, is known (see Burke [11]) to be a Poisson process of rate λ, which is of course a trivial *MAP*. A question which arises is whether, for more general queues, there exists a finite state Markov chain and a set of transitions for which the counting process of the observed transitions is statistically identical to the departure process of the original queue.

In a 1994 paper [42], Olivier and Walrand presented an argument to show that there exists no such finite state chain for an *MMPP/M/1* queue and conjectured that this is also true for a *MAP/M/1* queue. Unfortunately there is an algebraic error in the argument of Olivier and Walrand. This error is explained here (see also [7]). We also show that equation (3.2.3), which Olivier and Walrand claimed did not have a solution, does, in fact, have a solution for a large class of *MAP*s. Some numerical examples are given.

3.2 A necessary condition

Let Q be the transition rate matrix which represents a *MAP/M/1* queue, given in equation (2.2.1). Partition Q into Q_0 and Q_1 as follows,

$$Q_0 = \begin{bmatrix} D_0 & D_1 & 0 & \cdots \\ 0 & A_1 & A_0 & 0 \\ \vdots & & \ddots & \ddots \\ & & & \end{bmatrix} \quad \text{and} \quad Q_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ A_2 & 0 & 0 & \\ 0 & A_2 & 0 & \\ \vdots & & \ddots & \ddots \end{bmatrix}, \tag{3.2.1}$$

where $A_0 = D_1$, $A_2 = \mu I_m$ and $A_1 = D_0 - A_2$. The matrices Q_0 and Q_1 are filtration matrices, since they satisfy equations (2.4.1), (2.4.2) and (2.4.3). Let $J(t)$ be the process which counts the observed transitions with rates Q_1 corresponding to departures from the *MAP/M/1* queue. From Chapter 2, the process $J(t)$ is a *MAP*, if and only if there exists another counting process $J(t)^*$ defined on a *finite* state Markov chain with rate matrix Q^* and corresponding filtration matrices Q_0^* and Q_1^* such that $Q^* = Q_0^* + Q_1^*$ and

$$\boldsymbol{\Psi}(t) Q_1 \boldsymbol{e} = \boldsymbol{\Psi}(t)^* Q_1^* \boldsymbol{e}, \text{ for all } t \in \mathbb{R}^+. \tag{3.2.2}$$

Olivier and Walrand [42] considered the special case of an *MMPP/M/1* queue where the departures from the queue are observed. A necessary (and sufficient) condition for (3.2.2) to hold is given in Olivier and Walrand [42] to be

$$\boldsymbol{\Psi}_t (Q_1)^y (Q_0)^x \boldsymbol{e} = \boldsymbol{\Psi}_t^* (Q_1^*)^y (Q_0^*)^x \boldsymbol{e}, \text{ for all } x, y \geq 0. \tag{3.2.3}$$

Let the *MMPP* have the $m \times m$ descriptive matrices D_0 and D_1, as described in (2.1.1), (2.1.2) and (2.1.3), with D_1 diagonal, and let μ be the service rate. Using the known stationary distribution $\boldsymbol{\Psi}$ for the *MMPP/M/1* queue, Olivier and Walrand proceeded to obtain "the explicit computation of the terms $\boldsymbol{\Psi}(Q_1)^y(Q_0)^x\boldsymbol{e}$." Olivier and Walrand let

$$\alpha(y,x) = \boldsymbol{\Psi}(Q_1)^y(Q_0)^x\boldsymbol{e}, \tag{3.2.4}$$

and then considered the difference

$$\frac{\alpha(y,x+1)}{\mu^y(-\mu)^{x+1}} - \frac{\alpha(y,x)}{\mu^y(-\mu)^x}, \tag{3.2.5}$$

which they showed was equal to

$$-\boldsymbol{\pi}_0 R^y \left(\frac{-A_0+D}{-\mu}\right)^x \boldsymbol{e},$$

where R and $\boldsymbol{\pi}_0$ are defined by equations (2.2.5) and (2.2.6). Summing over all y they derived the following expression:

$$\sum_{y=0}^{\infty}\left(\frac{\alpha(y,x+1)}{\mu^y(-\mu)^{x+1}} - \frac{\alpha(y,x)}{\mu^y(-\mu)^x}\right) = -\boldsymbol{\pi}_0(I-R)^{-1}\left(\frac{-A_0+D}{-\mu}\right)^x \boldsymbol{e}, \quad \text{for all } x \geq 0. \tag{3.2.6}$$

Recall that $D\boldsymbol{e} = 0$ and observe that $\boldsymbol{\nu} = \boldsymbol{\pi}_0(I-R)^{-1}$ is the stationary distribution admitted by D, so that $\boldsymbol{\nu}D = 0$. Olivier and Walrand erroneously concluded in their Equation (18), that the right hand side of (3.2.6) is equal to

$$\boldsymbol{\nu}\left(\frac{A_0}{\mu}\right)^x \boldsymbol{e}, \text{ for all } x \geq 0. \tag{3.2.7}$$

From this point they argued by contradiction that no finite-state equivalent Markov chain Q^* could exist, unless the *MMPP* is Poisson.

However, equation (3.2.7) is incorrect. For example, a value of $x = 3$ yields

$$\boldsymbol{\nu}\left(\frac{-A_0+D}{-\mu}\right)^3 \boldsymbol{e} = \boldsymbol{\nu}\left(\left(\frac{A_0}{\mu}\right)^3 + \left(\frac{A_0DA_0}{\mu^3}\right)\right)\boldsymbol{e} \neq \boldsymbol{\nu}\left(\frac{A_0}{\mu}\right)^3 \boldsymbol{e}\ .$$

3.3 The necessary condition satisfied

We shall now construct the equivalent form for the finite-state Markov chain Q^* of (3.2.6) and show that this equivalence can be satisfied for an *MMPP/M/1* queue. We shall then show that this equivalence can also be satisfied for another very large class of *MAP/M/1* queues.

Let $\mathbf{\Psi}^*$ be the stationary distribution of the conservative generator Q^*. We can construct the equivalent expression for (3.2.6) involving Q^*, to give

$$-\boldsymbol{\pi}_0(I-R)^{-1}\left(\frac{-A_0+D}{-\mu}\right)^x \boldsymbol{e} \;=\; \mathbf{\Psi}^*\left(I-\frac{Q_1^*}{\mu}\right)^{-1}\left(\left(\frac{Q_0^*}{-\mu}\right)^{x+1}-\left(\frac{Q_0^*}{-\mu}\right)^x\right)\boldsymbol{e}, \quad \text{for all } x \geq 0, \tag{3.3.1}$$

which by the erroneous result of [42] should not be able to be satisfied by any finite-state Markov chain Q^*, when the left hand side is an *MMPP/M/1* queue.

We will now show that it is in fact possible for equation (3.3.1) to be satisfied for an *MMPP/M/1* queue. Recall that an *MMPP* has a characteristic matrix D_1 which is diagonal. Let D_0 be such that only $(D_0\boldsymbol{e})_i$ is non-zero, so that D_1 may be written as

$$D_1 = -D_0\boldsymbol{e}\boldsymbol{\gamma},$$

where $\boldsymbol{\gamma}$ is a row vector of zeros except that $[\gamma]_i = 1$. This *MMPP* is also a *PH*-renewal process, where $\boldsymbol{\alpha} \equiv \boldsymbol{\gamma}$. In fact, it is also known as an *IPP* or interrupted Poisson process.

Consider such an *MMPP* to be the arrival process to a positive recurrent single server queue of service rate μ, and let $\boldsymbol{\nu} = \boldsymbol{\pi}_0(I-R)^{-1}$ be the stationary probability vector of $D = D_0 + D_1$, such that $\boldsymbol{\nu}D = \mathbf{0}$. Let $\boldsymbol{x}_0$ be the stationary probability vector whose entries $\boldsymbol{x}_{0i}$ give the probability that a departure leaves the system empty and the phase of the arrival process in state i, for $i \in \{1,\ldots,m\}$. This is calculated from the stationary distribution for the *MMPP/M/1* by (see [31]),

$$\boldsymbol{x}_0 = \left[\boldsymbol{\pi}_0(I-R)^{-1}A_0\boldsymbol{e}\right]^{-1}\boldsymbol{\pi}_0 R A_2. \tag{3.3.2}$$

Proposition 3.1 *The service rate μ, matrices $D_1 = A_0$, $D_0 = -A_0 + D$ and associated matrix R, probability vector $\boldsymbol{\pi}_0$, together with the $(m+1) \times (m+1)$ matrices*

$$Q_0^* = \begin{bmatrix} D_0 & D_1\boldsymbol{e} \\ 0 & -\mu \end{bmatrix} \quad \text{and} \quad Q_1^* = \begin{bmatrix} 0 & 0 \\ \mu\boldsymbol{x}_0 & (1-\boldsymbol{x}_0\boldsymbol{e})\mu \end{bmatrix}, \tag{3.3.3}$$

and

$$\boldsymbol{\Psi}^* = (\boldsymbol{\pi}_0, 1 - \boldsymbol{\pi}_0\boldsymbol{e}) \ ,$$

satisfy equation (3.3.1).

Proof:

It is trivial to see that the rate matrix $Q^* = (Q_0^* + Q_1^*)$ is conservative and that $\boldsymbol{\Psi}^*\boldsymbol{e} = 1$, so we first show that $\boldsymbol{\Psi}^*$ is the unique stationary probability vector for Q^*, that is $\boldsymbol{\Psi}^*Q^* = \boldsymbol{0}$. Multiplying Q^* on the left by $\boldsymbol{\Psi}^*$, we get

$$\begin{aligned} \boldsymbol{\Psi}^*Q^* &= (\boldsymbol{\pi}_0, 1-\boldsymbol{\pi}_0\boldsymbol{e}) \begin{bmatrix} D_0 & D_1\boldsymbol{e} \\ \mu\boldsymbol{x}_0 & -\mu\boldsymbol{x}_0\boldsymbol{e} \end{bmatrix} \\ &= (\boldsymbol{\pi}_0 D_0 + (1-\boldsymbol{\pi}_0\boldsymbol{e})\mu\boldsymbol{x}_0, \boldsymbol{\pi}_0 D_1\boldsymbol{e} - (1-\boldsymbol{\pi}_0\boldsymbol{e})\mu\boldsymbol{x}_0\boldsymbol{e}). \end{aligned} \tag{3.3.4}$$

We will now show that the right hand side of (3.3.4) is zero, by showing that $\boldsymbol{\pi}_0 D_0 = -(1-\boldsymbol{\pi}_0\boldsymbol{e})\mu\boldsymbol{x}_0$. Consider the expression for the traffic intensity of the *MMPP/M/1* queue, given by (see [31]),

$$\rho = (1 - \boldsymbol{\pi}_0\boldsymbol{e}) = \frac{\boldsymbol{\pi}_0(I-R)^{-1}D_1\boldsymbol{e}}{\mu}.$$

Re-arranging, we get

$$(1-\boldsymbol{\pi}_0\boldsymbol{e})\mu(\boldsymbol{\pi}_0(I-R)^{-1}D_1\boldsymbol{e})^{-1} = 1. \tag{3.3.5}$$

Recalling that $D_1 = A_0$, and post-multiplying (3.3.5) by $\boldsymbol{\pi}_0 R A_2$, we see that

$$\boldsymbol{\pi}_0 R A_2 = (1-\boldsymbol{\pi}_0\boldsymbol{e})\mu\left[\boldsymbol{\pi}_0(I-R)^{-1}A_0\boldsymbol{e}\right]^{-1}\boldsymbol{\pi}_0 R A_2. \tag{3.3.6}$$

From (2.2.6), $\boldsymbol{\pi}_0 D_0 = -\boldsymbol{\pi}_0 R A_2$, so that using (3.3.2), equation (3.3.6) can be re-written as

$$\boldsymbol{\pi}_0 D_0 = -(1-\boldsymbol{\pi}_0\boldsymbol{e})\mu\boldsymbol{x}_0.$$

Post-multiplying by $\boldsymbol{e}$ on both sides and using the fact that $D_0\boldsymbol{e} = -D_1\boldsymbol{e}$, we get

$$\boldsymbol{\pi}_0 D_1\boldsymbol{e} = (1-\boldsymbol{\pi}_0\boldsymbol{e})\mu\boldsymbol{x}_0\boldsymbol{e}.$$

Substitution of this result into (3.3.4) yields

$$\boldsymbol{\Psi}^* Q^* \;=\; \mathbf{0},$$

and hence that $\boldsymbol{\Psi}^*$ is the unique stationary distribution for Q^*.

Now it remains to show that

$$-\boldsymbol{\pi}_0(I-R)^{-1}\left(\frac{D_0}{-\mu}\right)^x \boldsymbol{e} \;=\; \boldsymbol{\Psi}^*\left(I-\frac{Q_1^*}{\mu}\right)^{-1}\left(\left(\frac{Q_0^*}{-\mu}\right)^{x+1} - \left(\frac{Q_0^*}{-\mu}\right)^{x}\right)\boldsymbol{e}.$$

We re-write the right hand side as

$$(\boldsymbol{\pi}_0, 1-\boldsymbol{\pi}_0\boldsymbol{e})\begin{bmatrix} I & \mathbf{0} \\ -\boldsymbol{x}_0 & \boldsymbol{x}_0\boldsymbol{e}\end{bmatrix}^{-1}\left(\begin{bmatrix} \frac{D_0}{-\mu} & \frac{D_1\boldsymbol{e}}{-\mu} \\ \mathbf{0} & 1\end{bmatrix}^{x+1} - \begin{bmatrix} \frac{D_0}{-\mu} & \frac{D_1\boldsymbol{e}}{-\mu} \\ \mathbf{0} & 1\end{bmatrix}^{x}\right)\boldsymbol{e}$$

$$= (\boldsymbol{\pi}_0, 1-\boldsymbol{\pi}_0\boldsymbol{e})\begin{bmatrix} I & \mathbf{0} \\ -\boldsymbol{x}_0 & \boldsymbol{x}_0\boldsymbol{e}\end{bmatrix}^{-1}\left(\begin{bmatrix} \frac{D_0}{-\mu} & \frac{D_1\boldsymbol{e}}{-\mu} \\ \mathbf{0} & 1\end{bmatrix}^{x}\left\{\begin{bmatrix} \frac{D_0}{-\mu} & \frac{D_1\boldsymbol{e}}{-\mu} \\ \mathbf{0} & 1\end{bmatrix} - \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & 1\end{bmatrix}\right\}\right)\boldsymbol{e}$$

$$= (\boldsymbol{\pi}_0, 1-\boldsymbol{\pi}_0\boldsymbol{e})\begin{bmatrix} I & \mathbf{0} \\ -\boldsymbol{x}_0 & \boldsymbol{x}_0\boldsymbol{e}\end{bmatrix}^{-1}\left(\begin{bmatrix} \frac{D_0}{-\mu} & \frac{D_1\boldsymbol{e}}{-\mu} \\ \mathbf{0} & 1\end{bmatrix}^{x}\begin{bmatrix} \frac{D_0+I\mu}{-\mu} & \frac{D_1\boldsymbol{e}}{-\mu} \\ \mathbf{0} & 0\end{bmatrix}\right)\boldsymbol{e}$$

$$= (\boldsymbol{\pi}_0, 1-\boldsymbol{\pi}_0\boldsymbol{e}) \begin{bmatrix} I & \mathbf{0} \\ \frac{\boldsymbol{x}_0}{\boldsymbol{x}_0\boldsymbol{e}} & \frac{1}{\boldsymbol{x}_0\boldsymbol{e}} \end{bmatrix} \left(\begin{bmatrix} \frac{D_0}{-\mu} & \frac{D_1\boldsymbol{e}}{-\mu} \\ \mathbf{0} & 1 \end{bmatrix}^x \begin{bmatrix} \left(\frac{D_0}{-\mu} - I\right)\boldsymbol{e} + \frac{D_1\boldsymbol{e}}{-\mu} \\ 0 \end{bmatrix} \right),$$

but $(D_1 + D_0)\boldsymbol{e} \equiv \mathbf{0}$, so that the right hand side becomes

$$-(\boldsymbol{\pi}_0, 1-\boldsymbol{\pi}_0\boldsymbol{e}) \begin{bmatrix} I & \mathbf{0} \\ \frac{\boldsymbol{x}_0}{\boldsymbol{x}_0\boldsymbol{e}} & \frac{1}{\boldsymbol{x}_0\boldsymbol{e}} \end{bmatrix} \begin{pmatrix} \left(\frac{D_0}{-\mu}\right)^x \boldsymbol{e} \\ 0 \end{pmatrix} = -(\boldsymbol{\pi}_0, 1-\boldsymbol{\pi}_0\boldsymbol{e}) \begin{pmatrix} \left(\frac{D_0}{-\mu}\right)^x \boldsymbol{e} \\ \frac{\boldsymbol{x}_0}{\boldsymbol{x}_0\boldsymbol{e}} \left(\frac{D_0}{-\mu}\right)^x \boldsymbol{e} \end{pmatrix}$$

$$= -\left(\boldsymbol{\pi}_0 + (1-\boldsymbol{\pi}_0\boldsymbol{e})\frac{\boldsymbol{x}_0}{\boldsymbol{x}_0\boldsymbol{e}}\right)\left(\frac{D_0}{-\mu}\right)^x \boldsymbol{e}.$$

Recalling that $\boldsymbol{x}_0 = \left[\boldsymbol{\pi}_0(I-R)^{-1}A_0\boldsymbol{e}\right]^{-1} \boldsymbol{\pi}_0 R A_2$ and that, in this case, $A_2 = \mu I$, we see that

$$\begin{aligned} \frac{\boldsymbol{x}_0}{\boldsymbol{x}_0\boldsymbol{e}} &= \frac{\mu \left[\boldsymbol{\pi}_0(I-R)^{-1}A_0\boldsymbol{e}\right]^{-1} \boldsymbol{\pi}_0 R}{\mu \left[\boldsymbol{\pi}_0(I-R)^{-1}A_0\boldsymbol{e}\right]^{-1} \boldsymbol{\pi}_0 R\boldsymbol{e}} \\ &= \frac{\boldsymbol{\pi}_0 R}{\boldsymbol{\pi}_0 R\boldsymbol{e}}, \end{aligned}$$

so that

$$\begin{aligned} & -\left(\boldsymbol{\pi}_0 + (1-\boldsymbol{\pi}_0\boldsymbol{e})\left(\frac{\boldsymbol{x}_0}{\boldsymbol{x}_0\boldsymbol{e}}\right)\right)\left(\frac{D_0}{-\mu}\right)^x \boldsymbol{e} \\ &= -\left(\boldsymbol{\pi}_0 + (1-\boldsymbol{\pi}_0\boldsymbol{e})\left(\frac{\boldsymbol{\pi}_0 R}{\boldsymbol{\pi}_0 R\boldsymbol{e}}\right)\right)\left(\frac{D_0}{-\mu}\right)^x \boldsymbol{e} \\ &= -\left(\boldsymbol{\pi}_0 + \boldsymbol{\pi}_0((I-R)^{-1} - I)\boldsymbol{e}\left(\frac{\boldsymbol{\pi}_0 R}{\boldsymbol{\pi}_0 R\boldsymbol{e}}\right)\right)\left(\frac{D_0}{-\mu}\right)^x \boldsymbol{e} \\ &= -\left(\boldsymbol{\pi}_0 + \boldsymbol{\pi}_0(R + R^2 + R^3 + \ldots)\boldsymbol{e}\left(\frac{\boldsymbol{\pi}_0 R}{\boldsymbol{\pi}_0 R\boldsymbol{e}}\right)\right)\left(\frac{D_0}{-\mu}\right)^x \boldsymbol{e} \end{aligned}$$

$$= -\left(\boldsymbol{\pi}_0 + \boldsymbol{\pi}_0 R\boldsymbol{e}\left(\frac{\boldsymbol{\pi}_0(R + R^2 + R^3 \ldots)}{\boldsymbol{\pi}_0 R\boldsymbol{e}}\right)\right)\left(\frac{D_0}{-\mu}\right)^x \boldsymbol{e} \qquad \clubsuit$$

$$= -\boldsymbol{\pi}_0(I - R)^{-1}\left(\frac{D_0}{-\mu}\right)^x \boldsymbol{e}.$$

Sufficient justification for the step at $\clubsuit$, that is,

$$\boldsymbol{\pi}_0(R + R^2 + R^3 + \ldots)\boldsymbol{e}\left(\frac{\boldsymbol{\pi}_0 R}{\boldsymbol{\pi}_0 R\boldsymbol{e}}\right) = \boldsymbol{\pi}_0(R + R^2 + R^3 \ldots),$$

is that R be rank one. The matrix R is non-negative and has $sp(R) < 1$ (see Neuts [31]). That is, its eigenvalues all have modulus less than 1. From Theorem 3 in Chapter XIII of Gantmacher [22], a non-negative matrix R has a non-negative eigenvalue η of maximal real part, and associated non-negative right and left eigenvectors $\boldsymbol{v}$ and $\boldsymbol{u}$ respectively. If we further assume R is of rank one (so that its eigenvalues are η and 0), and we normalise $\boldsymbol{v}$ and $\boldsymbol{u}$ such that $\boldsymbol{ue} = 1$ and $\boldsymbol{uv} = 1$, then R may be written using its spectral expansion as

$$R = \eta\boldsymbol{vu}. \tag{3.3.7}$$

Hence for all $n \geq 1$, we may write

$$\begin{aligned}
(\boldsymbol{\pi}_0 R^n \boldsymbol{e})\boldsymbol{\pi}_0 R &= (\boldsymbol{\pi}_0\eta\boldsymbol{v}(\eta\boldsymbol{uv})^{n-1}\boldsymbol{ue})\boldsymbol{\pi}_0\eta\boldsymbol{vu} \\
&= (\boldsymbol{\pi}_0\eta\boldsymbol{vue})\boldsymbol{\pi}_0\eta\boldsymbol{v}(\eta\boldsymbol{uv})^{n-1}\boldsymbol{u} \\
&= (\boldsymbol{\pi}_0 R\boldsymbol{e})\boldsymbol{\pi}_0 R^n, \text{ for } n \geq 1.
\end{aligned}$$

From Latouche [25], it can be seen that for any *QBD*, the matrix $R = A_0(-U)^{-1}$. The interpretation of the matrix $(-U)^{-1}$ is not significant here and is omitted. For any *PH/M/1* queue, we have from $D_1 = (-D_0\boldsymbol{e})\boldsymbol{\alpha}$ that the matrix $A_0 = D_1$ is rank one. This implies that the matrix R for all *PH/M/1* queues is, in fact, rank one. In our example, the *MMPP* is also a *PH*-renewal process and hence R is rank one.

■

We now give a simple numerical example of an *MMPP* which is also a *PH*-renewal process and an *IPP* for which (3.3.1) holds. Consider the following *MMPP* arrival process described by the matrices

$$D_0 = \begin{bmatrix} -4 & 3 & 1 \\ 1 & -5 & 2 \\ 4 & 3 & -7 \end{bmatrix} \tag{3.3.8}$$

and

$$D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.3.9}$$

The stationary probability distribution of phases for this *MMPP* is given by $\boldsymbol{\nu} = \left(\left(\frac{5}{16}\right), \left(\frac{1}{2}\right), \left(\frac{3}{16}\right)\right)$. When this *MMPP* is used as arrival process to a single negative exponential server of rate $\mu > 1$ with infinite buffer, a possible candidate *MAP* for the departure process is given by the matrices Q_0^* and Q_1^* given in (3.3.3). It can be easily shown that the matrices D_0, D_1, Q_0^* and Q_1^* along with $\mu > 1$ satisfy equation (3.3.1). It must be noted however that equation (3.3.1) is necessary but not sufficient for the output process to be equivalent to a *MAP*. A necessary and sufficient condition is given in Chapter 4 in equation (4.1.1). This equation is not satisfied by Q_0^* and Q_1^* as given in (3.3.3) for the matrices D_1 and D_0 given in (3.3.8) and (3.3.9). Hence the departure process of the above *MMPP/M/1* queue is not given by the *MAP* represented by the matrices Q_0^* and Q_1^*.

Note that there are examples of *MAP/M/1* queues, not just *MMPP/M/1* queues, for which (3.3.1) can be shown to be valid. In fact in the proof of Proposition 3.1 we have shown that (3.3.1) can be satisfied by every *PH/M/1* queue, using those same constructions.

3.3.1 Three *MAP/M/1* queues having a *MAP* output

Consider now the three non-*MMPP/M/1* queues, with service rate $\mu > 1$, defined by

$$D_0 = \begin{bmatrix} -2 & (\frac{1}{2}) & (\frac{1}{2}) \\ 1 & -4 & 1 \\ (\frac{1}{2}) & 1 & -2 \end{bmatrix}, D_1 = \begin{bmatrix} (\frac{1}{2}) & 0 & (\frac{1}{2}) \\ 1 & 1 & 0 \\ 0 & 0 & (\frac{1}{2}) \end{bmatrix}, \quad (3.3.10)$$

with $\boldsymbol{\nu} = \left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right)$,

$$D_0 = \begin{bmatrix} -4 & 2 & 1 \\ 5 & -8 & 2 \\ 1 & 2 & -4 \end{bmatrix}, D_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad (3.3.11)$$

with $\boldsymbol{\nu} = \left(\frac{9}{20}, \frac{1}{5}, \frac{7}{20}\right)$, and

$$D_0 = \begin{bmatrix} -3 & 3 & 0 \\ 0 & -6 & 4 \\ 0 & 0 & -1 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ (\frac{1}{2}) & (\frac{1}{2}) & 0 \end{bmatrix}, \quad (3.3.12)$$

with $\boldsymbol{\nu} = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right)$.

All of these examples with the matrices Q_0^* and Q_1^* as described in (3.3.3), satisfy equation (3.3.1). They also satisfy the necessary and sufficient condition (4.1.1) for a *MAP/M/1* queue to have a process of departures which under stationary conditions is a *MAP*. This seems to indicate that the output from a single server queue can have a simple *MAP* description. However, things are not as simple as they seem.

Although not obvious at first glance, all three of the above arrival processes given by D_0 and D_1 under stationary conditions are a complicated description of a Poisson process of rate 1. This then implies by Burke's Theorem [11] that the output process is in fact a Poisson process of rate 1, a very simple MAP.

The question of being able to recognise a Poisson process from its MAP description is therefore central to the current discussion, since the process of departures from such a *MAP/M/1* queue is a simple MAP. These simple three state examples given above highlight the non-trivial nature of this question of being able to recognise a Poisson process from its MAP description. As the dimension of the MAP descriptions increases, the difficulty of recognising a Poisson process seemingly would also increase.

In the next chapter we address this question of when a MAP is Poisson.

Chapter 4

When is a stationary *MAP* Poisson?

4.1 Introduction

The question of when a stationary *MAP* is in fact a Poisson process appears not to have been discussed in the literature prior to [5], even though it is a very natural question. Here we consider this problem using the techniques of non-linear filtering which were briefly introduced in Chapter 2 and can be found in greater detail in Walrand [50].

From Walrand [50] it can be seen that two point processes defined by the filtration matrices $Q_0 + Q_1 = Q$ and $Q_0^* + Q_1^* = Q^*$ have the same finite dimensional distributions, if and only if for any given initial distributions of states $\boldsymbol{\pi}$ and $\boldsymbol{\pi}^*$ respectively, we have

$$\frac{\boldsymbol{\pi} e^{Q_0 t_1} Q_1 e^{Q_0 t_2} \dots Q_1 e^{Q_0 t_k} Q_1 \boldsymbol{e}}{\boldsymbol{\pi} e^{Q_0 t_1} Q_1 e^{Q_0 t_2} \dots Q_1 e^{Q_0 t_k} \boldsymbol{e}} = \frac{\boldsymbol{\pi}^* e^{Q_0^* t_1} Q_1^* e^{Q_0^* t_2} \dots Q_1^* e^{Q_0^* t_k} Q_1^* \boldsymbol{e}}{\boldsymbol{\pi}^* e^{Q_0^* t_1} Q_1^* e^{Q_0^* t_2} \dots Q_1^* e^{Q_0^* t_k} \boldsymbol{e}},$$

$$\text{for all } k \geq 1, \text{ and } t_i \in [0, \infty), \text{ for } i \in \{1, 2, \dots, k\}. \qquad (4.1.1)$$

Until now we have only considered the case where Q is an infinite matrix (the rate matrix of a *MAP/M/1* queue) and Q^* is a finite matrix. However equation (4.1.1) can be used to compare any two point processes. For this chapter, equation (4.1.1) is re-written for the case where the left-hand side is a *MAP* and the right-hand side is a Poisson process of rate $\lambda > 0$, given by the filtration matrices $Q_0^* = -\lambda$, $Q_1^* = \lambda$ so that $Q^* = 0$. Thus equation (4.1.1) becomes

$$\frac{\boldsymbol{\pi} e^{Q_0 t_1} Q_1 e^{Q_0 t_2} \ldots Q_1 e^{Q_0 t_k} Q_1 \boldsymbol{e}}{\boldsymbol{\pi} e^{Q_0 t_1} Q_1 e^{Q_0 t_2} \ldots Q_1 e^{Q_0 t_k} \boldsymbol{e}} = \lambda,$$
$$\text{for all } k \geq 1 \text{ and } t_i \in [0, \infty), \text{ for } i \in \{1, 2, \ldots, k\}. \tag{4.1.2}$$

Any point process generated by the observed transitions of $Q = Q_0 + Q_1$ which is equivalent to a Poisson process of rate $\lambda > 0$ must satisfy equation (4.1.2). One case where this clearly occurs is when $Q_1 \boldsymbol{e} = \lambda \boldsymbol{e}$. This result is not affected by the initial distribution $\boldsymbol{\pi}$ and corresponds to the case where the *MAP* has the same arrival rate λ in every phase.

We will consider the matrix Q_0 in its spectral form, first looking at the more general case when Q_0 is not assumed to be diagonalisable, and then consider the diagonalisable special case. To avoid over complicating matters, the general result is motivated in stages by first considering the equivalence of a *PH*-random variable to an exponential random variable. We then consider the equivalence of a *PH*-renewal process to a Poisson process and finally generalise to the equivalence of a *MAP* to a Poisson process. As the title of this Chapter suggests, we will investigate the equivalence of a stationary *MAP* to a Poisson process. That is, we will assume that the initial distribution $\boldsymbol{\pi}$ is in fact the stationary distribution of phases of the *MAP* under consideration.

4.2 Jordan canonical form of Q_0

Consider the matrix Q_0 in upper Jordan canonical form, written as (see (2.5.4))

$$Q_0 = T \left(\sum_{j=1}^{g} \lambda_j E_j + N_j \right) T^{-1} \quad , \tag{4.2.1}$$

where

- g is the number of Jordan blocks
- λ_j is the eigenvalue corresponding to the j^{th} Jordan block
- T is the transformation matrix for the Jordan canonical form
- E_j is an idempotent matrix descriptor of the j^{th} Jordan block
- N_j is a nilpotent matrix descriptor of the j^{th} Jordan block.

It follows that

$$e^{Q_0 t} = T \sum_{j=1}^{g} \sum_{v=1}^{p_j} \left(\frac{t^{v-1}}{(v-1)!} \right) e^{\lambda_j t} E_j {N_j}^{v-1} E_j T^{-1}, \tag{4.2.2}$$

where p_j is the order of the nilpotent matrix N_j, that is, $N_j^{p_j} = 0$ but $N_j^{p_j - 1} \neq 0$.

As a preliminary, we set up some notation for the transformation matrix T and also for T^{-1}. Let

$$T = \left(\left[\boldsymbol{\tau}_{1_1}, \ldots, \boldsymbol{\tau}_{1_{p_1}} \right], \left[\boldsymbol{\tau}_{2_1}, \ldots, \boldsymbol{\tau}_{2_{p_2}} \right], \ldots, \left[\boldsymbol{\tau}_{g_1}, \ldots, \boldsymbol{\tau}_{g_{p_g}} \right] \right),$$

where the columns $\boldsymbol{\tau}_{j_1}$ are right eigenvectors corresponding to the j^{th} eigenvalue λ_j of Q_0. For $v \in \{2, \ldots, p_j\}$, $\boldsymbol{\tau}_{j_v}$ are the generalised right eigenvectors corresponding to the j^{th} Jordan block. Similarly we write T^{-1} with rows $\bar{\boldsymbol{\tau}}_{j_v}$ as

$$T^{-1} = \begin{pmatrix} \begin{bmatrix} \bar{\boldsymbol{\tau}}_{1_1} \\ \vdots \\ \bar{\boldsymbol{\tau}}_{1_{p_1}} \end{bmatrix} \\ \\ \begin{bmatrix} \bar{\boldsymbol{\tau}}_{2_1} \\ \vdots \\ \bar{\boldsymbol{\tau}}_{2_{p_2}} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \bar{\boldsymbol{\tau}}_{g_1} \\ \vdots \\ \bar{\boldsymbol{\tau}}_{g_{p_g}} \end{bmatrix} \end{pmatrix},$$

where the rows $\bar{\boldsymbol{\tau}}_{j_{p_j}}$ are left eigenvectors corresponding to the j^{th} eigenvalue λ_j of Q_0. For $v \in \{1, \ldots, p_j - 1\}$, $\bar{\boldsymbol{\tau}}_{j_v}$ are the generalised left eigenvectors corresponding to the j^{th} Jordan block.

If there are $s \leq g$ distinct eigenvalues then we let $\mathcal{A}_j$, for $j = 1, 2, \ldots, s$ contain the indices for the Jordan blocks with λ_j on their diagonal. Then Q_0 may also be written as

$$Q_0 = T\left(\sum_{j=1}^{s}\left[\sum_{i\in\mathcal{A}_j} \lambda_j E_{(j,i)} + N_{(j,i)}\right]\right)T^{-1}, \tag{4.2.3}$$

with

$$e^{Q_0 t} = T\sum_{j=1}^{s}\left[\sum_{i\in\mathcal{A}_j}\sum_{v=1}^{p_{(j,i)}}\left(\frac{t^{v-1}}{(v-1)!}\right)e^{\lambda_j t}E_{(j,i)}{N_{(j,i)}}^{v-1}E_{(j,i)}\right]T^{-1}, \tag{4.2.4}$$

where $E_{(j,i)}$ and $N_{(j,i)}$ are the idempotent and nilpotent matrices respectively of the Jordan canonical form description which correspond to the i^{th} Jordan block in set $\mathcal{A}_j$, corresponding to the distinct eigenvalue λ_j. In (4.2.3) and (4.2.4), for each distinct eigenvalue λ_j within the square brackets, there is a unique collection of indices $i \in \mathcal{A}_j$ for that λ_j. Therefore, we can without ambiguity reduce the number of subscripts by an abuse of notation and write

$$Q_0 = T\left(\sum_{j=1}^{s}\sum_{i\in\mathcal{A}_j}\lambda_j E_i + N_i\right)T^{-1}, \tag{4.2.5}$$

with

$$e^{Q_0 t} = T\sum_{j=1}^{s}\sum_{i\in\mathcal{A}_j}\sum_{v=1}^{p_i}\left(\frac{t^{v-1}}{(v-1)!}\right)e^{\lambda_j t}E_i{N_i}^{v-1}E_iT^{-1}. \tag{4.2.6}$$

It is also convenient in the statement of theorems and their proofs, to define

$$P_j \equiv \max_{i\in A_j}(p_i).$$

Then because of the fact that $N_i^{p_i+n} \equiv 0$ for each i, for all $n \geq 0$, we may re-write (4.2.6) as

$$e^{Q_0 t} = T\sum_{j=1}^{s}\sum_{i\in\mathcal{A}_j}\sum_{v=1}^{P_j}\left(\frac{t^{v-1}}{(v-1)!}\right)e^{\lambda_j t}E_i{N_i}^{v-1}E_iT^{-1}. \tag{4.2.7}$$

We will also assume that the eigenvalues λ_j are ordered, such that

$$\Re(\lambda_1) \geq \Re(\lambda_2) \geq \Re(\lambda_3) \geq \ldots,$$

where $\Re(\lambda_j)$ is the real part of λ_j.

We shall state the main theorems of this chapter for irreducible Q_0. A theorem for reducible Q_0, however, is given for the equivalence of a *PH*-random variable to a negative exponential random variable. This theorem is not very informative because the reducible Q_0 case is far more complex in general. We shall address some particular sub-cases which highlight this complexity. Analogous results for the subsequent *PH*-renewal process and *MAP* equivalence to the Poisson process are omitted.

It is important here to make the distinction between an irreducible representation $(\boldsymbol{\alpha}, Q_0)$ of a *PH*-distribution and an irreducible matrix Q_0. Following the definition in Neuts [31], a representation $(\boldsymbol{\alpha}, Q_0)$ of a *PH*-distribution is called irreducible if each state of the corresponding chain has a positive probability of being visited when the initial distribution is $\boldsymbol{\alpha}$. It is shown in [31] that we may always restrict our attention to irreducible representations for both *PH*-distributions and their associated *PH*-renewal processes. The matrix Q_0 is called irreducible if, given any initial state in Q_0, every other state of the corresponding chain has a positive probability of being visited.

The matrices Q_0, are known as *ML*-matrices in Seneta [48]. They have important properties, proven by extension of the Perron-Frobenius structure. In Theorem 2.6 of [48], the irreducible *ML*-matrices are shown to have a unique eigenvalue λ_1 of maximal real part, with associated positive right and left eigenvectors $\boldsymbol{\tau}_1$ and $\bar{\boldsymbol{\tau}}_1$. Therefore, if Q_0 is irreducible, $\mathcal{A}_1 \equiv \{1\}$. We also assume the following normalisation for $\boldsymbol{\tau}_1$ and $\bar{\boldsymbol{\tau}}_1$.

$$\bar{\boldsymbol{\tau}}_1 \boldsymbol{\tau}_1 = \bar{\boldsymbol{\tau}}_1 \boldsymbol{e} = 1.$$

4.2.1 Equivalence of a *PH*-random variable to an exponential random variable

To investigate the equivalence of a *PH*-random variable with a negative exponential random variable, we assume that $\boldsymbol{\alpha e} = 1$ (that is, $\alpha_{m+1} \equiv 0$). Otherwise, there would be a positive atom of probability at $t = 0$, which would be a contradiction to an exponential random variable. Henceforth we make this assumption.

Theorem 4.1 *A PH $(\boldsymbol{\alpha}, Q_0)$ random variable, where the matrix Q_0 is irreducible, is negative exponential with parameter $\lambda > 0$ if and only if $\lambda = -\lambda_1$, and for all $(j, v) \in \{2, \ldots, s\} \times \{1, \ldots, P_j\}$,*

$$\boldsymbol{\alpha} T \left(\sum_{i \in \mathcal{A}_j} E_i N_i^{v-1} E_i \right) T^{-1} \boldsymbol{e} \;=\; 0.$$

Proof:

If $1 - \boldsymbol{\alpha} e^{Q_0 t} \boldsymbol{e} = 1 - e^{-\lambda t}$, then from (4.2.7) we have

$$\boldsymbol{\alpha} T \left(e^{\lambda_1 t} E_1 + \sum_{j=2}^{s} \sum_{i \in \mathcal{A}_j} \sum_{v=1}^{P_j} \left(\frac{t^{v-1}}{(v-1)!} \right) e^{\lambda_j t} E_i N_i^{v-1} E_i \right) T^{-1} \boldsymbol{e} = e^{-\lambda t}. \tag{4.2.8}$$

This is of the form

$$c_1 e^{\lambda_1 t} + \sum_{j=2}^{s} \sum_{v=1}^{P_j} c_{(j,v)} t^{v-1} e^{\lambda_j t} = e^{-\lambda t}, \text{ where } c_1 = \boldsymbol{\alpha} T E_1 T^{-1} \boldsymbol{e} \tag{4.2.9}$$

and for $(j, v) \in \{2, \ldots, s\} \times \{1, \ldots, P_j\}$,

$$c_{(j,v)} = \boldsymbol{\alpha} T \sum_{i \in \mathcal{A}_j} \left(\frac{1}{(v-1)!} \right) E_i N_i^{v-1} E_i T^{-1} \boldsymbol{e}.$$

Because the λ_j are distinct, for $(j, v) \in \{2, \ldots, s\} \times \{1, \ldots, P_j\}$, the functions $c_1 e^{\lambda_1 t}$ and $c_{(j,v)} t^{v-1} e^{\lambda_j t}$ on the left hand side of equation (4.2.9) are linearly independent. This follows, for example, from Theorem 12 in Chapter 2 of Coddington [13]. Now, c_1 can be re-written as $c_1 = \boldsymbol{\alpha} \boldsymbol{\tau}_1 \bar{\boldsymbol{\tau}}_1 \boldsymbol{e}$ by using the definitions of T and T^{-1} and the fact that E_1 consists entirely of zeros except for a one in the top left-hand corner.

From Seneta [48], $\boldsymbol{\tau}_1$ and $\bar{\boldsymbol{\tau}}_1$ are positive and since $\boldsymbol{\alpha} > 0$, we have that $c_1 > 0$. Equation (4.2.8) is true for all $t \in [0, \infty)$, therefore as $e^{\lambda_j t} \neq 0$ for $j \in \{1, 2, 3, \ldots, s\}$ and for all $t \in [0, \infty)$, we must have that $\lambda = -\lambda_1$, and so for all $j \in \{2, \ldots, s\}$, we have that

$$\boldsymbol{\alpha} T \left(\sum_{i \in \mathcal{A}_j} E_i N_i^{v-1} E_i \right) T^{-1} \boldsymbol{e} = 0, \text{ for all } v \in \{1, \ldots, P_j\}. \tag{4.2.10}$$

■

We now consider what information can be drawn from this theorem in the following corollary for the special case when all the eigenvalues are distinct.

Corollary 4.2 *A PH $(\boldsymbol{\alpha}, Q_0)$ random variable where Q_0 is irreducible and has distinct eigenvalues in each Jordan block, is negative exponential with parameter $\lambda > 0$, if and only if $\lambda = -\lambda_1$ and for all $j \in \{2, \ldots, g\}$, we have*

$$\begin{aligned} \boldsymbol{\alpha} \boldsymbol{\tau}_{j_v} &= 0, \text{ for all } v \in \{1, \ldots, p_j\} \\ \text{or } \bar{\boldsymbol{\tau}}_{j_v} \boldsymbol{e} &= 0, \text{ for all } v \in \{1, \ldots, p_j\}. \end{aligned}$$

Proof:

We use the result from the previous theorem. The assumption that there are distinct eigenvalues in each Jordan block means that in this case, $s = g$, and hence each set $\mathcal{A}_j$ has only one element. From equation (4.2.10) we see that for each $j \in \{2, 3, \ldots, g\}$, we have

$$\begin{array}{lrcl} (v = 1) & \boldsymbol{\alpha} T \left(E_j \right) T^{-1} \boldsymbol{e} & = & 0 \\ (v = 2) & \boldsymbol{\alpha} T \left(E_j N_j E_j \right) T^{-1} \boldsymbol{e} & = & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (v = p_j - 1) & \boldsymbol{\alpha} T \left(E_j N_j^{p_j - 2} E_j \right) T^{-1} \boldsymbol{e} & = & 0 \\ (v = p_j) & \boldsymbol{\alpha} T \left(E_j N_j^{p_j - 1} E_j \right) T^{-1} \boldsymbol{e} & = & 0. \end{array} \tag{4.2.11}$$

The nilpotent matrix N_j is such that $E_j N_j^{p_j-1} E_j$ has only one non-zero entry so that

$$\boldsymbol{\alpha} T \left(E_j N_j^{p_j-1} E_j \right) = \begin{pmatrix} \underset{\downarrow}{\overset{\text{position}}{j_{p_j}}} \\ 0, \ldots, 0, \ (\boldsymbol{\alpha}\boldsymbol{\tau}_{j_1}) \ , 0, \ldots, 0 \\ \\ \end{pmatrix}$$

and the left hand side of the last equation in (4.2.11) becomes

$$\begin{pmatrix} \underset{\downarrow}{\overset{\text{position}}{j_{p_j}}} \\ 0, \ldots, 0, \ (\boldsymbol{\alpha}\boldsymbol{\tau}_{j_1}) \ , 0, \ldots, 0 \\ \\ \end{pmatrix} \begin{pmatrix} (\bar{\boldsymbol{\tau}}_{1_1} \boldsymbol{e}) \\ \vdots \\ (\bar{\boldsymbol{\tau}}_{j_{p_j}} \boldsymbol{e}) \\ \vdots \\ (\bar{\boldsymbol{\tau}}_{g_{p_g}} \boldsymbol{e}) \end{pmatrix} \begin{matrix} \\ \\ \leftarrow \text{position } j_{p_j} \\ \\ \\ \end{matrix} .$$

Hence we can write for all $j \in \{2, 3, \ldots, g\}$, that

$$(\boldsymbol{\alpha}\boldsymbol{\tau}_{j_1})(\bar{\boldsymbol{\tau}}_{j_{p_j}} \boldsymbol{e}) = 0, \tag{4.2.12}$$

and for all $j \in \{2, 3, \ldots, g\}$, that

- $\boldsymbol{\alpha}\boldsymbol{\tau}_{j_1} = 0$ or
- $\bar{\boldsymbol{\tau}}_{j_{p_j}} \boldsymbol{e} = 0$.

Now there are three cases which must be considered in further detail for each $j \in \{2, 3, \ldots, g\}$.

1. $\bar{\boldsymbol{\tau}}_{j_{p_j}} \boldsymbol{e} \neq 0$
2. $\boldsymbol{\alpha}\boldsymbol{\tau}_{j_1} \neq 0$
3. $\boldsymbol{\alpha}\boldsymbol{\tau}_{j_1} = 0$ and $\bar{\boldsymbol{\tau}}_{j_{p_j}} \boldsymbol{e} = 0$.

Case 1: When $\bar{\boldsymbol{\tau}}_{j_{p_j}}\boldsymbol{e} \neq 0$, we have $\boldsymbol{\alpha}\boldsymbol{\tau}_{j_1} = 0$, so that from (4.2.11), the left hand side corresponding to $v = p_j - 1$ can be written as

$$\begin{pmatrix} & \text{position} & \\ & j_{p_j} & \\ & \downarrow & \\ 0,\ldots,0,(\boldsymbol{\alpha}\boldsymbol{\tau}_{j_1}), & (\boldsymbol{\alpha}\boldsymbol{\tau}_{j_2}) & ,0,\ldots,0 \end{pmatrix} \begin{pmatrix} (\bar{\boldsymbol{\tau}}_{1_1}\boldsymbol{e}) \\ \vdots \\ (\bar{\boldsymbol{\tau}}_{j_{p_j-1}}\boldsymbol{e}) \\ (\bar{\boldsymbol{\tau}}_{j_{p_j}}\boldsymbol{e}) \\ \vdots \\ (\bar{\boldsymbol{\tau}}_{g_{p_g}}\boldsymbol{e}) \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow \text{position } j_{p_j}, \\ \\ \\ \end{matrix}.$$

Hence

$$(\boldsymbol{\alpha}\boldsymbol{\tau}_{j_1})(\bar{\boldsymbol{\tau}}_{j_{p_j-1}}\boldsymbol{e}) + (\boldsymbol{\alpha}\boldsymbol{\tau}_{j_2})(\bar{\boldsymbol{\tau}}_{j_{p_j}}\boldsymbol{e}) = 0. \tag{4.2.13}$$

Then, using the fact that $\boldsymbol{\alpha}\boldsymbol{\tau}_{j_1} = 0$ and $\bar{\boldsymbol{\tau}}_{j_{p_j}}\boldsymbol{e} \neq 0$, we get

$$\boldsymbol{\alpha}\boldsymbol{\tau}_{j_2} = 0.$$

This procedure can be repeated for each $v = p_j - 2, p_j - 3, \ldots, 1$ in (4.2.11), to give

$$\boldsymbol{\alpha}\boldsymbol{\tau}_{j_v} = 0, \text{ for all } v \in \{1, \ldots, p_j\}.$$

Case 2: When $\boldsymbol{\alpha}\boldsymbol{\tau}_{j_1} \neq 0$ we have $\bar{\boldsymbol{\tau}}_{j_{p_j}}\boldsymbol{e} = 0$. Looking at the left hand side of the equation in (4.2.11) corresponding to $v = p_j - 1$, which has been re-written in (4.2.13), we must have that

$$\bar{\boldsymbol{\tau}}_{j_{p_j-1}}\boldsymbol{e} = 0.$$

This procedure can also be repeated for each $v = p_j - 2, p_j - 3, \ldots, 1$ in (4.2.11), to give

$$\bar{\boldsymbol{\tau}}_{j_v}\boldsymbol{e} \quad = \quad 0, \text{ for all } v \in \{1, \ldots, p_j\}.$$

Case 3: In the case when both $\boldsymbol{\alpha}\boldsymbol{\tau} j_1 = 0$ and $\bar{\boldsymbol{\tau}}_{j_{p_j}}\boldsymbol{e} = 0$, the choice of $v = p_j - 1$ yields

$$(\boldsymbol{\alpha}\boldsymbol{\tau}_{j_1})(\bar{\boldsymbol{\tau}}_{j_{p_j-1}}\boldsymbol{e}) + (\boldsymbol{\alpha}\boldsymbol{\tau}_{j_2})(\bar{\boldsymbol{\tau}}_{j_{p_j}}\boldsymbol{e}) = 0,$$

which gives no further information, so we look at $v = p_j - 2$. This yields

$$\begin{pmatrix} \begin{matrix} & & & \text{position} & \\ & & & j_{p_j} & \\ & & & \downarrow & \\ 0,\ldots,0, & (\boldsymbol{\alpha\tau}_{j_1}), & (\boldsymbol{\alpha\tau}_{j_2}), & (\boldsymbol{\alpha\tau}_{j_3}) & ,0,\ldots,0 \end{matrix} \end{pmatrix} \begin{pmatrix} (\bar{\boldsymbol{\tau}}_{1_1}\boldsymbol{e}) \\ \vdots \\ (\bar{\boldsymbol{\tau}}_{j_{p_j-2}}\boldsymbol{e}) \\ (\bar{\boldsymbol{\tau}}_{j_{p_j-1}}\boldsymbol{e}) \\ (\bar{\boldsymbol{\tau}}_{j_{p_j}}\boldsymbol{e}) \\ \vdots \\ (\bar{\boldsymbol{\tau}}_{g_{p_g}}\boldsymbol{e}) \end{pmatrix} \begin{matrix} \\ \\ \\ \\ \leftarrow \text{position } j_{p_j} \\ \\ \\ \end{matrix},$$

so that we get

$$(\boldsymbol{\alpha\tau}_{j_1})(\bar{\boldsymbol{\tau}}_{j_{p_j-2}}\boldsymbol{e}) + (\boldsymbol{\alpha\tau}_{j_2})(\bar{\boldsymbol{\tau}}_{j_{p_j-1}}\boldsymbol{e}) + (\boldsymbol{\alpha\tau}_{j_3})(\bar{\boldsymbol{\tau}}_{j_{p_j}}\boldsymbol{e}) = 0,$$

which reduces to

$$(\boldsymbol{\alpha\tau}_{j_2})(\bar{\boldsymbol{\tau}}_{j_{p_j-1}}\boldsymbol{e}) = 0,$$

and hence

$$\boldsymbol{\alpha\tau}_{j_2} = 0 \text{ or } \bar{\boldsymbol{\tau}}_{j_{p_j-1}}\boldsymbol{e} = 0.$$

This is a similar scenario to that in equation (4.2.12) and therefore it is only necessary to consider the situation when either $\bar{\boldsymbol{\tau}}_{j_{p_j}}\boldsymbol{e} \neq 0$ or $\boldsymbol{\alpha\tau}_{j_1} \neq 0$.

Therefore we have that for all $j \in \{2, 3, \ldots, g\}$,

$$\begin{aligned} \boldsymbol{\alpha\tau}_{j_v} &= 0, \text{ for all } v \in \{1, \ldots, p_j\}, \\ \text{or } \bar{\boldsymbol{\tau}}_{j_v}\boldsymbol{e} &= 0, \text{ for all } v \in \{1, \ldots, p_j\}. \end{aligned}$$

■

Before continuing on with the reducible Q_0 case, it is worthy to note that for irreducible Q_0, there will always be a term involving the exponential $e^{\lambda_1 t}$ in $\boldsymbol{\alpha}e^{Q_0 t}\boldsymbol{e}$, with a strictly positive coefficient. This implies for example, that an Erlang distribution (of order greater than 1) cannot have a representation that has an irreducible matrix parameter Q_0, since an order n Erlang distribution function is of the form

$$F(t) = 1 - \sum_{i=0}^{n} \left(\frac{(\lambda t)^i}{i!}\right) e^{-\lambda t}, \text{ for all } t \geq 0. \tag{4.2.14}$$

For the case when Q_0 is reducible, it can be seen from Gantmacher [22] that a reducible *ML*-matrix has an eigenvalue λ_1 of maximal real part (not necessarily unique), with an associated pair of non-negative (not necessarily positive) right and left eigenvectors.

Theorem 4.3 *A PH $(\boldsymbol{\alpha}, Q_0)$ random variable, where the matrix Q_0 is reducible, is negative exponential with parameter $\lambda > 0$ if and only if there exists λ_j such that $\lambda = -\lambda_j$, with*

$$\boldsymbol{\alpha} T \left(\sum_{i \in \mathcal{A}_j} E_i \right) T^{-1} \boldsymbol{e} \;=\; 1,$$

and

$$\boldsymbol{\alpha} T \left(\sum_{i \in \mathcal{A}_j} E_i N_i^{v-1} E_i \right) T^{-1} \boldsymbol{e} \;=\; 0, \;\textit{for all}\; v \in \{2, \ldots, P_j\}.$$

Then for all $k \neq j$, we have

$$\boldsymbol{\alpha} T \left(\sum_{i \in \mathcal{A}_k} E_i N_i^{v-1} E_i \right) T^{-1} \boldsymbol{e} \;=\; 0, \;\textit{for all}\; v \in \{1, \ldots, P_k\}.$$

Proof:

The proof follows by the same method used in the proof of Theorem 4.1, but noting that we do not in general have the condition, $\boldsymbol{\alpha}\boldsymbol{\tau}_1 > 0$. We shall now highlight two special sub-cases for reducible Q_0.

1. If λ_1 is unique, and $\boldsymbol{\alpha}\boldsymbol{\tau}_1 \neq 0$, then a similar Theorem 4.1 and Corollary 4.2 can be written for the irreducible Q_0 with these properties. The condition $\boldsymbol{\alpha}\boldsymbol{\tau}_1 \neq 0$, at least occurs when $\boldsymbol{\alpha}$ is strictly positive, but otherwise must be directly investigated.

2. If λ_1 is not unique, then it is possible that there exists

$$\boldsymbol{\alpha} T \left(\sum_{i \in \mathcal{A}_1} E_i N_i^{v-1} E_i \right) T^{-1} \boldsymbol{e} \neq 0,$$

for some $v > 1$. The term $\dfrac{t^{v-1} e^{\lambda_1 t}}{(v-1)!}$ makes equivalence to a negative exponential random variable impossible.

An extra significance of 2 above is that the term $\dfrac{t^{v-1} e^{\lambda_1 t}}{(v-1)!}$ occurs in the Erlang distribution (see (4.2.14)), from which we can conclude the following: an Erlang distribution of order n not only must necessarily have a reducible representation, but must have a Q_0 with a corresponding Jordan form, which has at least one Jordan block of dimension n with the Erlang parameter on its diagonal. For example, the irreducible *PH*-type distribution (note that Q_0 is reducible)

$$Q_0 = \begin{bmatrix} -3 & 0 & 0 & 0 & 0 & 3 \\ 0 & -3 & 3 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 & 4 \\ 0 & 1 & 1 & -6 & 4 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\text{with } \boldsymbol{\alpha} = \left(\frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0 \right),$$

is a complicated representation of an order 2 Erlang distribution of parameter $\lambda = 1$. It has a Jordan form with a Jordan block of unique maximal dimension 2, having eigenvalue -1 on its diagonal.

The possibilities within the reducible Q_0 case are many and should be best handled on an individual basis.

For the rest of this chapter, we shall consider the equivalence of *PH*-renewal processes and more generally *MAP*s to a Poisson process under stationary conditions. That is, the initial distribution of states will be taken to be the stationary distribution of phases for the *MAP*. This requires all of the *MAP* representations to be irreducible in the same sense as for the *PH*-renewal process representations. That is, there exists a unique positive probability vector $\boldsymbol{\nu}$ such that $\boldsymbol{\nu}(Q_0 + Q_1) = 0$.

4.2.2 A *PH*-renewal process

For the special case of a *PH*-renewal process we have $Q_1 = -Q_0\boldsymbol{e}\boldsymbol{\alpha}$, so that for $k = 1$, equation (4.1.2) becomes

$$\frac{-\boldsymbol{\pi} e^{Q_0 t_1} Q_0 \boldsymbol{e}}{\boldsymbol{\pi} e^{Q_0 t_1} \boldsymbol{e}} = \lambda, \text{ for } t_1 \in [0, \infty), \tag{4.2.15}$$

and for $k > 1$,

$$\frac{-(\boldsymbol{\pi} e^{Q_0 t_1} Q_0 \boldsymbol{e})(\boldsymbol{\alpha} e^{Q_0 t_2} Q_0 \boldsymbol{e}) \ldots \boldsymbol{\alpha} e^{Q_0 t_k} Q_0 \boldsymbol{e}}{(\boldsymbol{\pi} e^{Q_0 t_1} Q_0 \boldsymbol{e})(\boldsymbol{\alpha} e^{Q_0 t_2} Q_0 \boldsymbol{e}) \ldots \boldsymbol{\alpha} e^{Q_0 t_k} \boldsymbol{e}} = \frac{-\boldsymbol{\alpha} e^{Q_0 t_k} Q_0 \boldsymbol{e}}{\boldsymbol{\alpha} e^{Q_0 t_k} \boldsymbol{e}} = \lambda,$$

$$\text{for all } t_i \in [0, \infty) \text{ and } i \in \{1, 2, 3, \ldots\}. \tag{4.2.16}$$

We shall state the theorems and associated special case corollaries for irreducible Q_0 and then the reducible case can be handled in an analogous way to that given in the previous section.

Theorem 4.4 *A stationary PH-renewal process* $(\boldsymbol{\alpha}, Q_0)$*, where* Q_0 *is irreducible, is a Poisson process of rate* $\lambda > 0$ *if and only if* $\lambda = -\lambda_1$*, and for all* $(j, v) \in \{2, \ldots, s\} \times \{1, \ldots, P_j\}$*,*

$$\boldsymbol{\alpha} T \left(\sum_{i \in \mathcal{A}_j} E_i N_i^{v-1} E_i \right) T^{-1} \boldsymbol{e} \;=\; 0.$$

Proof:

Using equations (4.2.5) and (4.2.7), equation (4.2.16) can be re-written and re-arranged to get

$$\boldsymbol{\alpha} T \left(e^{\lambda_1 t}(\lambda_1 + \lambda) E_1 + \sum_{j=2}^{s} \sum_{i \in \mathcal{A}_j} \sum_{v=1}^{P_j} \left(\frac{t^{v-1}}{(v-1)!} e^{\lambda_j t} E_i N_i^{v-1} E_i \right) [(\lambda_j + \lambda) E_i + N_i] \right) T^{-1} \boldsymbol{e} = 0.$$

Now, $\boldsymbol{\tau}_1$ and $\bar{\boldsymbol{\tau}}_1$ are positive, so that $\boldsymbol{\alpha}\boldsymbol{\tau}_1\bar{\boldsymbol{\tau}}_1\boldsymbol{e} > 0$. Therefore as $e^{\lambda_j t} \neq 0$ for $j \in \{1, 2, 3, \dots, s\}$ for all $t \in [0, \infty)$, it must be that $\lambda = -\lambda_1$. Then by the same argument as for Theorem 4.1, it can be seen that for each $j \in \{2, 3, \dots, s\}$

$$\boldsymbol{\alpha}T\left(\sum_{i\in\mathcal{A}_j}\left(E_iN_i^{v-1}E_i\right)[(\lambda_j-\lambda_1)E_i+N_i]\right)T^{-1}\boldsymbol{e}=0,\ \text{for each } v\in\{1,2,\dots,P_j\}. \tag{4.2.17}$$

Now consider equation (4.2.15). Since we are considering the stationary process, we have that the initial distribution $\boldsymbol{\pi} = \boldsymbol{\nu}$, the stationary distribution admitted by the *PH*-renewal process. Then as Q_0 and $e^{Q_0 t}$ commute, the necessary condition $\lambda = -\lambda_1$ yields

$$\frac{-\boldsymbol{\nu}Q_0e^{Q_0t}\boldsymbol{e}}{\boldsymbol{\nu}e^{Q_0t}\boldsymbol{e}} = -\lambda_1 \text{ for all } t\in[0,\infty). \tag{4.2.18}$$

The next step is to establish a relationship between the stationary distribution $\boldsymbol{\nu}$ and the (initial) renewal probability vector $\boldsymbol{\alpha}$. Consider

$$\boldsymbol{\nu}Q = \boldsymbol{\nu}(Q_0 - Q_0\boldsymbol{e}\boldsymbol{\alpha}) = 0,$$

from which it can be seen that

$$\boldsymbol{\nu}Q_0 = (\boldsymbol{\nu}Q_0\boldsymbol{e})\boldsymbol{\alpha}.$$

It has been shown in Neuts [31] that we can always restrict our attention to the irreducible *PH*-renewal process representations, and from page 45 of [31] we also see that this implies that Q_0 is non-singular, and so we may write

$$\boldsymbol{\nu} = (\boldsymbol{\nu}Q_0\boldsymbol{e})\boldsymbol{\alpha}Q_0^{-1}. \tag{4.2.19}$$

Substituting this into equation (4.2.18), and noticing that $(\boldsymbol{\nu}Q_0\boldsymbol{e})$ is a scalar quantity, yields

$$\frac{-\boldsymbol{\alpha}e^{Q_0t}\boldsymbol{e}}{\boldsymbol{\alpha}Q_0^{-1}e^{Q_0t}\boldsymbol{e}} = -\lambda_1 \text{ for all } t\in[0,\infty). \tag{4.2.20}$$

Equations (4.2.5) and (4.2.7), imply that

$$Q_0^{-1}e^{Q_0 t} = T\left(\sum_{j=1}^{s}\sum_{i\in\mathcal{A}_j}\sum_{v=1}^{P_j}\left(\frac{e^{\lambda_j t}}{\lambda_j}\right)\mathcal{P}(\lambda_j,t,v)E_iN_i^{v-1}E_i\right)T^{-1},$$

where

$$\mathcal{P}(\lambda_j,t,v) \;=\; \sum_{z=0}^{v-1}\frac{t^z}{z!}\left(\frac{-1}{\lambda_j}\right)^{v-1-z}.$$

Equation (4.2.20) then can be written as

$$\frac{-\boldsymbol{\alpha}T\left(\sum_{j=1}^{s}\sum_{i\in\mathcal{A}_j}\sum_{v=1}^{P_j}\left(\frac{t^{v-1}}{(v-1)!}\right)e^{\lambda_j t}E_iN_i^{v-1}E_i\right)T^{-1}\boldsymbol{e}}{\boldsymbol{\alpha}T\left(\sum_{j=1}^{s}\sum_{i\in\mathcal{A}_j}\sum_{v=1}^{P_j}\left(\frac{e^{\lambda_j t}}{\lambda_j}\right)\mathcal{P}(\lambda_j,t,v)E_iN_i^{v-1}E_i\right)T^{-1}\boldsymbol{e}} = -\lambda_1.$$

Re-arranging we get

$$\begin{aligned}
&\boldsymbol{\alpha}T\left(e^{\lambda_1 t}E_1+\sum_{j=2}^{s}\sum_{i\in\mathcal{A}_j}\sum_{v=1}^{P_j}\left(\frac{t^{v-1}}{(v-1)!}\right)e^{\lambda_j t}E_iN_i^{v-1}E_i\right)T^{-1}\boldsymbol{e}\\
=\;&\lambda_1\boldsymbol{\alpha}T\left(\left(\frac{e^{\lambda_1 t}}{\lambda_1}\right)E_1+\sum_{j=2}^{s}\sum_{i\in\mathcal{A}_j}\sum_{v=1}^{P_j}\left(\frac{e^{\lambda_j t}}{\lambda_j}\right)\mathcal{P}(\lambda_j,t,v)E_iN_i^{v-1}E_i\right)T^{-1}\boldsymbol{e},
\end{aligned}$$

or

$$\boldsymbol{\alpha}T\left(\sum_{j=2}^{s}\sum_{i\in\mathcal{A}_j}\sum_{v=1}^{P_j}\left(\left(\frac{t^{v-1}}{(v-1)!}\right)-\left(\frac{\lambda_1}{\lambda_j}\right)\mathcal{P}(\lambda_j,t,v)\right)e^{\lambda_j t}E_iN_i^{v-1}E_i\right)T^{-1}\boldsymbol{e} \;=\; 0. \tag{4.2.21}$$

Now consider the polynomial,

$$\begin{aligned}
&\left(\frac{t^{v-1}}{(v-1)!}\right)-\left(\frac{\lambda_1}{\lambda_j}\right)\mathcal{P}(\lambda_j,t,v)\\
&=\left(\frac{t^{v-1}}{(v-1)!}\right)-\left(\frac{\lambda_1}{\lambda_j}\right)\sum_{z=0}^{v-1}\frac{t^z}{z!}\left(\frac{-1}{\lambda_j}\right)^{v-1-z}
\end{aligned}$$

$$= \begin{cases} 1 - \left(\frac{\lambda_1}{\lambda_j}\right) & \text{for } v = 1 \\ \\ \left(\frac{t^{v-1}}{(v-1)!}\right)\left(1 - \frac{\lambda_1}{\lambda_j}\right) \; - \; \left(\frac{\lambda_1}{\lambda_j}\right)\left(\sum_{z=0}^{v-2} \frac{t^z}{z!}\left(\frac{-1}{\lambda_j}\right)^{v-1-z}\right) & \text{for } v \geq 2. \end{cases} \tag{4.2.22}$$

Note that $\left(\frac{\lambda_1}{\lambda_j}\right) \neq 1$ for $j \in \{2, 3, \ldots, s\}$, because Q_0 is an irreducible matrix. Hence $1 - \left(\frac{\lambda_1}{\lambda_j}\right) \neq 0$, and so in equation (4.2.22), the term containing t^{v-1} has a non-zero coefficient. Therefore the polynomial of degree $v - 1$ is not identically zero for all $t \geq 0$. Thus, as (4.2.21) holds for all $t \in [0, \infty)$, we can use the same linear independence argument as in Theorem 4.1, to deduce that for each $j \in \{2, 3, \ldots, s\}$

$$\boldsymbol{\alpha} T \left(\sum_{i \in \mathcal{A}_j} E_i N_i^{v-1} E_i\right) T^{-1} \boldsymbol{e} \;=\; 0, \text{ for each } v \in \{1, 2, \ldots, P_j\}. \tag{4.2.23}$$

It is clear that these conditions are sufficient to satisfy (4.2.17), and hence the proof is complete. ■

Corollary 4.5 *A stationary PH-renewal process* $(\boldsymbol{\alpha}, Q_0)$, *where* Q_0 *is irreducible, and the Jordan canonical form has distinct eigenvalues in each Jordan block, is a Poisson process of rate* $\lambda > 0$, *if and only if* $\lambda = -\lambda_1$ *and for all* $j \in \{2, \ldots, g\}$,

$$\boldsymbol{\alpha} \boldsymbol{\tau}_{j_v} \;=\; 0, \text{ for all } v \in \{1, \ldots, p_j\} \tag{4.2.24}$$

$$\text{or } \bar{\boldsymbol{\tau}}_{j_v} \boldsymbol{e} \;=\; 0, \text{ for all } v \in \{1, \ldots, p_j\}. \tag{4.2.25}$$

Proof:

We proceed by starting with equation (4.2.23). In the case where each Jordan block has a distinct eigenvalue, (4.2.23) becomes, for each $j \in \{2, 3, \ldots, g\}$,

$$
\begin{array}{llrcl}
(v=1) & & \boldsymbol{\alpha} T\left(E_j\right) T^{-1} \boldsymbol{e} & = & 0 \\
(v=2) & & \boldsymbol{\alpha} T\left(E_j N_j E_j\right) T^{-1} \boldsymbol{e} & = & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
(v=p_j-1) & & \boldsymbol{\alpha} T\left(E_j N_j^{p_j-2} E_j\right) T^{-1} \boldsymbol{e} & = & 0 \\
(v=p_j) & & \boldsymbol{\alpha} T\left(E_j N_j^{p_j-1} E_j\right) T^{-1} \boldsymbol{e} & = & 0,
\end{array}
\tag{4.2.26}
$$

which is the same set of equations as (4.2.11), and so the result follows from the proof of Corollary 4.2.

■

4.2.3 General *MAP*s

For the general *MAP*, we consider the matrix parameters $Q_0 = D_0$ and $Q_1 = D_1$, in order to maintain the same notation. The proofs for the *PH*-renewal process equivalence were made much simpler because of the rank one nature of the matrix parameter $Q_1 = -Q_0 \boldsymbol{e}\boldsymbol{\alpha}$. This caused the infinite number of conditions in (4.1.2) to collapse to the two simple conditions (4.2.15) and (4.2.16). The same situation does not generally apply here because of the greater generality allowed for the matrix parameter Q_1. This means we have a more complex product form for $k \geq 2$.

Theorem 4.6 *A stationary general MAP, where the matrix Q_0 is irreducible, is Poisson of rate $\lambda > 0$, if and only if $\lambda = -\lambda_1$ and for all $k \in \mathbb{Z}^+$ with $(j(n), v(n)) \in \{1, \ldots, s\} \times \{1, \ldots, P_j(n)\}$, and for each $n \in \{1, 2, \ldots, k-1\}$, we have*

$$
\boldsymbol{\nu} \prod_{n=1}^{k-1} \left(T \sum_{i(n) \in \mathcal{A}_j(n)} E_{i(n)} N_{i(n)}^{v(n)-1} E_{i(n)} T^{-1} Q_1 \right) T \sum_{i(k) \in \mathcal{A}_j(k)} E_{i(k)} N_{i(k)}^{v(k)-1} E_{i(k)} T^{-1} \boldsymbol{e} \;=\; 0,
\tag{4.2.27}
$$

for all $(j(k), v(k)) \in \{2, \ldots, s\} \times \{1, \ldots, P_j(k)\}$.

Proof:

We will consider (4.1.2), first checking equivalence for the initial time interval $t = t_1$ and then for the subsequent time intervals t_k for $k \in \{2, 3, \ldots\}$. Re-arranging equation (4.1.2) for the case $k = 1$, and using the fact that $Q_0\boldsymbol{e} = -Q_1\boldsymbol{e}$, we get

$$\boldsymbol{\nu} e^{Q_0 t_1}(Q_0\boldsymbol{e} + \lambda\boldsymbol{e}) = 0.$$

Using equations (4.2.5) and (4.2.7), this can be re-written as

$$\boldsymbol{\nu}T\left(e^{\lambda_1 t_1}(\lambda_1 + \lambda)E_1\right)T^{-1}\boldsymbol{e} \quad +$$

$$\boldsymbol{\nu}T\left(\sum_{j=2}^{s}\sum_{i\in\mathcal{A}_j}\sum_{v=1}^{P_j}\left(\frac{t_1^{v-1}}{(v-1)!}e^{\lambda_j t_1}E_iN_i^{v-1}E_i\right)[(\lambda_j + \lambda)E_i + N_i]\right)T^{-1}\boldsymbol{e} = 0, \tag{4.2.28}$$

which by the same reasoning as for the proof of Theorem 4.1, yields the necessary condition that $\lambda = -\lambda_1$, and for all $j \in \{2, 3, \ldots, s\}$,

$$\boldsymbol{\nu}T\sum_{i\in\mathcal{A}_j}E_iN_i^{v-1}E_i\left[(\lambda_j - \lambda_1)E_i + N_i\right]T^{-1}\boldsymbol{e} = 0, \text{ for all } v \in \{1, 2, 3, \ldots, P_j\}.$$

The nilpotent matrices N_i are such that for each $j \in \{2, 3, \ldots, s\}$, this equation can be re-written as

$$\begin{aligned}
&(v = P_j) && \boldsymbol{\nu}T\sum_{i\in\mathcal{A}_j}E_iN_i^{P_j-1}E_i(\lambda_j - \lambda_1)T^{-1}\boldsymbol{e} = 0\\
&(v = P_j - 1) && \boldsymbol{\nu}T\left(\sum_{i\in\mathcal{A}_j}E_iN_i^{P_j-2}E_i(\lambda_j - \lambda_1) + E_iN_i^{P_j-1}E_i\right)T^{-1}\boldsymbol{e} = 0\\
&\quad\vdots && \qquad\vdots\\
&(v = 1) && \boldsymbol{\nu}T\left(\sum_{i\in\mathcal{A}_j}E_i(\lambda_j - \lambda_1) + E_iN_iE_i\right)T^{-1}\boldsymbol{e} = 0.
\end{aligned} \tag{4.2.29}$$

Therefore, noting that $\lambda_j - \lambda_1 \neq 0$ for all $j \in \{2, 3, \ldots, s\}$, we have for all $(j, v) \in \{2, 3, \ldots, s\} \times \{1, 2, 3, \ldots, P_j\}$, that

$$\boldsymbol{\nu}T\sum_{i\in\mathcal{A}_j}E_iN_i^{v-1}E_iT^{-1}\boldsymbol{e} = 0. \tag{4.2.30}$$

Equation (4.1.2), for the case $k \geq 2$, can be re-arranged, again using the fact that $Q_0\boldsymbol{e} = -Q_1\boldsymbol{e}$, to get

$$\boldsymbol{\nu}\left(\prod_{n=1}^{k-1} e^{Q_0 t_n} Q_1\right) e^{Q_0 t_k}(Q_0\boldsymbol{e} + \lambda\boldsymbol{e}) = 0.$$

Using equations (4.2.5) and (4.2.7), with the above necessary condition that $\lambda = -\lambda_1$, it follows that for all $k \in \mathbb{Z}^+$, with $(j(n), v(n)) \in \{1, \dots, s\} \times \{1, \dots, P_j(n)\}$ for each $n \in \{1, 2, \dots, k-1\}$, we have for

$$\boldsymbol{\Lambda} = \boldsymbol{\nu}\prod_{n=1}^{k-1}\left[T\sum_{j(n)=1}^{s}\sum_{i(n)\in\mathcal{A}_{j(n)}}\sum_{v(n)=1}^{P_{j(n)}}\frac{{t_n}^{v(n)-1}}{(v(n)-1)!}e^{\lambda_{j(n)}t_n}E_{i(n)}{N_{i(n)}}^{v(n)-1}E_{i(n)}T^{-1}Q_1\right]$$

and

$$\boldsymbol{\Xi} = T\sum_{j(k)=2}^{s}\sum_{i(k)\in\mathcal{A}_{j(k)}}\sum_{v(k)=1}^{P_{j(k)}}\frac{{t_k}^{v(k)-1}}{(v(k)-1)!}e^{\lambda_{j(k)}t_k}E_{i(k)}{N_{i(k)}}^{v(k)-1}E_{i(k)}\left[(\lambda_{j(k)}-\lambda_1)E_{i(k)} + N_{i(k)}\right]T^{-1}\boldsymbol{e},$$

for $(j(k), v(k)) \in \{2, \dots, s\} \times \{1, \dots, P_j(k)\}$, that

$$\boldsymbol{\Lambda}\boldsymbol{\Xi} = 0.$$

The functions

$$\frac{{t_\ell}^{v(\ell)-1}}{(v(\ell)-1)!}e^{\lambda_{j(\ell)}t_\ell},$$

for any choice of $v(\ell) \in \{1, 2, \dots, P_{j(\ell)}\}$ and $j(\ell) \in \{1, 2, \dots, s\}$, are clearly linearly independent for any choice of $\ell \in \{1, 2, \dots, k\}$. This is because the variables t_ℓ for $\ell \in \{1, 2, \dots, k\}$ are independent and for each ℓ, the $\lambda_{j(\ell)}$ are distinct for each $j(\ell) \in \{1, 2, \dots, s\}$. This follows, by the independence argument presented in the proof of Theorem 4.1. We also note that

$$\lambda_{j(k)} - \lambda_1 \neq 0, \text{ for all } j(k) \in \{2, 3, \dots, s\},$$

and

$$e^{\lambda_{j(\ell)}t_\ell} > 0, \text{ for all } t_\ell \in [0, \infty) \text{ and } \ell \in \{1, 2, \dots, k\}.$$

Using these facts, the coefficient terms of the functions

$$\frac{{t_\ell}^{v(\ell)-1}}{(v(\ell)-1)!}e^{\lambda_{j(\ell)}t_\ell}$$

in $\mathbf{\Lambda\Xi}$ must be zero. That is, for all $k \in \mathbb{Z}^+$, with $(j(n), v(n)) \in \{1, \ldots, s\} \times \{1, \ldots, P_j(n)\}$ for each $n \in \{1, 2, \ldots, k-1\}$, we have for

$$\mathbf{\Upsilon} = \boldsymbol{\nu} \prod_{n=1}^{k-1} \left(T \sum_{i(n) \in \mathcal{A}_{j(n)}} E_{i(n)} {N_{i(n)}}^{v(n)-1} E_{i(n)} T^{-1} Q_1 \right),$$

and

$$\mathbf{\Phi} = T \sum_{i(k) \in \mathcal{A}_{j(k)}} E_{i(k)} {N_{i(k)}}^{v(k)-1} E_{i(k)} \left[(\lambda_{j(k)} - \lambda_1) E_{i(k)} + N_{i(k)} \right] T^{-1} \boldsymbol{e}, \quad (4.2.31)$$

for $(j(k), v(k)) \in \{2, \ldots, s\} \times \{1, \ldots, P_j(k)\}$, that

$$\mathbf{\Upsilon\Phi} = 0.$$

The nilpotent matrices N_i are such that for each $j(k) \in \{2, 3, \ldots, s\}$, (4.2.31) can be re-written for each $v(k)$ as

$$(v(k) = P_{j(k)}) \qquad T \sum_{i \in \mathcal{A}_{j(k)}} E_i N_i^{P_{j(k)}-1} E_i (\lambda_{j(k)} - \lambda_1) T^{-1} \boldsymbol{e} = 0$$

$$(v(k) = P_{j(k)} - 1) \qquad T \left(\sum_{i \in \mathcal{A}_{j(k)}} E_i N_i^{P_{j(k)}-2} E_i (\lambda_{j(k)} - \lambda_1) + E_i N_i^{P_{j(k)}-1} E_i \right) T^{-1} \boldsymbol{e} = 0$$

$$\vdots \qquad\qquad \vdots$$

$$(v(k) = 1) \qquad T \left(\sum_{i \in \mathcal{A}_{j(k)}} E_i (\lambda_{j(k)} - \lambda_1) + E_i N_i E_i \right) T^{-1} \boldsymbol{e} = 0.$$

Therefore, we have for all $k \in \mathbb{Z}^+$, with $(j(n), v(n)) \in \{1, \ldots, s\} \times \{1, \ldots, P_j(n)\}$ for each $n \in \{1, 2, \ldots, k-1\}$, that

$$\boldsymbol{\nu} \left(\prod_{n=1}^{k-1} T \sum_{i(n) \in \mathcal{A}_j(n)} E_{i(n)} N_{i(n)}^{v(n)-1} E_{i(n)} T^{-1} Q_1 \right) T \sum_{i(k) \in \mathcal{A}_j(k)} E_{i(k)} N_{i(k)}^{v(k)-1} E_{i(k)} T^{-1} \boldsymbol{e} = 0, \quad (4.2.32)$$

for all $(j(k), v(k)) \in \{2, \ldots, s\} \times \{1, \ldots, P_j(k)\}$.

■

Corollary 4.7 *A stationary general MAP, where Q_0 is irreducible and the Jordan canonical form has distinct eigenvalues in each Jordan block, is Poisson of rate $\lambda > 0$, if and only if $\lambda = -\lambda_1$ and*

$$\begin{aligned} \boldsymbol{\nu}\boldsymbol{\tau}_{j_v} &= 0, \text{ for all } v \in \{1,2,3,\ldots,p_j\}, && (4.2.33) \\ \text{or } \bar{\boldsymbol{\tau}}_{j_v}\boldsymbol{e} &= 0, \text{ for all } v \in \{1,2,3,\ldots,p_j\} && (4.2.34) \end{aligned}$$

for all $j \in \{2,\ldots,g\}$ and for $k \geq 2$, we have

$$\boldsymbol{\nu}\left(\prod_{n=1}^{k-1} TE_{j(n)}{N_{j(n)}}^{v(n)-1}E_{j(n)}T^{-1}Q_1\right)TE_{j(k)}{N_{j(k)}}^{v(k)-1}E_{j(k)}T^{-1}\boldsymbol{e} = 0, \quad (4.2.35)$$

for all $(j(k), v(k)) \in \{2,\ldots,s\} \times \{1,\ldots,p_{j(k)}\}$ and $n \in \{1,2,\ldots,k-1\}$ with $(j(n), v(n)) \in \{1,\ldots,s\} \times \{1,\ldots,p_{j(n)}\}$.

Proof:

We will consider the appropriate equations from the proof of Theorem 4.6, following the same method of approach, whilst noting that each $\mathcal{A}_j$ has only one element in this case.

For $k = 1$, we get the necessary condition from equation (4.2.28), that

$$\lambda = -\lambda_1, \quad (4.2.36)$$

and from equation (4.2.30), for each $j \in \{2,3,\ldots,g\}$, we get

$$\begin{aligned} (v=1) \qquad \boldsymbol{\nu}T\left(E_j\right)T^{-1}\boldsymbol{e} &= 0 \\ (v=2,\ldots,p_j) \quad \boldsymbol{\nu}T\left(E_jN_j^{p_j-1}E_j\right)T^{-1}\boldsymbol{e} &= 0. \end{aligned}$$

These are similar equalities to (4.2.11), so that for all $j \in \{2,3,\ldots,g\}$, we have

$$\begin{aligned} \boldsymbol{\nu}\boldsymbol{\tau}_{j_v} &= 0, \text{ for all } v \in \{1,2,3,\ldots,p_j\} \\ \text{or } \bar{\boldsymbol{\tau}}_{j_v}\boldsymbol{e} &= 0, \text{ for all } v \in \{1,2,3,\ldots,p_j\}. \end{aligned}$$

When $k \geq 2$, we have from equation (4.2.32) that

$$\boldsymbol{\nu}\left(\prod_{n=1}^{k-1} TE_{j(n)}{N_{j(n)}}^{v(n)-1}E_{j(n)}T^{-1}Q_1\right)TE_{j(k)}{N_{j(k)}}^{v(k)-1}E_{j(k)}T^{-1}\boldsymbol{e} = 0, \quad (4.2.37)$$

for all $(j(k), v(k)) \in \{2,\ldots,s\} \times \{1,\ldots,p_{j(k)}\}$ and $n \in \{1,2,\ldots,k-1\}$ with $(j(n), v(n)) \in \{1,\ldots,s\} \times \{1,\ldots,p_{j(n)}\}$.

■

4.3 Special case of diagonalisable Q_0

If the $m \times m$ matrix Q_0 is diagonalisable, then it has m independent eigenvectors and may be written in spectral form as

$$Q_0 = \sum_{j=1}^{m} \lambda_j \boldsymbol{r}_j \boldsymbol{l}_j, \tag{4.3.1}$$

where λ_j are the eigenvalues of Q_0 with corresponding left eigenvectors $\boldsymbol{l}_j$ and corresponding right eigenvectors $\boldsymbol{r}_j$.

If there are $s < m$ distinct eigenvalues, let $\mathcal{A}_j$, for $j = 1, 2, \ldots, s$ be sets of indices for which the corresponding eigenvalue is λ_j.

Then Q_0 may also be written as

$$Q_0 = \sum_{j=1}^{s} \sum_{i \in \mathcal{A}_j} \lambda_j \boldsymbol{r}_i \boldsymbol{l}_i, \tag{4.3.2}$$

where $\boldsymbol{r}_i$ is the right eigenvector and $\boldsymbol{l}_i$ is the left eigenvector corresponding to the i^{th} eigenvalue λ_j in set $\mathcal{A}_j$.

Note: Because Q_0 is assumed to be an irreducible non-negative matrix, we have that $\mathcal{A}_1 \equiv \{1\}$ since λ_1 is the eigenvalue of Q_0 of maximal real part, which is unique (see Seneta [48]).

The proofs of all of the subsequent Theorems and Corollaries follow directly from their counterparts in the general case by re-writing the result such that each Jordan block is 1×1.

4.3.1 A *PH*-random variable

Theorem 4.8 *A PH-random variable* $(\boldsymbol{\alpha}, Q_0)$, *where* Q_0 *is irreducible and diagonalisable, is negative exponential with parameter* $\lambda > 0$, *if and only if* $\lambda = -\lambda_1$, *and for all* $j \in \{2, 3, \ldots, s\}$,

$$\boldsymbol{\alpha} \sum_{i \in \mathcal{A}_j} \boldsymbol{r}_i \boldsymbol{l}_i \boldsymbol{e} = 0.$$

Corollary 4.9 *A PH-random variable* $(\boldsymbol{\alpha}, Q_0)$*, where* Q_0 *is irreducible and has distinct eigenvalues, is negative exponential with parameter* $\lambda > 0$*, if and only if* $\lambda = -\lambda_1$*, and for all* $j \in \{2, 3, \dots, m\}$*,*

$$\boldsymbol{\alpha}\boldsymbol{r}_j = 0, \ \textit{or}\ \boldsymbol{l}_j\boldsymbol{e} = 0 \ .$$

4.3.2 A *PH*-renewal process

Theorem 4.10 *A PH-renewal process* $(\boldsymbol{\alpha}, Q_0)$*, where* Q_0 *is irreducible and diagonalisable, is Poisson with parameter* $\lambda > 0$*, if and only if* $\lambda = -\lambda_1$*, and for all* $j \in \{2, 3, \dots, s\}$*,*

$$\boldsymbol{\alpha} \sum_{i \in \mathcal{A}_j} \boldsymbol{r}_i \boldsymbol{l}_i \boldsymbol{e} = 0 \ .$$

Corollary 4.11 *A stationary PH-renewal process* $(\boldsymbol{\alpha}, Q_0)$*, where* Q_0 *is irreducible and has distinct eigenvalues, is a Poisson process of rate* $\lambda > 0$*, if and only if* $\lambda = -\lambda_1$*, and for all* $j \in \{2, 3, \dots, m\}$*,*

$$\boldsymbol{\alpha}\boldsymbol{r}_j = 0, \ \textit{or}\ \boldsymbol{l}_j\boldsymbol{e} = 0 \ .$$

Note, it can be easily shown that for a given j we have

$$\boldsymbol{\nu}\boldsymbol{r}_j = 0, \ \text{if and only if}\ \boldsymbol{\alpha}\boldsymbol{r}_j = 0,$$

by using $\boldsymbol{\nu} = (\boldsymbol{\nu} Q_0 \boldsymbol{e})\boldsymbol{\alpha} Q_0^{-1}$ shown in (4.2.19).

4.3.3 General *MAP*s

Theorem 4.12 *A stationary general MAP, where* Q_0 *is irreducible and diagonalisable, is Poisson of rate* $\lambda > 0$*, if and only if* $\lambda = -\lambda_1$*, and for all* $(k, j(n), j(k)) \in \mathbb{Z}^+ \times \{1, 2, \dots, s\} \times \{2, 3, \dots, s\}$*,*

$$\boldsymbol{\nu}\prod_{n=1}^{k-1}\left(\sum_{i(n)\in\mathcal{A}_{j(n)}}\boldsymbol{r}_{i(n)}\boldsymbol{l}_{i(n)}Q_1\right)\sum_{i(k)\in\mathcal{A}_{j(k)}}\boldsymbol{r}_{i(k)}\boldsymbol{l}_{i(k)}\boldsymbol{e} \quad = \quad 0.$$

For the following Corollary, the next definitions are useful.

$$\begin{aligned} I_R(0) &\stackrel{def}{=} \{j \geq 1 : \boldsymbol{\nu}\boldsymbol{r}_j \neq 0\}, \\ I_L(0) &\stackrel{def}{=} \{j \geq 2 : \boldsymbol{l}_j\boldsymbol{e} \neq 0\}, \end{aligned} \qquad (4.3.3)$$

and for all $i \geq 1$

$$I_R(i) \stackrel{def}{=} \{x : \boldsymbol{l}_j Q_1 \boldsymbol{r}_x \neq 0, \text{ for some } j \in I_R(i-1)\} \qquad (4.3.4)$$

$$I_L(i) \stackrel{def}{=} \{y : \boldsymbol{l}_y Q_1 \boldsymbol{r}_j \neq 0, \text{ for some } j \in I_L(i-1)\} \qquad (4.3.5)$$

Corollary 4.13 *A stationary general MAP, where Q_0 is irreducible and has distinct eigenvalues, is Poisson of rate $\lambda > 0$, if and only if $\lambda = -\lambda_1$,*

$$\boldsymbol{\nu}\boldsymbol{r}_j = 0 \text{ or } \boldsymbol{l}_j\boldsymbol{e} = 0, \text{ for all } j \in \{2, 3, \ldots m\} \qquad (4.3.6)$$

and for all $i \geq 0$ and $x \in I_R(i), y \in I_L(i)$, the following hold:

$$\boldsymbol{l}_x Q_1 \boldsymbol{r}_y = 0, \text{ and} \qquad (4.3.7)$$

for all $z \notin \{I_L(i) \cup I_R(i)\}$ either

$$\boldsymbol{l}_x Q_1 \boldsymbol{r}_z = 0, \text{ or } \boldsymbol{l}_z Q_1 \boldsymbol{r}_y = 0. \qquad (4.3.8)$$

Proof:

The requirement that $\lambda = -\lambda_1$ and (4.3.6) follow directly from Corollary 4.7. We may re-write equation (4.2.37) for the case where we have distinct eigenvalues as

$$\boldsymbol{\nu}\left(\prod_{n=1}^{k-1}\boldsymbol{r}_{j(n)}\boldsymbol{l}_{j(n)}Q_1\right)\boldsymbol{r}_{j(k)}\boldsymbol{l}_{j(k)}\boldsymbol{e} = 0, \qquad (4.3.9)$$

for all $(j(k)) \in \{2, \ldots, s\}$ and $n \in \{1, 2, \ldots, k-1\}$ with $(j(n)) \in \{1, \ldots, s\}$.

Consider $k = 2$ in (4.3.9), which yields

$$\boldsymbol{l}_{j(1)} Q_1 \boldsymbol{r}_{j(2)} = 0, \text{ for all } j(1) \in I_R(0), j(2) \in I_L(0).$$

Now consider $k = 3$ in (4.3.9), we can see that we require

$$\boldsymbol{l}_{j(1)} Q_1 \boldsymbol{r}_{j(2)} \boldsymbol{l}_{j(2)} Q_1 \boldsymbol{r}_{j(3)} = 0,$$

for all $j(1) \in I_R(0)$, $j(3) \in I_L(0)$. This along with the deduction from $k = 2$ implies that either

$$\boldsymbol{l}_{j(1)} Q_1 \boldsymbol{r}_{j(2)} = 0, \text{ or } \boldsymbol{l}_{j(2)} Q_1 \boldsymbol{r}_{j(3)} = 0,$$
$$\text{for all } j(1) \in I_R(0), j(3) \in I_L(0), j(2) \notin \{I_R(0) \cup I_L(0)\}.$$

If we then consider $k = 4$ in (4.3.9), we can see that we require

$$\boldsymbol{l}_{j(1)} Q_1 \boldsymbol{r}_{j(2)} \boldsymbol{l}_{j(2)} Q_1 \boldsymbol{r}_{j(3)} \boldsymbol{l}_{j(3)} Q_1 \boldsymbol{r}_{j(4)} = 0,$$

for all $j(1) \in I_R(0)$ and $j(4) \in I_L(0)$. Using the above results and the definitions in (4.3.5), we see that

$$\boldsymbol{l}_{j(1)} Q_1 \boldsymbol{r}_{j(2)} \neq 0 \text{ for all } j(2) \in I_R(1) \text{ and}$$
$$\boldsymbol{l}_{j(3)} Q_1 \boldsymbol{r}_{j(4)} \neq 0 \text{ for all } j(3) \in I_L(1),$$

which implies that we must have

$$\boldsymbol{l}_{j(2)} Q_1 \boldsymbol{r}_{j(3)} = 0,$$
$$\text{for all } j(2) \in I_R(1), j(3) \in I_L(1).$$

For $k = 5$ we have

$$\boldsymbol{l}_{j(1)} Q_1 \boldsymbol{r}_{j(2)} \boldsymbol{l}_{j(2)} Q_1 \boldsymbol{r}_{j(3)} \boldsymbol{l}_{j(3)} Q_1 \boldsymbol{r}_{j(4)} \boldsymbol{l}_{j(4)} Q_1 \boldsymbol{r}_{j(5)} = 0,$$

for all $j(1) \in I_R(0)$ and $j(5) \in I_L(0)$. Again using the previous result, it can be seen that this reduces to

$$\boldsymbol{l}_{j(2)} Q_1 \boldsymbol{r}_{j(3)} \boldsymbol{l}_{j(3)} Q_1 \boldsymbol{r}_{j(4)} = 0,$$

for all $j(2) \in I_R(1)$ and $j(4) \in I_L(1)$. These conditions are similar to those for the case $k = 3$ and in fact similarly imply that either

$$\boldsymbol{l}_{j(2)} Q_1 \boldsymbol{r}_{j(3)} = 0, \text{ or } \boldsymbol{l}_{j(3)} Q_1 \boldsymbol{r}_{j(4)} = 0,$$

for all $j(2) \in I_R(1)$,$j(4) \in I_L(1)$ and $j(3) \notin \{I_R(1) \cup I_L(1)\}$.

By further considering $k = 6, 7, \ldots$ the result can easily be established as the conditions repeat.

■

It is not clear from the statement of Corollary 4.13 whether there are finitely or infinitely many conditions in (4.3.7) and (4.3.8). However, there can only be a finite number of *unique* conditions, since there is a finite number of eigenvectors for any given finite matrix Q_0. Furthermore, if there exists a K such that $I_L(K) = I_R(K) = \emptyset$, then there will be no conditions in equations (4.3.7) and (4.3.8) for $i > K$. In fact if there exists a K' such that only one of $I_L(K') = \emptyset$ or $I_R(K') = \emptyset$, there will be no conditions in (4.3.7) for $i > K$. and by considering (4.3.9), it is easy to see that there are no *new* conditions in (4.3.8) for $i > K$.

At this point it is worth noting that (4.3.7) and (4.3.8) are not a consequence of $\lambda = -\lambda_1$ and (4.3.6). For example, let us consider the following non-Poisson example which satisfies $\lambda = -\lambda_1$ and (4.3.6).

$$Q_0 = \begin{bmatrix} -3 & 3 & 0 \\ 0 & -6 & 4 \\ 0 & 0 & -1 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}, \text{ so that } \boldsymbol{\nu} = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right).$$

It can easily be verified that $I_R(0) = \{1, 2\}$ and $I_L(0) = \{3\}$ while

$$\boldsymbol{l}_1 Q_1 \boldsymbol{r}_3 \neq 0, \text{ and } \boldsymbol{l}_2 Q_1 \boldsymbol{r}_3 \neq 0.$$

We will now recall the three *MAP* examples, given at the end of Chapter 3, which are complicated representations of a Poisson process of rate 1 under stationary conditions. Each example exhibits a distinct characteristic form covered in

Corollary 4.13. In (3.3.10), we had

$$D_0 = \begin{bmatrix} -2 & (\frac{1}{2}) & (\frac{1}{2}) \\ 1 & -4 & 1 \\ (\frac{1}{2}) & 1 & -2 \end{bmatrix}, D_1 = \begin{bmatrix} (\frac{1}{2}) & 0 & (\frac{1}{2}) \\ 1 & 1 & 0 \\ 0 & 0 & (\frac{1}{2}) \end{bmatrix},$$

$$\text{with } \boldsymbol{\nu} = \left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right),$$

in which $\boldsymbol{\nu}$ is the left eigenvector of the matrix Q_0 corresponding to λ_1.
In (3.3.11), we had

$$D_0 = \begin{bmatrix} -4 & 2 & 1 \\ 5 & -8 & 2 \\ 1 & 2 & -4 \end{bmatrix}, D_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\text{with } \boldsymbol{\nu} = \left(\frac{9}{20}, \frac{1}{5}, \frac{7}{20}\right),$$

in which $\boldsymbol{e}$ is the right eigenvector of the matrix Q_0 corresponding to λ_1.
Finally in (3.3.12), we had

$$D_0 = \begin{bmatrix} -3 & 3 & 0 \\ 0 & -6 & 4 \\ 0 & 0 & -1 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ (\frac{1}{2}) & (\frac{1}{2}) & 0 \end{bmatrix},$$

$$\text{with } \boldsymbol{\nu} = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right),$$

in which neither $\boldsymbol{\nu}$ or $\boldsymbol{e}$ is an eigenvector corresponding to λ_1, but $\boldsymbol{\nu}\boldsymbol{r}_2 = 0$ and $\boldsymbol{l}_3\boldsymbol{e} = 0$.

Chapter 5

Minimal order Phase representation

Our discussion in Chapter 4 involves a consideration of the higher order representations for the negative exponential distribution. This issue is closely allied to the question of finding a representation with the least possible number of states for any given *PH*-distribution. As noted by Neuts in [31], any given *PH*-distribution may have many distinct irreducible representations. This question of minimal order representation is non-trivial and has been addressed in the literature using many different approaches. Here we include a short section on some of the literature which considers the problem of minimal order phase representations as well as of that of non-uniqueness.

Some of our results in Chapter 4 will be expressed in terms of the framework of the results of other authors cited in this chapter.

In [37], O'Cinneide introduced two properties of *PH*-generators, called *PH*-simplicity and *PH*-majorization, useful in the study of *PH*-type distributions and their representations. *PH*-simplicity concerns the possibility of a given distribution having more than one representation in terms of the same *PH*-generator. The *PH*-generator Q_0 is described as being *PH*-simple if there is a unique initial vector $\boldsymbol{\alpha}$ for every distribution representable by this *PH*-generator. *PH*-majorization concerns the possibility that a *PH*-generator Q_{0_a} may provide representations for all the distributions representable by another *PH*-generator Q_{0_b}, in which case Q_{0_a} is said to *PH*-majorize Q_{0_b}.

One result which O'Cinneide states using these ideas is the fact that every *PH*-distribution which has an upper triangular representation also has a bi-diagonal representation. If we consider the set of bi-diagonal *PH*-generators and the set of upper triangular *PH*-generators, then in terms of *PH*-majorization, the set of bi-diagonal *PH*-generators *PH*-majorizes the set of upper triangular *PH*-generators. In fact these two constructs turn out to give rise to the same family of distributions, as proven in [37].

A theorem on minimal order follows, using our results and O'Cinneide's work on *PH*-simplicity.

Theorem 5.1 *A representation* $(\boldsymbol{\alpha}, Q_0)$ *of a PH-type distribution, where the* $n \times n$ *matrix* Q_0 *is PH-simple and has distinct eigenvalues, is of minimal order if* $\boldsymbol{\alpha}$ *is not orthogonal to any of the right eigenvectors of* Q_0*.*

Proof:

The representation $(\boldsymbol{\alpha}, Q_0)$ has distribution function given by $1 - \boldsymbol{\alpha} e^{Q_0 t} \boldsymbol{e}$ which can be written as

$$1 - \boldsymbol{\alpha} \sum_{j=1}^{n} e^{\lambda_j t} \boldsymbol{r}_j \boldsymbol{l}_j \boldsymbol{e}.$$

From Theorem 2 of [37], because Q_0 is *PH*-simple there is no left eigenvector of Q_0 which is orthogonal to $\boldsymbol{e}$. Therefore, since $\boldsymbol{\alpha}$ is not orthogonal to any right eigenvector of Q_0, all of the coefficients $\boldsymbol{\alpha} \boldsymbol{r}_j \boldsymbol{l}_j \boldsymbol{e}$ of $e^{\lambda_j t} > 0$ are non-zero. From Theorem 12 in Chapter 2 of Coddington [13], it can be seen that because the λ_j are distinct, the functions $\boldsymbol{\alpha} e^{\lambda_j t} \boldsymbol{r}_j \boldsymbol{l}_j \boldsymbol{e} \neq 0$ for each $j \in \{1, 2, \ldots n\}$ are linearly independent. This representation must therefore be unique, and has minimum order.

∎

For a interesting example of $(\boldsymbol{\alpha}, Q_0)$ which satisfies Theorem 5.1, let us consider the following bi-diagonal PH-distribution

$$Q_0(1) = \begin{pmatrix} -1 & 1 \\ 0 & -r \end{pmatrix},$$

with $\boldsymbol{\alpha}(1) = (1, 0)$, for any given finite $r > 0$. This distribution can be shown to have an infinite number of alternative irreducible bi-diagonal representations of the following form

$$Q_0(n) = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -2 & 2 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -n & n \\ 0 & \cdots & \cdots & 0 & -r \end{pmatrix},$$

with $\boldsymbol{\alpha}(n) = (\frac{1}{n}, \ldots, \frac{1}{n}, 0)$ for $n \in \{1, 2, \ldots\}$. These $(\boldsymbol{\alpha}(n), Q_0(n))$ representations demonstrate the contra-positive of Theorem 5.1 since they are not of minimal order and $\boldsymbol{\alpha}\boldsymbol{\tau}_\ell \equiv 0$ for all $\ell \in \{2, 3, \ldots, n\}$. Although this example displays that a reduction of order is possible, even while retaining the bi-diagonal form, this is not always the case. Consider the following example, taken from [37]. Let

$$Q_0 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & -4 \end{pmatrix} \tag{5.0.1}$$

and

$$\boldsymbol{\alpha} = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right).$$

The eigenvalues of Q_0 are obviously $-1, -2, -3, -4$. Also, $\boldsymbol{\alpha}$ is orthogonal to the right eigenvector of Q_0 which corresponds to the eigenvalue -4. This might indicate a possible reduction of order for the representation, and an obvious attempt could be to let

$$\hat{Q}_0 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{pmatrix}$$

and find an appropriate $\hat{\boldsymbol{\alpha}}$ such that

$$\boldsymbol{\alpha} e^{Q_0 t}\boldsymbol{e} = \hat{\boldsymbol{\alpha}} e^{\hat{Q}_0 t}\boldsymbol{e} \text{ for all } t \geq 0.$$

The solution for $\hat{\boldsymbol{\alpha}}\boldsymbol{e} = 1$, however, is $\hat{\boldsymbol{\alpha}} = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$. This does not give a valid phase-type description and therefore this distribution has a minimal bi-diagonal representation of order 4. The further question of minimal order in some form other than a bi-diagonal representation is not clear, demonstrating the non-trivial nature of minimal order.

One of our earlier results concerning the higher order representations for the negative exponential distribution, can be expressed in terms of *PH*-simplicity, as follows.

Corollary 5.2 *A PH* $(\boldsymbol{\alpha}, Q_0)$ *random variable, where the PH-generator* Q_0 *is irreducible, PH-simple, and its Jordan canonical form has distinct eigenvalues in each Jordan block, is negative exponential with parameter* $\lambda > 0$, *if and only if* $\lambda = -\lambda_1$ *and* $\boldsymbol{\alpha}$ *is the left eigenvector of* Q_0 *corresponding to* λ_1.

Proof: From Corollary 4.2, it can be seen that for all $j \in \{2, \ldots, g\}$, we require

$$\begin{aligned} \boldsymbol{\alpha}\boldsymbol{\tau}_{j_v} &= 0, \text{ for each } v \in \{1, \ldots, p_j\}, \\ \text{or } \bar{\boldsymbol{\tau}}_{j_v}\boldsymbol{e} &= 0, \text{ for each } v \in \{1, \ldots, p_j\}, \end{aligned}$$

in order for the distribution to be a negative exponential distribution. However, by Theorem 2, of [37] $\bar{\boldsymbol{\tau}}_{j_v}\boldsymbol{e} \neq 0$, for all $j \in \{2, \ldots, g\}$, since Q_0 is *PH*-simple. Hence $\boldsymbol{\alpha}\boldsymbol{\tau}_{j_v} = 0$ for all $(j, v) \in \{2, \ldots, g\} \times \{1, \ldots, p_j\}$, and so for equivalence to a negative exponential distribution of rate λ we need

$$\begin{aligned} e^{-\lambda t} &= \boldsymbol{\alpha} e^{Q_0 t}\boldsymbol{e} \\ &= \boldsymbol{\alpha} e^{\lambda_1 t}\boldsymbol{\tau}_1\bar{\boldsymbol{\tau}}_1\boldsymbol{e}. \end{aligned}$$

The eigenvectors $\boldsymbol{\tau}_1$ and $\bar{\boldsymbol{\tau}}_1$ can always be normalised such that

$$\bar{\boldsymbol{\tau}}_1\boldsymbol{\tau}_1 = \bar{\boldsymbol{\tau}}_1\boldsymbol{e} = 1,$$

so that

$$= \quad \boldsymbol{\alpha}\boldsymbol{\tau}_1 e^{\lambda_1 t},$$

and hence $\boldsymbol{\alpha}\boldsymbol{\tau}_1 \equiv 1$ and $\lambda_1 = -\lambda$. This then implies that

$$\boldsymbol{\alpha}Q_0 = \boldsymbol{\alpha}\boldsymbol{\tau}_1\bar{\boldsymbol{\tau}}_1\lambda_1 = \bar{\boldsymbol{\tau}}_1\lambda_1,$$

but as $\bar{\boldsymbol{\tau}}_1$ is the left eigenvector of Q_0 corresponding to the eigenvalue of maximal real part, we may write

$$\bar{\boldsymbol{\tau}}_1\lambda_1 = \bar{\boldsymbol{\tau}}_1 Q_0.$$

The matrix Q_0 is a non-conservative generator and so does not have a zero eigenvalue. Combining the last two expressions,

$$(\boldsymbol{\alpha} - \bar{\boldsymbol{\tau}})Q_0 \quad = \quad \mathbf{0},$$

requiring $\boldsymbol{\alpha} \equiv \bar{\boldsymbol{\tau}}$. ■

Corollary 5.3 *For any given irreducible PH-simple PH-generator Q_0, choosing $\boldsymbol{\alpha}$ to be the left eigenvector of the eigenvalue of maximal real part λ_1 of Q_0 yields a unique representation of a negative exponential distribution of rate $-\lambda_1$ for that PH-generator Q_0.*

Proof:

Since $\boldsymbol{\alpha}$ is the left eigenvector of the eigenvalue of maximal real part, then the conditions of Theorem 4.1 are satisfied, yielding a negative exponential distribution of rate $-\lambda_1$. Then from [37], as the *PH*-generator Q_0 is *PH*-simple, the negative exponential distribution of rate $-\lambda_1$ uniquely defines the initial vector $\boldsymbol{\alpha}$.

■

In [38], O'Cinneide considered the Laplace-Stieltjes transform of the complementary distribution function and used an approach to PH-type distributions which he called the invariant polytope approach. This approach firstly relies on the Invariant Polytope Lemma, which is described as a geometric characterisation of PH-type distributions, and secondly on a collection of techniques developed to construct R-invariant polytopes (see [38]). The Laplace-Stieltjes transform of the complementary distribution function for a PH-type distribution is a rational function and may be written as the ratio of two coprime polynomials $p(s)$ and $q(s)$. That is,

$$-\boldsymbol{\alpha}(sI - Q_0)^{-1}Q_0\boldsymbol{e} + \boldsymbol{\alpha}_{n+1} = \frac{p(s)}{q(s)}, \text{ for } s \in [0, \infty). \tag{5.0.2}$$

The degree of the denominator polynomial $q(s)$ is known as the algebraic degree of the distribution. The order of a PH-distribution is at least as great as its degree [38], and O'Cinneide proposed a conjecture that the minimal representation order is equal to this algebraic degree. However, he subsequently disproved this conjecture using his invariant polytope approach, but in doing so, provided a method of characterising PH-type distributions. This characterisation states that,

> A distribution on $[0, \infty)$ with rational Laplace-Stieltjes transform is of phase type if and only if it is either the point mass at zero, or (a) it has a continuous positive density on the positive reals, and (b) its Laplace-Stieltjes transform has a unique pole of maximal real part.

It was shown by construction that the order of some PH-type distributions is in fact greater than their algebraic degree. In Theorem 3.1 of [39], O'Cinneide gave a lower bound on the minimal number of states required to represent a PH-distribution based on poles of its Laplace-Stieltjes transform, for those PH-distributions whose Laplace-Stieltjes transforms have non-real poles. The lower bound is established by proving that the order n of a PH-distribution satisfies

$$n \geq \frac{\pi\theta}{\lambda_2 - \lambda_1}. \tag{5.0.3}$$

Here $\pi = 3.1415\ldots$ and $-\lambda_1$ is the pole of maximal real part of the Laplace-Stieltjes transform of the complementary distribution function of the *PH*-distribution. This Laplace-Stieltjes transform also has poles at $-\lambda_2 \pm i\theta$, such that $\theta > 0$. Note that the poles of the Laplace-Stieltjes transform of a *PH*-distribution can be seen to be eigenvalues of any *PH*-generator Q_0 which can be used to represent it, either directly from equation (5.0.2), or see [38]. O'Cinneide described a collection of *PH*-distributions of algebraic degree 3, that have arbitrarily large order, again showing that the algebraic degree is not always the minimal order of a distribution. For those distributions whose Laplace-Stieltjes transforms only have real poles, it was shown that they may be represented by a *PH*-generator which is bi-diagonal or Coxian.

The discussion continued in [41] where O'Cinneide considered the more specific upper or lower triangular *PH*-generator form (*TPH*-distribution). The *TPH*-distributions give rise to the same family of distributions as the Coxian distributions and as was shown in [37], each triangular representation has a bi-diagonal representation of the same order. The minimum number of states required to represent a *TPH*-distribution using a triangular *PH*-generator is defined as the triangular order. O'Cinneide established some interesting results using triangular order including giving a formula for the triangular order of the collection of *PH*-distributions of algebraic degree 3, which he had already shown to have order in excess of the algebraic degree. The discussion of triangular order is fairly complete, but it is noted that the triangular order of a *TPH*-distribution is at least as great as its order, and in fact may exceed its order.

Commault and Chemla in [15] used a dual representation to consider *PH*-type distributions. Using control theory and the idea of *PH*-simplicity, they proved that a *PH*-type distribution representation $(\boldsymbol{\alpha}, Q_0)$, where Q_0 is $n \times n$, has order equal to its algebraic degree if and only if one of the two following conditions hold.

a) $rank[Q_0\boldsymbol{e}, Q_0^2\boldsymbol{e}, \ldots, Q_0^n\boldsymbol{e}] = n$ and $rank \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\alpha}Q_0 \\ \boldsymbol{\alpha}Q_0^{n-1} \end{bmatrix} = n,$ or

b) the representation and its dual are both *PH*-simple.

Commault and Chemla pointed out that given a relation as in (5.0.2), the question of finding real $\boldsymbol{\alpha}$ and Q_0 for a given $\frac{p(s)}{q(s)}$ is a classical control theory problem that has received a complete solution. From control theory, the first condition is true if and only if n is equal to the degree of the polynomial $q(s)$ in (5.0.2). The second condition uses the concept of a dual which in this case is the time-reverse of the original representation.

Commault and Chemla also noted that for real Q_0 and $\boldsymbol{\alpha}$, a minimal order solution equal to the degree of $q(s)$ is always possible to find using a reduction procedure. The parameters $\boldsymbol{\alpha}$ and Q_0, however, are constrained by the fact that they represent a *PH*-distribution. That is, $\boldsymbol{\alpha} \geq 0$, $\boldsymbol{\alpha e} = 1$ and Q_0 is a non-singular $n \times n$ matrix with $[Q_0]_{ii} < 0$, $[Q_0]_{ij} \geq 0$ for all $i \neq j$, such that $Q_0\boldsymbol{e} \leq 0$ with not all row sums being identically zero. Constrained in this way, finding a minimal order solution equal to the degree of $q(s)$ in (5.0.2) becomes difficult because there is no known reduction procedure for reducing a non-*PH*-simple representation to a *PH*-simple one.

In [16], Commault and Chemla proved that the minimal number of states which are visited before absorption is equal to the difference between the degree of $q(s)$ and $p(s)$ as defined in (5.0.2). They proved that for a *PH*-distribution where $q(s)$ is of degree n with real roots and $p(s)$ is of degree less than or equal to 1, the given distribution has order n. In their proof, they exhibited a class of *PH*-type distributions whose order is equal to their algebraic degree. This result does not extend to the case where $q(s)$ has non-real roots or to where $p(s)$ has degree 2.

Ryden [47] also addressed the problem of minimal order. He gave a lower bound on the minimal order of *PH*-type distributions by representing them as aggregated Markov chains and using uniformisation to convert them to discrete time. His lower bound $n^E(\mu) - 1$, was given in terms of the dimension of a linear space $n^E(\mu)$ defined on the aggregated Markov chain representation of the *PH*-type distribution μ. Ryden verified O'Cinneide's result by showing the equivalence of $n^E(\mu) - 1$ to

$dim(Span(\mu)) - 1$. O'Cinneide showed in Theorem 2.1 of [38] that this is equal to the algebraic degree of the PH-type distribution μ. $Span(\mu)$, defined by O'Cinneide, is a linear space which "loosely speaking, is spanned by μ and all residual lifetime distributions associated with μ" [47].

Summarising the above discussion, there does not seem to be a simple method of being able to determine the minimal order of a PH-distribution except in a few particular cases. Theorem 4.1 and Corollary 4.2 showed that the exponential distribution may have a representation $(\boldsymbol{\alpha}, Q_0)$, where Q_0 may take any dimension $n \geq 1$. We have also shown that every irreducible PH-simple generator Q_0 of dimension $n \geq 1$ can be used as a complicated representation of a negative exponential distribution of rate $-\lambda_1$. In this case, $\boldsymbol{\alpha} \equiv \bar{\boldsymbol{\tau}}_1$, the left eigenvector of the eigenvalue of maximum real part λ_1. Trivially, this also implies that the negative exponential generator $-\lambda$ is PH-majorized by all irreducible PH-simple generators Q_0 having $-\lambda$ as the eigenvalue of maximal real part. In general, the minimal order is at least as great as the algebraic degree of the distribution, and is clearly no larger than the order of any of its representations.

Chapter 6

MAP/M/1 level and phase independence

A stable $M/M/1$ queue has a geometric stationary distribution involving a scalar $\rho = \frac{\lambda}{\mu} < 1$, which is just the traffic coefficient. That is,

$$\boldsymbol{\pi} = \pi_0[1, \rho, \rho^2, \rho^3, \ldots].$$

A stable $MAP/M/1$ queue has a matrix geometric stationary distribution of the form

$$\boldsymbol{\Psi} = \boldsymbol{\pi}_0[I, R, R^2, R^3, \ldots] \ , \tag{6.0.1}$$

where the matrix R, defined in equation (2.2.5), has a maximal eigenvalue η which satisfies $0 < \eta < 1$. The phase of the arrival process j, for $j \in \{1, 2, \ldots, n\}$, corresponds to the j^{th} entry of each of the vectors $\boldsymbol{\pi}_0 R^m$. The vector $\boldsymbol{\pi}_m = \boldsymbol{\pi}_0 R^m$, for $m \in \{0, 1, 2, \ldots\}$, is the stationary distribution of level m corresponding to m customers in the system. It is of interest to investigate conditions under which the matrix-geometric distribution (6.0.1) is exactly level and phase independent. That is, when

$$\boldsymbol{\Psi} = \boldsymbol{\pi}_0[1, \zeta, \zeta^2, \zeta^3, \ldots], \tag{6.0.2}$$

and $0 < \zeta < 1$ is a scalar.

Our interest is to study whether exact level and phase independence of a *MAP/M/1* queue implies that the stationary *MAP* is a Poisson process. The answer to this question is shown to be in the affirmative. However, it is then shown that not all *MAP/M/1* queues, where the stationary *MAP* is in fact Poisson, have an exact level and phase independent stationary distribution of the form (6.0.2).

We will also address another distinct form of level and phase independence exhibited by *MAP/PH/1* queues. In particular, we show that all *PH/M/1* queues are shift-one level and phase independent. That is, the stationary distribution is given by

$$\boldsymbol{\Psi} = \left[\boldsymbol{\pi}_0, \boldsymbol{\xi}_0 \left[\chi, \chi^2, \chi^3, \ldots\right]\right], \tag{6.0.3}$$

where $\chi\boldsymbol{\xi}_0 = \boldsymbol{\pi}_0 R$ and $0 < \chi < 1$ is a scalar.

Another form of level and phase independence was shown in Latouche and Taylor [26]. In the more general setting of a quasi-birth-and-death process (*QBD*), they showed that every *QBD* and hence every *MAP/M/1* queue has an ubiquitous form of level and phase independence which is asymptotic. They used the spectral expansion of the matrix R in the expression for the stationary distribution of level m, and showed that

$$\lim_{m\to\infty} \boldsymbol{\pi}_m = (\boldsymbol{\pi}_0\boldsymbol{v})\eta^m\boldsymbol{u} + o(\eta^m).$$

Here $\boldsymbol{v}$ is the positive right eigenvector of the matrix R corresponding to the unique maximal eigenvalue η. The term $(\boldsymbol{\pi}_0\boldsymbol{v})$ is a positive scalar and hence the level is asymptotically independent of the phase as m becomes large.

Ramaswami and Taylor [44] have shown that the level is exactly independent of the phase if and only if $\boldsymbol{\pi}_0 = (1-\eta)\eta\boldsymbol{u}$. Here $\boldsymbol{u}$ is the positive left eigenvector of R, corresponding to η and normalised so that $\boldsymbol{u}\boldsymbol{e} = 1$. In this case, the parameter ζ in (6.0.2) is in fact given by η. Latouche and Taylor in [26] have shown that exact level and phase independence can be achieved in any *QBD* by just a modification of the boundary transition behaviour. They have also shown that this result extends to the more general *GI/M/1*-type queues. This property of exact level and phase independence is a useful condition for quasi-reversibility as shown in Bean, Latouche and Taylor [34]. Quasi-reversibility is sufficient to imply that a network of such *QBD*s would have a product-form stationary distribution.

We first consider exact level and phase independence as defined in (6.0.2). Consider the conservative rate matrix for the *MAP/M/1* queue, given in (2.2.1). Assuming that it exhibits exact level and phase independence, we pre-multiply this rate matrix by its stationary distribution $\mathbf{\Psi} = \boldsymbol{\pi}_0 \left[1, \eta, \eta^2, \eta^3, \ldots\right]$. This yields

$$\mathbf{\Psi} Q = \boldsymbol{\pi}_0 \left[1, \eta, \eta^2, \eta^3, \ldots\right] \begin{bmatrix} D_0 & D_1 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix} = 0. \tag{6.0.4}$$

From (6.0.4), we can see that

$$\boldsymbol{\pi}_0(D_0 + \eta A_2) = 0, \tag{6.0.5}$$

and hence $\boldsymbol{\pi}_0 D_0 = -\boldsymbol{\pi}_0 \eta A_2$. Recall from Chapter 2 that the matrix A_2 has a simple form μI, since we only have a single server with a constant service rate μ. This implies that $\boldsymbol{\pi}_0 D_0 = -\eta\mu\boldsymbol{\pi}_0$, and so $\boldsymbol{\pi}_0$ is a left eigenvector of D_0 corresponding to eigenvalue $-\eta\mu$. We will now show that $\boldsymbol{\pi}_0$ is also a left eigenvector of D_1, which we will then use to show that the *MAP* is Poisson. Consider equation (6.0.4) which yields

$$\boldsymbol{\pi}_0(D_1 + \eta A_1 + \eta^2 A_2) = 0 \text{ and } \boldsymbol{\pi}_0 \eta^m (A_0 + \eta A_1 + \eta^2 A_2) = 0, \text{ for } m \in \{1, 2, \ldots\}. \tag{6.0.6}$$

Recall from Chapter 2 again that $(A_1 + A_2) = D_0$ and $A_0 = D_1$, so that (6.0.5) and (6.0.6) yield

$$\begin{aligned} \boldsymbol{\pi}_0(A_1 + A_2 + \eta A_2) \quad &+ \quad \boldsymbol{\pi}_0(A_0 + \eta A_1 + \eta^2 A_2) + \sum_{m=1}^{\infty} \boldsymbol{\pi}_0 \eta^m (A_0 + \eta A_1 + \eta^2 A_2) \\ &= \frac{\boldsymbol{\pi}_0}{(1-\eta)}(A_0 + A_1 + A_2) \\ &= \frac{\boldsymbol{\pi}_0}{(1-\eta)}(D_1 + D_0) \\ &= 0. \end{aligned}$$

This implies that $\boldsymbol{\pi}_0$ is also a left eigenvector of D_1 corresponding to eigenvalue $\eta\mu$. Having now established the fact that $\boldsymbol{\pi}_0$ is also the left eigenvector corresponding to eigenvalue $-\eta\mu$ of D_0 and eigenvalue $\eta\mu$ of D_1, then $\frac{\boldsymbol{\pi}_0}{\boldsymbol{\pi}_0\boldsymbol{e}}$ is the stationary distribution for the *MAP*. We will use this result in the series expansion of

$$\begin{aligned}\boldsymbol{\pi}_0 e^{D_0 t} D_1 &= \boldsymbol{\pi}_0 \left(\sum_{i=0}^{\infty} \frac{(D_0 t)^i}{i!}\right) D_1 \\ &= \boldsymbol{\pi}_0 \left(\sum_{i=0}^{\infty} \frac{(-\eta\mu t)^i}{i!}\right) D_1 \\ &= \boldsymbol{\pi}_0 D_1 e^{-\eta\mu t} \\ &= \boldsymbol{\pi}_0 (\eta\mu) e^{-\eta\mu t}. \end{aligned} \tag{6.0.7}$$

Recall that any point process defined by the filtration matrices Q_0 and Q_1 is Poisson of rate λ, if and only if it satisfies equation (4.1.2) for all initial distribution of states $\boldsymbol{\pi}$. That is,

$$\frac{\boldsymbol{\pi} e^{Q_0 t_1} Q_1 e^{Q_0 t_2} \dots Q_1 e^{Q_0 t_k} Q_1 \boldsymbol{e}}{\boldsymbol{\pi} e^{Q_0 t_1} Q_1 e^{Q_0 t_2} \dots Q_1 e^{Q_0 t_k} \boldsymbol{e}} = \lambda, \text{ for all } k \geq 1, \text{ and } t_i \in [0, \infty), \text{ for } i \in \{1, 2, \dots, k\}. \tag{6.0.8}$$

Using the stationary distribution $\boldsymbol{\nu} = \boldsymbol{\pi}_0 (I - R)^{-1}$ of the *MAP*, and (6.0.7), we will now show that the stationary *MAP* is in fact Poisson. Recall that we have already shown that $\boldsymbol{\nu}$ is a normalised version of $\boldsymbol{\pi}_0$, that is $\boldsymbol{\nu} = \boldsymbol{\pi}_0 (I - R)^{-1} = \frac{\boldsymbol{\pi}_0}{\boldsymbol{\pi}_0\boldsymbol{e}}$. The filtration of the *MAP* which gives the arrival transitions is trivially

$$Q_0 = D_0, Q_1 = D_1.$$

Substituting into the left hand side of (6.0.8) gives

$$\frac{\boldsymbol{\nu} e^{D_0 t_1} D_1 e^{D_0 t_2} \dots D_1 e^{D_0 t_k} D_1 \boldsymbol{e}}{\boldsymbol{\nu} e^{D_0 t_1} D_1 e^{D_0 t_2} \dots D_1 e^{D_0 t_k} \boldsymbol{e}} = \frac{(\eta\mu)^{k+1} e^{-\eta\mu t_1} e^{-\eta\mu t_2} \dots e^{-\eta\mu t_k} \boldsymbol{\nu}\boldsymbol{e}}{(\eta\mu)^{k} e^{-\eta\mu t_1} e^{-\eta\mu t_2} \dots e^{-\eta\mu t_k} \boldsymbol{\nu}\boldsymbol{e}}$$

$$= \eta\mu, \text{ for all } k \geq 1, \text{ and } t_i \in [0, \infty), \text{ for } i \in \{1, 2, \dots, k\}.$$

This implies that any *MAP/M/1* queue which admits an exact level and phase independent stationary distribution must have a Poisson arrival process of rate $\lambda =$

$\eta\mu$. Then by Burke [11] the output process of this particular *MAP/M/1* queue should also be Poisson of rate $\eta\mu$.

This can also be shown directly by considering the *MAP/M/1* queue, with a filtration defined by Q_0 and Q_1, which gives us the departure process. That is,

$$Q_0 = \begin{bmatrix} D_0 & D_1 & 0 & 0 & \cdots \\ 0 & A_1 & A_0 & 0 & \cdots \\ 0 & 0 & A_1 & A_0 & \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix} \text{ and } Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ A_2 & 0 & 0 & 0 & \cdots \\ 0 & A_2 & 0 & 0 & \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}.$$

From $\mathbf{\Psi} Q = \mathbf{0}$, we see that

$$\begin{aligned} \mathbf{\Psi} Q_0 &= -\mathbf{\Psi} Q_1 \\ &= -(\boldsymbol{\pi}_0 \eta A_2, \boldsymbol{\pi}_0 \eta^2 A_2, \boldsymbol{\pi}_0 \eta^3 A_2, \ldots) \\ &= -(\boldsymbol{\pi}_0 \eta\mu, \boldsymbol{\pi}_0 \eta^2 \mu, \boldsymbol{\pi}_0 \eta^3 \mu, \ldots) \\ &= -\eta\mu(\boldsymbol{\pi}_0, \boldsymbol{\pi}_0 \eta, \boldsymbol{\pi}_0 \eta^2, \ldots) \\ &= -\eta\mu\mathbf{\Psi}, \end{aligned}$$

and hence that

$$\begin{aligned} \mathbf{\Psi} e^{Q_0 t} &= \mathbf{\Psi} \sum_{i=0}^{\infty} \frac{Q_0^i}{i!} t^i \\ &= \mathbf{\Psi} \sum_{i=0}^{\infty} \frac{(\eta\mu)^i}{i!} t^i \\ &= e^{-\eta\mu t} \mathbf{\Psi}. \end{aligned}$$

Then

$$\begin{aligned} \mathbf{\Psi} e^{Q_0 t} Q_1 &= e^{-\eta\mu t} \mathbf{\Psi} Q_1 \\ &= e^{-\eta\mu t} \mu(\boldsymbol{\pi}_0 \eta, \boldsymbol{\pi}_0 \eta^2, \ldots) \\ &= e^{-\eta\mu t} \eta\mu \mathbf{\Psi}. \end{aligned}$$

From here we see that the left hand side of equation (6.0.8) for this filtration yields

$$\frac{\mathbf{\Psi} e^{Q_0 t_1} Q_1 e^{Q_0 t_2} \ldots Q_1 e^{Q_0 t_k} Q_1 \boldsymbol{e}}{\mathbf{\Psi} e^{Q_0 t_1} Q_1 e^{Q_0 t_2} \ldots Q_1 e^{Q_0 t_k} \boldsymbol{e}} = \frac{(\eta\mu)^{k+1} e^{-\eta\mu t_1} e^{-\eta\mu t_2} \ldots e^{-\eta\mu t_k} \mathbf{\Psi} \boldsymbol{e}}{(\eta\mu)^k e^{-\eta\mu t_1} e^{-\eta\mu t_2} \ldots e^{-\eta\mu t_k} \mathbf{\Psi} \boldsymbol{e}}$$

$$= \eta\mu, \text{ for all } k \geq 1, \text{ and } t_i \in [0, \infty), \text{ for } i \in \{1, 2, \ldots, k\},$$

which is Poisson of rate $\eta\mu$.

We will now show that shift-one level and phase independence is a characteristic of all *PH/M/1* queues. We will subsequently show that not all *MAP/M/1* queues which have a stationary *MAP* which is Poisson have an exact level and phase independent stationary distribution.

Theorem 6.1 *All stable PH/M/1 queues have a shift-one level and phase independent stationary distribution. This distribution is given by*

$$\boldsymbol{\Psi} = \left[\boldsymbol{\pi}_0, (\boldsymbol{\pi}_0 \boldsymbol{v})\boldsymbol{u}\left[\eta, \eta^2, \eta^3, \ldots\right]\right], \tag{6.0.9}$$

where η is the eigenvalue of maximal real part of the matrix R defined in (2.2.5), and $\boldsymbol{v}$ and $\boldsymbol{u}$ are the associated positive right and left eigenvectors respectively, normalised such that $\boldsymbol{uv} = \boldsymbol{ue} = 1$.

Proof:

Recall from Chapter 3 (or see Latouche [25]), that for a *PH/M/1* queue, R is rank one. From equation (3.3.7), the matrix R may be written using its spectral expansion as

$$R = \eta \boldsymbol{vu}.$$

The stationary distribution of a stable *MAP/M/1* queue is given in equation (6.0.1), which in this case may be written as

$$\boldsymbol{\Psi} = \boldsymbol{\pi}_0 \left[I, \boldsymbol{vu}\left[\eta, \eta^2, \eta^3, \ldots\right]\right].$$

This is in the same form as equation (6.0.3), where $\chi = \eta$ and $\boldsymbol{\xi}_0 = (\boldsymbol{\pi}_0 \boldsymbol{v})\boldsymbol{u}$, and hence the stationary distribution is shift-one level and phase independent as given in (6.0.9). ■

The following simple *PH/M/1* example will demonstrate that not all *MAP/M/1* queues which have a stationary Poisson *MAP* have an exact level and phase independent stationary distribution. Consider

$$D_0 = \begin{bmatrix} -3 & 3 & 0 \\ 0 & -6 & 4 \\ 0 & 0 & -1 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ (\frac{1}{2}) & (\frac{1}{2}) & 0 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

In this example, the matrix geometric stationary distribution is parameterised by

$$R = \begin{bmatrix} 0 & 0 & 0 \\ (\frac{1}{4}) & (\frac{1}{4}) & (\frac{1}{2}) \\ (\frac{1}{8}) & (\frac{1}{8}) & (\frac{1}{4}) \end{bmatrix} = \eta \boldsymbol{v}\boldsymbol{u} = \frac{1}{2} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right),$$

and

$$\boldsymbol{\pi}_0 = \left[\frac{5}{120}, \frac{5}{120}, \frac{5}{12}\right].$$

This is the *MAP/M/1* example given at the end of Chapter 3 in equation (3.3.12), which is a stationary Poisson process of rate 1. The stationary distribution of the *MAP*, which is given by the matrices D_0 and D_1, is

$$\boldsymbol{\nu} = \boldsymbol{\pi}_0 (I - R)^{-1} = \left[\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right].$$

The matrix D_0 has distinct eigenvalues and in spectral form is given by

$$D_0 = [\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \\ \boldsymbol{l}_3 \end{bmatrix} =$$

$$\begin{bmatrix} (\frac{6}{5}) & -1 & (\frac{4}{5}) \\ (\frac{4}{5}) & 0 & -(\frac{4}{5}) \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & 2 \\ 0 & -(\frac{5}{4}) & 1 \end{bmatrix}$$

Corollary 4.11 is satisfied as $\boldsymbol{\nu}\boldsymbol{r}_3 = 0$ and $\boldsymbol{l}_2\boldsymbol{e} = 0$, which implies that the stationary *MAP* is a Poisson process of rate $-\lambda_1 = 1$ and hence by Burke's Theorem the stationary *MAP/M/1* queue also has a Poisson output of rate 1. Here $\boldsymbol{\pi}_0\boldsymbol{v} = \frac{1}{2}$ so that the stationary distribution for the *MAP/M/1* queue may be written in the modified geometric form, or shift-one level and phase independent form:

$$\left[\boldsymbol{\pi}_0, \frac{\boldsymbol{u}}{2}\left[\left(\frac{1}{2}\right), \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \ldots\right]\right].$$

We note here that the stationary distribution for this *MAP/M/1* queue cannot be written as $\boldsymbol{\pi}_0[1, \zeta, \zeta^2, \ldots]$ for any scalar ζ, and hence is not exactly level and phase independent. Therefore a necessary but not sufficient condition for exact level and phase independence is that a stationary *MAP* be Poisson. From Ramaswami and Taylor [44], we have that the level is exactly independent of the phase if and only if $\boldsymbol{\pi}_0 = (1-\eta)\eta\boldsymbol{u}$.

Chapter 7

Approximations to *MAP/PH/1* departure processes

7.1 Introduction

The departure process of a queue is important in the analysis of networks of queues as it may be the arrival process to another queue in the network. There have been many papers (see [11], [12], [17], [18], [28], [46]) dealing with the output process of a single queue. In 1956, Burke [11] and later Reich [45], independently proved the 1955 claim by Morse [29] that "A little thought will convince one that the efflux from a single-channel, fed by Poisson arrivals, must be Poisson with the same rate as the arrivals". This result simplified much of the analysis for tandems that had already been undertaken, for example by R.R.P. Jackson [24].

For a *MAP/PH/1* queue with a finite buffer, that is, a *MAP/PH/1/k* queue, there exists an exact finite *MAP* description of the output process. This *MAP* description has dimension kmn, where k is maximum buffer size, m is the dimension of the matrix descriptor of the input *MAP* and n is the dimension of the matrix descriptor of the *PH*-server. This exact representation suffers from the "curse of dimensionality", especially when considered as a tool in the analysis of a network.

Approximate techniques which reduce the size of these representations therefore become necessary. An extensive list of references for methods of analysis of various tandem queues with a finite intermediate buffer is given in [32].

In the infinite buffer case, with which this chapter is primarily concerned, no such exact finite description has been found. In fact, it appears that there does not exist a finite *MAP* description for the output process of a stationary *MAP/PH/1* queue in which the *MAP* is not a Poisson process. The possibility of a finite description was addressed in Olivier and Walrand [42], who claimed that the output from an *MMPP/M/1* queue can not have a *MAP* description. Although their proof was shown to be flawed in Chapter 3, and also in [7], the author believes the result to be true. In Chapter 4 and also in [5], we addressed the question of when a *MAP* is just a complicated description of a Poisson process, in which case the output of the *MAP/M/1* queue must be a Poisson process.

In this chapter, as in [6], a family of *MAP* approximations to the departure process of the *MAP/PH/1* queue is proposed. To check the viability of these approximations, they are used as input to another single server queue, and the second queue length distributions are compared with their exact counterparts, calculated using matrix-analytic techniques. Other techniques to approximate point processes have been given in, for example, [1], [2], [3], [9], [20], [49], [51] and [52]. Where possible, the approximations given here are compared to the results of these publications.

The structure of the defining processes is exploited in the approximations used in this chapter. As a direct result of this, all of the approximations yield the exact output process for the trivial situation of Poisson arrivals to a negative-exponential first server.

7.2 The *MAP/PH/1* Queue

The notation for the *MAP/PH/1* queue is established by considering the Markov chain (x, y) defined for the *MAP/M/1* queue and its associated rate matrix given in (2.2.1) of Chapter 2. That is, the rate matrix

$$Q = \begin{bmatrix} D_0 & D_1 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Recall for the *MAP/M/1* queue description that x represents the number of customers in the queue at time t and y represents the phase of the arrival process at time t. The service times can be generalised by allowing them to be distributed according to an n-state *PH*-distribution $(\boldsymbol{\beta}, S)$ as defined in Chapter 2. To model the *MAP/PH/1* queue we need to keep track of the phase of both the arrival process and the service distribution, when $x \geq 1$. That is, for $x \geq 1$, we need to take $y = (a, s)$, where a and s represent the phase of the arrival process and service distribution respectively at time t. The Markovian simplicity is preserved since the sojourn times within each phase are still negative exponentially distributed. Let D_0 and D_1 be the $m \times m$ matrix descriptors of the *MAP*, defined by (2.1.1), (2.1.2) and (2.1.3) and let $\boldsymbol{\nu} > \mathbf{0}$ be the stationary distribution for the *MAP* such that $\boldsymbol{\nu}(D_0 + D_1) = \mathbf{0}$ and $\boldsymbol{\nu}\boldsymbol{e} = 1$. Let I_n and I_m be the identity matrices having the same dimensions as S and D_0 respectively. The *MAP/PH/1* queue then has conservative rate matrix

$$Q = \begin{bmatrix} B_1 & B_0 & & & & \\ B_2 & A_1 & A_0 & & & \\ & A_2 & A_1 & A_0 & & \\ & & A_2 & A_1 & A_0 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}, \tag{7.2.1}$$

where

$$B_0 = D_1 \otimes \boldsymbol{\beta},$$

$$\begin{aligned} B_1 &= D_0, \\ B_2 &= I_m \otimes \boldsymbol{S}^0, \\ A_0 &= D_1 \otimes I_n, \\ A_1 &= I_m \otimes S + D_0 \otimes I_n, \\ A_2 &= I_m \otimes \boldsymbol{S}^0 \boldsymbol{\beta}, \end{aligned}$$

and $\otimes$ is the Kronecker product as defined in Chapter 2. This rate matrix is in the form of a quasi-birth and death (QBD) process as given in Neuts [31]. Assume it defines an irreducible, regular Markov chain. Therefore this Markov chain has at most one stationary distribution $\boldsymbol{\Psi}$ such that $\boldsymbol{\Psi}Q = 0$. This stationary distribution is given by the following modified matrix geometric form

$$\boldsymbol{\Psi} = \boldsymbol{\pi}_0[I, R_0, R_0R, R_0R^2, \ldots], \tag{7.2.2}$$

where the $mn \times mn$ matrix R is the minimal non-negative solution to the matrix quadratic equation

$$R^2A_2 + RA_1 + A_0 = 0. \tag{7.2.3}$$

The $m \times mn$ matrix R_0 is given by

$$R_0 = -B_0\left(A_1 + RA_2\right)^{-1} \tag{7.2.4}$$

and the vector $\boldsymbol{\pi}_0$ is the unique positive solution to the system of equations

$$\boldsymbol{\pi}_0(B_1 + R_0B_2) = \mathbf{0} \quad \text{and} \quad \boldsymbol{\pi}_0\boldsymbol{e} + \boldsymbol{\pi}_0R_0(I - R)^{-1}\boldsymbol{e} = 1. \tag{7.2.5}$$

The inverse $(A_1 + RA_2)^{-1}$ always exists (see, for example, equation (16) of [25]).

7.3 A Family of Approximations

In this section we consider the departure process from a *MAP/PH/1* queue. We partition Q given in (7.2.1) into the filtration matrices Q_0 and Q_1 given by

$$Q_0 = \begin{bmatrix} B_1 & B_0 & & & & \\ & A_1 & A_0 & & & \\ & & A_1 & A_0 & & \\ & & & A_1 & A_0 & \\ & & & & \ddots & \ddots \end{bmatrix} \text{ and } Q_1 = \begin{bmatrix} 0 & & & & & \\ B_2 & 0 & & & & \\ & A_2 & 0 & & & \\ & & A_2 & 0 & & \\ & & & \ddots & \ddots & \end{bmatrix}. \quad (7.3.1)$$

The observed transitions recorded by Q_1 are departure transitions and the observed process is the departure process. However, this is not a *MAP* because there are infinitely many states.

One obvious method to obtain a *MAP* approximation for the departure process is to truncate the matrices Q_0 and Q_1 in (7.3.1) at a "sufficiently" large level ℓ and to use these as the *MAP* descriptors. A major problem with this method is that as the traffic intensity increases, the value of ℓ which gives an accurate approximation grows quickly with ℓ, making the size of the matrix descriptors extremely large. We also need to think about how to adjust the transition rates at level ℓ in order to ensure that the generator remains conservative. Two common ways to do this are to add diag$(A_0\boldsymbol{e})$ to A_1 or to add A_0 to A_1, both at level ℓ. The first of these effectively assumes that transitions to level $\ell+1$ do not happen, while the second allows these transitions to occur but assumes that the process remains at level ℓ. However, it was pointed out in Theorem 1 on page 362 of Bright and Taylor [10] that the best way to adjust the transition rates is to add RA_2 to A_1 at level ℓ. This takes into account the influence of the structure of the *QBD* beyond level ℓ on the invariant measure at level ℓ and below. In fact, the stationary distribution of this truncation is a renormalised version of the stationary distribution of the whole process.

Another way to obtain a *MAP* approximation for the departure process can be obtained by thinking physically about what is happening in the *QBD*. During the busy period, the departure process from the queue is just the *PH*-renewal process $(\boldsymbol{\beta}, S)$. It is only when the queue becomes empty that this simplicity is lost. However, if we knew the phase of the arrival process when the busy period finishes, we could model the time until the next busy period. To characterise the departure process exactly we also need to know the length of the busy period. Ramaswami in [43] has studied the busy period of queues with a matrix-geometric steady state probability vector. He gives expressions for the Laplace-Stieltjes transform for the distribution function of the busy period which are complex, even in the case of the simple *M/M/1* queue. The actual distribution function of the busy period of an *M/M/1* queue is given explicitly in equation (2.33) of Cohen [14] as an integral of a mixture of Bessel functions. Neither of these expressions leads easily to a *PH*-type description of the busy period. Thus in this thesis we approximate the busy period with a *PH*-type random variable in order to get a *MAP* description of the departure process. Below we construct a family of approximations indexed by a parameter k. The k^{th} approximation assumes that

1. the phase of the arrival process when the *QBD* moves from level k to level $k-1$ is given by its correct marginal distribution, and
2. the number of services during a sojourn at level k and above is geometrically distributed with the mean chosen such that the sojourn at level k and above has the correct mean.

Thus the $k = 1$ approximation assumes that

1. the phase of the arrival process when a busy period ends has the correct marginal distribution, and
2. the number of services during a busy period is geometrically distributed with the mean chosen such that the busy period has the correct mean.

Physically, the k^{th} approximation amalgamates levels k and above into a super level $\bar{k}$, approximates the distribution of the sojourn in level $\bar{k}$ by a geometric mixture of convolutions of *PH*-type distributions, and also approximates the phase on return to level $k-1$ by its correct marginal distribution. What is lost in this approximation is the exact distribution of the sojourn at and above level k and correlations between the return phases and sojourn times.

Although it must be true that the stationary rate of departures from a *MAP/PH/1* queue is equivalent to its stationary rate of arrivals, we prove this in the following lemma. We will use this result in the construction of the distribution of the *QBD* at level $k-1$, conditional on a departure having just occurred.

Lemma 7.1 *The expected stationary rate of departures is given by* $\boldsymbol{\nu} D_1 \boldsymbol{e}$.

Proof:

$$\begin{aligned}\boldsymbol{\Psi} Q_1 \boldsymbol{e} &= \boldsymbol{\pi}_0 R_0 B_2 \boldsymbol{e}_m + \boldsymbol{\pi}_0 R_0 \sum_{k\geq 1} R^k A_2 \boldsymbol{e}_{mn} \\ &= \boldsymbol{\pi}_0 R_0 B_2 \boldsymbol{e}_m + \boldsymbol{\pi}_0 R_0 R (I_{mn} - R)^{-1} A_2 \boldsymbol{e}_{mn}.\end{aligned}$$

From Neuts [31] we have that $RA_2\boldsymbol{e}_{mn} = A_0\boldsymbol{e}_{mn}$ and from $\boldsymbol{\Psi}(Q_0 + Q_1) = \boldsymbol{0}$ it can be shown that $\boldsymbol{\pi}_0 R_0 B_2 = \boldsymbol{\pi}_0 D_1$, so that

$$\begin{aligned}\boldsymbol{\Psi} Q_1 \boldsymbol{e} &= \boldsymbol{\pi}_0 D_1 \boldsymbol{e}_m + \boldsymbol{\pi}_0 R_0 (I_{mn} - R)^{-1} A_0 \boldsymbol{e}_{mn} \\ &= \boldsymbol{\pi}_0 D_1 \boldsymbol{e}_m + \boldsymbol{\pi}_0 R_0 (I_{mn} - R)^{-1} (D_1 \otimes \boldsymbol{e}_n) \boldsymbol{e}_{mn} \\ &= \boldsymbol{\pi}_0 D_1 \boldsymbol{e}_m + \boldsymbol{\pi}_0 R_0 (I_{mn} - R)^{-1} (I_m \otimes \boldsymbol{e}_n) D_1 \boldsymbol{e}_m \\ &= \left(\boldsymbol{\pi}_0 + \boldsymbol{\pi}_0 R_0 (I_{mn} - R)^{-1} (I_m \otimes \boldsymbol{e}_n)\right) D_1 \boldsymbol{e}_m \\ &= \boldsymbol{\nu} D_1 \boldsymbol{e}_m.\end{aligned}$$

■

The distribution of the *QBD* at level $k-1$, conditional on a departure having just occurred, can be calculated from the stationary distribution of the *QBD* by (see [31])

$$\boldsymbol{x}_{k-1} = \begin{cases} \boldsymbol{\pi}_0 R_0 B_2 (\boldsymbol{\nu} D_1 \boldsymbol{e})^{-1} & \text{for } k = 1 \\ \boldsymbol{\pi}_0 R_0 R^{k-1} A_2 (\boldsymbol{\nu} D_1 \boldsymbol{e})^{-1} & \text{for } k > 1, \end{cases} \qquad (7.3.2)$$

where we recall that $\boldsymbol{\nu}$ is the unique stationary distribution of the *MAP*. In our approximation, the probability of return to level $k-1$ after each service in the super-level $\bar{k}$ is

$$\frac{\boldsymbol{x}_{k-1}\boldsymbol{e}}{(\sum_{j=k-1}^{\infty}\boldsymbol{x}_j\boldsymbol{e})},$$

and the distribution of the return phase given that a return occurs is given by

$$\frac{\boldsymbol{x}_{k-1}}{\boldsymbol{x}_{k-1}\boldsymbol{e}}.$$

Thus the unconditional distribution of return phase at level $k-1$ is given by

$$\boldsymbol{y}_{k-1} = \frac{\boldsymbol{x}_{k-1}}{(\sum_{j=k-1}^{\infty}\boldsymbol{x}_j\boldsymbol{e})}. \tag{7.3.3}$$

The k^{th} *MAP* approximation for the departure process of the *MAP/PH/1* queue is thus given by

$$Q_0(k) = \begin{bmatrix} B_1 & B_0 & & & & \\ & A_1 & A_0 & & & \\ & & A_1 & \ddots & & \\ & & & \ddots & A_0 & \\ & & & & A_1 & E_0 \\ & & & & & E_1 \end{bmatrix}, Q_1(k) = \begin{bmatrix} 0 & & & & & \\ B_2 & 0 & & & & \\ & A_2 & \ddots & & & \\ & & \ddots & 0 & & \\ & & & A_2 & 0 & \\ & & & & E_2 & E_3 \end{bmatrix}. \tag{7.3.4}$$

Here the matrices are as defined previously with

$$\begin{aligned} E_0 &= D_1\boldsymbol{e} \otimes I_n, \\ E_1 &= S, \\ E_2 &= \boldsymbol{S}^0\boldsymbol{y}_{k-1}, \\ E_3 &= (1-\boldsymbol{y}_{k-1}\boldsymbol{e})\boldsymbol{S}^0\boldsymbol{\beta}. \end{aligned}$$

The stationary distribution of this *MAP* approximation can be shown to be given by

$$\boldsymbol{\nu}(k) = \boldsymbol{\pi}_0[I, R_0, R_0R, \ldots, R_0R^{k-2}, R_0R^{k-1}(I_{mn}-R)^{-1}(\boldsymbol{e}_m \otimes I_n)],$$

where $\boldsymbol{e}_m$ is an $m \times 1$ column of 1's.

It seems intuitive that neither of the approximations discussed above should be too drastic. The test of this, however, will come with the numerical results that we discuss in Section 5.

For the special case of $k = 1$, the MAP approximation to the departure process of the *MAP/PH/1* queue reduces to a *PH*-renewal process with the exact stationary inter-departure time distribution. In fact, all of our approximations have this exact stationary inter-departure time distribution. We prove this analytically in Chapter 8.

For $k = 1$, the MAP approximation is given by

$$Q_0(1) = \begin{pmatrix} D_0 & D_1\boldsymbol{e}\boldsymbol{\beta} \\ 0 & S \end{pmatrix} \text{ and } Q_1(1) = \begin{pmatrix} 0 & 0 \\ \boldsymbol{S}^0\boldsymbol{x}_0 & (1-\boldsymbol{x}_0\boldsymbol{e})\boldsymbol{S}^0\boldsymbol{\beta} \end{pmatrix}.$$

This can also be represented as a *PH*-distribution $(\boldsymbol{\alpha}, Q_0(1))$, where $Q_0(1)$ is as above and

$$\boldsymbol{\alpha} = (\boldsymbol{x}_0, (1-\boldsymbol{x}_0\boldsymbol{e})\boldsymbol{\beta}). \tag{7.3.5}$$

Here, $\boldsymbol{x}_0$ is the distribution of phases of the arrival process immediately after a departure that leaves the queue empty, calculated using (7.3.2). Bitran and Dasu [9] showed that this distribution was in fact the stationary inter-departure time distribution of the *MAP/PH/1* queue. The stationary distribution of this MAP approximation can be shown to be given by

$$\boldsymbol{\nu}(1) = [\boldsymbol{\pi}_0, \boldsymbol{\pi}_0 R_0 (I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n)].$$

This approximation precisely captures the distribution of the inter-departure times but ignores any correlation structures between these times. This $(m+n) \times (m+n)$ MAP approximation is satisfactory in a great many situations, as is shown by the results in Section 5. For the case of Poisson arrivals of rate $\lambda > 0$ to a negative exponential server of rate $\mu > 0$, such that $\lambda < \mu$, the approximation is exact as can be seen by the following. Let

$$D_0 = -\lambda, D_1 = \lambda, S = -\mu, \boldsymbol{\beta} = 1,$$

so that

$$\boldsymbol{\alpha} = \left(\frac{\mu - \lambda}{\mu}, \frac{\lambda}{\mu}\right), \quad Q_0(1) = \begin{pmatrix} -\lambda & \lambda \\ 0 & -\mu \end{pmatrix}.$$

In this case, because the arrivals are Poisson and service is negative exponential, $\boldsymbol{\pi}_0 = \boldsymbol{x}_0$. The vector $\boldsymbol{\beta} = 1$, so that from (7.3.5), the vector $\boldsymbol{\alpha}$ is identical to the stationary distribution $\boldsymbol{\nu}(1)$. The stationary probability vector $\boldsymbol{\nu}(1)$ of this process is also the left eigenvector of the matrix $Q_0(1)$ corresponding to eigenvalue λ, which by Corollary 4.11 implies that the departure process under stationary conditions is Poisson of rate λ.

Increasing the value of k clearly increases the accuracy of the approximation, but also the complexity of the *MAP*. This then requires a trade off be made by the modeller between the degree of accuracy required and the time taken to do the analysis.

7.4 The tandem queueing models

To measure the accuracy of the approximations of the departure process of the *MAP/PH/1* queue, we used the approximations as input to another single server queue and the stationary distribution of the second server queue length compared to that of a tandem queueing system. The tandem queueing system of two *FIFO* servers in series can be represented by a *QBD* as outlined in [8], by setting the number in the first and second queues to be the level and part of the phase description respectively. Under this regime, the size of one of the queues must be truncated at a "sufficiently large" value that will not affect the calculation of the stationary distribution of the that queue length. We refer to the results calculated for this *QBD* model as "exact" throughout this paper.

For comparison of numerical results, we calculated the probability distributions of the stationary second queue length so that any queue length probability being less than 10^{-14} was considered as 0. This distribution was then used to calculate the mean and variance for the stationary queue length.

Six different arrival processes as given in Appendix A.1 were used to feed the tandem queueing system, including three *PH*-renewal processes and three different correlated *MAP*s. The first *PH*-type server was one of the four given in Appendix A.2 and the second *PH*-type server was one of the four given in Appendix A.3. The service distributions were scaled appropriately to give the correct traffic intensity at each queue. Each of the two FIFO servers had an infinite buffer with the traffic intensity at each queue being one of $(0.1, 0.25, 0.5, 0.75)$.

All permutations of the above were investigated, except the situation where the fifth arrival process was fed to the first queue with traffic intensities 0.5 and 0.75 and with traffic intensity 0.75 at the second queue. The "exact" results were not able to be evaluated for these traffic intensities because of the very bursty nature of the arrival process; the required level of truncation for the *QBD* models according to the "sufficiently large" regime proved too large. All of our approximations, however, still deliver a result. Overall we used 1504 different queueing tandems to check the approximation methods via the moments of the second queue.

In Appendix A.1, the squared coefficient of variation $\frac{\sigma^2}{\mu^2}$ is given for each of the arrival processes. For those processes which have a non-zero lag-correlation structure, an indication of the level of this structure is given by the first two lag correlation coefficients c_1 and c_2, calculated by (see [33])

$$c_i = \frac{\frac{2}{\boldsymbol{\nu} D_1 \boldsymbol{e}}\boldsymbol{\nu}[(-D_0)^{-1}D_1]^i(-D_0)^{-1}\boldsymbol{e} - \frac{1}{(\boldsymbol{\nu} D_1 \boldsymbol{e})^2}}{\frac{2}{\boldsymbol{\nu} D_1 \boldsymbol{e}}\boldsymbol{\nu}(-D_0)^{-1}\boldsymbol{e} - \frac{1}{(\boldsymbol{\nu} D_1 \boldsymbol{e})^2}}, \tag{7.4.1}$$

where $\boldsymbol{\nu}$ is the stationary probability vector of the arrival process. Note that all *PH*-renewal arrival processes have a zero lag-correlation structure by their very nature.

7.5 The results

The results are compiled for the approximations with four variants for the aggregated state approximation. These are

- AGG1: $k = 1$
- AGG2: $k = 2$
- AGG3: $k = 3$
- AGG-3: $k = \max(j : p(j) > 10^{-3})$,

where $p(j)$ is the probability of there being j customers in the first queue of the tandem. Overall results for the 1504 different queueing tandems are broken up into 16 sets which specify the traffic intensities for both queues, then subsequent smaller groupings are given which tend to show the particular applicability of each variant.

The sets $A, B, \ldots, K$ and M, N, O, given in Table 7.5.1 each comprise 96 different combinations of the 6 arrival processes, 4 first server and 4 second server processes for the values of traffic intensity at each queue specified by ρ_1 and ρ_2. Sets L and P both comprise 80 different combinations of the 5 arrival processes (these do not include the fifth arrival process given in Appendix A.1), 4 first server distributions given in Appendix A.2 and 4 second server distributions given in Appendix A.3 for the values of traffic intensity at each queue specified by ρ_1 and ρ_2.

The second group of 16 sets $A^*, B^*, \ldots, P^*$, given in Table 7.5.2, are subsets of the sets $A, B, \ldots, P$ given in Table 7.5.1. They are comprised of 48 different combinations of the 3 PH-renewal arrival processes given in Appendix A.1, 4 first server distributions given in Appendix A.2 and 4 second server distributions given in Appendix A.3, for the values of traffic intensity at each queue specified by ρ_1 and ρ_2.

The results given in Tables 7.5.1 and 7.5.2 are the average over each set of the absolute percentage difference from the exact result given by

$$\frac{100\ |\text{exact- approximation}|}{\text{exact}}$$

for the first and second central moments of the queue length of the stationary second queue.

Set	*Traffic intensities*		*Averaged absolute percentage differences for the stationary second queue length of the tandem*								
			AGG1		AGG2		AGG3		AGG-3		
	ρ_1	ρ_2	E[μ]	E[σ^2]	E[μ]	E[σ^2]	E[μ]	E[σ^2]	E[μ]	E[σ^2]	k
A	0.1	0.1	0.3501	1.1388	0.0154	0.0650	0.0018	0.0090	0.0000	0.0002	3.6
B	0.1	0.25	2.6681	6.3460	0.1127	0.3817	0.0097	0.0484	0.0003	0.0021	3.6
C	0.1	0.5	5.6243	10.6426	0.1986	0.5438	0.0103	0.0440	0.0003	0.0017	3.6
D	0.1	0.75	6.3970	11.2535	0.2196	0.5240	0.0114	0.0352	0.0004	0.0014	3.6
E	0.25	0.1	0.2119	0.7201	0.0100	0.0414	0.0011	0.0052	0.0000	0.0000	6.0
F	0.25	0.25	2.2317	5.8062	0.2839	1.0148	0.0766	0.3366	0.0001	0.0009	6.0
G	0.25	0.5	6.1140	12.4001	0.8809	2.7517	0.2898	1.1332	0.0036	0.0211	6.0
H	0.25	0.75	7.4304	13.9464	1.1038	2.9392	0.3747	1.1412	0.0084	0.0428	6.0
I	0.5	0.1	0.0706	0.2601	0.0038	0.0166	0.0004	0.0018	0.0000	0.0000	12.3
J	0.5	0.25	0.6097	1.9343	0.0666	0.2604	0.0128	0.0579	0.0000	0.0000	12.3
K	0.5	0.5	4.6695	11.3070	1.4906	4.6422	0.8400	2.8534	0.0033	0.0206	12.3
L	0.5	0.75	2.7466	8.5090	1.2067	4.2924	0.5655	2.2551	0.0027	0.0189	♡ 8.9
M	0.75	0.1	0.0234	0.0887	0.0014	0.0061	0.0001	0.0007	0.0000	0.0000	26.3
N	0.75	0.25	0.1761	0.6198	0.0202	0.0839	0.0037	0.0170	0.0000	0.0000	26.3
O	0.75	0.5	1.3195	3.9419	0.3408	1.2225	0.1401	0.5490	0.0000	0.0000	26.3
P	0.75	0.75	2.8511	9.0210	1.6456	5.7547	0.9628	3.7110	0.0005	0.0040	♢ 18.6
Absolute percentage difference			2.7184	6.1210	0.4750	1.5338	0.2063	0.7624	0.0012	0.0071	

Table 7.5.1:

The results show a marked difference in the accuracy for both the mean queue length and its variance between the AGG1 approximation and the AGG2 approximation, with still further improvement for the AGG3 approximation. The AGG-3 approximation is not a fixed level aggregation but is dynamically decided by a specific first queue length probability truncation, making some of the *MAP* approximations smaller or larger than those of the AGG3 approximation. With this type

Set	*Traffic intensities*		*Averaged absolute percentage differences for the stationary second queue length of the tandem*								
			AGG1		AGG2		AGG3		AGG-3		
	ρ_1	ρ_2	E[μ]	E[σ^2]	E[μ]	E[σ^2]	E[μ]	E[σ^2]	E[μ]	E[σ^2]	k
A^*	0.1	0.1	0.0172	0.0715	0.0013	0.0064	0.0001	0.0006	0.0000	0.0000	3.3
B^*	0.1	0.25	0.0850	0.3544	0.0059	0.0311	0.0006	0.0039	0.0000	0.0003	3.3
C^*	0.1	0.5	0.2074	0.7835	0.0148	0.0694	0.0017	0.0092	0.0001	0.0007	3.3
D^*	0.1	0.75	0.3126	0.9396	0.0213	0.0751	0.0024	0.0095	0.0001	0.0006	3.3
E^*	0.25	0.1	0.0339	0.1357	0.0030	0.0142	0.0004	0.0020	0.0000	0.0000	4.9
F^*	0.25	0.25	0.1733	0.6955	0.0231	0.1112	0.0041	0.0223	0.0000	0.0001	4.9
G^*	0.25	0.5	0.6910	2.5918	0.1175	0.5370	0.0264	0.1406	0.0002	0.0019	4.9
H^*	0.25	0.75	1.3269	4.1176	0.2637	0.9543	0.0641	0.2633	0.0008	0.0044	4.9
I^*	0.5	0.1	0.0354	0.1384	0.0030	0.0138	0.0004	0.0018	0.0000	0.0000	8.5
J^*	0.5	0.25	0.1670	0.6470	0.0278	0.1267	0.0062	0.0315	0.0000	0.0000	8.5
K^*	0.5	0.5	1.0137	3.7074	0.2724	1.1610	0.0851	0.4099	0.0001	0.0008	8.5
L^*	0.5	0.75	3.0406	9.7222	1.1981	4.3764	0.5236	2.1395	0.0024	0.0168	8.5
M^*	0.75	0.1	0.0183	0.0709	0.0014	0.0064	0.0002	0.0008	0.0000	0.0000	18.0
N^*	0.75	0.25	0.1121	0.4237	0.0175	0.0760	0.0036	0.0173	0.0000	0.0000	18.0
O^*	0.75	0.5	0.6099	2.1485	0.1757	0.7126	0.0595	0.2666	0.0000	0.0000	18.0
P^*	0.75	0.75	3.0753	9.9312	1.6636	5.9560	0.9312	3.6546	0.0004	0.0032	18.0
Absolute percentage difference			0.6825	2.2799	0.2381	0.8892	0.1069	0.4358	0.0003	0.0018	

Table 7.5.2:

of parameterisation, excellent results can be gained as is seen by Tables 7.5.1 and 7.5.2, but the *MAP* approximations become large as ρ_1 becomes large. The average value of k is given for the AGG-3 approximation in the extreme right hand column. Note that the value of k at $\heartsuit$ for the set L is significantly less than for the sets I, J and K, because, as mentioned above, the bursty arrival process (the fifth arrival process given in Appendix A.1) for this combination of traffic intensities was not investigated. For the same reason, the value of k at $\diamondsuit$ for the set P is significantly less than that of sets M, N and O.

The actual times taken will vary according to differing software and platforms. Since investigation of the hardware and programming language parameters was not intended, the average times were recorded across all runs (including all variations, without exception) and given relative to the value for $k = 1$. These relative times are given in Table 7.5.3. For completeness, the actual average times are also recorded in Table 7.5.4 for the following computer configuration, with the code implemented using Matlab version 5.0.0.4064.

- Platform: SUNW Ultra-1, sun4u, sparc,
- Memory size: 192 Megabytes,
- Operating system: SunOS 5.5.1, Generic_103640-08 ,

Relative time taken with respect to $k = 1$			
AGG1	AGG2	AGG3	AGG-3
1.000	1.5817	1.9664	155.3203

Table 7.5.3:

Actual average time taken in seconds			
AGG1	AGG2	AGG3	AGG-3
.2752	.4353	.5411	42.7434

Table 7.5.4:

7.6 Comparison to other work

Whitt [52] used the work of Shimshak [49] on expected waiting times of queues in series to compare his work on approximating departure processes. Shimshak used simulation results for the total expected waiting time in two single server queues in series. The Fraker, Page and Marchal approximations are approximations constructed by Shimshak using previous approximations. The *M/M/1*, *M/G/1*, *QNA* and modified *QNA* approximations are described in Section 5 of Whitt [52].

We produce results for Shimshak's experiments **I**, **III** and **IV** using the methods that we have proposed in this paper. In Shimshak's experiment **II**, the second queue had ten servers and was therefore not considered in Whitt [52] or in this thesis. Experiments **I**, **III** and **IV** involved three different renewal arrival processes used to feed two queues in series. The arrival processes were a Poisson process, a two state hyper-exponential process with squared coefficient of variation 4 and an Erlang renewal process of order 10. The actual form of the hyper-exponential arrival process was constructed using the reference given in Shimshak [49] to the definition in Morse [30]. The service distributions were either negative-exponential (E_1) or Erlang of order 10 (E_{10}), with the traffic intensities being either 0.6 or 0.8.

Shimshak's experiment **I**				
System	*Queue 1*		*Queue 2*	
number	ρ_1	distr.	ρ_2	distr.
1	0.6	E_{10}	0.6	E_1
2	0.8	E_{10}	0.6	E_1
3	0.6	E_{10}	0.6	E_{10}
4	0.8	E_{10}	0.6	E_{10}
5	0.6	E_{10}	0.8	E_1
6	0.8	E_{10}	0.8	E_1
7	0.6	E_{10}	0.8	E_{10}
8	0.8	E_{10}	0.8	E_{10}

Table 7.6.1:

Shimshak's experiments **III** *and* **IV**				
System	*Queue 1*		*Queue 2*	
number	ρ_1	distr.	ρ_1	distr.
1	0.8	E_1	0.6	E_1
2	0.8	E_{10}	0.6	E_1
3	0.8	E_1	0.6	E_{10}
4	0.8	E_{10}	0.6	E_{10}
5	0.8	E_1	0.8	E_1
6	0.8	E_{10}	0.8	E_1
7	0.8	E_1	0.8	E_{10}
8	0.8	E_{10}	0.8	E_{10}

Table 7.6.2:

Experiment **I** has a Poisson arrival process of rate one with the first server fixed as E_{10}. In experiments **III** and **IV**, the arrival processes of rate one are the two state hyper-exponential and Erlang (E_{10}) process respectively with the first queue traffic intensity set to 0.8 for both cases. Shimshak's experiment **IV** has the same traffic intensities and service time distribution as his experiment **III**. The traffic intensity and service time distributions for Shimshak's experiments **I** and **III** are given in Tables 7.6.1 and 7.6.2.

The results for Shimshak's experiments **I**, **III** and **IV** are given respectively in Tables 7.6.3, 7.6.4 and 7.6.5. The simulation estimates of the total expected waiting time at the first and second queues were obtained by Shimshak and have a 95%

Shimshak's experiment **I***, with Poisson arrival stream*										
		Approximations								
Sys. *no.*	*Sim.* *est.*	M/M/1	M/G/1	Fraker	Page	Marchal	QNA (st. int.)	Mod. QNA	AGG1	AGG2
1	1.20	1.80	1.40	1.19	1.20	1.18	1.25	1.30	1.11	1.15
	(±0.09)	(0.50)	(0.17)	(-0.01)	(0.00)	(-0.02)	(0.04)	(0.08)	(-0.08)	(-0.04)
2	2.27	4.10	2.66	2.30	2.31	2.28	2.38	2.42	2.23	2.25
	(±0.23)	(0.81)	(0.17)	(0.01)	(0.02)	(0.01)	(0.05)	(0.07)	(-0.02)	(-0.01)
3	0.78	1.80	0.99	0.77	0.84	0.84	0.84	0.89	0.66	0.68
	(±0.06)	(1.31)	(0.27)	(-0.02)	(0.08)	(0.07)	(0.08)	(0.14)	(-0.15)	(-0.13)
4	1.83	4.10	2.26	1.90	1.99	1.98	1.98	2.01	1.83	1.83
	(±0.22)	(1.24)	(0.23)	(0.04)	(0.09)	(0.08)	(0.08)	(0.10)	(0.00)	(0.00)
5	3.41	4.10	3.70	3.07	3.10	3.06	3.21	3.27	2.98	3.18
	(±0.43)	(0.20)	(0.09)	(-0.10)	(-0.09)	(-0.10)	(-0.06)	(-0.04)	(-0.13)	(-0.07)
6	4.33	4.33	6.40	4.96	3.85	3.70	3.84	4.07	3.78	3.91
	(±0.60)	(0.48)	(0.15)	(-0.14)	(-0.10)	(-0.11)	(-0.06)	(-0.05)	(-0.13)	(-0.10)
7	1.93	4.10	2.26	1.60	1.73	1.72	1.77	1.83	1.48	1.64
	(±0.27)	(1.12)	(0.17)	(-0.17)	(-0.10)	(-0.11)	(-0.08)	(-0.05)	(-0.23)	(-0.15)
8	2.48	6.40	3.52	2.43	2.58	2.57	2.63	2.68	2.27	2.33
	(±0.29)	(1.58)	(0.42)	(-0.02)	(0.04)	(0.04)	(0.06)	(0.08)	(-0.08)	(-0.06)
Average relative error		0.91	0.21	-0.05	-0.01	-0.02	0.01	0.04	-0.10	-0.07
Average absolute relative error		0.91	0.21	0.06	0.07	0.07	0.06	0.08	0.10	0.07

Table 7.6.3:

confidence interval given in brackets below each estimate. The tables were produced in the same form as that of [52] for easy comparison. We give the total expected waiting times at both queues as calculated by each of the approximation methods, together with the relative error (in brackets). These relative errors were calculated by taking the difference of the approximation to the simulation estimate and dividing by the simulation estimate. Following the form of the tables in [52], we have given the average relative and average absolute relative errors for each approximation method across the eight systems. In experiment **III**, the Page method could not be applied to the system and for the Fraker approximation, the blanks indicate that a negative waiting time was recorded.

Note that across the three experiments, the two columns labelled QNA are in

Shimshak's experiment **III**, *with hyper-exponential arrival stream*									
		Approximations							
Sys. no.	*Sim. est.*	M/M/1	M/G/1	Fraker	Marchal	QNA (st. int.)	Mod. QNA	AGG1	AGG2
1	9.08	4.10	4.10	10.30	10.39	9.39	9.56	9.02	9.08
	(±1.38)	(-.055)	(-0.55)	(0.13)	(0.14)	(0.03)	(0.05)	(0.01)	(0.00)
2	6.49	4.10	6.49	–	7.91	7.72	7.94	6.78	6.79
	(±0.73)	(-0.37)	(-0.51)		(0.22)	(0.19)	(0.22)	(0.04)	(0.05)
3	8.55	4.10	3.18	10.17	9.86	8.98	9.16	8.53	8.53
	(±1.20)	(-0.52)	(-0.63)	(0.19)	(0.15)	(0.05)	(0.07)	(0.00)	(0.00)
4	6.01	4.10	2.26	–	7.43	7.31	7.54	6.25	6.25
	(±0.73)	(-0.32)	(-0.62)		(0.24)	(0.22)	(0.25)	(0.04)	(0.04)
5	12.31	6.40	6.40	13.50	13.54	12.92	13.09	12.38	12.55
	(±2.26)	(-0.48)	(-0.48)	(0.10)	(0.10)	(0.05)	(0.06)	(0.01)	(0.02)
6	9.64	6.40	4.96	–	10.78	10.67	10.88	9.30	9.44
	(±1.33)	(-0.34)	(-0.49)		(0.12)	(0.11)	(0.13)	(-0.04)	(-0.02)
7	11.13	6.40	4.96	12.55	11.90	11.49	11.65	10.68	10.79
	(±1.37)	(-0.42)	(-0.55)	(-0.13)	(0.07)	(0.03)	(0.05)	(-0.04)	(-0.03)
8	7.40	6.40	3.52	–	9.21	9.23	9.44	7.17	7.25
	(±0.95)	(-0.14)	(-0.52)		(0.24)	(0.25)	(0.28)	(-0.03)	(-0.02)
Average relative error		-0.39	-0.54	0.14	0.16	0.12	0.14	0.00	0.00
Average absolute relative error		0.39	0.54	0.14	0.16	0.12	0.14	0.03	0.02

Table 7.6.4:

general superior to all the columns to the left. Therefore we shall confine our comparison to those marked QNA. It is interesting to note that our approximations do not show their best results with a Poisson arrival stream, but this is not the case with the QNA methods. When the arrival stream is hyper-exponential, it appears that our methods strongly outperform the QNA methods as well as the others. When the arrival stream is Erlang, our method again outperforms the QNA methods with the method labelled AGG2 being the best. When the arrival process is Poisson, it is hard to separate any of the methods and so the simplest method should be chosen. It would be interesting to understand why both of our methods perform so much worse with a Poisson arrival stream.

Whitt's results in [52] are compared to simulation results obtained by Shimshak

Shimshak's experiment **IV**, *with Erlang* (E_{10}) *arrival stream*										
		Approximations								
Sys. no.	*Sim. est.*	M/M/1	M/G/1	Fraker	Page	Marchal	QNA (st. int.)	Mod. QNA	AGG1	AGG2
1	2.30	4.10	4.10	2.24	2.30	2.21	2.29	2.25	2.30	2.24
	(±0.19)	(0.78)	(0.78)	(-0.03)	(0.00)	(-0.04)	(0.00)	(-0.02)	(0.00)	(-0.03)
2	0.59	4.10	3.18	0.58	0.65	0.59	0.56	0.56	0.58	0.57
	(±0.04)	(5.95)	(4.39)	(-0.02)	(0.10)	(0.00)	(-0.05)	(-0.05)	(-0.02)	(-0.03)
3	1.95	4.10	3.18	1.81	1.92	1.84	1.89	1.85	1.92	1.89
	(±0.45)	(1.10)	(0.63)	(-0.07)	(-0.02)	(-0.06)	(-0.03)	(-0.05)	(-0.02)	(-0.03)
4	0.25	4.10	2.26	0.27	0.38	0.35	0.20	0.20	0.25	0.25
	(±0.02)	(15.40)	(8.04)	(0.08)	(0.52)	(0.40)	(-0.20)	(-0.20)	(0.00)	(0.00)
5	3.84	6.40	6.40	4.27	4.30	4.26	4.21	4.17	4.31	4.02
	(±0.33)	(0.67)	(0.67)	(0.11)	(0.12)	(0.11)	(0.10)	(0.09)	(0.12)	(0.05)
6	1.82	6.40	4.96	1.77	1.85	1.75	1.85	1.85	1.83	1.77
	(±0.19)	(2.52)	(1.73)	(-0.03)	(0.02)	(-0.04)	(0.02)	(0.02)	(0.01)	(-0.03)
7	2.68	6.40	4.96	2.79	2.92	2.88	2.77	2.73	2.90	2.68
	(±0.52)	(1.39)	(0.84)	(0.04)	(0.09)	(0.07)	(0.03)	(0.02)	(0.08)	(0.00)
8	0.46	6.40	3.52	0.50	0.61	0.58	0.43	0.43	0.49	0.45
	(±0.02)	(12.91)	(6.65)	(0.09)	(0.33)	(0.26)	(-0.06)	(-0.06)	(0.07)	(-0.02)
Average relative error		5.09	2.97	0.02	0.15	0.09	-0.02	-0.03	0.03	-0.01
Average absolute relative error		5.09	2.97	0.06	0.15	0.12	0.06	0.06	0.04	0.02

Table 7.6.5:

[49], and a future intention is to remove any simulation discrepancy by using the "exact" technique as outlined in Section 7.4. This, however, will consume a great deal of computer time and at present must be left as future work. The fitting of a *PH*-type approximation to the busy period which takes into account higher moments is also being investigated. The possible adaption and application to more general queues, such as the *MAP/G/1* and a feedback queue, will also be investigated in future work.

7.7 Summary

In this chapter we have proposed a family of methods to approximate the output process of *MAP/PH/1* queues. We compared the methods with exact results and also with results given in Whitt [52] for the experiments of Shimshak and found that they performed very well. In the next chapter, it is shown that our approximations for $k \geq 2$ have the property of matching the exact lag-correlation coefficients of the departure process up to order $k - 1$.

Chapter 8

Correlation structure of *MAP/PH/1* departure processes and the family of approximations

8.1 Introduction

Following the notation of Chapter 7, we let D_0 and D_1 be the $m \times m$ matrix descriptors of the *MAP* such that $D_1 \geq 0$, and we let $(\boldsymbol{\beta}, S)$ be the n-state *PH*-renewal service distribution. We also assume that the *QBD* process defines an irreducible regular Markov chain, so that it has a unique stationary distribution and both D_0^{-1} and $(D_0 \oplus S)^{-1}$ exist.

In this chapter we first establish the form of the stationary distribution given in Chapter 7 for the k^{th} approximation. We then prove that the stationary inter-event time distribution for each of our approximations is in fact the same as the stationary inter-departure time distribution for the *MAP/PH/1* queue. For $k \geq 2$ we then show that the lag-correlation coefficients $c_1(k), \ldots, c_{k-1}(k)$ for the stationary inter-event times of the k^{th} approximation are identical to the lag-correlation coefficients for the stationary inter-departure times of the *MAP/PH/1* queue.

All of our approximations thus have the exact stationary inter-event time distribution of the departure process from the *MAP/PH/1* queue. The correlation structure between successive inter-event times must therefore be all that separates our approximations from the actual departure process. This exact correlation structure would seem to be a very important characteristic to capture when modelling departure processes. One measure of this correlation structure is in fact given by the lag-correlation coefficients. Our k^{th} approximation captures exactly all of these lag-correlation coefficients up to the $k-1^{st}$ for the stationary inter-departure times of the *MAP/PH/1* queue. This claim is supported by the results given in Chapter 7, where the accuracy of our approximations increased as more of the lag-correlation coefficients were captured.

For $k = 1$ the *MAP* approximation is a *PH*-renewal process, which has been shown in [9] to have an inter-event time distribution which is the exact inter-departure time distribution of the *MAP/PH/1* queue. Of course, this process contains no correlation information about the departures from the *MAP/PH/1* queue. The results claimed are trivially true for the case $k = 1$.

Although it has already been discussed in [9] we will use this case to illustrate our method. To avoid ambiguity in the following Kronecker algebra, appropriate columns of ones and identity matrices will be given a subscript denoting their dimension.

Recall the filtration matrices which give the departure process of the *MAP/PH/1* queue, that is,

$$Q_0 = \begin{bmatrix} B_1 & B_0 & & & & \\ & A_1 & A_0 & & & \\ & & A_1 & A_0 & & \\ & & & A_1 & A_0 & \\ & & & & \ddots & \ddots \end{bmatrix} \text{ and } Q_1 = \begin{bmatrix} 0 & & & & \\ B_2 & 0 & & & \\ & A_2 & 0 & & \\ & & A_2 & 0 & \\ & & & \ddots & \ddots \end{bmatrix}. \quad (8.1.1)$$

From (7.2.2), the stationary distribution for this *QBD* is given by

$$\boldsymbol{\Psi} = [\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots]$$

$$= \boldsymbol{\pi}_0[I, R_0, R_0R, R_0R^2, \ldots]. \tag{8.1.2}$$

Recall also that the $k = 1$ *MAP* approximation to the departure process is given by

$$Q_0(1) = \begin{pmatrix} D_0 & D_1\boldsymbol{e}_m\boldsymbol{\beta} \\ 0 & S \end{pmatrix} \text{ and } Q_1(1) = \begin{pmatrix} 0 & 0 \\ \boldsymbol{S}^0\boldsymbol{x}_0 & (1-\boldsymbol{x}_0\boldsymbol{e}_m)\boldsymbol{S}^0\boldsymbol{\beta} \end{pmatrix}, \tag{8.1.3}$$

where

$$\boldsymbol{x}_0 = \boldsymbol{\pi}_0 R_0 B_2(\boldsymbol{\nu} D_1 \boldsymbol{e}_m)^{-1},$$

and $\boldsymbol{\nu}$ is the unique stationary distribution of the *MAP*. Also from equation (7.3.4), for $k \geq 2$, the *MAP* approximations are given by

$$Q_0(k) = \begin{bmatrix} B_1 & B_0 & & & & \\ & A_1 & A_0 & & & \\ & & A_1 & \ddots & & \\ & & & \ddots & A_0 & \\ & & & & A_1 & E_0 \\ & & & & & E_1 \end{bmatrix}, Q_1(k) = \begin{bmatrix} 0 & & & & & \\ B_2 & 0 & & & & \\ & A_2 & \ddots & & & \\ & & \ddots & 0 & & \\ & & & A_2 & 0 & \\ & & & & E_2 & E_3 \end{bmatrix}, \tag{8.1.4}$$

where

$$\begin{aligned} A_0 &= D_1 \otimes I_n, \\ A_1 &= D_0 \oplus S = \Big((I_m \otimes S) + (D_0 \otimes I_n)\Big), \\ A_2 &= I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}), \\ B_0 &= D_1 \otimes \boldsymbol{\beta}, \\ B_1 &= D_0, \\ B_2 &= I_m \otimes \boldsymbol{S}^0, \\ E_0 &= (D_1 \otimes I_n)(\boldsymbol{e}_m \otimes I_n), \\ E_1 &= S, \\ E_2 &= \boldsymbol{S}^0\boldsymbol{y}_{k-1} \text{ and} \\ E_3 &= (1-\boldsymbol{y}_{k-1}\boldsymbol{e}_{mn})\boldsymbol{S}^0\boldsymbol{\beta}. \end{aligned} \tag{8.1.5}$$

Recall also that $\boldsymbol{S}^0 = -S\boldsymbol{e}_n$ and that $\boldsymbol{y}_{k-1}$ is given in equation (7.3.3) by

$$\boldsymbol{y}_{k-1} = \frac{\boldsymbol{x}_{k-1}}{(\sum_{j=k-1}^{\infty} \boldsymbol{x}_j \boldsymbol{e}_{mn})} \text{ for } k \geq 2, \tag{8.1.6}$$

where

$$\boldsymbol{x}_{k-1} = \boldsymbol{\pi}_k A_2(\boldsymbol{\nu} D_1 \boldsymbol{e}_m)^{-1} = \boldsymbol{\pi}_0 R_0 R^{k-1} A_2(\boldsymbol{\nu} D_1 \boldsymbol{e}_m)^{-1} \text{ for } k \geq 2. \tag{8.1.7}$$

The following results used throughout this chapter, are given as a lemma. See Appendix B for some rules on the Kronecker manipulations used in their proofs. For reference, Appendix C contains the dimensions of common matrices used in this chapter.

Lemma 8.1 *The following relationships hold:*

a. $(\boldsymbol{e}_m \otimes I_n)\boldsymbol{e}_n = \boldsymbol{e}_{mn}$,

b. $(\boldsymbol{e}_m \otimes I_n)\boldsymbol{S}^0 = (I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}$,

c. $(\boldsymbol{e}_m \otimes I_n)(-S)^{-1} = (D_0 \oplus S)^{-1}(D_1 \otimes I_n)(\boldsymbol{e}_m \otimes I_n)S^{-1} - (D_0 \oplus S)^{-1}(\boldsymbol{e}_m \otimes I_n)$,

d. $\boldsymbol{\pi}_k(I_{mn} - R)^{-1} = \boldsymbol{\pi}_k + \boldsymbol{\pi}_{k+1}(I_{mn} - R)^{-1}$, *for all* $k \geq 1$,

e. $\boldsymbol{\pi}_k(I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n)\boldsymbol{S}^0\boldsymbol{y}_{k-1} = \boldsymbol{\pi}_k(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))$ *for all* $k \geq 2$.

Proof:

a.

$$\begin{aligned}(\boldsymbol{e}_m \otimes I_n)\boldsymbol{e}_n &= (\boldsymbol{e}_m \otimes \boldsymbol{e}_n) \\ &= \boldsymbol{e}_{mn}.\end{aligned}$$

■

b. Using the fact that $\boldsymbol{\beta}\boldsymbol{e}_n = 1$,

$$\begin{aligned}(\boldsymbol{e}_m \otimes I_n)\boldsymbol{S}^0 &= (I_m\boldsymbol{e}_m \otimes \boldsymbol{S}^0) \\ &= (I_m\boldsymbol{e}_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}\boldsymbol{e}_n)) \\ &= (I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))(\boldsymbol{e}_m \otimes \boldsymbol{e}_n) \\ &= (I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}.\end{aligned}$$

■

c. Consider

$$\begin{aligned}(D_0 \oplus S)(\boldsymbol{e}_m \otimes I_n)(-S)^{-1} &= \Big((D_0 \otimes I_n) + (I_m \otimes S)\Big)(\boldsymbol{e}_m \otimes I_n)(-S)^{-1} \\ &= (D_0\boldsymbol{e}_m \otimes I_n)(-S)^{-1} + (\boldsymbol{e}_m \otimes S)(-S)^{-1}.\end{aligned}$$

From $(D_0 + D_1)\boldsymbol{e}_m = \mathbf{0}$ we have $D_0\boldsymbol{e}_m = -D_1\boldsymbol{e}_m$, which gives

$$\begin{aligned}(D_0 \oplus S)(\boldsymbol{e}_m \otimes I_n)(-S)^{-1} &= (D_1\boldsymbol{e}_m \otimes I_n)S^{-1} - (\boldsymbol{e}_m \otimes I_n) \\ &= (D_1 \otimes I_n)(\boldsymbol{e}_m \otimes I_n)S^{-1} - (\boldsymbol{e}_m \otimes I_n),\end{aligned}$$

so that pre-multiplying both sides by $(D_0 \oplus S)^{-1}$ yields the result. ■

d. By definition, we have for all $k \geq 1$ that

$$\begin{aligned}\boldsymbol{\pi}_k(I_{mn} - R)^{-1} &= \boldsymbol{\pi}_k \sum_{i=0}^{\infty} R^i \\ &= \boldsymbol{\pi}_k + \boldsymbol{\pi}_k R \sum_{i=0}^{\infty} R^i \\ &= \boldsymbol{\pi}_k + \boldsymbol{\pi}_{k+1}(I_{mn} - R)^{-1}.\end{aligned}$$

■

e. Using equation (8.1.7) in (8.1.6), noting that $(\boldsymbol{\nu} D_1\boldsymbol{e}_m)^{-1}$ is a scalar and that $\boldsymbol{\pi}_{k+i} = \boldsymbol{\pi}_k R^i$ for all $k \geq 1$ and $i \geq 0$, we see that

$$\begin{aligned}\boldsymbol{y}_{k-1} = \frac{\boldsymbol{x}_{k-1}}{\sum_{j=k-1}^{\infty} \boldsymbol{x}_j \boldsymbol{e}_{mn}} &= \frac{\boldsymbol{\pi}_k A_2 (\boldsymbol{\nu} D_1 \boldsymbol{e}_m)^{-1}}{\sum_{j=k-1}^{\infty} \boldsymbol{\pi}_{j+1} A_2 (\boldsymbol{\nu} D_1 \boldsymbol{e}_m)^{-1} \boldsymbol{e}_{mn}} \\ &= \frac{\boldsymbol{\pi}_k A_2}{\boldsymbol{\pi}_k (I_{mn} - R)^{-1} A_2 \boldsymbol{e}_{mn}} \quad \text{for all } k \geq 2. \qquad (8.1.8)\end{aligned}$$

Substituting the definition of A_2 from (8.1.5) into equation (8.1.8), it can be seen that

$$\boldsymbol{y}_{k-1} = \frac{\boldsymbol{\pi}_k (I_m \otimes (\boldsymbol{S}^0 \boldsymbol{\beta}))}{\boldsymbol{\pi}_k (I_{mn} - R)^{-1} (I_m \otimes (\boldsymbol{S}^0 \boldsymbol{\beta})) \boldsymbol{e}_{mn}} \quad \text{for all } k \geq 2. \qquad (8.1.9)$$

Substitution of (8.1.9) into the left hand side of Lemma 8.1e gives

$$\boldsymbol{\pi}_k(I_{mn}-R)^{-1}(\boldsymbol{e}_m\otimes I_n)\boldsymbol{S}^0\frac{\boldsymbol{\pi}_k(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))}{\boldsymbol{\pi}_k(I_{mn}-R)^{-1}(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}},$$

which by using Lemma 8.1b yields

$$\boldsymbol{\pi}_k(I_{mn}-R)^{-1}(\boldsymbol{e}_m\otimes I_n)\boldsymbol{S}^0\frac{\boldsymbol{\pi}_k(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))}{\boldsymbol{\pi}_k(I_{mn}-R)^{-1}(\boldsymbol{e}_m\otimes I_n)\boldsymbol{S}^0}$$

$$=\boldsymbol{\pi}_k(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta})),\ \text{for all } k\geq 2.$$

■

8.2 The stationary distribution

To establish the unique form of the stationary distribution for the k^{th} MAP approximation, we will first introduce the following lemma.

Lemma 8.2 *The following relationship holds for all $k \geq 2$:*

$$\boldsymbol{\pi}_{k-1}(D_1\boldsymbol{e}_m\otimes I_n)+\boldsymbol{\pi}_k(I_{mn}-R)^{-1}(\boldsymbol{e}_m\otimes I_n)\Big(S+(1-\boldsymbol{y}_{k-1}\boldsymbol{e}_{mn})(\boldsymbol{S}^0\boldsymbol{\beta})\Big)=\boldsymbol{0}. \tag{8.2.1}$$

Proof:

Using (8.1.9) and Lemma 8.1d, it follows that

$$\begin{aligned}1-\boldsymbol{y}_{k-1}\boldsymbol{e}_{mn} &= 1-\frac{\boldsymbol{\pi}_k(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}}{\boldsymbol{\pi}_k(I_{mn}-R)^{-1}(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}}\\ &= \frac{\boldsymbol{\pi}_k(I_{mn}-R)^{-1}(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}}{\boldsymbol{\pi}_k(I_{mn}-R)^{-1}(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}}-\frac{\boldsymbol{\pi}_k(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}}{\boldsymbol{\pi}_k(I_{mn}-R)^{-1}(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}}\\ &= \frac{\boldsymbol{\pi}_{k+1}(I_{mn}-R)^{-1}(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}}{\boldsymbol{\pi}_k(I_{mn}-R)^{-1}(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}}\ \text{for all } k\geq 2. \qquad (8.2.2)\end{aligned}$$

The quantity given in (8.2.2) is a scalar which we will substitute into the left hand side of (8.2.1) to give

$$\begin{aligned}&\boldsymbol{\pi}_{k-1}(D_1\boldsymbol{e}_m\otimes I_n)\\&+\ \boldsymbol{\pi}_k(I_{mn}-R)^{-1}(\boldsymbol{e}_m\otimes I_n)S\\&+\ \boldsymbol{\pi}_k(I_{mn}-R)^{-1}(\boldsymbol{e}_m\otimes I_n)\boldsymbol{S}^0\frac{\boldsymbol{\pi}_{k+1}(I_{mn}-R)^{-1}(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}\boldsymbol{\beta}}{\boldsymbol{\pi}_k(I_{mn}-R)^{-1}(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}}.\end{aligned}$$

Using Lemma 8.1b, this yields

$$\begin{aligned}&\boldsymbol{\pi}_{k-1}(D_1\boldsymbol{e}_m\otimes I_n)\\&+\ \boldsymbol{\pi}_k(I_{mn}-R)^{-1}(\boldsymbol{e}_m\otimes I_n)S\\&+\ \boldsymbol{\pi}_k(I_{mn}-R)^{-1}(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}\frac{\boldsymbol{\pi}_{k+1}(I_{mn}-R)^{-1}(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}\boldsymbol{\beta}}{\boldsymbol{\pi}_k(I_{mn}-R)^{-1}(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}}\\[1em]&=\boldsymbol{\pi}_{k-1}(D_1\otimes I_n)(\boldsymbol{e}_m\otimes I_n)\\&\quad+\ \boldsymbol{\pi}_k(I_{mn}-R)^{-1}(I_m\otimes S)(\boldsymbol{e}_m\otimes I_n)\\&\quad+\ \boldsymbol{\pi}_{k+1}(I_{mn}-R)^{-1}(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}\boldsymbol{\beta},\end{aligned}$$

which by using Lemma 8.1b again yields

$$\begin{aligned}&\boldsymbol{\pi}_{k-1}(D_1\otimes I_n)(\boldsymbol{e}_m\otimes I_n)\\&+\ \boldsymbol{\pi}_k(I_{mn}-R)^{-1}(I_m\otimes S)(\boldsymbol{e}_m\otimes I_n)\\&+\ \boldsymbol{\pi}_{k+1}(I_{mn}-R)^{-1}(\boldsymbol{e}_m\otimes I_n)\boldsymbol{S}^0\boldsymbol{\beta}\\[1em]&=\boldsymbol{\pi}_{k-1}(D_1\otimes I_n)(\boldsymbol{e}_m\otimes I_n)\\&\quad+\ \boldsymbol{\pi}_k(I_{mn}-R)^{-1}(I_m\otimes S)(\boldsymbol{e}_m\otimes I_n)\\&\quad+\ \boldsymbol{\pi}_{k+1}(I_{mn}-R)^{-1}(I_m\otimes(\boldsymbol{S}^0\boldsymbol{\beta}))(\boldsymbol{e}_m\otimes I_n).\end{aligned}$$

This can be re-arranged using Lemma 8.1d to give

$$\begin{aligned}
&\boldsymbol{\pi}_{k-1}(D_1 \otimes I_n)(\boldsymbol{e}_m \otimes I_n) \\
&\quad + \ \boldsymbol{\pi}_k(I_m \otimes S)(\boldsymbol{e}_m \otimes I_n) \\
&\quad\ + \ \boldsymbol{\pi}_{k+1}(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))(\boldsymbol{e}_m \otimes I_n) \\
&\quad\ \ + \ \boldsymbol{\pi}_{k+1}(I_{mn} - R)^{-1}(I_m \otimes S)(\boldsymbol{e}_m \otimes I_n) \\
&\quad\ \ \ + \ \boldsymbol{\pi}_{k+2}(I_{mn} - R)^{-1}(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))(\boldsymbol{e}_m \otimes I_n).
\end{aligned} \tag{8.2.3}$$

From $\boldsymbol{\Psi}(Q_0 + Q_1) = \boldsymbol{0}$, we see for all $j \in \{0, 1, 2, \ldots\}$ and $k \geq 1$ that

$$\boldsymbol{\pi}_{k+j}A_0 + \boldsymbol{\pi}_{k+1+j}A_1 + \boldsymbol{\pi}_{k+2+j}A_2 = \boldsymbol{0},$$

or by using the definitions of A_0, A_1 and A_2 given in (8.1.5) that

$$\boldsymbol{\pi}_{k+j}(D_1 \otimes I_n) + \boldsymbol{\pi}_{k+1+j}\Big((I_m \otimes S) + (D_0 \otimes I_n)\Big) + \boldsymbol{\pi}_{k+2+j}(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta})) = \boldsymbol{0} \quad \text{for all } k \geq 1. \tag{8.2.4}$$

Note that the first three terms in (8.2.3) can be written as

$$\Big(\boldsymbol{\pi}_{k-1}(D_1 \otimes I_n) + \boldsymbol{\pi}_k(I_m \otimes S) + \boldsymbol{\pi}_{k+1}(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))\Big)(\boldsymbol{e}_m \otimes I_n),$$

which by (8.2.4) is equal to

$$-\boldsymbol{\pi}_k(D_0 \otimes I_n)(\boldsymbol{e}_m \otimes I_n).$$

We can now re-arrange (8.2.4) to give

$$\boldsymbol{\pi}_{k+1+j}(I_m \otimes S) + \boldsymbol{\pi}_{k+2+j}(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta})) = -\boldsymbol{\pi}_{k+j}(D_1 \otimes I_n) - \boldsymbol{\pi}_{k+1+j}(D_0 \otimes I_n)$$
$$\text{for all } k \geq 1 \text{ and } j \in \{0, 1, 2, \ldots\}.$$

Summing over all j yields

$$\begin{aligned}
&\boldsymbol{\pi}_{k+1}(I_{mn} - R)^{-1}(I_m \otimes S) + \boldsymbol{\pi}_{k+2}(I_{mn} - R)^{-1}(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta})) \\
&\quad = -\boldsymbol{\pi}_k(I_{mn} - R)^{-1}(D_1 \otimes I_n) - \boldsymbol{\pi}_{k+1}(I_{mn} - R)^{-1}(D_0 \otimes I_n).
\end{aligned}$$

Post-multiplying by $(\boldsymbol{e}_m \otimes I_n)$ yields an alternative expression for the last two terms in (8.2.3). That is,

$$\begin{aligned}
&\boldsymbol{\pi}_{k+1}(I_{mn} - R)^{-1}(I_m \otimes S)(\boldsymbol{e}_m \otimes I_n) + \boldsymbol{\pi}_{k+2}(I_{mn} - R)^{-1}(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))(\boldsymbol{e}_m \otimes I_n) \\
&\quad = -\boldsymbol{\pi}_k(I_{mn} - R)^{-1}(D_1 \otimes I_n)(\boldsymbol{e}_m \otimes I_n) - \boldsymbol{\pi}_{k+1}(I_{mn} - R)^{-1}(D_0 \otimes I_n)(\boldsymbol{e}_m \otimes I_n),
\end{aligned}$$

which by using the fact that $D_0\boldsymbol{e}_m = -D_1\boldsymbol{e}_m$ yields

$$\begin{aligned}
&\boldsymbol{\pi}_{k+1}(I_{mn} - R)^{-1}(I_m \otimes S)(\boldsymbol{e}_m \otimes I_n) + \boldsymbol{\pi}_{k+2}(I_{mn} - R)^{-1}(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))(\boldsymbol{e}_m \otimes I_n) \\
&\quad = \boldsymbol{\pi}_k(I_{mn} - R)^{-1}(D_0 \otimes I_n)(\boldsymbol{e}_m \otimes I_n) - \boldsymbol{\pi}_{k+1}(I_{mn} - R)^{-1}(D_0 \otimes I_n)(\boldsymbol{e}_m \otimes I_n) \\
&\quad = \boldsymbol{\pi}_k(D_0 \otimes I_n)(\boldsymbol{e}_m \otimes I_n).
\end{aligned}$$

Therefore (8.2.3) may be written as

$$-\boldsymbol{\pi}_k(D_0 \otimes I_n)(\boldsymbol{e}_m \otimes I_n) + \boldsymbol{\pi}_k(D_0 \otimes I_n)(\boldsymbol{e}_m \otimes I_n) = \mathbf{0}.$$

■

Theorem 8.1 *The stationary distribution of phases of the k^{th} MAP approximation to the departure process of the MAP/PH/1 queue is given by*

$$\boldsymbol{\nu}(k) = \Big(\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_{k-1}, \boldsymbol{\pi}_k(I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n)\Big) \text{ for all } k \geq 1, \tag{8.2.5}$$

where $(\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots)$ is the stationary distribution of the MAP/PH/1 queue.

Proof:

To prove that the probability vector given in (8.2.5) is the unique stationary probability vector for the k^{th} *MAP* approximation, it is only necessary to show that $\boldsymbol{\nu}(k)(Q_0(k) + Q_1(k)) = \mathbf{0}$. This is because we have

$$\begin{aligned}
\boldsymbol{\nu}(k)\boldsymbol{e} &= \boldsymbol{\pi}_0\boldsymbol{e}_m + \sum_{i=1}^{k-1}\boldsymbol{\pi}_i\boldsymbol{e}_{mn} + \boldsymbol{\pi}_k(I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n)\boldsymbol{e}_n \\
&= \boldsymbol{\pi}_0\boldsymbol{e}_m + \sum_{i=1}^{k-1}\boldsymbol{\pi}_i\boldsymbol{e}_{mn} + \boldsymbol{\pi}_k(I_{mn} - R)^{-1}\boldsymbol{e}_{mn} \\
&= \boldsymbol{\pi}_0\boldsymbol{e}_m + \boldsymbol{\pi}_1(I_{mn} - R)^{-1}\boldsymbol{e}_{mn} \\
&= \boldsymbol{\Psi}\boldsymbol{e} = 1,
\end{aligned}$$

where $\boldsymbol{\Psi}$ is the stationary distribution of the *MAP/PH/1* queue.

For $k \geq 2$, the k^{th} *MAP* approximation given by the filtration matrices $Q_0(k)$ and $Q_1(k)$ in equation (8.1.4) has identical non-zero entries to the *QBD* forms Q_0 and Q_1 given in equation (8.1.1) for levels $0, \ldots, k-2$. Because of this as well as because of the tri-diagonal structure, it is only necessary to prove that the last two entries of the row vector $\boldsymbol{\nu}(k)(Q_0(k)+Q_1(k))$, which correspond to levels $k-1$ and k, are zero. That is, for level $k-1$ we need

$$\begin{cases} \boldsymbol{\pi}_0 B_0 + \boldsymbol{\pi}_1 A_1 + \boldsymbol{\pi}_2 (I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n) E_2 = \mathbf{0}, & \text{if } k = 2, \\ \boldsymbol{\pi}_{k-2} A_0 + \boldsymbol{\pi}_{k-1} A_1 + \boldsymbol{\pi}_k (I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n) E_2 = \mathbf{0}, & \text{if } k > 2, \end{cases} \tag{8.2.6}$$

and for level k we need

$$\begin{aligned} &\boldsymbol{\pi}_{k-1} E_0 + \boldsymbol{\pi}_k (I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n)(E_1 + E_3) \\ &= \boldsymbol{\pi}_{k-1}(D_1 \boldsymbol{e}_m \otimes I_n) + \boldsymbol{\pi}_k (I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n)\Big(S + (1 - \boldsymbol{y}_{k-1}\boldsymbol{e}_{mn})\boldsymbol{S}^0\boldsymbol{\beta}\Big) \\ &= \mathbf{0}. \end{aligned} \tag{8.2.7}$$

Note that equation (8.2.7) is the result given by Lemma 8.2. Now from $\boldsymbol{\Psi}(Q_0+Q_1) = \mathbf{0}$, it can be seen that

$$\begin{aligned} &\boldsymbol{\pi}_0 B_0 + \boldsymbol{\pi}_1 A_1 + \boldsymbol{\pi}_2 A_2 = \mathbf{0} && \text{and} \\ &\boldsymbol{\pi}_{k-2} A_0 + \boldsymbol{\pi}_{k-1} A_1 + \boldsymbol{\pi}_k A_2 = \mathbf{0}, && \text{for } k > 2. \end{aligned}$$

Using this, we see that for equation (8.2.6) to hold requires

$$\boldsymbol{\pi}_k (I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n) E_2 = \boldsymbol{\pi}_k A_2, \text{for all } k \geq 2,$$

or, by using the definitions of E_2 and A_2 that

$$\boldsymbol{\pi}_k (I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n)\boldsymbol{S}^0 \boldsymbol{y}_{k-1} = \boldsymbol{\pi}_k (I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta})), \text{for all } k \geq 2,$$

which follows directly from Lemma 8.1e. ■

We have just shown that the stationary distribution of the k^{th} approximation is identical to the stationary distribution of the *MAP/PH/1* queue up to the $k-1^{st}$ level with the super level $\bar{k}$ taking up the remaining probability $\boldsymbol{\pi}_k(I_{mn} - R)^{-1}$. We will show in the next section that exactly matching this stationary distribution up to the $k-1^{st}$ level imparts more important properties.

8.3 The stationary inter-event time distribution

We have now established the common form of the stationary distribution for the *MAP* approximations. In this section we will show how this specific form translates to the stationary inter-event time distribution for each of the *MAP* approximations. In fact these inter-event time distributions are exactly the same. Moreover, they are identical to the stationary inter-departure time distribution for the *MAP/PH/1* queue. This statement is given in the following theorem.

Theorem 8.2 *The stationary inter-event time distributions for each of the MAP approximations to the departure process of the MAP/PH/1 queue are identical to the stationary inter-departure time distribution of the MAP/PH/1 queue.*

Proof:

For the $k = 1$ approximation, we have a *PH*-renewal process given by $(\boldsymbol{\alpha}, Q_0(1))$, where

$$\boldsymbol{\alpha} = (\boldsymbol{x}_0, (1 - \boldsymbol{x}_0\boldsymbol{e}_m)\boldsymbol{\beta}) \text{ and } Q_0(1) = \begin{pmatrix} D_0 & D_1\boldsymbol{e}_m\boldsymbol{\beta} \\ 0 & S \end{pmatrix}.$$

Recall that $\boldsymbol{x}_0$ is the stationary distribution of phases of the arrival process immediately after a departure that leaves the queue empty (when a busy period ends).

The stationary inter-event time interval for the $k = 1$ approximation is, with probability $\boldsymbol{x}_0\boldsymbol{e}_m$, the convolution of the idle period and the *PH*-type service time and is with probability $(1 - \boldsymbol{x}_0\boldsymbol{e}_m)$, just the *PH*-type service time. Thus the $k = 1$ approximation has exactly the same stationary inter-event time distribution as does the stationary departure process of the *MAP/PH/1* queue.

We will now show that this same argument can be applied in the case $k \geq 2$, by proving that the stationary distribution of the system being empty immediately after a departure (for all $k \geq 2$) is given by the same vector $\boldsymbol{x}_0$. The stationary distribution of the k^{th} *MAP* approximation is shown in Theorem 8.1 to be given by

$$\boldsymbol{\nu}(k) = \Big(\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_{k-1}, \boldsymbol{\pi}_k(I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n)\Big) \text{ for all } k \geq 2,$$

so that the stationary distribution of being in level 1 is the same for all MAP approximations for $k \geq 2$. Therefore from equation (7.3.2), we have that the stationary distribution of the system being empty immediately after a departure in the k^{th} MAP approximation is, for all $k \geq 2$, given by

$$\boldsymbol{\pi}_1 B_2 (\boldsymbol{\nu} D_1 \boldsymbol{e}_m)^{-1} = \boldsymbol{x}_0. \tag{8.3.1}$$

We must justify the use of $\boldsymbol{\nu} D_1 \boldsymbol{e}_m$ to normalise equation (8.3.1), by showing that $\boldsymbol{\Psi}(k) Q_1(k) \boldsymbol{e} = \boldsymbol{\nu} D_1 \boldsymbol{e}_m$.

$$\begin{aligned}
&\boldsymbol{\Psi}(k) Q_1(k) \boldsymbol{e} \\
= \;& \boldsymbol{\pi}_0 R_0 B_2 \boldsymbol{e}_m + \boldsymbol{\pi}_0 R_0 \sum_{i=1}^{k-2} R^k A_2 \boldsymbol{e}_{mn} \\
& + \boldsymbol{\pi}_0 R_0 R^{k-1} (I_{mn} - R)^{-1} (\boldsymbol{e}_m \otimes I_n) \left(E_2 \boldsymbol{e}_{mn} + E_3 \boldsymbol{e}_m \right) \\
= \;& \boldsymbol{\pi}_0 R_0 B_2 \boldsymbol{e}_m + \boldsymbol{\pi}_0 R_0 \sum_{i=1}^{k-2} R^k A_2 \boldsymbol{e}_{mn} \\
& + \boldsymbol{\pi}_0 R_0 R^{k-1} (I_{mn} - R)^{-1} (\boldsymbol{e}_m \otimes I_n) \left(\boldsymbol{S}^0 \boldsymbol{y}_{k-1} \boldsymbol{e}_{mn} + (1 - \boldsymbol{y}_{k-1} \boldsymbol{e}_{mn}) \boldsymbol{S}^0 \boldsymbol{\beta} \boldsymbol{e}_m \right) \\
= \;& \boldsymbol{\pi}_0 R_0 B_2 \boldsymbol{e}_m + \boldsymbol{\pi}_0 R_0 \sum_{i=1}^{k-2} R^k A_2 \boldsymbol{e}_{mn} \\
& + \boldsymbol{\pi}_0 R_0 R^{k-1} (I_{mn} - R)^{-1} (\boldsymbol{e}_m \otimes I_n) \boldsymbol{S}^0 \\
= \;& \boldsymbol{\pi}_0 R_0 B_2 \boldsymbol{e}_m + \boldsymbol{\pi}_0 R_0 \sum_{i=1}^{k-2} R^k A_2 \boldsymbol{e}_{mn} \\
& + \boldsymbol{\pi}_0 R_0 R^{k-1} (I_{mn} - R)^{-1} (\boldsymbol{e}_m \otimes \boldsymbol{S}^0 \boldsymbol{\beta}) \boldsymbol{e}_{mn}, \text{ by Lemma 8.1b} \\
= \;& \boldsymbol{\pi}_0 R_0 B_2 \boldsymbol{e}_m + \boldsymbol{\pi}_0 R_0 (I_{mn} - R)^{-1} A_2 \boldsymbol{e}_{mn} \\
= \;& \boldsymbol{\nu} D_1 \boldsymbol{e}_m, \text{ by Lemma 7.1.}
\end{aligned}$$

Hence for all MAP approximations with $k \geq 1$, we have that the stationary inter-event time interval, is with probability $\boldsymbol{x}_0 \boldsymbol{e}_m$, the convolution of the idle period and the PH-type service time and with probability $(1 - \boldsymbol{x}_0 \boldsymbol{e}_m)$, just the PH-type service time. ■

8.4 The mean and variance of the stationary inter-event times

The previous result of course implies that the mean and variance of the stationary inter-event time distribution of each *MAP* approximation are both identical to the mean and variance of the stationary inter-departure time distribution of the *MAP/PH/1* queue. However, we will give an alternative direct proof of the equivalence of mean and variance for all of our approximations as motivation for the proof of another result on lag-correlation coefficients.

Theorem 8.3 *The mean of the stationary inter-event times for the k^{th} MAP approximation to the departure process of the MAP/PH/1 queue is identical to the mean stationary inter-departure time of the MAP/PH/1 queue, for all $k \geq 1$.*

Proof:

The proof will proceed in terms of the stationary rates (which are the inverse of the mean inter-event times). The average arrival rate for the k^{th} *MAP* approximation is given by (see [33])

$$\boldsymbol{\nu}(k)Q_1(k)\boldsymbol{e},$$

so that the mean inter-event time is given by

$$\frac{1}{\boldsymbol{\nu}(k)Q_1(k)\boldsymbol{e}}. \tag{8.4.1}$$

For $k = 1$ the stationary arrival rate can be seen directly to be

$$\boldsymbol{\nu}_1 Q_1(1)\boldsymbol{e} = \boldsymbol{\pi}_1(I-R)^{-1}(\boldsymbol{e}_m \otimes I_n)\boldsymbol{S}^0.$$

For $k > 1$, $\boldsymbol{\nu}(k)Q_1(k)\boldsymbol{e}$ is given by

$$\begin{cases} \boldsymbol{\pi}_1(I_m \otimes \boldsymbol{S}^0)\boldsymbol{e}_m + \boldsymbol{\pi}_2(I-R)^{-1}(\boldsymbol{e}_m \otimes I_n)\boldsymbol{S}^0, & \text{if } k = 2, \\ \boldsymbol{\pi}_1(I_m \otimes \boldsymbol{S}^0)\boldsymbol{e}_m + \sum_{i=2}^{k-1} \boldsymbol{\pi}_i(I_m \otimes \boldsymbol{S}^0\boldsymbol{\beta})\boldsymbol{e}_{mn} + \boldsymbol{\pi}_k(I-R)^{-1}(\boldsymbol{e}_m \otimes I_n)\boldsymbol{S}^0 & \text{if } k \geq 3, \end{cases}$$

which using Lemma 8.1b and 8.1d, yields the same result as for $k = 1$.

Consider the *MAP/PH/1* queue represented by the *QBD* model given in (7.3.1), from which it can be seen that the stationary departure rate is given by

$$\boldsymbol{\pi}_1(I_m \otimes \boldsymbol{S}^0)\boldsymbol{e}_m + \sum_{i=2}^{\infty} \boldsymbol{\pi}_i(I_m \otimes \boldsymbol{S}^0\boldsymbol{\beta})\boldsymbol{e}_{mn},$$

which, again using Lemma 8.1b and 8.1d, can be re-written in exactly the same way as for the case $k = 1$. ■

To show equivalence of variance for the stationary inter-event times for all of our approximations, and also to show equivalence to the variance of the stationary inter-departure times of the *MAP/PH/1* queue, we need the following lemma on the form of $(-Q_0(k))^{-1}$, for $k \geq 2$.

Lemma 8.3 *For all $k \geq 2$, the matrix $-Q_0(k)^{-1}$ has the form*

$$\begin{pmatrix} X(0) & X_1 & X_2 & X_3 & \cdots & X_{k-2} & X_{k-1} & Z(k) \\ 0 & Y_0 & Y_1 & Y_2 & \cdots & Y_{k-3} & Y_{k-2} & Z_{k-1} \\ & 0 & Y_0 & Y_1 & \ddots & & Y_{k-3} & Z_{k-2} \\ & & 0 & Y_0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & 0 & \ddots & \ddots & Y_2 & Z_3 \\ & & & & \ddots & \ddots & Y_1 & Z_2 \\ & & & & & \ddots & Y_0 & Z_1 \\ 0 & & & \cdots & & & 0 & Z(0) \end{pmatrix}, \tag{8.4.2}$$

where

$$\begin{aligned} X(0) &= -D_0^{-1}, \\ X_i &= D_0^{-1}(D_1 \otimes \boldsymbol{\beta})\Big(-(D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^{i-1}(D_0 \oplus S)^{-1}, \ \textit{for } i \in \{1, 2, \ldots, k-1\}, \\ Y_i &= -\Big(-(D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^{i}(D_0 \oplus S)^{-1}, \ \textit{for } i \in \{0, 1, \ldots, k-2\}, \\ Z(0) &= -S^{-1}, \\ Z_i &= -\Big(-(D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^{i}(\boldsymbol{e}_m \otimes I_n)S^{-1} \ \textit{for } i \in \{1, 2, \ldots, k-1\} \textit{ and} \\ Z(k) &= D_0^{-1}(D_1 \otimes \boldsymbol{\beta})\Big(-(D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^{k-1}(\boldsymbol{e}_m \otimes I_n)S^{-1}. \end{aligned} \tag{8.4.3}$$

Proof:

We will show that $Q_0(k)(-Q_0(k))^{-1} = -I$, by considering each non-trivial term. Using (8.1.4), (8.4.2) and the definitions of the sub-matrices of $Q_0(k)$ given in (8.1.5), $Q_0(k)(-Q_0(k))^{-1}$ yields the following non-trivial terms.

$$B_1X(0) = D_0(-D_0)^{-1} = -I_m,$$

$$B_1X_1 + B_0Y_0 = D_0D_0^{-1}(D_1 \otimes \boldsymbol{\beta})(D_0 \oplus S)^{-1} - (D_1 \otimes \boldsymbol{\beta})(D_0 \oplus S)^{-1} = 0,$$

$$\begin{aligned} B_1X_i + B_0Y_{i-1} &= D_0D_0^{-1}(D_1 \otimes \boldsymbol{\beta})\Big(-(D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^{i-1}(D_0 \oplus S)^{-1} \\ &\quad -(D_1 \otimes \boldsymbol{\beta})\Big(-(D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^{i-1}(D_0 \oplus S)^{-1} \\ &= 0, \text{ for } 1 \leq i < k, \end{aligned}$$

$$\begin{aligned} B_1Z(k) + B_0Z_{k-1} &= D_0D_0^{-1}(D_1 \otimes \boldsymbol{\beta})\Big(-(D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^{k-1}(\boldsymbol{e}_m \otimes I_n)S^{-1} \\ &\quad -(D_1 \otimes \boldsymbol{\beta})\Big(-(D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^{k-1}(\boldsymbol{e}_m \otimes I_n)S^{-1} \\ &= 0, \end{aligned}$$

$$A_1Y_0 = (D_0 \oplus S)(-(D_0 \oplus S)^{-1}) = -I_{mn},$$

$$\begin{aligned} A_1Y_i + A_0Y_{i-1} &= -(D_0 \oplus S)\Big(-(D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^{i}(D_0 \oplus S)^{-1} \\ &\quad -(D_1 \otimes I_n)\Big(-(D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^{i-1}(D_0 \oplus S)^{-1} \\ &= 0, \text{ for } 1 \leq i < k-1, \end{aligned}$$

$$\begin{aligned} A_1Z_i + A_0Z_{i-1} &= (D_0 \oplus S)\Big(-(D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^{i}(\boldsymbol{e}_m \otimes I_n)S^{-1} \\ &\quad +(D_1 \otimes I_n)\Big(-(D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^{i-1}(\boldsymbol{e}_m \otimes I_n)S^{-1} \end{aligned}$$

$$= \quad 0, \text{ for } 2 \le i < k,$$

$$\begin{aligned} A_1 Z_1 + E_0 Z(0) &= (D_0 \oplus S)\Big(-(D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)(\boldsymbol{e}_m \otimes I_n)S^{-1} \\ &\quad +(D_1 \otimes I_n)(\boldsymbol{e}_m \otimes I_n)S^{-1} \\ &= 0 \text{ and} \end{aligned}$$

$$E_1 Z(0) = S(-S)^{-1} = -I_n.$$

■

Theorem 8.4 *The variance of the stationary inter-event times for the k^{th} MAP approximation to the departure process of the MAP/PH/1 queue is identical to the variance of the stationary inter-departure times for the MAP/PH/1 queue.*

Proof:

Written in the current context, page 396 of Neuts [33] implies that the variance of the stationary inter-arrival times of the approximation MAPs is given by

$$\sigma^2 = 2\frac{\boldsymbol{\nu}(k)(-Q_0(k))^{-1}\boldsymbol{e}}{\boldsymbol{\nu}(k)Q_1(k)\boldsymbol{e}} - \frac{1}{(\boldsymbol{\nu}(k)Q_1(k)\boldsymbol{e})^2}. \tag{8.4.4}$$

Therefore from (8.4.4) it can be seen that for the variances of the approximations to be identical it is necessary and sufficient to show that

$$\boldsymbol{\nu}(k)(-Q_0(k))^{-1}\boldsymbol{e}$$

is the same for all $k \ge 1$.

The equivalence of variance for all of the approximations will be shown by induction. First we show that $\boldsymbol{\nu}(1)(-Q_0(1))^{-1}\boldsymbol{e} = \boldsymbol{\nu}(2)(-Q_0(2))^{-1}\boldsymbol{e}$. In the case where $k = 1$,

$$Q_0(1) = \begin{pmatrix} D_0 & (D_1\boldsymbol{e}_m \otimes \boldsymbol{\beta}) \\ 0 & S \end{pmatrix}$$

and

$$(-Q_0(1))^{-1} = \begin{pmatrix} (-D_0)^{-1} & D_0^{-1}(D_1 \otimes \boldsymbol{\beta})(\boldsymbol{e}_m \otimes I_n)S^{-1} \\ 0 & (-S)^{-1} \end{pmatrix}.$$

For $k = 2$,

$$Q_0(2) = \begin{pmatrix} D_0 & (D_1 \otimes \boldsymbol{\beta}) & 0 \\ 0 & (D_0 \oplus S) & (D_1 \boldsymbol{e}_m \otimes I_n) \\ 0 & 0 & S \end{pmatrix}$$

and from (8.4.2), we have

$$(-Q_0(2))^{-1} = \begin{pmatrix} (-D_0)^{-1} & X_1 & Z(2) \\ 0 & -(D_0 \oplus S)^{-1} & Z_1 \\ 0 & 0 & (-S)^{-1} \end{pmatrix}.$$

Using these results, we may then write

$$\begin{aligned} \boldsymbol{\nu}(1)(-Q_0(1))^{-1}\boldsymbol{e} &= \boldsymbol{\pi}_0(-D_0)^{-1}\boldsymbol{e}_m + \boldsymbol{\pi}_0 D_0^{-1}(D_1 \otimes \boldsymbol{\beta})(\boldsymbol{e}_m \otimes I_n)S^{-1}\boldsymbol{e}_n \\ &\quad + \boldsymbol{\pi}_1(I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n, \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{\nu}(2)(-Q_0(2))^{-1}\boldsymbol{e} &= \boldsymbol{\pi}_0(-D_0)^{-1}\boldsymbol{e}_m + \boldsymbol{\pi}_0 X_1 \boldsymbol{e}_{mn} + \boldsymbol{\pi}_0 Z(2)\boldsymbol{e}_n \\ &\quad - \boldsymbol{\pi}_1(D_0 \oplus S)^{-1}\boldsymbol{e}_{mn} + \boldsymbol{\pi}_1 Z_1 \boldsymbol{e}_n \\ &\quad + \boldsymbol{\pi}_2(I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n. \end{aligned}$$

Hence using Lemma 8.1d, for equivalence of variance for $k = 1$ and $k = 2$ it is both necessary and sufficient to show that

$$\begin{aligned} &\boldsymbol{\pi}_0 D_0^{-1}(D_1 \otimes \boldsymbol{\beta})(\boldsymbol{e}_m \otimes I_n)S^{-1}\boldsymbol{e}_n + \boldsymbol{\pi}_1(\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n \\ &\qquad = \boldsymbol{\pi}_0 X_1 \boldsymbol{e}_{mn} + \boldsymbol{\pi}_0 Z(2)\boldsymbol{e}_n - \boldsymbol{\pi}_1(D_0 \oplus S)^{-1}\boldsymbol{e}_{mn} + \boldsymbol{\pi}_1 Z_1 \boldsymbol{e}_n. \quad (8.4.5) \end{aligned}$$

Using Lemma 8.1c, the term on the left hand side of (8.4.5) involving $\boldsymbol{\pi}_0$ becomes

$D_0^{-1}(D_1 \otimes \boldsymbol{\beta})(\boldsymbol{e}_m \otimes I_n)S^{-1}\boldsymbol{e}_n$

$$
\begin{aligned}
&= -D_0^{-1}(D_1 \otimes \boldsymbol{\beta})\Big((D_0 \oplus S)^{-1}(D_1 \otimes I_n)(\boldsymbol{e}_m \otimes I_n)S^{-1} - (D_0 \oplus S)^{-1}(\boldsymbol{e}_m \otimes I_n)\Big)\boldsymbol{e}_n \\
&= -D_0^{-1}(D_1 \otimes \boldsymbol{\beta})\Big((D_0 \oplus S)^{-1}(D_1 \otimes I_n)(\boldsymbol{e}_m \otimes I_n)S^{-1}\boldsymbol{e}_n - (D_0 \oplus S)^{-1}\boldsymbol{e}_{mn}\Big) \\
&= Z(2)\boldsymbol{e}_n + X_1\boldsymbol{e}_{mn}. \qquad (8.4.6)
\end{aligned}
$$

Using Lemma 8.1c again, consider the term on the left hand side of (8.4.5) involving $\boldsymbol{\pi}_1$, which yields

$$
\begin{aligned}
(\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n &= (D_0 \oplus S)^{-1}(D_1 \otimes I_n)(\boldsymbol{e}_m \otimes I_n)\boldsymbol{S}^0 - (D_0 \oplus S)^{-1}\boldsymbol{e}_{mn} \\
&= Z_1\boldsymbol{e}_n - (D_0 \oplus S)^{-1}\boldsymbol{e}_{mn}. \qquad (8.4.7)
\end{aligned}
$$

Hence equation (8.4.5) holds, and so the variance of the inter-event times is equal for approximations $k = 1$ and $k = 2$.

Now let us assume that the variances of the inter-event time distributions are equal up to an approximation $k = g \geq 2$. We show that the variance for approximation $k = g + 1$ is identical to the previous variances. From (8.4.2) it can be seen that for $g \geq 2$, we have

$$
-Q_0(g)^{-1} = \begin{pmatrix} & & & Z(g) \\ & \begin{pmatrix} N \end{pmatrix} & & Z_{g-1} \\ & & & \vdots \\ & & & Z_1 \\ 0 & \cdots & 0 & (-S)^{-1} \end{pmatrix} \qquad (8.4.8)
$$

and

$$
-Q_0(g+1)^{-1} = \begin{pmatrix} & & & X_g & Z(g+1) \\ & \begin{pmatrix} N \end{pmatrix} & & Y_{g-1} & Z_g \\ & & & \vdots & \vdots \\ & & & Y_1 & Z_2 \\ 0 & \cdots & 0 & -(D_0 \oplus S)^{-1} & Z_1 \\ 0 & \cdots & 0 & 0 & (-S)^{-1} \end{pmatrix}. \qquad (8.4.9)
$$

Note that it is not necessary here to give an explicit description for the square matrix N. Using the stationary distribution given in (8.2.5), and the matrix forms given in

(8.4.8) and (8.4.9), the equivalence of the variances requires

$$\begin{aligned}
&\boldsymbol{\pi}_0 Z(g)\boldsymbol{e}_n + \boldsymbol{\pi}_1 Z_{g-1}\boldsymbol{e}_n + \cdots + \boldsymbol{\pi}_{g-1} Z_1 \boldsymbol{e}_n + \boldsymbol{\pi}_g (I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n \\
&\quad = \boldsymbol{\pi}_0\Big(X_g \boldsymbol{e}_{mn} + Z(g+1)\boldsymbol{e}_n\Big) + \boldsymbol{\pi}_1\Big(Y_{g-1}\boldsymbol{e}_{mn} + Z_g \boldsymbol{e}_n\Big) + \cdots + \boldsymbol{\pi}_{g-1}\Big(Y_1 \boldsymbol{e}_{mn} + Z_2 \boldsymbol{e}_n\Big) \\
&\qquad + \boldsymbol{\pi}_g\Big(-(D_0 \oplus S)^{-1}\boldsymbol{e}_{mn} + Z_1 \boldsymbol{e}_n\Big) + \boldsymbol{\pi}_{g+1}(I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n,
\end{aligned}$$

which by Lemma 8.1d, reduces to

$$\begin{aligned}
&\boldsymbol{\pi}_0 Z(g)\boldsymbol{e}_n + \boldsymbol{\pi}_1 Z_{g-1}\boldsymbol{e}_n + \cdots + \boldsymbol{\pi}_{g-1} Z_1 \boldsymbol{e}_n + \boldsymbol{\pi}_g (\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n \\
&\quad = \boldsymbol{\pi}_0\Big(X_g \boldsymbol{e}_{mn} + Z(g+1)\boldsymbol{e}_n\Big) + \cdots + \boldsymbol{\pi}_{g-1}\Big(Y_1 \boldsymbol{e}_{mn} + Z_2 \boldsymbol{e}_n\Big) \\
&\qquad + \boldsymbol{\pi}_g\Big(-(D_0 \oplus S)^{-1}\boldsymbol{e}_{mn} + Z_1 \boldsymbol{e}_n\Big). \qquad (8.4.10)
\end{aligned}$$

Using Lemma 8.1c, the term involving $\boldsymbol{\pi}_g$ on the left hand side of (8.4.10) becomes

$$\begin{aligned}
&(\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n \\
&\quad = (D_0 \oplus S)^{-1}(D_1 \otimes I_n)(\boldsymbol{e}_m \otimes I_n)S^{-1}\boldsymbol{e}_n - (D_0 \oplus S)^{-1}(\boldsymbol{e}_m \otimes I_n)\boldsymbol{e}_n \\
&\quad = (D_0 \oplus S)^{-1}(D_1 \otimes I_n)(\boldsymbol{e}_m \otimes I_n)S^{-1}\boldsymbol{e}_n - (D_0 \oplus S)^{-1}\boldsymbol{e}_{mn} \\
&\quad = Z_1 \boldsymbol{e}_n - (D_0 \oplus S)^{-1}\boldsymbol{e}_{mn}. \qquad (8.4.11)
\end{aligned}$$

The term involving $\boldsymbol{\pi}_{g-1}$ on the left hand side of (8.4.10) now yields

$$Z_1 \boldsymbol{e}_n = (D_0 \oplus S)^{-1}(D_1 \otimes I_n)(\boldsymbol{e}_m \otimes I_n)S^{-1}\boldsymbol{e}_n,$$

which, by Lemma 8.1c, gives

$$\begin{aligned}
Z_1 \boldsymbol{e}_n &= (D_0 \oplus S)^{-1}(D_1 \otimes I_n)(D_0 \oplus S)^{-1}\boldsymbol{e}_{mn} \\
&\qquad - \Big((D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^2 (\boldsymbol{e}_m \otimes I_n)S^{-1}\boldsymbol{e}_n \\
&= Y_1 \boldsymbol{e}_{mn} + Z_2 \boldsymbol{e}_n.
\end{aligned}$$

It then follows similarly for the terms involving $\boldsymbol{\pi}_r$, with $r = 1, \ldots, g-2$, so that we have

$$Z_r \boldsymbol{e}_n = Y_r \boldsymbol{e}_{mn} + Z_{r+1}\boldsymbol{e}_n, \text{ for } r = 1, \ldots, g-1. \qquad (8.4.12)$$

Lastly, the term containing $\boldsymbol{\pi}_0$ on the left hand side of (8.4.10) yields

$$Z(g)\boldsymbol{e}_n = D_0^{-1}(D_1 \otimes \boldsymbol{\beta})\Big(- (D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^{g-1}(\boldsymbol{e}_m \otimes I_n)S^{-1}\boldsymbol{e}_n,$$

which by using Lemma 8.1c, gives

$$\begin{aligned} Z(g)\boldsymbol{e}_n &= D_0^{-1}(D_1 \otimes \boldsymbol{\beta})\Big(- (D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^{g}(\boldsymbol{e}_m \otimes I_n)S^{-1}\boldsymbol{e}_n \\ &\quad + D_0^{-1}(D_1 \otimes \boldsymbol{\beta})\Big(- (D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^{g-1}(D_0 \oplus S)^{-1}\boldsymbol{e}_{mn} \\ &= Z(g+1)\boldsymbol{e}_n + X_g\boldsymbol{e}_{mn}. \end{aligned} \tag{8.4.13}$$

By substituting (8.4.11), (8.4.12) and (8.4.13) into (8.4.10), it is seen that the g^{th} and the $g+1^{st}$ approximations have the same variance for their respective stationary inter-event times. Therefore by induction all of the approximations for $k \geq 1$ have the same variance.

8.5 The lag-correlation coefficients of the approximating MAPs

Now we turn to the study of something which is not covered by Theorem 8.2, that is, the correlation between successive inter-event times. In this section we use methods similar to those which were used in Section 8.4, to study the lag-correlation between successive inter-event times in our approximations.

We note here that all of our MAP approximations are defined by a conservative generator matrix $Q(k) = Q_0(k) + Q_1(k)$. The filtration matrix $Q_0(k)$ (the unobserved process) is a non-conservative generator matrix by definition, and $Q_1(k)$ (the observed process) is non-negative. Hence the $(i,j)^{th}$ entry of the $(a,b)^{th}$ block of the matrix $-Q_0(k)^{-1}$ is the expected sojourn in phase j of level b before an observed event from the k^{th} MAP approximation, given that the process starts in phase i of level a. Moreover, the matrix $-Q_0(k)^{-1}Q_1(k)$ is the probability transition matrix of the embedded Markov Chain at observed epochs of the k^{th} MAP approximation.

To assist with the proof of the next theorem concerning lag-correlation coefficients, we first establish the following lemma.

Lemma 8.4 *With*

$$\begin{aligned}
\boldsymbol{\sigma} &= (\boldsymbol{S}^0\boldsymbol{\beta})(-S)^{-1}\boldsymbol{e}_n, \\
\boldsymbol{\theta} &= (\boldsymbol{e}_m \otimes I_n)\boldsymbol{\sigma} \text{ and} \\
\boldsymbol{\epsilon} &= -(I_m \otimes (-\boldsymbol{S}^0))D_0^{-1}\boldsymbol{e}_m + \boldsymbol{\theta},
\end{aligned} \tag{8.5.1}$$

the following relationship holds:

$$-Q_1(k)Q_0(k)^{-1}\boldsymbol{e} = \begin{pmatrix} \boldsymbol{0} & \\ \boldsymbol{\epsilon} & \\ \left.\begin{matrix}\boldsymbol{\theta} \\ \vdots \\ \boldsymbol{\theta}\end{matrix}\right\} & \begin{matrix} k-2 \\ \text{repeats}\end{matrix} \\ \boldsymbol{\sigma} & \end{pmatrix}, \text{ for all } k \geq 2. \tag{8.5.2}$$

Proof:

It has been shown in the proof of Theorem 8.4 that for all $g \leq k$ with $k \in \{1, 2, \ldots\}$, the matrices $-Q_0(g)^{-1}$ and $-Q_0(k)^{-1}$ have the same row sum for level 0. We have also shown by symmetry, that the row sums are identical for all levels $1, \ldots, g-1$. Therefore we may use the row sum for level 0 from the matrix $-Q_0(1)^{-1}$ as the row sum for level 0 in $-Q_0(k)^{-1}$, for all $k \geq 1$. Similarly, we may use the row sum for level 1 of the matrix $Q_0(2)^{-1}$ as the row sum for levels $1, 2, \ldots, k-1$ in $-Q_0(k)^{-1}$ for all $k \geq 2$. Using this result and the fact from (8.4.7) that

$$-(D_0 \oplus S)^{-1}\boldsymbol{e}_{mn} + Z_1\boldsymbol{e}_n = (\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n,$$

we have for all $k \geq 2$ that

$$-Q_0(k)^{-1}\boldsymbol{e} = \begin{pmatrix} -D_0^{-1}\boldsymbol{e}_m + D_0^{-1}(D_1 \otimes \boldsymbol{\beta})(\boldsymbol{e}_m \otimes I_n)S^{-1}\boldsymbol{e}_n & \\ \left.\begin{matrix} (\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n \\ \vdots \\ (\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n \end{matrix}\right\} & \begin{matrix} k-1 \\ \text{repeats} \end{matrix} \\ (-S)^{-1}\boldsymbol{e}_n & \end{pmatrix}. \quad (8.5.3)$$

Then using (8.1.4) and (8.5.3), we see that

$$-Q_1(k)Q_0(k)^{-1}\boldsymbol{e} = \begin{pmatrix} \boldsymbol{0} & \\ \boldsymbol{\varepsilon} & \\ \left.\begin{matrix} \boldsymbol{\vartheta} \\ \vdots \\ \boldsymbol{\vartheta} \end{matrix}\right\} & \begin{matrix} k-2 \\ \text{repeats} \end{matrix} \\ \boldsymbol{\varsigma}(k) & \end{pmatrix} \text{ for all } k \geq 2,$$

where

$$\begin{aligned}
\boldsymbol{\varepsilon} &= B_2\Big(- D_0^{-1}\boldsymbol{e}_m + D_0^{-1}(D_1 \otimes \boldsymbol{\beta})(\boldsymbol{e}_m \otimes I_n)S^{-1}\boldsymbol{e}_n\Big) \\
&= B_2\Big(- D_0^{-1}\boldsymbol{e}_m + D_0^{-1}(D_1\boldsymbol{e}_m \otimes \boldsymbol{\beta})S^{-1}\boldsymbol{e}_n\Big) \\
&= B_2\Big(- D_0^{-1}\boldsymbol{e}_m + D_0^{-1}(-D_0\boldsymbol{e}_m \otimes \boldsymbol{\beta})S^{-1}\boldsymbol{e}_n\Big) \\
&= B_2\Big(- D_0^{-1}\boldsymbol{e}_m + (-\boldsymbol{e}_m \otimes \boldsymbol{\beta})S^{-1}\boldsymbol{e}_n\Big) \\
&= B_2\Big(- D_0^{-1}\boldsymbol{e}_m + \boldsymbol{e}_m\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n\Big) \\
&= -(I_m \otimes \boldsymbol{S}^0)D_0^{-1}\boldsymbol{e}_m + (I_m \otimes \boldsymbol{S}^0)\boldsymbol{e}_m\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n, \\
\boldsymbol{\vartheta} &= A_2(\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n \\
&= (I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))(\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n \text{ and} \\
\boldsymbol{\varsigma}(k) &= E_2(\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n + E_3(-S)^{-1}\boldsymbol{e}_n \\
&= \boldsymbol{S}^0\boldsymbol{y}_{k-1}(\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n + (1 - \boldsymbol{y}_{k-1}\boldsymbol{e}_{mn})(\boldsymbol{S}^0\boldsymbol{\beta})(-S)^{-1}\boldsymbol{e}_n.
\end{aligned}$$

Using the expression for $\boldsymbol{y}_{k-1}$ for all $k \geq 2$ given in (8.1.9), noting that $\boldsymbol{y}_{k-1}\boldsymbol{e}_{mn}$ is a scalar and that $\boldsymbol{\beta}\boldsymbol{e}_n = 1$, we may write

$$\begin{aligned}
\boldsymbol{S}^0\boldsymbol{y}_{k-1}(\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n &= \boldsymbol{S}^0\frac{\boldsymbol{\pi}_k(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))(\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n}{\boldsymbol{\pi}_k(I_{mn} - R)^{-1}(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}} \\
&= \boldsymbol{S}^0\frac{\boldsymbol{\pi}_k(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}\boldsymbol{e}_n\boldsymbol{\beta}))(\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n}{\boldsymbol{\pi}_k(I_{mn} - R)^{-1}(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}} \\
&= \boldsymbol{S}^0\frac{\boldsymbol{\pi}_k(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))(I_m \otimes \boldsymbol{e}_n\boldsymbol{\beta})(\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n}{\boldsymbol{\pi}_k(I_{mn} - R)^{-1}(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}} \\
&= \boldsymbol{S}^0\frac{\boldsymbol{\pi}_k(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))(\boldsymbol{e}_m \otimes \boldsymbol{e}_n\boldsymbol{\beta})(-S)^{-1}\boldsymbol{e}_n}{\boldsymbol{\pi}_k(I_{mn} - R)^{-1}(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))\boldsymbol{e}_{mn}} \\
&= \boldsymbol{S}^0\boldsymbol{y}_{k-1}(\boldsymbol{e}_m \otimes \boldsymbol{e}_n\boldsymbol{\beta})(-S)^{-1}\boldsymbol{e}_n \\
&= \boldsymbol{S}^0\boldsymbol{y}_{k-1}\boldsymbol{e}_{mn}\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n \\
&= \boldsymbol{y}_{k-1}\boldsymbol{e}_{mn}(\boldsymbol{S}^0\boldsymbol{\beta})(-S)^{-1}\boldsymbol{e}_n,
\end{aligned}$$

so that

$$\begin{aligned}
\varsigma(k) &= \boldsymbol{S}^0\boldsymbol{y}_{k-1}(\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n + (1 - \boldsymbol{y}_{k-1}\boldsymbol{e}_{mn})(\boldsymbol{S}^0\boldsymbol{\beta})(-S)^{-1}\boldsymbol{e}_n \\
&= \boldsymbol{y}_{k-1}\boldsymbol{e}_{mn}(\boldsymbol{S}^0\boldsymbol{\beta})(-S)^{-1}\boldsymbol{e}_n - \boldsymbol{y}_{k-1}\boldsymbol{e}_{mn}(\boldsymbol{S}^0\boldsymbol{\beta})(-S)^{-1}\boldsymbol{e}_n + (\boldsymbol{S}^0\boldsymbol{\beta})(-S)^{-1}\boldsymbol{e}_n \\
&= (\boldsymbol{S}^0\boldsymbol{\beta})(-S)^{-1}\boldsymbol{e}_n, \text{ for all } k \geq 2.
\end{aligned}$$

We may also re-write $\boldsymbol{\vartheta}$ to get

$$\begin{aligned}
\boldsymbol{\vartheta} &= (I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))(\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n \\
&= (\boldsymbol{e}_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))(-S)^{-1}\boldsymbol{e}_n \\
&= (\boldsymbol{e}_m \otimes I_n)(\boldsymbol{S}^0\boldsymbol{\beta})(-S)^{-1}\boldsymbol{e}_n.
\end{aligned}$$

Hence

$$\begin{aligned}
\varsigma(k) &= \boldsymbol{\sigma} = (\boldsymbol{S}^0\boldsymbol{\beta})(-S)^{-1}\boldsymbol{e}_n, \\
\boldsymbol{\vartheta} &= \boldsymbol{\theta} = (\boldsymbol{e}_m \otimes I_n)\boldsymbol{\sigma} \text{ and} \\
\boldsymbol{\varepsilon} &= \boldsymbol{\epsilon} = -(I_m \otimes \boldsymbol{S}^0)D_0^{-1}\boldsymbol{e}_m + \boldsymbol{\theta}.
\end{aligned}$$

■

Theorem 8.5 *For each $k \geq 2$ and for each $i \in \{1, 2, \ldots, k-1\}$, the lag-correlation coefficients $c_i(k+r)$ are identical for all $r \in \{0, 1, 2, \ldots\}$.*

Proof: We first note that for $k = 1$ we have a PH-renewal approximation to the inter-departure time distribution and, as a result, the stationary inter-event times are non-correlated.

Recall from (7.4.1) that the lag-correlation coefficients $c_i(k)$ of these approximation MAPs are given by

$$\frac{\frac{2}{\boldsymbol{\nu}(k)Q_1(k)\boldsymbol{e}}\boldsymbol{\nu}(k)[(-Q_0(k))^{-1}Q_1(k)]^i(-Q_0(k))^{-1}\boldsymbol{e} - \frac{1}{(\boldsymbol{\nu}(k)Q_1(k)\boldsymbol{e})^2}}{\frac{2}{\boldsymbol{\nu}(k)Q_1(k)\boldsymbol{e}}\boldsymbol{\nu}(k)(-Q_0(k))^{-1}\boldsymbol{e} - \frac{1}{(\boldsymbol{\nu}(k)Q_1(k)\boldsymbol{e})^2}}, \text{ for } i \geq 1, \tag{8.5.4}$$

where $\boldsymbol{\nu}(k)$ is the stationary distribution of the k^{th} MAP approximation given by Theorem 8.1. The denominator in (8.5.4) and the term $\frac{1}{\boldsymbol{\nu}(k)Q_1(k)\boldsymbol{e}}$ are the expressions for the variance and the mean of the stationary inter-event times respectively, given in equations (8.4.4) and (8.4.1). We have already shown in Theorem 8.2 that the stationary inter-event time distributions for all of our approximations are identical. Hence the proof can be reduced to proving equivalence for each $i = 1, \ldots, k-1$ and $r = 0, 1, 2, \ldots$, of

$$\boldsymbol{\nu}(k+r)[(-Q_0(k+r))^{-1}Q_1(k+r)]^i(-Q_0(k+r))^{-1}\boldsymbol{e}. \tag{8.5.5}$$

We first show that $c_1(k)$ for the stationary inter-event times is common to all MAP approximations for $k \geq 2$. The general case for $c_g(k)$ is then shown for all MAP approximations with $k > g \geq 2$.

Case $c_1(k)$: Using Lemmas 8.3 and 8.4, for $k = 2$ and $k = 3$ respectively, we have that

$$Q_0(2)^{-1}Q_1(2)Q_0(2)^{-1}\boldsymbol{e} = \begin{pmatrix} X_1\boldsymbol{\epsilon} + Z(2)\boldsymbol{\sigma} \\ Y_0\boldsymbol{\epsilon} + Z_1\boldsymbol{\sigma} \\ Z(0)\boldsymbol{\sigma} \end{pmatrix} \text{ and}$$

$$Q_0(3)^{-1}Q_1(3)Q_0(3)^{-1}\boldsymbol{e} \;=\; \begin{pmatrix} X_1\boldsymbol{\epsilon} + X_2\boldsymbol{\theta} + Z(3)\boldsymbol{\sigma} \\ Y_0\boldsymbol{\epsilon} + Y_1\boldsymbol{\theta} + Z_2\boldsymbol{\sigma} \\ Y_0\boldsymbol{\theta} + Z_1\boldsymbol{\sigma} \\ Z(0)\boldsymbol{\sigma} \end{pmatrix}.$$

From (8.5.5), equivalence of $c_1(k)$ for the $k=2$ and $k=3$ *MAP* approximations necessitates that

$$\begin{aligned}\boldsymbol{\pi}_0\Big(X_1\boldsymbol{\epsilon} + Z(2)\boldsymbol{\sigma}\Big) + \boldsymbol{\pi}_1\Big(Y_0\boldsymbol{\epsilon} + Z_1\boldsymbol{\sigma}\Big) + \boldsymbol{\pi}_2(I_{mn}-R)^{-1}(\boldsymbol{e}_m\otimes I_n)Z(0)\boldsymbol{\sigma}& \\ = \boldsymbol{\pi}_0\Big(X_1\boldsymbol{\epsilon} + X_2\boldsymbol{\theta} + Z(3)\boldsymbol{\sigma}\Big)& \\ +\boldsymbol{\pi}_1\Big(Y_0\boldsymbol{\epsilon} + Y_1\boldsymbol{\theta} + Z_2\boldsymbol{\sigma}\Big)& \\ +\boldsymbol{\pi}_2\Big(Y_0\boldsymbol{\theta} + Z_1\boldsymbol{\sigma}\Big)& \\ +\boldsymbol{\pi}_3(I_{mn}-R)^{-1}(\boldsymbol{e}_m\otimes I_n)Z(0)\boldsymbol{\sigma},&\end{aligned}$$

which reduces to

$$\begin{aligned}&\boldsymbol{\pi}_0 Z(2)\boldsymbol{\sigma} + \boldsymbol{\pi}_1 Z_1\boldsymbol{\sigma} + \boldsymbol{\pi}_2(\boldsymbol{e}_m\otimes I_n)Z(0)\boldsymbol{\sigma} \\ &\quad = \boldsymbol{\pi}_0\Big(X_2\boldsymbol{\theta} + Z(3)\boldsymbol{\sigma}\Big) + \boldsymbol{\pi}_1\Big(Y_1\boldsymbol{\theta} + Z_2\boldsymbol{\sigma}\Big) + \boldsymbol{\pi}_2\Big(Y_0\boldsymbol{\theta} + Z_1\boldsymbol{\sigma}\Big).\end{aligned} \tag{8.5.6}$$

We will consider seperately each of the terms on the left hand side of (8.5.6). That is, for the term involving $\boldsymbol{\pi}_0$, we have

$$\begin{aligned}Z(2)\boldsymbol{\sigma} &= -D_0^{-1}(D_1\otimes\boldsymbol{\beta})(D_0\oplus S)^{-1}(D_1\boldsymbol{e}_m\otimes I_n)S^{-1}\boldsymbol{\sigma} \\ &= D_0^{-1}(D_1\otimes\boldsymbol{\beta})(D_0\oplus S)^{-1}(D_1\otimes I_n)(\boldsymbol{e}_m\otimes I_n)(-S)^{-1}\boldsymbol{\sigma},\end{aligned}$$

which by Lemma 8.1c yields

$$\begin{aligned}Z(2)\boldsymbol{\sigma} &= D_0^{-1}(D_1\otimes\boldsymbol{\beta})\Big((D_0\oplus S)^{-1}(D_1\otimes I_n)\Big)^2(\boldsymbol{e}_m\otimes I_n)S^{-1}\boldsymbol{\sigma} \\ &\quad -D_0^{-1}(D_1\otimes\boldsymbol{\beta})(D_0\oplus S)^{-1}(D_1\otimes I_n)(D_0\oplus S)^{-1}(\boldsymbol{e}_m\otimes I_n)\boldsymbol{\sigma} \\ &= Z(3)\boldsymbol{\sigma} + X_2\boldsymbol{\theta}.\end{aligned}$$

Also, by Lemma 8.1c, for the term involving $\boldsymbol{\pi}_1$ we have that

$$\begin{aligned} Z_1\boldsymbol{\sigma} &= -(D_0 \oplus S)^{-1}(D_1 \otimes I_n)(\boldsymbol{e}_m \otimes I_m)(-S)^{-1}\boldsymbol{\sigma} \\ &= -\Big((D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^2(\boldsymbol{e}_m \otimes I_n)S^{-1}\boldsymbol{\sigma} \\ &\quad +(D_0 \oplus S)^{-1}(D_1 \otimes I_n)(D_0 \oplus S)^{-1}(\boldsymbol{e}_m \otimes I_n)\boldsymbol{\sigma} \\ &= Z_2\boldsymbol{\sigma} + Y_1\boldsymbol{\theta}. \end{aligned}$$

Again using Lemma 8.1c, for the term involving $\boldsymbol{\pi}_2$, we have

$$\begin{aligned} (\boldsymbol{e}_m \otimes I_n)Z(0)\boldsymbol{\sigma} &= (\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{\sigma} \\ &= (D_0 \oplus S)^{-1}(D_1 \otimes I_n)(\boldsymbol{e}_m \otimes I_n)S^{-1}\boldsymbol{\sigma} \\ &\quad -(D_0 \oplus S)^{-1}(\boldsymbol{e}_m \otimes I_n)\boldsymbol{\sigma} \\ &= Z_1\boldsymbol{\sigma} + Y_0\boldsymbol{\theta}. \end{aligned} \tag{8.5.7}$$

Substitution of these results into (8.5.6) shows the equivalence of $c_1(k)$ for the $k = 2$ and $k = 3$ *MAP* approximations.

Assuming $c_1(2) \equiv c_1(3) \equiv \ldots \equiv c_1(k)$ for some $k \geq 3$, we now show that this implies that $c_1(k) = c_1(k+1)$. For $k \geq 3$, we have that

$$Q_0(k)^{-1}Q_1(k)Q_0(k)^{-1}\boldsymbol{e} = \begin{pmatrix} X_1\boldsymbol{\epsilon} + \sum_{i=2}^{k-1} X_i\boldsymbol{\theta} + Z(k)\boldsymbol{\sigma} \\ Y_0\boldsymbol{\epsilon} + \sum_{i=1}^{k-2} Y_i\boldsymbol{\theta} + Z_{k-1}\boldsymbol{\sigma} \\ Y_0\boldsymbol{\theta} + \sum_{i=1}^{k-3} Y_i\boldsymbol{\theta} + Z_{k-2}\boldsymbol{\sigma} \\ \vdots \\ Y_0\boldsymbol{\theta} + \sum_{i=1}^{1} Y_i\boldsymbol{\theta} + Z_2\boldsymbol{\sigma} \\ Y_0\boldsymbol{\theta} + \sum_{i=1}^{0} Y_i\boldsymbol{\theta} + Z_1\boldsymbol{\sigma} \\ Z(0)\boldsymbol{\sigma} \end{pmatrix}, \tag{8.5.8}$$

and therefore from equation (8.5.5), equivalence of $c_1(k)$ and $c_1(k+1)$ requires

$$\begin{aligned} &\boldsymbol{\pi}_0\Big(X_1\boldsymbol{\epsilon} + \sum_{i=2}^{k-1} X_i\boldsymbol{\theta} + Z(k)\boldsymbol{\sigma}\Big) + \boldsymbol{\pi}_1\Big(Y_0\boldsymbol{\epsilon} + \sum_{i=1}^{k-2} Y_i\boldsymbol{\theta} + Z_{k-1}\boldsymbol{\sigma}\Big) \\ &\quad + \sum_{j=2}^{k-1}\boldsymbol{\pi}_j\Big(\sum_{i=0}^{k-j-1} Y_i\boldsymbol{\theta} + Z_{k-j}\boldsymbol{\sigma}\Big) + \boldsymbol{\pi}_k(I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n)Z(0)\boldsymbol{\sigma} \end{aligned}$$

$$= \boldsymbol{\pi}_0\Big(X_1\boldsymbol{\epsilon} + \sum_{i=2}^{k} X_i\boldsymbol{\theta} + Z(k+1)\boldsymbol{\sigma}\Big) + \boldsymbol{\pi}_1\Big(Y_0\boldsymbol{\epsilon} + \sum_{i=1}^{k-1} Y_i\boldsymbol{\theta} + Z_k\boldsymbol{\sigma}\Big)$$
$$+\sum_{j=2}^{k}\boldsymbol{\pi}_j\Big(\sum_{i=0}^{k-j} Y_i\boldsymbol{\theta} + Z_{k-j+1}\boldsymbol{\sigma}\Big) + \boldsymbol{\pi}_{k+1}(I_{mn}-R)^{-1}(\boldsymbol{e}_m\otimes I_n)Z(0)\boldsymbol{\sigma},$$

which reduces to

$$\boldsymbol{\pi}_0 Z(k)\boldsymbol{\sigma} + \sum_{j=1}^{k-1}\boldsymbol{\pi}_j Z_{k-j}\boldsymbol{\sigma} + \boldsymbol{\pi}_k(\boldsymbol{e}_m\otimes I_n)Z(0)\boldsymbol{\sigma}$$
$$= \boldsymbol{\pi}_0\Big(X_k\boldsymbol{\theta} + Z(k+1)\boldsymbol{\sigma}\Big) + \sum_{j=1}^{k-1}\boldsymbol{\pi}_j\Big(Y_{k-j}\boldsymbol{\theta} + Z_{k+1-j}\boldsymbol{\sigma}\Big) + \boldsymbol{\pi}_k\Big(Y_0\boldsymbol{\theta} + Z_1\boldsymbol{\sigma}\Big). \tag{8.5.9}$$

The equivalence of the terms involving $\boldsymbol{\pi}_k$ has already been shown in (8.5.7), and the equivalences for each term involving $\boldsymbol{\pi}_i$ for $i \in \{0, 1, 2, \ldots, k-1\}$, follow directly by the same arguments that were used in the previous case for $k = 2$ and $k = 3$. For completeness, we now give this argument for the term involving $\boldsymbol{\pi}_0$, and then for the general term $\boldsymbol{\pi}_j$ for $j \in \{1, 2, 3, \ldots, k-1\}$. Using Lemma 8.1c, we have

$$\begin{aligned}
Z(k)\boldsymbol{\sigma} &= D_0^{-1}(D_1\otimes\boldsymbol{\beta})\Big(-(D_0\oplus S)^{-1}(D_1\otimes I_n)\Big)^{k-1}(\boldsymbol{e}_m\otimes I_n)S^{-1}\boldsymbol{\sigma} \\
&= D_0^{-1}(D_1\otimes\boldsymbol{\beta})\Big(-(D_0\oplus S)^{-1}(D_1\otimes I_n)\Big)^{k-1}(D_0\oplus S)^{-1}\boldsymbol{\sigma} \\
&\quad + D_0^{-1}(D_1\otimes\boldsymbol{\beta})\Big(-(D_0\oplus S)^{-1}(D_1\otimes I_n)\Big)^{k}(\boldsymbol{e}_m\otimes I_n)S^{-1}\boldsymbol{\sigma} \\
&= X_k\boldsymbol{\theta} + Z(k+1)\boldsymbol{\sigma},
\end{aligned} \tag{8.5.10}$$

and for all $j \in \{1, 2, 3, \ldots, k-1\}$,

$$\begin{aligned}
Z_{k-j}\boldsymbol{\sigma} &= -\Big(-(D_0\oplus S)^{-1}(D_1\otimes I_n)\Big)^{k-j}(\boldsymbol{e}_m\otimes I_n)S^{-1}\boldsymbol{\sigma} \\
&= -\Big(-(D_0\oplus S)^{-1}(D_1\otimes I_n)\Big)^{k-j+1}(\boldsymbol{e}_m\otimes I_n)S^{-1}\boldsymbol{\sigma} \\
&\quad -\Big(-(D_0\oplus S)^{-1}(D_1\otimes I_n)\Big)^{k-j}(D_0\oplus S)^{-1}(\boldsymbol{e}_m\otimes I_n)\boldsymbol{\sigma} \\
&= Z_{k-j+1}\boldsymbol{\sigma} + Y_{k-j}\boldsymbol{\theta}.
\end{aligned} \tag{8.5.11}$$

Hence by induction all $c_1(k)$'s are identical for $k \geq 2$.

Case $c_g(k) : 2 \leq g < k$:

It remains to be shown that for each $k \geq 3$, that

$$c_g(k) = c_g(k+r), \text{ for all } g \in \{2,3,\ldots,k-1\} \text{ and } r \in \{0,1,2,\ldots\}.$$

We will first simplify the form of the column $Q_0(k)^{-1}Q_1(k)Q_0(k)^{-1}\boldsymbol{e}$, given in (8.5.8). From (8.5.10), we see that

$$\begin{aligned} Z(2)\boldsymbol{\sigma} &= X_2\boldsymbol{\theta} + Z(3)\boldsymbol{\sigma} \\ &= X_3\boldsymbol{\theta} + X_2\boldsymbol{\theta} + Z(4)\boldsymbol{\sigma} \\ &= \quad \vdots \\ &= \sum_{i=2}^{k-1} X_i\boldsymbol{\theta} + Z(k)\boldsymbol{\sigma}, \end{aligned}$$

so that

$$X_1\boldsymbol{\epsilon} + \sum_{i=2}^{k-1} X_i\boldsymbol{\theta} + Z(k)\boldsymbol{\sigma} = X_1\boldsymbol{\epsilon} + Z(2)\boldsymbol{\sigma}. \tag{8.5.12}$$

Similarly, using the result of (8.5.11), we can write

$$\begin{aligned} Z_1\boldsymbol{\sigma} &= Z_2\boldsymbol{\sigma} + Y_1\boldsymbol{\theta} \\ &= Z_3\boldsymbol{\sigma} + Y_2\boldsymbol{\theta} + Y_1\boldsymbol{\theta} \\ &= \quad \vdots \\ &= Z_{k-r+1}\boldsymbol{\sigma} + \sum_{i=1}^{k-r} Y_i\boldsymbol{\theta}. \end{aligned}$$

Therefore for $r = 2$, we have

$$Y_0\boldsymbol{\epsilon} + \sum_{i=1}^{k-2} Y_i\boldsymbol{\theta} + Z_{k-1}\boldsymbol{\sigma} = Y_0\boldsymbol{\epsilon} + Z_1\boldsymbol{\sigma}, \tag{8.5.13}$$

and for $r \in \{3,4,\ldots,k-1\}$, we have

$$Y_0\boldsymbol{\theta} + \sum_{i=1}^{k-r} Y_i\boldsymbol{\theta} + Z_{k-r+1}\boldsymbol{\sigma} = Y_0\boldsymbol{\theta} + Z_1\boldsymbol{\sigma}. \tag{8.5.14}$$

Note that $\boldsymbol{S}^0 = -S\boldsymbol{e}_n$ and that $(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n)$ is a scalar, so that using (8.5.7) in (8.5.14), we see that for all $r \in \{3,4,\ldots,k\}$,

$$Y_0\boldsymbol{\theta} + \sum_{i=1}^{k-r} Y_i\boldsymbol{\theta} + Z_{k-r+1}\boldsymbol{\sigma} = Y_0\boldsymbol{\theta} + Z_1\boldsymbol{\sigma}$$

$$\begin{aligned}
&= (\boldsymbol{e}_m \otimes I_n) Z(0) \boldsymbol{\sigma} \\
&= (\boldsymbol{e}_m \otimes I_n)(-S)^{-1}(\boldsymbol{S}^0 \boldsymbol{\beta})(-S)^{-1} \boldsymbol{e}_n \\
&= (\boldsymbol{e}_m \otimes I_n)(-S)^{-1}(-S)\boldsymbol{e}_n \Big(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n\Big) \\
&= (\boldsymbol{e}_m \otimes I_n)\boldsymbol{e}_n \Big(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n\Big) \\
&= \Big(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n\Big)\boldsymbol{e}_{mn}. \qquad (8.5.15)
\end{aligned}$$

Substitution of (8.5.12), (8.5.13) and (8.5.15) into equation (8.5.8) yields

$$Q_0(k)^{-1}Q_1(k)Q_0(k)^{-1}\boldsymbol{e} = \begin{pmatrix} X_1\boldsymbol{\epsilon} + Z(2)\boldsymbol{\sigma} \\ Y_0\boldsymbol{\epsilon} + Z_1\boldsymbol{\sigma} \\ \left.\begin{matrix} \Big(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n\Big)\boldsymbol{e}_{mn} \\ \vdots \\ \Big(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n\Big)\boldsymbol{e}_{mn} \end{matrix}\right\} \ k-2 \text{ repeats} \\ \Big(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n\Big)\boldsymbol{e}_n \end{pmatrix}, \qquad (8.5.16)$$

where we emphasise the fact that $\Big(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n\Big)$ is a scalar.

Now using (8.4.2) and the form of $Q_1(k)$ given in (8.1.4), we may write $-Q_0(k)^{-1}Q_1(k)$ as

$$\begin{pmatrix}
X_1B_2 & X_2A_2 & X_3A_2 & \cdots & X_{k-1}A_2 & Z(k)E_2 & Z(k)E_3 \\
Y_0B_2 & Y_1A_2 & Y_2A_2 & \cdots & Y_{k-2}A_2 & Z_{k-1}E_2 & Z_{k-1}E_3 \\
0 & Y_0A_2 & Y_1A_2 & & Y_{k-3}A_2 & Z_{k-2}E_2 & Z_{k-2}E_3 \\
 & 0 & Y_0A_2 & \ddots & & \vdots & \vdots \\
\vdots & & \ddots & \ddots & Y_1A_2 & Z_2E_2 & Z_2E_3 \\
 & & & \ddots & Y_0A_2 & Z_1E_2 & Z_1E_3 \\
0 & & \cdots & & 0 & Z(0)E_2 & Z(0)E_3
\end{pmatrix}, \qquad (8.5.17)$$

where we recall from (8.1.5) that

$$\begin{aligned}
B_2 &= (I_m \otimes \boldsymbol{S}^0), \\
A_2 &= (I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta})),
\end{aligned}$$

$$\begin{aligned} E_2 &= \boldsymbol{S}^0 \boldsymbol{y}_{k-1} \text{ and} \\ E_3 &= (1 - \boldsymbol{y}_{k-1}\boldsymbol{e}_{mn})\boldsymbol{S}^0\boldsymbol{\beta}. \end{aligned}$$

Important here is that the matrix $(-Q_0(k))^{-1}Q_1(k)$ is stochastic, since it is the probability transition matrix of the embedded Markov chain immediately after observed epochs of the k^{th} *MAP* approximation. Therefore, using (8.5.17) and (8.5.16) we may write $\Big((-Q_0(k))^{-1}Q_1(k)\Big)^2(-Q_0(k))^{-1}\boldsymbol{e}$ as

$$\begin{pmatrix} \mathbb{C}_{2,0} & \\ \mathbb{C}_{2,1} & \\ \mathbb{C}_{2,2} & \\ (\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n)\boldsymbol{e}_{mn} & \multirow{3}{*}{$\Bigg\}\; k-3 \text{ repeats}$} \\ \vdots & \\ (\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n)\boldsymbol{e}_{mn} & \\ (\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n)\boldsymbol{e}_n & \end{pmatrix}. \tag{8.5.18}$$

Here using the fact that all row sums of the matrix in (8.5.17) are identically 1, we have for all $k \geq 3$, that

$$\begin{aligned} \mathbb{C}_{2,0} &= X_1B_2\Big(X_1\boldsymbol{\epsilon} + Z(2)\boldsymbol{\sigma}\Big) + X_2A_2\Big(Y_0\boldsymbol{\epsilon} + Z_1\boldsymbol{\sigma}\Big) \\ &\quad +(\boldsymbol{e}_m - X_1B_2\boldsymbol{e}_m - X_2A_2\boldsymbol{e}_{mn})(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n), \\ \mathbb{C}_{2,1} &= Y_0B_2\Big(X_1\boldsymbol{\epsilon} + Z(2)\boldsymbol{\sigma}\Big) + Y_1A_2\Big(Y_0\boldsymbol{\epsilon} + Z_1\boldsymbol{\sigma}\Big) \\ &\quad +(\boldsymbol{e}_{mn} - Y_0B_2\boldsymbol{e}_m - Y_1A_2\boldsymbol{e}_{mn})(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n) \text{ and} \\ \mathbb{C}_{2,2} &= Y_0A_2\Big(Y_0\boldsymbol{\epsilon} + Z_1\boldsymbol{\sigma}\Big) + (\boldsymbol{e}_{mn} - Y_0A_2\boldsymbol{e}_{mn})(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n). \end{aligned}$$

Similarly for all $k \geq 3$ and $g < k$, we may write $\Big((-Q_0(k))^{-1}Q_1(k)\Big)^g(-Q_0(k))^{-1}\boldsymbol{e}$

as

$$\begin{pmatrix} \mathbb{C}_{g,0} \\ \mathbb{C}_{g,1} \\ \vdots \\ \mathbb{C}_{g,g} \\ \left.\begin{matrix} (\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n)\boldsymbol{e}_{mn} \\ \vdots \\ (\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n)\boldsymbol{e}_{mn} \end{matrix}\right\} k-g-1 \text{ repeats} \\ (\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n)\boldsymbol{e}_n \end{pmatrix}, \tag{8.5.19}$$

where the $\mathbb{C}_{g,i}$ are common to all *MAP* approximations with $k > g$. Therefore to show $c_g(k) \equiv c_g(k+1)$, for all $k \geq 2$ and for all $g \in \{2, 3, \ldots, k-1\}$, we see from (8.5.5) that it is only necessary to prove that

$$\begin{aligned} &\boldsymbol{\pi}_k(I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n)\Big(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n\Big)\boldsymbol{e}_n \\ &\qquad = \boldsymbol{\pi}_k\Big(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n\Big)\boldsymbol{e}_{mn} + \boldsymbol{\pi}_{k+1}(I_{mn} - R)^{-1}(\boldsymbol{e}_m \otimes I_n)\Big(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n\Big)\boldsymbol{e}_n, \end{aligned}$$

or, more simply, that

$$\boldsymbol{\pi}_k(\boldsymbol{e}_m \otimes I_n)\Big(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n\Big)\boldsymbol{e}_n = \boldsymbol{\pi}_k\Big(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n\Big)\boldsymbol{e}_{mn},$$

which is trivially true, as $\Big(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n\Big)$ is just a scalar.

Hence by induction $c_i(k) \equiv c_i(k+r)$ for all $k \geq 2$ and $0 < i < k$, with $r \in \{0, 1, 2, \ldots\}$.

■

8.6 *MAP/PH/1* departure process lag-correlations

By considering the filtration of the *MAP/PH/1* queue given by the matrices Q_0 and Q_1 in (8.1.1), we will establish the equation for the lag-correlation coefficients of the stationary inter-departure times. The matrix $Q = Q_0 + Q_1$ has finite, negative diagonal entries and finite, non-negative off diagonal entries. The row sums are all equal to zero, so that Q is a stable and in fact uniform conservative generator matrix. The matrix Q_1 is also strictly non-negative, with some positive entries at every level greater than zero (as departures are possible from every occupied level). Using the same methods that were used to establish the form of the matrix $-Q_0(k)^{-1}$ given in Lemma 8.3, it can readily be seen that the matrix $-Q_0^{-1}$ is given by

$$-Q_0^{-1} = \begin{pmatrix} X(0) & X_1 & X_2 & X_3 & \cdots \\ 0 & Y_0 & Y_1 & Y_2 & \cdots \\ & 0 & Y_0 & Y_1 & \\ \vdots & & 0 & Y_0 & \ddots \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}, \tag{8.6.1}$$

where

$$\begin{aligned} X(0) &= -D_0^{-1}, \\ X_i &= D_0^{-1}(D_1 \otimes \boldsymbol{\beta})\Big(-(D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^{i-1}(D_0 \oplus S)^{-1} \text{ and} \\ Y_i &= -\Big(-(D_0 \oplus S)^{-1}(D_1 \otimes I_n)\Big)^{i}(D_0 \oplus S)^{-1}. \end{aligned}$$

All of the sub-matrices in equation (8.6.1) are finite as the matrix

$$-(D_0 \oplus S)^{-1}(D_1 \otimes I_n)$$

is sub-stochastic. This can be shown by considering the filtration of the conservative rate matrix Q^*, given by

$$Q_0^* = (D_0 \oplus S) \text{ and } Q_1^* = (D_1 \oplus \boldsymbol{S}^0\boldsymbol{\beta}),$$

where Q_1^* is the observed process. Note that $(-Q_0^*)^{-1}Q_1^*$ is the probability transition matrix of the embedded Markov chain immediately after observed transition epochs for the process given by Q^*. In this case,

$$\begin{aligned} (-Q_0^*)^{-1}Q_1^* &= (-(D_0 \oplus S))^{-1}(D_1 \oplus (\boldsymbol{S}^0\boldsymbol{\beta})) \\ &= (-(D_0 \oplus S))^{-1}\Big((D_1 \otimes I_n) + (I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))\Big), \end{aligned}$$

which is stochastic. Therefore

$$(-(D_0 \oplus S))^{-1}(D_1 \otimes I_n)$$

is sub-stochastic, since $-(D_0 \oplus S)^{-1}$, $(D_1 \otimes I_n)$ and $(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))$ are non-negative and non-zero. Hence, $(-(D_0 \oplus S)^{-1}(D_1 \otimes I_n))^i$ exists for all $i \geq 1$, and in fact

$$\lim_{i \longrightarrow \infty} (-(D_0 \oplus S)^{-1}(D_1 \otimes I_n))^i \longrightarrow 0. \tag{8.6.2}$$

Given that $\boldsymbol{\Psi}$ is the stationary distribution of the *MAP/PH/1* queue, the distribution of states in the queue immediately after an arbitrary departure under the stationary regime is given by

$$\boldsymbol{\Psi}_{dep} = (\boldsymbol{\Psi} Q_1 \boldsymbol{e})^{-1} \boldsymbol{\Psi} Q_1 = \frac{1}{\lambda} \boldsymbol{\Psi} Q_1. \tag{8.6.3}$$

Here $\lambda = \boldsymbol{\Psi} Q_1 \boldsymbol{e}$ (see page 320 of Walrand [50]) is the average departure rate from the queue.

The distribution of time between two consecutive departures under the stationary regime is therefore given by

$$F(t) = \boldsymbol{\Psi}_{dep} \Big(\int_{x_1=0}^{t_1} e^{Q_0 x} Q_1 dx_1 \Big) \boldsymbol{e}, \tag{8.6.4}$$

where the exponential is defined by (see Cohen [14] or Walrand [50])

$$e^{Q_0 x} = \sum_{i=0}^{\infty} \frac{x^i}{i!} Q_0^i.$$

Hence the joint distribution of n consecutive inter-departure times can be written as

$$F(t_1, \dots t_n) = \boldsymbol{\Psi}_{dep} \Big(\int_{x_1=0}^{t_1} e^{Q_0 x_1} Q_1 dx_1 \dots \int_{x_n=0}^{t_n} e^{Q_0 x_n} Q_1 dx_n \Big) \boldsymbol{e}. \tag{8.6.5}$$

Taking the Laplace-Stieltjes transform of this, we get

$$\begin{aligned} \phi(s_1, \dots, s_n) &= \int_{t_1=0}^{\infty} \dots \int_{t_n=0}^{\infty} e^{-s_1 t_1} \dots e^{-s_n t_n} dF(t_1, \dots, t_n) \\ &= \boldsymbol{\Psi}_{dep} \Big(\int_{t_1=0}^{\infty} e^{-s_1 t_1} e^{Q_0 t_1} Q_1 dt_1 \dots \int_{t_n=0}^{\infty} e^{-s_n t_n} e^{Q_0 t_n} Q_1 dt_n \Big) \boldsymbol{e} \\ &= \boldsymbol{\Psi}_{dep} \Big((s_1 I - Q_0)^{-1} Q_1 \dots (s_n I - Q_0)^{-1} Q_1 \Big) \boldsymbol{e}, \\ &\qquad \text{for } s_1, \dots, s_n \geq 0. \end{aligned} \tag{8.6.6}$$

All of the above integrals are well defined for $s_1, \ldots, s_n > 0$, as can be seen from the definition of matrix resolvents in Chapter 1.3 of Anderson [4]. We also note, since our matrix Q_0 is strictly non-conservative, that the matrix resolvent

$$\Big((s_1 I - Q_0)^{-1} Q_1 \ldots (s_n I - Q_0)^{-1} Q_1\Big)$$

is in fact defined for $s_i = 0$, for all $i \in \{1, 2, \ldots n\}$.

Let X_1 and X_n be the random variables representing the first and n^{th} inter-departure time respectively, immediately after an arbitrary departure under the stationary regime. The n^{th} lag-correlation coefficient, c_n, is then given by

$$c_n = \frac{Cov(X_1, X_n)}{\Big(Var(X_1)Var(X_n)\Big)^{\frac{1}{2}}}. \tag{8.6.7}$$

The derivatives of the matrix resolvent given in (8.6.6) exist and are defined in the proof of Theorem 3.3 of Anderson [4], so that by differentiating the Laplace-Stieltjes transform (8.6.6), for $n = 1$, we can get an expression for the mean and variance of the stationary inter-departure times. That is, the mean inter-departure time is given by

$$\begin{aligned}
E[X] &= -\frac{\partial \phi(s_1)}{\partial s_1}\Big|_{s_1=0} \\
&= -\boldsymbol{\Psi}_{dep}(s_1 I - Q_0)^{-2}\Big|_{s_1=0} Q_1 \boldsymbol{e} \\
&= -\frac{\boldsymbol{\Psi} Q_1}{\lambda}(-Q_0)^{-2} Q_1 \boldsymbol{e} \\
&= -\frac{\boldsymbol{\Psi} Q_0}{\lambda}(-Q_0)^{-2} Q_0 \boldsymbol{e} \\
&= -\frac{\boldsymbol{\Psi}}{\lambda}\boldsymbol{e} \\
&= \frac{1}{\lambda},
\end{aligned}$$

since $\boldsymbol{\Psi} Q_1 = -\boldsymbol{\Psi} Q_0$ and $Q_1 \boldsymbol{e} = -Q_0 \boldsymbol{e}$. The variance σ^2, is given by

$$\begin{aligned}
\sigma^2 &= E[X^2] - (E[X])^2 \\
&= \frac{\partial^2 \phi(s_1)}{\partial s_1^2}\Big|_{s_1=0} - \Big(\frac{1}{\lambda}\Big)^2
\end{aligned}$$

$$
\begin{aligned}
&= \boldsymbol{\Psi}_{dep} 2(s_1 I - Q_0)^{-3}\Big|_{s_1=0} Q_1 \boldsymbol{e} - \left(\frac{1}{\lambda}\right)^2 \\
&= \frac{2\boldsymbol{\Psi} Q_1}{\lambda}(-Q_0)^{-3} Q_1 \boldsymbol{e} - \left(\frac{1}{\lambda}\right)^2 \\
&= \frac{2\boldsymbol{\Psi} Q_0}{\lambda}(-Q_0)^{-3} Q_0 \boldsymbol{e} - \left(\frac{1}{\lambda}\right)^2 \\
&= \frac{2\boldsymbol{\Psi}}{\lambda}(-Q_0)^{-1} \boldsymbol{e} - \left(\frac{1}{\lambda}\right)^2 .
\end{aligned}
$$

We can also get an expression for the covariance between the first and n^{th} inter-departure time, after an arbitrary departure in the stationary version. This is given for $i = 1, \ldots, n$, by

$$Cov(X_1, X_n) = E[X_1 X_n] - E[X_1]E[X_n].$$

The term $E[X_1]E[X_n]$ is trivially given by $\left(\frac{1}{\lambda}\right)^2$, and the joint expectation $E[X_1 X_n]$ is given by

$$
\begin{aligned}
E[X_1 X_n] &= \frac{\partial^2 \phi(s_1, \ldots s_n)}{\partial s_1 \, \partial s_n}\Big|_{s_i=0} \\
&= \boldsymbol{\Psi}_{dep}(s_1 I - Q_0)^{-2} Q_1 (s_2 I - Q_0)^{-1} Q_1 \ldots (s_{n-1} I - Q_0)^{-1} Q_1 (s_n I - Q_0)^{-2} Q_1 \boldsymbol{e}\Big|_{s_i=0} \\
&= \boldsymbol{\Psi}_{dep}(-Q_0)^{-2} Q_1 \Big((-Q_0)^{-1} Q_1\Big)^{n-2} (-Q_0)^{-2} Q_1 \boldsymbol{e} \\
&= \frac{\boldsymbol{\Psi}}{\lambda}(-Q_0)^{-1} Q_1 \Big((-Q_0)^{-1} Q_1\Big)^{n-2} (-Q_0)^{-1} \boldsymbol{e} \\
&= \frac{\boldsymbol{\Psi}}{\lambda}\Big((-Q_0)^{-1} Q_1\Big)^{n-1} (-Q_0)^{-1} \boldsymbol{e}. \qquad (8.6.8)
\end{aligned}
$$

Equation (8.6.7) can therefore be written as

$$c_n = \frac{\frac{\boldsymbol{\Psi}}{\lambda}\Big((-Q_0)^{-1} Q_1\Big)^{n-1} (-Q_0)^{-1} \boldsymbol{e} - \left(\frac{1}{\lambda}\right)^2}{\frac{2\boldsymbol{\Psi}}{\lambda}(-Q_0)^{-1} \boldsymbol{e} - \left(\frac{1}{\lambda}\right)^2},$$

or more simply as

$$c_n = \frac{\frac{\boldsymbol{\Psi}}{\lambda}\Big((-Q_0)^{-1} Q_1\Big)^{n-1} (-Q_0)^{-1} \boldsymbol{e} - \left(\frac{1}{\lambda}\right)^2}{\sigma^2}. \qquad (8.6.9)$$

For an insight into the physical meaning of the terms in (8.6.8), we will construct the same expression using a probabilistic argument involving the embedded Markov chain immediately after departure epochs. Under the stationary regime, the inter-departure times have the same expected mean and variance. However, we must consider the the correlation which exists between X_1 and X_n. The expectation $E[X_1X_n]$ in (8.6.8), clearly involves two dependent random variables. However, these random variables are conditionally independent given the state j_1 at the end of X_1 and the state i_{n-1} at the beginning of X_n. Thus we can write

$$\begin{aligned} E[X_1X_n] &= \sum_{j_1,i_{n-1}} E[X_1X_n|j_1,i_{n-1}]P(j_1,i_{n-1}) \\ &= \sum_{j_1,i_{n-1}} E[X_1|j_1]P(j_1,i_{n-1})E[X_n|i_{n-1}] \\ &= \sum_{j_1,i_{n-1}} E[X_1|j_1]P(j_1)P(i_{n-1}|j_1)E[X_n|i_{n-1}]. \qquad (8.6.10) \end{aligned}$$

This is illustrated in the following diagram.

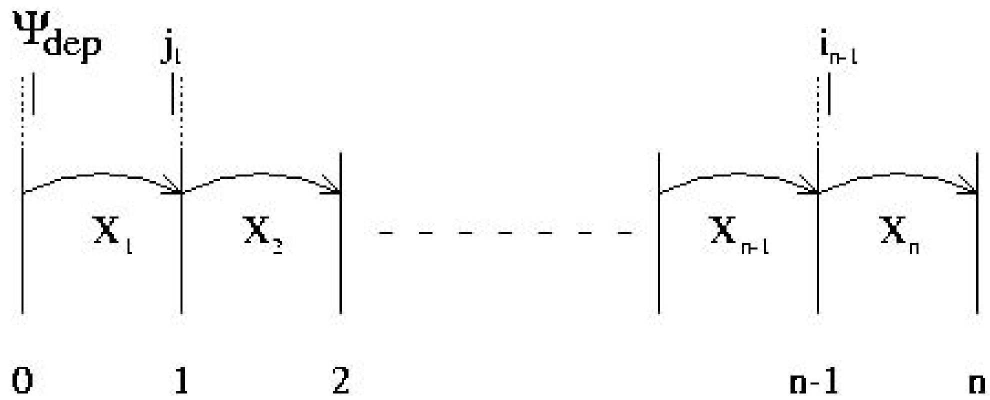

Figure 8.6.1: Stationary inter-departure intervals.

Recall that in our filtration of the *MAP/PH/1* queue, the matrix Q_0 governs transitions which are un-observed and the matrix Q_1 governs those transitions which are observed (the departure process). We will construct individual expressions for each of the terms $P(i_{n-1}|j_1)$, $E[X_n|i_{n-1}]$, $E[X_1|j_1]$ and $P(j_1)$.

The matrix $-Q_0^{-1}Q_1$ is the probability transition matrix of the embedded Markov chain immediately after departure epochs of the *MAP/PH/1* queue. Given that the process is in state j_1 immediately before the observed transition which ends X_1, the probability that it is in state i_1 immediately after this transition is given by

$$\xi_{j_1,i_1} = \begin{cases} \dfrac{[Q_1]_{j_1,i_1}}{\sum\limits_{\ell}[Q_1]_{j_1,\ell}} & \text{if } \sum\limits_{\ell}[Q_1]_{j_1,\ell} > 0, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Let Ξ be the matrix of terms $\xi_{j,i}$. This then allows us to use the probability transition matrix embedded immediately after departures to find $P(i_{n-1}|j_1)$. Hence if the process is in state j_1 immediately before the first arrival transition, then the probability of being in state i_{n-1} immediately after the $n-1^{st}$ arrival transition is given by

$$P(i_{n-1}|j_1) = \left[\Xi\Big(-Q_0^{-1}Q_1\Big)^{n-2}\right]_{(j_1,i_{n-1})}. \tag{8.6.11}$$

Given that the process is in state i_{n-1} immediately after the $n-1^{st}$ transition, the expected sojourn time before the n^{th} observed transition is given by

$$E[X_n|i_{n-1}] = \left[-Q_0^{-1}\boldsymbol{e}\right]_{i_{n-1}}. \tag{8.6.12}$$

In order to evaluate $P(j_1)$ and $E[X_1|j_1]$, since j_1 is the state immediately before the transition at the end of X_1, we will need to use the time reverse process to give us this probability and expected sojourn time. If we let $\Phi = diag(\boldsymbol{\Psi})$, then the time reverse matrix corresponding to Q_0 is given by

$$Q_0^R = \Phi^{-1}Q_0^T\Phi, \tag{8.6.13}$$

where the superscript T indicates the transpose. The j_1^{th} term of the row vector

$$\left[\left(-Q_0^R\right)^{-1}\boldsymbol{e}\right]^T$$

contains the expected sojourn time before an observed transition in the reverse time process, given that it starts in state j_1. Thus

$$E[X_1|j_1] = \left[\left(-Q_0^R\right)^{-1}\boldsymbol{e}\right]^T_{j_1}$$

$$
\begin{aligned}
&= \left[\left(-\Phi^{-1} Q_0^T \Phi \right)^{-1} \boldsymbol{e} \right]_{j_1}^T \\
&= \left[-\Phi^{-1} \left(Q_0^{-1} \right)^T \Phi \boldsymbol{e} \right]_{j_1}^T \\
&= \left[-\Phi^{-1} \left(Q_0^{-1} \right)^T \boldsymbol{\Psi}^T \right]_{j_1}^T \\
&= \left[\boldsymbol{\Psi} (-Q_0)^{-1} \left(\Phi^{-1} \right) \right]_{j_1}.
\end{aligned}
$$

This expected sojourn time corresponds to the first inter-arrival time X_1, given that the process ends in j_1. Similarly $P(j_1)$ is given by the j_1^{th} term of the distribution embedded immediately after departures in the reverse time process, that is, $\left[\boldsymbol{\Psi}_{dep}^R\right]_{j_1}$. This distribution is given by

$$
\boldsymbol{\Psi}_{dep}^R = \frac{\boldsymbol{\Psi} Q_1^R}{\boldsymbol{\Psi} Q_1^R \boldsymbol{e}} = \frac{\boldsymbol{\Psi} \left(\Phi^{-1} Q_1^T \Phi \right)}{\boldsymbol{\Psi} \left(\Phi^{-1} Q_1^T \Phi \right) \boldsymbol{e}} = \frac{\boldsymbol{e}^T Q_1^T \Phi}{\boldsymbol{e}^T Q_1^T \boldsymbol{\Psi}^T} = \frac{\left(\Phi Q_1 \boldsymbol{e} \right)^T}{\boldsymbol{\Psi} Q_1 \boldsymbol{e}},
$$

hence

$$
P(j_1) = \frac{[\Phi Q_1 \boldsymbol{e}]_{j_1}}{\boldsymbol{\Psi} Q_1 \boldsymbol{e}}. \tag{8.6.14}
$$

Therefore, we have

$$
P(j_1) P(i_{n-1}|j_1) = \frac{[\Phi Q_1 \boldsymbol{e}]_{j_1}}{\boldsymbol{\Psi} Q_1 \boldsymbol{e}} \left[\Xi \Big(-Q_0^{-1} Q_1 \Big)^{n-2} \right]_{(j_1, i_{n-1})},
$$

which by the definition of Ξ gives us

$$
P(j_1) P(i_{n-1}|j_1) = \frac{1}{\boldsymbol{\Psi} Q_1 \boldsymbol{e}} \left[\Phi Q_1 \Big(-Q_0^{-1} Q_1 \Big)^{n-2} \right]_{(j_1, i_{n-1})}.
$$

Hence we may write the joint expectation $E[X_1 X_n]$ as

$$
\sum_{j_1, i_{n-1}} E[X_1|j_1] P(j_1) P(i_{n-1}|j_1) E[X_{n-1}|i_{n-1}]
$$

$$
\begin{aligned}
&= \sum_{j_1, i_{n-1}} \left[\boldsymbol{\Psi} (-Q_0)^{-1} \left(\Phi^{-1} \right) \right]_{j_1} \frac{1}{\boldsymbol{\Psi} Q_1 \boldsymbol{e}} \left[\Phi Q_1 \Big(-Q_0^{-1} Q_1 \Big)^{n-2} \right]_{(j_1, i_{n-1})} \left[-Q_0^{-1} \boldsymbol{e} \right]_{i_{n-1}} \\
&= \frac{\boldsymbol{\Psi} \Big(-Q_0^{-1} Q_1 \Big)^{n-1} (-Q_0)^{-1} \boldsymbol{e}}{\boldsymbol{\Psi} Q_1 \boldsymbol{e}}.
\end{aligned} \tag{8.6.15}
$$

This is the same expression as given in (8.6.8), since $\lambda = \boldsymbol{\Psi} Q_1 \boldsymbol{e}$.

Theorem 8.6 *The lag-correlation coefficients $c_i(k)$ of the stationary inter-event times for the k^{th} MAP approximation to the departure process of the MAP/PH/1 queue are identical to the lag-correlation coefficients c_i of the stationary inter-departure times for the MAP/PH/1 queue, for each $i \in \{1, 2, \ldots k-1\}$, for all $k \geq 2$.*

Proof:

From the proof of Theorem 8.5, we only need to prove for all $k \geq 2$ that

$$\boldsymbol{\Psi}\Big(- Q_0^{-1}Q_1\Big)^i (-Q_0)^{-1}\boldsymbol{e} = \boldsymbol{\nu}(k)\Big(- Q_0(k)^{-1}Q_1(k)\Big)^i (-Q_0(k))^{-1}\boldsymbol{e},$$
$$\text{for all } i \in \{1, 2, \ldots k-1\}. \quad (8.6.16)$$

Using the form of $-Q_0^{-1}$ given in (8.6.1), and the form of Q_1 given in (8.1.1), it may be seen that

$$-Q_0^{-1}Q_1 = \begin{pmatrix} X_1B_2 & X_2A_2 & X_3A_2 & X_4A_2 & \cdots \\ Y_0B_2 & Y_1A_2 & Y_2A_2 & Y_3A_2 & \cdots \\ 0 & Y_0A_2 & Y_1A_2 & Y_2A_2 & \\ 0 & 0 & Y_0A_2 & Y_1A_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \quad (8.6.17)$$

and that

$$(-Q_0)^{-1}\boldsymbol{e} = \begin{pmatrix} -D_0^{-1}\boldsymbol{e}_m + \sum_{i=1}^{\infty} X_i\boldsymbol{e}_{mn} \\ \sum_{i=0}^{\infty} Y_i\boldsymbol{e}_{mn} \\ \vdots \end{pmatrix}, \quad (8.6.18)$$

where the sub-matrices are all as previously defined. From equations (8.4.12) and (8.4.13), we see that

$$\begin{aligned} Z_r\boldsymbol{e}_n &= Y_r\boldsymbol{e}_{mn} + Z_{r+1}\boldsymbol{e}_n, \text{ for all } r \geq 1, \text{ and} \\ Z(k)\boldsymbol{e}_n &= X_k\boldsymbol{e}_{mn} + Z(k+1)\boldsymbol{e}_n, \text{ for all } k \geq 2. \end{aligned}$$

Using the recursive definitions of Z_r and $Z(k)$ given in (8.4.3) and the statement given in (8.6.2), we can see that

$$\begin{aligned} \lim_{r\to\infty} Z_r\boldsymbol{e}_n &= 0, \text{ and} \\ \lim_{k\to\infty} Z(k)\boldsymbol{e}_n &= 0, \end{aligned}$$

so that

$$\begin{aligned} Z_r \boldsymbol{e}_n &= \sum_{j=r}^{\infty} Y_j \boldsymbol{e}_{mn}, \text{ for all } r \geq 1, & (8.6.19) \\ Z(k)\boldsymbol{e}_n &= \sum_{i=k}^{\infty} X_i \boldsymbol{e}_{mn}, \text{ for all } k \geq 2. & (8.6.20) \end{aligned}$$

From (8.6.19) and (8.6.20), we can re-write (8.6.18) as

$$-Q_0^{-1}\boldsymbol{e} = \begin{pmatrix} -D_0^{-1}\boldsymbol{e}_m + X_1\boldsymbol{e}_{mn} + Z(2)\boldsymbol{e}_n \\ Y_0\boldsymbol{e}_{mn} + Z_1\boldsymbol{e}_n \\ \vdots \end{pmatrix}. \tag{8.6.21}$$

Hence, using the definition of Y_0 given in (8.4.3) and equations (8.4.6) and (8.4.7), (8.6.21) may be re-written as

$$-Q_0^{-1}\boldsymbol{e} = \begin{pmatrix} -D_0^{-1}\boldsymbol{e}_m + D_0^{-1}(D_1 \otimes \boldsymbol{\beta})(\boldsymbol{e}_m \otimes I_n)S^{-1}\boldsymbol{e}_n \\ (\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n \\ \vdots \end{pmatrix}. \tag{8.6.22}$$

From (8.1.1) and (8.6.22) we may write

$$-Q_1Q_0^{-1}\boldsymbol{e} = \begin{pmatrix} 0 \\ B_2\Big(- D_0^{-1}\boldsymbol{e}_m + D_0^{-1}(D_1 \otimes \boldsymbol{\beta})(\boldsymbol{e}_m \otimes I_n)S^{-1}\boldsymbol{e}_n\Big) \\ A_2\Big((\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n\Big) \\ A_2\Big((\boldsymbol{e}_m \otimes I_n)(-S)^{-1}\boldsymbol{e}_n\Big) \\ \vdots \end{pmatrix},$$

which from the proof of Lemma 8.4 is equal to

$$-Q_1Q_0^{-1}\boldsymbol{e} = \begin{pmatrix} 0 \\ \boldsymbol{\epsilon} \\ \boldsymbol{\theta} \\ \boldsymbol{\theta} \\ \vdots \end{pmatrix}, \tag{8.6.23}$$

where we recall from (8.5.1) that

$$\begin{aligned}\boldsymbol{\theta} &= (\boldsymbol{e}_m \otimes I_n)(\boldsymbol{S}^0\boldsymbol{\beta})(-S)^{-1}\boldsymbol{e}_n \text{ and} \\ \boldsymbol{\epsilon} &= -(I_m \otimes (-\boldsymbol{S}^0))D_0^{-1}\boldsymbol{e}_m + \boldsymbol{\theta}.\end{aligned}$$

Using (8.6.1) and (8.6.23), we may then write

$$Q_0^{-1}Q_1Q_0^{-1}\boldsymbol{e} = \begin{pmatrix} X_1\boldsymbol{\epsilon} + \sum_{i=2}^{\infty} X_i\boldsymbol{\theta} \\ Y_0\boldsymbol{\epsilon} + \sum_{i=1}^{\infty} Y_i\boldsymbol{\theta} \\ \sum_{i=0}^{\infty} Y_i\theta \\ \sum_{i=0}^{\infty} Y_i\theta \\ \vdots \end{pmatrix}. \tag{8.6.24}$$

We will now simplify the three terms in (8.6.24) and establish the similarity to the finite case. From (8.5.12) we have that

$$X_1\boldsymbol{\epsilon} + \sum_{i=2}^{k-1} X_i\boldsymbol{\theta} + Z(k)\boldsymbol{\sigma} = X_1\boldsymbol{\epsilon} + Z(2)\boldsymbol{\sigma},$$

which by equation (8.6.20) yields

$$X_1\boldsymbol{\epsilon} + \sum_{i=2}^{\infty} X_i\boldsymbol{\theta} = X_1\boldsymbol{\epsilon} + Z(2)\boldsymbol{\sigma}. \tag{8.6.25}$$

Then from (8.5.13), we have that

$$Y_0\boldsymbol{\epsilon} + \sum_{i=1}^{k-2} Y_i\boldsymbol{\theta} + Z_{k-1}\boldsymbol{\sigma} = Y_0\boldsymbol{\epsilon} + Z_1\boldsymbol{\sigma},$$

which by equation (8.6.19) yields

$$Y_0\boldsymbol{\epsilon} + \sum_{i=1}^{\infty} Y_i\boldsymbol{\theta} = Y_0\boldsymbol{\epsilon} + Z_1\boldsymbol{\sigma}. \tag{8.6.26}$$

Finally from (8.5.14), we have that

$$\sum_{i=0}^{k-3} Y_i\theta + Z_{k-2}\boldsymbol{\sigma} = Y_0\boldsymbol{\theta} + Z_1\boldsymbol{\sigma},$$

which by equation (8.6.19) yields

$$\sum_{i=0}^{\infty} Y_i\boldsymbol{\theta} = Y_0\boldsymbol{\theta} + Z_1\boldsymbol{\sigma},$$

which by the result in (8.5.15) yields

$$\sum_{i=0}^{\infty} Y_i \boldsymbol{\theta} = (\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n)\boldsymbol{e}_{mn}. \tag{8.6.27}$$

Hence, using (8.6.25), (8.6.26) and (8.6.27) in (8.6.24), we have that

$$Q_0^{-1} Q_1 Q_0^{-1} \boldsymbol{e} = \begin{pmatrix} X_1 \boldsymbol{\epsilon} + Z(2)\boldsymbol{\sigma} \\ Y_0 \boldsymbol{\epsilon} + Z_1 \boldsymbol{\sigma} \\ (\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n)\boldsymbol{e}_{mn} \\ (\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n)\boldsymbol{e}_{mn} \\ \vdots \end{pmatrix}. \tag{8.6.28}$$

Then since $-Q_0^{-1} Q_1$ is stochastic, using (8.6.17) and (8.6.28) we may write for all $g \geq 1$ that

$$\left(-Q_0^{-1} Q_1\right)^g (-Q_0)^{-1} \boldsymbol{e} \;=\; \begin{pmatrix} \mathbb{C}_{g,0} \\ \mathbb{C}_{g,1} \\ \vdots \\ \mathbb{C}_{g,g} \\ (\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n)\boldsymbol{e}_{mn} \\ \vdots \end{pmatrix}, \tag{8.6.29}$$

where the terms $\mathbb{C}_{g,i}$, for all $0 \leq i \leq g$, are the same as those for all *MAP* approximations with $k > g$. This can be readily seen from the fact that for all $k \geq 2$, the first $k-1$ entries of the first k rows of the matrix $-Q_0(k)^{-1} Q_1(k)$ are identical to those of the matrix $-Q_0^{-1} Q_1$, and both are stochastic. Comparing (8.6.29) and (8.5.19), all that remains to be proven for

$$\boldsymbol{\Psi}\Big(- Q_0^{-1} Q_1\Big)^i (-Q_0)^{-1} \boldsymbol{e} \;=\; \boldsymbol{\nu}(k)\Big(- Q_0(k)^{-1} Q_1(k)\Big)^i (-Q_0(k))^{-1} \boldsymbol{e}, \text{ for all } k > i \geq 2,$$

to hold true is that

$$\boldsymbol{\pi}_k (I_{mn} - R)^{-1} (\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n)\boldsymbol{e}_{mn} \;=\; \boldsymbol{\pi}_k (I_{mn} - R)^{-1} (\boldsymbol{e}_m \otimes I_n)(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n)\boldsymbol{e}_n, \quad \text{for all } k > 2.$$

This is trivial since $(\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}_n)$ is just a scalar. ∎

8.7 Summary

We have shown that all of our approximations have the same stationary inter-event time distribution, and that this distribution is in fact the same as the stationary inter-departure time distribution for the *MAP/PH/1* queue. Furthermore, we have shown that the k^{th} approximation has the same lag-correlation coefficient structure as the *MAP/PH/1* queue, up to the $k-1^{st}$ lag-correlation coefficient. Both of these properties contribute to the accuracy of our approximations, with the increase in the level k delivering a substantial increase in accuracy.

Chapter 9

Summary

We have investigated the departure process from *MAP/PH/1* queues using matrix analytic methods in a variety of different ways. Our initial investigation was from the perspective of finding a *MAP* description of this process.

In a 1994 paper, Olivier and Walrand conjectured that the departure process of a *MAP/PH/1* queue is not a *MAP* unless the queue is a stationary *M/M/1* queue. Their conjecture was based on a claim that the departure process of an *MMPP/M/1* queue is not *MAP* unless the queue is a stationary *M/M/1* queue. We showed that their proof had an algebraic error, which left open the above question of whether the departure process of an *MMPP/PH/1* queue is a *MAP* or not.

A fundamental question arising from our investigations, was that of identifying stationary *M/M/1* queues in the class of *MAP/PH/1* queues. We addressed this question, using ideas from non-linear filtering theory and the Jordan canonical form for matrices, to give a characterisation as to when a stationary *MAP* is a Poisson process.

This consideration of higher order representations of the Poisson process naturally leads to the related question of minimal order representations for other processes. In particular, we considered the question of minimal order representations for *PH*-type distributions, an issue which has attracted much interest in the literature. We gave a short summary of some authors' work in this field and related their techniques to our work presented on the *PH*-type distributions.

We showed that if a *MAP/M/1* queue has an exact level and phase independent stationary distribution, the the *MAP* is Poisson. On the other hand, we have shown by example that a stationary *MAP/M/1* queue in which the *MAP* is Poisson does not necessarily have an exact level and phase independent stationary distribution. We have also proven that all *PH/M/1* queues exhibit what we have termed *shift-one level and phase independence*.

The question of whether the departure process of an *MAP/PH/1* queue is a *MAP* or not is a particularly difficult one, and remains unanswered except in the case where the stationary arrival *MAP* to the queue is Poisson.

We have proposed a family of *MAP* approximations to the departure process of *MAP/PH/1* queues indexed by a parameter k. We have demonstrated the capabilities of this family of approximations using our own test methods and also by comparison to other approximation methods in the literature. Our family of approximations performed very well. We have shown that the entire family of approximations has a stationary inter-event time distribution which is identical to the stationary inter-departure time distribution of the *MAP/PH/1* queue. The distributions of the inter-departure times being identical, there remains a distinct difference between the various approximations; that being the correlation between successive departure intervals. One measure of this structure is given by the lag-correlation coefficients. In this respect, we have shown that our k^{th} approximation accurately captures the first $k-1$ lag-correlation coefficients of the stationary departure process.

Further investigation of the possibility of placing a metric on the difference between our approximations and the exact departure process has been undertaken, and will be reported in future work. In particular, the notion of placing a measure on the correlation structure which exists between inter-event times shows good promise. As mentioned before, one measure of this structure is given by the lag-correlation coefficients. From our results, a simple question therefore might be: "How many of these lag-correlation coefficients have to be taken into account before the approximation becomes essentially indistinguishable from the actual departure process?" A simple modification of the approximation to the busy period in our $k = 1$ approximation has also shown good results, to be reported also at a later date. The generalisation of these approximation techniques to feedback queues and to more general service times is also reserved for future work. There are many avenues for future research, including the possible incorporation of our techniques in a package for network analysis. Here, for example, a fixed point iteration could be used in conjunction with our methods to gain some approximate solutions.

Appendix A

Tandem queue processes

A.1 The arrival processes.

1. Poisson of rate 1, $\dfrac{\sigma^2}{\mu^2} = 1.0000$

2. $$B_0 = \begin{pmatrix} -3 & 2 \\ 4 & -6 \end{pmatrix}, A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$
$$\frac{\sigma^2}{\mu^2} = 1.0969, c_1 = -0.0044, c_2 = 0.0004.$$

3. Erlang (E_2) of rate 1, $\dfrac{\sigma^2}{\mu^2} = 0.5000.$

4. Hyper-exponential
$$B_0 = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \boldsymbol{\alpha} = (0.75, 0.25), \frac{\sigma^2}{\mu^2} = 1.2400.$$

5. Markov Modulated Poisson process *MMPP*
$$B_0 = \begin{pmatrix} -10.0 & 1.0 \\ 0.4 & -0.8 \end{pmatrix}, A_0 = \begin{pmatrix} 9.0 & 0.0 \\ 0.0 & 0.4 \end{pmatrix},$$
$$\frac{\sigma^2}{\mu^2} = 4.9721, c_1 = 0.1892, c_2 = 0.0896.$$

6.

$$B_0 = \begin{pmatrix} -4 & 1 \\ 1 & -6 \end{pmatrix}, A_0 = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix},$$
$$\frac{\sigma^2}{\mu^2} = 1.1183, c_1 = 0.0046, c_2 = 0.0004.$$

A.2 The first server.

a Negative-exponential, $\frac{\sigma^2}{\mu^2} = 1.0000.$

b Erlang E_2, $\frac{\sigma^2}{\mu^2} = 0.5000.$

c Hyper-exponential

$$S = \begin{pmatrix} -6 & 0 \\ 0 & -12 \end{pmatrix}, \boldsymbol{\beta} = (0.2, 0.8), \frac{\sigma^2}{\mu^2} = 1.2222.$$

d Mixed-Erlang

$$\boldsymbol{\beta} = (0.6, 0.4), \frac{\sigma^2}{\mu^2} = 0.7188.$$

A.3 The second server.

i Negative-exponential, $\frac{\sigma^2}{\mu^2} = 1.0000.$

ii Hyper-exponential

$$S = \begin{pmatrix} -15 & 0 \\ 0 & -8.75 \end{pmatrix}, \boldsymbol{\beta} = (0.3, 0.7), \frac{\sigma^2}{\mu^2} = 1.0952.$$

iii Erlang E_2, $\frac{\sigma^2}{\mu^2} = 0.5000.$

iv Mixed-Erlang

$$\boldsymbol{\beta} = (0.25, 0.75), \frac{\sigma^2}{\mu^2} = 0.9200.$$

Appendix B

Kronecker manipulations

B.1 Kronecker product and sum

For the $m \times n$ matrix A and $r \times s$ matrix B the Kronecker product is the $mr \times ns$ matrix defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & & & a_{2n}B \\ \vdots & & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

For the $m \times m$ matrix A and $n \times n$ matrix B the Kronecker sum is the $mn \times mn$ matrix defined by

$$A \oplus B = A \otimes I_n + I_m \otimes B,$$

where the matrices I_m and I_n are the identity matrices of order m and n respectively.

B.2 Some properties and rules

The proofs for all of the following may be found in Graham [23]. The following is given with respect to the Kronecker product, since the Kronecker sum may be dealt with along the same lines by considering it as the addition of two Kronecker products. The dimensions of the matrices A, B, C and D are arbitrary unless specified.

1. For a scalar α, we have

$$A \otimes (\alpha B) = \alpha(A \otimes B).$$

2. Distributive rule with respect to addition.

$$(A + B) \otimes C = A \otimes C + B \otimes C,$$

   where the operation $A + B$ must be well defined.

3. Associative rule.

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C.$$

4. The mixed product rule.

$$(A \otimes B)(C \otimes D) = AC \otimes BD,$$

   where the products AC and BD must be well defined.

5. For an $m \times m$ matrix A and $n \times n$ matrix B

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1},$$

   provided the inverses exist.

Appendix C

Matrix dimensions

$m \times m$	D_0	
$m \times m$	D_1	
$n \times n$	S	
$1 \times n$	$\boldsymbol{\beta}$	
$n \times 1$	$\boldsymbol{S}^0$	
$n \times n$	$\boldsymbol{S}^0\boldsymbol{\beta}$	
$mn \times mn$	R	
$m \times mn$	R_0	
$1 \times m$	$\boldsymbol{\pi}_0$	
$1 \times mn$	$\boldsymbol{\pi}_i$	for all $i \geq 1$
$1 \times m$	$\boldsymbol{y}_0$	
$1 \times mn$	$\boldsymbol{y}_i$	for all $i \geq 1$

$mn \times n$	$(\boldsymbol{e}_m \otimes I_n)$
$mn \times n$	$(D_1\boldsymbol{e}_m \otimes I_n)$
$mn \times mn$	$(I_m \otimes (\boldsymbol{S}^0\boldsymbol{\beta}))$
$mn \times mn$	$(D_0 \otimes I_n)$
$mn \times mn$	$(I_m \otimes S)$

$$
\begin{aligned}
mn \times mn \quad & A_0 &&= (D_1 \otimes I_n) \\
mn \times mn \quad & A_1 &&= (D_0 \oplus S) \\
mn \times mn \quad & A_2 &&= (I_n \otimes (\boldsymbol{S}^0 \boldsymbol{\beta})) \\
m \times mn \quad & B_0 &&= (D_1 \otimes \boldsymbol{\beta}) \\
m \times m \quad & B_1 &&= D_0 \\
mn \times m \quad & B_2 &&= (I_m \otimes \boldsymbol{S}^0) \\
mn \times n \quad & E_0 &&= (D_1 \otimes I_n)(\boldsymbol{e}_m \otimes I_n) \\
n \times n \quad & E_1 &&= S \\
n \times mn \quad & E_2 &&= \boldsymbol{S}^0 \boldsymbol{y}_{k-1} \text{ for all } k \geq 2 \\
n \times n \quad & E_3 &&= (1 - \boldsymbol{y}_{k-1} \boldsymbol{e}_{mn}) \boldsymbol{S}^0 \boldsymbol{\beta} \text{ for all } k \geq 2
\end{aligned}
$$

$$
\begin{aligned}
m \times m \quad & X(0) & & \\
m \times mn \quad & X_i & & \text{for all } i \geq 1 \\
mn \times mn \quad & Y_i & & \text{for all } i \geq 0 \\
n \times n \quad & Z(0) & & \\
mn \times n \quad & Z_i & & \text{for } 0 < i < k \\
m \times n \quad & Z(k) & &
\end{aligned}
$$

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