
List of publications

Primary refereed journal articles are appended.

- *Scale-free avalanche dynamics in the stock market*
M. Bartolozzi, D.B. Leinweber and A.W. Thomas,
Proceeding to Econophysics Colloquium, in press Physica A
Canberra, 14-18 November 2005
Preprint: <http://arxiv.org/abs/physics/060171>
- *Spin glass behaviour of the antiferromagnetic Ising model on a scale-free network*
M. Bartolozzi, T. Surungan, D.B. Leinweber and A.G. Williams,
in press Physical Review B
Preprint: <http://arxiv.org/abs/cond-mat/0512488>
- *Scale-free networks in complex systems*
M. Bartolozzi, T. Surungan, D.B. Leinweber, A.W. Thomas and A.G. Williams,
Proceeding of SPIE International Symposium on Microelectronics, Mems
and Nanotechnology, Complex Systems **6039**, 249 (2005).
Brisbane, 12-14 December 2005,
Preprint: <http://arxiv.org/abs/cond-mat/0511273>
- *Symbiosis in the Bak-Sneppen model with economic applications*
M. Bartolozzi, D.B. Leinweber and A.W. Thomas,
Physica A **365**, 499 (2006).
Preprint: <http://arxiv.org/abs/cond-mat/0503421>
- *Stochastic opinion formation in scale-free networks*
M. Bartolozzi, D.B. Leinweber and A.W. Thomas,
Physical Review E **72**, 046113 (2005)
Preprint: <http://arxiv.org/abs/physics/0504168>.
- *Self-similar log-periodic structures in western stock markets from 2000*
M. Bartolozzi, S. Drozd, D.B. Leinweber J. Speth and A.W. Thomas,
International Journal of Modern Physics C, **16**, No. 9, 1-15 (2005).
Preprint: <http://arxiv.org/abs/cond-mat/0503421>

- *Self-organized criticality and stock market dynamics: an empirical study*
M. Bartolozzi, D.B. Leinweber and A.W. Thomas,
Physica A **350**, 421 (2005).
Preprint: <http://arxiv.org/abs/cond-mat/0405253>
- *Cellular automata model for financial market dynamics*
M. Bartolozzi and A. W. Thomas,
Physical Review E, **69**, 046112 (2004).
Preprint: <http://arxiv.org/abs/cond-mat/0311372>

Bibliography

- [AB02] R. Albert and A.-L. Barabasi. Statistical mechanics of complex networks. *Rev. Mod. Phys.*, 74:47, 2002.
- [ABS94] P. Ashwin, J.B. Buescu, and I. Stewart. Bubbling of attractors and synchronisation of chaotic oscillators. *Phys. Lett. A*, 193:126, 1994.
- [ACP04] M. Ausloos, M. Clippe, and A. Pekalski. Evolution of economic entities under heterogeneous political/environmental conditions within a bak-sneppen like dynamics. *Physica A*, 332:394, 2004.
- [Aea00] L.A.N. Amaral and et al. Classes of small-world networks. *Proc. Natl. Acad. Sci. U.S.A.*, 97:11149, 2000.
- [Aea01] V. Antoni and et al. Transport process in reversed-field-pinch plasmas: inconsistency with the self-organized criticality paradigm. *Phys. Rev. Lett.*, 87:045001, 2001.
- [AH05] R.F.S. Andrade and H.J. Herrmann. Magnetic moments on apollonian networks. *Phys. Rev. E*, 71:056131, 2005.
- [AHS02] A. Aleksiejuk, J.A. Holyst, and D. Stauffer. Ferromagnetic phase transition in barabási-albert networks. *Physica A*, 310:260, 2002.
- [AI02] M. Ausloos and K. Ivanova. Multifractal nature of stock exchange prices. *Comput. Phys. Commun.*, 147:582, 2002.
- [AI03] M. Ausloos and K. Ivanova. Dynamical model and nonextensive statistical mechanics of a market index on large time windows. *Phys. Rev. E*, 68:046122, 2003.
- [AJB99] R. Albert, H. Jeong, and A.-L. Barabási. Diameter of the world-wide web. *Nature*, 401:130, 1999.
- [AJB00] R. Albert, H. Jeong, and A.-L. Barabási. Attack and error tolerance of complex networks. *Nature*, 406:378, 2000.
- [AS97] P. Ashwin and S. Stone. Influence of noise near blowout bifurcation. *Phys. Rev. E*, 56:1635, 1997.
- [Axt01] R.L. Axtell. Zipf distribution of u.s. firm size. *Nature*, 293:1818, 2001.
- [BA99] A.-L. Barabási and R. Albert. Emergence of scaling in random networks. *Science*, 286:509, 1999.
- [Bac00] L. Bachelier. Theorie de la speculation. *Ann. Sci. Ec. Normale Super.*, III:21, 1900.
- [Bak99] P. Bak. *How Nature Works: The Science of Self-Organized Criticality*. Springer-Verlag, Berlin, 1999.
- [BCFV02] G. Boffetta, M. Cencini, M. Falcioni, and A. Vulpiani. Predictability: a way to characterize complexity. *Phys. Rep.*, 356:367, 2002.

- [BCG82] E.R. Berlekamp, J.H. Conway, and R.K. Guy. *Winning ways for your mathematical plays*. Academic Press, London, 1982.
- [BCG+99] G. Boffetta, V. Carbone, P. Giuliani, P. Veltri, and A. Vulpiani. Power laws in solar flares: Self-organized criticality or turbulence? *Phys. Rev. Lett.*, 83:4662, 1999.
- [BCLM03] G. Bonanno, G. Caldarelli, F. Lillo, and R.N. Mantegna. Topology of correlation-based minimal spanning trees in real and model markets. *Phys. Rev. E*, 68:046130, 2003.
- [BCSW93] P. Bak, K. Chen, J. Scheinkman, and M. Woodford. Aggregate fluctuations from independent sectoral shocks: self-organized criticality in a model of production and inventory dynamics. *Ric. Econ.*, 47:3, 1993.
- [BDL+05] M. Bartolozzi, S. Drożdż, D.B. Leinweber, J.M. Speth, and A.W. Thomas. Self-similar log-periodic structures in western stock markets since 2000. *Int. J. Mod. Phys. C*, 16 (9):1347, 2005.
- [Bec01] C. Beck. Dynamical foundations of nonextensive statistical mechanics. *Phys. Rev. Lett.*, 87:180601, 2001.
- [Bec03] C. Beck. Superstatistics. *Physica A*, 322:267, 2003.
- [Bec05] C. Beck. Superstatistics: recent developments and applications. *cond-mat/0502306*, 2005.
- [BJPV98] T. Bohr, M. Jensen, G. Paladin, and A. Vulpiani. *Dynamical systems approach to turbulence*. Cambridge University Press, Cambridge, 1998.
- [BL80] M.V. Berry and Z.V. Lewis. On the Weierstrass-Mandelbrot fractal function. *Proc. R. Soc. Lond. A*, 370:459, 1980.
- [BLT05a] M. Bartolozzi, D.B. Leinweber, and A.W. Thomas. Self-organized criticality and stock market dynamics: an empirical study. *Physica A*, 350:451, 2005.
- [BLT05b] M. Bartolozzi, D.B. Leinweber, and A.W. Thomas. Stochastic opinion formation in scale-free networks. *Phys. Rev. E*, 72:046113, 2005.
- [BLT06] M. Bartolozzi, D.B. Leinweber, and A.W. Thomas. Symbiosis in the bak-sneppen model for biological evolution with economic applications. *Physica A*, 365:499, 2006.
- [BM05] M. Baiesi and C. Maes. Realistic time correlations in self-organized criticality from a random walk driving. In *preprint: cond-mat/0505274*, 2005.
- [Bol85] B. Bollobás. *Random graphs*. Academic, London, 1985.
- [BP97] R. Badii and A. Politi. *Complexity: hierarchical structures and scaling in physics*. Cambridge University Press, Cambridge, 1997.
- [BP99] J.-P. Bouchaud and M. Potters. *Theory of financial risk*. Cambridge University Press, Cambridge, 1999.
- [BPS97] P. Bak, M. Paczuski, and M. Shubik. Price variations in a stock market with many agents. *Physica A*, 246:430, 1997.

- [BS73] F. Black and M. Scholes. The pricing of options and corporate liabilities. *J. Political Economy*, 81:637, 1973.
- [BS93] P. Bak and K. Sneppen. Punctuated equilibrium and criticality in a simple model of evolution. *Phys. Rev. Lett.*, 71:4083, 1993.
- [BSLW05] M. Bartolozzi, T. Surungan, D.B. Leinweber, and A.G. Williams. Spin glass behaviour of the antiferromagnetic ising model on a scale-free network. In press *Physical Review B*, preprint: cond-mat/0512488, 2005.
- [BT89] P. Bak and C. Tang. Earthquakes as a self-organized critical phenomenon. *J. Geophys. Res.*, 94:15635, 1989.
- [BT04] M. Bartolozzi and A.W. Thomas. Stochastic cellular automata model for stock market dynamics. *Phys. Rev. E*, 69:046112, 2004.
- [BTW87] P. Bak, C. Tang, and K. Wiesenfeld. Self-organized criticality: an explanation of $1/f$ noise. *Phys. Rev. Lett.*, 59:381, 1987.
- [BTW88] P. Bak, C. Tang, and K. Wiesenfeld. Self-organized criticality. *Phys. Rev. A*, 38:364, 1988.
- [BY86] K. Binder and A.P. Young. Spin glasses: experimental facts, theoretical concepts, and open questions. *Rev. Mod. Phys.*, 58:801, 1986.
- [BY88] R.N. Bhatt and A.P. Young. Numerical studies of ising spin glasses in two, three, and four dimensions. *Phys. Rev. B*, 37:5606, 1988.
- [Can93] E. Canessa. Singularity spectrum of self-organized criticality. *Phys. Rev. E*, 47:R5, 1993.
- [Can00] E. Canessa. Multifractality in time series. *J. Phys. A*, 33:3637, 2000.
- [CB00] R. Cont and J.-P. Bouchaud. Herd behaviour and aggregate fluctuations in financial markets. *Macroecon. Dyn.*, 4:170, 2000.
- [CC92] H.F. Chau and K.S. Cheng. Relations of $1/f$ and $1/f^2$ power spectra to self-organized criticality. *Phys. Rev. A*, 46:R2981, 1992.
- [Cea02] V. Carbone and et al. To what extent can dynamical models describe statistical features of turbulent flows? *Europhys. Lett.*, 58 (3):349, 2002.
- [Cea03] T. Chang and et. al. *Advances in space environmental research, Vol. I*. Kluwer Academic Publisher, AH Dordrecht, The Netherlands, 2003.
- [CEbAH00] R. Cohen, K. Erez, D. ben Avraham, and S. Havlin. Resilience of the internet to random breakdowns. *Phys. Rev. Lett.*, 85:4626, 2000.
- [CGA02] J. Camacho, R. Guimera, and L.A.N. Amaral. Robust patterns in food webs structures. *Phys. Rev. Lett.*, 88:228102, 2002.
- [Cha66] G.J.. Chaitin. On the length of programs for computing binary sequences. *J. Assoc. Comp. Math.*, 13:547, 1966.
- [CL96] E. Caglioti and V. Loreto. Dynamical properties and predictability of a class of self-organized critical models. *Phys. Rev. E*, 53:2953, 1996.
- [CLM⁺97] P. Cizeau, Y. Liu, M. Meyer, C.-K. Peng, and H.E. Stanley. Volatility distribution in the S&P500 stock index. *Physica A*, 245:441, 1997.

- [CNSW00] D.S. Callaway, M.E.J. Newman, S.H. Strogatz, and D.J. Watts. Network robustness and fragility: percolation on random graphs. *Phys. Rev. Lett.*, 85:5468, 2000.
- [CO92] K. Christensen and Z. Olami. Variation of the Gutenberg-Richter b values and nontrivial temporal correlations in a spring-block model for earthquakes. *Geophys. Res. Lett.*, 97:8729, 1992.
- [Cor04] A. Corral. Long-term clustering, scaling and universality in the temporal occurrence of earthquakes. *Phys. Rev. Lett.*, 92:108501, 2004.
- [CP00] J.X. Carvalho and C.P.C. Prado. Self-organized criticality in the Olami-Feder-Christensen model. *Phys. Rev. Lett.*, 84:4006, 2000.
- [CVV01] G. Cuniberti, A. Valleriani, and J.L. Vega. Effects of regulation on a self-organized market. *Quant. Finance*, 1:332, 2001.
- [Dau88] I. Daubechies. Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.*, 41 (7):909, 1988.
- [Day04] R.H. Day. *The divergent dynamics of economic growth*. Cambridge University Press, Cambridge, 2004.
- [DEB02] J. Davidsen, H. Ebel, and S. Bornholdt. Emergence of a small world from local interactions: modeling acquaintance networks. *Phys. Rev. Lett.*, 88:128701, 2002.
- [DGM02] S.N. Dogorovtsev, A.V. Goltsev, and J.F.F. Mendes. Ising model on networks with an arbitrary distribution of connections. *Phys. Rev. E*, 66:016104, 2002.
- [DGRS01] S. Drożdż, J. Grümmmer, F. Ruf, and J. Speth. Towards identifying the world stock market cross-correlations: Dax versus Dow Jones. *Physica A*, 294:226, 2001.
- [DGRS03] S. Drożdż, J. Grümmmer, F. Ruf, and J. Speth. Log-periodic self-similarity: an emerging financial law? *Physica A*, 324:174, 2003.
- [DLRVV97] P. De Los Rios, A. Valleriani, and J.L. Vega. Self-organized criticality driven by deterministic rules. *Phys. Rev. E*, 56:4876, 1997.
- [DM02] S.N. Dorogovtsev and J.F.F. Mendes. Evolution of networks. *Adv. Phys.*, 51:1079, 2002.
- [DMAD03] T. Di Matteo, T. Aste, and M.M. Dacorogna. Scaling behaviors in differently developed markets. *Physica A*, 324:183, 2003.
- [DMWC94] A. Davis, A. Marshak, W. Wiscombe, and R. Cahalan. Multifractal characterizations of nonstationarity and intermittency in geophysical fields: observed, retrieved or simulated. *J. Geophys. Res.*, [Atmos.], 99:8055, 1994.
- [Dot01] V. Dotsenko. *Introduction to the Replica Theory of Disordered Statistical Systems*. Cambridge University Press, Cambridge, 2001.
- [Dro01] B. Drossel. Biological evolution and statistical physics. *Adv. Phys.*, 50:209, 2001.
- [DRS99] S. Drożdż, F. Ruf, and J. Speth. Imprints of log-periodic self-similarity in the stock market. *Eur. Phys. J. B*, 10:589, 1999.

- [EA75] S.F. Edwards and P.W. Anderson. Theory of spin glasses. *J. Phys. F: Metal Phys.*, 5:965, 1975.
- [EG72] N. Eldredge and S.J. Gould. *Models in paleobiology*. Freeman, San Francisco, 1972.
- [Far92] M. Farge. Wavelets transform and their applications to turbulence. *Annu. Rev. Fluid Mech.*, 24:395, 1992.
- [FBS93] H. Flyvbjerg, P. Bak, and K. Sneppen. Mean field theory for a simple model of evolution. *Phys. Rev. Lett.*, 71:4087, 1993.
- [Fed88] J. Feder. *Fractals*. Plenum Press, New York & London, 1988.
- [Fei01] J.A. Feigenbaum. A statistical analysis of log-periodic precursors to financial crashes. *Quant. Finance*, 1:346, 2001.
- [Fei03] J.A. Feigenbaum. Financial physics. *Rep. Prog. Phys.*, 66:1611, 2003.
- [Fel68] W. Feller. *An introduction to probability theory and its applications, Vol.II*. Wiley & Sons, New York, 1968.
- [FF96] J.A. Feigenbaum and P.G.O. Freund. Discrete scale invariance in stock markets before crashes. *Int. J. Mod. Phys. B*, 10:3737, 1996.
- [FFF99] M. Faloutsos, P. Faloutsos, and C. Faloutsos. On power-law relationships of the internet topology. *Comp. Commun. Rev.*, 29:251, 1999.
- [Fri95] U. Frisch. *Turbulence*. Cambridge University Press, Cambridge, 1995.
- [FSK99] M. Farge, K. Schneider, and N. Kevlahan. Nongaussianity and coherent vortex simulations for two-dimensional turbulence using an adaptive orthogonal wavelet basis. *Phys. Fluids*, 11:2187, 1999.
- [FWR00] M.P. Freeman, N.W. Watkins, and D.J. Riley. Power law distributions of burst duration and interburst interval in the solar wind: Turbulence or dissipative self-organized criticality? *Phys. Rev. E*, 62:8794, 2000.
- [GBP⁺96] S. Ghashghaie, W. Breymann, J. Peinke, P. Talkner, and Y. Dodge. Turbulent cascade in foreign exchange markets. *Nature*, 381:767, 1996.
- [GC98] P. Giuliani and V. Carbone. A note on shell models for MHD turbulence. *Europhys. Lett.*, 43:527, 1998.
- [GC02] H.M. Gupta and J.R. Campanha. Tsallis statistics and gradually truncated Levy distribution of an economical index. *Physica A*, 309:381, 2002.
- [GDS02] A.Z. Gorski, S. Drożdż, and J. Speth. Financial multifractality and its subtleties: an example of DAX. *Physica A*, 316:496, 2002.
- [GGP03] E. Gaffeo, M. Gallegati, and A. Palestini. On the size distribution of firms: additional evidence from the g7 countries. *Physica A*, 324:47, 2003.
- [GPA⁺97] P. Gopikrishnan, V. Plerou, L.A.N. Amaral, M. Mayer, and H.E. Stanley. Scaling of the distribution of fluctuations of financial market indices. *Phys. Rev. E*, 60:5305, 1997.
- [Gra95] P. Grassberger. The bak-sneppen model for punctuated equilibrium. *Phys. Lett. A.*, 200:277, 1995.

- [GS02] S. Gluzman and D. Sornette. Log-periodic route to fractal functions. *Phys. Rev. E*, 65:036142, 2002.
- [Hea00] D. Head. Temperature scaling, glassiness and stationarity in the bak-sneppen model. *Eur. Phys. J. B*, 17:289, 2000.
- [Hea02] D.A. Head. Universal persistence exponents in an extremally driven system. *Phys. Rev. E*, 65:027104, 2002.
- [Her04] C.P. Herrero. Ising model in scale-free networks: A monte carlo simulation. *Phys. Rev. E*, 69:067109, 2004.
- [HJK⁺86] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia, and B.I. Shraiman. Fractal measures and their singularities: The characterization of strange sets. *Phys. Rev. A*, 33:1141, 1986.
- [HK92] T. Hwa and M. Kardar. Avalanches, hydrodynamics, and discharge events in models of sandpiles. *Phys. Rev. A*, 45:7002, 1992.
- [HK02] P. Holme and B.J. Kim. Growing scale-free networks with tunable clustering. *Phys. Rev. E*, 65:026107, 2002.
- [HN96] K. Hukushima and K. Nemoto. Exchange monte carlo method and application in spin glass simulations. *J. Phys. Soc. Japan*, 65:1863, 1996.
- [HP83] H.G.E. Hentschel and I. Procaccia. The infinite number of generalized dimensions of fractals and strange attractors. *Physica D*, 8:435, 1983.
- [HSSS98] J. Huang, H. Sauler, C. Sammis, and D. Sornette. Precursors, aftershocks, criticality and self-organized criticality. *Europhys. Lett.*, 41:43, 1998.
- [Hur51] H. Hurst. Long-term storage capacity of reservoirs. *Trans. Am. Soc. Civ. Eng.*, 116:770, 1951.
- [Ili99] K. Ilinski. Critical crashes? *Int. J. Mod. Phys. C*, 10:741, 1999.
- [ISC⁺02] K. Ivanova, H.N. Shirer, E.E. Clothiaux, N. Kitova, M.A. Mikhalev, T.P. Ackerman, and M. Ausloos. A case study of stratus cloud base height multifractal fluctuations. *Physica A*, 308:518, 2002.
- [IT02] F. Iglói and L. Turban. First- and second-order phase transitions in scale-free networks. *Phys. Rev. E*, 66:036140, 2002.
- [Jea01] H. Jeong and et al. Lethality and centrality in protein networks. *Nature (London)*, 411:41, 2001.
- [Jen98] H.J. Jensen. *Self-organized criticality: emergent complex behavior in physical and biological systems*. Cambridge University Press, Cambridge, 1998.
- [Jet al.00] H. Jeong and et al. The large scale organization of metabolic networks. *Nature (London)*, 407:651, 2000.
- [JSL99] A. Johansen, D. Sornette, and O. Ledoid. Predicting financial crashes using discrete scale invariance. *J. Risk*, 1:5, 1999.
- [Kai00] T. Kaizoji. Speculative bubbles and crashes in stock markets: an interacting-agent model of speculative activity. *Physica A*, 287:493, 2000.

- [KCV01] P. Kovacs, V. Carbone, and Z. Voros. Wavelet-based filtering of intermittent events from geomagnetic time-series. *Planet. Space Sci.*, 49:1219, 2001.
- [Kea94] G.G. Katul and et al. *Wavelets in geophysics*. Academic, San Diego, 1994.
- [Kea00] R. Kumar and et al. The web as a graph. *Proceeding of the 19th ACM Symposium on Principles of Database systems (unpublished)*, page 1, 2000.
- [KF03] N. Kozuki and N. Fuchikami. Dynamical model of financial markets: fluctuating 'temperature' causes intermittent behavior of price changes. *Physica A*, 329:222, 2003.
- [KHH02] A. Krawiecki, J.A. Holyst, and D. Helbing. Volatility clustering and scaling for financial time series due to attractor bubbling. *Phys. Rev. Lett.*, 89:158701, 2002.
- [Kol65] A.N. Kolmogorov. Three approaches to the quantitative definition of information. *Problems of information transmission*, 1:4, 1965.
- [KR04] N. Kawashima and H. Rieger. *Frustrated spin systems*. World Scientific, Singapore, 2004.
- [KRKK05] D.-H. Kim, G.J. Rodgers, B. Kahng, and D. Kim. Spin-glass phase transition on scale-free networks. *Phys. Rev. E*, 71:056115, 2005.
- [KY96] N. Kawashima and A.P. Young. Phase transition in the three-dimensional $\pm j$ Ising spin glass. *Phys. Rev. B*, 53:R484, 1996.
- [LdAG05] E. Lippiniello, L. de Arcangelis, and C. Godano. Memory in self-organized criticality. *Europhys. Lett.*, 72:678, 2005.
- [Lea99] Y. Liu and et al. Statistical properties of the volatility of price fluctuations. *Phys. Rev. E*, 60:1390, 1999.
- [Lea01] F. Liljeros and et al. The web of human sexual contacts. *Nature (London)*, 411:907, 2001.
- [LeB01] B. LeBaron. Stochastic volatility as a simple generator of apparent financial power laws and long memory. *Quant. Finance*, 1:621, 2001.
- [LH88] J. Lee and Stanley H.E. Phase transition in the multifractal spectrum of diffusion-limited aggregation. *Phys. Rev. Lett.*, 61:2945, 1988.
- [LH91] E.T. Lu and R.J. Hamilton. Avalanches and distribution of solar flares. *Astrophys. J.*, 380:L89, 1991.
- [LHMB93] E.T. Lu, R.J. Hamilton, J.M. McTieran, and K.R. Bromund. Solar flares and avalanches in driven dissipative systems. *Astrophys. J.*, 412:841, 1993.
- [LLS00] H. Levy, M. Levy, and S. Solomon. *Microscopic simulations of financial markets*. Academic press, New York, 2000.
- [LPC+99] L. Laloux, M. Potters, R. Cont, Aguilar, J.P., and Bouchaud J.P. Are financial crashes predictable? *Europhys. Lett.*, 45:1, 1999.
- [Lux01] T. Lux. Turbulence in financial markets: the surprising explanatory power of simple cascade models. *Quant. Finance*, 1:632, 2001.
- [Man97] B. Mandelbrot. *Fractals and scaling in finance*. Springer, New York, 1997.

- [Man01] B. Mandelbrot. Scaling in financial prices: Iv. Multifractal concentration. *Quant. Finance*, 1:641, 2001.
- [MB73] R.C. Merton and J. Bell. Rational theory of option pricing. *Econom.*, 41:141, 1973.
- [MJ03] F. Michael and M.D. Johanson. Financial market dynamics. *Physica A*, 320:525, 2003.
- [MPV87] M. Mezard, G. Parisi, and M.A. Visaroso. *Spin glass theory and beyond*. World Scientific, Singapore, 1987.
- [MRR⁺53] N. Metropolis, A.W. Rosenbluth, M.N. Rosenbluth, A.M. Teller, and E. Teller. Self-organized criticality: an explanation of 1/f noise. *J. Chem. Phys.*, 21:1087, 1953.
- [MS95] R.N. Mantegna and H.E. Stanley. Scaling behaviour in the dynamics of an economic index. *Nature*, 376:46, 1995.
- [MS97] R.N. Mantegna and H.E. Stanley. Stock market dynamics and turbulence: parallel analysis of fluctuation phenomena. *Physica A*, 239:225, 1997.
- [MS99] R.N. Mantegna and H.E. Stanley. *An introduction to econophysics: correlation and complexity in finance*. Cambridge University Press, Cambridge, 1999.
- [MS02] J.M. Montoya and R.V. Sole. Small world patterns in food webs. *J. Theor. Biol.*, 214:405, 2002.
- [Nag85] T. Nagatani. Self-organized criticality in 1D traffic flow model with inflow or outflow. *J. Phys. A: Math. Gen.*, 28:L119, 1985.
- [Nag95] K. Nagel. Emergent traffic jams. *Phys. Rev. E*, 51:2909, 1995.
- [Nag96] T. Nagatani. Self-organized criticality in 1D traffic flow. *Fractals*, 4:279, 1996.
- [Nak98] H. Nakao. Asymptotic power law of moments in a random multiplicative process with weak additive noise. *Phys. Rev. E*, 58:1591, 1998.
- [NCML01] J.P. Norman, P. Charbonneau, S.W. McIntosh, and H.-L. Liu. Waiting-time distributions in lattice models of solar flares. *Astrophys. J.*, 557:891, 2001.
- [NH93] K. Nagel and H.J. Herrmann. Deterministic model for traffic jams. *Physica A*, 199:254, 1993.
- [NP99] M.E.J. Newman and R.G. Palmer. Models of extinction: a review. In *preprint: adap-org/9908002*, 1999.
- [NTG95] W.I. Newman, D.L. Turcotte, and A.M. Gabrielov. Log-periodic behaviour of a hierarchical failure model with applications to precursory seismic activation. *Phys. Rev. E*, 52:4827, 1995.
- [OC92] Z. Olami and K. Christensen. Temporal correlations, universality, and multifractality in a spring-block model of earthquakes. *Phys. Rev. A*, 46:R1720, 1992.
- [OCK⁺03] J.-P. Onnela, A. Chakraborti, K. Kaski, J. Kertesz, and A. Kanto. Dynamics of market correlations: taxonomy and portfolio analysis. *Phys. Rev. E*, 68:056110, 2003.

- [Ogi85] A.T. Ogielski. Dynamics of three-dimensional ising spin glasses in thermal equilibrium. *Phys. Rev. B*, 32:7384, 1985.
- [OS94] E. Ott and J.C. Sommerer. Blowout bifurcations: the occurrence of riddled basins and on-off intermittency. *Phys. Lett. A*, 188:39, 1994.
- [PA00] A. Ponzi and Y. Aizawa. Criticality and punctuated equilibrium in spin system model of financial market. *Chaos, Solitons & Fractals*, 11:1739, 2000.
- [Par83] G. Parisi. Order parameter for spin-glasses. *Phys. Rev. Lett.*, 50:1946, 1983.
- [PB99] W. Paul and Baschnagel. *Stochastic processes: from physics to finance*. Springer, Berlin, 1999.
- [Pea99] V. Plerou and et al. Universal and nonuniversal properties of cross correlations in financial time series. *Phys. Rev. Lett.*, 83:1471, 1999.
- [PHH94] N. Platt, S.M. Hammel, and J.F. Heagy. Effects of additive noise on on-off intermittency. *Phys. Rev. Lett.*, 72:3498, 1994.
- [PMB96] M Paczuski, S. Mlasov, and P. Bak. Avalanche dynamics in evolution, growth and depinning models. *Phys. Rev. E*, 53:414, 1996.
- [PST93] N. Platt, E.A. Spiegel, and C. Tresser. On-off intermittency: a mechanism for bursting. *Phys. Rev. Lett.*, 70:279, 1993.
- [PSV01] R. Pastor-Satorras and A. Vespignani. Epidemic spreading in scale-free networks. *Phys. Rev. Lett.*, 86:3200, 2001.
- [PTVF94] W.H. Press, S.A. Teulosky, W.T. Vetterlong, and B.P. Flannery. *Numerical recipes in Fortran*. Cambridge University Press, Cambridge, 1994.
- [Red98] S. Redner. How popular is your paper? an empirical study of citation distribution. *Eur. Phys. J. B*, 4:131, 1998.
- [RNZR⁺01] C. Rodriguez Neto, A. Zanandrea, F.M. Ramos, R.R. Rosa, M.J.A. Bolzan, and L.D.A. Sa. Multiscale analysis from turbulent time series with wavelet transform. *Physica A*, 295:215, 2001.
- [RRN⁺01] F.M. Ramos, F.M. Rosa, C.R. Neto, M.J.A. Bolzan, and L.D. Abreu Sa. Nonextensive thermostatics description of intermittency in financial markets. *Non-linear Anal.: Real World App.*, 47:3521, 2001.
- [RS82] D.M. Raup and J.J. Sepkoski. Mass extinctions in marine fossil records. *Science*, 215:1501, 1982.
- [RT98] D.C. Roberts and D.L. Turcotte. Fractality and self-organized criticality of wars. *Fractals*, 6:351, 1998.
- [SDS90] D. Sornette, P. Davy, and A. Sornette. Structuration of the lithosphere in plate tectonics as a self-organized critical phenomenon. *J. Geophys. Res.*, 95:17353, 1990.
- [Sea01] E. Spada and et al. Search of self-organized criticality processes in magnetically confined plasmas: hints from the reversed field pinch configuration. *Phys. Rev. Lett.*, 86:3032, 2001.

- [SFA04] W. Souma, Y. Fujiwara, and H. Aoyama. Heterogeneous economic networks. *9th Workshop on economics and heterogeneous interacting agents (WEHIA2004) May 27-29, Kyoto, (physics/0502005)*, 2004.
- [SJ97] D. Sornette and A. Johansen. Large financial crashes. *Physica A*, 245:411, 1997.
- [SJB96] D. Sornette, A. Johansen, and J.-P. Bouchaud. Stock market crashes, precursors and replicas. *J. Phys. I France*, 6:167, 1996.
- [SNC02] R. Sanchez, D.E. Newman, and B.A. Carreras. Waiting-time statistics of self-organized-criticality systems. *Phys. Rev. Lett.*, 88:068302, 2002.
- [sof05] Pajek software. Download at <http://vlado.fmf.uni-lj.si/pub/networks/pajek>. In *University of Lubijana*, 2005.
- [Sol64] R.J. Solomonoff. A formal theory of inductive inference. *Inform. Contr.*, 7:1 and 224, 1964.
- [Sor98] D. Sornette. Discrete scale invariance and complex dimension. *Phys. Rep.*, 297:239, 1998.
- [Sor03a] D. Sornette. Critical market crashes. *Phys. Rep.*, 378:1, 2003.
- [Sor03b] D. Sornette. *Why stock market crash*. Princeton University Press, Princeton, 2003.
- [Sor04] D. Sornette. *Critical phenomena in natural sciences*. Springer-Verlag, Berlin, 2004.
- [SOT04] T. Surungan, Y. Okabe, and Y. Tomita. Study of the fully frustrated clock model using the wang-landau algorithm. *J. Phys. A*, 37:4219, 2004.
- [SP98] D. Stauffer and T.J.P. Penna. crossover in the Cont-Bouchaud model for market fluctuations. *Physica A*, 256:284, 1998.
- [SP01] H.E. Stanley and V. Plerou. Scaling and universality in economics: empirical results and theoretical interpretation. *Quant. Finance*, 1:563, 2001.
- [SS89] A. Sornette and D. Sornette. Self-organized criticality and earthquakes. *Europhys. Lett.*, 9:197, 1989.
- [SST99] D. Sornette, D. Stauffer, and H. Takayasu. Market fluctuations: multiplicative and percolation models, size effects and predictions. *cond-mat/9909439*, 1999.
- [Sta71] H.E. Stanley. *Introduction to phase transitions and critical phenomena*. Oxford University Press, Oxford, 1971.
- [Sta85] D. Stauffer. *Introduction to percolation theory*. Taylor & Francis, London, 1985.
- [SZ02] D. Sornette and W.-X. Zhou. The US 2000-2002 market descent: how much longer and deeper? *Quant. Finance*, 2:468, 2002.
- [SZ03] D. Sornette and W.-X. Zhou. The US 2000-2002 market descent: clarification. *Quant. Finance*, 3:C39, 2003.
- [Tab01] K. Tabelow. Gap function in the finite bak-sneppen model. *Phys. Rev. E*, 65:047101, 2001.

- [TABO03] C. Tsallis, C. Anteneodo, L. Borland, and R. Osorio. Nonextensive statistical mechanics and economics. *Physica A*, 324:89, 2003.
- [Tsa88] C. Tsallis. Possible generalization of Boltzmann-Gibbs statistics. *J. Stat. Phys.*, 52:479, 1988.
- [Tsa99] C. Tsallis. Nonextensive statistics: theoretical, experimental and computational evidences and connections. *Braz. J. Phys.*, 29:1, 1999.
- [TST97] H. Takayasu, A.-H. Sato, and M. Takayasu. Stable infinite variance fluctuations in randomly amplified Langevin systems. *Phys. Rev. Lett.*, 79:966, 1997.
- [TT99] H. Takayasu and M. Takayasu. Critical fluctuations of demand and supply. *Physica A*, 269:24, 1999.
- [Tur36] A.M. Turing. On computable numbers with an application to the entscheidungs problem. *Proc. London Math. Soc.*, 2:230, 1936.
- [Tur99] D.L. Turcotte. Self-organized criticality. *Rep. Prog. Phys.*, 62:1377, 1999.
- [Vaz01] A. Vazquez. Statistics of citation networks. *Europhys. Lett.*, 54:430, 2001.
- [VBMA98] N. Vandewalle, P. Boveroux, A. Minguet, and M. Ausloos. The crash of october 1987 seen as a phase transition: amplitude and universality. *Physica A*, 255:201, 1998.
- [VBT01] N. Vandewalle, F. Brisbois, and X. Tordoir. Non-random topology of stock markets. *Quant. finance*, 1:372, 2001.
- [VdZ50] A. Van der Ziel. On the noise spectra of semi-conductor noise and of flicker effect. *Physica (Amsterdam)*, 16:359, 1950.
- [Vea03] A. Valdiva and et. al. *Advances in space environmental research, Vol. I*. Kluwer Academic Publisher, AH Dordrecht, The Netherlands, 2003.
- [VFHG02] L. Vlahos, T. Fragos, H. Hisliker, and M. Georgoulis. Statistical properties of the energy release in emerging and evolving active regions. *Astrophys. J. Lett.*, 575:L87, 2002.
- [vN66] J. von Neumann. *Theory of self-reproducing automata*. A.W. Burks, Univ. of Illinois Press, Champaign, IL, 1966.
- [Vol97] D. Vollhardt. Characteristic crossing points in specific heat curves of correlated systems. *Phys. Rev. Lett.*, 78:1307, 1997.
- [Wea02] R.J. Williams and et al. Two degrees of separation in complex food webs. *Proc. Natl. Acad. Sci. U.S.A.*, 99:12913, 2002.
- [Wil79] K.G. Wilson. Problems in physics with many length scales. *American Scientific*, 241:140, 1979.
- [Wol86] S. Wolfram. *Theory and applications of cellular automata*. World Scientific, Singapore, 1986.
- [WS92] D.G. Wentzel and P.E. Seiden. Solar active regions as a percolation phenomenon. I. *Astrophys. J.*, 390:280, 1992.
- [WS96] D.G. Wentzel and P.E. Seiden. Solar active regions as a percolation phenomenon. II. *Astrophys. J.*, 460:522, 1996.

- [WS98] D.J. Watts and S.H. Strogatz. Collective dynamics of small-world networks. *Nature (London)*, 393:440, 1998.
- [WSM98] M.S. Wheatlans, P.A. Sturrock, and J.M. McTiernan. The waiting-time distribution of solar flare hard X-ray bursts. *Astrophys. J.*, 509:448, 1998.
- [Yam01] T. Yamano. Regulation effects on market with bak-sneppen model in high dimensions. *Int. J. Mod. Phys. C*, 9:1329, 2001.
- [YJB02] S. Yook, H. Jeong, and A.-L. Barabási. Modeling the internet's large-scale topology. *Proc. Natl. Acad. Sci. U.S.A.*, 99:13382, 2002.
- [ZS02] W.-X. Zhou and D. Sornette. Statistical significance of periodicity with heavy tailed correlated noise. *Int. J. Mod. Phys. C*, 13(2):137, 2002.
- [ZS03a] W.-X. Zhou and D. Sornette. Evidence of a worldwide stock market log-periodic anti-bubble since mid-2000. *Physica A*, 330:543, 2003.
- [ZS03b] W.-X. Zhou and D. Sornette. Renormalization group analysis of the 2000-2002 anti-bubble in the US S&P500 index: explanation of the hierarchy of five crashes and prediction. *Physica A*, 330:584, 2003.
- [ZS04] W.-X. Zhou and D. Sornette. Causal slaving of the US treasury bond yield antibubble by the stock market antibubble of august 2000. *Physica A*, 337:586, 2004.
- [ZS05] W.-X. Zhou and D. Sornette. Testing the stability of the 2000 US stock market 'antibubble'. *Physica A*, 348:428, 2005.

Bartolozzi, M. & Thomas, A.W. (2004) Stochastic cellular automata model for stock market dynamics

Physical Review E v. 69 pp. 046112-1 - 046112-7

NOTE: This publication is included in the print copy of the thesis held in the University of Adelaide Library.

It is also available online to authorised users at:

<http://dx.doi.org/10.1103/PhysRevE.69.046112>



Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

Physica A 350 (2005) 451–465

PHYSICA A

www.elsevier.com/locate/physa

Self-organized criticality and stock market dynamics: an empirical study

M. Bartolozzi^a, D.B. Leinweber^{a,*}, A.W. Thomas^{a,b}

^a*Special Research Centre for the Subatomic Structure of Matter (CSSM) and Department of Physics, University of Adelaide, Adelaide, SA 5005, Australia*

^b*Jefferson Laboratory, 12000 Jefferson Ave., Newport News, VA 23606, USA*

Available online 1 January 2005

Abstract

The stock market is a complex self-interacting system, characterized by intermittent behaviour. Periods of high activity alternate with periods of relative calm. In the present work we investigate empirically the possibility that the market is in a self-organized critical state (SOC). A wavelet transform method is used in order to separate high activity periods, related to the avalanches found in sandpile models, from quiescent. A statistical analysis of the filtered data shows a power law behaviour in the avalanche size, duration and laminar times. The memory process, implied by the power law distribution of the laminar times, is not consistent with classical conservative models for self-organized criticality. We argue that a “near-SOC” state or a time dependence in the driver, which may be chaotic, can explain this behaviour.

© 2004 Elsevier B.V. All rights reserved.

PACS: 05.65.+b; 05.45.Tp; 02.70.Hm; 45.70.Ht; 02.70.Rr

Keywords: Complex systems; Econophysics; Self-organized criticality; Wavelets

1. Introduction

Since the publication of the articles of Bak, Tang and Wiesenfeld (BTW) [1], the concept of self-organized criticality (SOC) has been invoked to explain the

*Corresponding author.

E-mail address: dleinweb@physics.adelaide.edu.au (D.B. Leinweber).

dynamical behaviour of many complex systems, from physics to biology and the social sciences [2,3]. The key concept of SOC is that complex systems, that is systems constituted by many interacting elements, although obeying different microscopic physics, may exhibit similar dynamical behaviour. In particular, the statistical properties of these systems can be described by power laws, reflecting a lack of any characteristic scale. These features are equivalent to those of physical systems during a phase transition, that is at the critical point. It is worth emphasizing that the original idea [1] was that the critical state was reached “naturally”, without any external tuning. This is the origin of the adjective *self-organized*. In reality a certain degree of tuning is necessary: implicit tunings like local conservation laws and specific boundary conditions seem to be important ingredients for the appearance of power laws [2].

The classical example of a system exhibiting SOC behaviour is the 2D sandpile model [1–3]. Here the cells of a grid are randomly filled, by an external random driver, with “sand”. When the gradient between two adjacent cells exceeds a certain threshold a redistribution of the sand occurs, leading to more instabilities and further redistributions. The benchmark of this system, indeed of all systems exhibiting SOC, is that the distribution of the avalanche sizes, their duration and the energy released, obey power laws.

The framework of self-organized criticality has been claimed to play an important role in solar flaring [4], space plasmas [5] and earthquakes [6] in the context of both astrophysics and geophysics. In the biological sciences, SOC, has been related, for example, with biodiversity and evolution/extinction [7]. Some work has also been carried out in the social sciences. In particular, traffic flow and traffic jams [8], wars [9] and stock-market [3,10–12] dynamics have been studied. A more detailed list of subjects and references related to SOC can be found in the review paper of Turcotte [3].

In the present work we will provide empirical evidence for connections between self-organized criticality and the stock market, considered as a complex system constituted of many interacting individuals. We analyze the tick-by-tick behaviour of the Nasdaq100 index, $P(t)$, from 21/6/1999 to 19/6/2002 for a total of 2^{19} data. A sample of this data is illustrated in Fig. 1(a). In particular, we study the logarithmic returns of this index, which are defined as $R(t) = \ln(P(t+1)) - \ln(P(t))$ and plotted in Fig. 1(b).

To examine the extent to which our findings apply to other stock market indices we also studied the S&P ASX50 (for the Australian stock market) at intervals of 30 min over the period 20/1/1998 to 1/5/2002, for a total of 2^{14} data points. Possible differences between daily and high-frequency data have also been taken into consideration though the analysis of the Dow Jones daily closures from 2/2/1939 to 13/4/2004. The results are presented in Section 3.

From a visual analysis of the time series of returns, Fig. 1(b), we observe long periods of relative tranquility, characterized by small fluctuations, and periods in which the index goes through very large fluctuations, equivalent to avalanches, clustered in relatively short time intervals. These may be viewed as a consequence of a build-up process leading the system to an extremely unstable state. Once this

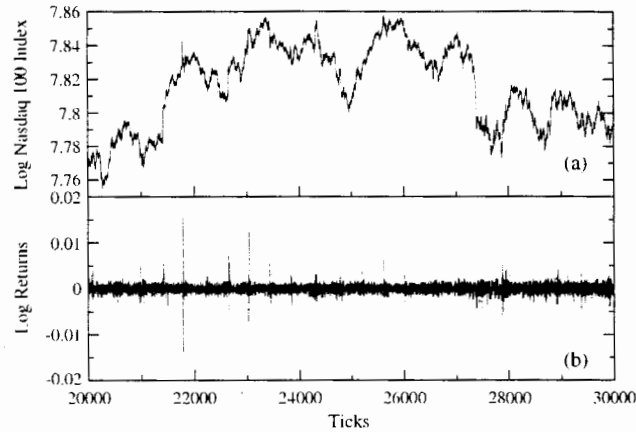


Fig. 1. Sample of the tick-by-tick time series of the Nasdaq100 (a), as well as the corresponding returns (b).

critical point has been reached, any small fluctuation can, in principle, trigger a chain reaction, similar to an avalanche, which is needed to stabilize the system again.

2. Wavelet method

With the recent development of the interdisciplinary area of complexity, many physicists have started to study the dynamical properties of stock markets [13,14]. Empirical results have shown that the time series of financial returns show a behaviour similar to hydrodynamic turbulence [15,16]—although differences have also been pointed out [16]. Both the spatial velocity fluctuations in turbulent flows and the stock market returns show an intermittent behaviour, characterized by broad tails in the probability distribution function (PDF), and a non-linear multifractal spectrum [15]. The PDF for the normalized logarithmic returns,

$$r(t) = \frac{R(t) - \langle R(t) \rangle_l}{\sigma(R(t))}, \quad (1)$$

where $\langle \dots \rangle_l$ is the average over the length of the sample, l , and σ the standard deviation, is plotted in Fig. 2. The departure from a Gaussian behaviour is evident, in particular, in the peak of the distribution and in the broad tails, which are related to extreme events.

The empirical analogies between turbulence and the stock market may suggest the existence of a temporal information cascade for the latter [15]. This is equivalent to assuming that various traders require different information according to their specific strategies. In this way different time scales become involved in the trading process. In the present work we use a wavelet method in order to study multi-scale market dynamics.

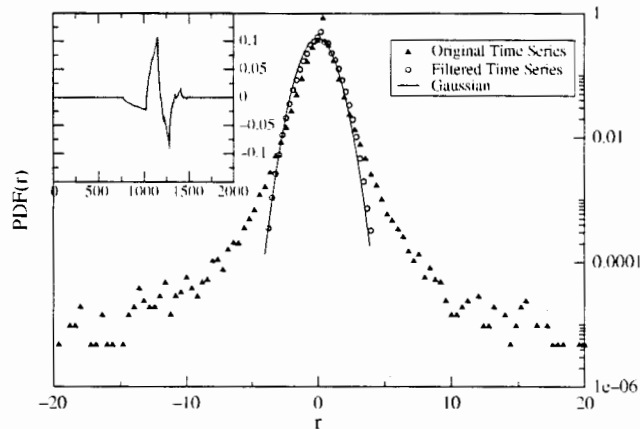


Fig. 2. PDF of the logarithmic returns of the Nasdaq100 index before (triangles) and after filtering (circles), with $C = 2$. The original time series is reduced to the level of noise. A Gaussian distribution is plotted for comparison. The inset shows the fourth member of the Daubechies wavelets used in the filtering.

The wavelet transform is a relatively new tool for the study of intermittent and multifractal signals [19]. The approach enables one to decompose the signal in terms of scale and time units and so to separate its coherent parts—that is, the bursty periods related to the tails of the PDF—from the noise-like background, thus enabling an independent study of the intermittent and the quiescent intervals [20].

The continuous wavelet transform (CWT) is defined as the scalar product of the analyzed signal, $f(t)$, at scale λ and time t , with a real or complex “mother wavelet”, $\psi(t)$:

$$W_T f(t) = \langle f, \psi_{\lambda,t} \rangle = \int f(u) \bar{\psi}_{\lambda,t}(u) du = \frac{1}{\sqrt{\lambda}} \int f(u) \bar{\psi}\left(\frac{u-t}{\lambda}\right) du. \quad (2)$$

The idea behind the wavelet transform is similar to that of windowed Fourier analysis and it can be shown that the scale parameter is indeed inversely proportional to the classic Fourier frequency. The main difference between the two techniques lies in the resolution in the time–frequency domain. In the Fourier analysis the resolution is scale independent, leading to aliasing of high- and low-frequency components that do not fall into the frequency range of the window. However, in the wavelet decomposition the resolution changes according to the scale (i.e., frequency). At smaller scales the temporal resolution increases at the expense of frequency localization, while for large scales we have the opposite. For this reason the wavelet transform is considered a sort of mathematical “microscope”. While the Fourier analysis is still an appropriate method for the study of harmonic signals, where the information is equally distributed, the wavelet approach becomes fundamental when the signal is intermittent and the information localized.

The CWT of Eq. (2) is a powerful tool to graphically identify coherent events, but it contains a lot of redundancy in the coefficients. For a time-series analysis it is often

preferable to use a discrete wavelet transform (DWT). The DWT can be seen as a appropriate sub-sampling of Eq. (2) using dyadic scales. That is, one chooses $\lambda = 2^j$, for $j = 0, \dots, L - 1$, where L is the number of scales involved, and the temporal coefficients are separated by multiples of λ for each dyadic scale, $t = n2^j$, with n being the index of the coefficient at the j th scale. The DWT coefficients, $W_{j,n}$, can then be expressed as

$$W_{j,n} = \langle f, \psi_{j,n} \rangle = 2^{-j/2} \int f(u) \psi(2^{-j}u - n) du, \quad (3)$$

where $\psi_{j,n}$ is the discretely scaled and shifted version of the mother wavelet. The wavelet coefficients are a measure of the correlation between the original signal, $f(t)$, and the mother wavelet, $\psi(t)$ at scale j and time n . In order to be a wavelet, the function $\psi(t)$ must satisfy some conditions. First, it has to be well localized in both real and Fourier space and second the following relation:

$$C_\psi = 2\pi \int_{-\infty}^{+\infty} \frac{|\hat{\psi}(k)|^2}{k} dk < \infty, \quad (4)$$

must hold, where $\hat{\psi}(k)$ is the Fourier transform of $\psi(t)$. The requirement expressed by Eq. (4) is called *admissibility* and it guarantees the existence of the inverse wavelet transform. The previous conditions are generally satisfied if the mother wavelet is an oscillatory function around zero, with a rapidly decaying envelope. Moreover, for the DWT, if the set of the mother wavelet and its translated and scaled copies form an orthonormal basis for all functions having a finite squared modulus, then the energy of the starting signal is conserved in the wavelet coefficients. This property is, of course, extremely important when analyzing physical time series [21]. More comprehensive discussions on the wavelet properties and applications are given in Refs. [22,19]. Among the many orthonormal bases known, in our analysis we use the fourth member of the Daubechies wavelets [22], shown in the inset of Fig. 2. The spiky form of this wavelet insures a strong correlation for the bursty events in the time series. The following method of analysis has also been tested with other wavelets and the results are qualitatively unchanged.

The importance of the wavelet transform in the study of turbulent signals lies in the fact that the large amplitude wavelet coefficients are related to the extreme events in the tails of the PDF, while the laminar or quiescent periods are related to the ones with smaller amplitude [21]. In this way it is possible to define a criterion whereby one can filter the time series of the coefficients depending on the specific needs. In our case we adopt the method used in Ref. [21] and originally proposed by Katul et al. [23]. In this method wavelet coefficients that exceed a fixed threshold are set to zero, according to

$$\tilde{W}_{j,n} = \begin{cases} W_{j,n} & \text{if } W_{j,n}^2 < C \cdot \langle W_{j,n}^2 \rangle_n, \\ 0 & \text{otherwise,} \end{cases} \quad (5)$$

here $\langle \dots \rangle_n$ denotes the average over the time parameters at a certain scale and C is the threshold coefficient. In the next section we will see that the precise value of the

parameter C is not critical for our analysis. However, it is possible to tune C such that only Gaussian noise is filtered. Once we have filtered the wavelet coefficients $\tilde{W}_{j,n}$ we perform an inverse wavelet transform, obtaining a *smoothed* version, Fig. 3(b), of the original time series, Fig. 3(a). The residuals of the original time series with the filtered one correspond to the bursty periods which we aim to study, Fig. 3(c).

3. Data analysis

In the previous section we have introduced the wavelet method in order to distinguish periods of high activity and periods of low or noise-like activity. The results are shown in Fig. 3 for $C = 2$. In order to choose an appropriate cut-off for the wavelet energy, that is to fix a proper C , we tune this parameter until the statistics on the kurtosis and the skewness of the filtered time-series approach the noise levels, namely 3 and 0, respectively. Once we have isolated the noise part of the time series we are able to perform a reliable statistical analysis on the *coherent events* of the residual time series, Fig. 3(c). In particular, we define coherent events as the periods of the residual time series in which the volatility, $v(t) \equiv |R(t)|$, is above a small threshold, $\varepsilon \approx 0$. The smoothing procedure is emphasized by the change in the PDFs before and after the filtering—as shown in Fig. 2. From this plot it is clear how the broad tails, related to the high-energy events that we want to study, and the associated central peak are cut-off by the filtering procedure. The filtered time series is basically a Gaussian, related to a noise process.

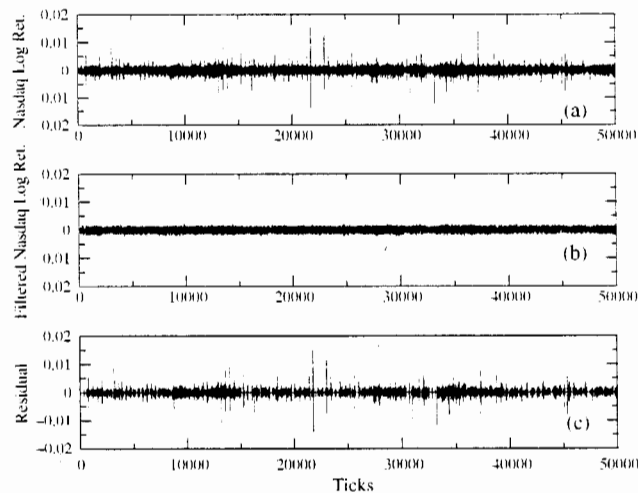


Fig. 3. A sample of the original time series of logarithmic returns for the Nasdaq100 is shown in (a), same as Fig. 1(b). The filtered version is shown in (b). The noise-like behaviour of this time series is evident. The residual time series is shown in (c). This corresponds to the high activity periods of the time series, related to the broad wings of the PDF. The cut-off parameter in this case is $C = 2$.

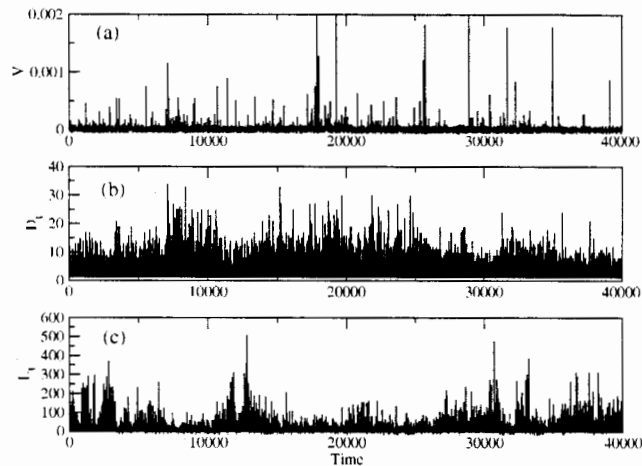


Fig. 4. Time series for the avalanche volumes, V , for the Nasdaq100. (a); duration, D_t , of the avalanches. (b); and laminar times, L_t (c). The plots are obtained using $C = 2$ as the filtering parameter.

A parallel between avalanches in the classical sandpile models (BTW models) exhibiting SOC [1] and the previously defined coherent events in the stock market is straightforward. In order to test the relation between the two, we make use of some properties of the BTW models. In particular, we use the fact that the avalanche size distribution and the avalanche duration are distributed according to power laws, while the laminar, or waiting times between avalanches are exponentially distributed, reflecting the lack of any temporal correlation between them [24,25]. This is equivalent to stating that the triggering process has no memory.

Similar to the dissipated energy in a turbulent flow, we define an avalanche, V , in the market context as the integrated value of the squared volatility, over each coherent event of the residual time series. The duration, D_t , is defined as the interval of time between the beginning and the end of a coherent event, while the laminar time, L_t , is the time elapsing between the end of an event and the beginning of the next one. The time series for V , D_t and L_t are plotted in Fig. 4 for $C = 2$.

The results for the statistical analysis for the Nasdaq100 index are shown in Figs. 5, 6 and 7, respectively, for the avalanche size, duration and laminar times. The robustness of our method has been tested against the energy threshold. we perform the same analysis with different values of C .

A power law relation is clearly evident for all the quantities investigated, largely independent of the specific value of C . At this point is important to stress the difference in the distribution of laminar times between the BTW model and the data analyzed. As explained previously, the BTW model shows an exponential distribution for the latter, derived from a Poisson process with no memory [24,25]. The power law distribution found for the stock market instead implies the existence of temporal correlations between coherent events. This empirical result rules-out the hypothesis that the stock market is in a SOC state, at least in relation to the classical sandpile models.

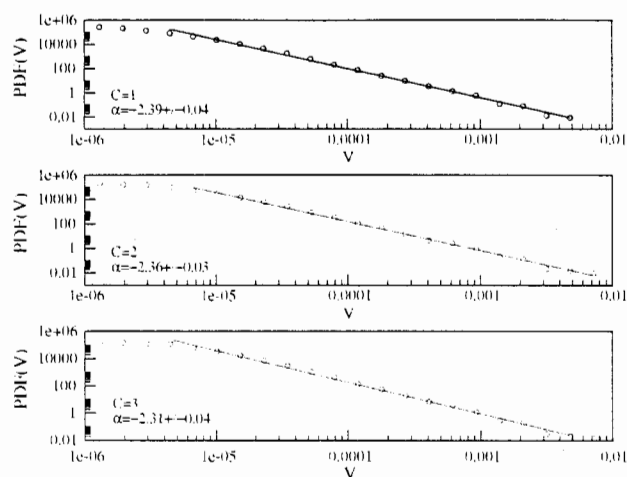


Fig. 5. Probability distribution function for the avalanche sizes tested against several values of C . The power law behaviour is robust with respect to this parameter.

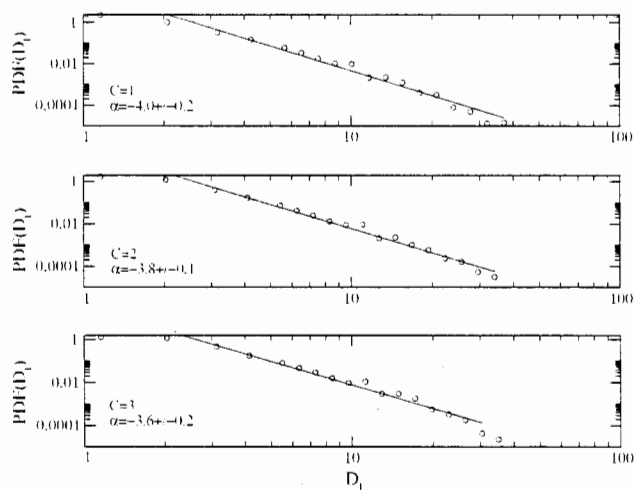


Fig. 6. The duration of the high activity periods show a power law distribution, independent of the value of C .

In order to extend the study of the avalanche behaviour to different markets, we perform the same analysis over the 30 min returns for the S&P ASX50. The results are shown in Figs. 8–10. While the power law scaling for the laminar times is still very clear, the power law for the other quantities is less precise, perhaps reflecting a different underlying dynamics compared to the Nasdaq100. On the other hand, it is also important to stress the difference in length of the two time series analyzed.

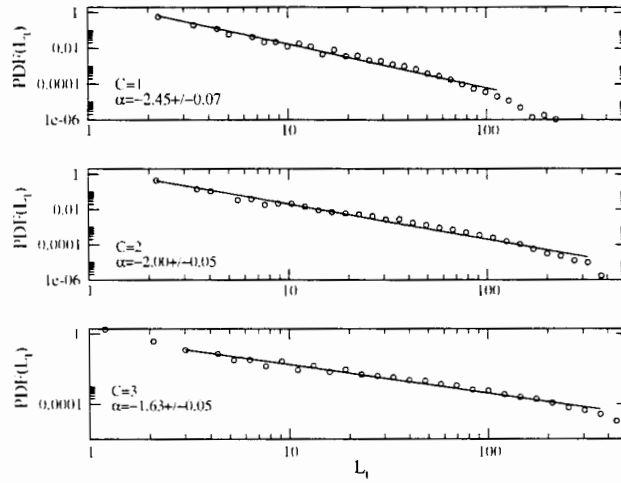


Fig. 7. Power law distribution of laminar times for different values of C.

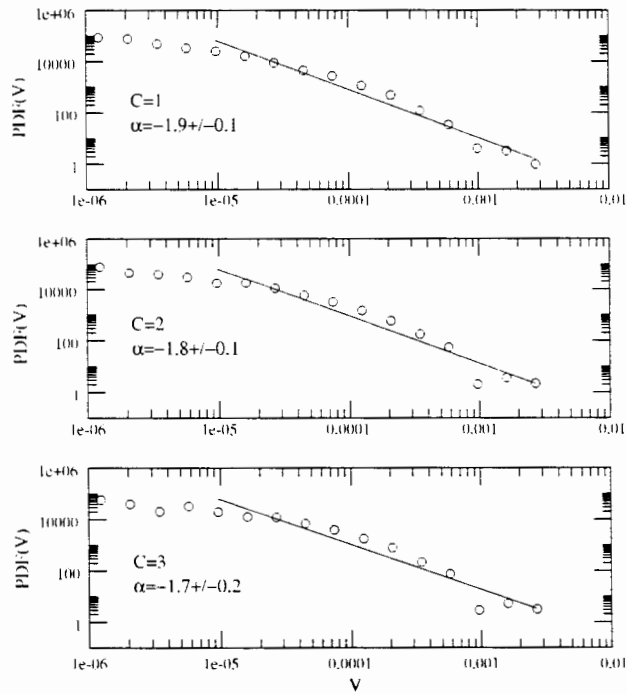


Fig. 8. Probability distribution function for the avalanche sizes for the S&P ASX50, from 20/1/1988 to 1/5/2002.

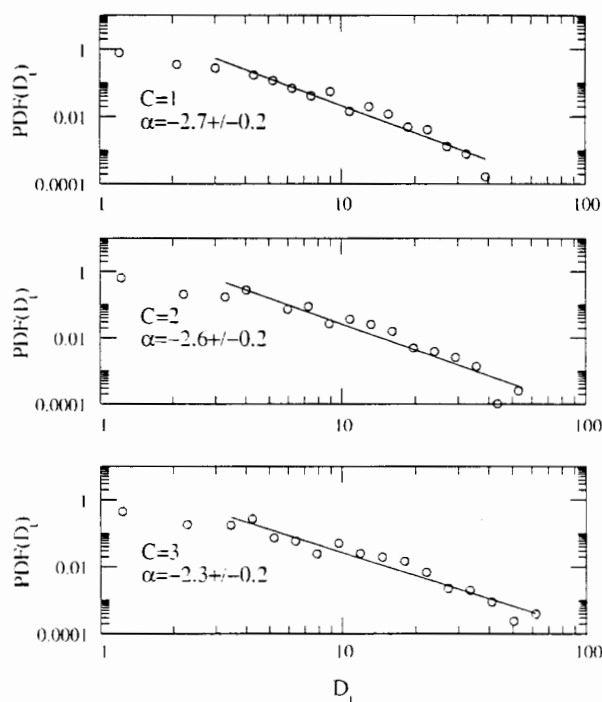


Fig. 9. Distribution of the duration of the *coherent events* for the S&P ASX50.

While for the Nasdaq100 we used 2^{19} data points, only 2^{14} were available for the S&P ASX50, making the first study statistically more reliable.

We also investigate the possibility of differences between high-frequency data and daily closures by considering a sample of 2^{14} daily closures of the Dow Jones index, from 2/2/1939 to 13/4/2004. The power law behaviour is consistent with that found for the high-frequency data, as shown in Figs. 11–13.

4. Discussion

Similar power law behaviour for V , D_t and L_t has been found in the context of solar flaring [24] and in geophysical time series [21]. In the case of solar flaring, Boffetta et al. [24] have shown that the characteristic distributions found empirically are more similar to the dissipative behaviour of the shell model for turbulence [26,27] than to SOC. On the other hand the intermittency in turbulent flows discussed in Section 2 is believed to be the result of a non-linear energy cascade that generates non-Gaussian events at small scales [17] where the shape of the PDF is extremely leptokurtic. At larger scales the spatial correlation decreases and the PDF converges toward a Gaussian. These features imply the absence of global self-similarity—which, as we have noted, is an intrinsic component of SOC models [18].

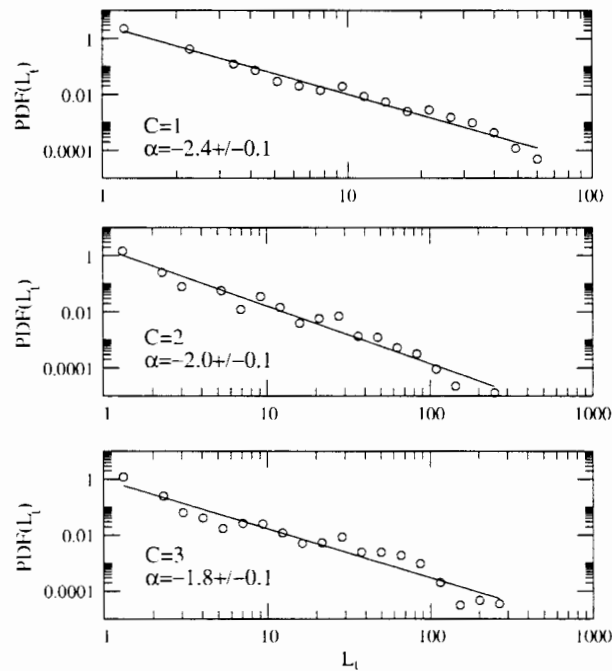


Fig. 10. Distribution of laminar times for S&P ASX50 index.

Freeman et al. [28] have argued that, although an exponential distribution holds for classical sandpile models, there exist some non-conservative modifications of the BTW models in which departures from an exponential behaviour for the L_t distribution [29–32] are observed in the presence of scale-free dynamics for other relevant parameters. The question remains whether or not these systems are still in a SOC state [28]. If we assume that the power law scaling of the laminar times corresponds to a breakdown of self-organized criticality, then we face the problem of how to explain the observed scale-free behaviour of the non-conservative models. This ambiguity can be resolved if we assume that the system is in a *near-SOC* state, that is the scaling properties of the system are kept even if it is not exactly critical and temporal correlations may be present [28,33]. Another possible scenario is related with the existence of temporal correlations in the driver [34,35]. In this case the power law behaviour of the waiting time distribution would be explained and the realization of a SOC state preserved [34,35].

5. Conclusions

In the present work we have investigated empirically the possible relations between the theory of self-organized criticality and the stock market. The existence

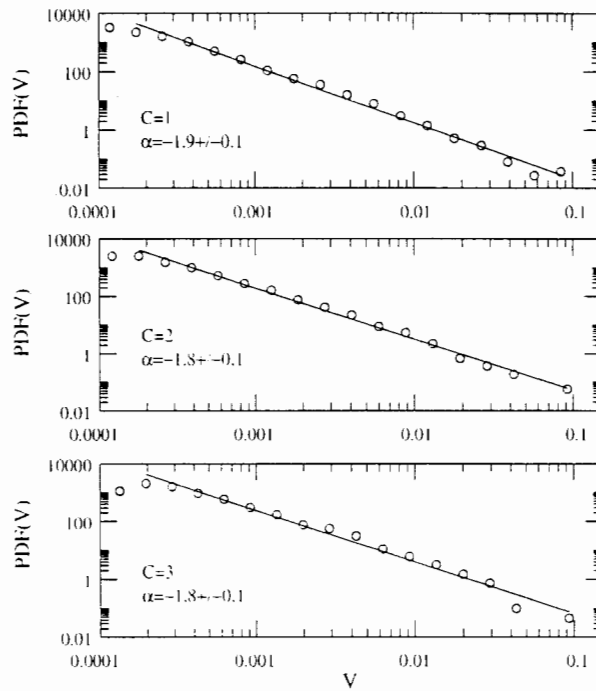


Fig. 11. Probability distribution function for the avalanche sizes for the Dow Jones daily closures, from 2/2/1939 to 13/4/2004.

of a SOC state for the market would be of great theoretical importance, as this would impose some constraints on the dynamics of this complex system. A bounded attractor in the state space would be implied. Moreover, we would have a better understanding of business cycles.

From the wavelet analysis on a sample of high-frequency data for the Nasdaq100 index, we have found that the behaviour of high activity periods, or avalanches, is characterized by power laws in the size, duration and laminar times. The power laws in the avalanche size and duration are a characteristic feature of a critical underlying dynamics in the system, but this is not enough to claim the self-organized critical state. In fact, the power law behaviour in the laminar time distribution implies a memory process in the triggering driver that is absent in the classical BTW models, where an exponential behaviour is expected. This does not rule out completely the hypothesis of underlying self-organized critical dynamics in the market. Non-conservative systems, as for the case of the stock market, near the SOC state can still show power law dynamics even in presence of temporal correlations of the avalanches [28,33]. Another possible explanation is that the memory process, possibly chaotic, is intrinsic in the driver. In this case the power law behaviour of the waiting time distribution would be explained and the realization of a SOC state preserved [34,35].

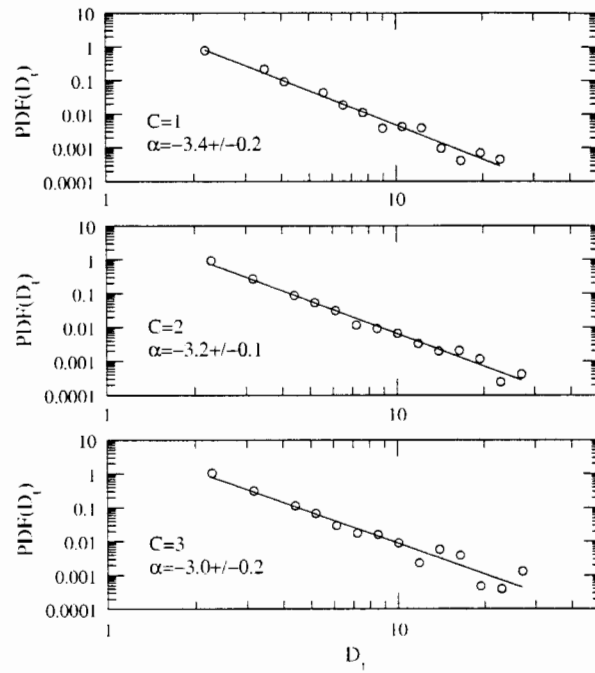


Fig. 12. Distribution of the duration of the Dow Jones index.

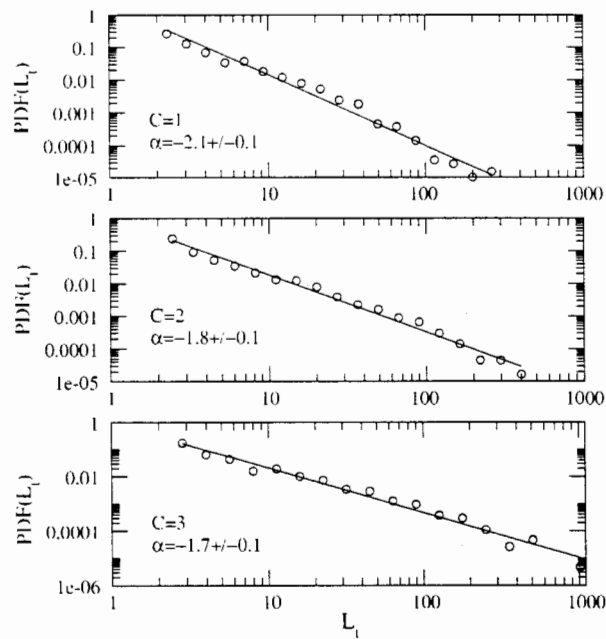


Fig. 13. Distribution of laminar times for the Dow Jones index.

These findings extend beyond the Nasdaq100 index analysis. A similar quantitative behaviour has been observed in the S&P ASX50 high-frequency data for the Australian market and the daily closures of the Dow Jones index for the American market. At this point it is important to stress that this does not imply that all the markets must display the same identical characteristics. In the case of a near-SOC dynamics, for example, the power law shape of the distribution can be influenced by the degree of dissipation of the system which may change from market to market, implying a non-universal behaviour.

In conclusion, a definitive relation between SOC theory and the stock market has not been found. Rather, we have shown that a memory process is related with periods of high activity. The memory could result from some kind of dissipation of information, similar to turbulence, or possibly a chaotic driver applied to the self-organized critical system. Of course, a combination of the two processes can also be possible. Our future work will be devoted to the study of new tests for self-organized criticality and the implementation of numerical models [36].

Acknowledgements

This work was supported by the Australian Research Council.

References

- [1] P. Bak, et al., *Phys. Rev. Lett.* 59 (1987) 381;
P. Bak, et al., *Phys. Rev. A* 38 (1988) 364.
- [2] H.J. Jensen, *Self-Organized Criticality: Emergent Complex Behavior in Physical and Biological Systems*, Cambridge University Press, Cambridge, 1998.
- [3] D.L. Turcotte, *Rep. Prog. Phys.* 62 (1999) 1377.
- [4] E.T. Lu, R.J. Hamilton, *Astrophys. J.* 380 (1991) L89;
E.T. Lu, et al., *Astrophys. J.* 412 (1993) 841.
- [5] T. Chang, et al., in: *Advances in Space Environmental Research*, vol. I, Kluwer Academic Publisher, AH Dordrecht, The Netherlands, 2003;
A. Valdivia, et al., in: *Advances in Space Environmental Research*, vol. I, Kluwer Academic Publisher, AH Dordrecht, The Netherlands, 2003.
- [6] P. Bak, C. Tang, *J. Geophys. Res.* 94 (15) (1989) 635;
A. Sornette, D. Sornette, *Europhys. Lett.* 9 (1989) 197;
D. Sornette, et al., *J. Geophys. Res.* 95 (17) (1990) 353;
J. Huang, et al., *Europhys. Lett.* 41 (1998) 43.
- [7] P. Bak, K. Snappen, *Phys. Rev. Lett.* 71 (1993) 4083.
- [8] K. Nagel, H.J. Herrmann, *Physica A* 199 (1993) 254;
K. Nagel, M. Paczuski, *Phys. Rev. E* 51 (1995) 2909;
T. Nagatani, *J. Phys. A* 28 (1995) L119;
T. Nagatani, *Fractals* 4 (1996) 279.
- [9] D.C. Roberts, D.L. Turcotte, *Fractals* 6 (1998) 351.
- [10] P. Bak, et al., *Ric. Econ.* 47 (1993) 3.
- [11] P. Bak, et al., *Physica A* 246 (1997) 430.
- [12] J. Feigenbaum, *Rep. Prog. Phys.* 66 (2003) 1611.

- [13] R.N. Mantegna, H.E. Stanley, *An Introduction to Econophysics: Correlation and Complexity in Finance*, Cambridge University Press, Cambridge, 1999;
- J.-P. Bouchard, M. Potters, *Theory of Financial Risk*, Cambridge University Press, Cambridge, 1999.
- [14] W. Paul, J. Baschnagel, *Stochastic Processes: From Physics to Finance*, Springer, Berlin, 1999.
- [15] S. Ghashghaie, et al., *Nature* 381 (1996) 767.
- [16] R.N. Mantegna, H.E. Stanley, *Physica A* 239 (1997) 225.
- [17] U. Frisch, *Turbulence*, Cambridge University Press, Cambridge, 1995.
- [18] V. Carbone, et al., *Europhys. Lett.* 58 (3) (2002) 349.
- [19] M. Farge, *Annu. Rev. Fluid Mech.* 24 (1992) 395.
- [20] M. Farge, et al., *Phys. Fluids* 11 (1999) 2187.
- [21] P. Kovács, et al., *Planet. Space Sci.* 49 (2001) 1219.
- [22] I. Daubechies, *Comm. Pure Appl. Math.* 41 (7) (1988) 909.
- [23] G.G. Katul, et al., *Wavelets in Geophysics*, Academic, San Diego, CA, 1994, pp. 81–105.
- [24] G. Boffetta, et al., *Phys. Rev. Lett.* 83 (1999) 4662.
- [25] M.S. Wheatland, et al., *Astrophys. J.* 509 (1998) 448.
- [26] T. Bohr, et al., *Dynamical System Approach to Turbulence*, Cambridge University Press, Cambridge, 1998.
- [27] P. Giuliani, V. Carbone, *Europhys. Lett.* 43 (1998) 527.
- [28] M.P. Freeman, et al., *Phys. Rev. E* 62 (2000) 8794.
- [29] K. Christensen, Z. Olami, *Geophys. Res. Lett.* 97 (1992) 8729.
- [30] Z. Olami, K. Christensen, *Phys. Rev. A* 46 (1992) R1720.
- [31] T. Hwa, M. Kardar, *Phys. Rev. A* 45 (1992) 7002.
- [32] H.F. Chau, K.S. Cheng, *Phys. Rev. A* 46 (1992) R2981.
- [33] J.X. Carvalho, C.P.C. Prado, *Phys. Rev. Lett.* 84 (2000) 4006.
- [34] P. De Los Rios, et al., *Phys. Rev. E* 56 (1997) 4876.
- [35] R. Sanchez, et al., *Phys. Rev. Lett.* 88 (2002) 068302-1.
- [36] M. Bartolozzi, A.W. Thomas, *Phys. Rev. E* 69 (2004) 046112.

Bartolozzi, M., Drozd, S., Leinweber, D.B., Speth, J., & Thomas, A.W. (2005) Self-similar log-periodic structures in western stock markets from 2000.
International Journal of Modern Physics C v. 16 (9) pp. 1347-1361

NOTE: This publication is included in the print copy of the thesis held in the University of Adelaide Library.

It is also available online to authorised users at:

<http://dx.doi.org/10.1142/S0129183105007972>

Bartolozzi, M., Leinweber, D.B., & Thomas, A.W. (2005) Stochastic opinion formation in scale-free networks.

Physical Review E v. 72 pp. 046113-1 - 046113-10

NOTE: This publication is included in the print copy of the thesis held in the University of Adelaide Library.

It is also available online to authorised users at:

<http://dx.doi.org/10.1103/PhysRevE.72.046113>



Symbiosis in the Bak–Sneppen model for biological evolution with economic applications

M. Bartolozzi^{a,b,*}, D.B. Leinweber^{a,b}, A.W. Thomas^{a,b,c}

^aSpecial Research Centre for the Subatomic Structure of Matter (CSSM), University of Adelaide, Adelaide, SA 5005, Australia

^bDepartment of Physics, University of Adelaide, Adelaide, SA 5005, Australia

^cJefferson Laboratory, 12000 Jefferson Ave., Newport News, VA 23606, USA

Received 5 April 2005; received in revised form 26 August 2005

Available online 21 October 2005

Abstract

In the present work we extend the Bak–Sneppen model for biological evolution by introducing local interactions between species. This “environmental” perturbation modifies the intrinsic fitness of each element of the ecology, leading to higher survival probability, even for the less fit. While the system still self-organizes toward a critical state, the distribution of fitness broadens, losing the classical step-function shape. A possible application in economics is discussed, where firms are represented as evolving species linked by mutual interests.

© 2005 Elsevier B.V. All rights reserved.

Keywords: Complex systems; Evolution/extinction; Self-organized criticality; Econophysics

In the past two decades several studies have been devoted to the investigation of the ubiquitous presence of power laws in natural and social systems. An important contribution to this field of research has been given by Bak, Tang and Wiesenfeld (BTW) [1,2], who developed the concept of self-organized criticality (SOC). The key idea behind SOC is that complex systems, that is systems constituted of many interacting elements, although obeying different microscopic physics, may exhibit similar dynamical behaviour, statistically described by the appearance of power laws in the distributions of their characteristic features. The lack of a characteristic scale, indicated by the power laws, is equivalent to those of physical systems during a phase transition—that is, at the critical point. It is worth emphasizing that the original idea [1,2] was that the critical state is reached “naturally”, without any external tuning. This is the origin of the adjective *self*-organized. In reality a certain degree of tuning is necessary: implicit tunings like local conservation laws and specific boundary conditions seem to be important ingredients for the appearance of power laws [3].

The classical example of a system exhibiting SOC behaviour is the two dimensional sandpile model [1–4]. Here the cells of a grid are randomly filled, by an external driver, with “sand”. When the gradient between two adjacent cells exceeds a certain threshold a redistribution of the sand occurs, leading to more instabilities and further redistributions. The avalanche dynamics that drives the system from one metastable state to another is

*Corresponding author. CSSM, Rm. 126, Lvl 1 Physics Building, University of Adelaide, SA 5005, Australia.

E-mail address: mbartolo@physics.adelaide.edu.au (M. Bartolozzi).

the benchmark of all systems exhibiting SOC. In particular, the distribution of the avalanche sizes, their duration and the energy released, all obey power laws.

The framework of self-organized criticality has been claimed to play an important role in solar flaring [5,6], space plasmas [7,8] and earthquakes [9–13] in the context of both astrophysics and geophysics. In biology, SOC has been linked to the punctuated equilibrium [12] in species evolution [14]. Some work has also been carried out in the social sciences. In particular, traffic flow and traffic jams [15–18], wars [19], as well as stock-market [4,20–22] dynamics, have been studied. A more detailed list of subjects and references related to SOC can be found in the review paper of Turcotte [4].

In the present work, we extend the Bak–Sneppen (BS) model for evolution [14] by introducing explicit coupling terms in the fitness of each species of the ecology. We find that the equilibrium configuration of the model can be deeply influenced by the environmental forces, leading to a wider survival probability also for species with a lower degree of adaptation. A possible application of our extension of the BS model to the economic world is that the distribution of global-fitness can be related to the size distribution of firms in the most developed markets. In this respect, the evolution of firms is seen as a punctuated equilibrium process in which the convolution of mutual interest can justify the spreading in size of the firms themselves.

The toy model proposed in 1993 by Bak and Sneppen [14] is one of the most popular models for biological evolution. The main idea behind this model is that each species can be uniquely characterized by a single parameter called *fitness*. The fitness of a species represents its degree of adaptation with respect to the external environment. Highly adapted species will hardly undergo any successful, spontaneous mutations. At the opposite end of the scale, if a species has a very low degree of fitness it needs to mutate in order to survive and its mutation automatically influences the other species belonging to the same environment. These concepts can be easily formulated as a simple one dimensional model. Suppose that the ecology can be represented by a periodic array of N cells and each cell, i , is assigned a fitness, $B(i)$, taken from a uniform distribution between 0 and 1. Once we have fixed the initial condition, for each discrete time-step, the dynamical evolution of the system works as follows:

- (a) locate the species with minimum fitness—that is, the one most likely to mutate, k ,
- (b) change the fitness of k and that of its neighbours (species related) according to

$$\begin{aligned} B(k-1) &\rightarrow u_1, \\ B(k) &\rightarrow u_2, \\ B(k+1) &\rightarrow u_3, \end{aligned} \tag{1}$$

where the new fitness value, u_i , is a random number taken from a uniform distribution bounded between 0 and 1.

From numerical [14] and analytical [23] studies it has been shown that the values of the fitness evolve to a step function, in the thermodynamic limit ($N \rightarrow \infty$), characterized by a single value, B_c . For $B < B_c$ the distribution of fitness, $P(B)$, is uniformly equal to zero while for $B > B_c$ we have $P(B) = 1/(1 - B_c)$, determined by the normalization condition. An example is shown in Fig. 1(a) and (b).

In this model, it is also possible to define a burst-like avalanche dynamics. Suppose we fix a threshold for the fitness, B_0 and consider $B_m(t)$ as the minimal fitness at time step t . If at a certain time step, t_1 , it happens that $B_m(t_1) < B_0$ then we can measure the interval of time, T , needed for having again $B_m(t_1 + T) > B_0$. In this case an avalanche of duration, or size, T has taken place in order to restore a minimal fitness in the system. If $B_0 = B_c$ then we have $P(T) \sim T^{-\gamma}$: the system is critical, see Fig. 1(c).

According to this model the great mass extinctions of species, like dinosaurs for example, can be explained in terms of burst-like dynamics. A small perturbation in a critical self-connected system can trigger a chain reaction that may influence a great part of the species in the ecosystem. Time series of fossils samples seem to be in agreement with this avalanche dynamics in the extinction/evolution of species. A more detailed discussion of the BS model goes beyond the scope of the present work. For a general review see Ref. [24].

Despite its simplicity, the BS model evolves according to a complex dynamics and it is able to explain some empirical features of the biological evolution [14]. An implicit assumption in the model is that every species is

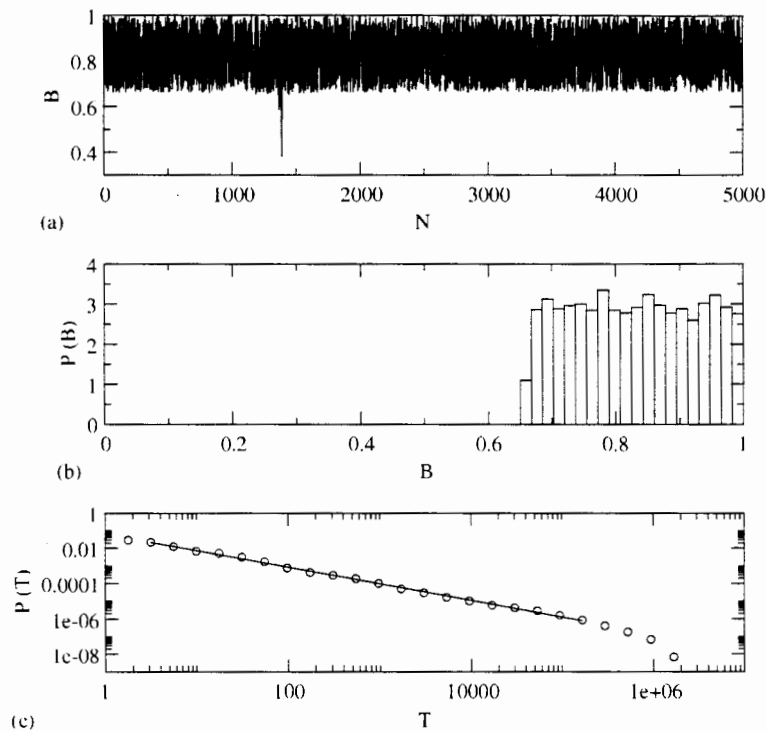


Fig. 1. (a) Snapshot of the fitness, B , after 8×10^7 time steps in a stable configuration. All the values are above a critical threshold, except for those around 1500, where an avalanche is clearly visible. (b) Probability distribution of the fitness, $P(B)$, shown in (a). While the distribution is equal to zero on the left hand side, a plateau is evident for $B > B_c \sim 0.667$. In the thermodynamic limit the plateau will become exactly a constant. In this case the simulation has been carried out for $N = 5000$ species. (c) Probability distribution, $P(T)$, of avalanche duration, T , in the stationary regime for the Bak–Sneppen model. The avalanche time series has been recorded in the stationary state of the system, after $\sim 10^7$ iterations for $N = 2000$ species. The power law exponent is $\gamma \sim 1$.

deeply connected to its environment. A mutation on a single element automatically triggers a mutation in its neighbours.

But is this approximation always appropriate? Consider, for example, three species in a one dimensional array and suppose that $B_i = B_{min}$ while $B_{i-1} \cong B_{i+1} \cong 1$. In the standard BS model the i th cell undergoes to mutation that also triggers a change in B_{i-1} and B_{i+1} . From a biological point of view it means that two extremely well-adapted species have to mutate in order to cope with the mutation of the i th species. This can be interpreted as a very particular (pessimistic) case—such as, for example, the case where the i th species is the main source of food for both the other species.

In order to stress this idea, we use some examples from different areas in which a similar evolutionary dynamics can be applied. Suppose that a new unfit or unskilled player joins a strong team. Will this player trigger a regression in the team performance or will the team compensate for this lack of skills? This is a small perturbation after all.

Another example comes from economics. In this case, it has been shown [25–27], that the dynamics of different firms is correlated. In fact, it is not unusual for a company to own large amounts of stock of other companies and so on. The result is an entangled environment, where the evolution of a firm is, in a way, linked to the evolution of its network of interaction. Is it then possible, in this case, for an wealthy environment to sustain an unfit element, or will its lack of “fitness” bring to the brink of the financial collapse all the other partners, as the BS model would suggest?

We provide an answer to these questions using a modified version of the BS model that takes into account the feedback of the environment on the single element. We refer to this model incorporating local interactions in the BS model as the LIBS model. For the sake of simplicity we do not consider the topology of the

interaction, that may be very complex; rather, we use a simple one dimensional array. The influence of the network structure on the dynamics of the model will be discussed in our future work.

As a first approximation we consider our species to be arranged on a one dimensional array with nearest neighbour interactions. This means that the micro-environment is composed of three cells. The value of the fitness, B , for each cell is taken, according to the BS model, from a uniform distribution between zero and one. The fitness parameter, B_i , of the i th cell represents the *self-fitness* of the species. Motivated by the aforementioned examples, we add an environmental contribution to the self-fitness that leads to a *global-fitness*, F_i , according to

$$F_i = B_i + A_{i,i-1}B_{i-1} + A_{i,i+1}B_{i+1}, \quad (2)$$

where $A_{i,i-1}$ and $A_{i,i+1}$ are the fractions of fitness that the i th cell shares with its neighbours. The matrix of A s is not symmetric, reflecting the fact that the contribution in one direction can be very different from the contribution in the other. This is equivalent to considering a directed *weighted graph* with a trivial necklace topology.

In the sport example, the global fitness corresponds to the fit players that contribute to sustaining the unskilled team-mate. From the economic point of view, it represents the capability of a firm to gain benefits from its partnerships with other firms. In this particular case, B_i represents the wealth generated by the firm itself, while the other two terms represent the contribution, in different forms, from the linked firms. In general, we can consider the new terms in the definition of F_i as short ranged random forces acting on the i th cell.

At the beginning of the simulation, the self-fitness is drawn from a uniform distribution between zero and one. The same is done for the link weights, A_{ij} . It is worth emphasizing that, in general, for two cells i and j , $A_{ij} \neq A_{ji}$.

Assuming that the neighbours can cooperate in defining the fitness of a species (optimistic view), the extremal dynamics is moved from B_{min} to F_{min} . Once the site with minimum global fitness, k , is located, then the self-fitness and the interactions of this species are redrawn according to the following rules:

$$\begin{aligned} A_{k,k-1} &\rightarrow u_1, \\ A_{k-1,k} &\rightarrow u_2, \\ B(k) &\rightarrow u_3, \\ A_{k,k+1} &\rightarrow u_4, \\ A_{k+1,k} &\rightarrow u_5, \end{aligned} \quad (3)$$

where the new values for the changed quantities are taken from a uniform distribution between zero and one, as in the BS model. However, in contrast to the BS model, a change in the fitness of the k th species does not automatically trigger a change in the neighbours. Only the interactions are changed.

In order to test the stability of the model, we monitor the *average fitness* and the *gap function*, $G(t)$, for both B and F . The gap function is nothing but the tracking function of the minimum of $B_{min}(t)$ (or $F_{min}(t)$). At $t = 0$ we have $G(0) = B_{min}(0)$ (or $G(0) = F_{min}(0)$). As the evolution proceeds eventually for a certain t_1 , we will have $B_{min}(t_1) > B_{min}(0)$ (or $F_{min}(t_1) > F_{min}(0)$) as the minimum values are converging toward the critical value. The gap function is then updated as $G(t_1) = B_{min}(t_1)$ (or $G(t_1) = F_{min}(t_1)$) and so on. It is easy to see that in the stationary state the gap function converges toward the critical value.¹

¹For simulations on finite BS systems, a perfectly stationary state can never be achieved during a finite number of mutations. This drawback, discussed in Refs. [28–30], is due to spurious correlations in the dynamics of the avalanches induced by the finiteness of the lattice as $G(t)$ gets closer to the critical value and, therefore, their average duration is supposed to diverge. As soon as we get very close to this point, an artefact regime sets in and the gap function starts to saturate toward $B = 1$. The phase in which $G(t) \sim B_c$ can be regarded as a transition point for the physically meaningful state: the larger the system is, that is, the closer it is to the thermodynamic limit, the slower is the drift from this point and the system can be regarded, to a good approximation, as stable. An accurate study of this phenomenon in relation to the LIBS model, although very interesting, is not of fundamental importance in the context of the present work. Therefore, we will consider the system to be stable as soon as the gap function and the average reach a plateau.

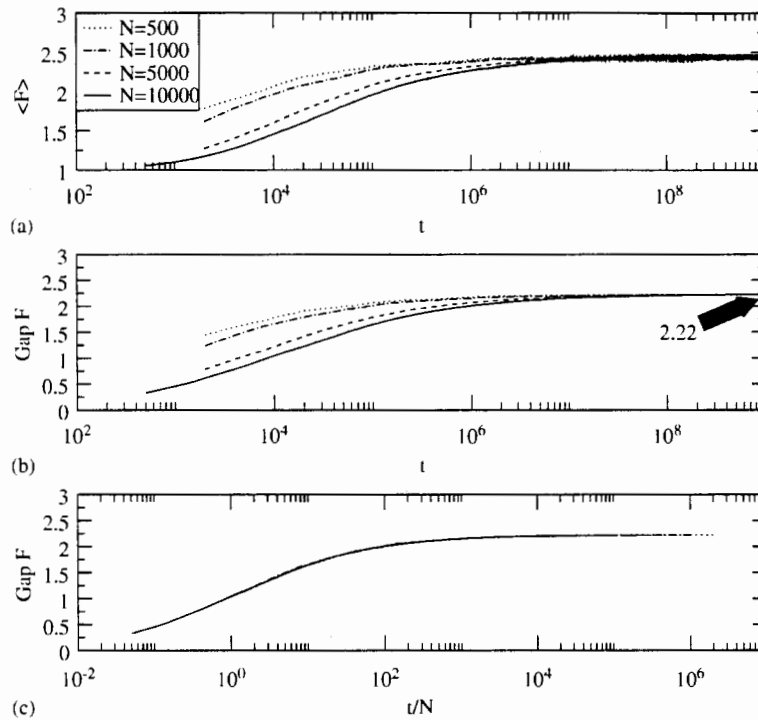


Fig. 2. (a) Average value ($\langle \dots \rangle$) and (b) gap (Gap) function for the global fitness F and different system sizes for simulations up to 10^9 mutations, t . Note that the gap function converges approximately to the value $F_c \sim 2.22$ that corresponds to the critical threshold for this model. A simple rescaling of the time, $t \rightarrow t/N$, collapses the curves onto a single curve as shown in (c).

In Fig. 2, the time series of average values and the gap function of F are plotted for different number of species in the ecology. The time to reach the stable state depends strongly on the size: for $N = 10^4$, the largest system in our simulations, we need approximately $t \sim 10^8$ mutations to achieve the equilibrium. Note also that a simple rescaling, $t \rightarrow t/N$, leads to a collapse of these curves. The relaxation times in the BS model are, approximately, one order of magnitude lower compared to the LIBS model of the same size (or in rescaled time).

A snapshot of the grid in the stable configuration is shown in Fig. 3. We notice immediately that the local fitness is no longer distributed like a step function (as for BS). Rather a long, exponential, tail of low fitness is evident, as shown in Fig. 4 (left). The cells with a higher local fitness still have a greater probability to survive but the global-fitness, or the presence of environmental partnerships, widens the possibility of survival, even for some species with a lower degree of self-fitness.

If we examine the global-fitness, a single avalanche is present—as in the classical BS model. Moreover, the probability distribution function for the avalanche duration, shown in Fig. 5 and computed with respect to F , is power law distributed, in relation with the criticality of the model. The index of the distribution turns out to be different from that of BS: the change in the dynamics has also led to a change in the universality class of the model.

The distribution of global-fitness, shown in Fig. 4 (right) differs significantly from the step-function of the BS model. It displays a polynomial decay (fourth order fit in the plot) above a critical threshold, as a result of the convolution of stochastic variables. A similar non-trivial distribution can also be found in the size distribution of firms, suggesting a possible practical application of the LIBS model.

Axtell [31] analysed the size distribution of US companies, defined as the total number of employees, during 1997. He found that it could be well represented by a Zipf distribution, $P(s) \sim s^{-\alpha}$, with $\alpha \sim 1$ and s being the size of the firm. Further investigation of this issue has been carried out by Gaffeo et al. [32], who analysed a database of companies for the G7 countries from 1987 to 2000. To some extent, this analysis confirmed the

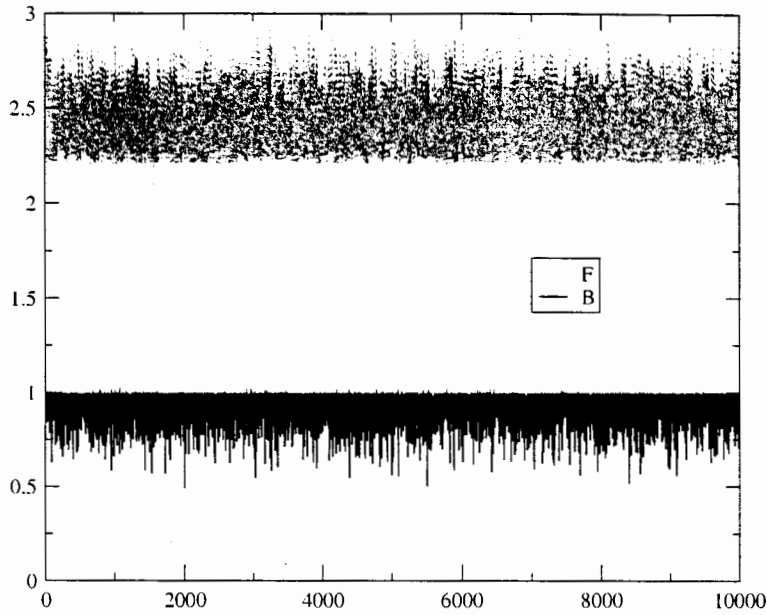


Fig. 3. Snapshot of the grid, $N = 10^4$, in a stable configuration for fitness B and global fitness F .

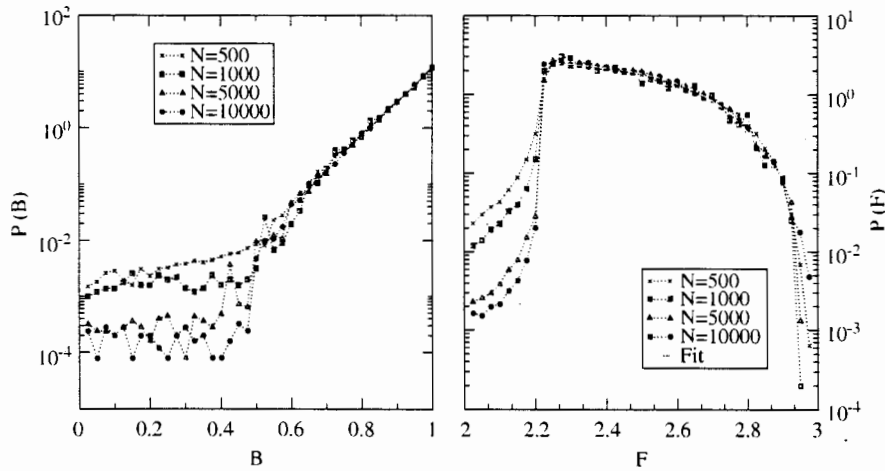


Fig. 4. Left: probability distribution function, $P(B)$, for fitness B . The step distribution of the standard BS model has been replaced with an exponential distribution with a cut-off at $B \sim 0.55$. Right: probability distribution, $P(F)$, for the global-fitness, F , and fit with a fourth order polynomial. In this case, a sharp threshold is visible, as in the standard BS model, indicating that poor fitness environments undergo mutation. As we consider larger N the transition, at $F_c \sim 2.22$, gets sharper and sharper as expected by a finite size analysis. The values of the $P(F)$ below this threshold are related to the recorded avalanches. The distributions shown in these plots are the results of an average over 50 different configurations in the stable regime.

findings of Axtell, namely a power law distribution with $\alpha \sim 1$. However, $\alpha \sim 1$ was obtained only for a particular definition of the firm size, and particular business periods. More generally, they found a robust power law behaviour. However, the index was seen to change with the time window analysed and the definition of firm size used.

The qualitative discrepancies between our $P(F)$ and the distribution empirically found for the size distribution of firms can be explained by allowing more complex topologies for interactions between species,

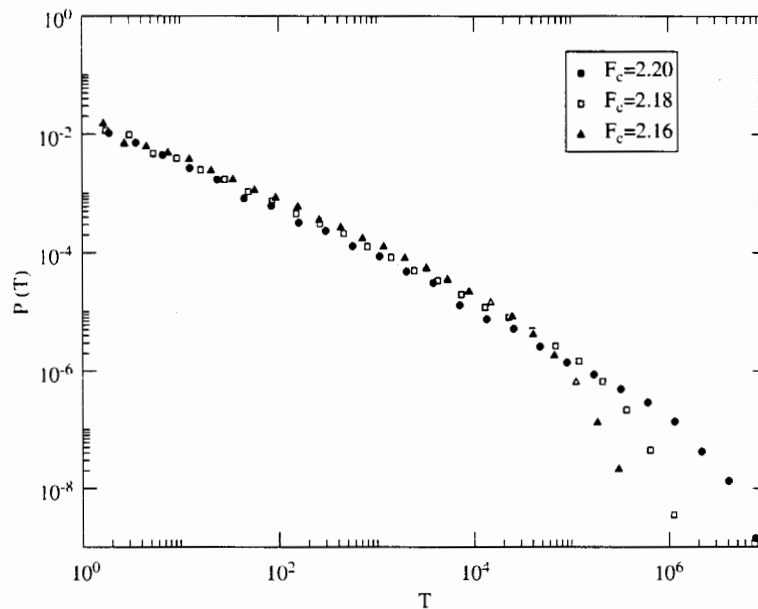


Fig. 5. Probability distribution, $P(T)$, of the avalanche duration, T , in the LIBS model. Three different thresholds during the stable regime are considered. The critical exponent is $\gamma \sim 0.83$. For $F_c = 2.20$ we are very close to the critical point and very large avalanches are present.

or economical entities in this case. Different kinds of convolution can generate a different shape in the distribution of the global-fitness. This can be easily deduced by writing Eq. (2) in a general form as

$$F_i = \sum_{j=1}^{k_i} A_{ij} B_{ij}, \quad (4)$$

where $A_{i,j} = 1$ and the sum over j is extended to all the k_i neighbours of the i th species. In Eq. (4), no particular topology is specified.

For an isotropic model on a D -dimensional lattice, k_i is equal for all the species and depends only on D and the definition of neighbourhood: the theoretical boundaries for F are equal for all the species and we can talk about a “democratic” model. However, recent studies have shown [33,34] that, in real biological and social systems the number of links per elementary unit, k , is not constant, but characterized by a non-trivial probability distribution function, $P(k)$, as a result of the complex nature of the interactions between species or individuals.² From Eq. (4), we can immediately see that, by adopting a complex network as the underlying structure for the interaction between species, we move to a model in which each species may have a different boundary value for the global-fitness, since $0 \leq F_i \leq k_i$. This inequality has a straightforward interpretation: species with a large number of connections will have a higher barrier against environmental changes because they can rely on numerous resources.

A simple way to obtain a complex network structure is by considering an open system, where the number of species is not fixed but grows in time, as for example, the firms in a dynamic economy. In this case, it will be more likely for new economic entities to be connected with a well established firm that has already a large number of connections (growth and preferential attachment are actually the two main ingredients for the Albert–Barabási model for scale-free networks [33]). According to our model these “hubs”, that is companies such as General Motors or Coca Cola, have a higher chance of surviving a turbulent period compared to

²Two widely studied networks in literature are *random* and *scale-free*, for which the degrees are, respectively, Poisson and power law distributed. Especially, the latter seem to describe quite well the topology of interactions in biological, social and technological systems. For a modern review of network theory see Refs. [33,34].

isolated nodes: they have a larger influence in the dynamics of the model.³ This simple consideration, although not exhaustive, shows how the underlying topology of the interactions can play an important role in the final distribution of the global fitness. Further analytical and numerical tests would be of great importance in order to understand the dynamics of the LIBS model and to what extent it can be applied to real systems.

It is also worth pointing out that another parameter related to $P(F)$ is the domain of values for B itself. In the present case we take B to be uniform in the interval $(0,1)$. In fact, a change in this distribution, while preserving the dynamics of the model, would lead to a different shape in the final distribution of F .

In summary, the quantitative shape of the distribution of the global fitness parameter depends deeply on the details of the self-fitness parameters and the network that defines their interactions. These important issues will be addressed in our future work.

The results obtained with the LIBS model confirm the relevance of self-organized criticality in complex systems and, in particular, economics. The concept of mutual cooperation, introduced via the global-fitness, can explain the ubiquity of broad tails in the distribution of characteristic quantities of physical and social systems in terms of a convolution of variables between elements of the network of interaction. In the economic context, this asymptotic behaviour can be related to the empirical findings concerning the distribution of the size of firms. The possible relevance of self-organized criticality in economics has already been suggested by recent theoretical and empirical studies [20,21,36,37], while possible applications of the BS model in this field can be found in Refs. [38–40]. The application of the SOC concept to social sciences can, in general, be motivated by empirical observations of the “intermittent” activity in the human dynamics at every level, from wars to revolutions and, in particular, intellectual production where moments of frenetic activity can alternate with long breaks, with lengths which cannot be predicted.

This process is, in a way, similar to the discharge, via avalanches, needed in the classical sandpile model, to restore the critical slope. In a real economic world a wide series of changes, similar to avalanches, can be triggered by exogenous or endogenous shocks related to structural changes at macroeconomic level, for example the creation and successive enlargement of the European community or the fall of the Soviet empire, or at microeconomic level, the invention of a new technology [41]. Since the shocks leading to avalanches are of different nature, we also expect the existence of different time scales involved in the self-organization process. In SOC systems, in general, the existence of a sharp separation between time scales, energy storage and relaxation, appears to be a strict prerequisite. In the BS model, as in the LIBS model, by mutating one unit at a time, we implicitly assume that the time to extinction, t_e , of a species depends exponentially on its global fitness, that is $t_e(i) \propto e^{F_i}$. The exponential separation of the extinction times is at the core of the “punctuation”. In economic terms we can still assume this behaviour: changes of poorly fitted firms can be simply related to small microeconomic fluctuations that can happen in time scales of the order of weeks or months while much longer times are needed to change the fitness of a highly adapted company. In the latter case, radical changes are needed, as for example a switch from one political regime to another, an event that may take centuries to happen.

On a final note, we consider now a further extension of the LIBS model by including second nearest-neighbour interactions in the simple one dimensional topology. The global-fitness of Eq. (2) becomes

$$F_i = B_i + A_{i,i-1}B_{i-1} + A_{i,i+1}B_{i+1} + A_{i,i-2}B_{i-2} + A_{i,i+2}B_{i+2}, \quad (5)$$

where the second order coefficients are not independent random numbers but are rather related via $A_{i,i\pm 2} = A_{i,i\pm 1} \cdot A_{i\pm 1,i\pm 2}$. The reason behind this choice, that can be easily extended to the n th order neighbours, is motivated by the assumption that the higher order interactions are damped by the “distance” between the two species and therefore $A_{i,i\pm 2} \leq A_{i,i\pm 1}$. By using this formulation, we attempt to mimic a hierarchical dependence of the global fitness in the ecology: species become explicitly related to their second nearest neighbours via the mediation of their first neighbours and so on. Using these constraints the mutation rules remain the same as in Eq. (3) since a change in the first order coefficients triggers automatically a change in the higher order ones.

³This situation of “freezing” of the large “hubs” has some analogies with spin glass theory where some species freeze in a random configuration leading to a rough landscape of energy levels at small temperatures [35].

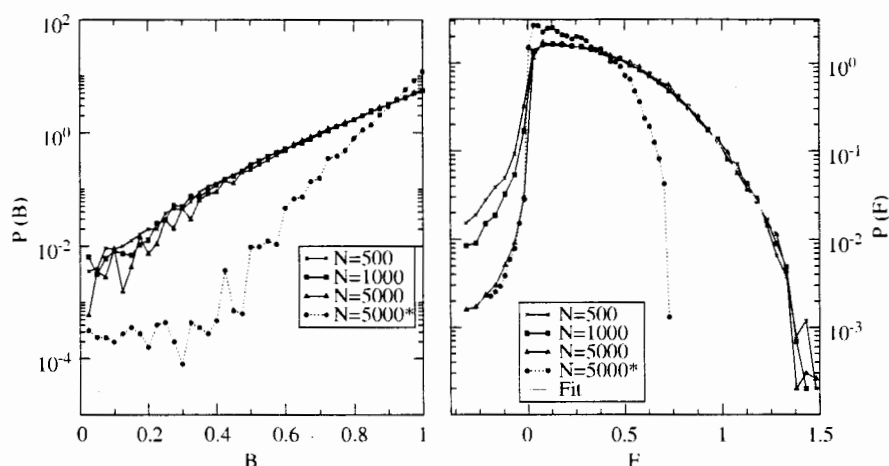


Fig. 6. Left: distribution of self-fitness, B , for the second order neighbours for various number of cells, N . The distribution for the first order neighbours is also plotted (*). Note that an extension of the neighbourhood leads to a slower decay rate. Moreover, a clear cut-off for low B is no longer evident. Right: distribution of global fitness F . In this case, the distributions have been rescaled according to $F \rightarrow F - F_c$. For the second order neighbours we have $F_c \sim 2.37$. The fit with a sixth order polynomial, also shown, produces a smaller χ^2 compared to the fourth order used for the first nearest neighbour case.

The dynamics that results from the numerical simulations is similar to the first-neighbour LIBS model. After an extensive transient we reach a critical stationary state characterized by avalanches of mutations with size, T , which are power law distributed. The distributions for B and F , in the stable regime, are shown in Fig. 6 along with the distribution obtained by considering just the first neighbour interaction.

In the case of B , we can notice that by enlarging the neighbourhood, the distributions show a slower decaying rate and they appear to be smoother. In this case, we have an exponential decay all the way down to zero, without any clear cut-off for low B . Despite their fitness, all the species have a chance of survival if sustained by a healthy environment. Regarding the global fitness, instead, a polynomial decay is still evident, although the order is higher compared to the first neighbours case. It is also important to note that a relatively large change in the theoretical range for F , for which the bounds are now $0 \leq F_i \leq 5$, does not lead to a consequent rise in the threshold value. It just moves slightly from $F_c \sim 2.22$ to $F_c \sim 2.37$. This is equivalent to saying that in the previous case, in order to be considered “fit”, a species had to exceed roughly 74% of the possible range for F . Now just 48% is sufficient! In conclusion, a hierarchical extension of the cooperation between species in the LIBS model leads to an easier adaptation and survival probability: the more compact the ecosystem is, the higher will be the chances of survival of each single species as long as they cooperate for their mutual interest.

In conclusion, we have extended the Bak–Sneppen model for biological evolution by introducing explicitly local interactions between elements of the ecology. Numerical simulations have shown how the dynamics of the model, while still leading to a self-organized critical state, can be largely effected by the environmental forces, leading to smoother distributions in both the intrinsic fitness, B , and the global fitness, F . As already pointed out by Grassberger [42] the BS model cannot be taken too seriously for describing the punctuated equilibrium of biological evolution. Nevertheless, because of its simplicity, it can easily be used as paradigm for other complex systems. In the present work, we suggest a possible application of our extension of the BS model to the economic world. In particular, the distribution of global-fitness can be related to the size distribution of firms in the most developed markets. In this respect, the evolution of firms is seen as a punctuated equilibrium process in which the convolution of mutual interest can justify the spreading in size of the firms themselves. We have emphasized that the actual shape of the distribution of global-fitness is related to the topology of the interaction. Future work will be devoted to the application of the model proposed in this paper to more complex topologies—as, for example, to scale-free networks [33,34], that better represent the real interactions between economic entities.

This work was supported by the Australian Research Council.

References

- [1] P. Bak, C. Tang, K. Wiesenfeld, Self-organized criticality: an explanation of $1/f$ noise, *Phys. Rev. Lett.* 59 (1987) 381.
- [2] P. Bak, C. Tang, K. Wiesenfeld, Self-organized criticality, *Phys. Rev. A* 38 (1988) 364.
- [3] H.J. Jensen, *Self-Organized Criticality: Emergent Complex Behavior in Physical and Biological Systems*, Cambridge University Press, Cambridge, 1998.
- [4] D.L. Turcotte, Self-organized criticality, *Rep. Prog. Phys.* 62 (1999) 1377.
- [5] E.T. Lu, R.J. Hamilton, Avalanches and the distribution of solar flares, *Astrophys. J.* 380 (1991) L89.
- [6] E.T. Lu, R.J. Hamilton, J.M. McTieran, K.R. Bromund, Solar flares and avalanches in driven dissipative systems, *Astrophys. J.* 412 (1993) 841.
- [7] T. Chang, et al., *Advances in Space Environmental Research*, vol. 1, Kluwer Academic Publisher, AH Dordrecht, The Netherlands, 2003.
- [8] A. Valdiva, et al., *Advances in Space Environmental Research*, vol. 1, Kluwer Academic Publisher, AH Dordrecht, The Netherlands, 2003.
- [9] P. Bak, C. Tang, Earthquakes as a self-organized critical phenomenon, *J. Geophys. Res.* 94 (15) (1989) 635.
- [10] A. Sornette, D. Sornette, Self-organized criticality and earthquakes, *Europhys. Lett.* 9 (1989) 197.
- [11] D. Sornette, P. Davy, A. Sornette, Structuration of the lithosphere in plate tectonics as a self-organized critical phenomenon, *J. Geophys. Res.* 95 (1990) 17353.
- [12] S.J. Gould, N. Eldredge, Punctuated equilibrium comes of age, *Nature* 366 (1993) 223.
- [13] J. Huang, H. Sauler, C. Sammis, D. Sornette, Precursors, aftershocks, criticality and self-organized criticality, *Europhys. Lett.* 41 (1998) 43.
- [14] P. Bak, K. Sneppen, Punctuated equilibrium and criticality in a simple model of evolution, *Phys. Rev. Lett.* 71 (1993) 4083.
- [15] K. Nagel, H.J. Herrmann, Deterministic models for traffic jams, *Physica A* 199 (1993) 254.
- [16] K. Nagel, M. Paczuski, Emergent traffic jams, *Phys. Rev. E* 51 (1995) 2909.
- [17] T. Nagatani, Self-organized criticality in 1D traffic flow model with inflow or outflow, *J. Phys. A: Math. Gen.* 28 (1995) L119.
- [18] T. Nagatani, Self-organized criticality in 1D traffic flow, *Fractals* 4 (1996) 279.
- [19] D.C. Roberts, D.L. Turcotte, Fractality and self-organised criticality of wars, *Fractals* 6 (1998) 351.
- [20] P. Bak, K. Chen, J. Scheinkman, M. Woodford, Aggregate fluctuations from independent sectoral shocks: self-organized criticality in a model of production and inventory dynamics, *Ric. Econ.* 47 (1993) 3.
- [21] P. Bak, M. Paczuski, M. Shubik, Price variations in a stock market with many agents, *Physica A* 246 (1997) 430.
- [22] J. Feigenbaum, Financial physics, *Rep. Prog. Phys.* 66 (2003) 1611.
- [23] H. Flyvbjerg, P. Bak, K. Sneppen, Mean field theory for a simple model of evolution, *Phys. Rev. Lett.* 71 (1993) 4087.
- [24] M. Paczuski, S. Maslov, P. Bak, Avalanche dynamics in evolution, growth and depinning models, *Phys. Rev. E* 53 (1996) 414.
- [25] N. Vanderwalle, F. Brisbois, X. Tordoir, Non-random topology of stock markets, *Quant. Finance* 1 (2001) 372.
- [26] G. Bonanno, G. Caldarelli, F. Lillo, R.N. Mantegna, Topology of correlation-based minimal spanning trees in real and model markets, *Phys. Rev. E* 68 (2003) 46130.
- [27] W. Souma, Y. Fujiwara, H. Aoyama, Ninth Workshop on Economics and Heterogeneous Interacting Agents, WEHIA, 2004, Kyoto University, Kyoto, Japan, May 27–29, 2004, preprint: physics/0502005.
- [28] D. Head, Temperature scaling, glassiness and stationarity in the Bak–Sneppen model, *Eur. Phys. J. B* 17 (2000) 289.
- [29] K. Tabelow, Gap function in finite Bak–Sneppen model, *Phys. Rev. E* 63 (2001) 047101.
- [30] D.A. Head, Universal persistence exponents in an extremally driven system, *Phys. Rev. E* 65 (2002) 027104.
- [31] R.L. Axtell, Zipf distribution of U.S. firm size, *Nature* 293 (2001) 1818.
- [32] E. Gaffeo, M. Gallegati, A. Palestrini, On the size distribution of firms: additional evidence from the G7 countries, *Physica A* 324 (2003) 117.
- [33] R. Albert, A.-L. Barabási, Statistical mechanics of complex networks, *Rev. Mod. Phys.* 74 (2002) 47.
- [34] S.N. Dorogovtsev, J.F.F. Mendes, Evolution of networks, *Adv. Phys.* 51 (2001) 1079.
- [35] M. Mezard, G. Parisi, M.A. Virasoro, *Spin Glass Theory and Beyond*, World Scientific, Singapore, 1987.
- [36] A. Ponzì, Y. Aizawa, Criticality and punctuated equilibrium in spin system model of financial market, *Chaos, Solitons Fractals* 11 (2000) 1739.
- [37] M. Bartolozzi, D.B. Leinweber, A.W. Thomas, Self-organized criticality and stock market dynamics: an empirical study, *Physica A* 350 (2005) 451.
- [38] G. Cuniberti, A. Valleriani, J.L. Vega, Effects of regulation on a self-organized market, *Quant. Finance* 1 (2001) 332.
- [39] T. Yamano, Regulation effects on market with Bak–Sneppen model in high dimensions, *Int. J. Mod. Phys. C* 9 (2001) 1329.
- [40] M. Ausloos, M. Clippe, A. Pekalski, Evolution of economic entities under heterogeneous political/environmental conditions within a Bak–Sneppen-like dynamics, *Physica A* 332 (2004) 394.
- [41] R.H. Day, *The Divergent Dynamics of Economic Growth*, Cambridge University Press, Cambridge, 2004.
- [42] P. Grassberger, The Bak–Sneppen model for punctuated equilibrium, *Phys. Lett. A* 200 (1995) 277.

Spin glass behavior of the antiferromagnetic Ising model on a scale-free network

M. Bartolozzi^{1,*} T. Surungan^{1,2,†} D. B. Leinweber^{1,‡} and A. G. Williams^{1§}

¹*Special Research Center for the Subatomic Structure of Matter (CSSM),
The University of Adelaide, Adelaide, SA 5005, Australia*

²*Department of Physics, Hasanuddin University, Makassar 90245, Indonesia*

(Dated: May 6, 2006)

Abstract

Antiferromagnetic Ising spins on the scale-free Barabási-Albert network are studied via the Monte Carlo method. Using the replica exchange algorithm, we calculate the temperature dependence of various physical quantities of interest including the overlap and the Binder parameters. We observe a transition between a paramagnetic phase and a spin glass phase and estimate the critical temperature for the phase transition to be $T \sim 4.0(1)$ in units of J/k_B , where J is the coupling strength between spins and k_B is the Boltzmann constant. Using the scaling behaviour of the Binder parameter, we estimate the scaling exponent to be $\nu \sim 1.10(2)$.

PACS numbers: 89.75.Hc, 75.10.Nr, 64.60.Cn, 05.70.Fh

Keywords: scale free network, spin glasses, Monte Carlo.

*Electronic address: mbartolo@physics.adelaide.edu.au

†Electronic address: tasrief@unhas.ac.id

‡Electronic address: dleinweber@physics.adelaide.edu.au

§Electronic address: anthony.williams@adelaide.edu.au

I. INTRODUCTION

In the last few years, the study of complex networks has found relevance in various fields including sociology, ecology, biology, economics and physics. In these networks, vertices do not have homogeneous links or connectivities. A particularly relevant structure found in several empirical studies is the so-called *scale-free network* (SFN), which is characterized by the power law distribution of the degree of connectivities, $P(k) \sim k^{-\gamma}$, with k the number of links for a node, and γ the decay exponent of the distribution. A network with $\gamma \rightarrow 0$ has nodes with a relatively homogeneous number of links (somewhat resembling the case on regular lattices), while large γ corresponds to the existence of “very famous” nodes (or hubs), i.e., those having direct links to the majority of vertices.

Many networks realized in Nature show scale-free structure. Some examples studied include food webs [1], power grids and neural networks [2, 3], cellular networks [4], sexual contacts [5], Internet routers [6], the World Wide Web [7], actor collaborations [2, 3, 8, 9], the citation network of scientists [10] and the stock market [11].

In addition to the scale-free behaviour, these networks are characterized by a high clustering coefficient, C , in comparison with random graphs [12]. The clustering coefficient, C , is computed as the average of local clustering, C_i , for the i^{th} node, defined as

$$C_i = \frac{2y_i}{z_i(z_i - 1)}, \quad (1)$$

where z_i is the total number of nodes linked to the site i and y_i is the total number of links between those nodes. As a consequence both C_i and C lie in the interval $[0,1]$. The high level of clustering found supports the idea that a *herding* phenomenon is a common feature in social and biological communities. The parameter C also represents the density of triangles, that is of elementary cells, associated with the network.

Numerical studies on SFNs have demonstrated how topology plays a fundamental role in infection spreading [13], opinion formation in large communities [14] and tolerance against random and preferential node removal [14, 15]. A detailed description of the progress in this emerging field of statistical mechanics can be found in the recent reviews of Refs. [16–18].

The aforementioned empirical findings have inspired physicists to investigate the dynamics of standard models in the new case where the interactions between elements are described by complex interactions. These include the study of various magnetic models such as the

Ising model. An intriguing issue concerns how the unusual topology acts to influence the cooperative behaviour of the spins. Studies of the ferromagnetic (FM) Ising model on a SFN, using several theoretical techniques [19–22] including the Monte Carlo (MC) method [22], have found the robustness of ferromagnetic ordering against thermal fluctuations for the degree distribution exponent $\gamma \leq 3$. This result is actually intuitive if we notice that, as γ gets smaller, nodes at the edge of the network will generally have more connections. In this situation, the system resembles the FM Ising model on a regular lattice which exceeds the lower critical spatial dimension, $d_l = 2$. There the ordered phase is very robust against thermal fluctuations. However, for the antiferromagnetic (AF) case with a SFN, the situation is different.

Two factors come to play a central role in the dynamics of the AF-SFN model; namely the competition induced by the AF interaction in the elementary triangles of the network and the randomness related to the non-regular connections. The abundance of elementary triangles in the network leads to frustration, as, for example, only two of the three spins can be anti-aligned. More generally, frustration refers to the inability of the system to remain in a single lowest energy state (ground state). These ingredients lead the AF SFN to belong to a class of randomly frustrated systems commonly referred to as spin glasses (SGs).

Most studies of SGs have been performed on regular lattices. These studies have shown that frustration and randomness are the key ingredients for SG behavior, characterized by a frozen random spin orientation at low temperatures [23]. Spin glasses on a SFN with mixed AF and FM bonds have been investigated recently by Kim *et al.* [24]. They found, for $\gamma \leq 3$ and even distributions of the two kinds of interaction, that the system is always in a SG state for any finite temperature. A study of the pure AF Ising model on a SFN is of great theoretical interest since, despite the homogeneity of the bonds, it inherits all the characteristics of a SG from the random frustration related to its geometry. General reviews on SG systems can be found in Refs. [23].

In this paper we consider the AF Ising model on a SFN, more precisely the Barabási-Albert (BA) network with tunable clustering [25]. Using the replica exchange algorithm [26] of the Monte Carlo method, we calculate the order parameters of spin glass behaviour, the so-called overlap parameter and its distribution. For an accurate determination of the critical temperature, we also evaluate the Binder parameter. The paper is organized as follows: Section II describes the model and the method. The results are discussed in Section III.

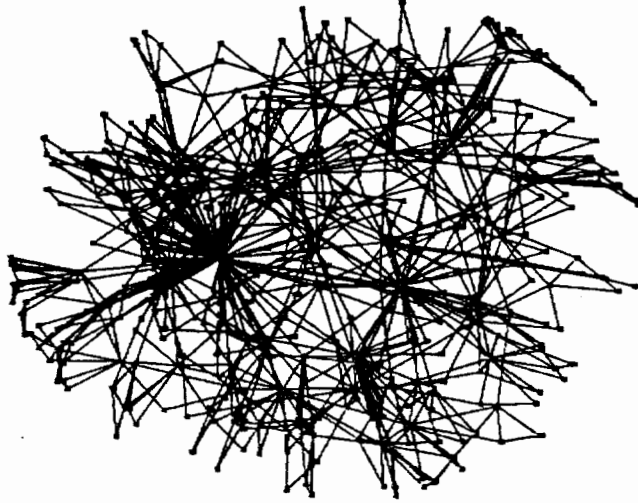


FIG. 1: (Color online). Example of a scale-free network. The number of nodes is 500 with clustering probability $\theta = 0.9$ and $m_0 = m = 2$. The number of nodes has been kept small in order to preserve the clarity of the plot. Note that, for such small networks, a large scale invariant range is obtained only if one considers the ensemble average over several realizations. This plot has been realized with the Pajek software [27].

Section IV is devoted to the concluding remarks.

II. MODEL AND SIMULATION METHOD

A. The model

In order to create the scale-free network topology we make use of the Barabási-Albert model [9]. This is based on two main considerations: (i) linear growth and (ii) preferential attachment. In practice the network is initialized with m_0 disconnected nodes. At each step a new node with m edges is added to the pre-existing network. The probability that an edge of the new node is linked with the i th node is expressed by $\Pi(k_i) = k_i / \sum_j k_j$. The iteration of this preferential growing process yields a scale free network, where the probability of having a node with k connections is $P(k) \sim k^{-\gamma}$ with $\gamma = 3$. This is an interesting value. In the thermodynamic limit, the second moment of the distribution diverges, $\langle k^2 \rangle = \infty$, for $\gamma \leq 3$. This leads to peculiar properties of theoretical models in this range of γ values [18]. In the present work we focus on the case in which $\gamma = 3$ and the divergence of $\langle k^2 \rangle$ is

logarithmic. An extensive investigation of the phase space for the AF model on SFN is left for future work.

It is also worth noting that the Barabási-Albert model cannot reproduce a high clustering coefficient. In fact, the value of this coefficient depends on the total number of nodes, N , in the network [16] and in the thermodynamic limit, $N \rightarrow \infty$, $C \rightarrow 0$.

In the AF Ising system the average cluster coefficient, C , plays a fundamental role in the dynamics. In fact, it represents the average number of triangles per node and, as a result, it is directly related to the degree of frustration in the network. In order to keep this parameter constant, on average, with the size of the network, we introduce a further step in the growth process, namely the triad formation proposed by Holme and Kim [25]. In this case, if the new added node is linked with an older node, i , having other links, then with a certain probability, θ , the next link of the new node, if any remain, will be added to a randomly selected neighbour of node i . This method of introducing friends to friends, while preserving the scale-free nature of the networks with $\gamma \sim 3$, generates high clustering coefficients that do not depend on N . The only tunable parameter that changes the value of the clustering coefficient is the *clustering probability* θ . An example of a SF network generated with this algorithm is shown in Fig. 1 for 500 nodes.

We simulate various sizes of the network with many different realizations and investigate the scaling behaviour of the various physical quantities we are interested in. All the simulations have been carried out fixing $\theta = 0.9$, corresponding to an average clustering coefficient of $C \sim 0.39$, close to the value found in many real systems [16]. On each SFN constructed at the beginning of the simulation, we assign to each vertex an Ising spin, and to each link an AF interaction. The Hamiltonian can be written as follows

$$H = - \sum_{\langle ij \rangle} J_{ij} s_i s_j. \quad (2)$$

Here the summation is performed over the connected spins s_i and s_j occupying sites i and j , respectively. The coupling interaction $J_{ij} = J = -1$ is AF. As previously mentioned, each vertex with the local cluster coefficient $C_i > 0$ together with its neighbours, compose elementary triangles. Due to the AF interactions the local system is frustrated.

It is worth pointing out that C is related to the degree of frustration of each network. Due to the probabilistic algorithm used for their construction, the value of C fluctuates around a mean value from one network to the next and, therefore, provides a source of randomness

that, as we will see, gives rise to the spin glass properties of the model. This probabilistic growth is not shared by other algorithms which use recursion formulas to generate scale-free structures, such as, for example, the Apollonian networks [28]. In this case, once one fixes the number of iterations of the algorithm, which is proportional to the number of nodes of the final network, one also fixes its topology. The element of randomness is therefore missing in the Apollonian procedure.

As a random system, each realization of a network of size N will differ in the “structure” of connectivities. Therefore, in order to have reliable statistics, we average over many realizations of the SF network for each specified size. The system sizes that we simulate are $N = 1024, 2048, 4096,$ and 8192 . In general, one takes into account more realizations for small system sizes and less for large system sizes as the latter tend to self-average. However, since the self-averaging of physical quantities for larger system sizes is interfered with by the increase of ground state degeneracy, we do not take less realizations. Instead all physical quantities of interest for each system size are averaged over 1000 network realizations. Moreover, for each realization of the network, we fix $m_0 = m = 5$, corresponding to a coordination number on a regular lattice of approximately 10. In the thermodynamic limit, the average connectivity for the BA network is $\langle k \rangle = 2m = 10$, emphasizing the fact that we are implicitly dealing with a high dimensional system.

Another peculiarity of SF networks is the existence of a broad distribution of “hubs”, that is nodes with a large number of connections, k . The energy difference in a spin flip actually depends on the number of connections of the spin itself, $\Delta E_i = -2s_i \sum_{j=1}^{k_i} s_j$. Thus in the AF case for the i th

Another peculiarity of SF networks is the existence of a broad distribution of “hubs”, that is nodes with a large number of connections, k . The energy difference in a spin flip actually depends on the number of connections of the spin itself, $\Delta E_i = -2s_i \sum_{j=1}^{k_i} s_j$. Thus in the AF case for the i th spin with k_i connections, the hubs are more likely to “freeze” into a particular configuration compared to the nodes with just a few links. This property resembles the spin glass behaviour of particular alloys where some elements freeze into a particular orientation at a higher temperature than others.

B. Simulation method

The calculation of the thermal averages of the physical quantities of interest is performed using the replica exchange MC method [26]. In this method the evolution of M replicas, each in equilibrium with a heat bath of inverse temperature β_m for the m^{th} replica, is simulated in parallel. Given a set of inverse temperatures, $\{\beta\}$, the probability distribution of finding the whole system in a state $\{X\} = \{X_1, X_2, \dots, X_M\}$ is

$$P(\{X, \beta\}) = \prod_{m=1}^M \tilde{P}(X_m, \beta_m), \quad (3)$$

with

$$\tilde{P}(X_m, \beta_m) = Z(\beta_m)^{-1} \exp(-\beta_m H(X_m)), \quad (4)$$

and $Z(\beta_m)$ is the partition function at the m^{th} temperature. We can then define an exchange matrix between the replicas in our Markov chain, $W(X_m, \beta_m | X_n, \beta_n)$, that is the probability to switch the configuration X_m at the temperature β_m with the configuration X_n at β_n . By using the detailed balance condition, required to keep the entire system at equilibrium, on the transition matrix

$$\begin{aligned} & P(\dots, \{X_m, \beta_m\}, \dots, \{X_n, \beta_n\}, \dots) \cdot W(X_m, \beta_m | X_n, \beta_n) \\ &= P(\dots, \{X_n, \beta_n\}, \dots, \{X_m, \beta_m\}, \dots) \cdot W(X_n, \beta_n | X_m, \beta_m), \end{aligned} \quad (5)$$

along with Eq. (4), we have that

$$\frac{W(X_m, \beta_m | X_n, \beta_n)}{W(X_n, \beta_n | X_m, \beta_m)} = \exp(-\Delta), \quad (6)$$

where $\Delta = (\beta_n - \beta_m)(H(X_m) - H(X_n))$. With the above constrains we can choose the matrix coefficients according to the standard Metropolis method and, therefore, we have

$$W(X_m, \beta_m | X_n, \beta_n) = \begin{cases} 1 & \text{if } \Delta < 0, \\ \exp(-\Delta) & \text{if } \Delta > 0. \end{cases} \quad (7)$$

In our simulation we restrict the exchange to temperatures next to each other; that is, we consider only the terms $W(X_m, \beta_m | X_{m+1}, \beta_{m+1})$. This choice is motivated by the fact that the acceptance ratio decays exponentially with $(\beta_n - \beta_m)$.

The replica exchange method is extremely efficient for simulating systems such as spin glasses, that can otherwise become frozen in some particular configuration at low temperatures when using a standard Metropolis algorithm for the configuration update. In this

case, as we lower the temperature, the system can become trapped into a local minimum of the free-energy where the barriers are so high that the time required for the system to move to another allowed region of the configuration space diverges to infinity as a function of the system size. If the system is trapped in a local minimum then the ergodicity condition is not fulfilled anymore and the measure that one makes become biased by the particular region of the configuration space that is being sampled. By using the exchange replica method, instead, we keep switching the temperatures between the M copies of the system and, as long as the higher temperature is in a hot phase (where, the system can easily explore all the configuration space), then we are in principle able to explore all the configuration space also for the lower temperatures. Another advantage of this method is that the replica exchange reduces drastically the temporal correlation in the system dynamics at each temperature. This enables one to collect more independent measures for the thermal averages of the physical quantities and, therefore, reduces the uncertainty.

It is important to stress that, before starting the actual simulations, some care is required in selecting the set of inverse temperatures, $\{\beta\}$. In fact, the method is efficient only when a fairly large transition probability is maintained in the range of interest. From Eq. 7, we can see that, in the hot phase, temperatures can be more coarsely spaced while in the cold phase the temperatures need to be closer to each other. An optimal set of temperatures can be obtained by iterating, in preliminary runs, the following map [26]:

$$\begin{aligned}\tilde{\beta}_1 &= \beta_1, \\ \tilde{\beta}_m &= \tilde{\beta}_{m-1} + (\beta_m - \beta_{m-1}) \cdot p_m/c,\end{aligned}\tag{8}$$

where p_m is the acceptance ratio for the switch between two configurations at the m th temperature and $c = \sum_{m=1}^M p_m / (M - 1)$ is a normalization factor. The initial value for the set $\{\beta\}$ is uniform in the interval of interest and we ensure that β_1 belongs to the hot phase. For each iteration of the map, a run of a few thousand MC steps is carried out to calculate the acceptance ratios, p_m , which are then plugged into Eq. (8) in order to obtain a new set of inverse temperatures. After a few iterations, the map of Eq. (8) converges to a fixed point, $\{\beta^*\}$, which sets the values of the temperatures to be used in our simulations. For each iteration of the map, a run of the system (a few thousand MC steps are enough) is carried out to calculate the acceptance ratios, p_m .

In using this method, we define a “local” MC (LMC) update as a MC update for each spin of each replica, either consecutively through all elements of the network or randomly.

Given that we can group the inverse temperatures in even and odd pairs, (β_m, β_{m+1}) , after each LMC update we alternate attempts to switch configurations from one temperature to the next. According to this procedure, we define a Monte Carlo step (MCS) as a LMC plus a half (m odd or even) exchange trial.

For each realization of the network we start from a random configuration of the spins and then perform 10^3 LMC updates in order to reach thermal equilibrium. After this transient period, we run the simulation for 3×10^5 MCSs while taking a total of 6×10^4 measures for the thermal averages, that is one every 5 MCSs (temporal correlations are lost very quickly by using the replica exchange method). We consider low temperatures in a search for the possible existence of a phase transition. The thermal averages obtained for each network are then averaged over the ensemble of networks. In the following, we indicate $\langle \dots \rangle$ as the thermal average and $[\dots]_{\text{av}}$ as the ensemble average. The statistical errors in the plots, where reported, are calculated via the bootstrap method.

III. RESULTS AND DISCUSSION

A. Spatial correlations and specific heat

As a first step we investigate the extent of spatial correlation of the spins in the SF network by making use of the spatial autocorrelation function which is defined on a regular lattice as

$$\xi(r) = \left[\frac{1}{L_d} \langle s_i s_{i+r} \rangle \right]_{\text{av}}, \quad (9)$$

where L_d is the total number of pairs at distance r and depends just on the dimension considered. In a SF network the situation is more complicated since there may be several paths leading from a certain node to another. We then define r as the *minimum* path between two nodes and the denominator of the Eq. (9) becomes dependent on r . The results, averaged over 50 configurations, between the temperatures of $T = 5.0$ and $T = 2.1$ are shown in Fig. 2 for $N = 1024$. All the temperatures in the present paper are expressed in units of J/k_B , where J is the coupling strength between spins and k_B is the Boltzmann constant.

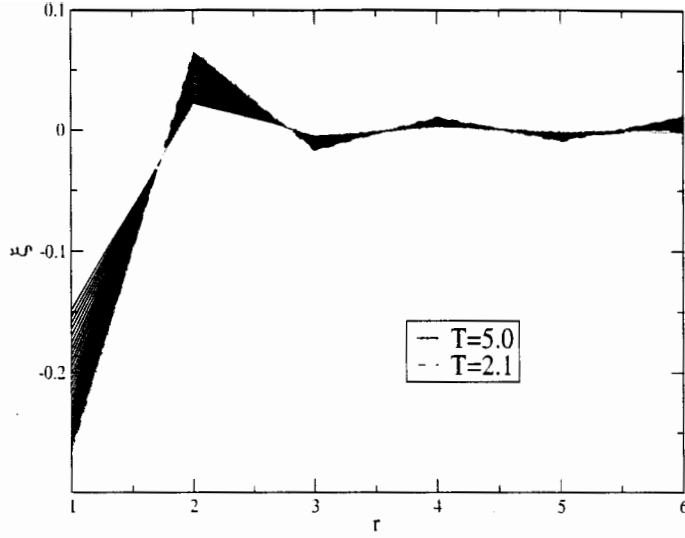


FIG. 2: (Color online). Spatial autocorrelation, $\xi(r)$, for $N = 1024$ averaged over 50 network configurations for temperatures between $T = 5.0$ and $T = 2.1$. The plot shows that next neighbour spins tend to be anti-parallel as in standard AF Ising model. The AF interaction in the triangular units of the system results in high frustration. Note that the number of nodes at large distances is much smaller than the ones at smaller distances and so the average calculated for $r = 5$ and $r = 6$ includes just few samples. This is a consequence of the “small-world” effects in SF networks.

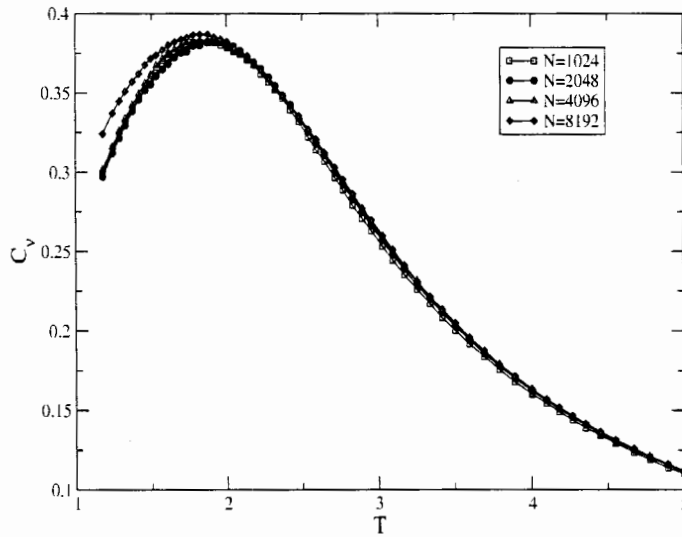


FIG. 3: Specific heat, C_ν , as a function of the temperature and system size. The plot has been obtained by averaging over 50 network configurations for each N . Note that the specific heat does not scale with the size of the system.

In order to give a better interpretation of the plot in Fig. 2 we remind the reader about an important propriety of SF networks; that is their “small world structure”. The “hubs”, in fact, play a fundamental role in linking sites otherwise very distant. Moreover, the average path length increases just logarithmically with the size of the network [16, 17]. In the plot of Fig. 2, for $N = 1024$ nodes, an upper limit of $r = 6$ is encountered. While all the 50 configurations reach $r = 6$, only a few networks exceed this limit.

The plot emphasizes how neighboring spins, on average, tend to be anti-correlated, as expected in the AF case. The autocorrelation decreases with the distance from the node under consideration. The temperature dependence is also in accord with the expectations. The absolute value of the correlation decreases with increasing temperature and vice versa. Indeed, the highest and lowest temperatures form a perfect boundary for all the curves. This is an expected result, since thermal effects always tend to reduce the correlation between the spin interactions.

We also study the behaviour of the specific heat, C_ν , defined as follows

$$C_\nu(T) = \left[\frac{1}{Nk_B T^2} (\langle E^2 \rangle - \langle E \rangle^2) \right]_{\text{av}}, \quad (10)$$

where k_B is the Boltzmann constant. Although no singularity is expected for this quantity in the spin-glass transition, it is interesting to compare its behaviour with other studies. The dependence of the specific heat on temperature is reported in Fig. 3. The statistical errors, in this case, are smaller than the size of the symbols and therefore are not reported. A common Schottky peak of the specific heat for a finite system is observed at the temperature of $T \sim 2.0$ independent of the system size. Below this point, we found that C_ν decreases and goes to zero as $T \rightarrow 0$.

This behaviour follows from simple entropy considerations. In fact, since we are dealing with a finite Ising system, the entropy is bounded at each finite temperature as well,

$$S(T) = \int_0^T \frac{C_\nu(T)}{T} dT < 2^N, \quad (11)$$

and, necessarily, $C_\nu \rightarrow 0$ for $T \rightarrow 0$.

The next section is dedicated to study of the SG behaviour and the phase transition of the system. In order to achieve this task, we evaluate the corresponding order parameters, the overlap parameter and the Binder parameter.

B. Observing spin glass behaviour

With the presence of frustration and randomness in the AF-SFN model, we expect to observe a spin glass transition, i.e., a transition from a temporal disordered to a temporal ordered phase at low temperatures.

This feature is not shared by the so-called fully frustrated systems [29]. This type of transition might be characterized by the order parameter such as that suggested by Edward and Anderson [30], defined as follows

$$q_{EA} = \left[\frac{1}{N} \sum_i \langle s_i \rangle^2 \right]_{\text{av}}. \quad (12)$$

However, an ergodic Markov chain of a system having Z_2 symmetry will ensure the thermal average of the i th spin vanishes. Therefore a finite value of this measure simply reflects the non-ergodicity in the MC update.

A more appropriate quantity that is often used to characterize the SG state is the overlap parameter, q , defined as [31, 32]

$$q = \frac{1}{N} \sum_i s_i^{(\alpha)} s_i^{(\beta)}, \quad (13)$$

where the superscripts α and β denote two copies of the same configuration of connectivity at the same temperature. The actual value of q is extracted from both the thermal and disorder average, $[\langle \dots \rangle]_{\text{av}}$.

Using the replica exchange MC simulation, the two copies, α and β , are allocated at each temperature of the parallel tempering. This means, if the measurement is performed on M points of temperatures, there are M pairs of replicas. The Metropolis spin update is performed on each node for every MC step. As a part of the equilibration steps of the algorithm described in the previous section, we exchange two α (and β) replicas of neighboring temperatures, according to a certain probability. Then, for each temperature, the α and β replicas are superimposed every 5 MCSs in order to measure the overlap parameters, as defined in Eq.(13).

In particular, for the Ising system, due to the Z_2 symmetry, it is important to evaluate the absolute value of the order parameter,

$$|q| \equiv \left[\left\langle \left| \frac{1}{N} \sum_i s_i^{(\alpha)} s_i^{(\beta)} \right| \right\rangle \right]_{\text{av}}, \quad (14)$$

to overcome the implication of the Z_2 symmetry of the Hamiltonian, that is the configurations s_i and $-s_i$ have equal Boltzmann weights. That is, if the system is at thermal equilibrium and if we take quite long MCS then the usual q should average to zero. The existence of a spin glass phase is indicated by the convergence of $|q|$ to a finite value as we increase the network size. At the same time, a convergence of $|q|$ to zero at high temperatures is anticipated. In the latter case the system is in the paramagnetic phase.

The temperature dependence of $|q|$, resulting from the simulations, is shown in Fig. 4. The existence of a SG phase is indicated by the finite value of $|q|$ in the low temperature region, and the approach of $|q|$ to zero at higher temperatures associated with the paramagnetic phase. For high temperatures and large networks, $|q|$ is approaching zero in accord with the thermodynamic limit where $|q| = 0$ [33].

The existence of these two different phases can also be observed from the distribution of q , as shown in Fig. 5. For higher temperatures we observe simple Brownian fluctuations of the values of q , leading to a singly peaked Gaussian distribution characteristic of a paramagnetic state. By decreasing the temperature, the distribution spreads out, reflecting the increasing number of metastable disordered states associated with a substantial frustration. At lower temperatures the distribution develops double peaks reflecting the Z_2 symmetry and a finite value of $|q|$, representative of the SG phase. We note that the shape of the observed distribution at low temperatures is different from that of the conventional Ising system where the double peaks approach delta-like double peaks reflecting a simple doubly degenerate ground state [34].

An accurate evaluation of critical temperature of the phase transition is achieved via the Binder parameter defined as follows

$$g_L = \frac{1}{2} \left(3 - \frac{[\langle q^4 \rangle]_{av}}{[\langle q^2 \rangle]_{av}^2} \right). \quad (15)$$

Here $\langle q^2 \rangle$ and $\langle q^4 \rangle$ are respectively the second and the fourth cumulant moment of q . In this calculation, in order to avoid systematic correlation errors that could bias the results if we were evaluating this average over g_L directly [35], the second and fourth order cumulants are averaged prior to taking their ratio. The Binder parameter is constrained in the range $0 \leq g_L \leq 1$. At high temperature, where thermal fluctuations overcome all cooperative interaction, the system is expected to exist in the paramagnetic phase where there is no spatial autocorrelation. As a result, the distribution of q should be Gaussian centered at

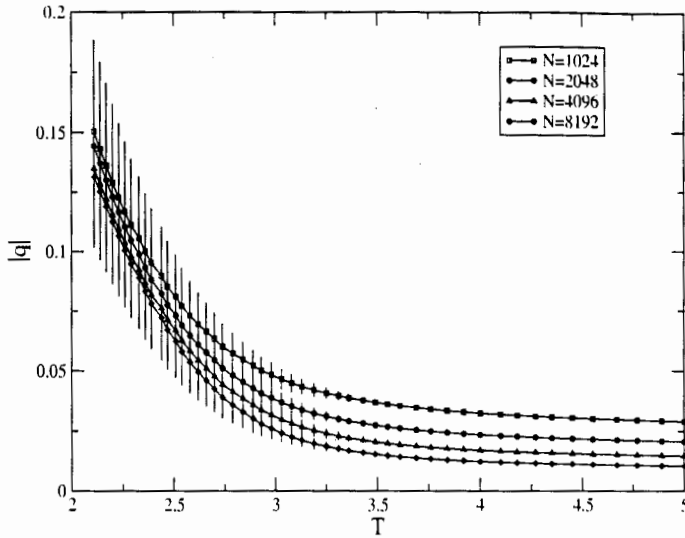


FIG. 4: Temperature dependence of the overlap parameter, q , for different system sizes N . The increasing value of q at low temperatures indicates a SG phase. For a given network size, 1000 realizations of the SFN are averaged over.

$q = 0$. In this case the ratio of the cumulants, $\langle q^4 \rangle / \langle q^2 \rangle^2 \rightarrow 3$, resulting in $g_L \rightarrow 0$. At low temperatures, the cooperative interaction becomes dominant and the ratio of the cumulants approaches unity so that $g_L \rightarrow 1$.

Fig. 6 (inset) displays the temperature dependence of the Binder parameter for a variety of network sizes. A spin-glass state is observed for lower temperatures where the Binder parameter deviates from zero, and increases with the system size while approaching to 1. In the thermodynamic limit, we expect $g_L \rightarrow 1$ just below the critical temperature. A crossing point in the size dependence of g_L indicates that the critical temperature for the SG phase transition is $T \sim 4.0$. Fig. 6 indicates that for temperatures above $T \sim 4.0$ the Binder parameter, while remaining always above zero, does indeed order in an opposite manner indicative of a genuine crossing of the curves and in accord with a genuine spin glass transition at finite temperature. This feature which is not observed for uniformly distributed AF and FM bonds, as $T_c = \infty$ in the thermodynamic limit [24]. However, the value of the transition temperature is not determined with high accuracy by the crossing of the Binder parameter. In fact, finite size effects seem to slightly distort the tendency for very small networks, as in the case of $N = 1024$. At the same time, the statistical errors in the paramagnetic phase for large networks, see $N = 8192$, appear to be significant and

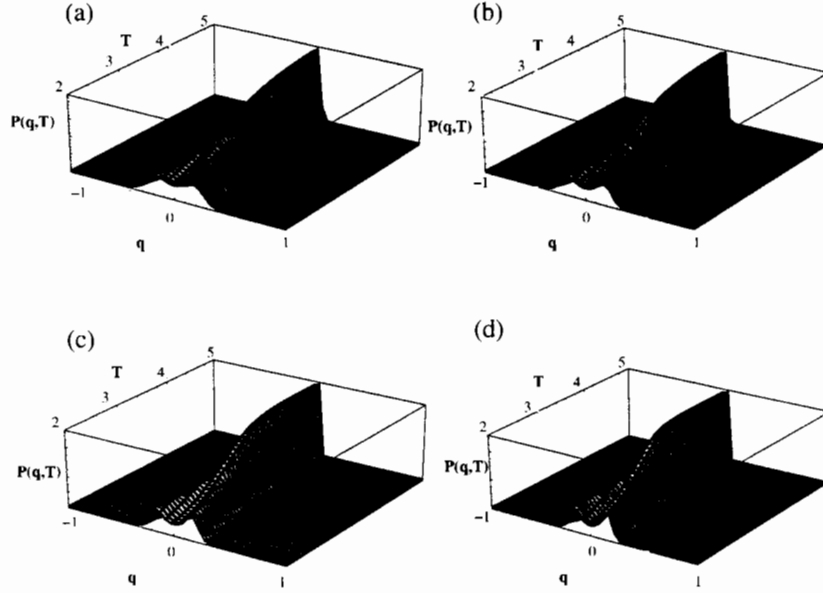


FIG. 5: (Color online). The distribution of q at various temperatures for different system sizes, including (a) $N = 1024$, (b) $N = 2048$, (c) $N = 4096$ and (d) $N = 8192$.

some points are scattered.

A more accurate estimate of the critical temperature, T_c , for finite size systems can be obtained using scaling arguments. For a SG system, the Binder parameter depends on the system size L as

$$g_L = \tilde{g}_L[(T - T_c)L^{1/\nu}], \quad (16)$$

being $\nu > 0$ the spin glass correlation length exponent, implying that at T_c the Binder cumulant does not depend on L . For the SFN, the system size scales logarithmically with the number of nodes N [16–18, 24] and therefore we take $L = \log(N)$. This slow increase in the diameter of the system, as well as the average path length, is a manifestation of the “small-world” property of this network, induced by the presence of a large number of highly connected hubs which create shortcuts between the nodes. An important implication of this feature is that we cannot embed the network in any finite dimensional lattice: we are

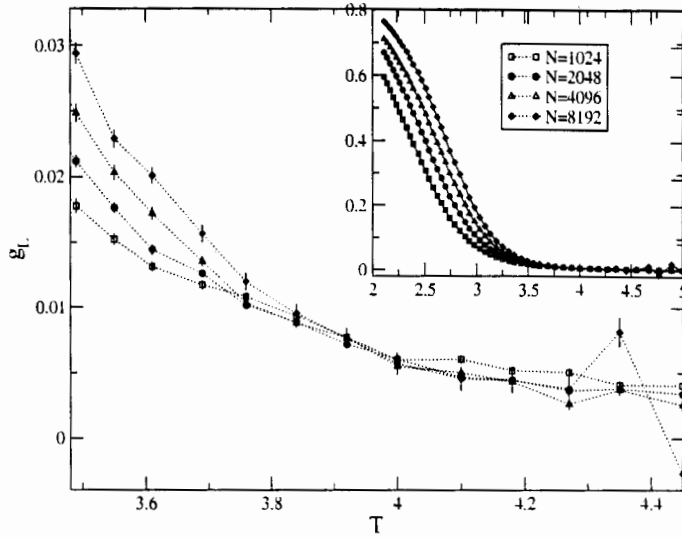


FIG. 6: Scaling behaviour of the Binder cumulant, g_L , for different system sizes. Each system size is averaged over 1000 realizations of the network configuration.

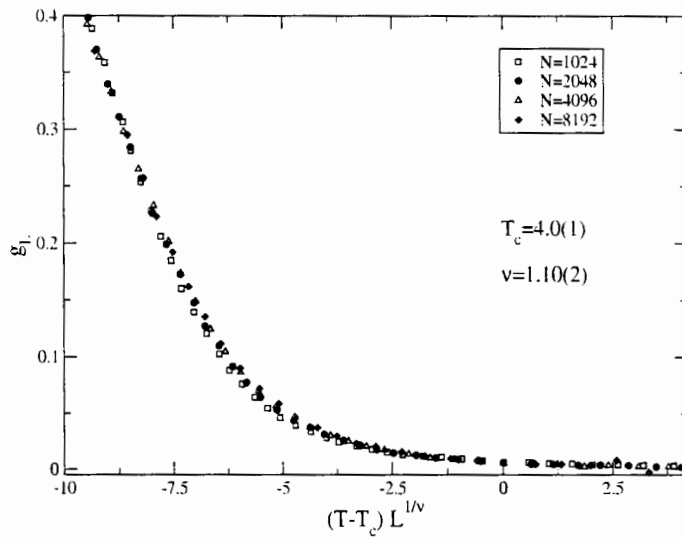


FIG. 7: Scaling plot of the data illustrated in Fig. 6, fitted to Eq. 16.

implicitly dealing with a high dimensional system. The correlation length, in this case, is still well defined although its value gets close to the densely-connected, mean field limit as we increase the average connectivity of the nodes, $\langle k \rangle = 2m$.

The parameters T_c and ν are determined by constraining the temperature dependence of the Binder parameter for each network size to lie on a single curve. The curves following

the scaling behaviour of Eq. (16) are shown in Fig. 7. From this fit we estimate the critical temperature $T_c \sim 4.0(1)$ and the exponent of the SG correlation length $\nu \sim 1.10(2)$. It is important to underline that this kind of behaviour is not observed for an AF system on a regular triangular lattice.

IV. CONCLUDING REMARKS

In summary, we have investigated the antiferromagnetic Ising model on a Barabási-Albert scale-free network using the replica exchange Monte Carlo method. Through the calculation of the overlap parameter we observe spin glass behaviour at low temperatures. Using the scaling behaviour of the Binder parameter the critical temperature separating the SG and the paramagnetic phases is found to be $T_c = 4.0(2)$ with a scaling exponent of SG correlation length $\nu \sim 1.10(2)$. Such behaviour is not observed for the AF Ising model on regular triangular lattices. Hence the topology of the interactions plays a critical role in the dynamics of the system.

Acknowledgments

The authors wish to thank Y. Okabe, E. Marinari and J.-S. Wang for valuable discussions. One of the authors (TS) is grateful for the hospitality of the Center for the Subatomic Structure of Matter (CSSM) at the University of Adelaide during his academic visit to the Center. The computation of this work has been done using the Hydra teraflop supercomputer facility of the South Australian Partnership for Advanced Computing (SAPAC).

-
- [1] R. J. Williams, E.L. Berlow, J.A. Dunne, A.-L. Barabási and W.D. Martinez, Proc. Nat. Acad. Sci. U.S.A. **99**, 12913 (2002); J. Camacho, R. Guimerà and L. A. N. Amaral, preprint cond-mat/0103114; J. M. Montoya and R. V. Solé J. Theor. Biol. **214**, 405 (2002).
 - [2] L. A. N. Amaral, A. Scala, M. Barthelemy and H.E Stanley, Proc. Nat. Acad. Sci. U.S.A. **97**, 11 (2000).
 - [3] D. J. Watts and S. H. Strogatz, Nature **393**, 440 (1998).

- [4] H. Jeong, B. Tombor, R. Albert, Z.N. Oltvai and A.-L. Barabási, *Nature* **407**, 651 (2000); H. Jeong, S.P. Mason, Z.N. Oltvai and A.-L. Barabási, *Nature* **411**, 41 (2001).
- [5] F. Liljeros, C.R. Edling, L.A.N. Amaral, H.E. Stanley and Y. Aberg, *Nature* **411**, 907 (2001).
- [6] M. Faloutsos, P. Faloutsos and C. Faloutsos *Comput. Commun. Rev.* **29**, 251 (1999); R. Pastor-Satorras, A. Vazquez, A. Vespignani, *Phys. Rev. Lett.* **87**, 258701 (2001); S. Yook, H. Jeong and A.-L. Barabási, *Proc. Nat. Acad. Sci. U.S.A.* **99**, 13382 (2002).
- [7] R. Albert, H. Jeong and A.-L. Barabási, *Nature* **401**, 130 (1999); R. Kumar, P. Raghavan, S. Rajagopalan, D. Sivakumar, A. Tomkins and E. Upfal, *Proceeding of the 9th ACM Symposium on Principles of Database Systems*, p. 1.
- [8] M. E. J. Newman, S. H. Strogatz and D. J. Watts, *Phys. Rev. E* **64**, 026118 (2001); R. Albert and A.-L. Barabási, *Phys. Rev. Lett.* **85**, 5234 (2000).
- [9] A.-L. Barabási and R. Albert, *Science* **286**, 509 (1999).
- [10] S. Redner, *Eur. Phys. J. B*, **4**, 131 (1998); A. Vazquez, *Europhys. Lett.* **54**, 430 (2001).
- [11] G. Bonanno, G. Caldarelli, F. Lillo and R.N. Mantegna, *Phys. Rev. E* **68**, 046130 (2003); J.-P. Onnela, A. Chakraborti, K. Kaski, J. Kertesz and A. Kanto, *Phys. Rev. E* **68**, 056110 (2003).
- [12] B. Bollobás, *Random Graphs*, 2nd. ed., (Cambridge University Press, Cambridge, 2001).
- [13] R. Pastor-Satorras and A. Vespignani, *Phys. Rev. Lett.*, **86**, 3200 (2001); R. Pastor-Satorras and A. Vespignani, *Phys. Rev. E*, **63**, 066117 (2001).
- [14] M. Bartolozzi, D.B. Leinweber and A.W. Thomas, *Phys. Rev. E*, **72**, 046113 (2005).
- [15] R. Albert, H. Jeong and A.-L. Barabási, *Nature* **406**, 378 (2000); R. Cohen, K. Erez, D. ben-Avraham and S. Havlin, *Phys. Rev. Lett.*, **85**, 4626 (2000); D.S. Callaway, M.E.J. Newman, S.H. Strogatz and D. J. Watts, *Phys. Rev. Lett.*, **85**, 5468 (2000).
- [16] R. Albert and A.-L. Barabási, *Rev. Mod. Phys.* **74**, 47 (2002).
- [17] S.N. Dorogovtsev and J.F.F. Mendes, *Adv. Phys.* **51**, 1079 (2002).
- [18] S.N. Dorogovtsev and J.F.F. Mendes, *Evolution of Networks*, (Oxford University Press, Oxford, 2003).
- [19] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, *Phys. Rev. E* **66**, 016104 (2002).
- [20] F. Iglói and L. Turban, *Phys. Rev. E* **66**, 036140 (2002).
- [21] A. Aleksiejuk, J. A. Holyst and D. Stauffer, *Physica A*, **310**, 260 (2002).
- [22] C. P. Herrero, *Phys. Rev. E* **69**, 067109 (2004).
- [23] K. Binder and A.P. Young, *Rev. Mod. Phys.* **58**, 801 (1986); N. Kawashima and H. Rieger in

Frustrated Spin Systems, edited by H.-T. Diep (World Scientific, 2004, Singapore)

- [24] D.-H. Kim, G.J. Rodgers, B. Kahng and D. Kim, *Phys. Rev. E* **71**, 056115 (2005).
- [25] P. Holme and B. J. Kim, *Phys. Rev. E* **65**, 026107 (2002).
- [26] K. Hukushima and K. Nemoto, *J. Phys. Soc. Japan* **65**, 1863 (1996).
- [27] download at <http://vlado.fmf.uni-lj.si/pub/networks/pajek/>
- [28] R.F.S. Andrade and H.J. Herrmann, *Phys. Rev. E* **71**, 056131 (2005).
- [29] T. Surungan, Y. Okabe, and Y. Tomita, *J. Phys. A* **37**, 4219 (2004).
- [30] S.F. Edwards and P.W. Anderson, *J. Phys. F: Metal Phys.* **5**, 965 (1975).
- [31] G. Parisi, *Phys. Rev. Lett.* **50**, 1946 (1983).
- [32] R. N. Bhatt and A. P. Young, *Phys. Rev. B* **37**, 5606 (1988).
- [33] A.T. Ogielski, *Phys. Rev. B* **32**, 7384 (1985).
- [34] V. Dotsenko, *Introduction to the Replica Theory of Disordered Statistical Systems*, (Cambridge Univ. Press, Cambridge, 2001).
- [35] N. Kawashima and A.P. Young, *Phys. Rev. B* **53**, R484 (1996).