

THE UNIVERSITY OF ADELAIDE

# Generalized Geometry

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## **Abstract**

Generalized geometry is a recently discovered branch of differential geometry that has received a reasonable amount of interest due to the emergence of several connections with areas of Mathematical Physics. The theory is also of interest because the different geometrical structures are often generalizations of more familiar geometries. We provide an introduction to the theory which explores a number of these generalized geometries.

After introducing the basic underlying structures of generalized geometry we look at integrability which offers some geometrical insight into the theory and this leads to Dirac structures. Following this we look at generalized metrics which provide a generalization of Riemannian metrics.

We then look at generalized complex geometry which is a generalization of both complex and symplectic geometry and is able to unify a number of features of these two structures. Beyond generalized complex geometry we also look at generalized Calabi-Yau and generalized Kähler structures which are also generalizations of the more familiar structures.

This work contains no material that has been accepted for the award of any other degree or diploma in any University or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

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The material in this thesis is heavily influenced by the theses of Marco Gualtieri [11] and Gil Cavalcanti [5] as well as (unpublished) notes from Nigel Hitchin [16] which I am grateful to have received. Although the presentation and proofs given in this thesis are my own work, most of the results are can be found in these references.

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