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Generalized Geometry

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Abstract

Generalized geometry is a recently discovered branch of differential geometry that has received a reasonable amount of interest due to the emergence of several connections with areas of Mathematical Physics. The theory is also of interest because the different geometrical structures are often generalizations of more familiar geometries. We provide an introduction to the theory which explores a number of these generalized geometries.

After introducing the basic underlying structures of generalized geometry we look at integrability which offers some geometrical insight into the theory and this leads to Dirac structures. Following this we look at generalized metrics which provide a generalization of Riemannian metrics.

We then look at generalized complex geometry which is a generalization of both complex and symplectic geometry and is able to unify a number of features of these two structures. Beyond generalized complex geometry we also look at generalized Calabi-Yau and generalized Kähler structures which are also generalizations of the more familiar structures.

This work contains no material that has been accepted for the award of any other degree or diploma in any University or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

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The material in this thesis is heavily influenced by the theses of Marco Gualtieri [11] and Gil Cavalcanti [5] as well as (unpublished) notes from Nigel Hitchin [16] which I am grateful to have received. Although the presentation and proofs given in this thesis are my own work, most of the results are can be found in these references.

Contents

Chapter 1

Introduction

Generalized geometry was created by Nigel Hitchin [15] originally as a way of characterizing special geometry in low dimensions [13],[14], and has been further developed by Hitchin's students M. Gualtieri [11], G. Cavalcanti [5] and F. Witt [20]. The initial motivations for the subject have since been overshadowed by the remarkable appearance of certain concepts from string theory and supergravity such as B-field symmetries, 3-form flux, D-branes, and connections with skew torsion.

This thesis is an introduction to generalized geometry with the intention of organizing the different geometrical structures encountered in generalized geometry into a consistent framework. In Chapter 2 we introduce the the fundamental structure underpinning all further structures, the so-called generalized tangent bundle.

The various structures of generalized geometry can often be described as a reduction of structure of the generalized tangent bundle together with an integrability condition. In Chapter 3 we look at the integrability of subbundles which naturally leads to the concept of Dirac structures, introduced by Courant and Weinstein [8],[9].

In Chapter 4 we investigate a natural way in which Riemannian geometry extends to generalized geometry. The result is that a number of related concepts such as the Levi-Civita connection and Hodge star also have natural extensions.

Chapter 5 looks at generalized complex structures, one of the most important structures in generalized geometry. These structures are simultaneously a generalization of complex structures and symplectic structures. Also in this chapter we consider the deformation theory for generalized complex structures as well a look at generalized Calabi-Yau manifolds which may possibly be an appropriate setting for mirror symmetry with torsion.

In Chapter 6 we look at generalized Kähler structures which are a natural extension of Kähler manifolds to generalized geometry. The remarkable feature of generalized Kähler manifolds is that there structure can be equivalently described by a bi-hermitean structure first discovered by Gates, Hull and Roček [10] arising from non-linear sigma models with $N = (2, 2)$ supersymmetry.

Generalized geometry has developed rapidly into a considerably large subject. There are numerous topics which this thesis does not touch upon such as generalized submanifolds [11],[1] and their relation to D-branes, T-duality in generalized geometry [5] following the framework of T-duality in [3],[2],[4] and moduli spaces [21] to name just a few.

Chapter 2

Generalized tangent bundle

Generalized tangent bundles play a role in Generalized Geometry that resembles the tangent bundle in more familiar geometries. In particular they come equipped with a naturally defined inner product as well as a skew-symmetric bracket acting on sections; moreover the exterior algebra of the cotangent bundle plays the role of spinors with the generalized tangent bundle acting on the exterior bundle by a naturally defined Clifford action. The untwisted generalized tangent bundle is simply the direct sum of the tangent and cotangent bundles. Beyond this there are the twisted generalized tangent bundles, so named because they are formed by twisting by a gerbe.

2.1 The generalized tangent bundle

2.1.1 Introduction

We present the fundamental structure of Generalized Geometry. Let M be a smooth manifold. The *generalized tangent bundle* is the bundle $EM = TM \oplus T^*M$ over M. When the manifold in question is understood we will often simply write T and T^* for the tangent and cotangent bundles and E for EM. There is a naturally defined bilinear form on this bundle arising from the pairing of dual vector spaces and is given by

$$(X + \xi, Y + \eta) = \frac{1}{2}(\eta(X) + \xi(Y)) \tag{2.1}$$

Where X,Y are tangent vectors and ξ,η are cotangent vectors, all over the same base point. This bilinear form has signature (n, n) where $n = \dim M$.

Each fibre of the generalized tangent bundle has an action on the corresponding fibre of the exterior bundle $\wedge T^*$ given by

$$(X + \xi) \cdot \omega = \xi \wedge \omega + \iota_X \omega \tag{2.2}$$

where ι_X denotes contraction by X. It then follows that

$$(X + \xi) \cdot ((X + \xi) \cdot \omega) = \xi(X)\omega = (X + \xi, X + \xi)\omega$$

and thus the exterior algebra is given the structure of a bundle of Clifford E modules with respect to the natural bilinear form on E. The corresponding bundle of Clifford algebras generated by the relation $(X + \xi)^2 = \xi(X)$ is denoted Cliff(E).

The form $(\ , \ )$ on E allows one to reduce the structure group of E to $O(n, n)$; in fact by considering first Stiefel-Whitney classes we can further reduce the structure to $SO(n, n)$ since $w_1(T \oplus T^*) = w_1(T) + w_1(T^*) = 0$. As $SO(n, n)$-bundles we have the isomorphism Cliff(E) $\simeq \wedge E \simeq \wedge T \otimes \wedge T^*$. A spin structure for E is then a lift of the transition functions of E from $SO(n, n)$ to $Spin(n, n)$ with the cocycle condition preserved. In the case of an $SO(n)$-bundle, the obstruction to such a lift is the second Stiefel-Whitney class w_2. However for an indefinite metric, as is the case here the obstruction is different. Given a bundle E with $SO(p, q)$ structure we can always reduce structure to the maximal compact subgroup $S(O(p) \times O(q))$ which corresponds to decomposing the bundle E into a sum of positive and negative subbundles, $E = E^+ \oplus E^-$ (this can be done by the Gram-Schmidt procedure). Then as worked out by Karoubi in [17], the obstruction to finding a lift of the structure of E from $SO(p, q)$ to $Spin(p, q)$ is precisely $w_2(E^+) - w_2(E^-)$. In the case of the generalized tangent bundle, $E = T \oplus T^*$, we find that if E is decomposed as $E = E^+ \oplus E^-$, then the projection $\pi : E \to T$ induces isomorphisms $\pi : E^+ \simeq T$, $\pi : E^- \simeq T$ and thus $w_2(E^+) - w_2(E^-) = 0$ showing that a lift of the structure of E to $Spin(n, n)$ is always possible.

The Clifford algebra Cliff(n, n) corresponding to a vector space of signature (n, n) has a unique irreducible representation. Then since $Spin(n, n) <$ Cliff(n, n) one can use this representation combined with a spin structure for the generalized tangent bundle E to construct the spin bundle $S(E)$ for this spin structure. It is shown in [11] that there is a lift of the $GL(n) < SO(n, n)$ structure of $T \oplus T^*$ to a spin structure such that $S(E) \simeq \wedge T^* \otimes |\mathrm{det} T|^{\frac{1}{2}}$, i.e., the forms tensored by a trivial line bundle. The effect of tensoring by the $|\mathrm{det} T|$ factor can be thought of as a change in how the transition functions act on $\wedge T^*$ by a multiplicative factor. However this line bundle is trivial so by an appropriate choice of transition functions it has no effect at all. Therefore we shall always consider $\wedge T^*$ as the space of spinors in generalized geometry. The representation of the Clifford algebra used to construct the spin bundle $S(E)$ also defines Clifford multiplication, a bundle homomorphism $c : \mathrm{Cliff}(E) \otimes S(E) \to S(E)$ which is nothing more than the action (2.2).

The decomposition of $S(E)$ into its two irreducible $Spin(n, n)$ representations $S(E)^{\pm}$ corresponds to the splitting of $\wedge T^*$ into $\wedge T^{ev \backslash odd}$, the even and odd forms which follows since the Clifford action (2.2) changes the parity of forms.

2.1.2 The Mukai pairing

The spinors have a bilinear form $\langle\ ,\ \rangle : S \otimes S \to \det T^*$ known as the *Mukai pairing*. Let α denote the main anti-automorphism of the Clifford algebra $\mathrm{Cliff}(E)$ given by $\alpha(e_1 e_2 \dots e_k) = e_k \dots e_2 e_1$. Then the form is given by

$$\langle s, t\rangle = [\alpha(s) \wedge t]_{\mathrm{top}} \tag{2.3}$$

where $[\ ]_{\mathrm{top}}$ denotes taking the top degree part of the form.

Proposition 2.1.1. *The bilinear form on spinors satisfies*

$$\langle vs, vt\rangle = (v, v)\langle s, t\rangle. \tag{2.4}$$

where $v \in E$ acts on spinors by Clifford multiplication.

Proof. For $v = X + \xi$ and s a form of degree k we have

$$\alpha(vs) = \alpha(\iota_X s + \xi \wedge s) = (-1)^{k-1}\iota_X \alpha(s) + \alpha(s) \wedge \xi$$

and so if t has degree $n - k$ where n is the top degree then

$$[\alpha(vs), vt]_{\mathrm{top}} = (-1)^{k-1}\iota_X \alpha(s) \wedge \xi \wedge t + \alpha(s) \wedge \xi \wedge \iota_X t.$$

Also

$$\begin{aligned} 0 &= \iota_X(\alpha(s) \wedge \xi \wedge t) \\ &= \iota_X \alpha(s) \wedge \xi \wedge t + (-1)^k \alpha(s) \wedge \xi(X) \wedge t + (-1)^{k+1}\alpha(s) \wedge \xi \wedge \iota_X t. \end{aligned}$$

Combining these two gives

$$\langle vs, vt\rangle = \xi(X)\langle s, t\rangle = (v, v)\langle s, t\rangle. \tag{2.5}$$

as required. □

We also have that $\langle s, t\rangle = (-1)^{\frac{n(n-1)}{2}}\langle t, s\rangle$. Recall that $Spin(E) = \{e_1 e_2 \dots e_{2k} \,|\, (e_j, e_j) = \pm 1\}$. It follows that $\langle\ ,\ \rangle$ is invariant under the connected component $Spin^+(E)$.

2.2 The Courant bracket

The third important piece of structure to be introduced is the *Courant bracket*, a bilinear, skew-symmetric bracket on the sections of E:

Definition 2.2.1. The *Courant bracket* is the bilinear form $[\ ,\ ]$ on sections of E given by

$$[u,v] = [X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}\,d(\eta(X) - \xi(Y)) \quad (2.6)$$

where $u = X+\xi$ and $v = Y+\eta$.

2.2.1 Motivation and properties

The Courant bracket is skew-symmetric but it does not satisfy the Jacobi identity and as such it may appear unnatural at first. The bracket was first introduced in [8],[9] in the study of Dirac structures. We will introduce Dirac structures in section (3.4), but here we will derive the form of the bracket using spinors. First recall the following relations

$$\mathcal{L}_X = d\iota_X + \iota_X d \quad (2.7)$$

$$\iota_{[X,Y]} = [\mathcal{L}_X, \iota_Y] \quad (2.8)$$

where the bracket in (2.8) is the ordinary Lie bracket. Combining (2.7) and (2.8) we have

$$\iota_{[X,Y]} = d\iota_X\iota_Y + \iota_X d\iota_Y - \iota_Y d\iota_X - \iota_Y\iota_X d. \quad (2.9)$$

Equation (2.9) uniquely defines the Lie bracket. Now just as vector fields have the natural action of contraction on forms, we have an extension of this action to generalized tangent vectors given by Clifford multiplication (2.2). Replacing the contractions in (2.9) by the Clifford action of sections of E will then uniquely define a bracket operation on sections of E. Skew symmetrization of this bracket will yield the Courant bracket. To determine this bracket we first determine

$$\begin{aligned}
\mathcal{L}_{X+\xi}d\omega &= (X+\xi)d\omega + d((X+\xi)\omega) \\
&= \iota_X d\omega + \xi\wedge d\omega + d(\iota_X\omega + \xi\wedge\omega) \\
&= \mathcal{L}_X\omega + d\xi\wedge\omega.
\end{aligned}$$

So we obtain

$$\begin{aligned}
[X+\xi, Y+\eta]\cdot\omega &= [\mathcal{L}_{X+\xi}, (Y+\eta)\cdot]\omega \\
&= \mathcal{L}_X(\iota_Y\omega + \eta\wedge\omega) + d\xi\wedge(\iota_Y\omega+\eta\wedge\omega) \\
&\quad -(Y+\eta)(\mathcal{L}_X\omega + d\xi\wedge\omega) \\
&= \iota_{[X,Y]}\omega + \mathcal{L}_X(\eta\wedge\omega) - \eta\wedge\mathcal{L}_X\omega + d\xi\wedge\iota_Y\omega \\
&\quad -\iota_Y(d\xi\wedge\omega) \\
&= \iota_{[X,Y]}\omega + \mathcal{L}_X(\eta)\wedge\omega - \iota_Y d\xi\wedge\omega.
\end{aligned}$$

This defines a bracket operation called the *Dorfman bracket*

$$[X+\xi, Y+\eta]_D = [X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi. \tag{2.10}$$

The Dorfman bracket is not skew symmetric; its skew-symmetrization $[u,v] = \frac{1}{2}[u,v]_D - \frac{1}{2}[v,u]_D$ is the Courant bracket. We now establish some properties of the Courant bracket. Actually it is easier to work out the properties of the Dorfman bracket first. First we have the relation

$$[u,v] = [u,v]_D - d(u,v) \tag{2.11}$$

which follows from

$$[u,v]_D + [v,u]_D = \mathcal{L}_X\eta + \mathcal{L}_Y\xi - \iota_X d\eta - \iota_Y d\xi = d(\iota_X\eta + \iota_Y\xi) = 2d(u,v)$$

where $u = X+\xi$ and $v = Y+\eta$. This relation shows that if u and v are orthogonal with respect to $(\ ,\ )$ then $[u,v]_D = [u,v]$. If $\pi : E \to T$ is projection onto the first factor then

$$\pi([u,v]_D) = [\pi(u), \pi(v)] \tag{2.12}$$

and similarly for the Courant bracket

$$\pi([u,v]) = [\pi u, \pi v]. \tag{2.13}$$

Proposition 2.2.1. *[11] For sections u,v,w of E we have*

$$\pi(u)(v,w) = ([u,v]_D, w) + (v, [u,w]_D). \tag{2.14}$$

Proof. Let $u = A+\alpha$, $v = B+\beta$, $w = C+\gamma$. We start with the right hand side

$$\begin{aligned}
& ([A,B] + \mathcal{L}_A\beta - \iota_B d\alpha, C+\gamma) + (B+\beta, [A,C] + \mathcal{L}_A\gamma - \iota_C d\alpha) \\
=\ & \tfrac{1}{2}(\iota_{[A,B]}\gamma + \iota_C(\mathcal{L}_A\beta - \iota_B d\alpha) + \iota_{[A,C]}\beta + \iota_B(\mathcal{L}_A\gamma - \iota_C d\alpha)) \\
=\ & \tfrac{1}{2}([\mathcal{L}_\alpha, \iota_B]\gamma + \iota_C\mathcal{L}_A\beta + [\mathcal{L}_A, \iota_C]\beta + \iota_B\mathcal{L}_A\gamma) \\
=\ & \tfrac{1}{2}(\mathcal{L}_A\iota_B\gamma + \mathcal{L}_A\iota_C\beta) \\
=\ & A(\iota_B\gamma + \iota_C\beta) \\
=\ & \pi(u)(v,w)
\end{aligned}$$

as required. □

Corollary 2.2.1.1. *For sections u,v,w of E we have*

$$\pi(u)(v,w) = ([u,v] + d(u,v), w) + (v, [u,w] + d(u,w)). \tag{2.15}$$

Let u, v be sections of E and f be a function. Then one verifies

$$[u, fv]_D = f[u,v]_D + (\pi(u)f)v \tag{2.16a}$$
$$[fv, u]_D = f[v,u]_D + (\pi(u)f)v - 2(u,v)df \tag{2.16b}$$

and it follows directly that

$$[u, fv] = f[u,v] + (\pi(u)f)v - (u,v)df. \tag{2.17}$$

2.2.2 Failure of the Jacobi identity

We have already mentioned that the Courant bracket fails to satisfy the Jacobi identity, but it will be useful for later to determine exactly what the failure is. Therefore we define the *Jacobiator*

$$\mathrm{Jac}(u,v,w) = [[u,v],w] + [[v,w],u] + [[w,u],v] \tag{2.18}$$

defined for sections u, v, w of E. To find an expression for the Jacobiator we first establish the following identity for the Dorfman bracket

$$[u, [v,w]_D]_D = [[u,v]_D, w]_D + [v, [u,w]_D]_D. \tag{2.19}$$

This identity says that $[u, \ ]_D$ acts as a derivation of the Dorfman bracket. If the Dorfman bracket were skew-symmetric this would be equivalent to the Jacobi identity. To establish (2.19) let $u = X + \xi, v = Y + \eta, w = Z + \phi$. Then

$$\begin{aligned}
& [[u,v]_D, w]_D + [v, [u,w]_D]_D \\
= \ & [[X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi, Z + \phi] + [Y + \eta, [X,Z] + \mathcal{L}_X\phi - \iota_Z d\xi] \\
= \ & [[X,Y],Z] + [Y,[X,Z]] + \mathcal{L}_{[X,Y]}\phi - \iota_Z d(\mathcal{L}_X\eta - \iota_Y d\xi) \\
& +\mathcal{L}_Y(\mathcal{L}_X\phi - \iota_Z d\xi) - \iota_{[X,Z]}d\eta \\
= \ & [X,[Y,Z]] + \mathcal{L}_X\mathcal{L}_Y\phi - \mathcal{L}_Y\mathcal{L}_X\phi - \iota_Z\mathcal{L}_X d\eta - \iota_Z\mathcal{L}_Y d\xi \\
& +\mathcal{L}_Y\mathcal{L}_X\phi - \mathcal{L}_Y\iota_Z d\xi - \iota_{[X,Z]}d\eta \\
= \ & [X,[Y,Z]] + \mathcal{L}_X\mathcal{L}_Y\phi - \iota_{[Y,Z]}d\xi - \mathcal{L}_X\iota_Z d\eta \\
= \ & [X,[Y,Z]] + \mathcal{L}_X(\mathcal{L}_Y\phi - \iota_Z d\eta) - \iota_{[Y,Z]}d\xi \\
= \ & [u, [v,w]_D]_D.
\end{aligned}$$

Next we relate $[[u,v]_D, w]_D$ to $[[u,v],w]$:

$$\begin{aligned}
[[u,v]_D, w]_D &= [[u,v] + d(u,v), w]_D \\
&= [[u,v], w]_D \\
&= [[u,v], w] + d([u,v], w)
\end{aligned}$$

where we have used the fact that $[a,b]_D = 0$ when a is a closed 1-form. Now we can finally calculate

$$\begin{aligned}
\mathrm{Jac}(u,v,w) &= [[u,v],w] + \text{cyclic permutations} \\
&= \tfrac{1}{4}([[u,v]_D, w]_D - [[v,u]_D, w]_D - [w,[u,v]_D]_D + [w,[v,u]_D]_D + \text{cp}) \\
&= \tfrac{1}{4}(([u,[v,w]_D]_D - [v,[u,w]_D]_D) + (-[v,[u,w]_D]_D + [u,[v,w]_D]_D) \\
&\quad -[w,[u,v]_D]_D + [w,[v,u]_D]_D + \text{cp})) \\
&= \tfrac{1}{4}(-[v,[u,w]_D]_D + [u,[v,w]_D]_D + \text{cp}) \\
&= \tfrac{1}{4}([[u,v]_D, w]_D + \text{cp}) \\
&= \tfrac{1}{4}([[u,v],w] + d([u,v],w) + \text{cp}) \\
&= \tfrac{1}{4}\mathrm{Jac}(u,v,w) + \tfrac{1}{4}d(([u,v],w) + ([v,w],u) + ([w,u],v)).
\end{aligned}$$

We thus have

$$\mathrm{Jac}(u,v,w) = d(\mathrm{Nij}(u,v,w)) \tag{2.20}$$

where we define $\mathrm{Nij}(u,v,w)$, the *Nijenhuis operator* by

$$\mathrm{Nij}(u,v,w) = \tfrac{1}{3}(([u,v],w) + ([v,w],u) + ([w,u],v)). \tag{2.21}$$

The reason behind the nomenclature will become clear later. Note however that the Nijenhuis operator is not tensorial.

2.3 Symmetries

Before we can extend the structure of section (2.1) to the more general setting we need to understand some symmetry properties of the natural pairing and of the Courant bracket. The issue of symmetries is further clarified in section (2.5).

We are interested in maps that preserve the structure of the generalized tangent bundle. We first turn to the linear theory. Given a vector space V one can form the vector space $E = V \oplus V^*$ with the natural bilinear form of signature (n,n). The linear endomorphisms of E preserving the bilinear form is the orthogonal group $O(E) = O(n,n)$. The Lie algebra $\mathfrak{so}(E)$ of

$O(E)$ (and $SO(E)$) is the Lie algebra of endomorphisms of E which are skew adjoint with respect to the bilinear form. Any such endomorphism L can be viewed as a skew-symmetric bilinear form via $u, v \mapsto (Lu, v)$. Under this identification $\mathfrak{so}(E) \simeq \wedge^2 E = \wedge^2(V \oplus V^*) \simeq \wedge^2 V \oplus \mathrm{End}(V) \oplus \wedge^2 V^*$. We now establish how these three components act on E. First consider a 2-form $B \in \wedge^2 V^*$. Then B acts as an endomorphism of E via

$$B(X + \xi) = BX = \iota_X B. \tag{2.22}$$

The exponential $e^B \in SO(E)$ then acts as

$$e^B(X + \xi) = (1 + B)(X + \xi) = X + \xi + \iota_X B. \tag{2.23}$$

The transformation 2.23 is known as a *B-transformation*. Similarly consider $\beta \in \wedge^2 V$. Then β acts on E via

$$\beta(X + \xi) = \iota_\xi \beta \tag{2.24}$$

and its exponential by

$$e^\beta(X + \xi) = X + \iota_\xi \beta + \xi \tag{2.25}$$

known as a *β-transformation*. Lastly consider $A \in \mathrm{End}(V)$. Then its action on E is

$$A(X + \xi) = AX - A^t \xi, \tag{2.26}$$

where the minus sign is used to yield a skew-symmetric action of A on E. The exponential is

$$e^A(X + \xi) = e^A X + e^{-A^t}\xi = e^A X + ((e^A)^t)^{-1}\xi. \tag{2.27}$$

More generally for any $P \in GL(V)$ we have a $SO(E)$ action

$$P(X + \xi) = PX + (P^t)^{-1}\xi. \tag{2.28}$$

We now return to setting of a smooth manifold M with generalized tangent bundle $\pi : E \to M$. The transformations established above still apply, only they become bundle endomorphisms, i.e., sections of $\mathrm{End}(T \oplus T^*)$. For example a B-transform now involves using a section of $\wedge^2 T^*$, a 2-form. These transformations preserve the bilinear form $(\ , \ )$ on each fibre. Nevertheless, it will turn out that B-transformations play a much more important role than the other 2 types of transformations. This fact will be established in section (2.5) but at present the following proposition will give an indication as to why this is so.

Proposition 2.3.1. *[11] Let $u = X + \xi$ and $v = Y + \eta$ be two sections of $T \oplus T^*$ and B a 2-form. Then*

$$[e^B u, e^B v] = e^B [u, v] - \iota_X \iota_Y dB. \tag{2.29}$$

Proof.

$$\begin{aligned}
[e^B u, e^B v] &= [X + \xi + \iota_X B, Y + \eta + \iota_Y B] \\
&= [u, v] + [X, \iota_Y B] + [\iota_X B, Y] \\
&= [u, v] + \mathcal{L}_X \iota_Y B - \frac{1}{2} d\iota_X \iota_Y B - \mathcal{L}_Y \iota_X B + \frac{1}{2} d\iota_Y \iota_X B \\
&= [u, v] + (\mathcal{L}_X \iota_Y - \mathcal{L}_Y \iota_X + d\iota_Y \iota_X) B \\
&= [u, v] + (\mathcal{L}_X \iota_Y - \iota_Y d\iota_X) B \\
&= [u, v] + (\mathcal{L}_X \iota_Y - \iota_Y d\iota_X) B - \iota_Y \iota_X dB + \iota_Y \iota_X dB \\
&= [u, v] + (\mathcal{L}_X \iota_Y - \iota_Y \mathcal{L}_X) B + \iota_Y \iota_X dB \\
&= [u, v] + \iota_{[X,Y]} B - \iota_X \iota_Y dB \\
&= e^B [u, v] - \iota_X \iota_Y dB.
\end{aligned}$$

□

Notice in particular that for B a closed 2-form, the Courant bracket is preserved.

2.4 Gerbes and twisting

We present two extensions to the theory of the generalized tangent bundle presented in section (2.1). In the first extension the generalized tangent bundle is twisted by a gerbe, while the second involves a modification to the Courant bracket and exterior derivative to twisted versions. The relation between the two types of twisting is shown.

2.4.1 Twisting by a gerbe

First we examine twisting by a gerbe. A *gerbe* is one of a number of objects in a hierarchy. Let M be a smooth manifold. The first object is a function $g : M \to U(1)$. Given an open cover $\{U_\alpha\}$ of M. We can also view it as a collection of functions $g_\alpha : U_\alpha \to U(1)$ such that on $U_\alpha \cap U_\beta$, $g_\alpha = g_\beta$, i.e., a 0-cocycle in $\mathrm{H}^0(M, U(1))$. The second object in the hierarchy is a $U(1)$ line bundle which can also be viewed as a 1-cocycle in $\mathrm{H}^1(M, U(1))$ or by the $U(1)$ transition functions $g_{\alpha\beta}$, and the third object is a gerbe, a

2-cocycle in $\mathrm{H}^2(M, U(1))$, a collection of $U(1)$ valued functions $g_{\alpha\beta\gamma}$ defined on the triple intersection $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$ satisfying $g_{\beta\gamma\delta} g^{-1}_{\alpha\gamma\delta} g_{\alpha\beta\delta} g^{-1}_{\alpha\beta\gamma} = 1$ on $U_{\alpha\beta\gamma\delta}$. Continuing the analogy between these objects, one defines a *connective structure* on such a gerbe as a collection of 1-forms $A_{\alpha\beta}$ defined on double intersections such that $A_{\alpha\beta} = -A_{\beta\alpha}$ and on triple intersections satisfying

$$A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma} \tag{2.30}$$

taking exterior derivative gives

$$dA_{\alpha\beta} + dA_{\beta\gamma} + dA_{\gamma\alpha} = 0. \tag{2.31}$$

A standard argument using a partition of unity shows that (possibly after passing to a refinement of the cover) we can find 2-forms B_α defined on the U_α such that on double intersections

$$B_\beta - B_\alpha = dA_{\alpha\beta} \tag{2.32}$$

and finally we have the globally defined 3-form $H = dB_\beta = dB_\beta$ called the *curvature* of the gerbe. Notice that $\frac{1}{2\pi i}[H]$ is integral.

Given a gerbe $\mathcal{G}$ with connective structure $A_{\alpha\beta}$ we can twist $E = T \oplus T^*$ to produce a new vector bundle $E(\mathcal{G})$. Over each U_α, the fibres of $E(G)$ are the same as $T \oplus T^*$ however in the transition from U_α to U_β the fibres are related via a B-transform

$$X + \xi \mapsto X + \xi + \iota_X dA_{\alpha\beta}. \tag{2.33}$$

Thus a section of $E(\mathcal{G})$ is given by a collection $\{X + \xi_\alpha\}$ such that on double intersections they satisfy $\xi_\beta = \xi_\alpha + \iota_X dA_{\alpha\beta}$. Note that by equation (2.31) this construction satisfies the cocycle condition, furthermore the transitions are B-transforms by the closed 2-forms $dA_{\alpha\beta}$ so both the form $(\ , \ )$ and Courant bracket $[\ , \ ]$ are well defined for $E(\mathcal{G})$. The bundle $E(\mathcal{G})$ is called a *generalized tangent bundle* or *twisted generalized tangent bundle* to emphasize the twisting.

Now we consider the relation between gerbe twisting and the spinors $\wedge T^*$. In order for the Clifford action (2.2) to remain well-defined under B-transformations it needs to be equivariant, i.e., the bundle map

$$\wedge E \otimes \wedge T^* \to \wedge T^*$$

given by Clifford multiplication is equivariant under B-transforms. Given a 2-form B, its action as a map $B : T \to T^*$ is an element of $\mathfrak{so}(E)$ and so uniquely lifts to an element of $\mathfrak{spin}(E)$ and then exponentiates to an element of $Spin(E)$ covering the B-transformation $e^B \in SO(E)$. As an element of $Spin(E)$ the action of B on a spinor ω is

$$\omega \mapsto e^{-B} \wedge \omega \tag{2.34}$$

where the exponential is $e^{-B} = 1 - B + \frac{1}{2} B \wedge B + \dots$ and this makes the Clifford action invariant as can be verified directly. Also, since this element of the spin group is in the connected component of the identity it leaves the form $\langle\ ,\ \rangle$ on spinors invariant.

Given a gerbe $\mathcal{G}$ with connective structure $A_{\alpha\beta}$, one wishes to have a well-defined Clifford action of the twisted generalized tangent bundle E on a spinor bundle. Recall E is constructed by patching together $T \oplus T^*$ over U_α to $T \oplus T^*$ over U_β by a B-transformation using $dA_{\alpha\beta}$. In exactly the same way we can use the corresponding action of B-fields on $\wedge T^*$ to construct a bundle of twisted spinors $S(E)$. Equivariance of the Clifford action under B-transforms implies that there is a well defined Clifford action of E on $S(E)$. A section of $S(E)$ is a collection of forms ω_α defined on U_α such that on the double intersection

$$\omega_\beta = e^{-dA_{\alpha\beta}} \omega_\alpha. \tag{2.35}$$

As the form $\langle\ ,\ \rangle$ is invariant under B-transforms the spinors $S(E)$ still retain this form. In addition the exterior derivative is well defined on sections of $S(E)$ for we have

$$d(\omega_\beta) = d(e^{-dA_{\alpha\beta}} \omega_\alpha) = e^{-dA_{\alpha\beta}} d\omega_\alpha$$

Thus after twisting by a gerbe we still retain all the structure of section 2.1. To summarize we have

- A vector bundle $\pi : E \to T$
- A symmetric bilinear form $(\ ,\ )$ on E of signature (n, n)
- A skew-symmetric bracket $[\ ,\ ]$ on sections of E
- A Clifford action of E on a bundle of spinors $S(E)$
- A bilinear form $\langle\ ,\ \rangle$ on the spinors
- A differential operator for spinors $\tilde{d} : \Gamma(S^\pm(E)) \to \Gamma(S^\mp(E))$.

Note that the bracket on E still satisfies equations (2.13), (2.15) and (2.17) (in (2.15) and (2.17), d is still the exterior derivative).

2.4.2 The twisted Courant bracket

We now show how all of this structure can be related back to the untwisted tangent bundle. Given a gerbe $\mathcal{G}$ with connective structure $A_{\alpha\beta}$ we can find 2-forms B_α such that on the double intersection $B_\beta - B_\alpha = dA_{\alpha\beta}$. Such 2-forms yield an isomorphism $\phi : T \oplus T^* \simeq E$ which over U_α is given by $\phi : X + \xi \mapsto X + \xi + \iota_X B_\alpha$. However we will see that E is distinguished from $T \oplus T^*$ by the additional structures. Locally the isomorphism ϕ is a B-transform by B_α. Thus for sections $u = X + \xi$ and $v = Y + \eta$ of E, $(\phi(u), \phi(v)) = (u, v)$ and $[\phi(u), \phi(v)] = \phi([u, v]) - \iota_X \iota_Y dB_\alpha = \phi([u, v]) - \iota_X \iota_Y H$ where $H = dB_\alpha$ over U_α is the curvature of the connective structure. Thus under the identification of $E(\mathcal{G})$ with E, the bracket $[\ ,\ ]$ on $E(\mathcal{G})$ does not get identified with the Courant bracket but rather with a modified bracket which we now define.

Definition 2.4.1. Let H be a closed 3-form. The *twisted Courant bracket* is the bracket on sections $u = X + \xi$ and $v = Y + \eta$ of E given by

$$[u, v]_H = [u, v] - \iota_X \iota_Y H. \tag{2.36}$$

Note that we require H to be closed but it is not necessary that $\frac{1}{2\pi i}[H]$ is integral. The isomorphism ϕ now gives

$$[\phi(u), \phi(v)] = \phi[u, v]_H.$$

The twisted Courant bracket allows us to rewrite equation 2.29 as

$$[e^B u, e^B v]_H = e^B [u, v]_{H+dB}. \tag{2.37}$$

Before examining the twisted Courant bracket any further, let us consider the spinors $S(E)$. Recall that a section of $S(E)$ is a collection of forms ω_α defined on U_α such that on the double intersection

$$\omega_\beta = e^{-dA_{\alpha\beta}} \omega_\alpha.$$

Then we have $e^{B_\beta}\omega_\beta = e^{B_\alpha}\omega_\alpha$ is a globally defined form ω with $\omega_\alpha = e^{-B_\alpha}\omega$. This gives as isomorphism $S(E) \simeq \wedge T^*$. Under this isomorphism we have that the exterior derivative on $S(E)$ becomes $e^{B_\alpha} d e^{-B_\alpha} = d - dB_\alpha = d - H$ where H is the curvature of the connective structure. We use the notation $d_H = d - H$. Thus spinors for E can be thought of as differential forms but with the exterior derivative d replaced by d_H.

Thus we have transported all the structures of the twisted generalized tangent bundle back to $E = T \oplus T^*$. In summary we have

- The bundle $\pi : E = T \oplus T^* \to T$

- A symmetric bilinear form $(\; , \;)$ on E of signature (n,n)
- The twisted Courant bracket $[\; , \;]_H$ on sections of E
- A Clifford action of E on a bundle of spinors $S(E) = \wedge T^*$
- A bilinear form $\langle \; , \; \rangle$ on the spinors
- A differential operator for spinors, $d_H : \Gamma(S^{\pm}(E)) \to \Gamma(S^{\mp}(E))$.

The twisted bracket $[\; , \;]_H$ still satisfies equations (2.13), (2.15) and (2.17), however it no longer agrees with the Lie bracket on vector fields as there is an additional term. Thus at times we will need to distinguish between the twisted Courant bracket and the Lie bracket.

The twisted Courant bracket means that ultimately one can avoid using gerbes and use only the untwisted tangent bundle $E = T \oplus T^*$ but with a twisted Courant bracket and twisted exterior derivative on spinors. The point of view we adopt will usually be the latter, as it does not require the 3-form H to be integral and it avoids having to introduce a local cover.

2.5 Courant automorphisms

At this point one is tempted to ask if there is a natural way to define morphisms for generalized tangent bundles so that they form a category, presumably with a forgetful functor back to the category of smooth manifolds. It is not clear if there is a fruitful definition of such morphisms between different manifolds yet there is a natural concept of automorphism which we call *Courant automorphism*. The definition and characterization of these maps is the subject of this section.

We are interested in diffeomorphisms of E which preserve all the structure of E. In particular this means that such a map φ must act on any given fibre as a linear map to another fibre, i.e., there is an underlying diffeomorphism f of M such that $\pi\varphi = f\pi$. The map φ must also preserve the inner product and the twisted Courant bracket, $[\varphi u, \varphi v]_H = \varphi[u, v]_H$. The following proposition characterizes such maps:

Proposition 2.5.1. *[11]*

Let $f : M \to M$ be a diffeomorphism and let $\varphi : E \to E$ be a diffeomorphism of the generalized tangent bundle $E = TM \oplus T^*M$ such that restricted to the fibre over any $x \in M$, $\varphi : E_x \to E_{f(x)}$ is a linear map from E_x to $E_{f(x)}$ and such that φ preserves $(\; , \;)$ and $[\; , \;]_H$. Then φ is a composition of the *map $f_* \oplus (f^{-1})^* : E \to E$ with a B-field transformation by a closed 2-form B.*

Proof. The map $f_* \oplus (f^{-1})^* : E \to E$ satisfies the assumptions in the proposition. Thus we may compose φ with the inverse of $f_* \oplus (f^{-1})^*$, hence it suffices to assume f is the identity, in which case φ is simply a section of $\mathrm{End}(E)$. We now use equation 2.17 to proceed. Let u, v be sections of E and h be a function. Then

$$\begin{aligned} \varphi[u, hv] &= [\varphi u, h\varphi v] \\ &= h[\varphi u, \varphi v] + (\pi\varphi(u)h)\varphi v - (\varphi u, \varphi v)dh \\ &= h\varphi[u, v] + (\pi\varphi(u)h)\varphi v - (u, v)dh. \end{aligned}$$

Yet also

$$\begin{aligned} \varphi[u, hv] &= \varphi(h[u, v] + (\pi(u)h)v - (u, v)dh) \\ &= h\varphi[u, v] + \varphi((\pi(u)h)v) - (u, v)\varphi(dh) \\ &= h\varphi[u, v] + (\pi(u)h)\varphi v - (u, v)\varphi dh. \end{aligned}$$

So equating these we have

$$(\pi\varphi(u)h)\varphi v - (u, v)dh = (\pi(u)h)\varphi v - (u, v)\varphi dh. \tag{2.38}$$

First consider the case when $u = X$ and $v = Y$ are vector fields. Substituting and applying π to equation (2.38) gives

$$(\pi\varphi(X)h)\pi\varphi Y = (Xh)\pi\varphi Y.$$

Note that $\pi\varphi : T \to T$ is an endomorphism of T which by (2.13) preserves the Lie bracket. So $\pi\varphi$ can not completely vanish and it follows that $\pi\varphi X = X$. Now letting $u = X$ and $v = \eta$, a 1-form and substituting into (2.38) gives

$$(Xh)\varphi\eta - \eta(X)dh = (Xh)\varphi\eta - \eta(X)\varphi dh$$

which implies that φ acts as the identity on 1-forms. As a matrix φ must therefore have the form

$$\varphi = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$$

and moreover since φ preserves the form $(\ ,\ )$ it follows that B is skew-symmetric. Thus φ is a B-transform and as it preserves $[\ ,\ ]_H$ we must also have $dB = 0$. $\square$

Thus we define a *Courant automorphism* as the composition of a diffeomorphism and a closed B-transform. There is however a larger class of maps which will sometimes be useful. If we transform by a B-field then equation (2.37) shows that when B is not closed then one twisted Courant bracket is mapped onto another twisted Courant bracket. In this sense such a B-transform is a morphism carrying the structure of one twisted bracket to another.

Chapter 3

Integrability

The Frobenius integrability theorem is a remarkable result because it connects the geometric notion of foliations to the analytic notion of involutive subbundles of the tangent space. It is natural therefore to question whether there is a geometric interpretation of subbundles of $T \oplus T^*$ which are involutive with respect to the Courant bracket. This is indeed the case and is a topic of considerable importance to generalized geometry. The Chapter is largely influenced by the [11].

Definition 3.0.1. Let V be a subbundle of $T \oplus T^*$ we say that V is *Courant involutive* or *Courant integrable* if the sections of V are closed under the Courant bracket.

We shall similarly use the terms *Frobenius involutive* or *Frobenius integrable* for subbundles of T that are involutive with respect to the Lie bracket.

3.1 Involutive subbundles

It turns out that Courant integrable subbundles are divided into two classes. This classification is closely related to the notion of isotropic subbundles which we now define.

Definition 3.1.1. Let V be a subbundle of $T \oplus T^*$. We say that V is *isotropic* if the restriction of the bilinear form $(\ ,\ )$ to V completely vanishes. An isotropic subbundle is further called *maximal isotropic* if it has the same rank as the tangent bundle which is the largest possible rank for an isotropic subbundle.

We argue that $n = \mathrm{rk}(T)$ is the maximal rank for an isotropic subbundle as follows: let I be an isotropic subbundle of E. Non-degeneracy of $(\ ,\ )$

gives an isomorphism $E \simeq E^*$ and upon restriction a surjective map $\delta : E \to I^*$. The inclusion $\iota : I \to E$ maps into the kernel of δ implying $\mathrm{rk}(E) = \mathrm{rk}(\ker(\delta)) + \mathrm{rk}(\delta) \geq \mathrm{rk}(I) + \mathrm{rk}(I^*)$, hence $2\,\mathrm{rk}(I) \leq \mathrm{rk}(E) = 2\,\mathrm{rk}(T) = 2n$. On the other hand we know that T and T^* are isotropics of dimension n so this is the maximal rank attained. We can now state a classification result for Courant involutive subbundles.

Proposition 3.1.1. *[11] Let V be a Courant involutive subbundle of $E = T \oplus T^*$. Then either V is isotropic or V has the form $V = U \oplus T^*$ where U is a non-zero subbundle of T which is Frobenius involutive.*

Proof. Suppose V is involutive but not isotropic, so there is a section $X + \xi$ such that at some point $m \in M$, $\xi(X)_m \neq 0$. But then for any function f, $[X + \xi, f(X + \xi)] = (Xf)(X + \xi) - \xi(X)df$ so by integrability of V we have $df_m \in V_m$ for every f and so $T^*_m \subseteq V_m$. The inclusion must be proper since T^*_m is an isotropic subspace. This shows that the rank of V exceeds the maximal rank for isotropics and hence every fibre of V fails to be isotropic. Therefore the above argument applies to every fibre of V showing that $T^* = \ker(\pi_T : V \to T)$ is a smooth subbundle of V. Then since T^* is isotropic the inclusion must be proper. Hence V can be written $V = U \oplus T^*$ where U us a non-zero subbundle of T. Lastly U must be Frobenius integrable since the Courant bracket agrees with the Lie bracket on vector fields. $\square$

We see that there are essentially two types of Courant integrable subbundles. Those of type $V = U \oplus T^*$ are uniquely determined by $U = \pi_T(V)$, a Frobenius integrable subbundle and as such they do not give us any new types of geometry. This leaves the second, more interesting type of Courant integrable subbundle, those that are isotropic. The most interesting case is when we have an integrable, maximal isotropic subbundle for we have the following proposition:

Proposition 3.1.2. *[11] Let V be a maximal isotropic subbundle of E. Then the following are equivalent:*

- *V is involutive*
- $\mathrm{Jac}|_V = 0$
- $\mathrm{Nij}|_V = 0$.

Remark 3.1.1. This also holds for maximal isotropic subbundles of $T \otimes \mathbb{C}$. These will be of particular interest later.

Proof. Suppose V is involutive. Then since V is also isotropic it follows from (2.21) that $\mathrm{Nij}|_V = 0$.

Now suppose $\mathrm{Nij}|_V = 0$. This immediately implies that $\mathrm{Jac}|_V = 0$. Lastly suppose $\mathrm{Jac}|_V = 0$. It remains to show V is involutive. Suppose to the contrary that there are sections u, v of V such that $[u, v]$ is not a section of V. Then since V is maximal isotropic there is a third section w of V such that $([u, v], w) \neq 0$. Note that by (2.15) we have that for any three section a, b, c of V

$$0 = \pi(a)(b, c) = ([a, b], c) + (b, [a, c]).$$

from which it follows that

$$\begin{aligned}
\mathrm{Nij}|_V(u, v, w) &= \tfrac{1}{3}((([u, v], w) + ([v, w], u) + ([w, u], v) \\
&= \tfrac{1}{3}((([u, v], w) - ([v, u], w) - ([u, w], v)) \\
&= \tfrac{1}{3}((([u, v], w) - ([v, u], w) + ([u, v], w)) \\
&= ([u, v], w).
\end{aligned}$$

Then for any function f we have

$$\begin{aligned}
0 &= \mathrm{Jac}|_V(u, v, fw) \\
&= d\mathrm{Nij}|_V(u, v, fw) \\
&= d(f\mathrm{Nij}|_V(u, v, w)) \\
&= df\mathrm{Nij}|_V(u, v, w) + f\mathrm{Jac}|_V(u, v, w) \\
&= df([u, v], w).
\end{aligned}$$

This is a contradiction and therefore V is integrable. $\square$

Note that in this proof we discovered that the Nijenhuis operator, when restricted to an isotropic subbundle V takes the form

$$\mathrm{Nij}|_V(u, v, w) = ([u, v], w) = ([v, w], u) = ([w, u], v) \tag{3.1}$$

and in particular its restriction to an isotropic bundle is tensorial, indeed it is a section of $\wedge^3 V^*$. The interesting conclusion to draw from this result is that for a maximal isotropic subbundle V, integrability is equivalent to the Courant bracket satisfying the Jacobi identity on sections of V.

3.2 Lie algebroids

We have seen that for integrable maximal isotropics, the Courant bracket satisfies the Jacobi identity on its sections. In addition equation (2.17) simplifies to

$$[u, fv] = f[u, v] + (\pi(u)f)v.$$

Bundles with such structure play an important role in generalized geometry so we now examine a suitable abstraction of this data.

Definition 3.2.1. Let M be a smooth manifold. A *Lie algebroid* is a vector bundle V over M equipped with the following structures:

- a bundle morphism $a : V \to TM$ called the *anchor*
- a bilinear form $[\ ,\ ] : \Gamma(V) \otimes \Gamma(V) \to \Gamma(V)$ making $(\Gamma(V), [\ ,\ ])$ a Lie algebra

such that $a : \Gamma(V) \to \Gamma(TM)$ is a Lie algebra homomorphism and

$$[u, fv] = f[u, v] + (a(u)f)v \tag{3.2}$$

where u, v are sections of V and f is a function on M.

Note that we can also consider complex Lie algebroids where TM is replaced by its complexification. The generalized tangent bundle E fails to be a Lie algebroid as it fails to satisfy two conditions in the definition, however when restricted to an integrable maximal isotropic subbundle both of these conditions are satisfied and provides the key example of a Lie algebroid. Another example is given by Frobenius integrable subbundles of T with the anchor being inclusion. Lie algebroids generalize the structure of the tangent bundle and keeping with this analogy we now devolop their differential geometry.

First we define the differential $d_V : \Gamma(\wedge^k V^*) \to \Gamma(\wedge^{k+1} V^*)$ by

$$\begin{aligned} d_V\omega(u_0, u_1, \ldots, u_k) = &\sum_{i=0}^{k} (-1)^i a(u_i)\omega(u_0, \ldots, \hat{u}_i, \ldots, u_k) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([u_i, u_j], u_0, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_k). \end{aligned} \tag{3.3}$$

It should be noted that $d_V\omega$ is tensorial because of property (3.2) and furthermore this satisfies $d_V^2 = 0$ and $d_V \circ a^* = a^* \circ d$ where d is the ordinary exterior derivative. Next we have contraction $\iota : \Gamma(V) \otimes \Gamma(\wedge^{k+1} V^*) \to \Gamma(\wedge^k V^*)$ which for $u \in \Gamma(V)$ is written $\iota_u : \Gamma(\wedge^{k+1} V^*) \to \Gamma(\wedge^k V^*)$ and is given by

$$(\iota_u\omega)(u_1, \ldots, u_k) = \omega(u, u_1, \ldots, u_k). \tag{3.4}$$

Naturally this leads to defining a Lie derivative $\mathcal{L} : \Gamma(V) \otimes \Gamma(\wedge^k V^*) \to \Gamma(\wedge^k V^*)$ given by

$$\mathcal{L}_u = d_V \iota_u + \iota_u d_V. \tag{3.5}$$

These three types of operators span a graded Lie algebra satisfying

$$[d_V, \mathcal{L}_u] = 0 \tag{3.6a}$$
$$\{d_V, \iota_v\} = \mathcal{L}_v \tag{3.6b}$$
$$[\mathcal{L}_u, \iota_v] = \iota_{[u,v]} \tag{3.6c}$$

where $\{\ ,\ \}$ denotes an anti-commutator. The graded algebra $\Gamma(\wedge V^*)$ is then a representation of this algebra. Furthermore we have that $d_V, \iota_v, \mathcal{L}_u$ are graded derivations of the wedge product of degrees $1, -1, 0$. That is we have:

$$d_V(\alpha \wedge \beta) = d_V\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d_V\beta \tag{3.7a}$$
$$\iota_v(\alpha \wedge \beta) = \iota_v\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \iota_v\beta \tag{3.7b}$$
$$\mathcal{L}_u(\alpha \wedge \beta) = \mathcal{L}_u\alpha \wedge \beta + \alpha \wedge \mathcal{L}_u\beta. \tag{3.7c}$$

We define the cohomology of a Lie algebroid to be the cohomology of the differential complex $(\Gamma(\wedge V^*), d_V)$ and shall be denoted $\mathrm{H}^k(V)$. Given a function f and section u of V we have by definition $(d_V f)(u) = (a(u))(f) = (df)(a(u))$ so that $d_V f = a^* df$. Now given a section w of $\wedge V^*$ we have that $[d_V, f]w = d_V(fw) - f d_V w = d_V(f) \wedge w = a^*(df) \wedge w$. This shows that d_V is a first order differential operator with principal symbol $s : T^*M \to \mathrm{End}(\wedge V^*)$ given by $s(\xi) = a^*(\xi)\wedge$, moreover it is clear that if $s(\xi)w = a^*(\xi) \wedge w = 0$, where $\xi \neq 0$ then w has the form $w = a^*(\xi) \wedge v$ and so the complex $(\Gamma(\wedge^* V^*), d_V)$ is elliptic and in particular for compact manifolds Lie algebroid cohomology is finite dimensional.

There is another particularly useful construction for Lie algebroids which is an extension of the bracket $[\ ,\ ]$ to a graded skew-symmetric bracket on sections of $\wedge V$ called the *Schouten bracket* defined by

$$\begin{aligned}[u_1 \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_q] \\ = \sum_{i,j} (-1)^{i+j}[u_i, v_j] \wedge u_1 \wedge \cdots \wedge \hat{u_i} \wedge \cdots \wedge u_p \wedge v_1 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_q\end{aligned} \tag{3.8}$$

and $[u, f] = -[f, u] = a(u)f$. This bracket makes $\Gamma(\wedge V)$ a graded Lie algebra with the degree k part being $\Gamma(\wedge^{k+1} V)$. That is, we have the following identities:

$$[u, v] = -(-1)^{(u-1)(v-1)}[v, u] \tag{3.9a}$$

$$[u, [v, w]] = [[u, v], w] + (-1)^{(u-1)(v-1)}[u, [v, w]] \tag{3.9b}$$

where we use the notation $(-1)^u = (-1)^k$ to denote the sign of the degree of $u \in \Gamma(\wedge^k V)$. The second relation says that $\mathrm{ad}_u = [u, \ ]$ is a derivation of $[\ ,\ ]$ of degree $\deg(u) - 1$. We also have that ad_u is a derivation of the wedge product of degree $\deg(u) - 1$, that is

$$[u, v \wedge w] = [u, v] \wedge w + (-1)^{(u-1)v} v \wedge [u, w]. \tag{3.10}$$

3.3 Generalized foliations

We have seen that isotropic Courant integrable subbundles of the generalized tangent bundle are Lie algebroids. In this section we explain the geometric implications of Lie algebroids on the underlying manifold.

We need the concept of a *generalized foliation.* A *leaf* of M is an injective immersion $l \subseteq M$ such that for each point $x \in l$ there is an open neighborhood U of x in M such that the connected component of x in $U \cap M$ is an embedded submanifold of M. A *generalized foliation* of M is a collection of leaves which form a disjoint cover for M. In the usual definition of a foliation the leaves all have the same dimension but for generalized foliations they are allowed to vary. A *distribution* $\triangle$ is a collection of subspaces $\{\triangle(x) | x \in M\}$ such that $\triangle(x)$ is a subspace of $T_x M$. We say that a distribution $\triangle$ is of *finite type* if for any $x \in M$ there is a neighborhood U of x in M such that over U there exist smooth vector fields $X_1, X_2, \ldots X_k$ such that for any $y \in U$, $\triangle_y$ is the space spanned by $X_1(y), \ldots X_k(y)$. Note that we are not assuming pointwise linear independence of the $X_1, \ldots X_n$ so that the dimensions of the $\triangle_y$ may vary. Note that a generalized foliation gives rise to a distribution by taking the tangent spaces of the leaves; however it need not be a finite type distribution. This is clear from the fact that the dimension of the leaves of a finite type distribution must be lower semi-continuous which in turn follows from the fact that the dimension of the span of vector fields $X_1, \ldots X_k$ is lower semi-continuous. The importance of finite type distributions is that, as worked out by Sussmann [19], the Frobenius theorem generalizes:

Proposition 3.3.1. *[19] Let $\triangle$ be a distribution of finite type. Then $\triangle$ is the distribution arising from a generalized foliation if and only if it is involutive, that is if X, Y are vector fields with values in $\triangle$ then so is $[X, Y]$.*

We now show how Lie algebroids give rise to generalized foliations. Let V be a real Lie algebroid. Associated to V is the distribution $\triangle = a(V)$ where $a : V \to T$ is the anchor. Let $X_1, \dots X_k$ be a local frame for V. Then $a(X_1), \dots a(X_k)$ locally span $a(V) = \triangle$ so we have that $\triangle$ is of finite type. Moreover, since $[a(u), a(v)] = a([u, v])$ it follows that $\triangle$ is involutive and therefore gives rise to a generalized foliation.

Now consider a complex Lie algebroid V with anchor $a : V \to T \otimes \mathbb{C}$. This time we can not just take $a(V)$ so first we define $K = \{X \in V | a(X) = \overline{a(X)}\}$ and we let $\triangle = a(K) \subseteq T$. We have that $\triangle$ is involutive using $\overline{[a(u), a(v)]} = [\overline{a(u)}, \overline{a(v)}]$. However we can not so easily conclude that $\triangle$ is of finite type. To proceed note that K is the kernel of $i(a - \overline{a}) : V \to T$ so when this map is surjective, K is a smooth subbundle of V and we can use a local frame of K to show $\triangle$ is of finite type. But $i(a - \overline{a}) : V \to T$ is surjective if and only if $a(V) + \overline{a(V)} = T \otimes \mathbb{C}$.

To summarise, a real Lie algebroid V always gives rise to a distribution $\triangle = a(V)$ which arises from a generalized foliation. A complex Lie algebroid V gives rise to a distribution $\triangle = \{a(X) | a(X) = \overline{a(X)}\}$ and in the case when $a(V) + \overline{a(V)} = T \otimes \mathbb{C}$ we have that $\triangle$ arises from a generalized foliation.

There is an additional feature for complex Lie algebroids. As before we assume $a(V) + \overline{a(V)} = T \otimes \mathbb{C}$. Let us also assume we are in a neighborhood in which the dimension of $\triangle \otimes \mathbb{C} = a(V) \cap \overline{a(V)}$ is constant. Thus $a(V)$ has constant dimension as well. Note that $a(V)$ is a complex integrable subbundle of $T \otimes \mathbb{C}$ and that since $a(V) + \overline{a(V)}$ is integrable we can use the Newlander-Nirenberg theorem to conclude that in a neighborhood there are complex valued functions $\{z_1, \dots z_m\}$ such that $\{dz_1, \dots dz_m\}$ are pointwise linearly independent and such that they annihilate $a(V)$. That is, if $a(V)$ has codimension k, then there are k transverse complex coordinate functions $\{z_1, \dots, z_k\}$.

Definition 3.3.1. Let $\triangle$ be a generalized distribution on M. A point $x \in M$ is called a *regular point* of the distribution if x has a neighborhood in which the dimension of $\triangle$ is constant.

Thus we have that a complex Lie algebroid V such that $a(V) + \overline{a(V)} = T \otimes \mathbb{C}$, then we have a generalized distribution such that in a neighborhood of a regular point there are transverse complex coordinates.

3.4 Dirac structures

We have seen that amongst the integrable subbundles, a particularly interesting class are those that are maximal isotropic. Therefore we make the following definition:

Definition 3.4.1. An *almost Dirac structure* is a maximal isotropic subbundle of the generalized tangent bundle. An almost Dirac structure is said to be integrable to a *Dirac structure* if it is Courant integrable.

Note that we can also define complex Dirac structures by considering subbundles of the complexified generalized tangent bundle. These will play a substantial role later on. Given an almost Dirac structure V, it follows from (3.1.2) that V is a Dirac structure if and only if the Nijenhius operator completely vanishes on V and moreover since V is isotropic the Nijenhius operator restricted to V is given by (3.1).

We give some examples of Dirac structures:

Example 3.4.1 (Foliations). Let $U \subseteq T$ be a smooth distribution of constant rank. Then U determines a maximal isotropic bundle $V = U \oplus \mathrm{Ann}(U)$. We claim that V is Courant involutive if and only if U is Frobenius involutive, i.e., if and only if U arises from a foliation of the manifold. First it is clear that if V is Courant involutive then U is Frobenius involutive. Conversely assume U is Frobenius involutive. From (3.3) it is clear that if $\xi \in \Gamma(\mathrm{Ann}(U))$ then $d\xi \in \Gamma(\mathrm{Ann}(U))$. This clearly implies V is involutive.

Example 3.4.2 (Pre-symplectic geometry). The tangent bundle is a Dirac structure. If we use a 2-form $\omega \in \Gamma(\wedge^2 T^*)$ to B-transform T we have that

$$e^\omega T = \{X + \omega X |\ X \in T\}$$

is an almost Dirac structure. In fact, using (2.29) we see that $e^\omega T$ is integrable if and only if $d\omega = 0$. More explicitly, we calculate Nijenhius operator for sections $e^\omega X, e^\omega Y, e^\omega Z \in \Gamma(e^\omega T)$.

$$\begin{aligned}
\mathrm{Nij}(e^\omega X, e^\omega Y, e^\omega Z) &= ([e^\omega X, e^\omega Y], e^\omega Z) \\
&= (e^\omega [X, Y] - \iota_X \iota_Y d\omega, e^\omega Z) \\
&= ([e^\omega [X, Y], e^\omega Z) - (\iota_X \iota_Y d\omega, Z + \omega Z) \\
&= ([X, Y], Z) - \tfrac{1}{2} \iota_Z \iota_X \iota_Y d\omega \\
&= \tfrac{1}{2} d\omega(X, Y, Z)
\end{aligned}$$

where we have used the fact that $e^\omega T$ is isotropic so that we may use the simpler expression for the Nijenhius operator on $e^\omega T$.

Example 3.4.3 (Poisson geometry). In contrast to the last example we can start with T^* which is also a Dirac structure and then perform a β-transform (see equation 2.25) by a bi-vector field $\beta \in \Gamma(\wedge^2 T)$. Thus we have that

$$e^{\beta}T^* = \{\xi + \beta\xi |\ \xi \in T^*\}$$

is an almost Dirac structure. The bi-vector β determines a bracket operation on functions f, g given by

$$\{f, g\} = \beta(df, dg). \tag{3.11}$$

We determine when this space is integrable by determining $\mathrm{Nij}|_{e^{\beta}T^*}$. Since $\mathrm{Nij}|_{e^{\beta}T^*}$ is tensorial, it suffices to consider three sections of the form $e^{\beta}df, e^{\beta}dg, e^{\beta}dh$ for functions f, g, h. We find that

$$\begin{aligned}
& \mathrm{Nij}|_{e^{\beta}T^*}(e^{\beta}df, e^{\beta}dg, e^{\beta}dh) \\
= \ & ([df + \beta df, dg + \beta dg], dh + \beta dh) \\
= \ & ([\{f,\ \} + df, \{g,\ \} + dg], \{h,\ \} + dh) \\
= \ & ([\{f,\ \}, \{g,\ \}] + d\{f,g\} - d\{g,f\} - \tfrac{1}{2}d(\{f,g\} - \{g,f\}), \{h,\ \} + dh) \\
= \ & ([\{f,\ \}, \{g,\ \}] + d\{f,g\}, \{h,\ \} + dh) \\
= \ & \tfrac{1}{2}(dh([\{f,\ \}, \{g,\ \}]) + \{h, \{f,g\}\}) \\
= \ & \tfrac{1}{2}(\{f, \{g,h\}\} - \{g, \{f,h\}\} + \{h, \{f,g\}\}) \\
= \ & \tfrac{1}{2}(\{f, \{g,h\}\} + \{g, \{h,f\}\} + \{h, \{f,g\}\}).
\end{aligned}$$

Therefore we find that the almost Dirac structure $e^{\beta}T^*$ is integrable if and only if the bracket $\{\ ,\ \}$ defined by β satisfies the Jacobi identity, i.e., β is a Poisson structure. One can also rephrase this using the Schouten bracket. Using (3.8) we find that

$$[\beta, \beta](df, dg, dh) = 2\{f, \{g,h\}\} + 2\{g, \{h,f\}\} + 2\{h, \{f,g\}\} \tag{3.12}$$

and it follows that

$$\mathrm{Nij}|(e^{\beta}\xi, e^{\beta}\eta, e^{\beta}\phi) = \tfrac{1}{4}[\beta, \beta](\xi, \eta, \phi) \tag{3.13}$$

so $e^{\beta}T^*$ is integrable if and only if $[\beta, \beta] = 0$.

Suppose we are given two transverse Dirac Structures V and V', that is $E = V \oplus V'$. Then the form $(\ ,\ )$ allows us to identify V' with V^* so we have $E = V \oplus V^*$. This generalizes the splitting $E = T \oplus T^*$. Notice that the bilinear form on E given by the pairing of V and V^* is the same as the form obtained by pairing T and T^*.

Proposition 3.4.1. *[18] Let $E = V \oplus V^*$ be a splitting of the generalized tangent bundle into transverse Dirac structures. Given sections u, v of E write them as $u = X + \xi$ and $v = Y + \eta$ where $X, Y \in \Gamma(V)$, $\xi, \eta \in \Gamma(V^*)$. Then the Courant bracket of u and v is given by*

$$\begin{aligned}[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d_V(\iota_X\eta - \iota_Y\xi) \\ + [\xi,\eta] + \mathcal{L}_\xi Y - \mathcal{L}_\eta X - \tfrac{1}{2}d_{V^*}(\iota_\xi Y - \iota_\eta X).\end{aligned} \tag{3.14}$$

Proof. It is clear that when u, v are sections of V then (3.14) simplifies to $[u, v]$ and similarly for sections of V^*. Thus we assume $u = X \in \Gamma(V)$ and $v = \eta \in \Gamma(V^*)$. Note that since $X \in \Gamma(V)$ we have $\pi(X) = a(X)$ where $a : V \to T$ is the anchor of the Lie algebroid V. Furthermore if $a_* : V^* \to T$ is the anchor for V^* then $a \oplus a_* : V \oplus V^* \to T$ is the projection $\pi : T \oplus T^* \to T$. We have seen that for a function f, we have $d_V f = f \circ a$ so that $(df, \ ) = \frac{1}{2}df \circ \pi = (d_V f + d_{V^*} f, \ )$. Using (2.15) we thus find

$$\begin{aligned}\pi(X)(\eta, Y) &= a(X)(\eta, Y) \\ &= ([X,\eta] + d(X,\eta), Y) + (\eta, [X,Y] + d(X,Y)) \\ &= ([X,\eta], Y) + (d_V(X,\eta), Y) + (\eta, [X,Y]) \\ &= \tfrac{1}{2}\iota_Y[X,\eta] + \tfrac{1}{4}\iota_Y d_V\eta(X) + \tfrac{1}{2}\eta([X,Y]) \\ &= \tfrac{1}{2}\iota_Y[X,\eta] + \tfrac{1}{4}a(Y)\eta(X) + \tfrac{1}{2}\eta([X,Y]).\end{aligned}$$

Therefore

$$\begin{aligned}\iota_Y[X,\eta] &= a(X)\eta(Y) - \tfrac{1}{2}a(Y)\eta(X) - \eta([X,Y]) \\ &= (d_V\eta)(X,Y) + \tfrac{1}{2}a(Y)\iota_X\eta \\ &= \iota_Y\iota_X d_V\eta + \tfrac{1}{2}\iota_Y d_V(\iota_X\eta) \\ &= \iota_Y(\mathcal{L}_X\eta - \tfrac{1}{2}d_V(\iota_X\eta)).\end{aligned}$$

Thus the V^* part of $[X, \eta]$ is $\mathcal{L}_X\eta - \frac{1}{2}d_V(\iota_X\eta)$. Interchanging V and V^* we have that the V part of $[\eta, X]$ is $\mathcal{L}_\eta X - \frac{1}{2}d_{V^*}(\iota_\eta X)$. Thus we have found

$$[X,\eta] = \mathcal{L}_X\eta - \tfrac{1}{2}d_V(\iota_X\eta) - \mathcal{L}_\eta X + \tfrac{1}{2}d_{V^*}(\iota_\eta X)$$

which completes the proof. $\square$

Now that we have the general formula for the Courant bracket we are ready to prove a key result for Dirac structures:

Proposition 3.4.2. *[18] Let $E = V \oplus V^*$ be a splitting of E into Dirac strucures and let $\epsilon \in \Gamma(\wedge^2 V^*)$. Then $V_\epsilon = \{X + \epsilon X |\ X \in V\}$ is an almost*

Dirac structure. V_ϵ is integrable if and only if ϵ satisfies the generalized Maurer-Cartan equation

$$d_V\epsilon + \tfrac{1}{2}[\epsilon,\epsilon] = 0 \tag{3.15}$$

where $d_V : \Gamma(\wedge^2 V^*) \to \Gamma(\wedge^3 V^*)$ is the differential of V and $[\ ,\ ] : \Gamma(\wedge^2 V^*) \to \Gamma(\wedge^3 V^*)$ is the Schouten bracket for V^*.

Proof. It is clear that V_ϵ is an almost Dirac structure. We will complete the proof by showing that for sections u, v, w of V we have

$$\mathrm{Nij}(e^\epsilon u, e^\epsilon v, e^\epsilon w) = \tfrac{1}{2}(d_V\epsilon + \tfrac{1}{2}[\epsilon,\epsilon])(u,v,w). \tag{3.16}$$

Using (3.14) we may write the Courant bracket as $[\ ,\ ] = [\ ,\ ]_V + [\ ,\ ]_{V^*}$ where

$$[X+\xi, Y+\eta]_V = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d_V(\iota_X\eta - \iota_Y\xi) \tag{3.17}$$
$$[X+\xi, Y+\eta]_{V^*} = [\xi,\eta] + \mathcal{L}_\xi Y - \mathcal{L}_\eta X - \tfrac{1}{2}d_{V^*}(\iota_\xi Y - \iota_\eta X). \tag{3.18}$$

Thus

$$\mathrm{Nij}|_{V_\epsilon}(e^\epsilon u, e^\epsilon v, e^\epsilon w) = ([e^\epsilon u, e^\epsilon v], e^\epsilon w) = ([e^\epsilon u, e^\epsilon v]_V, e^\epsilon w) + ([e^\epsilon u, e^\epsilon v]_{V^*}, e^\epsilon w).$$

Thus we need only show

$$([e^\epsilon u, e^\epsilon v]_V, e^\epsilon w)|_{V_\epsilon} = \tfrac{1}{2}(d_V\epsilon)(u,v,w)$$

and

$$([e^\epsilon u, e^\epsilon v]_{V^*}, e^\epsilon w)|_{V_\epsilon} = \tfrac{1}{4}[\epsilon,\epsilon](u,v,w).$$

But we have already seen this in the examples (3.4.2) and (3.4.3). The algebra here is essentially the same and so we have completed the proof. □

3.5 Integrability in the twisted case

So far in this chapter we have only considered using the untwisted Courant bracket. Generalizing the results to the twisted Courant bracket is straightforward and we summarise here. In this section we consider twisting as the modification of the Courant bracket by a closed 3-form H.

First of all one defines the Jacobiator and Nijenhuis operators by the same equations (2.18) and (2.21) and and one finds that

$$\mathrm{Nij}_H(u,v,w) = \mathrm{Nij}(u,v,w) + \tfrac{1}{2}H(\pi(u),\pi(v),\pi(w)) \tag{3.19a}$$
$$\mathrm{Jac}_H(u,v,w) = \mathrm{Jac}(u,v,w) - \tfrac{1}{2}d(\iota_{\pi(u)}\iota_{\pi(v)}\iota_{\pi(w)}H) \tag{3.19b}$$

where the subscript denotes that the operator corresponds to the twisted bracket. We have that Proposition (3.1.2) still holds in the twisted case. On isotropic subbundles we still have that Nij_H is tensorial and given by the simpler formula (3.1). Now twisted Courant involutive subbundles, just like the untwisted case are Lie algebroids. We can define a twisted Dirac structure, noting however that there is no difference between almost Dirac structures in the twisted and untwisted case. We now re-examine two of our examples of Dirac structures in the twisted case.

Example 3.5.1 (Twisted pre-symplectic geometry)*.* Let $\omega \in \Gamma(\wedge^2 T^*)$ and consider the almost Dirac structure $V = e^\omega = \{X + \omega X | \ X \in T\}$. With the aid of example (3.4.2) we see that

$$\mathrm{Nij}_H|_V(e^\omega X, e^\omega Y, e^\omega Z) = \tfrac{1}{2}(d\omega + H)(X,Y,Z).$$

Therefore V is twisted Courant integrable if and only if $d\omega = -H$.

Example 3.5.2 (Twisted Poisson geometry). Let $\beta \in \Gamma(\wedge^2 T)$ and consider the almost Dirac structure $V = e^\beta T^* = \{\xi + \beta\xi | \ \xi \in T^*\}$. Once again we refer back to the untwisted case, example (3.4.3) to see that

$$\begin{aligned}
\mathrm{Nij}_H|_V(e^\beta\xi, e^\beta\eta, e^\beta\phi) &= \tfrac{1}{4}[\beta,\beta](\xi,\eta,\phi) + \tfrac{1}{2}H(\pi(e^\beta\xi), \pi(e^\beta\eta), \pi(e^\beta\phi)) \\
&= \tfrac{1}{4}[\beta,\beta](\xi,\eta,\phi) + \tfrac{1}{2}H(\beta\xi, \beta\eta, \beta\phi) \\
&= \tfrac{1}{2}(\tfrac{1}{2}[\beta,\beta] + \beta^* H)(\xi,\eta,\phi),
\end{aligned}$$

where the bracket is the (untwisted) Schouten bracket and $\beta^* H$ is the pull-back of H by $\beta : T^* \to T$. Therefore by equation (3.12), we see that V is a twisted Dirac structure if and only if

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = -\beta^* H(df, dg, dh). \tag{3.20}$$

Finally, we can combine these two examples just as we did in the untwisted case to prove the following:

Proposition 3.5.1. *Let $H \in \Gamma(\wedge^3 T^*)$ be a closed form used to twist the Courant bracket. Let $E = V \oplus V^*$ be a splitting of E into twisted Dirac strucures and let $\epsilon \in \Gamma(\wedge^2 V^*)$. Then $V_\epsilon = \{X + \epsilon X | \ X \in V\}$ is an almost Dirac structure. V_ϵ is twisted Courant integrable if and only if ϵ satisfies the twisted generalized Maurer-Cartan equation*

$$d_V \epsilon + \tfrac{1}{2}[\epsilon,\epsilon] = -\pi^* H \tag{3.21}$$

where $d_V : \Gamma(\wedge^2 V^*) \to \Gamma(\wedge^3 V^*)$ is the (untwisted) differential of V, $[\ ,\ ] : \Gamma(\wedge^2 V^*) \to \Gamma(\wedge^3 V^*)$ is the (untwisted) Schouten bracket for V^* and $\pi : V \to T$ is the anchor.

Proof. Using $\pi(e^\epsilon u) = \pi(u)$ the proposition follows from equation (3.19a) and equation (3.16) of Proposition (3.4.2). □

Chapter 4

Generalized metrics

The addition of further structures associated to a generalized tangent bundle leads to various geometries. The first such structure is a generalization of a Riemannian metric and is thus called a *generalized metric*. One could therefore describe this geometry as generalized Riemannian geometry. From a generalized metric one produces generalizations of the Levi-Civita connection, the Hodge star, the inner product on forms and the Laplace-de Rham operator.

4.1 Generalized metrics

We motivate the definition of a generalized metric by the following observation. Suppose we have a Riemannian manifold (M, g). The metric g is completely determined by its graph $G = \{X + gX | X \in TM\} \subset TM \oplus T^*M$ where g is viewed as the map $g : TM \to T^*M$ given by $X \mapsto g(X, \ )$. One then finds that $(X+gX, Y+gY) = g(X, Y)$, i.e., the restriction of the natural form $(\ , \ )$ to G is positive definite.

Definition 4.1.1. Let E be a generalized tangent bundle for M. A *generalized metric* V is a positive definite subbundle of rank $n = \dim M$, that is, the restriction of the form $(\ , \ )$ to V is positive definite.

Given a generalized metric $V \subset E$, we define $V^+ = V$ and $V^- = V^\perp$, the orthogonal complement of V. Thus $E = V^+ \oplus V^-$, the form $(\ , \ )$ is positive definite on V^+, negative definite on V^-. Thus a generalized metric is equivalent to a reduction of the structure group of E from $O(n, n)$ to $O(n) \times O(n)$. To proceed further let us first consider the untwisted case $E = T \oplus T^*$. Note that as $T^* \subset E$ is isotropic, $T^* \cap V^+ = 0$. So just as in the motivating example one can write V^+ as a graph $\{X + tX | X \in T\}$

where $t : T \to T^*$. Splitting t into symmetric and anti-symmetric parts, $t = g + B$ we have that V^+ consists of elements of the form $X + gX + BX$ while V^- consists of elements of the form $X - gX + BX$. One finds that $(X + gX + BX, Y + gY + BY) = g(X, Y)$ and since V^+ is positive definite, g is an ordinary Riemannian metric. So in the untwisted case a generalized metric is equivalently given by a Riemannian metric g and a 2-form B which is known as the *B-field*.

In the twisted case one can still locally write V^+ as a graph $X+g_\alpha X+B_\alpha X$ which under the transition between coordinate patches $U_\alpha \to U_\beta$ changes to $X + g_\alpha X + B_\alpha X + \iota_X dA_{\alpha\beta} = X + g_\beta X + B_\beta X$. Thus on equating symmetric and skew-symmetric parts $g_\alpha = g_\beta = g$ still defines a Riemannian metric while $B_\beta = B_\alpha + dA_{\alpha\beta}$ so in the gerbe twisted case a generalized metric is equivalently given by a Riemannian metric g and a collection of 2-forms B_α transforming under $B_\beta = B_\alpha + dA_{\alpha\beta}$. Notice that we have a globally defined 3-form H given locally as $H = dB_\beta = dB_\alpha$, the *curvature* of the gerbe.

4.1.1 Generalized isometries

We now consider which Courant isomorphisms preserve the structure of a generalized metric. More generally we can consider the action of Courant automorphisms on generalized metrics. Suppose we have a generalized metric V^+ and a Courant automorphism φ which is the combination of a diffeomorphism f followed by a B-transform by the 2-form B. Thus the action of φ is given by

$$X + \xi \mapsto f_*X + (f^{-1})^*\xi + \iota_{f_*X}B. \tag{4.1}$$

Since φ preserves the form $(\ ,\ )$, the image $\varphi(V^+)$ is also a generalized metric. We write V^+ as a graph $X + gX + B^+X$ and the image under φ is $f_*X + (f^{-1})^*(gX + B^+X) + \iota_{f_*X}B$. Letting $Y = f_*X$ we see that a generic element of $\varphi(V^+)$ has the form $Y + ((f^{-1})^*(g + B^+))Y + BY$. Thus the metric g and B-field B^+ transform under φ according to

$$\varphi_* g = (f^{-1})^* g \tag{4.2a}$$
$$\varphi_* B^+ = (f^{-1})^* B^+ + B. \tag{4.2b}$$

Definition 4.1.2. Let V^+ be a generalized metric, φ a Courant automorphism. Then φ is a generalized isometry if $\varphi(V^+) = V^+$.

From the transformations (4.2), it follows that φ is a generalized isometry if and only if $g = (f^{-1})^*g$ and $B^+ = (f^{-1})^*B^+ + B$. In particular a generalized isometry always has an underlying isometry, in fact given any isometry f we could combine it with the B field $B = B^+ - (f^{-1})^* B^+$ to produce a morphism that carries V^+ onto itself, however we might not have that B is closed.

4.2 Connections

Associated to every generalized metric V^+ is a connection on V^+ compatible with the metric. Recall equation (2.15), a relation reminiscent of the condition for a connection to be compatible and will be used as motivation. We assume v and w are sections of V^+ and $X = \pi(u)$ is a vector field. If we choose $u = X^-$ to be the extension of X to V^- then (2.15) simplifies to

$$X(v, w) = ([X^-, v], w) + (v, [X^-, w]). \tag{4.3}$$

Therefore we tentatively define

$$\nabla_X v = [X^-, v]_+$$

where we use a superscript $\pm$ to denote extension to $V^\pm$ and a subscript $\pm$ to denote orthogonal projection onto $V^\pm$. We now verify this defines an affine connection on V^+. From equation (2.17) we see that

$$\nabla_X f v = (f[X^-, v] + (Xf)v)_+ = f\nabla_X v + (Xf)v$$

and also that

$$\nabla_{fX} v = [fX^-, v]_+ = f[X^-, v]_+ - ((\pi(v)f)X^-)_+ + (X^-, v)df_+ = f\nabla_X v$$

hence ∇ defines a connection on V^+. Furthermore it follows from (4.3) that ∇ is compatible with the metric on V^+. Although ∇ is a connection on V^+ we can use the projection π to identify V^+ with the tangent bundle T and we obtain a corresponding connection which will also be denoted ∇ on T defined by

$$(\nabla_X Y)^+ = \nabla_X Y^+ = [X^-, Y^+]_+. \tag{4.4}$$

Observe that (4.4) does define a compatible connection on T. Now $[X^-, Y^+]_+ = ([X^-, Y^+] - [X, Y]^-)_+$. The benefit of writing this is that $\pi([X^-, Y^+]) = \pi([X, Y]^-) = [X, Y]$ so that $[X^-, Y^+] - [X, Y]^-$ is a 1-form. Thus it has the form $2gZ$ for some vector field Z. Thus $(\nabla_X Y)^+ = (2gZ)_+ = Z^+$. The last equality holds since $2gZ = (Z + gZ + BZ) - (Z - gZ + BZ) = Z^+ - Z^-$ is the decomposition of $2gZ$. Thus $\nabla_X Y = Z$. Hence

$$2g\nabla_X Y = [X^-, Y^+] - [X, Y]^-. \tag{4.5}$$

In fact we can repeat the above using the twisted Courant bracket $[\ ,\ ]_H$, in which case the connection is given by

$$2g\nabla_X Y = [X^-, Y^+]_H - [X, Y]^-. \tag{4.6}$$

Note that the second bracket is a Lie bracket.

Proposition 4.2.1. *[16] The connection* ∇ *on* T *has torsion* $T = -(dB+H)$.

Proof. We have

$$2g(\nabla_X Y - \nabla_Y X - [X, Y]) = [X^-, Y^+]_H - [Y^-, X^+]_H - 2[X, Y]^- - 2g[X, Y]. \tag{4.7}$$

Note that the omission of the subscript H indicates the Lie bracket. To simplify this expression first observe that as the Courant bracket of two 1-forms is zero then $[X^+ - X^-, Y^+ - Y^-]_H = 0$, so expanding gives

$$[X^+, Y^+]_H + [X^-, Y^-]_H = [X^+, Y^-]_H + [X^-, Y^+]_H.$$

Thus $[X^+, Y^-]_H + [X^-, Y^+]_H = \frac{1}{2}([X^+ + X^-, Y^+ + Y^-]_H)$. But $X^+ + X^- = 2(X + \iota_X B)$ and the properties of the Courant bracket imply

$$\begin{aligned} [X + \iota_X B, Y + \iota_Y B]_H &= [X, Y]_H + \iota_{[X,Y]} B - \iota_X \iota_Y dB \\ &= [X, Y] + \iota_{[X,Y]} B - \iota_X \iota_Y (dB + H). \end{aligned}$$

Thus

$$[X^+, Y^-]_H + [X^-, Y^+]_H = 2[X, Y] + 2\iota_{[X,Y]} B - 2\iota_X \iota_Y (dB + H). \tag{4.8}$$

Substituting (4.8) into the (4.7) we find the right hand side becomes

$$\begin{aligned} & 2[X, Y] + 2\iota_{[X,Y]} B - 2\iota_X \iota_Y (dB + H) - 2[X, Y]^- - 2g[X, Y] \\ = \;& 2[X, Y] + 2\iota_{[X,Y]} B - 2\iota_X \iota_Y (dB + H) - [X, Y]^- - [X, Y]^+ \\ = \;& 2[X, Y] + 2B[X, Y] - 2\iota_X \iota_Y (dB + H) - 2([X, Y] + B[X, Y]) \\ = \;& -2\iota_X \iota_Y (dB + H). \end{aligned}$$

Thus the torsion is given by

$$g(\nabla_X Y - \nabla_Y X - [X, Y]) = -\iota_X \iota_Y (dB + H). \tag{4.9}$$

$\square$

Remark 4.2.1. One can similarly define a connection using the orthogonal complement V^-. All the algebra is the same except that g is replaced by $-g$ and so equation (4.9) shows that the connection obtained on T has torsion $+(dB + H)$. Thus there are actually two compatible connections $\nabla^\pm$ arising from a generalized metric, with torsion $\mp(dB + H)$. Thus $\frac{1}{2}(\nabla^+ + \nabla^-)$ is the Levi-Civita connection.

4.3 The Born-Infeld metric

4.3.1 Induced metric on the generalized tangent bundle

A generalized metric V^+ decomposes the (untwisted) generalized tangent bundle $E = V^+ \oplus V^-$ into positive and negative definite subbundles. By switching the sign of the metric on V^- we obtain a positive definite metric on E. Another way to view this is by defining the bundle endomorphism $G \in \text{End}(E)$ which is defined as multiplication by ± 1 on $V^\pm$. Then we have that the bilinear form $(G\_\_ \,, \_\_)$ on E is positive definite and symmetric and thus defines a metric on E. By restricting to T we obtain a Riemannian metric. Our presentation of this metric and the Hodge theory that follows is greatly influenced by [12].

The map G can be expressed in terms of the metric g and B-field B. First consider the case when $B = 0$. Then for a tangent vector X, $2X = X^+ + X^-$, so $2GX = X^+ - X^- = 2gX$. Similarly $G(gX) = X$ so in matrix form G is

$$G = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}. \tag{4.10}$$

Now we suppose there is a B-field B. Then $V^\pm = e^B V_0^\pm$ where $V_0^\pm$ are the corresponding subbundles when $B = 0$. It follows that $G = e^B G_0 e^{-B}$, where G_0 is as given in equation (4.10). In matrix form this is

$$G = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}. \tag{4.11}$$

The restriction of the metric $(G\_\_ \,, \_\_)$ to T is thus $(GX, Y) = (-g^{-1}BX + (g - Bg^{-1}B)X, Y) = (gX, Y) - (Bg^{-1}BX, Y) = g(X, Y) + (g^{-1}BX, BY) = g(X, Y) + g^{-1}(BX, BY)$, where we have used the fact that B is skew adjoint. We thus have

$$(GX, Y) = g(X, Y) + g^{-1}(BX, BY). \tag{4.12}$$

Note that we can also write $g - Bg^{-1}B = (g - B)g^{-1}(g + B)$. So $(GX, Y) = ((g-B)g^{-1}(g+B)X, Y) = (g^{-1}(g+B)X, (g+B)Y) = g^{-1}((g+B)X, (g+B)Y)$ is another expression for this new metric. We let $dvol_G$ denote the volume form corresponding to this metric.

4.3.2 The Born-Infeld metric

The next construction is a generalization of the Hodge star which in turn will allow us to define a metric called the *Born-Infeld metric* on spinors. Let

$* = e_1 e_2 \ldots e_n$ where $e_1, \ldots e_n$ is an oriented orthonormal basis for V^+. Then $*$ acts on $\wedge T^*$ by Clifford multiplication and we have that $*^2 = (-1)^{\frac{n(n-1)}{2}}$. Consider first the case when $B = 0$. Then an oriented orthonormal basis for V^+ is $\{\partial_1 + dx^1, \ldots \partial_n + dx^n\}$ where $\{\partial_\mu\}$ is an oriented orthonormal basis for T and $\{dx^\mu\}$ the dual basis. So $* = (\partial_1 + dx^1) \cdots (\partial_n + dx^n)$. Then if σ denotes the main anti-automorphism of the Clifford algebra then we have

$$\sigma(*)\omega = \sigma(\star\omega)$$

where $\star$ is the ordinary Hodge star for g. Let us also define $\tilde{\alpha} = \star\star\,\alpha$ so if α has degree k then $\tilde{\alpha} = (-1)^{k(n-k)}\alpha$. Now we see

$$\begin{aligned}
[\alpha \wedge \star\beta]_{\text{top}} &= [\star\alpha \wedge \star\star\,\beta]_{\text{top}} \\
&= [\star\alpha \wedge \tilde{\beta}]_{\text{top}} \\
&= (-1)^{\frac{n(n-1)}{2}} [\sigma(\star\alpha \wedge \tilde{\beta})]_{\text{top}} \\
&= (-1)^{\frac{n(n-1)}{2}} [\sigma(\tilde{\beta}) \wedge \sigma(\star\alpha)]_{\text{top}} \\
&= (-1)^{\frac{n(n-1)}{2}} [\sigma(\tilde{\beta}) \wedge \sigma(*)\alpha]_{\text{top}} \\
&= [\sigma(\tilde{\beta}) \wedge *\alpha]_{\text{top}} \\
&= \langle \tilde{\beta}, *\alpha \rangle \\
&= \langle *\tilde{\beta}, **\,\alpha \rangle \\
&= \langle \alpha, *\tilde{\beta} \rangle
\end{aligned}$$

where we have used $\sigma(*) = (-1)^{\frac{n(n-1)}{2}} *$. So we have

$$g(\alpha, \beta) dvol_g = \langle \alpha, *\tilde{\beta} \rangle$$

where $g(\beta, \alpha)$ denotes the metric on forms induced by g. In the general case where we have a B-field we define a bilinear form G on spinors by

$$G(\alpha, \beta) dvol_G = \langle \alpha, *\tilde{\beta} \rangle. \tag{4.13}$$

Now let $V^+ = e^B V_0^+$, where V_0^+ is the corresponding generalized metric when $B = 0$. If $\{e_\mu\}$ is an oriented orthonormal basis for V_0^+ then $\{e^B e_\mu\}$ gives such a basis for V^+ and hence $* = e^B *_0$. Recall equivariance of the B-field on the Clifford action $(e^B u)(e^{-B}\omega) = e^{-B}(u\omega)$ where u is in the Clifford algebra of E and ω is a spinor. Thus $(e^B u)(\omega) = e^{-B}(u e^B \omega)$. Therefore as spinor endomorphisms we have $* = e^{-B} *_0 e^B$. Now we find that

$$\begin{aligned} G(\alpha,\beta) dvol_G &= \langle \alpha, *\tilde{\beta}\rangle \\ &= \langle \alpha, e^{-B} *_0 e^B \tilde{\beta}\rangle \\ &= \langle e^B\alpha, *_0 e^B\tilde{\beta}\rangle \\ &= \langle e^B\alpha, *_0 \widetilde{e^B\beta}\rangle \\ &= g(e^B\alpha, e^B\beta) dvol_g \end{aligned}$$

where we have used the invariance of $\langle\ ,\ \rangle$ under B-transforms. Thus in particular we find that $G(\alpha,\beta)$ is symmetric and positive definite. By integration we have a metric on differential forms which, following [12] we define as the *Born-Infeld metric*.

Definition 4.3.1. Let M be compact. The *Born-Infeld metric* is the metric h on differential forms given by

$$h(\alpha,\beta) = \int G(\alpha,\beta) dvol_G = \int \langle \alpha, *\tilde{\beta}\rangle = \int g(e^B\alpha, e^B\beta) dvol_g. \tag{4.14}$$

4.3.3 Hodge theory for generalized metrics

Associated with the spinors is a differential complex

$$d_H : \Gamma(S^+) \to \Gamma(S^-) \tag{4.15}$$

where $S^{\pm} = \wedge^{en\backslash odd} T^*$. The main distinction compared to the de Rham complex is that in the generalized case we only have a $\mathbb{Z}_2$ grading. We now turn to the question of finding an adjoint for d_H. First some properties of the $\sim$ operation, $\widetilde{\alpha} = **\alpha$.

$$\widetilde{\alpha\wedge\beta} = \tilde{\alpha}\wedge\tilde{\beta} \tag{4.16a}$$
$$\widetilde{\sigma(\omega)} = \sigma(\tilde{\omega}) \tag{4.16b}$$
$$\widetilde{d_H\omega} = d_{\tilde{H}}\tilde{\omega} \tag{4.16c}$$
$$\widetilde{e^B\omega} = e^B\tilde{\omega} \tag{4.16d}$$
$$\widetilde{\star\omega} = \star\tilde{\omega} \tag{4.16e}$$
$$\widetilde{*\omega} = *\tilde{\omega} \tag{4.16f}$$
$$\langle\tilde{\alpha},\tilde{\beta}\rangle = \langle\alpha,\beta\rangle. \tag{4.16g}$$

Note that (4.16f) follows from (4.16d) and (4.16e) since $* = e^{-B} *_0 e^B$. Now if s has degree k then

$$\begin{aligned} d_H s \wedge t + (-1)^k s \wedge d_{-H} t &= ds \wedge t - H \wedge s \wedge t \\ &\quad + (-1)^k s \wedge dt + (-1)^k s \wedge H \wedge t \\ &= d(s \wedge t) \end{aligned}$$

So on replacing s by $\sigma(s)$ and noting that $d(\sigma(s)) = (-1)^k \sigma(ds)$, we find that the expression $\langle d_H s, t\rangle + \langle s, d_{-H} t\rangle$ is exact. Now we have

$$\begin{aligned} \int \langle d_H s, *\tilde{\beta}\rangle &= \int \langle \widetilde{d_H s}, *\beta\rangle \\ &= -\int \langle \tilde{s}, d_{-\tilde{H}}(*\beta)\rangle \\ &= -\int \langle *\tilde{s}, *d_{-\tilde{H}}(*\beta)\rangle \\ &= -\int \langle \sigma(*) d_{-\tilde{H}}(*\beta), *\tilde{s}\rangle. \end{aligned}$$

So the adjoint of d_H is $d_H^* = - *^{-1} d_{-\tilde{H}} *$. Note that in even dimensions $-\widetilde{H} = H$.

We can form the operator $\mathcal{D}_+ = d_H + d_H^*$. Now the principal symbol $s(\xi)$ of $\mathcal{D}_+$ is given by

$$s(\xi)\omega = \xi \wedge \omega - *^{-1}(\xi \wedge *\omega).$$

But using $* = e^{-B} *_0 e^B$ we find that

$$e^B s(\xi)(e^{-B}\omega) = \xi \wedge \omega - *_0^{-1}(\xi \wedge *_0\omega) = \xi \wedge \omega + \xi \llcorner \omega.$$

This last expression is just the principal symbol for $d + d^*$ and shows that $\mathcal{D}_+$ is elliptic and therefore so is $\Delta_{d_H} = \mathcal{D}_+^2 = d_H d_H^* + d_H^* d_H$. The index for $d_H : \Gamma(S^+) \to \Gamma(S^-)$ is clearly the Euler characteristic.

Chapter 5

Generalized complex geometry

Generalized complex structures are of central importance in generalized geometry. They are simultaneously a generalization of complex and symplectic geometry. Generalized complex complex manifolds have at each point an integer called the type which can jump discontinuously throughout the manifold. Symplectic manifolds have type zero throughout while complex manifolds have maximal type throughout equal to their complex dimension. More generally in an open set with constant type k, a generalized complex manifold admits a foliation, the leaves of which are symplectic and there are $2k$ transverse coordinates which have a complex structure.

5.1 Generalized almost complex structures

Just as in complex geometry there are generalized almost complex structures and an integrability condition for them to be generalized complex structures. In this section we focus on generalized almost complex structures.

5.1.1 Linear generalized complex structures

First we consider the linear theory consisting of $E = V \oplus V^*$ where V is a vector space. A complex structure on V is an endomorphism $J : V \to V$ satisfying $J^2 = -1$. A symplectic structure on V is a linear homomorphism $\omega : V \to V^*$ such that ω is skew-symmetric $(\omega X)Y = -(\omega Y)X$ and non-degenerate which translates to ω being invertible. Alternatively we can use the natural form $(\ ,\ )$ on E to rephrase this as $(\omega X, Y) + (X, \omega Y) = 0$. We can generalize both of these structures with the following definition

Definition 5.1.1. Let V be a vector space. A *linear generalized complex structure* is a linear homomorphism $J : T \oplus T^* \to T \oplus T^*$ such that

- J is a complex structure $J^2 = -1$

- J is skew-adjoint $(JX, Y) + (X, JY) = 0$

where $(\; , \;)$ is the natural bilinear form on $T \oplus T^*$.

Equivalently, the condition that J is skew-adjoint can be replaced with requiring J to be orthogonal for we have $JJ^* = 1 \Leftrightarrow J^2 J^* = J \Leftrightarrow J^* = -J$.

A complex structure $J : V \to V$ canonically defines a generalized complex structure

$$\mathcal{J} = \begin{pmatrix} -J & 0 \\ 0 & J^t \end{pmatrix} \tag{5.1}$$

where J^t denotes the dual map $J^t : V^* \to V^*$. Likewise a symplectic structure $\omega : V \to V^*$ canonically defines a generalized complex structure

$$\mathcal{J} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{5.2}$$

5.1.2 Relation to isotropics

We now consider alternative descriptions of generalized complex structures. Just as in the ordinary complex case we have that the complexification $E \otimes \mathbb{C}$ splits into the $\pm i$ eigenspaces of a generalized complex structure J:

$$E \otimes \mathbb{C} = V_{+i} \oplus V_{-i}$$

with the projections being $P_{\pm i} = \frac{1}{2}(1 \mp iJ)$. In addition we also have that the form $(\; , \;)$ extended to $E \otimes \mathbb{C}$ vanishes when restricted to the eigenspaces $V_{\pm i}$ for we have for $u, v \in V_i$

$$(u, v) = (Ju, Jv) = (iu, iv) = -(u, v).$$

Recall that we call such subspaces isotropic. Moreover since the eigenspaces have the same dimesion as V, they are maximal isotropics. Therefore we have that a generalized complex structure gives a splitting of $E \otimes \mathbb{C} = W \oplus \overline{W}$ where W is a maximal isotropic subspace. Conversely, given a maximal isotropic W of $E \otimes \mathbb{C}$ such that $W \cap \overline{W} = \{0\}$ then we can define a generalized complex structure J simply by specifying W to be the $+i$ eigenspace of J and $\overline{W}$ to be the $-i$ eigenspace. Therefore we have found an equivalent description of generalized complex structures, a maximal isotropic subspace W of $E \otimes \mathbb{C}$ with $W \cap \overline{W} = \{0\}$.

A generalized complex structure $\mathcal{J}$ defines the group $U(n,n) = GL(2n,\mathbb{C}) \cap O(2n,2n)$ where $GL(2n,\mathbb{C})$ is the group of linear automorphisms commuting with $\mathcal{J}$. Notice that this shows that generalized complex structures only exist in even dimensions. The eigenspaces $W, \overline{W}$ of $\mathcal{J}$ are $GL(2n,\mathbb{C})$-invariant. Thus the splitting of $E \otimes \mathbb{C}$ into $U(n,n)$-irreducible representations is simply the splitting $E \otimes \mathbb{C} \simeq W \oplus \overline{W}$ given by $\mathcal{J}$. Now $O(2n,2n)$ acts transitively on the space of maximal isotropics with real index zero by $W \mapsto TW$ and consequently there is also a transitive action of $O(2n,2n)$ on generalized complex structures given by conjugation $\mathcal{J} \mapsto T\mathcal{J}T^{-1}$. The stabilizer of this action on a given $\mathcal{J}$ is the group $U(n,n)$. Therefore a generalized complex structure is equivalent to a coset of $U(n,n)$ in $O(2n,2n)$ and the space of generalized complex structures is the homogeneous space $O(2n,2n)/U(n,n)$. This shows that a generalized complex structure is equivalent to a reduction from the group $O(2n,2n)$ to $U(n,n)$. Note however that the generalized complex structures $\mathcal{J}$ and $-\mathcal{J}$ lead to the same reduction of structure.

5.1.3 Description of maximal isotropics

We consider now the task of classifying the maximal isotropic subspaces. If U is a subspace of V then $U \oplus \mathrm{Ann}(U)$ is a maximal isotropic where $\mathrm{Ann}(U)$ is the Annihilator of U, $\mathrm{Ann}(U) = \{\xi \in V^* \,|\, \xi(U) = 0\}$. We denote the space $U \oplus \mathrm{Ann}(U)$ by $L(U)$. We can also perform a B-transform to $L(U)$ to obtain $e^B L(U)$, which is also maximal isotropic. In fact $e^B L(U)$ only depends on the restriction of B to U. For any $\epsilon \in \wedge^2 U^*$ we can form the space $L(U,\epsilon) = \{X + \xi \mid \ \xi|_U = \epsilon X\}$. Then if $\iota : U \to V$ is inclusion and $\iota^* B = \epsilon$ we have that $L(U,\epsilon) = e^B L(U)$ so that the space $L(U,\epsilon)$ is maximal isotropic. In fact we now show every maximal isotropic has this form.

Proposition 5.1.1. *[11] Every maximal isotropic subspace of $V \oplus V^*$ is of the form $L(U,\epsilon)$, where U is a subspace of V, $\epsilon \in \wedge^2 U^*$ and U, ϵ are unique.*

Proof. Let W be an isotropic space. Let $\pi_1 : V \oplus V^* \to V$ be projection onto the first factor and similarly π_2 for the second factor. Let $U = \pi_1(W)$. We define $\epsilon : U \to U^* \simeq V^*/\mathrm{Ann}(U)$ as follows. Given $X \in U$, let ξ be any element of V^* such that $X + \xi \in W$. We let $\epsilon X = \xi|_U$. To show this is well defined suppose that $X + \xi_1$ and $X + \xi_2$ are both elements of W lying above X. Then for any $Y \in U$, choose $Y + \eta \in W$. Now W is an isotropic subspace so $0 = ((X + \xi_1) - (X + \xi_2), Y + \eta) = (\xi_1 - \xi_2)(Y)$. Thus $\xi_1 - \xi_2 \in \mathrm{Ann}(U)$

as required. Also $(\epsilon X)(X) = \xi_1(X) = 0$ showing skew-symmetry $\epsilon \in \wedge^2 U^*$. Thus we have shown that W is a subspace of $\{X+\xi \mid \quad \xi|_U = \epsilon X\} = L(U,\epsilon)$. Finally, when W is maximal we must have $W = L(U,\epsilon)$ since $L(U,\epsilon)$ is also maximal. In this case U and ϵ are uniquely determined. $\square$

So we have an alternative characterization of maximal isotropics, as a subspace U of V (or $V\otimes\mathbb{C}$ in the complex case) together with a 2-form ϵ on U, $\epsilon \in \wedge^2 U^*$.

Definition 5.1.2. Let W be a maximal isotropic subspace of $V \oplus V^*$. Let $U = \pi(W)$ be the projection of W onto V. The *type* of W is the codimension of U in V.

Note that if W is a complex subspace of the complexification $E \otimes \mathbb{C}$ of $E = V \oplus V^*$ then the above definition still holds, taking the complex codimension of $(\pi \otimes 1)(W)$ in $V \otimes \mathbb{C}$. Also note that if $\iota : U \to V$ is inclusion then

$$e^B L(U,\epsilon) = L(U, \epsilon + \iota^* B) \tag{5.3}$$

and as such B-transforms preserve the type of maximal isotropics. We are lead to ask what the condition $W \cap \overline{W} = \{0\}$ for a maximal isotropic translates into in terms of U and ϵ.

Definition 5.1.3. Let W be a maximal isotropic subspace of $E \otimes \mathbb{C}$. The *real index* of W is the complex dimension of $W \cap \overline{W}$.

This definition is such that generalized complex structures are equivalent to maximal isotropics W of $E \otimes \mathbb{C}$ of real index zero.

Proposition 5.1.2. *[11] The maximal isotropic $W = L(U,\epsilon)$ of $E\otimes\mathbb{C}$ has real index zero if and only if $U + \overline{U} = V \otimes \mathbb{C}$ and $\omega = \mathrm{Im}(\epsilon)|_{U\cap\overline{U}}$ is non-degenerate.*

Proof. If W has index zero then $E \otimes \mathbb{C} = W \oplus \overline{W}$ and hence $U + \overline{U} = V \otimes \mathbb{C}$. Let $B \in \wedge^2 V^*$ be such that $\iota^* B = \epsilon$. Suppose there exists a real $0 \neq X \in U \cap \overline{U}$ such that $\omega X = 0$. Then $(B - \overline{B})X|_{U+\overline{U}} = 0$. Hence $(B-\overline{B})X = \alpha + \beta$ where $\alpha \in \mathrm{Ann}(U)$ and $\beta \in \mathrm{Ann}(\overline{U})$. Since $U + \overline{U} = V \otimes \mathbb{C}$, then $\mathrm{Ann}(U) \cap \mathrm{Ann}(\overline{U}) = \{0\}$. Then as $(B - \overline{B})X$ is imaginary we have $\beta = -\overline{\alpha}$. Thus $BX - \alpha = \overline{B}X - \overline{\alpha} = \xi$, say. Then ξ is real, $\xi|_U = BX|_U = \epsilon X$ and $\xi|_{\overline{U}} = \overline{\epsilon}X$. Thus $0 \neq X + \xi \in W \cap \overline{W}$, a contradiction so ω must be non-degenerate.

Conversely suppose $U + \overline{U} = V \otimes \mathbb{C}$ and ω is non-degenerate. Suppose $X + \xi \in W \cap \overline{W}$. Then $\xi|_U = \epsilon X$ and $\xi|_{\overline{U}} = \overline{\epsilon}X$. Thus $\omega X = 0$ so by non-degeneracy, $X = 0$. This in turn implies $\xi|_U = 0$ and $\xi|_{\overline{U}} = 0$ hence $\xi = 0$ and thus W has real index zero. $\square$

Thus we have determined precisely the conditions for an isotropic $L(U, \epsilon)$ to define a generalized complex structure.

5.1.4 Relation to spinors

There is a connection between isotropic subspaces and spinors and it will lead to another description of generalized complex structures. Given a spinor $\phi \in \wedge T^*$ consider the subspace of E defined by

$$L_\phi = \{v \in E \,|\, v\phi = 0\}. \tag{5.4}$$

If $u, v \in L_\phi$ then

$$0 = uv\phi + vu\phi = 2(u, v)\phi$$

so if $\phi \neq 0$ then $(u, v) = 0$ and thus L_ϕ is an isotropic subspace. Now the Clifford action is $Spin(V \oplus V^*)$ equivariant so for any element $g \in Spin(V \oplus V^*)$ we have $(gv)(g\phi) = g(v\phi)$. It follows that $L_{g\phi} = gL_\phi$. In particular for a B-transform we have $e^B L_\phi = L_{e^{-B}\phi}$.

Definition 5.1.4. A spinor ϕ is called *pure* if L_ϕ is a maximal isotropic subspace.

Now consider the maximal isotropic $W = L(U) = U \oplus \mathrm{Ann}(U)$.

Proposition 5.1.3. *The spinor ϕ satisfies $L_\phi = W = U \oplus \mathrm{Ann}(U)$ if and only if ϕ is a non-zero multiple of $e_1 \wedge \cdots \wedge e_k$ where $\{e_i\}$ is a basis for $\mathrm{Ann}(U)$.*

Proof. Let $\{u_i\}_1^t$ be a basis for U and let $\{v_j\}_1^k$ be a basis for a complement of U in V. Let $\{u_i^*\}_1^t \cup \{v_j^*\}_1^k$ be a dual basis so in particular $\{v_j^*\}$ is a basis for $\mathrm{Ann}(U)$. Suppose the spinor ϕ satisfies $L_\phi = W$. Then for each j, $0 = v_j^*\phi = v_j^* \wedge \phi$ and so $\phi = v_1^* \wedge \cdots \wedge v_k^* \wedge \omega = \Omega \wedge \omega$ where ω is a form involving only the $\{u_i^*\}$ and $\Omega = v_1^* \wedge \cdots \wedge v_k^*$. Furthermore $0 = u_i\phi = \iota_{u_i}\phi = (-1)^k \Omega \wedge \iota_{u_i}\omega$ for all i so that $\omega = c$ a non-zero constant. The converse is trivial. $\square$

Corollary 5.1.3.1. *The spinor ϕ satisfies $L_\phi = W = L(U, \epsilon)$ if and only if ϕ is a non-zero multiple of $e^{-B}\Omega$ where $B \in \wedge^2 V^*$ is any 2-form such that $\iota^* B = \epsilon$ and $\Omega = e_1 \wedge \cdots \wedge e_k$ is an orientation for $\mathrm{Ann}(U)$.*

Proof. Suppose ϕ is such that $L_\phi = W$. Then $L_{e^B\phi} = e^{-B}W = L(U, 0) = U \oplus \mathrm{Ann}(U)$. Thus $e^B\phi = c\Omega$ where c is a non-zero constant. Similarly the converse holds. $\square$

Corollary 5.1.3.2. *Every maximal isotropic W has, up to non-zero multiples, a unique pure spinor ρ such that $W = L_\rho$. If $W = L(U, \epsilon)$ and B is any 2-form with $\iota^* B = \epsilon$ then $\rho = ce^{-B}\Omega$ where c is a non-zero scalar and Ω is an orientation for* $\mathrm{Ann}(U)$. *The type of W is the degree of Ω.*

Proof. This follows since every maximal isotropic has the form $L(U, \epsilon)$. Also note that the type of W is the codimension of U which is the dimension of $\mathrm{Ann}(U)$, the degree of Ω. $\square$

Thus a spinor ρ is pure if and only if it has the form $\rho = e^B\Omega$, where $\Omega = e_1 \wedge \cdots \wedge e_k$. In the case of complex maximal isotropics we write $\rho = e^{B+i\omega}\Omega$ where B and ω are real.

We have established the relation between pure spinors and maximal isotropics. Now we wish to describe which pure spinors correspond to a generalized complex structure. This happens if and only if the maximal isotropic has real index zero.

Proposition 5.1.4. *Let ρ_1 and ρ_2 be pure spinors. Then $L_{\rho_1} \cap L_{\rho_2} = \{0\}$ if and only if $\langle \rho_1, \rho_2 \rangle \neq 0$, where $\langle\ ,\ \rangle$ is the bilinear form (2.3) on spinors.*

Proof. By the invariance of $\langle\ ,\ \rangle$ under B-transforms, it suffices to consider the case where $W_1 = L_{\rho_1} = L(U_1, \epsilon_1)$ and $W_2 = L_{\rho_2} = L(U_2, 0) = U_2 \oplus \mathrm{Ann}(U_2)$. Let B be any 2-form such that $\iota^* B = \epsilon$ so that $W_1 = e^B L(U_1, 0)$. After rescaling we can assume $\rho_1 = e^{-B}\Omega_1$ and $\rho_2 = \Omega_2$. Then

$$\begin{aligned} \langle \rho_1, \rho_2 \rangle &= [\sigma(e^{-B}\Omega_1) \wedge \Omega_2]_{\mathrm{top}} \\ &= [\sigma(\Omega_1) \wedge \sigma(e^{-B}) \wedge \Omega_2]_{\mathrm{top}} \\ &= \pm[e^B \wedge \Omega_1 \wedge \Omega_2]_{\mathrm{top}}. \end{aligned}$$

Consider first the case where $U_1 + U_2 \neq V \otimes \mathbb{C}$. Then $\mathrm{Ann}(U_1) \cap \mathrm{Ann}(U_2) \neq \{0\}$ and so $\Omega_1 \wedge \Omega_2 = 0$, thus $\langle \rho_1, \rho_2 \rangle = 0$. But since $\mathrm{Ann}(U_i) \subseteq W_i$ for $i = 1, 2$ we also have $W_1 \cap W_2 \neq \{0\}$, proving the result in this case.

Thus we now assume $U_1 + U_2 = V \otimes \mathbb{C}$. Choose a basis $f_1, \ldots f_r$ for $U_1 \cap U_2$. Let $e_1, \ldots e_k$ be a basis for a complement of $U_1 \cap U_2$ in U_1 and similarly $g_1, \ldots g_j$ for a complement of $U_1 \cap U_2$ in U_2. Let $\{f_1^*, \ldots f_r^*\}, \{e_1^*, \ldots e_k^*\}, \{g_1^*, \ldots g_j^*\}$ be corresponding dual bases. Then W_1 has basis

$$\{e^B f_1, \ldots e^B f_r, e^B e_1, \ldots e^B e_k, g_1^*, \ldots g_j^*\}$$

and W_2 has basis

$$\{f_1, \ldots f_r, g_1, \ldots g_j, e_1^*, \ldots e_k^*\}.$$

Then $W_1 \cap W_2 = \{0\}$ if and only if the wedge product of these two bases is a non-zero element of $\det(V \oplus V^*) \otimes \mathbb{C}$. Their wedge product is

$$\begin{aligned}
&(f_1 + Bf_1) \wedge \dots (f_r + Bf_r) \wedge (e_1 + Be_1) \wedge \dots \wedge (e_k + Be_k) \wedge g_1^* \wedge \dots \wedge g_j^* \\
&\wedge f_1 \wedge \dots \wedge f_r \wedge g_1 \wedge \dots g_j \wedge e_1^* \wedge \dots \wedge e_k^* \\
&= \pm c f_1 \wedge \dots \wedge f_r \wedge e_1 \wedge \dots \wedge e_k \wedge g_1 \wedge \dots \wedge g_j \wedge Bf_1 \wedge \dots Bf_r \wedge \Omega_1 \wedge \Omega_2
\end{aligned}$$

where c is a non-zero constant. Therefore this quantity is non-zero if and only if the quantity

$$Bf_1 \wedge \dots \wedge Bf_r \wedge \Omega_1 \wedge \Omega_2 \in \det V^* \tag{5.5}$$

is non-zero. We write $B = A + C$ where the two form A involves only the basis elements $\{f_1^*, \dots f_r^*\}$ while C contains all terms of B involving other basis elements. Then 5.5 is non-zero if and only if

$$Af_1 \wedge \dots \wedge Af_r \wedge \Omega_1 \wedge \Omega_2$$

is non-zero. Note that A is a skew-symmetric map from the span of $\{f_1, \dots f_r\}$ to the span of $\{f_1^*, \dots f_r^*\}$. The expression $Af_1 \wedge \dots \wedge Af_r$ can be thought of as the determinant of this map. Now as A is a skew-symmetric, we can instead use the expression $[e^A]_r$, which is essentially the Pfaffian of A. It follows that (5.5) is non-zero if and only if

$$[e^A]_r \wedge \Omega_1 \wedge \Omega_2 = [e^B \wedge \Omega_1 \wedge \Omega_2]_{\text{top}}$$

is non-zero. This last quantity is $\pm\langle \rho_1, \rho_2 \rangle$ which completes the proof. □

Corollary 5.1.4.1. *Let ρ be a pure spinor. Then L_ρ has index zero if and only if $\langle \rho, \overline{\rho} \rangle \neq 0$.*

Thus we have that a generalized complex structure is equivalently given by a pure spinor $\rho = e^{B+i\omega}\Omega$ such that $\langle \rho, \overline{\rho} \rangle \neq 0$. This in turn is equivalent to $[\sigma(e^{B+i\omega}\Omega) \wedge e^{B-i\omega}\overline{\Omega}]_{\text{top}} \neq 0$, that is $[e^{-2i\omega}\Omega \wedge \overline{\Omega} \neq 0]_{\text{top}} \neq 0$. Note that this expression is zero unless $\dim V = 2n$ is even. This is another way to see that generalized complex structures exist if and only if $\dim V$ is even. If the generalized complex structure has type k then Ω has degree k. Thus we have further shown that the type satisfies $0 \leq k \leq n = \frac{1}{2}\dim V$. And the condition $\langle \rho, \overline{\rho} \rangle \neq 0$ can finally be rewritten as $\omega^{n-k} \wedge \Omega \wedge \overline{\Omega} \neq 0$.

5.1.5 Examples

We have seen that generalized complex structures can be described in three different ways; as an endomorphism $\mathcal{J}$, as a maximal isotropic W and as a pure spinor ρ. We now illustrate these with our motivating examples.

In the case of a complex structure J, we have the associated generalized complex structure $\mathcal{J}$ given by (5.1). The $+i$ eigenspace of $\mathcal{J}$ is therefore $W = \wedge^{(0,1)}V \oplus \wedge^{(1,0)}V^*$. Thus $U = \pi_V(W) = \wedge^{(0,1)}V$ so if $\dim V = 2n$ then U has codimension n and hence W has maximal type $k = n$. Now the annihilator of $\wedge^{(0,1)}V$ is $\wedge^{(1,0)}V^*$ so $W = L(U, 0)$. Therefore the pure spinor ρ is given by $\rho = dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n = \Omega$.

In the case of a symplectic structure ω, we have the associated generalized complex structure $\mathcal{J}$ given by (5.2). The $+i$ eigenspace of $\mathcal{J}$ is $W = \{X - i\omega X \mid X \in V \otimes \mathbb{C}\}$. Thus $U = \pi_V(W) = V \otimes \mathbb{C}$. So U has codimension 0 and so W has minimal type $k = 0$. We have $W = L(V \otimes \mathbb{C}, -i\omega)$ and so the pure spinor ρ is given by $\rho = e^{-i\omega}$. Note that non-degeneracy of ω, $\omega^n \neq 0$, is precisely the condition for $\rho = e^{-i\omega}$ to satisfy $\langle \rho, \overline{\rho} \rangle \neq 0$ and hence define a generalized complex structure.

5.1.6 Generalized almost complex structures

Now that we have worked out the linear theory we can return to the setting of a manifold.

Definition 5.1.5. Let M be a smooth manifold. A *generalized almost complex structure* for M is an endomorphism $\mathcal{J} : T \oplus T^* \to T \oplus T^*$ such that $\mathcal{J}$ is a linear generalized complex structure on each fibre.

As in the linear theory we see that this is equivalent to a bundle splitting $E \otimes \mathbb{C} = W \otimes \overline{W}$, where W is a maximal isotropic subbundle of $E \otimes \mathbb{C}$. Thus a generalized almost complex structure is equivalent to a maximal isotropic subbundle W with $W \cap \overline{W} = 0$, where 0 denotes the zero bundle. In terms of spinors, a generalized almost complex structure is equivalent to a line subbundle $\mathcal{K}$ of $\wedge T^*$ consisting of pure spinors, i.e., such that for any non-zero $\phi \in \mathcal{K}_x$ we have $\langle \phi, \overline{\phi} \rangle \neq 0$. This line bundle is called the *canonical bundle* since in the case of an ordinary almost complex structure it is the canonical bundle $\wedge^{(n,0)}T^*$. Note that finding a globally-defined pure spinor ϕ is equivalent to finding a generalized almost complex structure with trivial canonical bundle. In particular this is the case for a symplectic manifold M, ω in which case the globally defined spinor is $e^{-i\omega}$. Lastly, a generalized almost complex structure is equivalent to a reduction of the structure group of E from $O(2n, 2n)$ to $U(n, n)$.

5.1.7 Topological conditions for almost structures

In this section we show that a manifold admits a generalized almost complex structure if and only if it admits an almost complex structure. Since any almost complex structure has an associated generalized almost complex structure it remains to show that given a generalized almost complex structure $\mathcal{J}$ we can construct an almost complex structure. The proof is in two steps. First we show that we can reduce the structure group of the generalized tangent bundle from $U(n,n)$ to $U(n) \times U(n)$ and then we identify one of the $U(n)$ factors with the tangent bundle so that we can transport $\mathcal{J}$ to an endomorphism on T.

The first step is to reduce the structure group from $U(n,n)$ to $U(n) \times U(n)$; that is we seek a splitting of E into a positive and a negative definite subbundle $E = V^+ \oplus V^-$ such that $V^\pm$ are $\mathcal{J}$-invariant.

Proposition 5.1.5. *A principal $U(n,n)$ bundle can always be reduced to a principal $U(n) \times U(n)$ bundle.*

Proof. We give two proofs, the first is a local construction using frames while the second is a more direct proof using algebraic topology.

We may think of a principal $U(n,n)$ bundle as a principal bundle of frames together with a signature (n,n)-metric and complex structure $\mathcal{J}$ preserving the metric. To reduce structure to $U(n) \times U(n)$ we must find local frames of the form $\{e_1, \mathcal{J}e_1, e_2, \mathcal{J}e_2, \dots, e_n, \mathcal{J}e_n, f_1, \mathcal{J}f_1, \dots, f_n, \mathcal{J}f_n\}$ such that $(e_i, e_j) = \delta_{ij}$, $(f_i, f_j) = -\delta_{ij}$, $(e_i, f_j) = 0$. Such a basis is easily constructed by a Gram-Schmidt style proces as follows: take an element v_1 of positive norm, then $v_1, \mathcal{J}v_1$ are linearly independent and since $\mathcal{J}$ preserves the metric, $\mathcal{J}v_1$ has positive norm as well. If they do not span a maximal positive definite subspace then we can find v_2 of positive norm and $v_1, \mathcal{J}v_1, v_2, \mathcal{J}v_2$ are linearly independent. Continue in this manner to obtain $v_1, \mathcal{J}v_1, \dots, v_n, \mathcal{J}v_n$ which span a maximal positive definite subspace. Then perform Gram-Schmidt on $v_1, \dots v_n$ to obtain $e_1, \mathcal{J}e_1, \dots, e_n, \mathcal{J}e_n$. Now we have a $\mathcal{J}$-invariant positive definite subspace V^+. Take the orthogonal complement V^- and repeat the above to obtain $f_1, \mathcal{J}f_1, \dots, f_n, \mathcal{J}f_n$ as required.

For the second proof note that a principal G-bundle P has a reduction of structure to a subgroup $H < G$ if and only if the fibre bundle $P \times_H (G/H)$ admits a section. In the case of $U(n) \times U(n) < U(n,n)$ we have that $U(n) \times U(n)$ is homotopic to $U(n,n)$ and so the quotient space is contractible. Thus the fibre bundle $P \times_H (G/H)$ has contractible fibres and so admits a section. $\square$

Thus given a generalized almost complex structure $\mathcal{J}$ we can find a splitting $E = V^+ \oplus V^-$ of E into positive and negative $\mathcal{J}$-invariant subbundles. Since V^+ is positive definite we have that the projection $\pi : E \to T$ gives an isomorphism $\pi : V^+ \to T$. Using this isomorphism we can identify $\mathcal{J}|_{V^+}$ with an almost complex structure on T. Thus we have proven the claim of this section.

5.2 Integrability

A generalized almost complex structure is a purely algebraic struture and so far there have been no analytic conditions. But just as with ordinary complex structures or symplectic structures there is an analytic condition. For complex structures the condition is closure of the space of eigen-sections of the complex structure while for symplectic structures the condition is that the symplectic form is closed. We introduce an integrability condition for generalized almost complex structures and show that this generalizes these two conditions for complex and symplectic structures.

Definition 5.2.1. Let J be a generalized almost complex structure and let W be the $+i$ eigenbundle of J. We say J is an *integrable generalized almost complex structure* or that J is a *generalized complex structure* if W is Courant involutive.

The results on integrability developed in Chapter 3 apply here and in particular integrability of a generalized almost complex structure is equivalent to the Courant bracket to satisfy the Jacobi identity when restricted to the corresponding maximal isotropic subbundles. We thus have that the maximal iostropic subbundles associated to integrable generalized complex structures are Lie algebroids.

Let us examine the integrability condition for the complex and symplectic cases to see that this does indeed generalize more familiar conditions. In the complex case the $+i$ eigenbundle is $W = \wedge^{(0,1)}T \oplus \wedge^{(1,0)}T^*$. If we take two vector fields $X, Y \in \wedge^{(0,1)}T$ then their Courant bracket is just the Lie bracket so this is the usual integrability condition for almost complex structures. It should be noted that the only term of a bracket $[X + \xi, Y + \eta]$ of sections of W, that might not remain in W is the $[X, Y]$ term so in fact Courant integrability is in this case equivalent to the usual integrability condition. In the symplectic case, the $+i$ eigenbundle is $W = \{X - i\omega X \mid X \in T \otimes \mathbb{C}\}$. Note that this is the same as in Example (3.4.2) except we are now using $i\omega$ in place of ω. Therefore the condition for integrability is that $d\omega = 0$.

The integrability condition in the general case can be viewed as a combination of the conditions in the complex and symplectic cases as the following proposition demonstrates:

Proposition 5.2.1. *[11] Let V be a subbundle of $T \otimes \mathbb{C}$ and $\epsilon \in \Gamma(\wedge^2 V^*)$. Then the maximal isotropic subbundle $L(V, \epsilon)$ is Courant integrable if and only if V is Frobenius integrable and $d_V \epsilon = 0$ where $d_V : \Gamma(\wedge^k V^*) \to \Gamma(\wedge^{k+1} V^*)$ is the differential associated to V which is a Lie algebroid when V is Frobenius integrable.*

Proof. Since the Courant bracket and Lie bracket agree for vector fields V must be Frobenius integrable for $W = L(V, \epsilon)$ to be Courant integrable. Therefore we assume V is Frobenius integrable. Consider now the case $\epsilon = 0$. Then $W = L(V, 0) = V \oplus \mathrm{Ann}(V)$. Let $u = X + \xi$ and $v = Y + \eta$ be sections of $L(V, 0)$. Since we are assuming V is Frobenius integrable and since 1-forms Courant commute, we need only consider the case where $u = X$ and $v = \eta$. We have

$$[u, v] = [X, \eta] = \mathcal{L}_X \eta - \frac{1}{2} d\eta(X) = \iota_X d\eta$$

since η is a section of $\mathrm{Ann}(V)$. Now if Z is any section of V then

$$\iota_X d\eta(Z) = d\eta(X, Z) = \eta(X) - \eta(Z) - \eta[X, Z] = 0$$

where we have made use of the Frobenius integrability of V. So we see that $L(V, 0)$ is Courant integrable if and only if V is Frobenius integrable. Now consider the case where we have a 2-form $\epsilon \in \Gamma(\wedge^2 V^*)$. Choose a smooth extension of ϵ, a smooth 2-form $B \in \Gamma(\wedge^2 V^*)$ with $\epsilon = \iota^* B$. Then $W = L(V, \epsilon) = e^B L(V, 0)$. Any two sections of W have the form $e^B u, e^B v$ where u, v are sections of $L(E, 0)$. Then

$$[e^B u, e^B v] = e^B [u, v] - \iota_X \iota_Y dB.$$

Thus W is Courant integrable if and only if $\iota_X \iota_Y dB = 0$ for all vector fields on V, i.e., $dB|_{V \otimes V} = 0$ or equivalently, $d_V \epsilon = 0$. $\square$

Note however that a generalized almost complex structure can only be written in the form $L(V, \epsilon)$, for V a subbundle of T in a neighborhood where the type is constant. In the more general case integrability is given by the vanishing of the Nijenhuis operator, as established in Section (3.4) on Dirac structures.

5.2.1 Geometry of regular points

Given an generalized almost complex structure $\mathcal{J}$, consider the i-eigenbundle V. This is a complex Lie algebroid such that $T \otimes \mathbb{C} = \pi(V) + \overline{\pi(V)}$. Let $U = \pi(V)$ so that $U + \overline{U} = T \otimes \mathbb{C}$ and let $\triangle = U \cap \overline{U}$ be the corresponding real distribution.

Recall that for a generalized distribution a regular point is a point with a neighborhood where the dimensions of the fibres of the distribution does not vary. In the present situation the dimension of the fibres of $\triangle = U \cap \overline{U}$ are $2m - 2k$ where T has dimension $n = 2m$ and k is the type of $\mathcal{J}$ at that point. Thus a point is regular if and only the type is constant in a neighborhood.

We are in the situation discussed in Section (3.3) where we have a generalized foliation corresponding to $\triangle$ such that at regular points there are k transverse complex coordinate functions where k is the codimension of $\pi(V) = U$, that is k is the type of $\mathcal{J}$ at this point. In this sense we can interpret the type of a generalized complex structure as the number of complex coordinates. However we can conclude even more. In the vicinity of a regular point we can write $V = L(U, \epsilon)$ and by Proposition (5.2.1) we have that U is a Frobenius integrable subbundle of the tangent bundle and $d_U \epsilon = 0$. By proposition (5.1.2) we have that the leaves of the foliation around a regular point inherit a non degenerate 2-form $\omega = \mathrm{Im}(\epsilon)|_{U \cap \overline{U}}$. Note that since $d_U \epsilon = 0$ we have that ω restricted to a given leaf is a closed non-degenerate 2-form. Thus the leaves are symplectic manifolds of dimension $2m - 2k$.

So far we have found that in a neighborhood of a regular point of type k a foliation consisting of symplectic leaves and complex transverse coordinates can be found. In fact even more than this can be said. First we note that there is a canonical example of a $2m$ dimensional generalized complex manifold M_k^{2m} of constant type k for $0 \leq k \leq m$. Let $(\mathbb{C}^n, \mathcal{J})$ denote $\mathbb{C}^n$ with the standard complex structure $\mathcal{J}$ and let $(\mathbb{R}^n, \omega)$ denote $\mathbb{R}^{2n}$ with the standard symplectic form $\omega = dx^0 \wedge dx^1 + \cdots + dx^{2n-1} \wedge dx^{2n}$. The symplectic form ω induces a generalized complex structure J_ω of type 0. Then we define M_k^{2m} as the generalized complex manifold $(\mathbb{C}^k \times \mathbb{R}^{2m-2k}, \mathcal{J} \oplus J_\omega)$. Likewise we can construct generalized complex manifolds of dimension $2m$ and constant type k by replacing $\mathbb{C}^k$ with open subsets of $\mathbb{C}^k$. The following theorem states that modulo B-tranforms, generalized complex manifolds always take this canonical form around regular points.

Proposition 5.2.2 (Generalized Darboux Theorem, [11]). *Let x be a regular point of type k in a generalized complex manifold of dimension $2m$. Then there exists a neighborhood U of x such that U is diffeomorphic to the product of an open set in $\mathbb{C}^k$ with the standard symplectic space $\mathbb{R}^{2m-2k}$ and such that* *after a B-transform on U, the generalized complex structure on U agrees with the generalized complex structure on the product space.*

The generalized Darboux theorem contains both the ordinary Darboux theorem and the Newlander-Nirenberg theorem as special cases.

5.2.2 Integrability and spinors

The connection between maximal isotropics and pure spinors furnishes an alternative description of integrability and moreover a $\mathbb{Z}$-grading to spinors.

Let $E = V \oplus \overline{V}$ be the splitting of E into the maximal isotropics of a generalized almost complex structure and let $\mathcal{K}$ be the corresponding canonical bundle. The pairing $(\ , \ )$ can be used to identify $\overline{V}$ with V^*. Clifford multiplication gives a bundle endomorphism $c : \mathrm{Cliff}(V \oplus V^*) \otimes \mathcal{K} \to \wedge T^*$. As a $\mathrm{SO}(V \oplus V^*)$ bundle $\mathrm{Cliff}(V \oplus V^*)$ is just the exterior bundle $\wedge(V \oplus V^*) \simeq \wedge V \otimes \wedge V^*$. It is clear that c is surjective and the kernel of c contains the left ideal generated by V so by dimension considerations c gives an isomorphism $c : \wedge V^* \otimes \mathcal{K} \to \wedge T^*$. This provides $\wedge T^*$ with a new $\mathbb{Z}$-grading $\wedge T^* = \oplus_{j=0}^n U_j$ where $U_j = \wedge^j V^* \cdot \mathcal{K}$. Notice that as Clifford multiplication changes the parity of forms and since pure spinors have either even or odd parity then this new $\mathbb{Z}$-grading refines the $\mathbb{Z}_2$-grading of spinors into even and odd forms. Also note that Clifford multiplication by elements of V^* increases the degree by 1 while Clifford multiplication by elements of V decreases the degree by 1.

Now we consider how integrability relates to spinors. Let u, v be sections of V, so $u \cdot \mathcal{K} = 0$ and $v \cdot \mathcal{K} = 0$. We have that $[u, v]$ is a section of V if and only if $[u, v] \cdot \mathcal{K} = 0$. Let ϕ be a local trivialization of $\mathcal{K}$, a pure spinor. We seek to evaluate $[u, v] \cdot \phi$. Recall that the Dorfman bracket was defined as a generalization of equation (2.9), that is we have

$$[u, v]_D \phi = d(u \cdot v \cdot \phi) + u \cdot d(v \cdot \phi) - v \cdot d(u \cdot \phi) - v \cdot u \cdot d\phi. \tag{5.6}$$

In the case where u and v are sections of V and ϕ is a pure spinor then we have $[u, v] \cdot \phi = [u, v]_D \cdot \phi = -v \cdot u \cdot d\phi = u \cdot v \cdot d\phi = (u \wedge v) \cdot d\phi$. Globally, this says that $[u, v]$ is a section of V if and only if $(u \wedge v) \cdot d(\Gamma(\mathcal{K})) = 0$. It is clear however that $\oplus_{j=0}^k U_j = \oplus_{j=0}^k \wedge^j V^* \cdot \mathcal{K}$ is the space of spinors that are annihilated by $\wedge^{k+1} V$. Thus V is integrable if and only if $d(\Gamma(U_0)) \subseteq \Gamma(U_0) \oplus \Gamma(U_1)$. Moreover, since d changes the parity of forms we have that V is integrable if and only if $d(\Gamma(U_0)) \subseteq \Gamma(U_1)$. We thus have

Proposition 5.2.3. *[11] Let $\mathcal{K}$ be the canonical bundle of a generalized almost complex strucure $\mathcal{J}$. Then $\mathcal{J}$ is integrable if and only if*

$$d(\Gamma(U_0)) \subseteq \Gamma(U_1) \tag{5.7}$$

where $U_j = \wedge^j V^* \cdot \mathcal{K}$ and V is the $+i$ eigenbundle of $\mathcal{J}$. Alternatively, $\mathcal{J}$ is integrable if and only if every local non-vanishing section $\phi \in \Gamma(\mathcal{K})$ (i.e., a pure spinor) satisfies

$$d\phi = v \cdot \phi = \iota_X \phi + \xi \wedge \phi \tag{5.8}$$

where $v = X + \xi \in \Gamma(V^*)$.

Note that in the twisted case d is replaced by d_H. We have noted that d_H changes the parity of spinors. Now that we have a finer grading we can split d_H into its constituent parts. In particular we make the following definitions

Definition 5.2.2. Let $\wedge T^* = \oplus_{j=0}^n U_j$ be the $\mathbb{Z}$-grading on spinors induced by a generalized almost complex structure. We define differential operators

$$\begin{aligned} \overline{\partial}_H &= \pi_{k+1} \circ d_H : \Gamma(U_k) \to \Gamma(U_{k+1}) \\ \partial_H &= \pi_{k-1} \circ d_H : \Gamma(U_k) \to \Gamma(U_{k-1}) \end{aligned}$$

where π_k denotes projection onto U_k.

The utility of this definition is that it allows us to rephrase the integrability condition in a familiar way:

Proposition 5.2.4. *[11] Let $\wedge T^* = \oplus_{j=0}^n U_j$ be the $\mathbb{Z}$-grading on spinors induced by a generalized almost complex structure. Then the generalized complex structure is twisted integrable if and only if*

$$d_H = \partial_H + \overline{\partial}_H. \tag{5.9}$$

Proof. Note that by Proposition (5.2.2) we have that integrability is equivalent to $d_H = \partial_H + \overline{\partial}_H$ as an operator $d_H : \Gamma(U_0) \to \Gamma(\wedge T^*)$. Therefore we need only show that this identity remains for all spinors when we have integrability. We use induction on spinor degree. The result is true for U_0. Now suppose the result is true for U_j for all $j < k$. Let $u, v \in \Gamma(V)$ and let $\phi \in \Gamma(U_k)$. Note that it suffices to show $v \cdot u \cdot d_H\phi \in \Gamma(U_{k-3}) \oplus \Gamma(U_{k-1})$. From (5.6) we have

$$v \cdot u \cdot d_H\phi = d_H(u \cdot v \cdot \phi) + u \cdot d_H(v \cdot \phi) - v \cdot d_H(u \cdot \phi) - [u, v]_{D,H}\phi.$$

Integrability shows $[u, v]_{D,H}$ is a section of V, and since Clifford multiplication by elements of V reduced the degree by 1, the result follows by induction. □

Note that we assumed (5.6) still holds in the twisted case. This is indeed true and it defines the *twisted Dorfman bracket*

$$[X + \xi, Y + \eta]_{D,H} = [X, Y] + \mathcal{L}_X\eta - \iota_Y d\xi - \iota_X\iota_Y H \tag{5.10}$$

the skew-symmetrization of which is the twisted Courant bracket.

We have the following alternative description of the spinor decomposition:

Proposition 5.2.5. *[5] Let $\mathcal{J}$ be a generalized almost complex structure acting on an $n = 2m$ dimensional space. Then U_k is the $i(m-k)$-eigenspace of $\mathcal{J}$, where the action of $\mathcal{J}$ is obtained by lifting $\mathcal{J}$ from $\mathfrak{so}(n,n)$ to $\mathfrak{spin}(n,n)$ and using Clifford multiplication.*

Proof. First recall that the action ρ of $Spin(E)$ on E is given by $\rho(x)e = x \cdot e \cdot x^{-1}$. This extends to an action of $Spin(E)$ on $\mathrm{Cliff}(E)$ given by the same formula. For $x \in \mathfrak{spin}(E)$ the associated Lie algebra action is the Clifford commutator $d\rho(x)e = [x, e] = x \cdot e - e \cdot x$. Now if ϕ is a spinor and $w \in V^*$ then

$$\begin{aligned} \mathcal{J}(w \cdot \phi) &= (\mathcal{J} \cdot w)\phi \\ &= [\mathcal{J}, w]\phi + w\mathcal{J}\phi. \end{aligned}$$

Now since the action ρ of $Spin(E)$ on $\mathrm{Cliff}(E)$ factors through to the fundamental representation of $SO(E)$ on E, we have that $[\mathcal{J}, w] = \mathcal{J}(w) = -iw$ since V^* is the $-i$-eigenspace of $\mathcal{J}$. Thus it follows that the elements of V^* act as lowering operators sending an eigenvector ϕ with eigenvalue $i\lambda$ to an eigenvector $w\phi$ with eigenvalue $i(\lambda - 1)$, similarly elements of V act as raising operators. Note that $\mathcal{J}$ acting on the complex spinors must have at least one eigenvector. Then by applying raising and lowering operators we see that U_n is an eigenspace of minimal weight while U_0 is an eigenspace of maximal weight. The eigenvalues of $\mathcal{J}$ acting on spinors are say $i\lambda, i(\lambda+1), \ldots, i(\lambda+n)$. We also have that $\mathcal{J}$ is a real operator so there must be symmetry of the eigenvalues under conjugation. Thus $\lambda = \frac{n}{2} = m$ and the result follows. □

Note that for an integrable generalized complex structure $d_H = \partial_H + \overline{\partial}_H$ and $d_H^2 = 0$ so we find $\partial_H^2 = 0$, $\overline{\partial}_H^2 = 0$ and $\partial_H\overline{\partial}_H + \overline{\partial}_H\partial_H = 0$. We also define an operator $d_H^{\mathcal{J}} = i(\overline{\partial}_H - \partial_H)$. Then by (5.2.5) we have that $d_H^{\mathcal{J}} = [d_H, \mathcal{J}]$.

5.2.3 More on the Mukai pairing

We now wish to consider how the $\mathbb{Z}$-grading on spinors relates to the Mukai pairing. There is an alternative way to understand the Mukai pairing which will be useful. First note that if $\omega \in \wedge T^*$ and $X \in T$ then the Clifford multiplication gives

$$X \cdot \omega + \omega \cdot X = \iota_X \omega$$

which can be shown by induction on the degree of ω. It follows that $X_k \wedge \cdots \wedge X_1 \cdot \omega \cdot X_1 \wedge \cdots \wedge X_k = \iota_{X_n} \ldots \iota_{X_1}\omega \cdot X_1 \wedge \cdots \wedge X_k$. If we let $f \in \det(T)$

be a non-zero determinant form then it follows that $\alpha(f)\cdot\omega\cdot f = \iota_f\omega\cdot f$ where α denotes the main anti-automorphism of the Clifford algebra. Thus

$$\begin{aligned}\iota_f\langle s,t\rangle f &= \iota_f[\alpha(s)\wedge t]_{\mathrm{top}}\cdot f\\ &= \iota_f(\alpha(s)\wedge t)\cdot f\\ &= \alpha(f)\cdot\alpha(s)\wedge t\cdot f\\ &= \alpha(s\cdot f)\cdot t\cdot f.\end{aligned}$$

The left ideal in $\mathrm{Cliff}(T\oplus T^*)$ generated by $\det(T)$ is a Clifford module isomorphic to $\wedge T^*$ where the Clifford action is simply left multiplication. In fact this action is the familiar action (2.2) which we shall denote by $c:\mathrm{Cliff}(T\oplus T^*)\to\mathrm{End}(\wedge T^*)$ which is given by

$$(c(u)\omega)\cdot f = u\cdot\omega\cdot f.$$

Thus if $s,t\in\wedge V^*$ and ϕ is a pure spinor then we have

$$\begin{aligned}\iota_f\langle c(s)\phi, c(t)\phi\rangle f &= \alpha(c(s)\phi\cdot f)\cdot c(t)\phi\cdot f\\ &= \alpha(s\cdot\phi\cdot f)\cdot t\cdot\phi\cdot f\\ &= \alpha(\phi\cdot f)\cdot\alpha(s)\cdot t\cdot\phi\cdot f\\ &= \alpha(\phi\cdot f)\cdot(\alpha(s)\wedge t)\cdot\phi\cdot f\end{aligned}$$

Now we claim that $\phi\cdot f$ has the form $\lambda\cdot\theta$ where λ is a non-zero determinant form for V and $\theta\in\wedge V^*$ is non zero. We have that if $v\in V$ then $v\cdot\phi\cdot f = c(v)\phi\cdot f = 0$. Thus we may write $\lambda\cdot\theta = \phi\cdot f$ then we have

$$\begin{aligned}\iota_f\langle c(s)\phi, c(t)\phi\rangle f &= \alpha(\theta)\cdot\alpha(\lambda)\cdot(\alpha(s)\wedge t)\cdot\lambda\cdot\theta\\ &= \alpha(\theta)\cdot\iota_\lambda\langle s,t\rangle\lambda\cdot\theta.\end{aligned}$$

Thus we have related the Mukai pairing on $\wedge T^*$ with the Mukai pairing on $\wedge V^*$. Now suppose that $s\in\wedge^j V^*$ and $t\in\wedge^k V^*$. Then $\langle s,t\rangle = 0$ unless $j+k=n$ where $n=2m$ is the top degree. But then $c(s)\phi\in U_j$ and $c(t)\phi\in U_k$ and it follows that the Mukai pairing $U_k\otimes U_j\to\wedge^n T^*$ is zero unless $k+j=n$. Now if ϕ is a pure spinor then $\langle\overline{\phi},\phi\rangle\neq 0$ so it follows that $\overline{\phi}\in U_n$ or as a global statement, $\overline{U_0}=\overline{\mathcal{K}}=U_n$. In fact since Clifford multiplication by elements of V lowers spinor degree by 1 while Clifford multiplication by elements of $V^*=\overline{V}$ raises degree by 1, it follows that $\overline{U_j}=U_{n-j}$. Then since d is a real operator we have that $\overline{\partial\phi}=\overline{\partial}(\overline{\phi})$.

The Mukai pairing gives an isomorphism $U_0\otimes U_n\to\det(T^*)$ but since a generalized almost complex structure implies existence of almost complex structures we have that M is orientable and so $\det(T^*)$ is trivial. We also have that $U_0=\mathcal{K}$ and $U_n=\det(V^*)\otimes\mathcal{K}$ so putting it altogether we have $\mathcal{K}\otimes\mathcal{K}\simeq\det(V)$.

5.3 Generalized Calabi-Yau manifolds

In this section we introduce a special class of generalized complex manifolds which have been proposed as a setting for mirror symmetry. Different geometrical structures in generalized geometry can often be understood as a reduction of structure of the generalized tangent bundle together with an integrability condition. We use this approach to arrive at a definition of generalized Calabi-Yau manifolds. However we will only briefly look at the problem of classifying them.

There actually are a number of inequivalent definitions of Calab-Yau manifolds but the definition we shall take is that a Calabi-Yau manifold is a manifold whose holonomy is contained in $SU(n)$. This in turn is equivalent to being a Kähler manifold such that the canonical line bundle is holomorphically trivial.

Recall that a generalized almost complex structure is equivalent to a reduction of structure of the generalized tangent bundle to $U(n,n)$. We shall think of almost Calabi-Yau structures as reductions of structure of the generalized tangent bundle to $SU(n,n)$. As usual, a generalized almost complex structure $\mathcal{J}$ provides a decomposition $E = V \oplus V^*$ of the generalized tangent bundle. Then the corresponding structure group $U(n,n)$ at a point x consists of the endomorphisms of E_x commuting with $\mathcal{J}_x$ and preserving the duality pairing. A $U(n,n)$ endomorphism of E_x is then an element of $SU(n,n)$ if it preserves a volume form on V, that is if its induced action on $\det(V)$ is trivial. However if $\mathcal{K}$ is the canonical bundle of $\mathcal{J}$ then we have seen that $\mathcal{K} \otimes \mathcal{K} \simeq \det(V)$ so that $c_1(V) = c_1(\det(V)) = 2c_1(\mathcal{K})$. A trivialization of $\mathcal{K}$ would imply a trivialization of $\det(V)$, however due to the possibility of torsion the converse can not be concluded. Nevertheless we have shown that a generalized almost complex structure with (topologically) trivial canonical bundle yields a reduction of structure to $SU(n,n)$. A trivialization of the canonical bundle is equivalent to a nowhere vanishing global section, that is a nowhere vanishing globally defined pure spinor Ω such that $\langle \Omega, \overline{\Omega} \rangle \neq 0$ at all points.

We now seek a suitable integrability condition. First we require that the generalized almost complex structure is integrable. Now we look for a condition that generalizes the notion of a holomorphically trivial canonical bundle. The obvious condition is to require that Ω is $\overline{\partial}$-closed, $\overline{\partial}\Omega = 0$. Note that since $\Omega \in \Gamma(U_0)$ we always have that $\partial\Omega = 0$. Thus, assuming $\mathcal{J}$ is integrable, the condition that $\overline{\partial}\Omega = 0$ is equivalent to $d\Omega = 0$. Conversely, if Ω is d-closed then by (5.8) we have that $\mathcal{J}$ is integrable. We are therefore ready to define generalized Calabi-Yau structures.

Definition 5.3.1. A *generalized Calabi-Yau structure* on a manifold M is a globally defined, nowhere vanishing, d-closed, pure spinor $\Omega \in \Gamma(\wedge T^*M)$ such that $\langle \Omega, \overline{\Omega} \rangle \neq 0$ at all points of M.

By replacing d with d_H we can also define twisted generalized Calabi-Yau structures. Although in the definition we say $\Omega \in \Gamma(\wedge T^*M)$, we have that since Ω is a pure spinor it is either an even or odd form. The type of the generalized complex structure at any point is the degree of lowest degree non-zero part of Ω.

The obvious examples of generalized Calabi-Yau manifolds represent the two extreme cases. First consider a generalized Calabi-Yau whose type is 0 at all points. Then the trivializing pure spinor must have the form $e^{B+i\omega}$ with B and ω real 2-forms. As we have already seen, the condition $\langle e^{B+i\omega}, e^{B-i\omega} \rangle \neq 0$ at all points implies that ω is a non-degenerate 2-form. Also since $e^{B+i\omega}$ is closed we have that B and ω are both closed. Thus we find that generalized Calabi-Yau manifolds of constant type 0 are precisely B-transformed symplectic manifolds, which also means that topologically they are exactly the symplectic manifolds.

Moving to the other extreme, consider generalized Calabi-Yau manifolds M of constant maximal type. Thus if M is $2m$ dimensional, then their pure spinor has the form $e^{\epsilon}\Omega$, where ϵ is a complex 2-form and Ω is a complex m-form. The Mukai-pairing yields the condition $\Omega \wedge \overline{\Omega} \neq 0$ at all points and since $e^{\epsilon}\Omega$ is d-closed, the lowest degree part shows that $d\Omega = 0$. This shows that Ω on its own is a Calabi-Yau structure and in fact is the Calabi-Yau struture corresponding to a complex manifold with holomorphically trivial canonical bundle. All that remains is to determine what possible ϵ are allowed. The condition $d(e^{\epsilon}\Omega) = 0$ is equivalent to $d\Omega = 0$ and $d\epsilon \wedge \Omega = 0$. Note that the only part of ϵ that has any effect in $e^{\epsilon}\Omega$ is the $(0, 2)$-part, where the bigrading on forms refers to the grading of the ordinary complex structure given by Ω. Thus we assume ϵ is a $(0, 2)$-form. The condition $d\epsilon \wedge \Omega = 0$ is then equivalent to $\overline{\partial}\epsilon = 0$. Thus we have established that generalized Calabi-Yau manifolds of constant maximal type are complex manifolds with trivial canonical bundle whose structure is transformed by a $\overline{\partial}$-closed $(0, 2)$-form ϵ.

5.4 Deformations of generalized complex structures

The deformation theory for generalized complex structures is in many ways a straightforward generalization of ordinary complex deformation theory. Deformations of a generalized complex structure can be viewed as perturbations in the defining endomorphism $\mathcal{J}$ preserving integrability. However to measure the distinct deformations one is lead to factor out the actions of diffeomorphisms and B-transforms. This leads to the conclusion that infinitesimal deformations are measured by the second cohomolgy $H^2(V)$ of the Lie algebroid V corresponding to $\mathcal{J}$. Moreover, it is shown in [11] that for compact manifolds, geniune deformations are realized as the zero set of an obstruction map $\Phi : H^2(V) \to H^3(V)$.

Let M be a manifold with generalized complex structure $\mathcal{J}$ and corresponding splitting $E \otimes \mathbb{C} = V \oplus \overline{V}$ into maximal isotropics. We are interested in changing from $\mathcal{J}$ to a new generalized complex structure on M. We can view this as a change of the isotropic V to a new isotropic V_1 and correspondingly $\overline{V}$ will change to $\overline{V_1}$. We assume that the deformation from V to V_1 is sufficiently small so that V_1 and $\overline{V}$ intersect only in the zero section. Therefore we can describe V_1 as a graph of a bundle endomorphism $\epsilon : V \to \overline{V}$, or after using the pairing to identify $\overline{V}$ with V^*, we view ϵ as an endomorphism from V to V^*. The graph of ϵ has the form $\{X + \epsilon X | X \in V\}$ that is, $V_1 = (1 + \epsilon)V$. Now for V_1 to define a generalized almost complex structure, we require that V_1 is maximal isotropic. It is clear that $V_1 \oplus V^* = E$ so that V_1 has the right dimension. For V_1 to be isotropic it follows that ϵ must be skew-adjoint with respect the the pairing, that is ϵ is a section of $\wedge^2 V^*$, a 2-form on V. So far we have that V_1 is a generalized almost complex structure. The integrability condition for V_1 follows from Proposition (3.4.2) and it is that ϵ satisfies the equation

$$d_V \epsilon + \tfrac{1}{2}[\epsilon, \epsilon] = 0. \tag{5.11}$$

Consider a smooth family of deformations ϵ_t of V with $\epsilon_0 = 0$. Writing $\epsilon_t = t\epsilon + t^2\epsilon^1 + \dots$ we see that for ϵ_t to satisfy equation (5.11) we require that the infinitesimal deformation ϵ must satisfy the linearization of (5.11), $d_V \epsilon = 0$. In this way we can think of infinitesimal deformations of V as d_V-closed 2-forms on V.

In order to measure distinct deformations we wish to factor out the actions of diffeomorphisms and closed B-field transformations. These are the transformations which we have called Courant automorphisms and we consider them to be the automorphisms of the generalized tangent bundle. In fact we shall consider only the cases of diffeomorphisms connected to the identity and B-transforms by exact B-fields. Such a Courant automorphism can be written as $F = e^B \circ e^X$ where $B = d\xi$ for some 1-form ξ and e^{tX} represent the 1-parameter subgroup of diffeomorphisms associated to the vector field X. In fact, as we are interested in infinitesimal deformations we shall be interested in the family of Courant automorphisms $F_t = e^{tB} \circ e^{tX}$. The following proposition taken from [11] shows how F_t acts on infinitesimal deformations:

Proposition 5.4.1. *[11] Let V be the i-eigenbundle of a generalized complex structure, let $\epsilon \in \mathcal{C}^\infty(\wedge^2 V^*)$ and $X + \xi \in \mathcal{C}^\infty(T \oplus T^*)$. Then then for all t in a sufficiently small neighborhood of zero, the action of $F_t = e^{td\xi} \circ e^{tX}$ on the graph of ϵ satisfies*

$$F_t \epsilon = \epsilon + t d_V((X+\xi)|_{V^*}) + t^2 R(\epsilon, X+\xi, t) \tag{5.12}$$

where $(X+\xi)|_{V^*}$ is the V^* component of $X+\xi$ and $R(\epsilon, X+\xi, t)$ is a smooth function of ϵ, $X+\xi$ and t.

Therefore infinitesimal deformations which differ only by the above Courant automorphisms will differ only by a d_V-exact term. Therefore it is reasonable to regard the second Lie algebroid cohomolgy $H^2(V)$ as representing the space of infinitesimal deformations of V. The following proposition taken from [11] makes this more precise:

Proposition 5.4.2. *[11] Let M be a compact generalized complex manifold. There exists an open neighbourhood $U \subset H^2(V)$ of zero and an analytic obstruction map $\Phi : U \to H^3(V)$ with $\Phi(0) = 0$, $d\Phi(0) = 0$. There is a smooth family $\widetilde{\mathcal{M}} = \{\epsilon_u | u \in U,\ \epsilon_0 = 0\}$, $\widetilde{\mathcal{M}} \subset \mathcal{C}^\infty(\wedge^2 V)$ of generalized almost complex deformations such that the integrable deformations in $\widetilde{\mathcal{M}}$ are precisely the sub-family $\mathcal{M} = \{\epsilon_z | z \in \mathcal{Z} = \Phi^{-1}(0)\}$. Furthermore any sufficiently small deformation of V is equivalent to at least one member in $\mathcal{M}$. When the obstruction map vanishes, then $\mathcal{M}$ is a smooth, locally complete family, that is, any family of deformations of V, when restricted to a sufficiently small open set in in its base, can be obtained by pull-back of a map to $\mathcal{M}$.*

Chapter 6

Generalized Kähler geometry

So far we have seen generalized metrics and generalized complex structures. By combining these two structures in a compatible way we naturally arrive at generalized Kähler structures. We then show that generalized Kähler structures can be equivalently described by a bi-Hermitian geometry, first discovered by Gates, Hull and Roček [10] as precisely the structure required for $N = (2, 2)$ supersymmetric non-linear sigma models. We also prove a generalized version of the Kähler identities.

6.1 Generalized Kähler Structures

We have seen that for a generalized metric $V = V^+$ we can define an endomorphism $G : E \to E$ which is multiplication by 1 on V and multiplication by -1 on $V^- = V^\perp$. Then $(G\ ,\ )$ is a positive definite metric on E. Moreover any endomorphism $G : E \to E$ such that $G^2 = 1$, $G^* = G$ and $(GX, X) > 0$ for $X \neq 0$ defines a generalized metric and in fact this gives an equivalent definition of generalized metrics. Now given a generalized almost complex structure $\mathcal{J}$, we can generalize the concept of a Hermitian metric

Definition 6.1.1. Let $\mathcal{J}$ be a generalized almost complex structure and a generalized metric G, we say G is a *generalized Hermitian metric* with respect to $\mathcal{J}$ if for all X, Y

$$(GJX, JY) = (GX, Y). \tag{6.1}$$

That is, the positive definite metric on E defined by G is compatible with $\mathcal{J}$.

A generalized metric is thus generalized Hermitian if and only if $\mathcal{J}^* G \mathcal{J} = G$ or since $\mathcal{J}^* = -\mathcal{J}$, if and only if $G\mathcal{J} = \mathcal{J}G$. So G is generalized Hermitian if and only if it commutes with $\mathcal{J}$. Let $V^\pm$ denote the ± 1 eigenspaces of G.

Since G and $\mathcal{J}$ commute, we have $V^{\pm}$ are $\mathcal{J}$-invariant. It follows that a generalized Hermitian metric is equivalent to a reduction of structure from $U(n,n)$ given by $\mathcal{J}$, to $U(n) \times U(n)$. We can thus rephrase Proposition (5.1.5) as follows:

Proposition 6.1.1. *Every generalized almost complex structure $\mathcal{J}$ has a compatible generalized Hermitian metric.*

Since $[G, \mathcal{J}] = 0$ we have $(G\mathcal{J})^2 = G^2\mathcal{J}^2 = -1$, and $(G\mathcal{J})^* = \mathcal{J}^*G^* = -\mathcal{J}G = -(G\mathcal{J})$. Thus $(G\mathcal{J})$ is a generalized almost complex structure. We now have two generalized almost complex structures which we denote as $\mathcal{J}_1 = \mathcal{J}$ and $\mathcal{J}_2 = G\mathcal{J}$. We have $[\mathcal{J}_1, \mathcal{J}_2] = 0$ and note that on V^+, $\mathcal{J}_1 = \mathcal{J}_2$ while on V^-, $\mathcal{J}_1 = -\mathcal{J}_2$. We also have that $\mathcal{J}_1\mathcal{J}_2 = -G$. To summarise, we have two commuting generalized almost complex structures $\mathcal{J}_1, \mathcal{J}_2$ such that $G = -\mathcal{J}_1\mathcal{J}_2$ is a generalized metric.

Example 6.1.1. Let M be a smooth manifold with almost complex structure J and Hermitian metric g. As usual J defines a generalized almost complex structure $\mathcal{J}$ given by (5.1) and a generalized metric G given by (4.10). We have

$$G\mathcal{J} \;=\begin{pmatrix} 0 & g^{-1}J^t \\ -gJ & 0 \end{pmatrix} \quad \mathcal{J}G \;=\begin{pmatrix} 0 & -Jg^{-1} \\ J^tg & 0 \end{pmatrix}.$$

Let $\omega(X, Y) = g(X, JY)$ be the associated Hermitian form. As a map $\omega : T \to T^*$ given by $X \mapsto \omega(X, \ )$ we have $\omega = J^tg$. Since g is Hermitian we also have $\omega = -gJ$. Similarly, $\omega^{-1} = Jg^{-1} = -g^{-1}J^t$. Thus $G\mathcal{J} = \mathcal{J}G$ and is the generalized almost complex structure associated to ω as in (5.2). Notice that integrability of J does not imply the integrability condition $d\omega = 0$ for ω. Conversely $d\omega = 0$ does not imply J is integrable. The Kähler condition is precisely that both J and ω are integrable.

Definition 6.1.2. A *generalized Kähler structure* is a pair $\mathcal{J}_1, \mathcal{J}_2$ of commuting, integrable generalized complex structures such that $-\mathcal{J}_1\mathcal{J}_2$ is a generalized metric.

Alternatively we have that a generalized Kähler structure is an integrable generalized structure $\mathcal{J}$ and a generalized Hermitian metric G, such that $G\mathcal{J}$ is an integrable generalized complex structure. This last condition that $G\mathcal{J}$ be integrable generalizes the condition $d\omega = 0$ for a complex structure to be Kähler.

Similarly we define twisted generalized Kähler structures by replacing Courant integrability by twisted integrability.

Proposition 6.1.2. *Let $(\mathcal{J}_1, \mathcal{J}_2, G)$ be a generalized Kähler structure. We have a splitting*

$$E \otimes \mathbb{C} = V_i^+ \oplus V_{-i}^+ \oplus V_i^- \oplus V_{-i}^-$$

into the simultaneous eigenbundles of $\mathcal{J}_1$ and G. Each of the four eigenbundles is isotropic and integrable.

Proof. Since $\mathcal{J}_1$ and G commute the above splitting occurs and the eigenbundles are isotropic. We show V_i^+ is integrable with the other bundles being similar. Let $u, v \in \Gamma(V_i^+)$. So $\mathcal{J}_1 u = iu$, $\mathcal{J}_1 v = iv$ and since $\mathcal{J}_1 = \mathcal{J}_2$ on V^+ the same applies for $\mathcal{J}_2$. Now since $\mathcal{J}_1$ and $\mathcal{J}_2$ are integrable $\mathcal{J}_1[u, v] = \mathcal{J}_2[u, v] = i[u, v]$. So $[u, v]$ is a $+i$ eigensection for both $\mathcal{J}_1$ and $\mathcal{J}_2$. The $+i$ eigenbundle for $\mathcal{J}_1$ is $V_i^+ \oplus V_i^-$ while for $\mathcal{J}_2 = \mathcal{J}_1 G$ the $+i$ eigenbundle is $V_i^+ \oplus V_{-i}^-$. Thus $[u, v]$ is a section of V_i^+ as required. □

6.2 Relation to bi-Hermitian geometry

The projection $\pi : E \to T$ allows us to identify T with either of the $V^\pm$ subbundles associated to a generalized metric. This will allow us to describe generalized Kähler structures in terms of data which describes a bi-Hermitian geometry.

Since $V^\pm$ are positive and negative definite subbundles of the same rank as T we have that $\pi : V^\pm \to T$ are bundle isomorphisms. Therefore we can identify $\mathcal{J}_1$ on $V^\pm$ with endomorphisms $\mathcal{J}_\pm$ on T. That is, $\mathcal{J}_\pm : T \to T$ are defined by

$$(\mathcal{J}_\pm X)^\pm = \mathcal{J}_1 X^\pm. \tag{6.2}$$

Which can also be written $\mathcal{J}_\pm X = \pi(\mathcal{J}_1 X^\pm)$. Note that replacing $\mathcal{J}_1$ by $\mathcal{J}_2$ in the above simply interchanges $\mathcal{J}_-$ with $-\mathcal{J}_-$. Thus we have two almost complex structures $\mathcal{J}_\pm$. Recall also that we have connections $\nabla^\pm$ defined on $V^\pm$ or on T after identification and are given (on T) by

$$2g\nabla_X^\pm Y = [X^\mp, Y^\pm] - [X, Y]^\mp. \tag{6.3}$$

Recall that a generalized metric gives a metric g on T given by

$$g(X, Y) = (X^+, Y^+). \tag{6.4}$$

Recall also that $\nabla^\pm$ are compatible with g.

Proposition 6.2.1. *[11, 16] Let $\mathcal{J}_\pm$,$\nabla^\pm$,g be the almost complex structures, connections and metric arising from a generalized Kähler structure. Then $\mathcal{J}_\pm$ are integrable and covariantly constant with respect to their corresponding connection $\nabla^\pm \mathcal{J}_\pm = 0$. Also g is Hermitian with respect to both $\mathcal{J}_+$ and $\mathcal{J}_-$.*

Proof. First we show $\mathcal{J}_+$ is integrable. Let $\mathcal{J}_+X = iX$, $\mathcal{J}_+Y = iY$. We thus have $\mathcal{J}_1X^+ = (\mathcal{J}_+X)^+ = iX^+$ and similarly $\mathcal{J}_1Y^+ = iY^+$. Integrability of $\mathcal{J}_1$ then implies $\mathcal{J}_1[X^+, Y^+] = i[X^+, Y^+]$. By Proposition (6.1.2) we have that $[X^+, Y^+]$ is a section of V^+ and moreover since $\pi[X^+, Y^+] = [X, Y]$ we have that $[X^+, Y^+] = [X, Y]^+$ and so $(\mathcal{J}_+[X, Y])^+ = \mathcal{J}_1[X, Y]^+ = i[X, Y]^+$. Thus $\mathcal{J}_+[X, Y] = i[X, Y]$ showing $\mathcal{J}_+$ is integrable. Similarly $\mathcal{J}_-$ is integrable.

Now we show g is Hermitian with respect to $\mathcal{J}_+$. This is a straightforward calculation

$$\begin{aligned} g(\mathcal{J}_+X, \mathcal{J}_+Y) &= ((\mathcal{J}_+X)^+, (\mathcal{J}_+Y)^+) \\ &= (\mathcal{J}_1X^+, \mathcal{J}_1Y^+) \\ &= (X^+, Y^+) \\ &= g(X, Y) \end{aligned}$$

where we have used that G is generalized Hermitian with respect to $\mathcal{J}_1$.

Now we show $\nabla^+\mathcal{J}_+ = 0$. That is for vector fields X, Y, Z we have $g(\nabla^+_X(\mathcal{J}_+Y), Z) = g(\mathcal{J}_+\nabla^+_XY, Z) = -g(\nabla^+_XY, \mathcal{J}_+Z)$. But using (6.3) this is equivalent to showing

$$([X^-, \mathcal{J}_1Y^+], Z^+) = -([X^-, Y^+], \mathcal{J}_1Z^+). \tag{6.5}$$

First note that by (2.11) we have that for orthogonal sections the Courant and Dorfman brackets agree and so (6.5) is equivalent to

$$([\mathcal{J}_1Y^+, X^-]_D, Z^+) = -([Y^+, X^-]_D, \mathcal{J}_1Z^+). \tag{6.6}$$

Note that by (2.14) we have

$$0 = ([Y^+, X^-]_D, \mathcal{J}_1Z^+) + (X^-, [Y^+, \mathcal{J}_1Z^+]_D) \tag{6.7a}$$
$$0 = ([\mathcal{J}_1Y^+, X^-]_D, Z^+) + (X^-, [\mathcal{J}_1Y^+, Z^+]_D). \tag{6.7b}$$

So equation (6.6) is equivalent to

$$(X^-, [\mathcal{J}_1Y^+, Z^+]_D) = -(X^-, [Y^+, \mathcal{J}_1Z^+]_D). \tag{6.8}$$

Let $Y^+ = Y^+_i + Y^+_{-i}$ and $Z^+ = Z^+_i + Z^+_{-i}$ be the decomposition of Y^+ and Z^+ into the eigenbundles of $\mathcal{J}_1$. Then

$$\begin{aligned} (X^-, [\mathcal{J}_1Y^+, Z^+]_D) &= (X^-, [\mathcal{J}_1Y^+, Z^+] + d(\mathcal{J}_1Y^+, Z^+)) \\ &= (X^-, [iY^+_i, Z^+_{-i}] + [-iY^+_{-i}, Z^+_i] + d(\mathcal{J}_1Y^+, Z^+)) \end{aligned}$$

where we have used Proposition (6.1.2) to drop terms orthogonal to X^-. We also have

$$\begin{aligned}(X^-, [Y^+, \mathcal{J}_1 Z^+]_D) &= (X^-, [Y^+, \mathcal{J}_1 Z^+] + d(Y^+, \mathcal{J}_1 Z^+)) \\ &= (X^-, [Y^+_{-i}, iZ^+_i] + [Y^+_i, -iZ^+_{-i}] - d(\mathcal{J}_1 Y^+, Z^+))\end{aligned}$$

which proves (6.6) as required. □

Note that all the results in this section apply equally well in the twisted and untwisted cases. If we only consider the untwisted case then the torsion of the connections $\nabla^\pm$ will be an exact 3-form while in the twisted case the torsion is closed but need not be exact.

6.3 Hodge Theory for generalized Kähler structures

We have seen that a generalized almost complex structure provides spinors with a $\mathbb{Z}$-grading and that a generalized metric yields the Born-Infeld metric on spinors. If the generalized metric is Hermitian with respect to a generalized almost complex structure then we get a second generalized almost complex structure and consequently a bi-grading on spinors.

Although we have already introduced the Born-Infeld metric, no reason for considering this metric was given. Now we can show that for a generalized Hermitian metric the $\mathbb{Z}$-grading on spinors is an orthogonal decomposition under the Born-Infeld metric. Let $\mathcal{J}$ be a generalized almost complex structure and G a generalized Hermitian metric with respect to $\mathcal{J}$. Let $* = e_1 e_2 \dots e_n$ be the product of an oriented orthonormal basis of V^+. Then $V^+ \otimes \mathbb{C} = V^+_i \oplus V^+_{-i}$ and since V^+_i and V^+_{-i} have the same rank, the action of $*$ on spinors preserves the grading because for every basis element that raises degree there is a corresponding one that lowers it. So we have $* : U_k \to U_k$. Recall that to define the Born-Infeld metric we also needed the map $\widetilde{\ }: \wedge T^* \to T^*$ defined by $\widetilde{\alpha} = (-1)^{k(n-k)}\alpha$ on a spinor of (form) degree k and with n the top degree and in this case $n = 2m$ is even so $\widetilde{\alpha} = (-1)^k \alpha$. Then for a pure spinor ρ of type k we have $\widetilde{\rho} = (-1)^k \rho$ and so for $s \in U_j$ we have $\widetilde{s} = (-1)^{k+j} s$. So in particular $\widetilde{\ }: U_j \to U_j$ preserves the grading. Therefore if $s \in U_j$ and $t \in U_k$ then $\overline{s} \in U_{n-k}$ and so $\langle \overline{s}, \widetilde{*t} \rangle$ is non-zero only when $j + (n-k) = n$, that is $j = k$. Thus the space of sections of spinors $\Gamma(\wedge T^*)$ decomposes orthogonally under the Born-Infeld metric into the $\Gamma(U_j)$. Moreover, since we have a second generalized almost complex structure $\mathcal{J}_2 = G\mathcal{J}$ which also commutes with G then we have a bi-grading on spinors which is orthogonal under the Born-Infeld metric.

Let the two almost complex structures be denoted $\mathcal{J}_1$ and $\mathcal{J}_2$. Then let $U_{j,k} = U_j^1 \cap U_k^2$ where the superscripts denotes to which generalized almost complex structure the grading belongs. Since the generalized almost complex structures commute these are the simultaneous eigenbundles and we have the orthogonal decomposition $T^* = \oplus_{j,k=0}^n U_{j,k}$. Note however that not all of these spaces are non-empty. The action of any element of E on spinors will must either raise or lower the $\mathcal{J}_1$-grading and also the $\mathcal{J}_2$-grading. Thus the parity of $j+k$ must always be the same. Now we shall show that the parity of $j+k$ must match that of m where $n = 2m$ is the dimension of the space involved. Let ϕ be a pure spinor for $\mathcal{J}_1$ let $E = V_i^+ \oplus V_{-i}^+ \oplus V_i^- \oplus V_{-i}^-$ be the simultaneous decomposition of E into $\mathcal{J}_1, G$ eigenspaces. Note that each of these four spaces has dimension m. Let $w \in \wedge^m V_{-i}^-$ be non-zero and consider $\rho = w \cdot \phi$. Now ρ is annihilated by $V_i^+ \oplus V_{-i}^-$ which is the $-i$-eigenspace for $\mathcal{J}_2$. But since the kernel of a spinor is isotropic it can not exceed dimension n so we have that ρ is indeed a pure spinor for $\mathcal{J}_2$. This shows $U_0^2 \subseteq U_m^1$ and thus $U_{j,k}$ is non-zero only when $j+k \equiv m(\mathrm{mod}(2))$. The bi-grading allows the differential d_H to be divided into constituent operators, in particular we define

$$\begin{aligned}
\partial_H^{\bar{1},\bar{2}} &= \pi^{j+1,k+1} \circ d_H : \Gamma(U_{j,k}) \to \Gamma(U_{j+1,k+1}) \\
\partial_H^{\bar{1},2} &= \pi^{j+1,k-1} \circ d_H : \Gamma(U_{j,k}) \to \Gamma(U_{j+1,k-1}) \\
\partial_H^{1,\bar{2}} &= \pi^{j-1,k+1} \circ d_H : \Gamma(U_{j,k}) \to \Gamma(U_{j-1,k+1}) \\
\partial_H^{1,2} &= \pi^{j-1,k-1} \circ d_H : \Gamma(U_{j,k}) \to \Gamma(U_{j-1,k-1})
\end{aligned}$$

where $\pi^{j,k}$ denotes projection onto $U_{j,k}$.

Proposition 6.3.1. *[12] Let $\mathcal{J}_1, \mathcal{J}_2$ be commuting generalized almost complex structures such that $G = -\mathcal{J}_1\mathcal{J}_2$ is a generalized metric. Then $(\mathcal{J}_1, \mathcal{J}_2, G)$ is a generalized (twisted) Kähler structure if and only if*

$$d_H = \partial_H^{\bar{1},\bar{2}} + \partial_H^{\bar{1},2} + \partial_H^{1,\bar{2}} + \partial_H^{1,2}.$$

Proof. We have that $(\mathcal{J}_1, \mathcal{J}_2, G)$ is Kähler if and only if $\mathcal{J}_1$ and $\mathcal{J}_2$ are integrable, that is if and only if $d_H\Gamma(U_j^1) \subseteq \Gamma(U_{j-1}^1) \oplus \Gamma(U_{j+1}^1)$ and $d_H\Gamma(U_k^2) \subseteq \Gamma(U_{k-1}^2) \oplus \Gamma(U_{k+1}^2)$. This is equivalent to $d_H\Gamma(U_{j,k}) \subseteq \Gamma(U_{j-1}^1 \oplus U_{j+1}^1) \cap \Gamma(U_{k-1}^2 \oplus U_{k+1}^2) = \Gamma(U_{j-1,k-1} \oplus U_{j-1,k+1} \oplus U_{j+1,k-1} \oplus U_{j+1,k+1})$. □

Chapter 7

Closing remarks

In this thesis we have seen that many familiar structures in differential geometry have a natural generalization in generalized geometry. In this way generalized geometry is able draw connections between seemingly different structures, as well as provide instances of new geometrical structures. However, it remains to be seen if this will offer any deep insights into the familiar structures like complex and symplectic geometry or whether it will simply yield generalizations. For example one can ask if generalized complex structures will prove useful in classifying smooth 4-manifolds. For example in [7], $3\mathbb{CP}^2\#19\overline{\mathbb{CP}^2}$ is shown to have a generalized complex structure, but admits no complex or symplectic structures.

As mentioned in the Introduction, there are many further topics is generalized geometry; submanifolds, group actions, deformation theory for generalized Kähler or Calabi-Yau structures, T-duality and other generalized structures arising from reduction of structure of the generalized tangent bundle. Much of the current interest in generalized geometry however is due to the emergence of connections with areas of Mathematical Physics, in particular Mirror Symmetry. The first step in this direction is the adaptation of T-duality to generalized geometry, as seen in [5]. Another step in this direction is seen in [21] where it is shown that an analog of the Tian-Todorov theorem holds for twisted generalized Calabi-Yau manifolds, that is, the moduli space of compact generalized complex structures is unobstructed and smooth.

In any case, generalized geometry has expanded into an interesting subject in its own right with many further questions to be answered.

Bibliography

[1] O. Ben-Bassat and M. Boyarchenko. *Submanifolds of generalized complex manifolds.* (2003). [math.DG/0309013].

[2] P. Bouwknegt, J. Evslin and V. Mathai, T-duality: Topology Change from H-flux, *Communications in Mathematical Physics*, **249** no.2 (2004) 383 - 415. [hep-th/0306062].

[3] P. Bouwknegt, J. Evslin and V. Mathai, On the Topology and Flux of T-Dual Manifolds, *Physical Review Letters*, **92**, 181601 (2004). [hep-th/0312052].

[4] P. Bouwknegt, K. Hannabuss and V. Mathai, T-duality for principal torus bundles. *Journal of High Energy Physics*, **03** (2004) 018, 10 pages. [hep-th/0312284].

[5] G.R. Cavalcanti. *New Aspects of the ddc-lemma*, DPhil thesis, University of Oxford, (2005). [math.DG/0501406].

[6] G.R. Cavalcanti. *Introduction to generalized complex geometry*, Lecture notes from conference *Mathematics of String Theory 2006 Australian National University, Canberra.*

[7] G.R. Cavalcanti, M. Gualtieri, *A Surgery for generalized complex structures on 4-manifolds.* [math.DG/0602333 v1].

[8] T. Courant. Dirac manifolds. *Trans. Amer. Math. Soc.*, **319**:631661, (1990).

[9] T. Courant and A. Weinstein. Beyond Poisson structures. *Action hamiltoniennes de groupes. Troisième théorème de Lie (Lyon, 1986)*, volume **27** of Travaux en Cours, pages 3949. Hermann, Paris, (1988).

[10] S. Gates Jr, C. Hull, and M. Rocek. Twisted multiplets and new supersymmetric nonlinear σ-models. *Nuclear Phys. B*, **248**(1):157.186, (1984).

[11] M. Gualtieri. *Generalized complex geometry*, DPhil thesis, University of Oxford, (2003). [math.DG/0401221].

[12] M. Gualtieri. *Generalized geometry and the Hodge decomposition.* [math.DG/0409093].

[13] N.J.Hitchin, The geometry of three-forms in six dimensions. *J. Differential Geometry* **55**, 547 576, (2000).

[14] N.J.Hitchin, Stable forms and special metrics. *Global Differential Geometry: The Mathematical Legacy of Alfred Gray*, M. Fernández and J. A. Wolf (eds.), *Contemporary Mathematics* **288**, American Mathematical Society, Providence, (2001).

[15] N. Hitchin, Generalized Calabi-Yau manifolds. *Quart.J.Math.Oxford* Ser. **54** (2003) 281-308. [math.DG/0209099].

[16] N. Hitchin. *Generalized Geometry - an introduction.* unpublished, (2006).

[17] M. Karoubi. Algèbres de Clifford et K-théorie. *Ann. Scient. Ec. Norm. Sup.*, **1**(1):161, (1968).

[18] Z. Liu, A. Weinstein and P. Xu, Manin Triples For Lie Bialgebroids. *J. Differential Geom*, **45** 547-574, (1997). [arXiv:dg-ga9508013 v3].

[19] H. Sussmann. Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.*, **180**:171188, (1973).

[20] F. Witt. *Closed forms and special metrics*, DPhil thesis, University of Oxford, (2004). [math.DG/0502443].

[21] Y. Li. *On Deformations of Generalized Complex Structures: the Generalized Calabi-Yau Case.* [hep-th/0508030].