THE DEVELOPMENT AND APPLICATION OF THE FINITE ELEMENT METHOD AND FINITE STRIP METHOD IN ENGINEERING ANALYSIS

VOLUME I

By

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CONTENTS

(The publications within each section follow a sectional title page).

PREFACE

The 45 papers herein (Volume I) are divided into three major sections, the Finite Element Method of engineering analysis, the Finite Strip Method of engineering analysis, and general topics, with the arrangement of the papers within each section primarily chronological. Books and chapters in books are presented in the fourth section.

The Finite Element Method has by now been firmly established as the most powerful and versatile tool of analysis and is applicable to a wide range of problems in civil, mechanical, chemical, aeronautical, electrical and other engineering disciplines. I started research on the Finite Element Method in 1962, collaborated with Professor O.C. Zienkiewicz on the first book on Finite Element Method in 1967, and I have maintained my interest in its developments up to the present day. The earlier part of my work had been submitted for my D.Sc. degree from the University of Wales and therefore only publications from late 1972 onwards are included here. Most of the finite element papers are concerned with thick, multilayered structures (Section I - 4, 7, 9, 12, 14, 16, 18, 20), flow and heat transfer problems (Section I - 5, 6, 10, 11, 17, 22), basic developments (Section I - 15, 19, 21, 23, 25, 26), and industrial applications (Section I - 6, 8, 17). Paper 6 came out of a consulting project for the prediction of groundwater level for the Crowchild Trail, a freeway in the city of Calgary, Alberta, Canada. Paper 8 deals with the analysis of the Eckofisk Oil Storage Tank which is a two acre concrete island in the North Sea, and the whole consulting project took 1½ years to complete. Finally, paper 17 resulted from the cooperative effort between the Civil Engineering Department at the

University of Adelaide and the Victoria Electricity Trust on the investigation of a real engineering problem in production.

The Finite Strip Method is a semi-analytical method which is often regarded as a special form of Finite Element Method and is applicable to a special class of structures such as folded plates (Section II - 6, 9), plates (Section II - 1, 2), tall buildings (Section II - 3, 5, 7, 8, 10, 11), etc. (Section II - 4), in which the shape of the section does not change in one or two directions. The method is less general than the standard Finite Element Method, but has the great advantage of having much smaller degrees of freedom in the modelling of a structure, with the consequential reduction in computer cost and computer storage requirements. The method has been applied to bridge analysis and design extensively (see Report and User Manual of STRIP, Department of The Environment Bridge Design Computer Programs, March 1972), and recently it was applied to the wind load analysis of the 33 storeyed Sunning Plaza Office Block in Hong Kong.

The third section contains primarily state-of-the-art papers on finite element and finite strip methods as well as other numerical methods of analysis (Section III - 1, 2, 3, 4).

The fourth section contains books published during the period 1973 - 79, and forms Volume II of this thesis.

ACKNOWLEDGEMENTS

I wish to offer grateful thanks to my students and to my research and academic colleagues who have worked together with me on the publications listed in this thesis during the last six to seven years. In addition, a special appreciation must be given to Professor J.D. Davies, my former colleague of the Department of Civil Engineering, University of Wales, Swansea, for his interest and enthusiasm in the Finite Strip Method which I started in Swansea and developed further in Calgary and in Adelaide.

STATEMENT

This thesis contains no information or material that I have previously submitted for a degree to any other University.

YAU KAI CHEUNG

Explanatory Notes

Paper No.

Section I

1. Joint authorship: Dr. Kajita was a postdoctoral fellow working on one of my research projects funded by National Research Council of Canada. The project was initiated and supervised by Y.K.C. and implemented by T.K. The manuscript was prepared by Y.K.C. in collaboration with T.K.

2. Joint authorship: A collaborative study carried out by Dr. Lichardus of the Czechoslovakian Academy of Sciences in Bratislava and Y.K.C. and Mrs. Watson, my programming assistant. Y.K.C. and M.W. provided the computer programs and carried out all the computations while S.L. prepared the manuscript.

3. Joint authorship: Miss Wang was a research student working under the supervision of Dr. Sargious and Y.K.C. Y.K.C. provided the basic computer program and directed Miss Wang in adapting it for this problem. The manuscript was prepared jointly by M.S. and Y.K.C.

4. Joint authorship: Dr. Khatua was a research student working on one of Y.K.C.'s research projects funded by N.R.C. of Canada. The project was initiated and supervised by Y.K.C. and implemented by T.K. The manuscript was prepared by Y.K.C. in collaboration with T.K.

5. Joint authorship: Mr. Harrison was a part-time lecturer working for his M.Sc. at the University of Calgary. Y.K.C. was responsible for initiating the project and advising D.G.H. on the adaptation of the higher order bending element to flow problems. D.G.H. carried out the implementation and the manuscript was prepared jointly.

6. Joint authorship: Dr. Clark was the Chief Engineer of Hardy and Associates of Calgary and this project came out of a consulting job for the firm. Dr. Clark provided the field measurement data for this project. Dr. Skjolingstad was a research student working on one of Y.K.C.'s research projects funded by N.R.C. of Canada. The project was initiated and supervised by Y.K.C. and implemented by L.S. The manuscript was prepared by Y.K.C. in collaboration with L.S.

7. Joint authorship: As in (4) above.

8. Joint authorship: This paper came out of a consulting job. Y.K.C. was responsible for the major part of the computing, A.G. for modelling the perforated shell and W.H.D. for general back up. The manuscript was prepared by A.G. in collaboration with Y.K.C. and W.H.D.

9. Joint authorship: [illegible]. Westbrook was a colleague in the Mathematics Department of the University of Calgary. His contribution was in initiating the project and

in indicating the functional. The computation was carried out by Dr. Chakrabarti under my supervision. The manuscript was written by D.R.W. in collaboration with Y.K.C.

10. Joint authorship: This paper came out of a contract research project awarded by the Department of Mines and Resources of Canada to Y.K.C. The project was supervised by Y.K.C. and implemented by L.S. The manuscript was prepared by Y.K.C. in collaboration with L.S.

11. Joint authorship: As in (4).

12. Joint authorship: As in (4) with Kwok replacing Khatua.

13. Joint authorship: As in (1).

14. Joint authorship: As in (12).

15. Joint authorship: This paper came out of the Eckofisk Tank consulting job (see(8)). The idea was developed by Y.K.C. and the program was implemented ty T.K. under Y.K.C.'s supervision. The manuscript was prepared jointly.

16. Joint authorship: Dr. Kasemset was a Postdoctoral Fellow working on my research project. The manuscript was prepared jointly.

17. Joint authorship: Mr. Meksham was a research student working under the joint supervision of Y.K.C. and M.F.Y. Many of the techniques used were suggested by Y.K.C. and implemented by M.J.M. The manuscript was

prepared by Y.K.C. in collaboration with M.J.M. and M.F.Y.

18. Joint authorship: As in (16).

19. Joint authorship: As in (12). Dr. Delcourt was a research student working under my supervision at the University of Adelaide. Her contribution was in implementing the non-self-adjoint part of the project.

20. Joint authorship: As in (9).

21. Joint authorship: Dr. Chan is a colleague in the Department of Civil Engineering, University of Hong Kong. The project was initiated and directed by Y.K.C., and the theoretical groundwork was laid out by Y.K.C. The manuscript was prepared jointly.

22. Joint authorship: Mr. Tham is a research student under my direct supervision. The project was initiated by Y.K.C. and implemented by L.G.T. The manuscript was prepared by Y.K.C. in collaboration with L.G.T.

23. Joint authorship: As in (21).

24. Joint authorship: As in (21).

25. Joint authorship: Mr. Wong was a research assistant supported by a University research grant awarded to Y.K.C. The project was supervised by Y.K.C. and implemented by P.W.W. while H.C.C. assisted in perfecting the computer programs. The manuscript was prepared by Y.K.C. in collaboration with P.W.W. and H.C.C.

26. Joint authorship: A collaborative study carried out by Dr. Carey of the University of Texas and Mr. Lau, my research student. The project was initiated and designed by G.C. but all the mathematical development and computational work were carried out under Y.K.C.'s supervision. The manuscript was prepared jointly by G.C. and Y.K.C.

27. Joint authorship: A collaborative study carried out with Y.T.L. of the University of Hong Kong.

Paper No.

Section II

1. Joint authorship: Dr. Chan was a postdoctoral fellow working on one of my projects funded by N.R.C. of Canada. The project was initiated and supervised by Y.K.C. and implemented by H.C.C. The manuscript was prepared by Y.K.C. in collaboration with H.C.C.

2. Joint authorship: As in (1), with Dr. Wu replacing Dr. Chan.

3. Joint authorship: A collaborative study carried out between Y.K.C., S.G.H. and Y.B.C., all of the University of Adelaide.

4. Joint authorship: A collaborative study carried out between Y.K.C., M.F.Y. and D.A.L., all of the University of Adelaide.

5. Joint authorship: Dr. Kasemset was a postdoctoral fellow working under my supervision. The project was initiated by Y.K.C. and implemented by C.K. and S.G.H. The manuscript was prepared jointly.

6. Joint authorship: Dr. Delcourt was a research student working under my supervision at the University of Adelaide. The project was initiated by Y.K.C. and implemented by C.D. The manuscript was prepared by Y.K.C. in collaboration with C.D.

7. Joint authorship: Dr. Kasemset was a postdoctoral fellow and Mr. Swaddiwudhipong is a research student working under my supervision. The project was initiated

by Y.K.C. and implemented by C.K. and S.S. The manuscript was prepared by Y.K.C. in collaboration with C.K.

8. Joint authorship: As in (7), with the deletion of Mr. Swaddiwudhipong.

9. Joint authorship: As in (6).

10. Joint authorship: As in (6), with Mr. Swaddiwudhipong replacing Dr. Delcourt.

11. Joint authorship: As in (10).

12. Joint authorship: As in (6), with Mr. Fan of the University of Hong Kong replacing Dr. Delcourt.

13. Joint authorship: A collaborative study carried out between Y.K.C. and S.G.H. at the University of Adelaide. The computer program was written by Mr. George Cookson of the University Computer Centre under our joint supervision.

Paper No.

Section III

1. Sole authorship: A review of the various methods of analysis of pavements and the presentation of some of my research work in this area.

2. Sole authorship: A review of the application of the finite strip method in different areas of structural analysis and the presentation of a up-to-date literature survey.

3. Sole authorship: As in (1) with tall buildings replacing pavements.

4. Joint authorship: Mr. Tham is a research student working under my supervision at the University of Hong Kong. The project was initiated by Y.K.C. and implemented by L.G.T. The manuscript was prepared by Y.K.C. in collaboration with L.G.T. The invited lecture was presented by Y.K.C.

5. Joint authorship: Dr. Jones was a Postdoctoral Fellow supported by an A.R.G.C. grant awarded to Dr. Mazumdar and Y.K.C. The project was initiated by J.M. and supervised jointly by J.M. and Y.K.C. The work was carried out by R.J. and the manuscript was prepared by R.J. in collaboration with J.M. and Y.K.C.

Book No.

Section IV

1. Sole authorship: A reference book in which a great part of the material was based on papers authored or co-authored by Y.K.C.

2. Sole authorship: A chapter of a reference book. The material presented included Russian and Chinese references and also some of my own research work on beams and plates on elastic foundations.

3. Joint authorship: This book was initiated by Y.K.C. and the contents were laid out by Y.K.C. M.F.Y. was responsible for the computer programs and the numerical examples, and he also contributed to the writing of a part of the text.

INDEX TO PUBLICATIONS

Section I - Finite Element Methods in Engineering Analysis
(27 publications)

1. T. Kajita and Y.K. Cheung, "Finite Element Analysis of Cable-stayed Bridges", Publications, International Association for Bridges and Structural Engineering, Vol. 32-II, 1972.

2. S. Lichardus, Y.K. Cheung and M. Watson, "Orthotropic Flat Plates Supported on Columns Elongated in Plane", ZAMM 52, T141-T142, 1972.

3. S.K. Wang, M.A. Sargious and Y.K. Cheung, "Effect of Openings on Stresses in Rigid Pavements", Transportation Engineering Journal, Proc. ASCE, Vol. 99, No. TE2, May 1973.

4. T.P. Khatua and Y.K. Cheung, "Stability Analysis of Multilayer Sandwich Structures", AIAA Journal, Vol. 11, No.9, September 1973.

5. D.G. Harrison and Y.K. Cheung, "A Higher-order Triangular Finite Element for the Solution of Field Problems in Orthotropic Media", Int. J. Num. Meth. in Engineering, Vol. 7, 287-295, 1973.

6. L. Skjolingstad, Y.K. Cheung and J.L. Clark, "Groundwater Predictions for a Highway Cut Using Finite Elements", 26th Canadian Geotechnical Conference, 18 October 1973, Toronto.

7. T.P. Khatus and Y.K. Cheung, "Finite Element Analysis of Multilayer Sandwich Plates and Shells", Proceedings, Conference on Finite Element Methods in Engineering, School of Civil Engineering, University of New South Wales, Sydney, Australia, 1974.

8. A. Ghali, W.H. Dilger and Y.K. Cheung, "Prestressed Concrete Underwater Oil Tank", Engineering Journal, May/June 1974, pp. 19-22 (Canada).

9. D.R. Westbrook, S. Chakrabarti and Y.K. Cheung, "A Three Dimensional Finite Element Method for Plate Bending", Int. J. Mech. Sciences, Vol. 16, pp. 479-487, 1974.

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11. Y.K. Cheung and L. Skjolingstad, "Two and Three Dimensional Ground-water Seepage by Finite Element", Proc. Conference on Finite Element Methods in Engineering, University of New South Wales, Sydney, August 1974, pp. 751-766.

12. W.L. Kwok and Y.K. Cheung, "Analysis of Circular and Annular Laminated Thick Plates", Proc. Conference on Finite Element Methods in Engineering, University of New South Wales, Sydney, August 1974, pp. 177-194.

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15. Y.K. Cheung and T.P. Khatua, "A Finite Element Solution Program for Large Structures", Int. J. Num. Method In Eng., Vol. 10, 1976, pp. 401-412.

16. C. Kasemset and Y.K. Cheung, "Axisymmetric Vibration of Multilayer Sandwich Plates and Shells", Proceedings, Int. Conf. on Finite Element Methods in Civil Engineering, University of Adelaide, December 1976.

17. M.J. Melksham, M.F. Yeo and Y.K. Cheung, "The Initial Effects of Water Jet Cleaning on Superheater Tubes", Proceedings, Int. Conf. on Finite Element Methods in Civil Engineering, University of Adelaide, December 1976.

18. C. Kasemset, Y.K. Cheung and T.P. Khatua, "Curved Multilayered Element for Axisymmetric Shells", J. Eng. Mech. Div., Proc. ASCE, Vol. 193, EMI, Feb. 1977, pp. 139-151.

19. W.L. Kwok, Y.K. Cheung and C. Delcourt, "Application of Least Square Collocation Technique in Finite Element and Finite Strip Formulation", Int. J. Num. Meths. in Eng., Vol. 11, 1977, pp. 1391-1404.

20. D.R. Westbrook, S. Chakrabarti and Y.K. Cheung, "A Three Dimensional or Penalty Finite Element Method for Plate Bending", Int. J. Mech. Sci. Vol. 18, 1976, pp. 347-350.

21. Y.K. Cheung and H.C. Chan, "A Family of Rectangular Bending Elements", Journal of Computer and Structures, Vol. 10, pp. 613-619, 1979.

22. Y.K. Cheung and L.G. Tham, "Time-space Finite Elements for Unsaturated Flow Through Porous Media", Vol. 1, proceedings of the 3rd International Conference on Numerical Methods in Geomechanics, Aachen April 1979, pp. 251-256.

23. Y.K. Cheung and H.C. Chan, "Generation of a Series of High Order Rectangular Plane Stress Elements", Vol. 2, proceedings of the Seventh Canadian Congress on Applied Mechanics, Sherbrooke, May 27-June 1, 1979, pp. 899-900A.

24. Y.K. Cheung and H.C. Chan, "Analysis of Shear Walls using Higher Order Finite Elements", Building & Environment, Scotland, 1979.

25. Y.K. Cheung, P.W. Wong and H.C. Chan, "Generation of Higher Order Subparametric Bending Elements", accepted for publication in Engineering Analysis.

26. Y.K. Cheung, G. Carey and S.L. Lau, "Mixed Operator Problems Using Least-Squares Finite Elements Collocation", accepted for publication in Journal of Computer Methods in Applied Mechanics and Engineering.

27. Y.T. Leung and Y.K. Cheung, "A New Frame Super-Element in Static and Dynamic Analysis", paper for Seventh Australasian Conference on the Mechanics of Structures and Materials, Perth, May 1980.

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1. H.C. Chan and Y.K. Cheung, "Static and Dynamic Analysis of Multi-layered Sandwich Plates", Int. J. Mech. Sci., Vol. 14, 1972.

2. Y.K. Cheung and C.I. Wu, "Frequency Analysis of Rectangular Plates Continuous in One or Two Directions, Int. J. Earthquake Engineering and Structural Dynamics", Vol. 3, 1974, pp. 3-14.

3. Y.B. Cheung and Y.K. Cheung and S.G. Hutton, "Frequency Analysis of Coupled Shear Walls by Finite Strip Method", Fifth Australasian Congress on the Mechanics of Structures and Materials, University of Melbourne, 1975.

4. Y.K. Cheung, M.F. Yeo and D.A. Cumming, "Three-dimensional Analysis of Flexible Pavements with Special Reference to Edge Loads", Proceedings, 1st Conf. of the Road Engineering Association of Asia and Australia, Bangkok, 1976.

5. Y.K. Cheung, C. Kasemset and S.G. Hutton, "Frequency Analysis of Coupled Shear Wall Assemblies", Int. J. Earthquake Engineering and Structural Dynamics, Vol. 5, 1977, pp. 191-201.

6. Y.K. Cheung and C. Delcourt, "Buckling and Vibration of Thin, Flat-Walled Structures Continuous over Several Spans", Paper No. 7959, Part 2, Proc. Inst. Civ. Engrs. January 1977, pp. 93-103.

7. Y.K. Cheung, C. Kasemset and S. Swaddiwudhipong, "Vibration and Stability of Tall Frame Structures", Invited paper for Int. Conf. on Computer Applications in Developing Countries, Vol. 2, August 22-25, 1977, pp. 977-989, Bangkok.

8. Y.K. Cheung and C. Kasemset, "Approximate Frequency Analysis of Shear Wall Frame Structures", Int. J. Earthquake Engineering and Structural Dynamics, Vol. 6, 1978, pp. 221-229.

9. C. Delcourt and Cheung, Y.K., "Finite Strip Analysis of Continuous Folded Plates", I.A.B.S.E., Periodica, May 1978.

10. Y.K. Cheung and S. Swaddiwudhipong, "Analysis of Frame Shear Wall Structures Using Finite Strip Elements", Proc. Instn. Civil Engineers, Part 2, Sept. 1978.

11. Y.K. Cheung and S. Swaddiwudhipong, "Free Vibration of Frame Shear Wall Structures on Flexible Foundations", Earthquake Engineering and Structural Dynamics, Vol. 7, No. 4, July - August, 1979, pp. 355-368.

12. Y.K. Cheung and S.C. Fan, "Analysis of Pavements and Layered Foundations by Finite Layer Method", Vol. 3, Proceedings of the 3rd International Conference on Numerical Methods in Geomechanics, Aachen April 1979, pp. 1129-1135.

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Section III - General Topics

(5 publications)

1. Y.K. Cheung, "Numerical Analysis of Pavements", Proceedings, Symposium on Recent Developments in the Analysis of Soil Behaviour and their Application to Geotechnical Structures, University of New South Wales, July 1975.

2. Y.K. Cheung, "Finite Strip Method and Its Applications", Special Lecture, Int. Conf. on Finite Element Methods in Civil Engineering, University of Adelaide, December 1976.

3. Y.K. Cheung, "Tall Buildings - Analysis and Design", H.K.I.E. Journal, May 16, 1979.

4. Y.K. Cheung and L.G. Tham, "On the Numerical Solution of Certain Initial Value Problems", to be published in Vol. 4 of the Proceedings of the 3rd Int. Conf. on Numerical Methods in Geomechanics, Aachen, Germany, 2-6, April, 1979.

5. Y.K. Cheung, J. Mazumdar, R. Jones, "Vibration and Buckling of Plates Elevated Temperatures", accepted for publication in Intern. Journal of Solids and Structures, 1979.

Section IV - Books and Chapters of Books

(3 publications)

1. Y.K. Cheung, "Finite Strip Method in Structural Analysis", Pergamon Press, 1976, pp. 1-232.

2. Y.K. Cheung, (C.S. Desai and J.T. Christian (eds.)), Chapter 5 of "Numerical Methods in Geotechnical Engineering", McGraw-Hill, 1977, pp. 176-210.

3. Y.K. Cheung & M.F. Yeo, "A Practical Introduction to Finite Element Analysis", Pitmans Publishing Ltd., London, May 1979, pp. 1-180.

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(27 publications)

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1. Kajita, T., & Cheung, Y. K. (1972). Finite element analysis of cable-stayed bridges. *International Association for Bridges and Structural Engineering, 32*(2), 101-112.

2. Lichardus, S., Cheung, Y. K., & Watson, M. (1972). Orthotropic flat plates supported on columns elongated in plane. *ZAMM, 52*, T141-T142.

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5. Harrison, D. G., & Cheung, Y. K. (1973). A higher-order triangular finite element for the solution of field problems in orthotropic media. *International Journal for Numerical Methods in Engineering*, *7*(3), 1097-0207. doi: http://dx.doi.org/10.1002/nme.1620070306

6. Skjolingstad, L., Cheung, Y. K., & Clark, J. L. (1973). *Groundwater predictions for a highway cut using finite elements*. Paper presented at the 26th Canadian Geotechnical Conference, Toronto. (included)

7. Khatua, T. P., & Cheung, Y. K. (1974, August). *Finite element analysis of multilayer sandwich plates and shells*. Paper presented at the Conference on Finite Element Method, University of New South Wales, Sydney.

8. Ghali, A., Dilger, W. H., & Cheung, Y. K. (1974). Prestressed concrete underwater oil tank. *Engineering Journal, May/June*, 19-22.

9. Westbrook, D. R., et al. (1974). "A three dimensional finite element method for plate bending." *International Journal of Mechanical Sciences, 16*(7): 479-487. (included)

10. Cheung, Y. K., & Skjolingstad, L. (1974, August). *Unsteady radial flow in gas reservoir by finite elements*. Paper presented at the Conference on Finite Element Methods in Engineering, University of New South Wales, Sydney.

11. Cheung, Y. K., & Skjolingstad, L. (1974, August). *Two and three dimensional ground water seepage by finite element*. Paper presented at the Conference on Finite Element Methods in Engineering, University of New South Wales, Sydney.

12. Kwok, W. L., & Cheung, Y. K. (1974, August). *Analysis of circular and annular laminated thick plates*. Paper presented at the Conference on Finite Element Methods in Engineering, University of New South Wales, Sydney.

13. Chan, H. C., & Cheung, Y. K. (1974). Contact pressure of rigid footings on elastic foundations. *Civil Engineering, Apr.*, 51-59.

14. Cheung, Y. K. & Kwok, W. L. (1975). Dynamic analysis of circular and sector thick, layered plates. *Journal of Sound and Vibration*, *42*(2), 147-158. doi: https://doi.org/10.1016/0022-460X(75)90212-6

15. Cheung, Y. K., & Khatua, T. P. (1976). A finite element solution program for large structures. *International Journal for Numerical Methods in Engineering*, *10*(2), 401-412. doi: http://dx.doi.org/10.1002/nme.1620100210

16. Kasemset, C., & Cheung, Y. K. (1976, December). *Axisymmetric vibration of multilayer sandwich plates and shells*. Paper presented at the International Conference on Finite Element methods in Civil Engineering, University of Adelaide, Adelaide.

17. Melksham, M. J., Yeo, M. F., & Cheung, Y. K. (1976, December). *The initial effects of water jet cleaning on superheater tubes*. Paper presented at the International Conference on Finite Element methods in Civil Engineering, University of Adelaide, Adelaide.

18. Kasemset, C., Cheung, Y. K., & Khatua, T. P. (1977). Curved multilayered element for axisymmetric shells. *Journal of the Engineering Mechanics Division*, *103*(EM1), 139-151.

19. Kwok, W. L., Cheung, Y. K., & Delcourt, C. (1977). Application of least square collocation technique in finite element and finite strip formulation. *International Journal for Numerical Methods in Engineering*, *11*(9), 1391-1404. doi: http://dx.doi.org/10.1002/nme.1620110905

20. Westbrook, D. R., Chakrabarti, S., & Cheung, Y. K. (1976). A three dimensional or penalty finite element method for plate bending. *International Journal of Mechanical Sciences*, *18*(7-8), 347-350. doi: https://doi.org/10.1016/0020-7403(76)90010-2

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24. Cheung, Y. K., & Chan, H. C. (1979). Analysis of shear walls using higher order finite elements. *Building & Environment*, *14*(3), 217-224. doi: https://doi.org/10.1016/0360-1323(79)90040-4

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26. Cheung, Y. K., Carey, G., & Lau, S. L. (1980). Mixed operator problems using least-squares finite elements collocation. *Computer Methods in Applied Mechanics and Engineering*, *22*(1), 121-130. doi: https://doi.org/10.1016/0045-7825(80)90054-7

27. Leung, Y. T., & Cheung, Y. K. (1980, May). *A new frame super-element in static and dynamic analysis*. Paper presented at the Seventh Australasian Conference on the Mechanics of Structures and Materials, Perth.

GROUNDWATER PREDICTIONS FOR A HIGHWAY CUT USING FINITE ELEMENTS

by

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ABSTRACT

This paper describes the results of a hydrotechnical investigation for a highway cut in Northwest Calgary. The cut extends below the water table in a fine grained glacial lake deposit. The transient and steady state behaviour of the groundwater for different drain configurations were analysed using a finite element program. The results of these computations showed that two drains were necessary to lower the water table below the excavation level. Two pump tests were used to determine the field permeabilities and confirmed the values used in the computer program which were based on index properties of the soil. Observation wells were installed and the variation in the groundwater levels after construction corresponded very closely to those predicted in the finite element analyses.

INTRODUCTION

The geotechnical engineer is very often confronted with groundwater seepage problems, and most of these are concerned with free surface flow. The method of constructing a flow net is satisfactory when analysing simple steady state problems, but if the boundary conditions become complicated or if transient seepage is important, the design has generally been excessively conservative.

The finite element method has been proven to be a powerful tool in steady state free surface groundwater seepage as documented by Zienkiewicz and Cheung (1965), Finn (1967) and Taylor and Brown (1967). The method has also been extended to analyze transient behaviour by Cheung et. al. (1970).

This technique was used to analyse transient groundwater conditions along the extension of the Crowchild Trail in Northwest ;ary. The construction involved excavation below the original water table in a fine glacial lakebed deposit consisting of silty sand and sandy or silty clay with lenses of silt and sand.

This application is to the authors knowledge the first time a transient finite element analysis has been used on a project and where water levels have been recorded during and after construction to check the calculations.

OBJECTIVES

The objectives of this study were:

1. To use a finite element analysis to determine the transient and steady state draw-down, in order that a drainage line configuration could be set.
2. To use the same method to predict the draw-down with time.
3. To determine the infiltration characteristics and permeability by means of a pump test.
4. To compare analytical results with field observations during and after construction.

SITE DESCRIPTION

Figure 1 shows a plan of the area which was investigated. The grounds along the route were considerably modified by the development of both the Banff Highway and Morley Trail. Road cuts and fills and continuing urban development mask the original character of the ground surface and affect both the surficial drainage system and groundwater conditions. Interpreted surficial drainage patterns prior to construction are shown in Figure 1.

Soil profiles along the highway extension are shown for two typical sections on Figure 2. The upper part of the profile consists of fine lake deposits. They originated during a second period of glaciation when the Eastern ice sheet blocked the Bow Valley to form a large lake called Lake Calgary. The upper part of the lake sediments in the area studied, basically consists of three distinct soil types. These are: sandy silt, sandy and clayey silt and clayey silt. The first two silt types are strongly stratified and interbedded with lenses of clay and sandy materials which range in thickness from several inches to several feet. The clayey silt encountered below these surficial layers is more uniform and is less permeable than the upper part of the soil profile.

The groundwater table along the studied route reflects the water retention properties of individual soil types. Because of the generally low permeability of the clayey silt a high groundwater table was encountered throughout the area. Air photo interpretation revealed that the natural runoff from this area was intercepted by the existing road. Both existing and anticipated groundwater conditions had an unfavourable implication with respect to the construction and performance of the proposed extension unless the area was properly drained.

METHOD

Because of the unfavourable groundwater conditions, a drainage system was required for the proper performance of the highway section on a long term basis. A finite element computer program developed at the University of Calgary was used to analyze both the transient and steady state behaviour of the water table after installation of the drains. The program is described in detail in a paper by Cheung et. al. (1970), and only a brief summary of the technique will be presented here.

The program utilizes a mesh of triangular elements to simulate the seepage region. In the steady state analysis the upper boundary is adjusted until the potential at a node is equal to its elevation. In the transient case the velocities of the top nodes are computed and used to determine how much the free surface moves in the given time interval. The mesh is then adjusted to this new configuration. All the adjustments in both the steady state and the transient analysis are done automatically by the program. In order to facilitate the data input the program also uses an automatic mesh generator. In addition the program is inexpensive to run; a transient analysis of a section can be done for between 10 and 20 dollars depending on the detail required.

ANALYSIS OF DRAINAGE SYSTEM

Two sections, namely Station 30 + 00 and Station 60 + 00 were selected as representative and analysed for the simplest case where the construction dewatering would be handled by a single line. The lower layer of clayey silt is less permeable than the other two strata. The bottom boundary in the analysis was therefore taken to be the top of this layer. The results are shown in Figures 3 to 5. The steady state analysis showed a satisfactory draw-down for Section 60 + 00 (Figure 3), but was unsatisfactory for Section 30 + 00 (Figure 4). The transient flow analysis indicated that a single line was inadequate to provide the drainage required during construction. (Figure 5).

A two line drainage system was therefore studied. The transient flow analysis in Section 30 + 00 (Figure 6) showed that the water table would not be drawn down below the road surface until a period of three months after the installation. At Section 60 + 00 (Figure 7) the water table would be below the road surface after a period of about six weeks.

PUMP TEST

The permeability of the soil profile was not tested prior to the analysis, but estimated on the basis of known index properties of representative samples, as related to results of other pump tests in the same soil strata close to the test site. It is therefore reasonable to assume that the estimated permeabilities were reliable. The permeability could, however, substantially affect the results of the sophisticated analyses and the actual performance of the proposed drainage system. A decision was therefore made to investigate the groundwater flow characteristics in the field by means of pump tests on two wells. The location of the wells are shown on Figure 1. Six observation wells were used for each pump test. These wells were postioned at 3, 10 and 25 radius in two directions. In addition there was an old well at 100 feet radius in one direction.

The pump tests were carried out over three days. When the required depressions in the pumped wells were reached, the pumps were operated at low discharge in order to simulate transient to steady flow conditions in their vicinities. The hydrograph for well 1 is shown in Figure 8.

Data obtained during both the second stage of pumping and the recovery stage were used to evaluate coefficients of permeability.

Both wells gave essentially the same results yielding a permeability of 3×10^{-4} cm/sec. for the surficial layers which was in good agreement with values used in the theoretical analyses. The stratification of the actual soil justifies a slightly higher horizontal and a slightly lower vertical permeability than the calculated coefficient. The shapes of the depression cones recorded during the field tests, indicated that the difference between the permeability in the two directions would probably not be greater than one order.

The analytical solutions for the draw-downs were compared with the depression cones recorded during the field pump tests. The correlation of calculated and measured gradients and action radii lead to the conclusion that the solution represented by Section 30 + 00 was likely the predominant case and that those conditions would prevail along the route.

CONCLUSIONS DERIVED FROM ANALYSIS AND FIELD TESTS

The field tests and steady state analysis indicated that the long-term sub-drainage requirements could be satisfied by a centre drain down the medium strip and parallel drains on the north and south shoulders. It was recommended that excavation be carried down to a depth within two feet of the water table at which time the centre and north drainage lines should be installed. Previous experience indicated that earth moving equipment would bog down as the water table was approached. The transient state analysis indicated that six weeks drainage time would be required before excavation could proceed without equipment difficulty due to groundwater, and a total of three months would be required before the excavation could be carried to the required depth over the entire length. This analysis did not take the effect of rainfall into account.

FIELD RECORDS OF WATER TABLE BEHAVIOUR

Construction of the expressway extension started in late June, 1971. Difficulty with the earthmoving equipment was encountered at a depth corresponding to about two feet above the water table. In order to expedite construction, the contractor chose to attempt stabilization of the soils to improve trafficability for excavation. The drainage lines were not operative until early fall. The drainage system within the area analyzed and mentioned were effected in October, 1971.

First observation wells were installed in September but were wiped out by construction equipment so that the first meaningful readings were secured in late October and early November, two weeks after the drainage system was activated. Readings have been continued to date. Figure 9 shows the groundwater fluctuation for Station 60 + 00 for the 1972 - 1973 period. The minimum water level agrees very well with the steady state conditions predicted. The records for the first few weeks are particularly useful for comparing the predicted transient analysis and recorded behaviour. Figure 10 shows the predicted transient state analysis and the recorded water levels form the time of installation of the observation wells until March of 1972, when it appears that the steady state condition was obtained. The agreement between recorded and predicted behaviour is very good. It would appear from these results that increased evaporation as a result of the excavation was offset by rainfall during the observation period.

CONCLUSIONS DERIVED FROM FIELD RECORDS DURING AND AFTER CONSTRUCTION

The rate of draw-down after installation of the drainage system was predicted with good accuracy by the transient analysis. Neither evaporation nor precipitation was accounted for in this analysis. These aspects appear to offset each other in this particular case. It is possible to extend the program to include surface infiltration. This might be necessary if the program is to be used in areas of high rainfall.

The minimum water levels recorded agree closely with the predicted steady state water level. The records suggest that the analytical method demonstrated is useful in determining the design of subdrainage systems for highways and also useful for determining construction requirements where the rate of draw-down is of significance. The annual fluctuations were not analyzed but are not of significance to the highway inasmuch as they are confined permanently to the area well outside the drainage lines.

ACKNOWLEDGEMENTS

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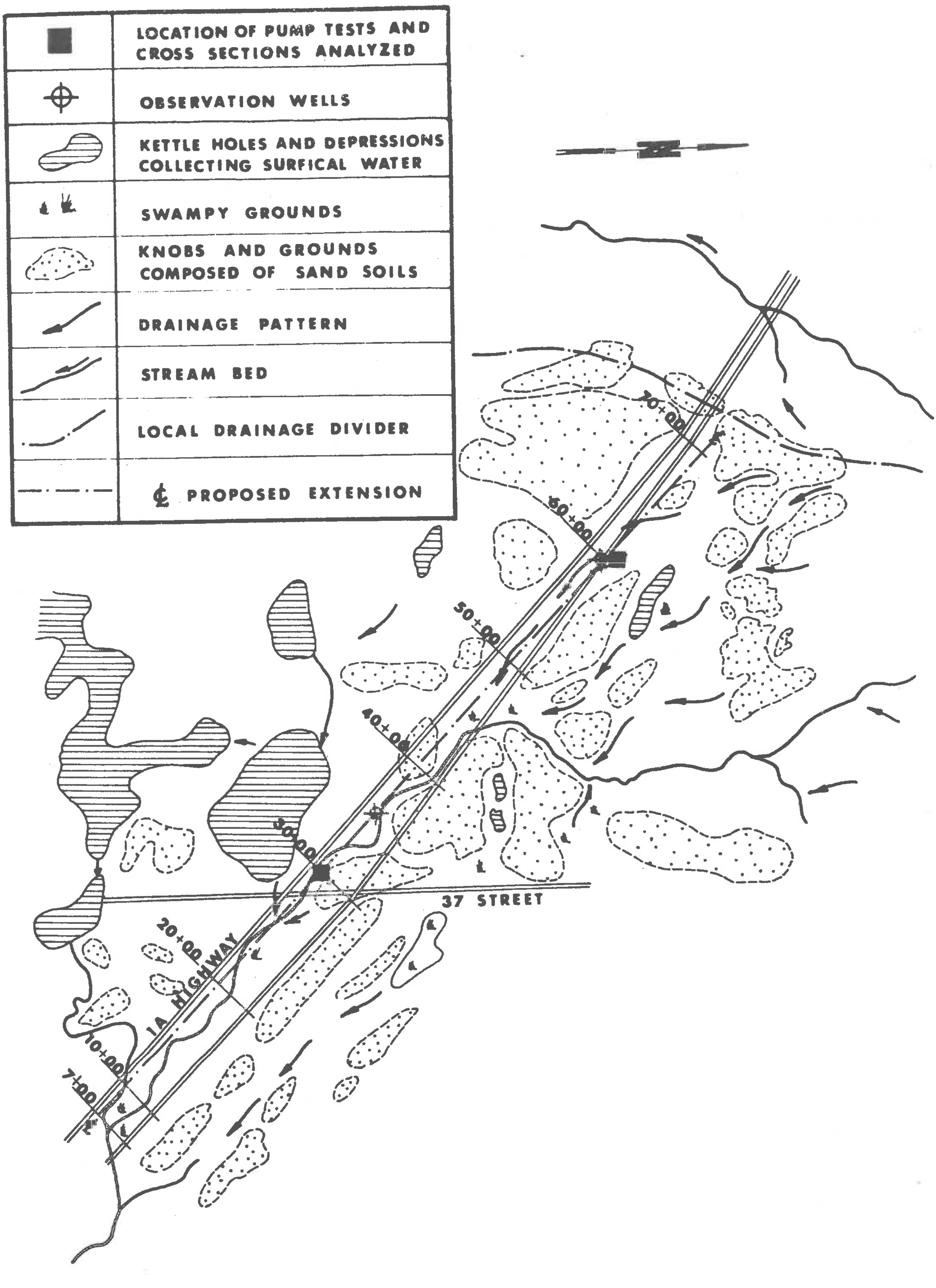

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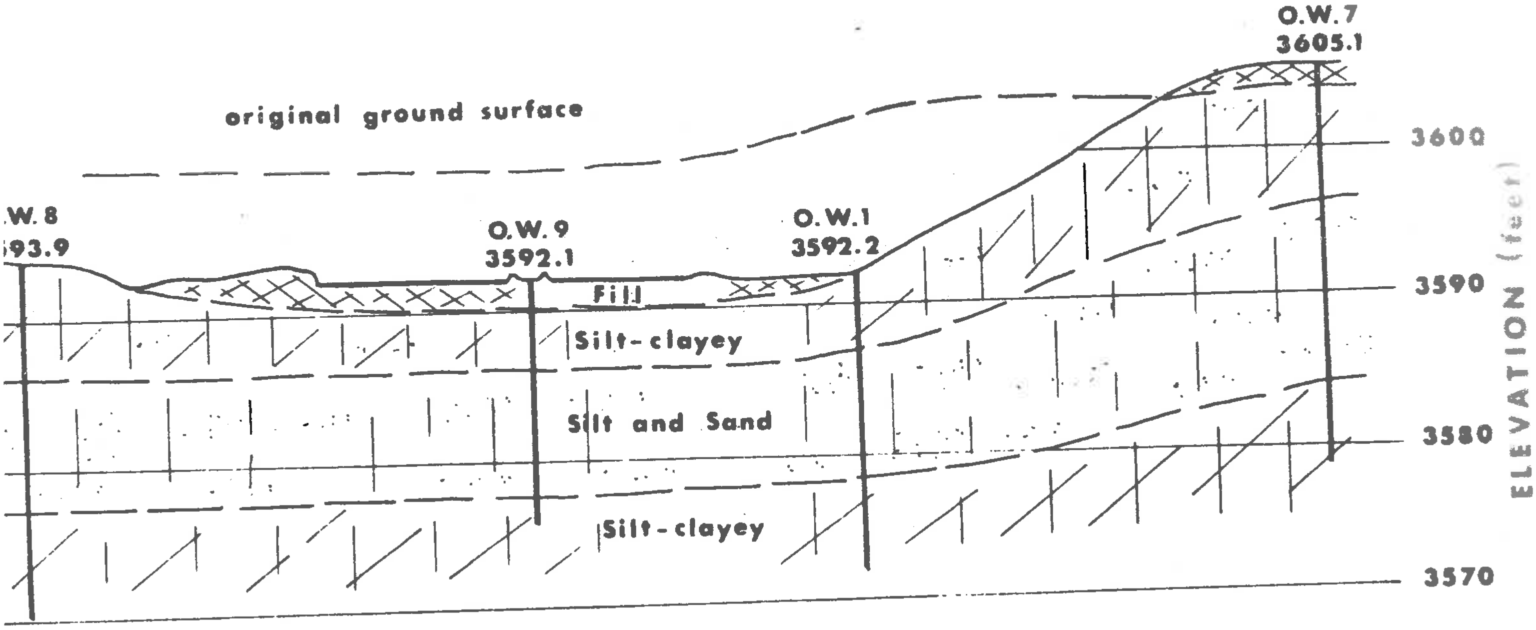

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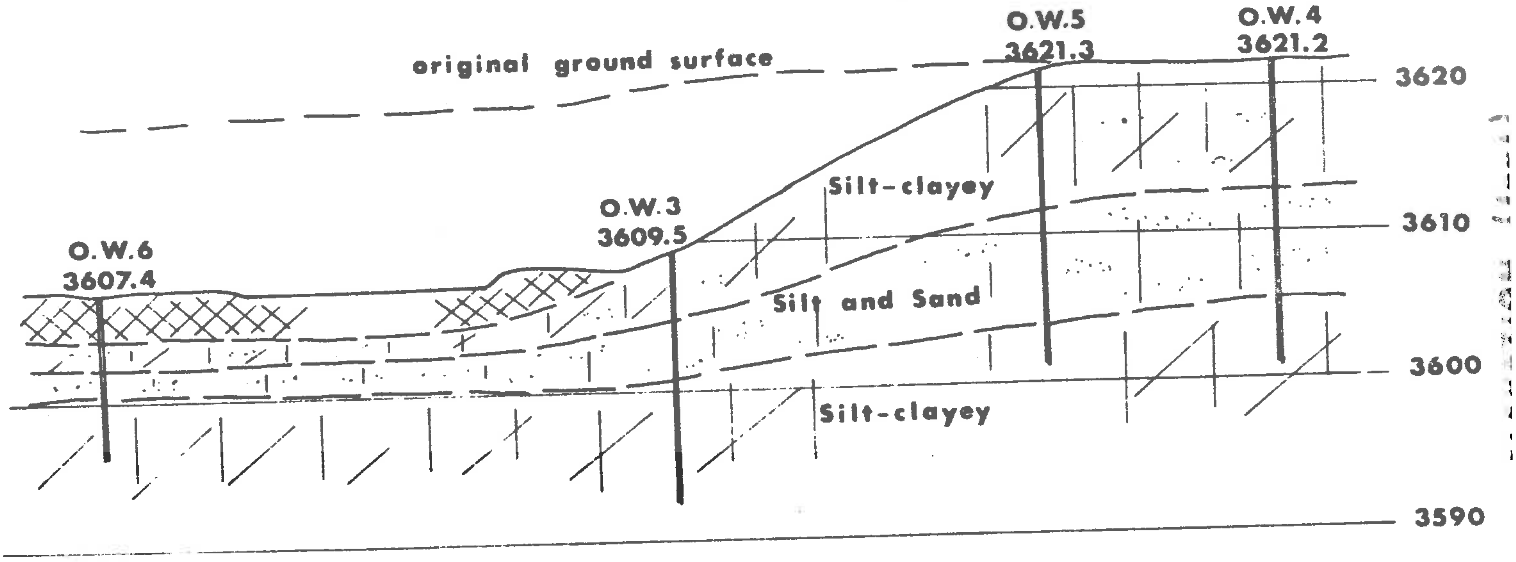

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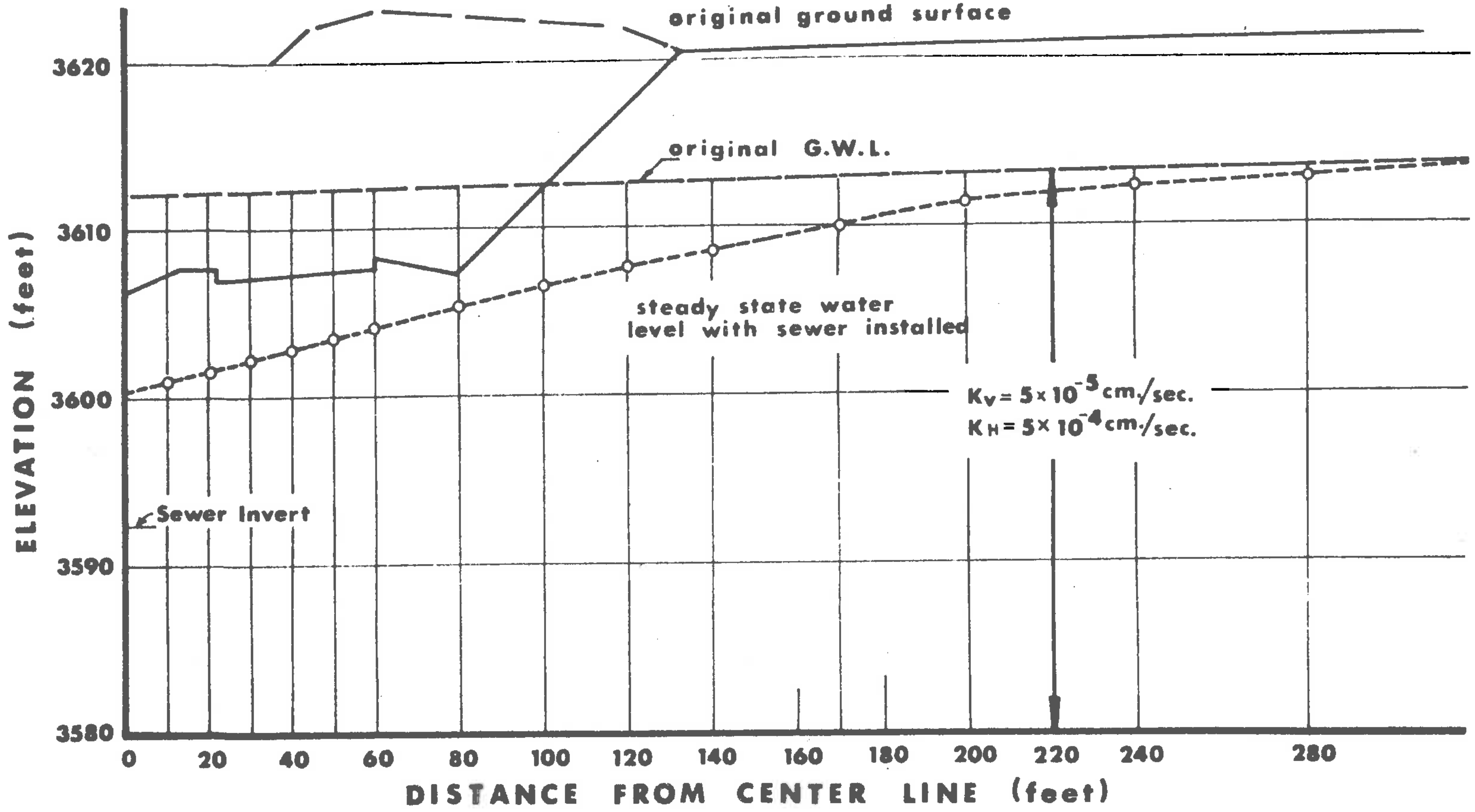

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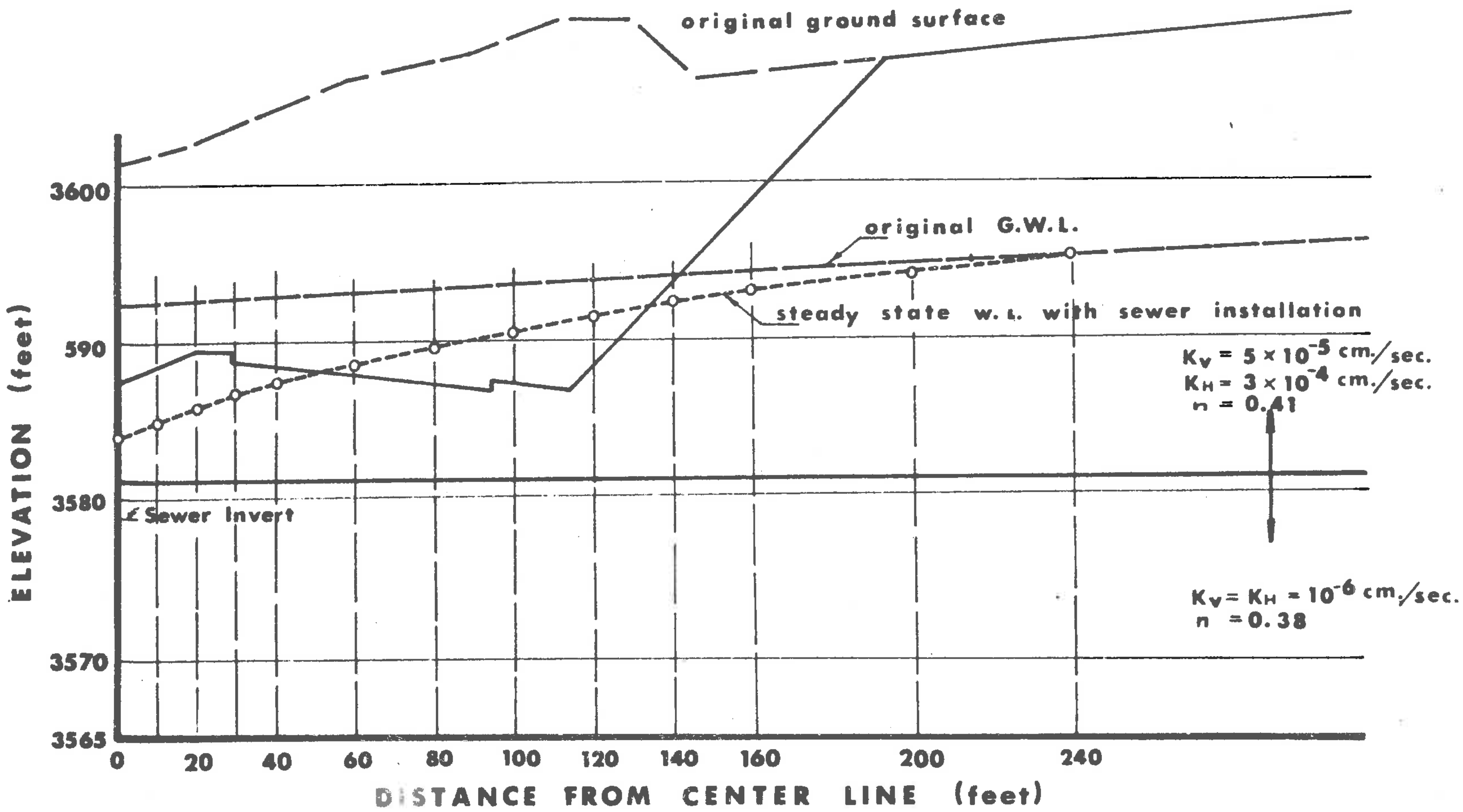

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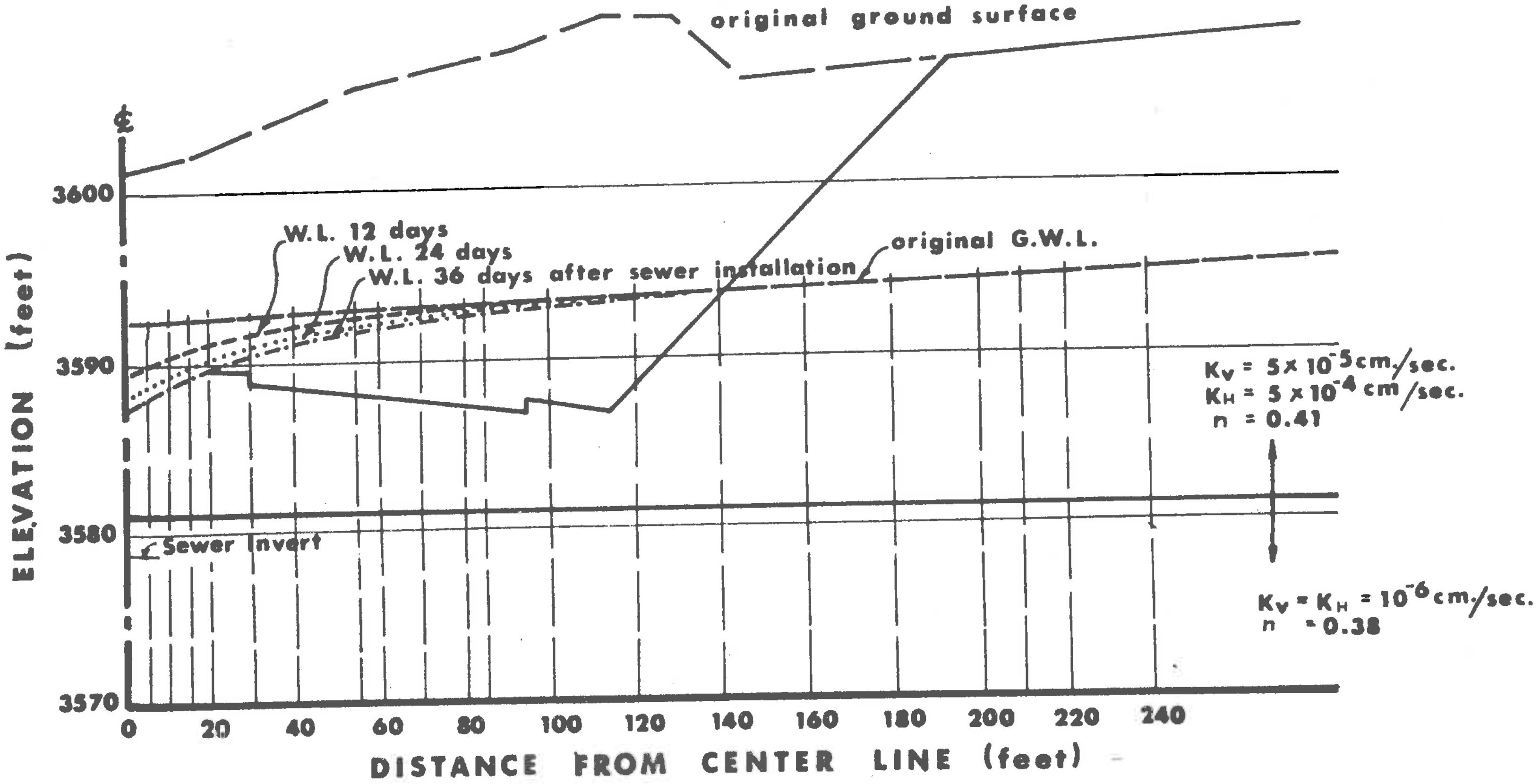

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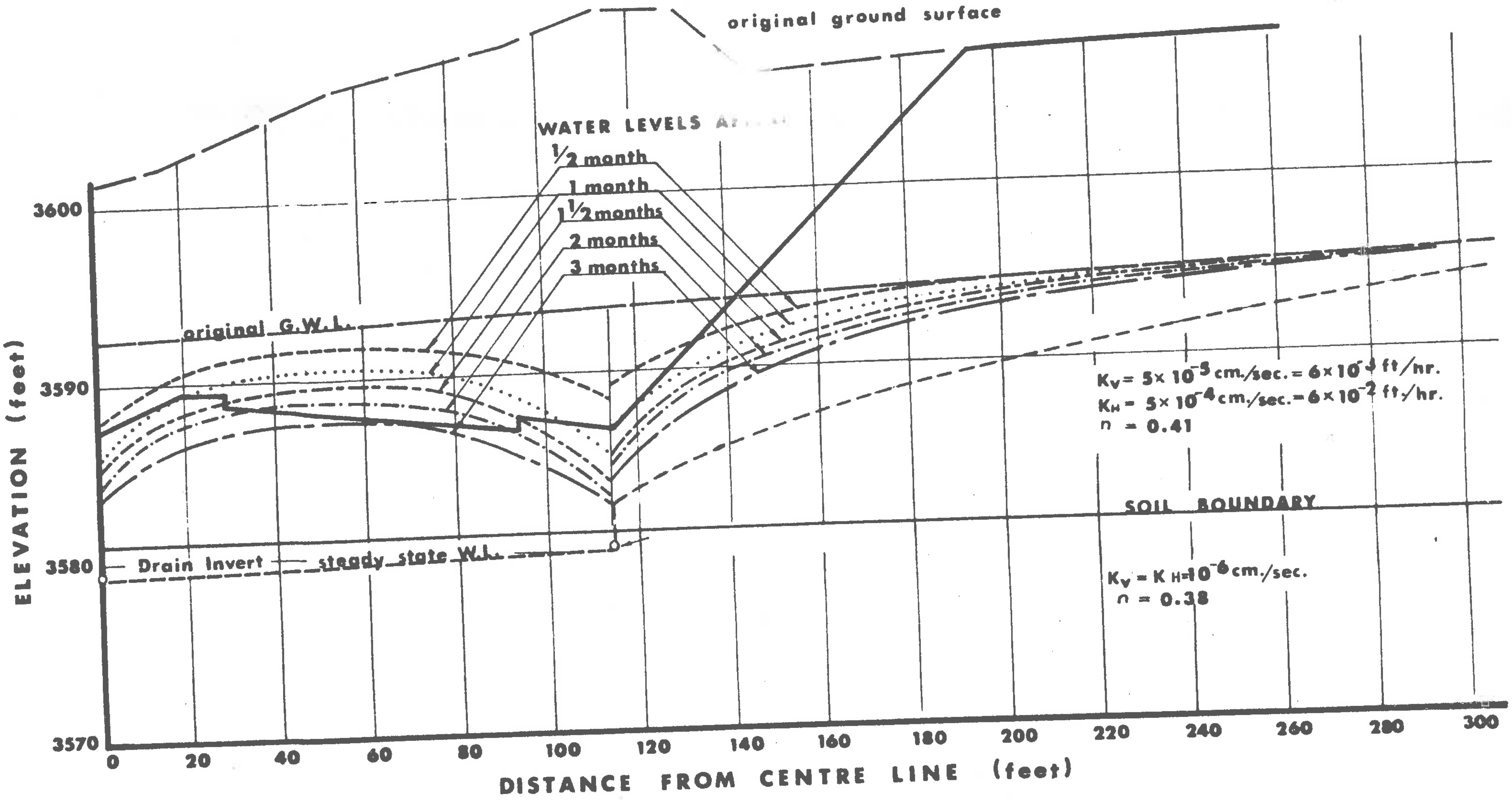

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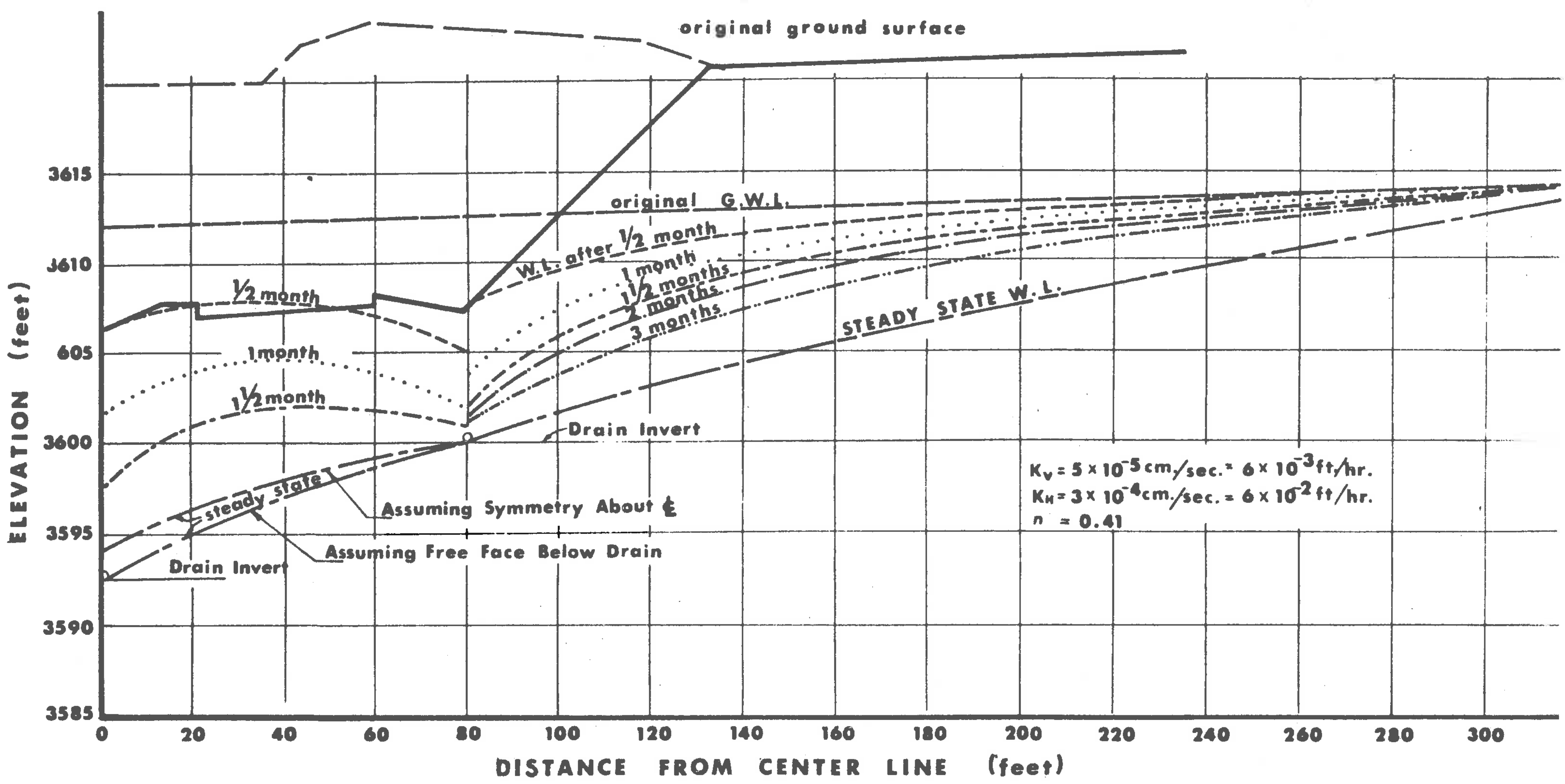

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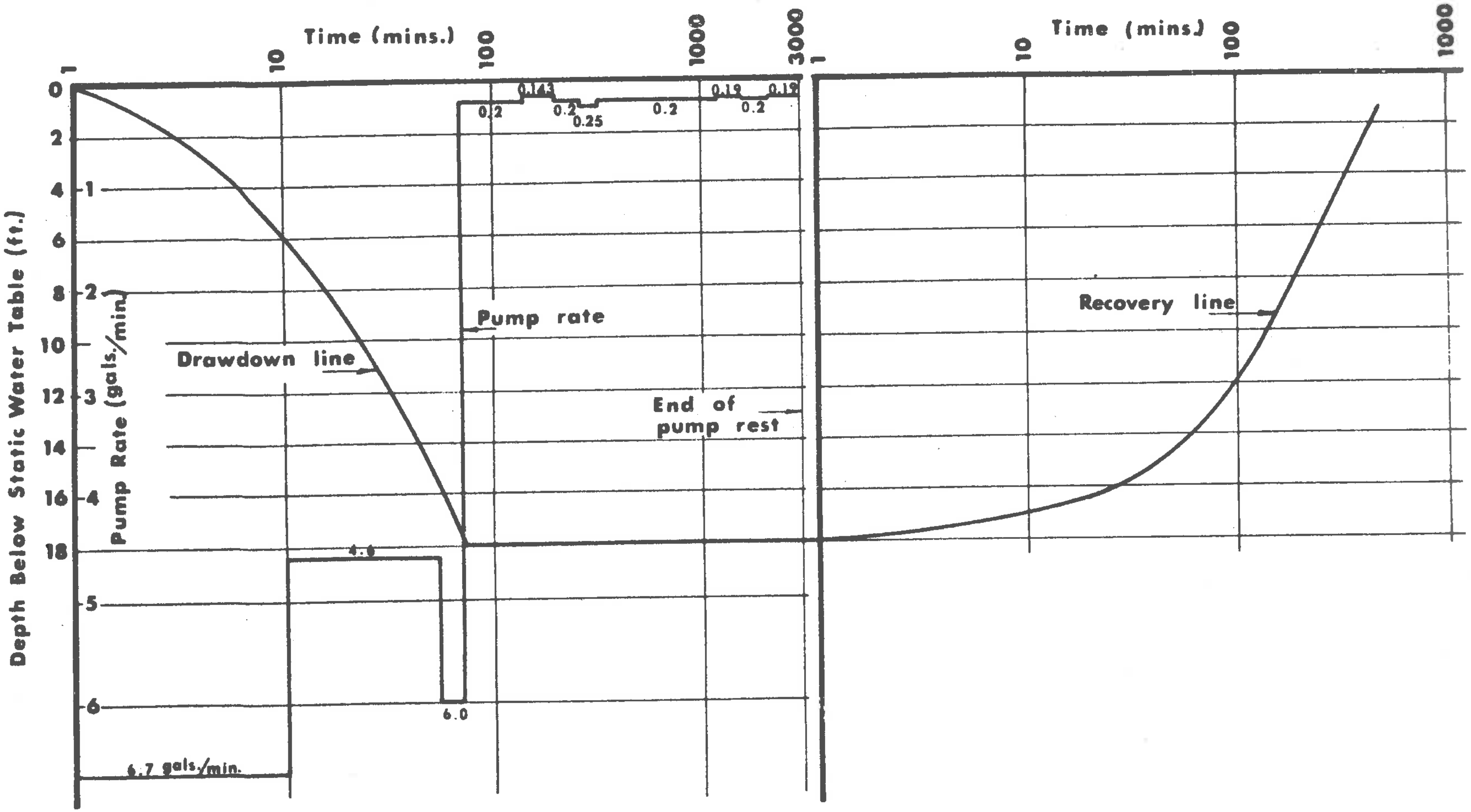

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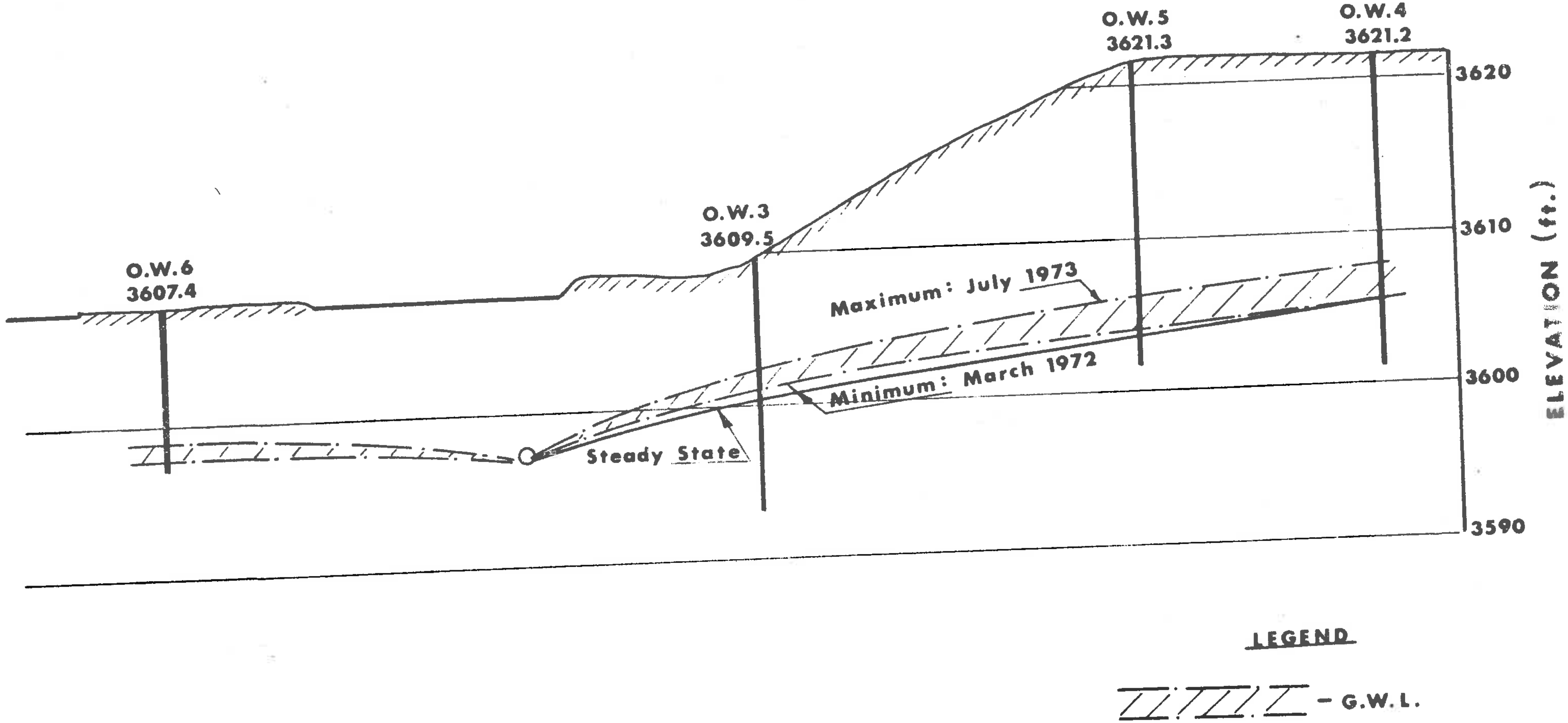

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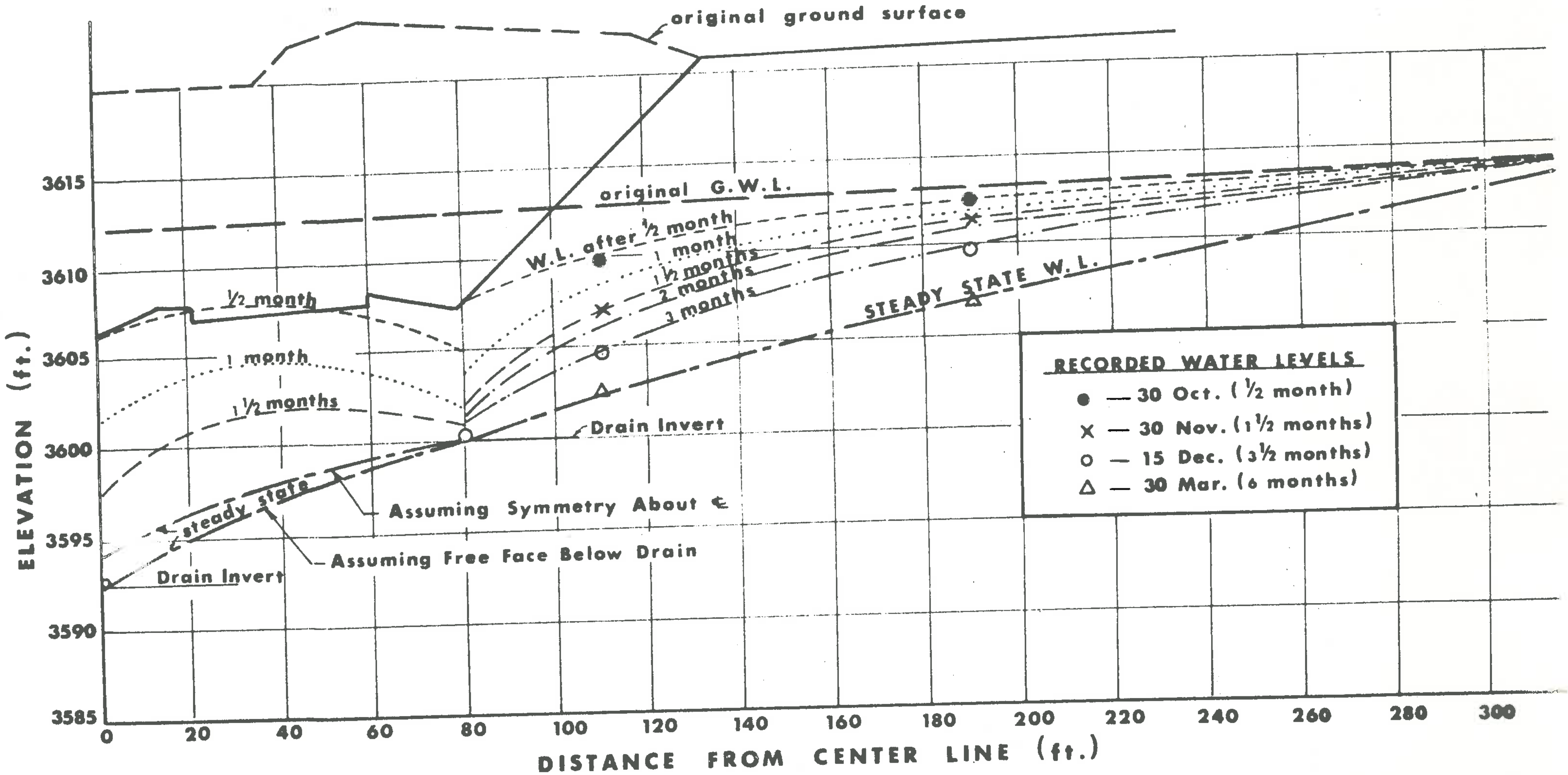

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A THREE-DIMENSIONAL FINITE ELEMENT METHOD FOR PLATE BENDING

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Summary—A three-dimensional finite element method is proposed for plate bending. The displacement function is assumed to have a particular form with respect to the thickness variable.

The assumed displacement function is substituted into the three-dimensional potential energy functional and a two-dimensional variational problem emerges.

The two-dimensional problem is treated by the finite element method and it is seen that for conforming solutions the only requirement is continuity of the basis functions.

The method includes transverse shear and thickness effects and may be used for both thin and moderately thick shells.

To illustrate the method and to compare it with other results, the problems of a square plate under point and distributed loads and with simply supported and clamped boundary are treated numerically.

NOTATION

$2h$	thickness of plate
L	typical length in plate (half side in the numerical examples)
x, y, z	Cartesian co-ordinates
p, T^{*i}	prescribed surface loads
$\sigma_x\ \sigma_y\ \sigma_z\ \tau_{xy}\ \tau_{zx}\ \tau_{yz}$	normal and shearing stresses
$\varepsilon_x\ \varepsilon_y\ \varepsilon_z\ \gamma_{xy}\ \gamma_{zx}\ \gamma_{yz}$	normal and shearing strains
u, v, w	displacements in x, y, z directions
E	Young's modulus
μ	shear modulus
ν	Poisson's ratio
$u_i, v_i, w_i, \overline{w}_i$	nodal displacements
L_i	area co-ordinates
δ^e	element nodal displacements
k^e	element stiffness matrix

INTRODUCTION

Nordgren[1] in a paper on the error in plate theory estimates the difference in energy between the plate theory solution and the three-dimensional solution of a related problem. He finds that the relative error in the energy is of order $(h/L)^2$ (h/L is the ratio of thickness to length). It is expected, however, that th error in the normal displacement is also of order $(h/L)^2$ and this is strengthened by results of Simmonds[2] who decreases the energy error to order $(h/L)^4$ without changing the normal displacement. In both cases displacements of a specific three-dimensional form which are based on the plate theory solution are used.

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In the present paper a displacement field with a similar three-dimensional form is chosen, Nordgren's displacement field being a special case of that given here. The displacement field is substituted into the three-dimensional potential energy functional and the integration with respect to the thickness variable z is carried out. The theorem of minimum potential energy leads to a variational problem for four unknown functions defined on the mid-plane. This is the variational problem considered here. Since Nordgren's displacement field is an admissible one for this variational problem, the solution of the present problem is closer in energy to that of the three-dimensional solution than Nordgren's plate theory solution.

Moreover, since we are approximating a three-dimensional solution our results should be relevant in the case of thick plate. This is illustrated by the comparison of our numerical results with results for Reissner plates by Pryor *et al.*[3] and with results of Irons[4] and Smith.[5]

The solution of the variational problem is approximated by means of the finite element method. It is here that the present formulation in three dimensions shows its advantage over the usual finite element formulation for plates. For a conforming solution only continuity of the displacement functions is required, whereas the usual conforming solution requires continuity of both the functions and the normal derivatives, a restriction not easily satisfied.

The method is illustrated in the present paper by its application to the problems of a square plate under constant pressure and under a central point load with both clamped and simply supported boundaries. The numerical results obtained for these problems are given in Section 4.

Other authors (for example, Utku[6] and Melosh[7]) have used elements in which variation of the transverse shears through the thickness is taken into account. Each author accounts for the transverse shear in his own way using his own *ad hoc* assumptions. Stricklin *et al.*[8] use an element in which the in-plane displacements are taken in a form similar to that used here. They do not, however, include transverse shear effects and therefore the method only applies to thin plates. The Euler equations for the minimization of the energy functional used by Stricklin *et al.* are not in fact related to plate theory. The approximation to plate theory comes about as a result of the choice of displacement functions.

The direct three-dimensional formulation used in this paper does not involve any hidden assumptions. Nordgren's work shows that the solution of our variational problem does in fact approximate the plate theory solution in the thin case and a proof of theoretical convergence in energy may be obtained along conventional lines (e.g. ref. (10)). The approximation to a three-dimensional problem also seems particularly relevant in the thick plate case.

2. THE VARIATIONAL PROBLEM

The plate is assumed to occupy the region in Cartesian three space bounded by a cylinder whose generators are parallel to the z axis and meet the xy plane in a closed curve Γ, and by the two planes $z = \pm h$. The loading on the plane surfaces is taken to be normal and is such that $\sigma_z = \pm \frac{1}{2}p$ on $z = \pm h$. (This choice is made to ensure that the three-dimensional problem has only bending terms.) On the curve surfaces or edges, tractions

T^{*i} may be specified on a part Γ^* of the boundary Γ for $-h \leqslant z \leqslant h$. In order that Nordgren's analysis should apply the traction may not be specified arbitrarily but should be of "classical plate theory" type. The solution of the three-dimensional problem is the displacement field which satisfies any specified displacement boundary conditions on the edges and minimizes the potential energy: for example, see ref. (12).

$$\int_{-h}^{h} \mathrm{d}z \int_D (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + 2\tau_{xy}\gamma_{xy} + 2\tau_{yz}\gamma_{yz} + 2\tau_{zx}\gamma_{zx})\,\mathrm{d}A$$
$$-\int_D p w_{z=h}\,\mathrm{d}A - \int_D p w_{z=-h}\,\mathrm{d}A - \int_{-h}^{h}\mathrm{d}z \int_{\Gamma^*} T^{*i} u_i\,\mathrm{d}s, \quad (2.1)$$

where D is the domain in the xy plane bounded by Γ.

In the present work we restrict the displacement vector to the form

$$\left.\begin{aligned} u &= z u_1(x,y), \\ v &= z v_1(x,y), \\ w &= w_0(x,y) + \tfrac{1}{6}(3z^2 - h^2)\, w_2(x,y). \end{aligned}\right\} \quad (2.2)$$

The inclusion of the w_2 term is in agreement with Nordgren's work where it is shown that the inclusion of such a term substantially improves the error estimate. The choice of the coefficient $\frac{1}{6}(3z^2-h^2)$ for w_2, in preference to z^2, is made to avoid product terms in w_0 and w_2 in the energy functional.

u_1, v_1, w_0, w_2 are functions of x and y only which satisfy any relevant displacement boundary conditions on $\Gamma - \Gamma^*$. This displacement field is substituted in the three-dimensional energy functional and the integration with respect to z is carried out leading to the variational problem for u_1, v_1, w_0, w_2 which is the basis of the present work. This problem is:

$$\text{Minimize} \quad 2\mu h \int_D \left[\left(u_1 + \frac{\partial w_0}{\partial x}\right)^2 + \left(v_1 + \frac{\partial w_0}{\partial y}\right)^2\right] \mathrm{d}x\,\mathrm{d}y$$
$$+\frac{4\mu h^3}{3}\int_D \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial v_1}{\partial y}\right)^2 + w_2^2 + \frac{\nu}{1-2\nu}\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + w_2\right)^2 + \left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right)^2\right] \mathrm{d}x\,\mathrm{d}y$$
$$+\frac{2\mu h^5}{45}\int_D \left[\left(\frac{\partial w_2}{\partial x}\right)^2 + \left(\frac{\partial w_2}{\partial y}\right)^2\right] \mathrm{d}x\,\mathrm{d}y - 2\int_D p\left(w_0 + \frac{h^2}{3} w_2\right) \mathrm{d}x\,\mathrm{d}y$$
$$-2\int_{\Gamma^*} (Q_0^* w_0 + M_{1n}^* u_1 + M_{2n}^* v_1 + Q_2^* w_2)\,\mathrm{d}s \quad (2.3)$$

[μ is the shear modulus $= E/2(1+\nu)$ where E is Young's modulus and ν is Poisson's ratio].

$Q_0^*, M_{1n}^*, M_{2n}^*, Q_2^*$ are determined from the prescribed surface tractions T^{*i} on Γ^* which may be all or part of Γ. As already stated their values cannot be given arbitrarily but must conform to certain "plate theory" type restrictions. In all of the cases considered in this paper either no boundary conditions are specified, so that Γ^* is empty or the boundary traction values are given as zero. In both cases the integral over Γ^* is zero and is not included. An examination of the Euler equations suggests that the terms

$$\frac{2\mu h^5}{45}\int_D \left[\left(\frac{\partial w_2}{\partial x}\right)^2 + \left(\frac{\partial w_2}{\partial y}\right)^2\right] \mathrm{d}x\,\mathrm{d}y \quad \text{and} \quad \frac{h^2}{3}\int_D p w_2\,\mathrm{d}x\,\mathrm{d}y$$

are small compared to the other terms of the functional and this expectation was confirmed in early computations. Consequently, these terms were omitted from the computations which are presented in the numerical results.

3. THE FINITE ELEMENT

An approximate solution of the variational problem is obtained by the finite element method. Fig. 1 shows a typical triangular element in the xy space which has been used.

The displacement field assumed for this element is

$$\left.\begin{aligned} u_1 &= L_i u_i + L_j u_j + L_m u_m, \\ v_1 &= L_i v_i + L_j v_j + L_m v_m, \\ w_0 &= L_i(2L_i - 1) w_i + L_j(2L_j - 1) w_j + L_m(2L_m - 1) w_m \\ &\quad + 4L_i L_j w_p + 4L_j L_m w_q + 4L_m L_i w_r, \\ w_2 &= L_i \overline{w}_i + L_j \overline{w}_j + L_m \overline{w}_m, \end{aligned}\right\} \tag{3.1}$$

where L_i, L_j and L_m are the area co-ordinates (see ref. (9)). The interpolation functions are those commonly used in plane stress triangles; that is, the constant strain and linear strain elements. It is seen that w_0 is specified at all six nodes while u_1, v_1 and w_2 are only specified at the vertices. Hence there are 15 degrees of freedom for each element. It

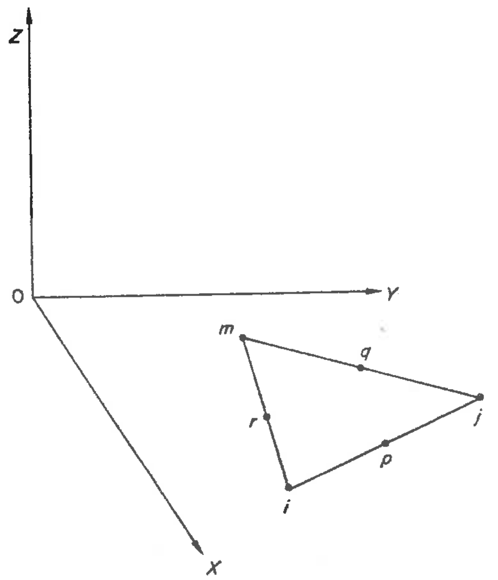

FIG. 1. A triangular element and the global co-ordinate system.

should be noted that the displacement w_0 is a polynomial of degree one higher than that for u_1 and v_1. This choice is motivated by the fact that in thin plates u_1 and v_1 are approximately equal to $-\partial w_0/\partial x$ and $-\partial w_0/\partial y$, respectively, and the element is required to model thin plate theory. It can be shown that the displacements (3.1) are continuous across the interelement boundaries giving a conforming solution.

Writing (3.1) in matrix notation

$$\begin{Bmatrix} u_1 \\ v_1 \\ w_0 \\ w_2 \end{Bmatrix} = [C]\{\delta^e\}, \tag{3.2}$$

with

$$\{\delta^e\}^T = \{\delta_i^T \, \delta_j^T \, \delta_m^T, \delta_p, \delta_q, \delta_r\},$$

and

$$\{\delta_i\}^T = \{u_i, v_i, w_i, \overline{w}_i\}$$

represents the displacement at a vertex and

$$\delta_p = w_p$$

the displacement at a mid-point.

The strain matrix may be obtained in the form

$$\{\varepsilon\} = [B]\{\delta^e\}. \tag{3.3}$$

The element stiffness matrix $[k^e]$ is then obtained in the usual way as

$$[k^e] = \int_V [B]^{\mathrm{T}}[D][B]\,\mathrm{d}V, \tag{3.4}$$

where $[D]$ is a 6×6 matrix of material constants. The size of the element stiffness matrix is 15×15.

4. NUMERICAL RESULTS

The results of the analysis of four well-known problems for a square plate are discussed in this section. The problems are listed in Table 1. Four different thickness to length ratios were considered for each problem resulting in 16 case studies. The length of the side is $2L$ in each case.

TABLE 1

	Boundary condition	Loading	h/L
1	Simply supported	Distributed load q	0·025, 0·05, 0·10, 0·125
2	Simply supported	Point load p	0·025, 0·05, 0·10, 0·125
3	Clamped	Distributed load q	0·025, 0·05, 0·10, 0·125
4	Clamped	Point load p	0·025, 0·05, 0·10, 0·125

Due to the double symmetry of the problems only a quarter of the plate was analysed in each case. The mesh size $n\times n$, therefore, refers to the number of elements on a half side of the plate giving a total of $2n^2$ triangles for the quarter plate. The triangulation used is shown in Fig. 4. Each mesh system was generated using identical elements. Poisson's ratio has been taken as 0·3 throughout. Nodal forces due to the distributed loading were obtained by the use of tributary areas.

Table 2 shows the convergence of the central deflections of the simply supported plates. The edge boundary conditions for these examples were taken to be the vanishing of all displacements except the normal displacement. It is observed, as expected, that the central deflections converge towards the results given by classical thin plate theory only for the lower values of the thickness to length ratio. The deflections are substantially higher for the larger thickness to length ratios. For $h/L = 0{\cdot}10$ the increase over the thin plate is about 4 per cent and for $h/L = 0{\cdot}125$ the increase is about 6 per cent. Convergence is monotonic in all cases. It may be noted that the results of the present finite element analysis are comparable to those of Pryor *et al.*[3] who use Reissner's plate theory with the shear coefficient $k = 1$ and a rectangular element with 20 degrees of freedom. The normal displacement, the total rotations and the shears are specified at each corner. The results quoted are for a 6×6 rectangular mesh. They are also comparable to those of Smith[5] who uses a thick plate theory in which the energy depends on higher order derivatives. Smith uses rectangular elements with 10 degrees of freedom at each node and polynomials of degree seven. The results here are also for a 6×6 rectangular mesh. For an $n\times n$ mesh, without taking boundary conditions or further symmetry into account, it is seen that the total number of degrees of freedom for the whole quarter plate is $5(n+1)^2$ for Pryor *et al.*, $10(n+1)^2$ for Smith and $7(n+1)^2-2n-3$ for the present work. Irons[4] obtains higher values using isoparametric elements and a shear coefficient $k = 1{\cdot}2$. The final row is the values for the best conforming element of Bazeley *et al.*[11] with a 6×6 mesh.

Convergence of the central deflections for the clamped plate are shown in Table 3. The boundary conditions for clamped plates were taken to be the vanishing of all displacements. Once again the central deflections converge to values higher than those given by classical thin plate theory when the thickness to length ratio is increased. This time, however, the increase in the central deflections are considerably higher than in the previous cases. This is probably due to an increased contribution from the shearing deformation. The convergence is again monotonic in all cases.

TABLE 2. CONVERGENCE OF CENTRAL DEFLECTION α FOR A SIMPLY SUPPORTED PLATE

Mesh ¼ plate	Distributed load q				Concentrated central load p			
	$h/L = 0{\cdot}025$	$h/L = 0{\cdot}05$	$h/L = 0{\cdot}1$	$h/L = 0{\cdot}125$	$h/L = 0{\cdot}025$	$h/L = 0{\cdot}05$	$h/L = 0{\cdot}1$	$h/L = 0{\cdot}125$
5 × 5	38·649	39·773	41·293	42·306	1·0796	1·1322	1·2632	1·3577
6 × 6	39·500	40·227	41·662	42·674	1·1109	1·1544	1·2876	1·3857
7 × 7	39·928	40·476	41·858	42·860	1·1283	1·1682	1·3044	1·4056
8 × 8	40·168	40·626	41·982	42·978	1·1391	1·1775	1·3169	1·4208
Thin plate theory	40·621	40·621	40·621	40·621	1·1568	1·1568	1·1568	1·1568
Smith[5]	—	40·987	42·051	—	—	1·174	1·200	—
Irons[4]	—	42·747	46·113	—	—	1·2412	1·4279	—
Pryor *et al.*[3]	—	40·924	42·234	—	—	1·219	1·353	—
Bazeley *et al.*[11]	38·4	38·4	38·4	38·4	1·117	1·117	1·117	1·117

$\alpha = Dw_c/q(2L)^4$ for the distributed load q. $\alpha = Dw_c/p(2L)^2$ for the central point load p, $2L = 10$, $\nu = 0{\cdot}3$ and w_c is the displacement at the centre.

TABLE 3. CONVERGENCE OF CENTRAL DEFLECTION α FOR A CLAMPED PLATE

Mesh $\frac{1}{4}$ plate	Distributed load q				Concentrated central load p			
	$h/L = 0{\cdot}025$	$h/L = 0{\cdot}05$	$h/L = 0{\cdot}1$	$h/L = 0{\cdot}125$	$h/L = 0{\cdot}025$	$h/L = 0{\cdot}05$	$h/L = 0{\cdot}1$	$h/L = 0{\cdot}125$
5×5	11·332	12·102	13·726	14·843	0·4827	0·5328	0·6676	0·7647
6×6	11·792	12·370	13·931	15·032	0·5104	0·5535	0·6911	0·7917
7×7	12·041	12·532	14·061	15·154	0·5265	0·5668	0·7076	0·8114
8×8	12·192	12·639	14·150	15·237	0·5368	0·5760	0·7200	0·8265
Thin plate theory	12·65	12·65	12·65	12·65	0·5679	0·5679	0·5679	0·5679
Smith[5]	—	13·015	13·717	—	—	0·5757	0·5974	—
Irons[4]	—	13·261	15·040	—	—	0·6207	0·7771	—
Bazeley *et al.*[11]	11·7	11·7	11·7	11·7	0·511	0·511	0·511	0·511

$\alpha = Dw_c/q(2L)^4$ for the distributed load, $\alpha = Dw_c/p(2L)^2$ for the point load. $2L = 10$, $\nu = 0{\cdot}3$.

Finally, the distribution of the bending moment M_x along the central line $y = 0$ is shown in Figs. 2 and 3. Because the normal stresses vary linearly across the thickness, the bending moments are given by $M_x = \frac{2}{3}h^3 \sigma_x$ where σ_x is the normal stress. Equations (3.1) and (3.3) show that the tangential strains in an element are constant with respect to x and y. Because of this the moments were first computed at the centre of gravity of each element and stresses at other points were obtained by interpolation.

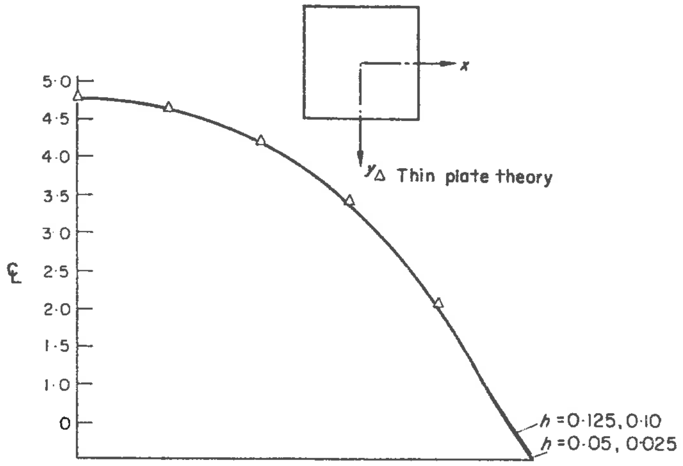

FIG. 2. M_x at $y = 0$ for simply supported plate under uniformly distributed load.

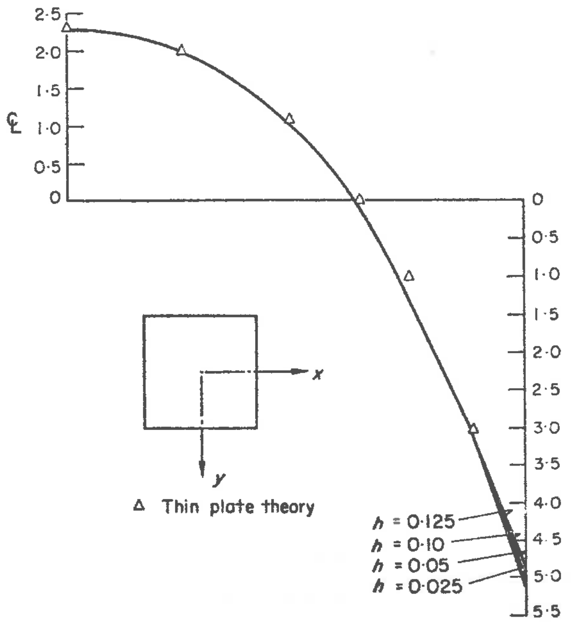

FIG. 3. M_x at $y = 0$ for clamped plate under uniformly distributed load.

INTERPOLATION ALONG $y = 0$

Stresses along the centre line $y = 0$ of the plate were obtained by using Lagrange's interpolation formula. First, the stresses along the line $y = 0$, for all points excluding the centre and the boundary of a plate, were obtained by interpolating and extrapolating along lines x = constant. Then the stresses at the centre and the boundary of the plates were obtained by repeating the process along $y = 0$.

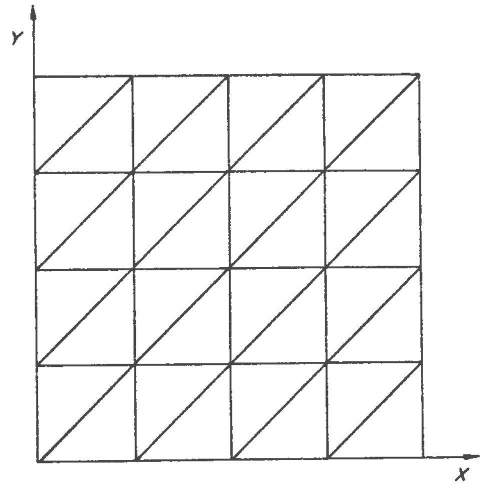

FIG. 4. Triangulation, 4 × 4 mesh.

CONCLUSION

The present method is suggested as a means to obtain simple conforming finite elements for plate bending problems. Numerical convergence for the thin plate problem has been demonstrated and a convergence proof along the usual lines may be easily obtained. Improved convergence should be achieved by the use of higher order elements and improvements in this direction are currently under examination.

In the case of thick plates, it is seen that the present formulation allows transverse shearing and the results for thick plates are comparable to those of other authors and in particular to those obtained for a Reissner plate theory by Pryor *et al.*[3] The rather wide spread of results for thick plates in the literature suggests that more work might be done in this area.

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GENERATION OF HIGHER ORDER SUBPARAMETRIC BENDING ELEMENTS

BY

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INTRODUCTION

Based on Mindlin's[1] plate bending theory that the curvatures of a plate under bending are given by the first derivatives of the rotations θ_x and θ_y (which are regarded as independent variables), instead of being represented by the second derivatives of the deflection w normal to the middle plane of the plate, Cheung and Chan[2] have developed a method for generating the stiffness matrices for higher order rectangular plate bending elements with any number of nodes. It has also been demonstrated that very accurate results are obtained for many rectangular plate bending problems by using just one or a few of the 17-node or 21-node elements. However, the shape of the elements is restricted to rectangular only and the elements so produced are unsuitable for modelling plates of arbitrary shapes. It is therefore desirable that similar higher order plate bending elements of more complex shapes and with straight or curved boundaries can be made available. So far bending elements of this type but with only 4 nodes and 8 nodes have been developed by Rock and Hinton[3]. In this paper the formulation of higher order subparametric "quadrilateral" bending elements bounded by straight or curved sides with 17, 21, 25 or any number of nodes is described. Examples of solutions are presented here to illustrate that plates of irregular shapes such as skewed, trapezoidal, curved plates etc., can be easily modelled by using only one or two of these higher order elements and the results are very accurate when compared with available solutions.

SUBPARAMETRIC FORMULATION OF STIFFNESS MATRICES FOR HIGHER ORDER ELEMENTS

The formulation of the stiffness matrix for a higher order "quadrilateral" element of any shape begins with the transformation of a square parent element in its natural coordinates (ξ,η) as shown in Fig. 1(a) into the actual shape of the "quadrilateral" in its local coordinates (x,y). Since higher order elements contain a fairly large number of nodes, in mapping the geometric shape of the element it is not necessary to use all the node points. A subparametric transformation is usually quite sufficient. By using 4 corner nodes it is possible to transform the parent element into a quadrilateral with straight sides as shown in Fig. 1(b); and by using 8 nodes, i.e. 4 corner nodes and 4 mid-side nodes, it is possible to transform the parent element into a distorted "quadrilateral" with parabolic curved sides as shown in Fig. 1(c).

The coordinates (x,y) of any point in the element can be expressed in terms of the coordinates of the nodal points for mapping (x_i, y_i) as follows:-

$$x = \sum_{1}^{m} N_i(\xi,\eta)x_i$$

$$y = \sum_{1}^{m} N_i(\xi,\eta)y_i$$

where $N_i(\xi,\eta)$ are the mapping functions in terms of the natural coordinates (ξ,η).

The mapping functions $N_i(\xi,\eta)$ are already well established for the cases being used here:

$m = 4$

$$N_i = \tfrac{1}{4}(1 + \xi.\xi_i)(1 + \eta.\eta_i)$$

with $\xi_i = -1, 1, 1, -1$

$\eta_i = -1, -1, 1, 1$

for $i = 1, 2, 3, 4$

and

$m = 8$

$$N_i = \tfrac{1}{4}(1 + \xi.\xi_i)(1 + \eta.\eta_i)(\xi.\xi_i + \eta.\eta_i - 1)$$

for $i = 1, 3, 5, 7$

$$N_i = \tfrac{1}{2}(1 - \xi^2)(1 + \eta.\eta_i) \qquad \text{for } i = 2, 6$$

$$N_i = \tfrac{1}{2}(1 + \xi.\xi_i)(1 - \eta^2) \qquad \text{for } i = 4, 8$$

with $\xi_i = -1, 0, 1, 1, 1, 0, -1, -1$

$$\eta_i = -1, -1, -1, 0, 1, 1, 1, 0 \qquad \text{for } i = 1 \text{ to } 8.$$

The displacements at any point in the element will be related to the displacements of all the n nodals in the element by

$$w = \sum_1^n L_i(\xi,\eta)\, w_i$$

$$\theta_x = \sum_1^n L_i(\xi,\eta)\, \theta_{x_i}$$

$$\theta_y = \sum_1^n L_i(\xi,\eta)\, \theta_{y_i}$$

where L_i are displacement relationship function in terms of the natural coordinates (ξ,η). They are given by

$$[L_1 \quad L_2 \quad L_3 \;\ldots\; L_i \;\ldots\; L_n]$$

$$= [1 \quad \xi \quad \eta \quad \xi^2 \quad \xi.\eta \quad \eta^2 \;\ldots][C_N^{-1}]$$

where $[1 \quad \xi \quad \eta \quad \xi^2 \quad \xi.\eta \quad \eta^2 \;\ldots]$ is an n-term polynomial in ξ and η and $[C_N^{-1}]$ is the inverse of the matrix of the natural coordinates of the nodes as described in reference (2).

Since the strains of the element will be given by the differentials of the displacements with respect to x and y; but the displacements have now been expressed as functions of ξ and η, the relation between the derivatives in the two coordinate systems is given by a Jacobian transformation, i.e.

$$\left\{\begin{matrix}\frac{\partial}{\partial\xi}\\ \frac{\partial}{\partial\eta}\end{matrix}\right\} = \begin{bmatrix}\frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi}\\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta}\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{bmatrix} = [J]\left\{\begin{matrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{matrix}\right\}$$

where

$$[J] = \begin{bmatrix}\frac{\partial N_1}{\partial\xi} & \frac{\partial N_2}{\partial\xi} & \cdots & \frac{\partial N_m}{\partial\xi}\\ \frac{\partial N_1}{\partial\eta} & \frac{\partial N_2}{\partial\eta} & \cdots & \frac{\partial N_m}{\partial\eta}\end{bmatrix}\left\{\begin{matrix}x_1 & y_1\\ x_2 & y_2\\ \vdots & \vdots\\ x_m & y_m\end{matrix}\right\} = \begin{pmatrix}J_{11} & J_{12}\\ J_{21} & J_{22}\end{pmatrix}_{(\xi,\eta)}$$

and inverting

$$\left\{\begin{matrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{matrix}\right\} = [J^{-1}]\left\{\begin{matrix}\frac{\partial}{\partial\xi}\\ \frac{\partial}{\partial\eta}\end{matrix}\right\} = \begin{bmatrix}I_{11} & I_{12}\\ I_{21} & I_{22}\end{bmatrix}\left\{\begin{matrix}\frac{\partial}{\partial\xi}\\ \frac{\partial}{\partial\eta}\end{matrix}\right\}$$

In Mindlin's bending theory, the strain vector is

$$\{\varepsilon\} = \left\{\begin{matrix}-\frac{\partial\theta_x}{\partial x}\\ -\frac{\partial\theta_y}{\partial y}\\ -\left(\frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x}\right)\\ \frac{\partial w}{\partial x} - \theta_x\\ \frac{\partial w}{\partial y} - \theta_y\end{matrix}\right\} = \sum_{1}^{n}\begin{bmatrix}0 & -\frac{\partial L_i}{\partial x} & 0\\ 0 & 0 & -\frac{\partial L_i}{\partial y}\\ 0 & -\frac{\partial L_i}{\partial y} & -\frac{\partial L_i}{\partial x}\\ \frac{\partial L_i}{\partial x} & -L_i & 0\\ \frac{\partial L_i}{\partial y} & 0 & -L_i\end{bmatrix}\left\{\begin{matrix}w_i\\ \theta_{x_i}\\ \theta_{y_i}\end{matrix}\right\}$$

$$= \sum_{1}^{n} [B_i]\ \{\delta_i\}$$

However the L_i's are functions of the natural coordinate (ξ,η) instead of (x,y); the Jacobian transformation must be applied in order that the differentiation can be carried out. Therefore, re-writing

$$\frac{\partial L_i}{\partial x} = I_{11}\frac{\partial L_i}{\partial \xi} + I_{12}\frac{\partial L_i}{\partial \eta} = \phi_i\ (\xi,\eta)$$

$$\frac{\partial L_i}{\partial y} = I_{21}\frac{\partial L_i}{\partial \xi} + I_{22}\frac{\partial L_i}{\partial \eta} = \psi_i\ (\xi,\eta)$$

Thus finally the strain transformation matrix can be expressed as functions of ξ and η as follows

$$B_i = \begin{bmatrix} 0 & -\phi_i & 0 \\ 0 & 0 & -\psi_i \\ 0 & -\psi_i & -\phi_i \\ \phi_i & -L_i & 0 \\ \psi_i & 0 & -L_i \end{bmatrix} (\xi,\eta)$$

and the stress vector can be written as

$$\{\sigma\} = [D] \sum_{1}^{n} [B_i]\{\delta_i\}$$

where [D] is the elasticity matrix for bending containing constants only.

The stiffness sub-matrix for any node i with respect to the displacements at node j is

$$k_{ij} = \iint B_i^T\, D\, B_j\, dxdy \qquad \text{over the whole region of the element}$$

$$= \int_{-1}^{1}\int_{-1}^{1} [B_i^T \cdot D \cdot B_j\ \det J]\, d\xi d\eta$$

For any given values of coordinates (ξ_k, η_ℓ) the value of the term in the brackets [ ] can be evaluated. Hence the coefficients in the stiffness sub-matrix k_{ij} can be obtained approximately by Gauss Legendre numerical integration method. From experience, it was found that with 5x5 gaussian points, accurate results were obtained for the stiffness matrices for high order elements with 17 and 21 nodes.

The load vector for a uniformly distributed load q is given by

$$\{F\} = \int_{-1}^{+1} \int_{-1}^{+1} [L]^T \det J.d\xi d\eta$$

and is also determined by numerical integration.

NUMERICAL EXAMPLES

A number of examples including skew cantilever plates, skew and curved slab bridges as well as trapezoidal slab bridges are presented to demonstrate the simplicity and versatility of the higher order sub-parametric bending elements. Accurate results were obtained for all cases in which only a few elements were used to model the structure.

Example 1. Uniformly loaded isotropic cantilever plates with different skew angles

A skewed cantilever plate with a skew angle of 20°, 40° or 60° subjected to a uniformly distributed load was analysed by using one 17-node or two 21-node subparametric bending elements.

The deflections at the free corners of the plates are compared with other numerical and experimental results as tabulated in Table 1, and the bending moments along the skewed centre line of the plates are compared with the results Ghali[10] using finite strip method as shown in Fig. 2. It is seen that using only one 17-node element the results are fairly close to other numerical solutions at low skew angles; but when the skew angle is more than 40° the results

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are not so good because of the sharp change of bending moment at the junction of the fixed edge and the skewed free edge. By using two 21-node elements, however, the results agree closely with other numerical solutions for all skew angles, although all of the numerical results quoted here are 0 to 20 per cent lower than the corresponding experimental values given in reference (7).

Example 2. Effects of the variation of skew angle and plate thickness on uniformly loaded rhombic slab simply supported along two opposite sides.

An investigation was carried out on a rhombic slab simply supported along two opposite sides with

(a) the skew angle varies from 10 to 70 degrees, and

(b) the thickness varies from 0.01 to 0.1 of the side length of the slab.

In case (a) the thin plate solutions (thickness to side length ratio taken as 0.01) were obtained and the deflections at centre and at mid-edge and the principal moments at centre are plotted against the skew angle and compared with the results given by Ramstad[12] as shown in Figures 3b, 3c and 3d. In general two 21-node elements are sufficient to give accurate results; except for large skew angles, then, four elements should be used to give a better answer. In case (b), when investigating the effect of shear deformation on the deflection of a moderately thick rhombic slab, three different values of the thickness to side length ratio, 0.01, 0.05 and 0.10 and three different skew angles, 30°, 40° and 50°, were assumed. Results are given in Table 2. It is seen that the effect of shear deformation on the central deflection of the plate becomes more predominant as the thickness of the plate increases and as the skew angle increases.

Example 3   Curved slab bridge simply supported on two radial edges subjected to point loads. A fan-shaped curved slab bridge with a subtending angle of 60° and with outer and inner radii of 660 and 330 mm respectively as shown in Fig. 4 was analysed by using two 21-node subparametric "quadrilateral" elements with curved sides. The bridge was simply supported on the radial edges and was loaded by point loads applied separately at the outer, mid and inner points along the centre radius. The deflection profile of the mid-span radius and the distributions of the bending moments at mid-span are shown in Figs. 5 and 6 in conjunction with Coull's theorectical and experimental results (13). Good agreement between the results clearly demonstrates the accuracy as well as the versatility of the subparametric elements in modelling curved slabs even with only two elements.

Example 4   Trapezoidal slab bridge simply supported along two parallel sides, and loaded with a concentrated load at mid-span.

The trapezoidal slab bridge of dimensions as shown in Fig. 7(a) is modelled by two 21-node elements, and the results are checked against those obtained from a simple triangular element solution (15) in which a total of 168 elements were used. Comparisons of deflections along the central section, and bending moment m_x along the centre line are given in Fig. 7(b) and 7(c). It can be seen that once again the agreement is very good.

CONCLUSION:

The method presented here will enable higher order bending elements of any shape and with any number of nodes to be formulated and the stiffness matrices generated automatically. The degree of variation of the moment field within the element will be one degree less than the highest power of the displacement polynomial assumed in the corresponding direction, and therefore for a 17-node or 21-node element the variation of bending moment within the element is a cubic function. Hence a host of problems of plates with irregular shapes can be modelled easily with just a few of these higher order elements and yet very accurate results can be obtained. The simplicity and versatility of these elements in the idealization of irregular-shaped plates have led to significant simplification in data preparation and output interpretation as well as considerable reduction in computing effort when compared with other methods hitherto employed.

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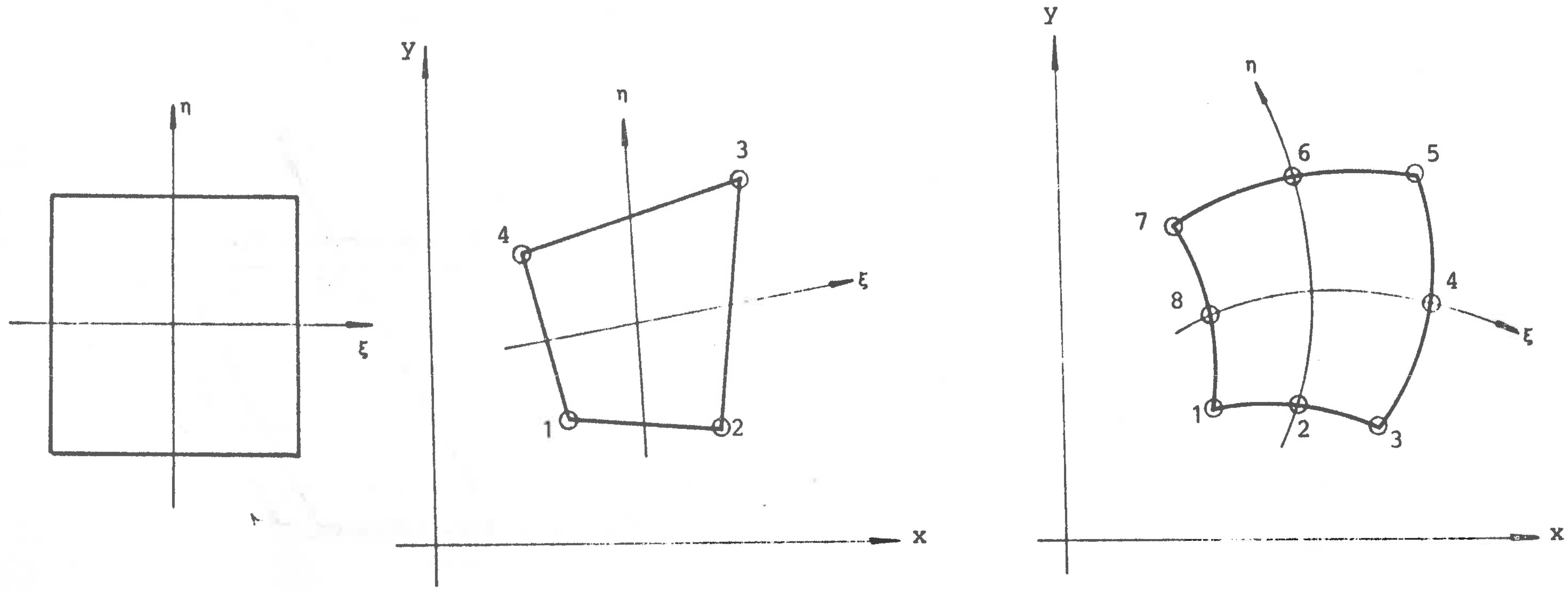
Deflection	Skew Angle β	Experimental	Finite Element Method (25 elements)		Finite Strip Method (8 strips, 7 terms)	Present Method	
		Ref. 7	Ref. 8	Ref. 9	Ref. 10	one 17 node	two 21-node
$\frac{w_A D}{qa^4}$	20°	.0994	.0952	.0950	.0950	.0930	.0944
	40°	.0529	.0496	.0493	.0499	.0498	.0492
	60°	.0159	.0143	.0140	.0151	.0129	.0140
$\frac{w_B D}{qa^4}$	20°	.1367	.1292	.1300	.1298	.1287	.1298
	40°	.1132	.1035	.1071	.1051	.0916	.1040
	60°	.0855	.0674	.0746	.0627	.0343	.0646

Table 1 Deflections of skewed cantilever plates under a uniformly distributed loading (Poisson's Rotio=0.3)

Skew Angle \ Thickness/Span	Central Deflection ($\frac{q\ell^3 \ell_x}{D}$) (Percentage of thin plate solution in brackets)		
	0.01	0.05	0.1
30^o	.00897 (100)	.00916 (102.1)	.00951 (106)
40^o	.00647 (100)	.00668 (103.2)	.00710 (109.7)
50^o	.00398 (100)	.00420 (105.5)	.00467 (117.2)

Table 2 Effect of shear deformation on central deflection of Rhombic Plate (Two 21-node elements; $\nu = 0.31$)



(a) Parent Element in Natural Coordinates

(b) 4-Node Mapping Element

(c) 8-Node Mapping Element

FIG. 1

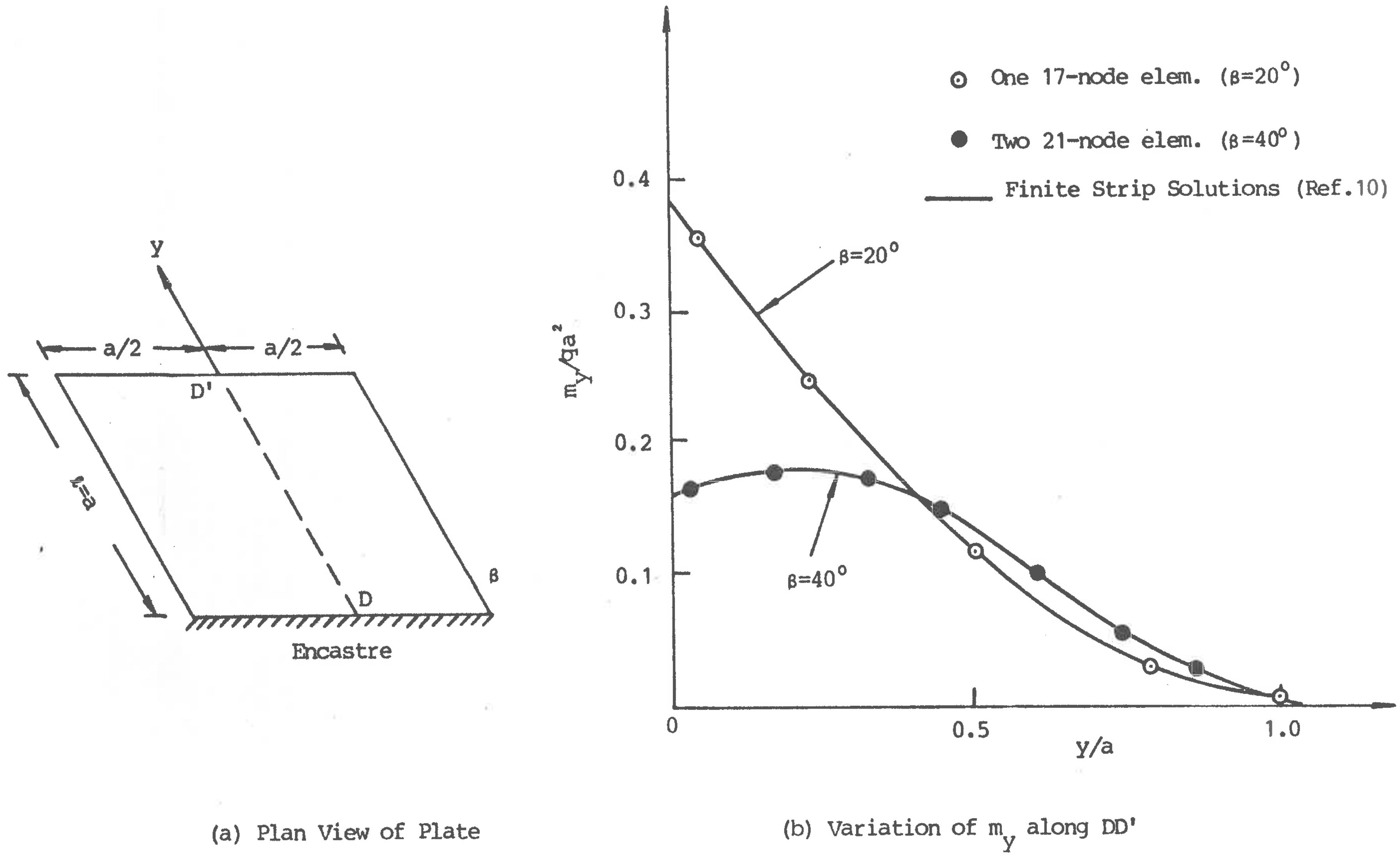


FIG. 2 Skewed Cantilever Plates

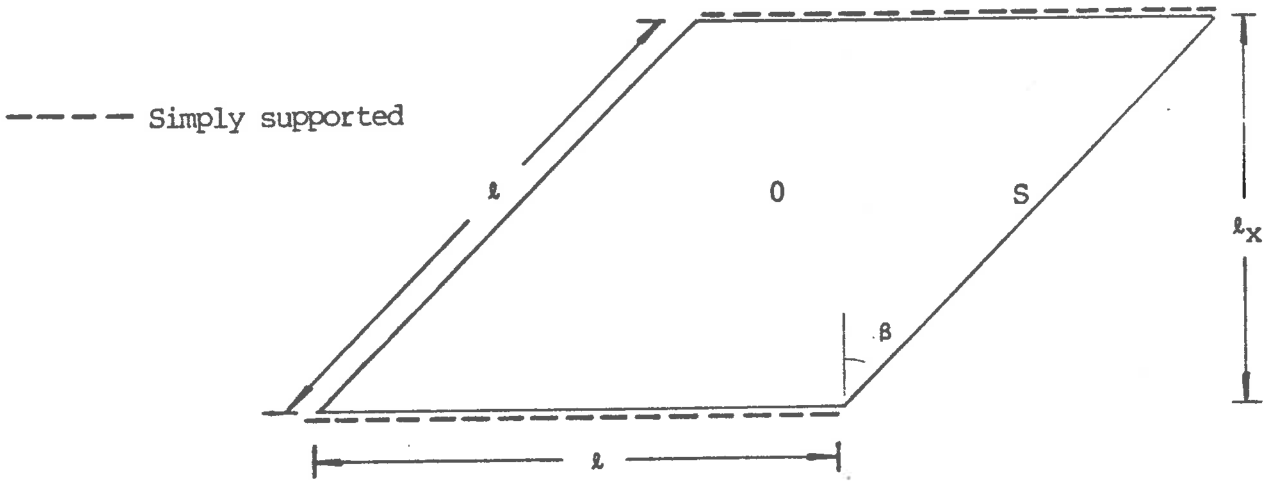


FIG. 3(a) Simply Supported Rhombic Plate

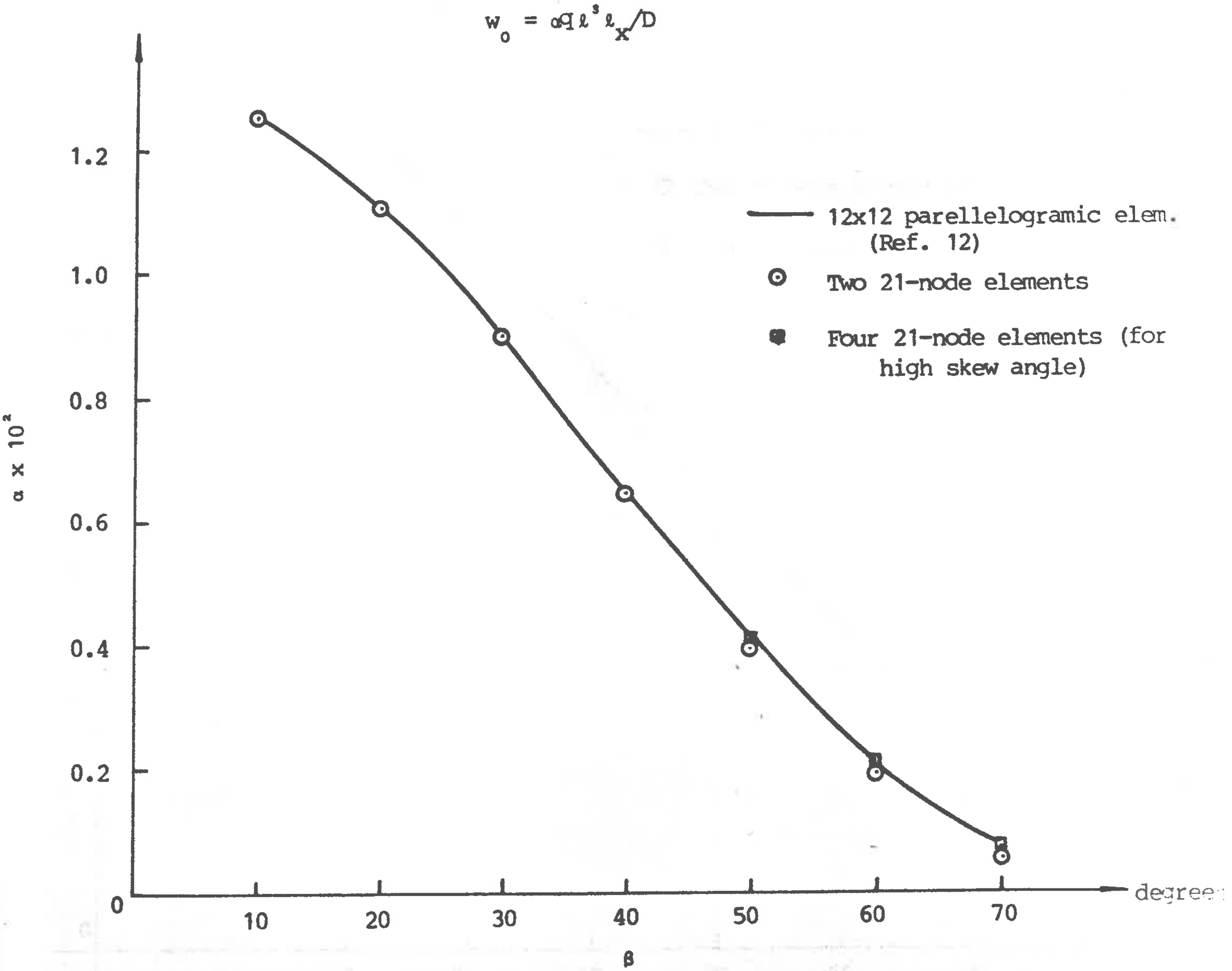


FIG. 3(b) Central Deflection Versus Skew Angle (ν = 0.31)

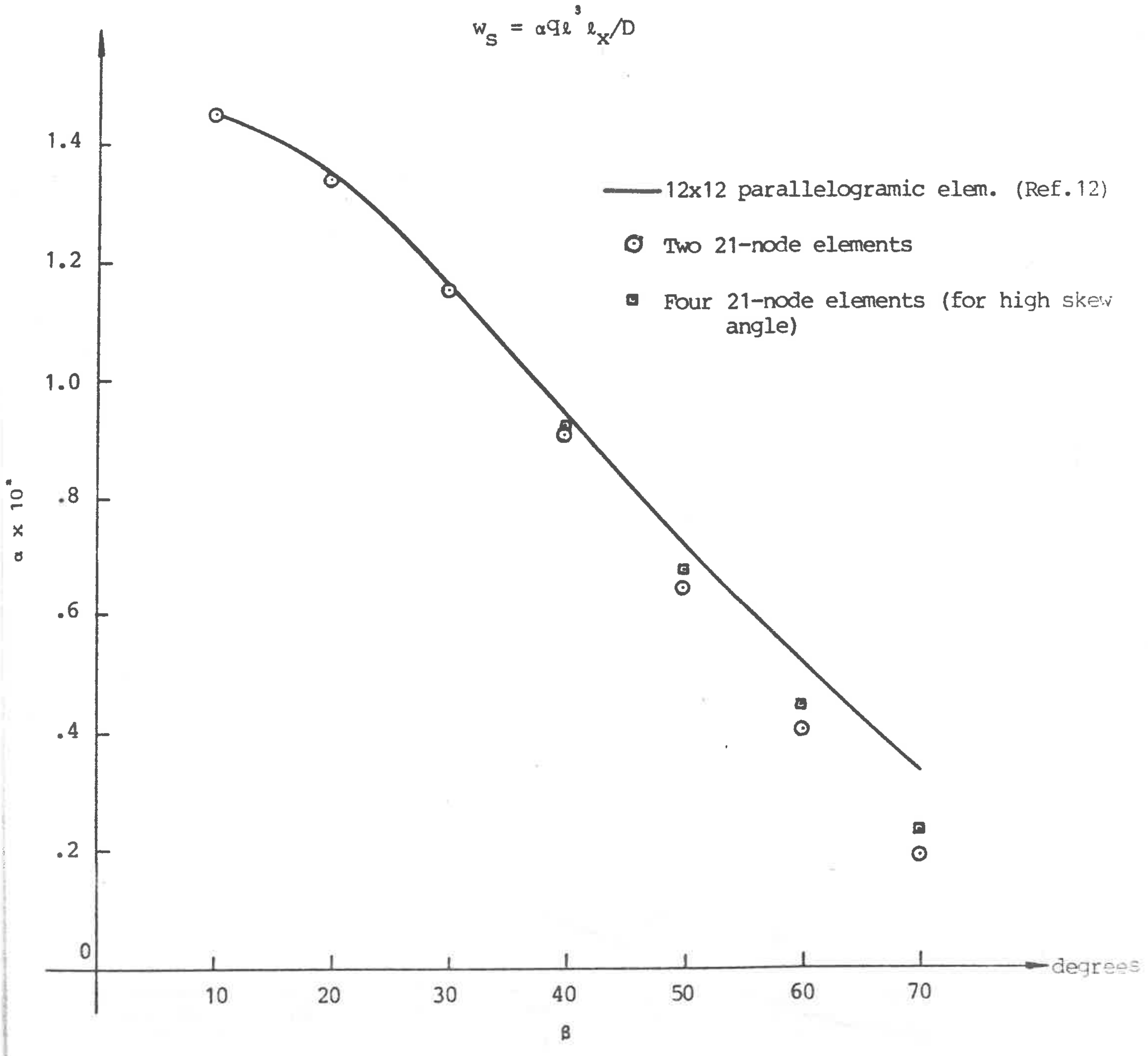


FIG. 3(c) Side Deflection (at point S) versus Skew Angle ($\nu = 0.31$)

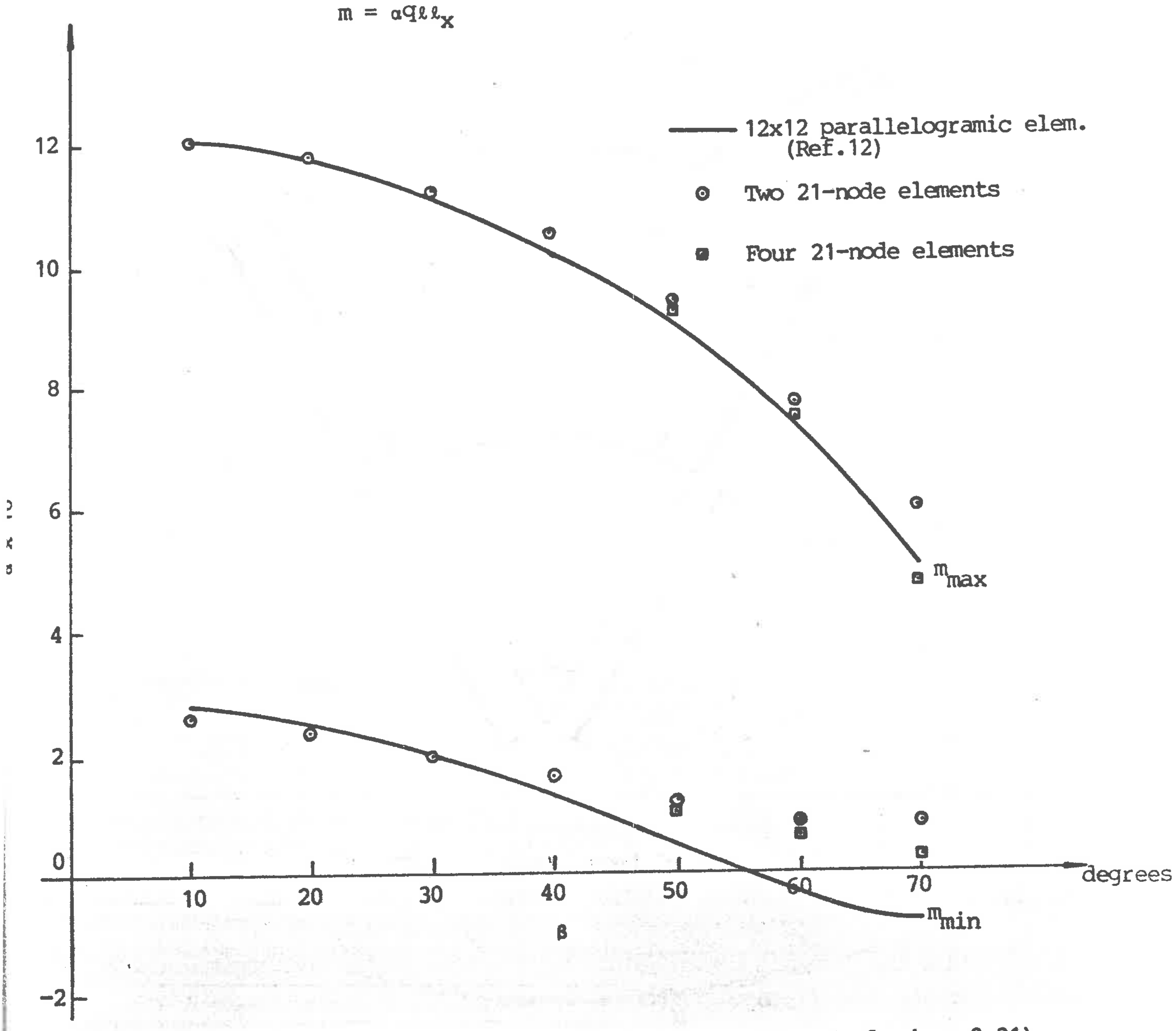


FIG. 3(d) Principal Moment at Centre Versus Skew Angle (ν = 0.31)

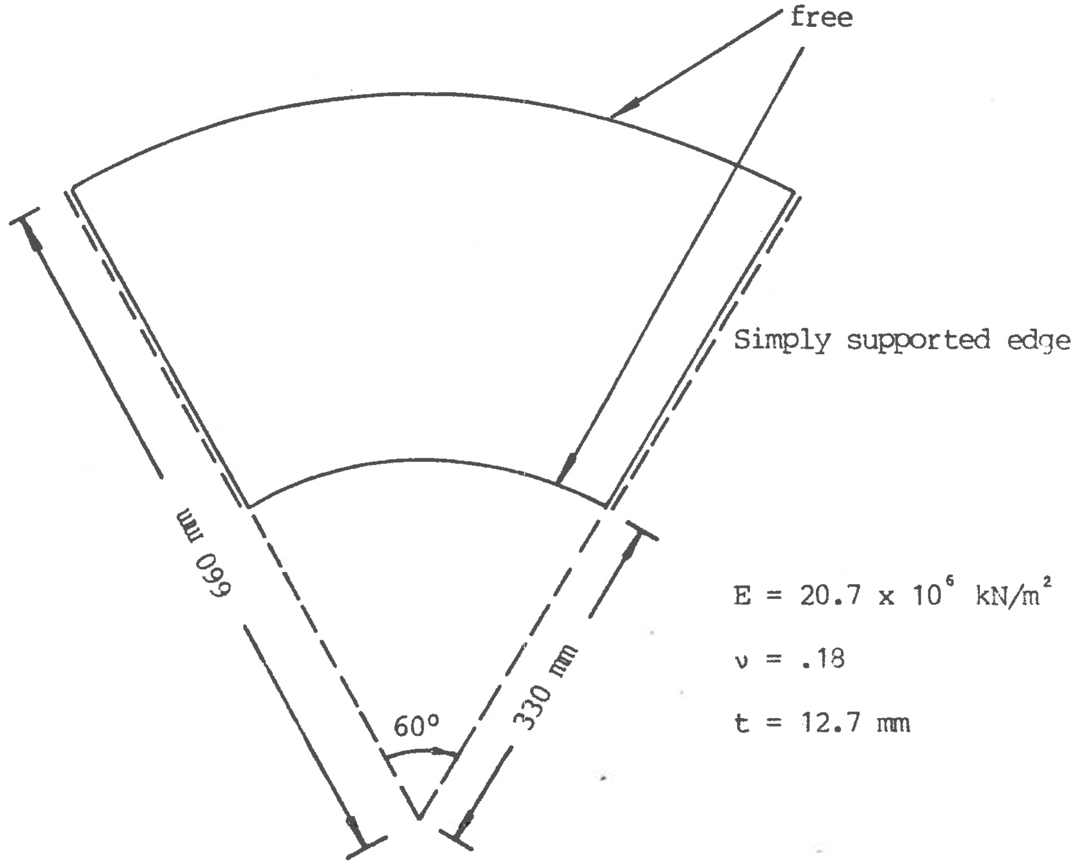
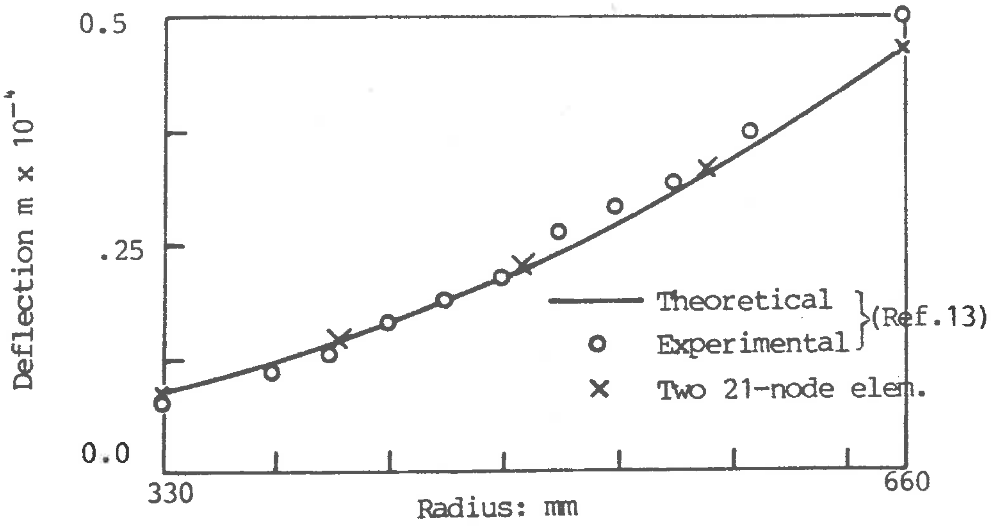
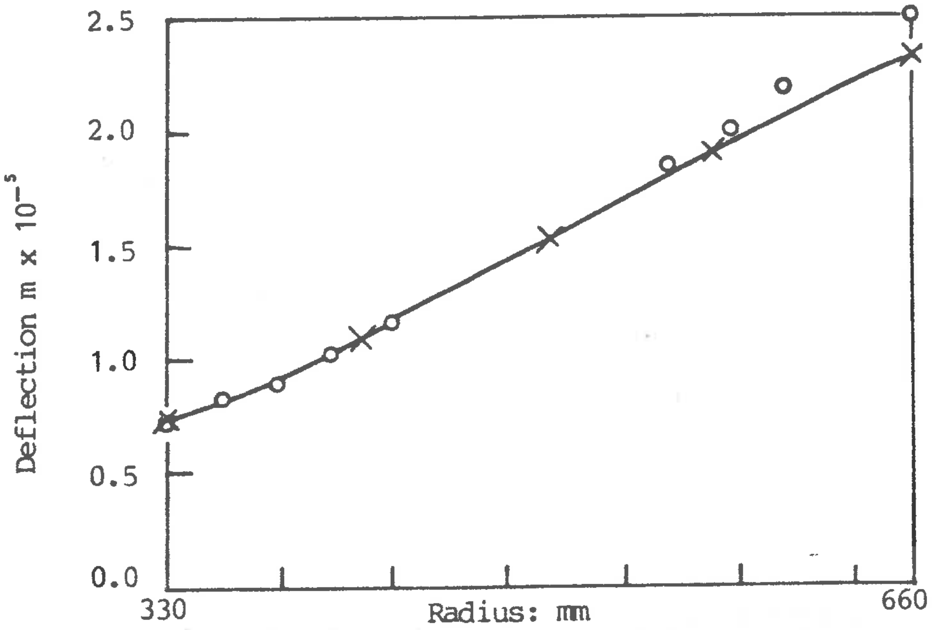


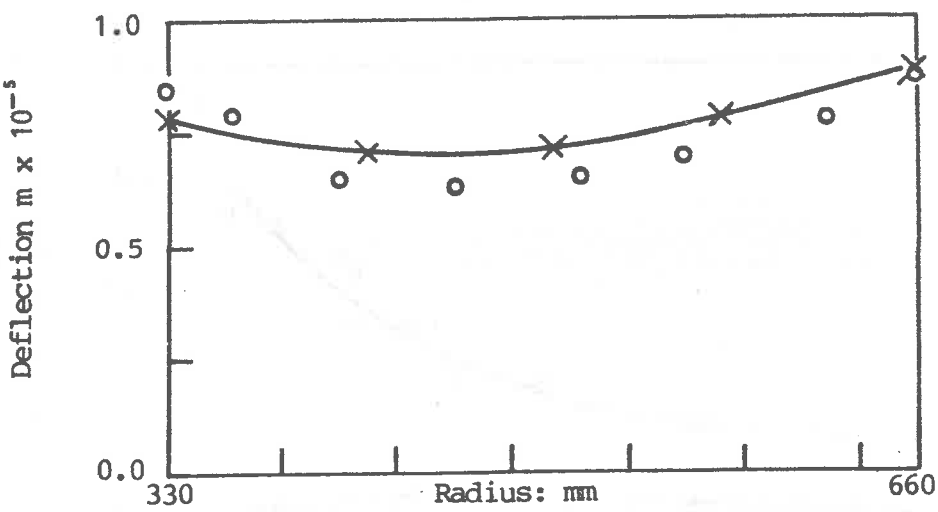
FIG. 4 Slab Curved in Plan



(a) Load at outer edge



(b) Load at mid-radius



(c) Load at inner edge

FIG. 5: Radial Distributions of Mid-span Deflections due to Unit Load

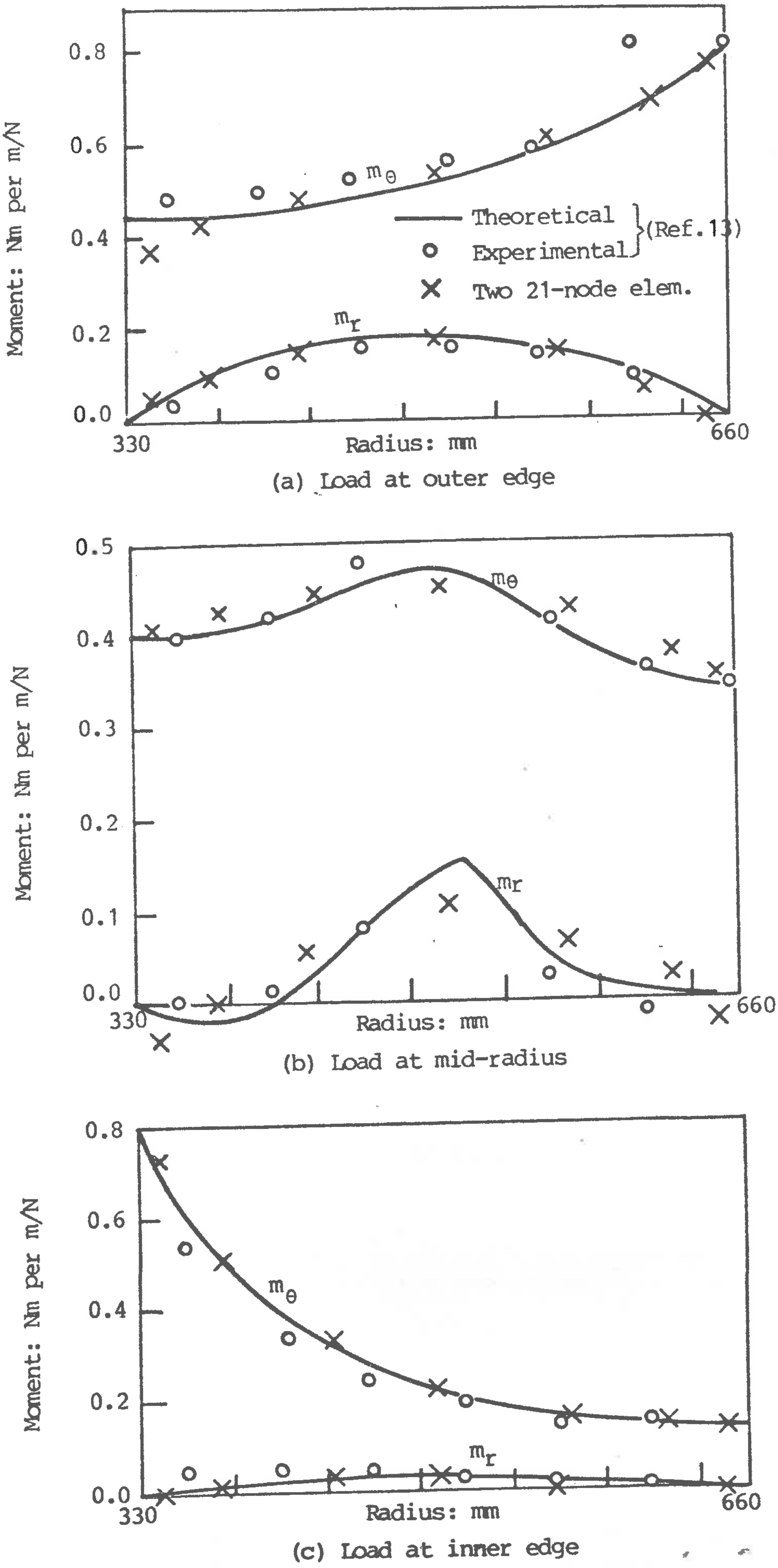


FIG. 6 Distributions of Tangential and Radial Bending Moments at Mid-span due to Unit Load

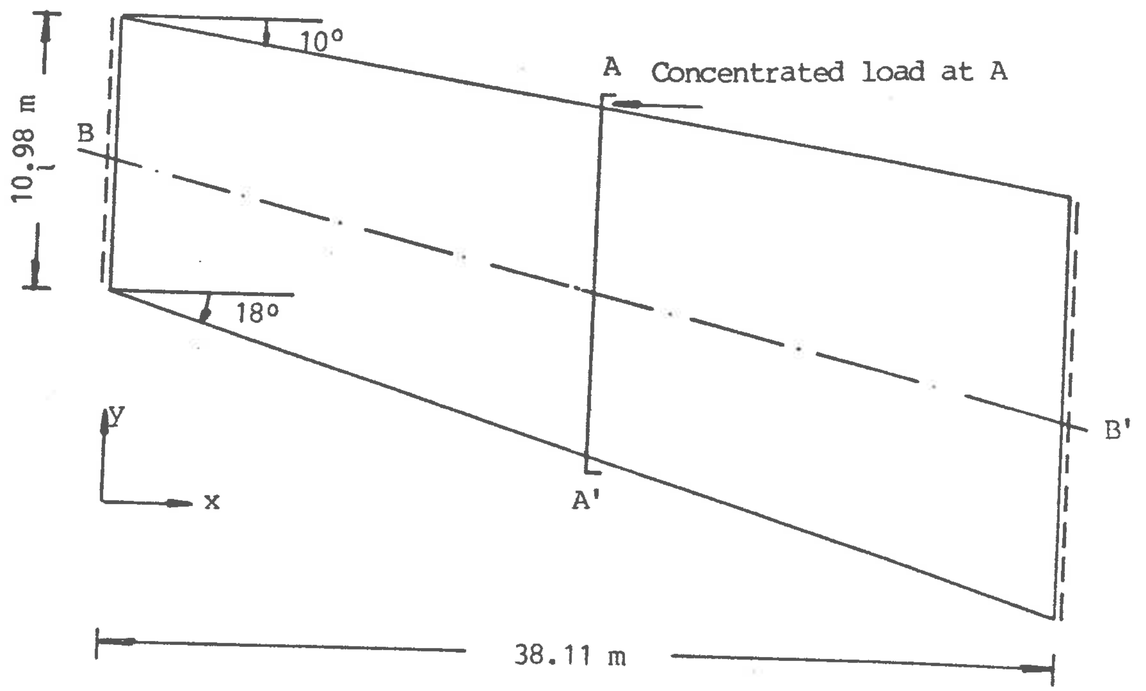


FIG. 7(a) Trapezoidal Slab Bridge

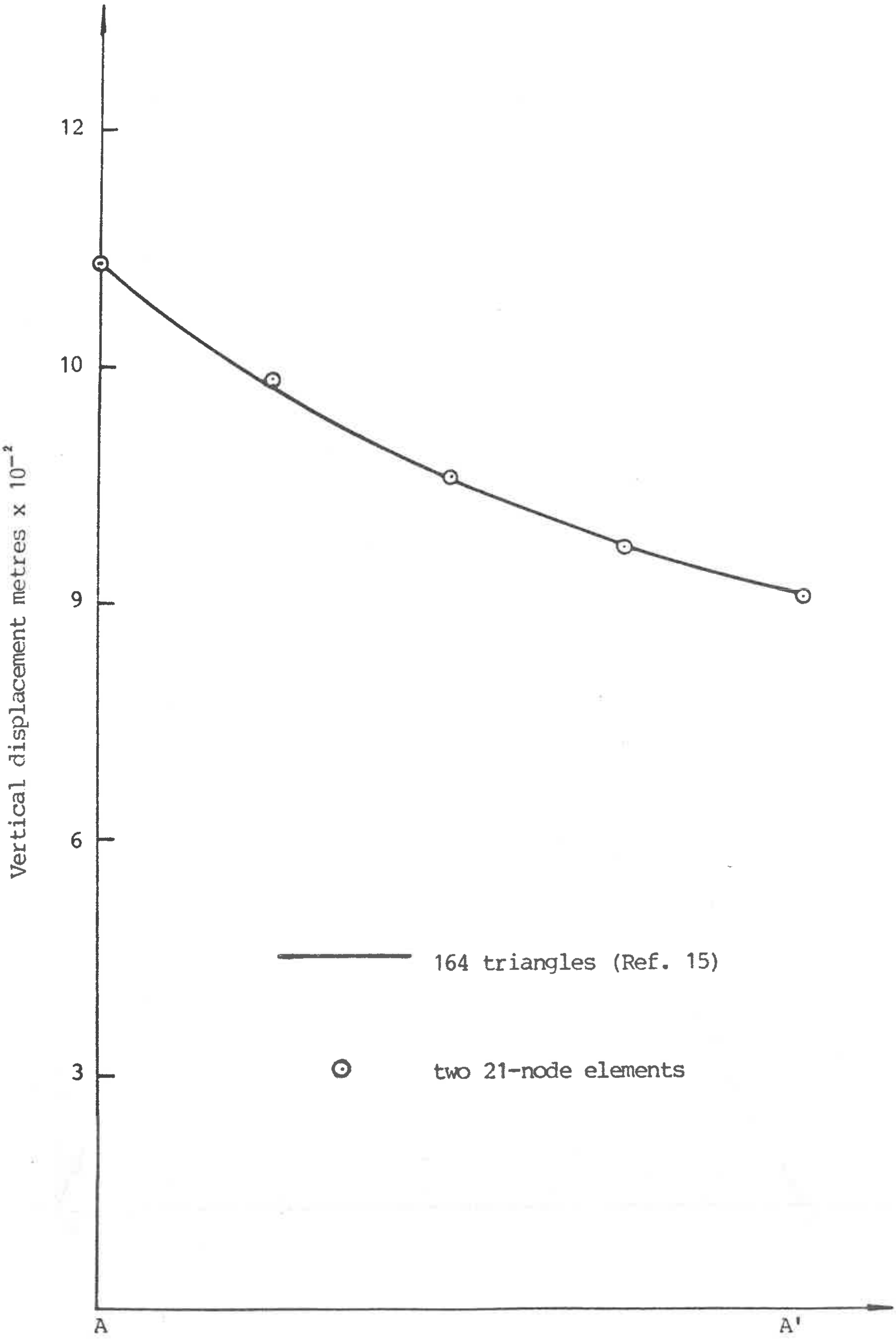


FIG. 7(b) Variation of Deflection along AA'

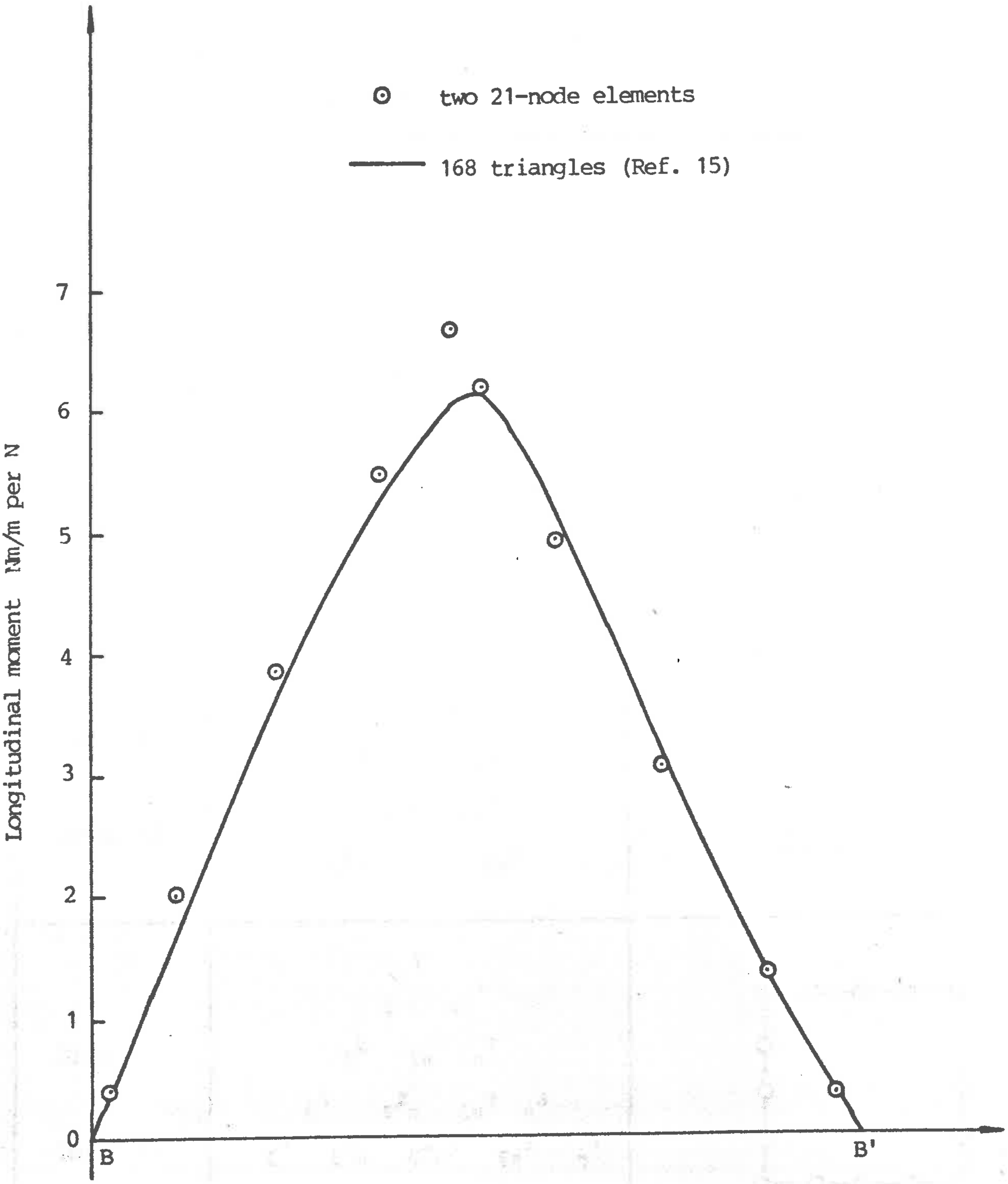
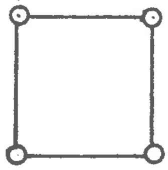
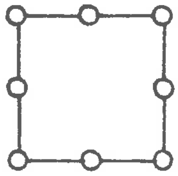
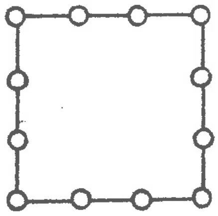
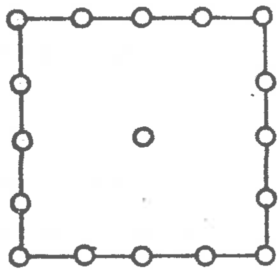
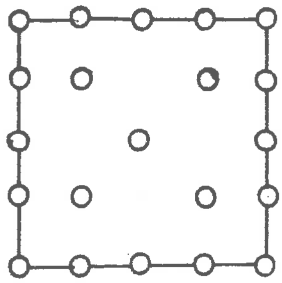
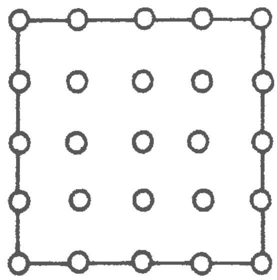


FIG. 7(c) Variation of m_x along BB'

Appendix Symmetrical Parent Elements

No. of nodes	Terms in polynomial	Element type
Side 2 Total 4	1 $\xi \quad \eta$ $\xi\eta$	
Side 3 Total 8	1 $\xi \quad \eta$ $\xi^2 \quad \xi\eta \quad \eta^2$ $\xi^2\eta \quad \xi\eta^2$	
Side 4 Total 12	1 $\xi \quad \eta$ $\xi^2 \quad \xi\eta \quad \eta^2$ $\xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3$ $\xi^3\eta \qquad \xi\eta^3$	
Side 5 Total 17	1 $\xi \quad \eta$ $\xi^2 \quad \xi\eta \quad \eta^2$ $\xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3$ $\xi^4 \quad \xi^3\eta \quad \xi^2\eta^2 \quad \xi\eta^3 \quad \eta^4$ $\xi^4\eta \qquad \xi\eta^4$	

Appendix (Cont.) Symmetrical Parent Elements

No. of nodes	Terms in polynomial	Element type
Side 5 Total 21	1 $\xi \quad \eta$ $\xi^2 \quad \xi\eta \quad \eta^2$ $\xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3$ $\xi^4 \quad \xi^3\eta \quad \xi^2\eta^2 \quad \xi\eta^3 \quad \eta^4$ $\xi^4\eta \quad \xi^3\eta^2 \quad \xi^2\eta^3 \quad \xi\eta^4$ $\xi^3\eta^3$ $\xi^4\eta^4$	
Side 5 Total 25	1 $\xi \quad \eta$ $\xi^2 \quad \xi\eta \quad \eta^2$ $\xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3$ $\xi^4 \quad \xi^3\eta \quad \xi^2\eta^2 \quad \xi\eta^3 \quad \eta^4$ $\xi^4\eta \quad \xi^3\eta^2 \quad \xi^2\eta^3 \quad \xi\eta^4$ $\xi^4\eta^2 \quad \xi^3\eta^3 \quad \xi^2\eta^4$ $\xi^4\eta^3 \quad \xi^3\eta^4$ $\xi^4\eta^4$	

INDEX TO PUBLICATIONS

Section II – Finite Strip Methods in Engineering Analysis

(13 publications)

All publications are included in the print copy of the thesis held in the University of Adelaide Library and unless indicated have been removed due to copyright regulations

1. Chan, H. C. & Cheung, Y. K. (1972). Static and dynamic analysis of multi-layered sandwich plates. *International Journal of Mechanical Sciences, 14*(6), 399-406. doi: https://doi.org/10.1016/0020-7403(72)90082-3

2. Wu, C. I. & Cheung, Y. K. (1974). Frequency analysis of rectangular plates continuous in one or two directions. *Earthquake Engineering and Structural Dynamics*, *3*(1), 3-14. doi: https://doi.org/10.1002/eqe.4290030102

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5. Cheung, Y. K., Hutton, S. G. & Kasemset, C. (1977). Frequency analysis of coupled shear wall assemblies. *Earthquake Engineering & Structural Dynamics, 5*(2), 191-201.

6. Cheung, Y.K. & Delcourt, C. (1977). Buckling and vibration of thin, flat•walled structures continuous over several spans. *Proceedings of the Institution of Civil Engineers, 63*(4), 943-946. (included)

7. Cheung, Y. K., Kasemset, C., & Swaddiwudhipong, S. (1977, August). *Vibration and stability of tall frame structures.* Paper presented at the International Conference on Computer Applications in Developing Countries, Bangkok.

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Proc. Instn Civ. Engrs, Part 2, 1977, **63**, Dec., 943–946

7959 DISCUSSION

Buckling and vibration of thin, flat-walled structures continuous over several spans

Y. K. CHEUNG & C. DELCOURT

Professor F. W. Williams, University of Wales Institute of Science and Technology

Paragraphs 6–18 of the Paper are devoted to finding the continuous beam eigenvalues and eigenfunctions. This seems unnecessary because a continuous beam is a very simple plane frame and the free vibration of such frames is extensively treated in the literature. Thus equation (9) is the no-sway member equation, with α and γ commonly being denoted by $(EI/l)S$ and $(EI/l)SC$, where S and SC are analogous to the well-known stability functions s and sc. Moreover, equation (11) is merely the stiffness matrix formulation of the simple plane frame which results from connecting two such members together to form a continuous beam. However, it is known that a structure can have natural frequencies in addition to those for which the determinant of its stiffness matrix is zero when, as in the Paper, the matrix coefficients are transcendental functions of μ_n. The existence of such frequencies is discussed in reference 24, which also gives a theoretically proven method which is guaranteed to find these and all other natural frequencies of a structure. The method has been extended to buckling problems.[25] Reference 26 gives a complete description of how the method can be used to find the natural frequencies of a plane frame.

31. Unfortunately, the Authors' method (§ 15) only finds those natural frequencies for which the determinant of the stiffness matrix (i.e. the matrix of equation (11) in their example) is zero. Therefore it seems that the Authors should adopt the method described in § 30 (which also overcomes the problem mentioned at the end of § 15) or an alternative, as otherwise eigenvalues will be missed for some problems.

32. The danger of missing eigenvalues does not arise for most problems, but is present for any beam with identical spans and clamped ends. Physical argument shows that such a beam will have natural frequencies coinciding with those of a single span with its ends clamped, and that the corresponding modes will involve zero rotations at the intermediate supports. Figs 3(b), 3(d) and 3(f) illustrate this situation for a beam with two identical spans. Moreover, the first six eigenvalues must clearly correspond to modes 1–6 in Fig. 3. Hence the method in the Paper will miss alternate eigenvalues by finding all those with anti-symmetric modes and missing all those with symmetric modes, because each symmetric mode corresponds to a non-trivial solution of an equation similar to equation (11) for which the vector of the θs is null. Specifically, equation (11) applies to the problem of Fig. 3 when $l_2 = l_1$ and $\theta_{1n}(0) = \theta_{2n}(l_2) = 0$. Equation (11) can then be simplified to the scalar form

$$2\alpha(l_1)\theta_{2n}(0) = 0 \qquad (18)$$

and equating the appropriate determinant to zero gives

$$\alpha(l_1) = 0 \qquad (19)$$

Paper published: *Proc. Instn Civ. Engrs*, Part 2, 1977, **63**, Mar., 93–103.

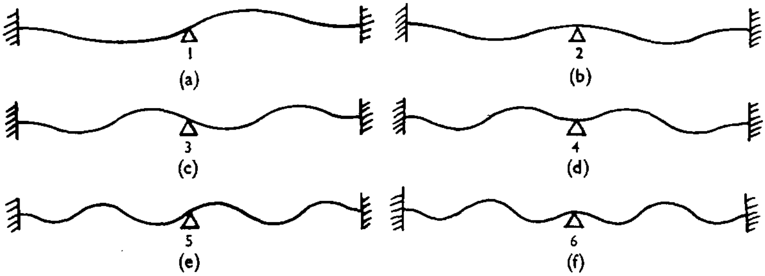

Fig. 3. Vibration modes for a beam with two identical spans and clamped ends

or, using equation (9)

$$s_1c_2 - s_2c_1 = 0 \qquad (20)$$

It can be shown that equation (20) gives the anti-symmetric modes of Fig. 3; the symmetric modes are given by

$$1 - c_1c_2 = 0 \qquad (21)$$

It follows, from the definition of α in equation (9), that $\alpha(l_1) = \infty$ when equation (21) is satisfied. Hence the symmetric modes correspond to the determinant of the stiffness matrix, which is equal to $2\alpha(l_1)$, passing through infinity instead of zero.

33. The Authors give the lowest natural frequency (Table 1, case 6) and the lowest critical load (Table 2, case 5) for problems related to Fig. 3. Have they tried to find the second eigenvalues? If so, were the results obtained in error, due to the omission of all symmetric continuous beam eigenfunctions from the set of eigenvalues used?

34. The Authors clearly envisage the finite strip method being used to find all the possible buckling or vibration modes of a plate assembly. The inclusion of modes involving several longitudinal half waves between supports (e.g. the local mode of Fig. 2(b)) is presumably the only reason why n has to be large. However, such modes can be found efficiently by classical methods[2-5,12] as the assumption of sinusoidal variation of displacement in the longitudinal direction causes negligible errors when the support conditions are incompatible with such variation, because of the relative remoteness of the supports. Indeed, these errors seem likely to be small compared with those resulting from the finite strip analysis, in view of the displacement function assumptions of equation (1), the need to keep n and the number of strips as low as possible, and the possible simplification of the problem, represented by the replacement of the cross-section of Fig. 2(a) by that of Fig. 2(c).

35. The classical method requires only 12 degrees of freedom to handle the cross-section of Fig. 2(c) using the method given in the appendix of reference 27, and 36 degrees of freedom to handle the cross-section of Fig. 2(a) using a sub-structure approach similar to that given in Fig. 7(c) of reference 27. Furthermore, the associated stiffness matrices have a band width (number of terms within the band and to the right of the leading diagonal) of 7 which is maintained throughout the computation. Increasing l and S in Fig. 2 will not alter these quantities, whereas the comparable finite strip quantities of Table 3 are increased. However, it has to be admitted that the classical method would involve repeating the stiffness matrix calculations. For instance, about 12 repetitions would be required, at each of three assumed half wavelengths of the sinusoidal longitudinal displacement pattern, to obtain the local mode eigenvalue for the problem of Fig. 2. Nevertheless the solution would be obtained rapidly, because of the small number of degrees of freedom and the narrow band width, and it seems certain that the Authors' method must take much longer.

36. A wise policy might be to use classical methods to find the eigenvalues for which

there are several half waves between supports and the finite strip method, with one strip per plate and a low value of n to reduce the number of degrees of freedom, to find the other eigenvalues.

Dr Cheung and Mrs Delcourt

In §§ 30–33 Professor Williams points out that, in the particular case of a multispan beam with both ends clamped and equal spans, the eigenfrequencies corresponding to a zero displacement vector are missing. We fully agree. This was effectively detected soon after submission of the Paper. Since, in this case, the structure degenerates into substructures which are the different spans clamped, these eigenfrequencies can be easily determined[28] and introduced where they need to be in the growing sequence of μ_n obtained by finding the zeros of the determinant of the stiffness matrix. The method suggested in reference 24 could effectively be applied to check the number of eigenvalues below each μ_n and then the missing eigenvalues corresponding to a zero displacement vector could be inserted. This is unnecessary in all the cases where one end at least is not clamped, because the solution corresponding to a zero displacement vector is the trivial solution; consequently it is not worthwhile increasing the computer time by calculating roots which will not be used, as the only eigenfrequencies will be those obtained by the method presented in the Paper.

38. We are aware that α and β are equivalent to $(EI/l)S$ and $(EI/l)SC$, but the method in the Paper gives a quick and automatic way of determining the eigenfunctions Y_n necessary for the derivation of a multispan finite strip having any boundary conditions, number of spans, length of spans and so on.

39. With reference to §§ 34 and 35, it was stated in the Paper that n is proportional to α (§ 27), α being the number of half-waves for local buckling; it is then natural that n has to be large if α is large.

40. We do not deny the efficiency of the classical methods. First, we compare two numerical methods—finite element and finite strip—but we do not present any comparison between the efficiency of an exact and an approximate method. Second, the influence of the supports can be important in the case of short structures, and in the case of an overall buckling mode which will appear automatically in the present method as well as a local or torsional buckling mode, as no assumption is made about the type of buckling. Furthermore, the present method can easily handle orthotropic structures.

41. As regards § 36, the ultimate aim of the finite strip method applied to multispan structures being the stability analysis and static analysis of multispan flat walled structures, the number n has to be held large enough in view of

(*a*) predicting an eventual local buckling mode in a stability analysis
(*b*) approaching accurately the displacement and load distributions in a static analysis.

However, in dynamic analysis, the policy of using a low value of n to find the other eigenvalues without involving too many degrees of freedom can be adopted.

42. Whenever a new approach is presented it is the privilege of the user to combine the advantageous features of the new and the old methods.

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Errata

The last displacement function of equations (1) should read

$$w = \sum_{n=1}^{r} [(1-3x^2/b^2+2x^3/b^3)w_{in}+(x-2x^2/b+x^3/b^2)\theta_{in}$$
$$+(3x^2/b^2-2x^3/b^3)w_{jn}+(x^3/b^2-x^2/b)\theta_{jn}] \sum_{s=1}^{S} Y_{sn}$$

44. Equation (4) should read

$$\frac{\mathrm{d}^4 Y_s}{\mathrm{d}y_s{}^4} - \frac{m_s\omega^2}{EI_s} Y_s = 0$$

45. The expression for C_s in the penultimate line of § 7 should read

$$C_s = (m_s/EI_s)^{1/4}(m/EI)^{-1/4}$$

46. The term (2, 1) of the matrix of equation (7) should read

$$\beta(1+s_1s_2-c_1c_2)$$

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IABSE PROCEEDINGS P-14/78

MÉMOIRES AIPC
IVBH ABHANDLUNGEN

Finite Strip Analysis of Continuous Folded Plates

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International Association for Bridge and Structural Engineering IABSE
Association Internationale des Ponts et Charpentes AIPC
Internationale Vereinigung für Brückenbau und Hochbau IVBH

The discussion of these Proceedings will be open until August 31, 1978.

La discussion de ce Mémoire est ouverte jusqu'au 31 août 1978.

Die Diskussion dieser Abhandlung wird bis 31. August 1978 möglich sein.

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Finite Strip Analysis of Continuous Folded Plates

Analyse par bandes finies de toits plissés continus

Endliche Streifenelemente für die Berechnung von durchlaufenden Faltwerken

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SUMMARY
The finite strip method is generalized to study folded plates continuous over any number of spans and submitted to uniformly distributed loads or line loads. The end supports are either clamped, simply supported or free. The results obtained agree well with experimental and theoretical values available in the literature and a comparison of computational efforts required between finite element and finite strip methods demonstrates that the latter method has a very definite advantage over the former.

RÉSUMÉ
La méthode par bandes finies est généralisée pour permettre l'étude de toits plissés continus, d'un nombre quelconque de travées, soumis à des charges uniformes ou linéaires. Les travées extrêmes sont soit encastrées, soit simplement appuyées, soit libres. Les résultats concordent bien avec les valeurs théoriques et expérimentales données dans la littérature. La comparaison des efforts livrés par l'ordinateur dans les méthodes des éléments finis et des bandes finies montre que cette dernière est nettement plus avantageuse que la précédente.

ZUSAMMENFASSUNG
Auf der Basis endlicher Streifenelemente wird eine Methode entwickelt für die Berechnung durchlaufender Faltwerke beliebiger Felderzahl unter verteilten Lasten oder Linienlasten. Die Enden des Faltwerks können eingespannt, frei drehbar gelagert oder frei sein. Die mit dieser Methode berechneten Ergebnisse stimmen gut mit Versuchsergebnissen und theoretischen Lösungen anderer Autoren überein. Vom Berechnungsaufwand her gesehen hat die vorgelegte Methode gegenüber Methoden endlicher Elemente eindeutige Vorteile.

Introduction

Traditionally, the analysis of folded plate structures are based either on the "ordinary theory" or on the "elasticity theory" and most of the literature has been concerned with the analysis of single span structures. [1] More recently attention has been focused on continuous folded plates. Beaufait [2] used the ordinary theory and developed a computer solution for folded plate with arbitrary end conditions, although some of the assumptions appear to be questionable [3]. Pultar et al [4] presented a solution based on elasticity theory for continuous folded plates with simply supported ends, in which a force method is applied subsequently to the standard analysis in order to determine the redundant reactive forces at the intermediate supports.

The technique was also used by Scordelis et al [5] for the analysis of box girder structures with internal rigid diaphragms. Continuous folded plate structures were also analysed by Lee [6], using a finite difference technique.

The most versatile tool of analysis is obviously the finite element method. Rockey and Evans [7] were the first to study the behaviour of folded plate structures by using a rectangular element [8], and Lo and Scordelis [9] developed a finite segment method (which is a special type of finite element method based on ordinary theory) in an attempt to reduce the excessive number of degrees of freedom involved in a problem.

Recently, the finite strip method pioneered by Cheung [10] was applied to rectangular [11], curved [12,13] and skew [14] folded plate structures, all with simply supported ends. The object of the present paper is to extend the finite strip method to the analysis of continuous folded plate structures with arbitrary end conditions, but without resorting to the standard procedure of carrying out a subsequent flexibility analysis.

Finite Strip Approach

It is now well known that a finite strip is a special finite element for which the boundary conditions of the structure, in the longitudinal direction, are *a priori* included in the approximation of the displacement field. In the present approach, such boundary conditions may include simply supported, clamped, or free edge conditions. However, as far as the loading is concerned the study is restricted to distributed loads or longitudinal line loads which are the dominant types of loadings for folded plate structures.

Stiffness matrix and force vector

The procedure of formulating the stiffness matrix and the force vector for a flat shell strip has been already given elsewhere [10] and will not be repeated here. However, the approximation used for the displacement field will be given below. For a flat shell strip bounded by sides i and j (Fig. 1) and continuous over several spans the displacement field is:

$$u = \sum_{n=1}^{N} \left[\left(1 - \frac{x}{b}\right) u_{in} + \left(\frac{x}{b}\right) u_{jn} \right] \sum_{s=1}^{S} Y_{sn}$$

$$v = \sum_{n=1}^{N} [(1 - \frac{x}{b}) v_{in} + (\frac{x}{b}) v_{jn}] \sum_{s=1}^{S} Y'_{sn}$$

$$w = \sum_{n=1}^{N} [(1 - 3x^2/b^2 + 2x^3/b^3)w_{in} + (x - 2x^2/b + x^3/b^2)\theta_{in} + (3x^2/b^2 - 2x^3/b^3)w_{jn} + (x^3/b^2 - x^2/b)\theta_{jn}] \sum_{s=1}^{S} Y_{sn}$$

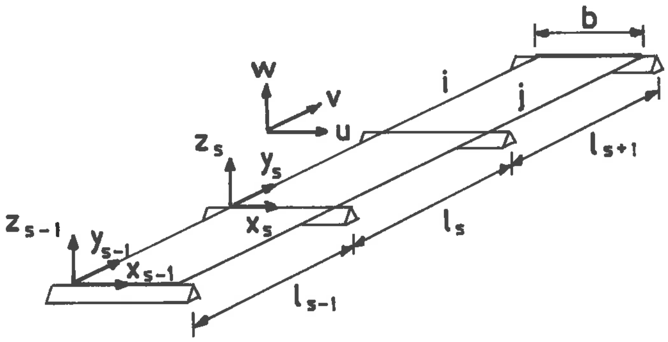


Fig. 1 A Typical continuous finite strip

where S and N are respectively the number of spans and the number of terms used in the series, Y' denotes the first derivative of Y with respect to y; x, y, u, v, w, θ are the coordinates and displacements indicated in Fig. 1, and b is the width of the strip. By definition, the boundary conditions in the longitudinal direction must *a priori* be satisfied by the functions Y_{sn}, which are described hereafter. It should be noted that, in the present analysis, a simple end support oran intermediate support is a diaphragm which is infinitely stiff in its own plane but perfectly flexible normal to its plane.

Basic Functions for Multispan Finite Strip

The basic functions Y_{sn} are the eigenfunctions of a corresponding continuous beam which has the same characteristics as the structure to be considered in terms of end conditions, number and length of spans, relative rigidities, etc. The complete computation of these eigenfunctions can be found in ref. 15 and is not repeated herein in detail. However a resume of the procedure is given in the following.

The continuous beam eigenfunctions are computed by using a general stiffness approach. For a typical s^{th} span, the differential equation for vibration is given by

$$\frac{d^4 Y_s}{dy^4} - \frac{m_s \omega^2}{E_s I_s} Y_s = 0 \qquad \text{...... (1)}$$

assuming harmonic motion as usual. In Eq. (1), Y_s is the amplitude of the mode with reference to a local set of coordinate system originated at the left end of s^{th} span. I_s, m_s, E_s are respectively the second moment of the area of the section, the mass per unit length and Young's modulus.

The general solution for this equation is

$$Y_{sn} = A_{sn} \sin G_s\mu_n y_s + B_{sn} \cos G_s\mu_n y_s + C_{sn} \sinh G_s\mu_n y_s + D_{sn} \cosh G_s\mu_n y_s \quad \text{......(2)}$$

where $\mu_n = \left(\frac{m\omega_n^2}{EI}\right)^{1/4}$ and $G_s = \left(\frac{m_s}{EI_s}\right)^{1/4} \left(\frac{m}{EI}\right)^{-1/4}$ denotes the common eigenvalue and the reciprocal of the relative wave propagation of s^{th} span. The unknowns, $\mu_n, A_{sn}, B_{sn}, C_{sn}, D_{sn}$ are determined by expressing the boundary conditions at both end supports and at all intermediate supports.

Solution for μ_n

By inserting the coordinates of the two end supports of the s^{th} span into Eq. (2), the following equations are derived:

$$\begin{Bmatrix} Y_{sn} & (0) \\ \theta_{sn} & (0) \\ Y_{sn} & (\ell_s) \\ \theta_{sn} & (\ell_s) \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -\beta & 0 & -\beta & 0 \\ s_1 & c_1 & s_2 & c_2 \\ -\beta c_1 & +\beta s_1 & -\beta c_2 & -\beta s_2 \end{bmatrix} \begin{Bmatrix} A_{sn} \\ B_{sn} \\ C_{sn} \\ D_{sn} \end{Bmatrix} \quad \text{......(3)}$$

The two end moments for this particular span can be written as:

$$\begin{Bmatrix} M_{sn}(0) \\ M_{sn}(\ell_s) \end{Bmatrix} = EI_s\beta^2 \begin{bmatrix} 0 & -1 & 0 & 1 \\ s_1 & c_1 & -s_2 & -c_2 \end{bmatrix} \begin{Bmatrix} A_{sn} \\ B_{sn} \\ C_{sn} \\ D_{sn} \end{Bmatrix} \quad \text{......(4)}$$

In eq. (3 and (4) $\theta_{sn} = -\frac{dY_{sn}}{dy_s}$ $\quad s_1 = \sin \beta\ell_s \quad c_1 = \cos \beta\ell_s$

$M_{sn} = \frac{d^2Y_{sn}}{dy_s^2}$ $\quad s_2 = \sinh \beta\ell_s \quad c_2 = \cosh \beta\ell_s$

$\beta = G_s\mu_n$

By inverting Eq. (3) and incorporating the result in Eq. (4), the moments can be written in terms of the deflections and the slopes. As all the intermediate supports are rigid supports, the condition

$$Y_{sn}(0) = Y_{sn}(\ell_s) = 0 \qquad 1 < s < S$$

is used to establish, for each span (except the first and last spans) a relation between moments and slopes only:

$$\begin{Bmatrix} M_{sn}(0) \\ M_{sn}(\ell_s) \end{Bmatrix} = \begin{bmatrix} \alpha & \gamma \\ \gamma & \alpha \end{bmatrix} \begin{Bmatrix} \theta_{sn}(0) \\ \theta_{sn}(\ell_s) \end{Bmatrix} \quad \text{......(5a)}$$

in which $\alpha = \frac{EI_s}{s}\,\beta(s_1\,c_2 - s_2\,c_1)/(1 - c_1\,c_2)$ ......(5b)

$\gamma = \frac{EI_s}{s}\,\beta(s_2 - s_1)/(1 - c_1\,c_2)$ ......(5c)

By establishing the moment equilibrium (i.e. $M_{s-1,n}\,(\ell_{s-1}) + M_{sn}(0) = 0$) and compatibility conditions (i.e. $\theta_{s-1,n}(\ell_{s-1}) = \theta_{sn}(0)$) at all intermediate supports through Eq. (5a, b, c) the following set of equations can be obtained.

$$\begin{Bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} = \begin{bmatrix} \gamma(\ell_1) & \alpha(\ell_1)+\alpha(\ell_2) & \gamma(\ell_2) & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & \gamma(\ell_2) & \alpha(\ell_2)+\alpha(\ell_3) & \gamma(\ell_3) & \cdot & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma(\ell_{S-1}) & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \gamma(\ell_{S-1}) & \alpha(\ell_{S-1})+\alpha(\ell_S) & 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_{1n}(0) \\ \theta_{2n}(0) \\ \theta_{3n}(0) \\ \cdot \\ \cdot \\ \cdot \\ \theta_{Sn}(0) \\ \theta_{Sn}(\ell_S) \\ Y_{1n}(0) \\ Y_{Sn}(\ell_S) \end{Bmatrix}$$

......(6a)

The matrix in Eq. (6a) is overdeterminate, involving S-1 equations and S+3 unknowns; the boundary conditions at the end supports should be incorporated to make the matrix square. A non-trivial solution of μ_n is obtained by equating the determinant of this matrix to zero. Eq. (6a) can be shortened as:

$\{o\} = [K]\ \{d\}$ ......(6b)

and then the solutions will be given by

$\det\,[K] = 0$ ......(6c)

[K] being a square matrix, including the boundary conditions, and {d} being the vector of displacements.

The coefficients of this determinental equation are transcendental functions of μ_n and can be solved by the Newton Raphson method as suggested in ref.[15]. Unfortunately it was found that at times certain roots might be missing and in order to make sure that all the roots up to the desired number N can be computed, an adaptation of the "modified regula falsi" method [16, 17] is used herein, although it has the disadvantage of being somewhat time-consuming. Attention should be drawn to the fact that for the case of both ends clamped only, Eq. (6c) will miss out the roots corresponding to the situation in which the whole displacement vector {d} is equal to zero, i.e., the continuous structure degenerates into a series of single-span substructures with both ends clamped. The eigenvalues of such beams are of course common knowledge [10] and they can be inserted into the correct positions in the acending array of μ_n obtained from

Eq. (6c). The position of the missing roots can also be determined by using a method proposed by Wittrick and Williams [20], in which a "sign count" of the matrix [K] is made.

Solution for A_{sn}, B_{sn}, C_{sn}, D_{sn}

If the ends are either supported or clamped all the supports are rigid supports; this leads to the condition $Y_{sn}(0) = 0$ which implies that $B_{sn} = - D_{sn}$.

The moment equilibrium, the compatibility condition and the zero deflection conditions at each intermediate support provides 3S-3 equations for the remaining 3S constants. The second condition at the left end support (i.e. $M_{1,n}(0) = 0$ if simply supported or $\theta_{1n} = 0$ if clamped) provides one more equation, while the zero deflection at the last support gives another equation. Finally if A_{1n} is taken as a unit reference constant, the set of 3S equation can be solved. The remaining condition at the right end support is automatically satisfied because it has been incorporated once already in the solution of μ_n. In the particular case of free ends, the process is exactly the same but 4 unknown must be kept for the first and last spans. The conditions of zero shear force and zero moment at the end will provide the necessary equations.

Convergence

In finite strip analysis, it is important to be able to predict the number of terms N required in the series so that results of reasonable accuracy can be obtained. It has been shown in ref. [11]that for single span folded plate structures, 5 non-zero Fourier series terms are sufficient to give accurate deflections, moments and membrane forces in a finite strip analysis. For continuous structures, the series will not converge as quickly and it is also difficult to fix a value N which is applicable to all problems involving different number of spans and span lengths.

A simple but approximate method of predicting the value N is to examine the convergence characteristics of the distributed loads and take N as the number of terms required for approximating the loads over a continuous beam with the same number of spans and span length ratio. As the functions Y_{sn} are eigenfunctions obtained by solving Eq. (1), they possess the property of orthogonality

$$\int_0^{\ell_s} Y_{sm} \, Y_{sn} \, dy = 0 \qquad \text{for } m \neq n$$

For this reason, any load $q_s(y)$ can be resolved into the same eigenfunction series as

$$q_s(y) = \Sigma_N \, q_{sn} \, Y_{sn} \qquad \text{......(7)}$$

$$\text{in which } q_{sn} = \frac{\int_o^{\ell_s} q_s(y)\, Y_{sn}\, dy}{\int_o^{\ell_s} Y_{sn}^2\, dy}$$

and the term by term convergence of the total integrated load can be examined. The above integration is carried out numerically.

An an example, the special case of q(y) = 80 kg/m on a two-span beam (either for equal spans or for unequal spans with $\ell_1/\ell_2 = 0.8$, which is characteristic of the examples treated later) is examined by using up to eleven terms of the series, while the convergence study for a single span beam (using Fourier series) with the same uniformly distributed loading is also made for comparison. In Fig. 2 (a,b,c), the approximate loads for specified values of N demonstrate clearly the faster convergence of the Fourier series. In Fig. 2d the percentage error of the total integrated load with respect to N is worked out for the three beams and it can be seen that for the unequal span continuous beam, the convergence becomes very slow for N > 10. It is therefore concluded that the small gain in accuracy obtained by increasing N even further cannot be justified, keeping in mind that the number of degrees of freedom per node is equal to 4N. For the simply supported beam, convergence becomes very slow after five terms, and this is in accordance with the findings given in reference 11.

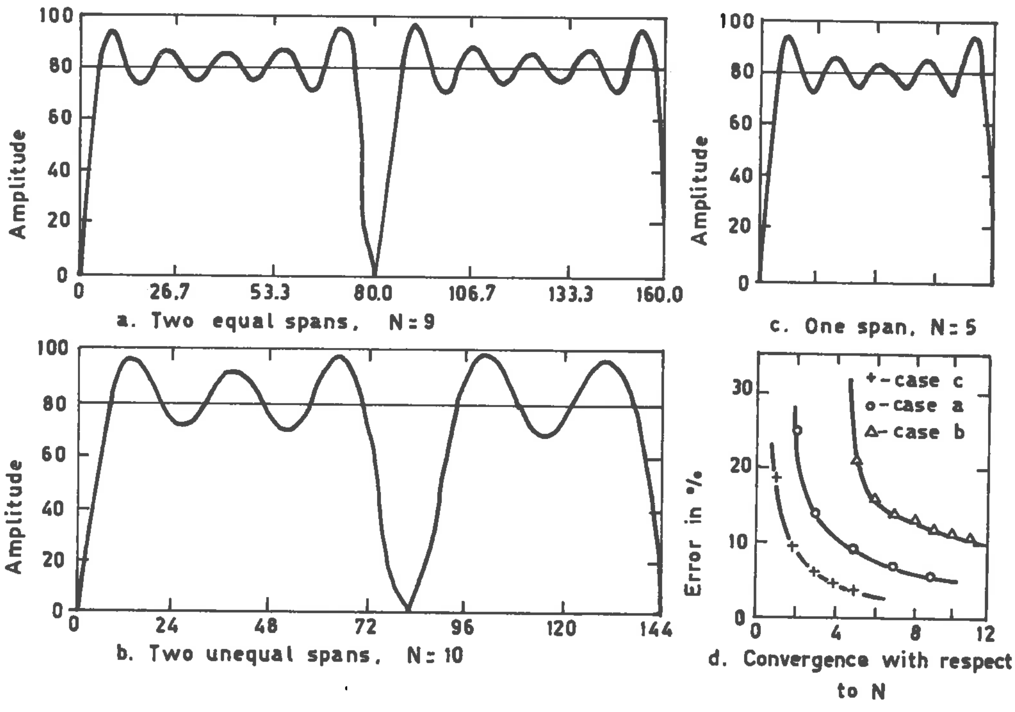


Fig. 2 Approximation of a uniform load of 80 kg/m on three different beams, with the series $\sum_N Y_{sn}$

Numerical Examples

The results for two-span continuous structures with simply supported ends and for a single-span structure with overhangs (treated as three-span structure in the finite strip analysis) are presented in this section.

The first example consists of a continuous folded plates structure initially studied by Beaufait [1] using the ordinary theory and subsequently by Scordelis and Lo [3,9], using both the elasticity theory and a finite segment ordinary method. The dimensions and loadings of the structures, which has two unequal spans, are shown in Fig. 3. The modulus of elasticity is taken as 25.25 x 10^6 kN/m^2 and Poisson's ratio as zero. The number of terms used in the series will be equal to 10 unless otherwise specified. Two different meshes using 9 and 18 strips for half of the section have been used in the analysis. Little difference can be detected in the results except for transverse bending moment; for this reason, results for the two meshes are only given here for the transverse bending moment.

The longitudinal variations of the vertical deflections, the longitudinal stresses and the transverse bending moments at B and D are listed in Table 1, 2, and 3 respectively, while the transverse distribution of longitudinal stresses and transverse bending moments for mid-section of span 1 is shown in Fig. 4a and 4b. From Tables 1, 2 and 3 it can be concluded that the agreement between the finite strip results and the values from ref.[9] (in particular those due to the elasticity method) is very good and the only point worth commenting concerns the transverse bending moments, in which a marked discrepancy exists for the case of the ordinary theory because of its one-way slab assumption.

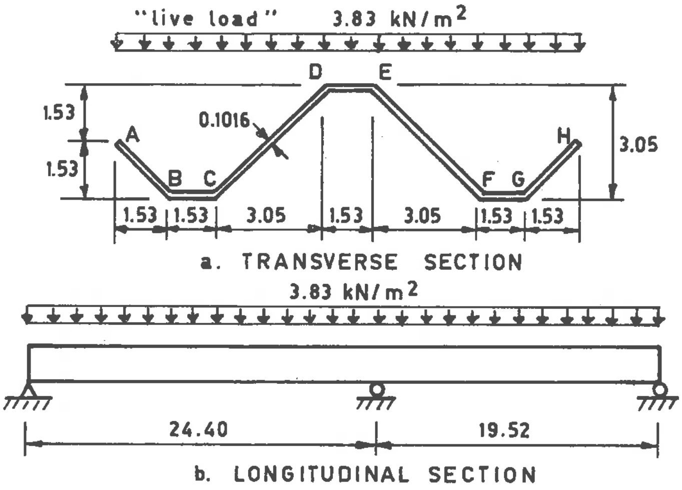


Fig.3 Dimensions (in m) and loading for Example 1

For completeness, the transverse distribution of shear stresses τ_{xy} at the left end support and at the intermediate support is shown in Fig. 5, These curves were obtained by extropoloting the values of τ_{xy} between the supports. Finally, in order to study the convergence of the results, the values of the vertical deflection at joint A and the longitudinal stress and transverse moment at joint B (mid-section of span 1) are given in Table 4 for N = 7, 8, 9 and 10. It can be observed that for N > 9, the convergence becomes very slow, as predicted by the test proposed previously.

Table 1

Comparison of vertical deflections, in metres times 10^4

x in metres	finite strip method 9 strips		finite segment ordinary theory		elasticity theory	
	B	D	B	D	B	D
3.66	42.40	17.39	47.28	17.69	42.70	18.00
8.54	76.25	31.72	84.49	32.64	77.47	32.64
10.98	80.52	33.55	88.45	34.77	81.74	34.47
13.42	75.34	31.42	82.96	32.94	76.86	32.64
18.30	44.23	18.91	48.50	20.13	45.45	19.83
30.50	15.56	3.36	20.13	3.97	16.47	3.97
35.38	28.37	7.32	35.38	7.63	29.59	7.93
37.82	27.15	7.32	33.55	7.63	27.76	7.93

Table 2

Comparison of longitudinal stresses (kN/m^2)

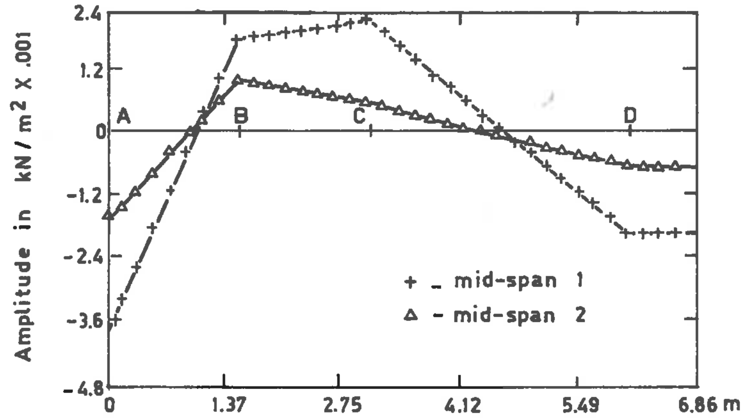
x in metres	finite strip method 9 strips		finite segment ordinary theory		elasticity theory	
	B	D	B	D	B	D
3.66	1311	-1242	1477	-1201	1311	-1256
10.98	1932	-2001	2056	-2015	1960	-2056
13.42	1691	-1704	1801	-1753	1739	-1787
18.30	476	- 400	518	- 469	518	- 483
23.18	-2360	1932	-2491	1691	-2256	1739
24.40	-3450	2691	-3636	2346	-3436	2843
25.62	-2415	2001	-2539	1801	-2312	1849
30.50	207	69	297	83	255	55
37.82	1256	- 897	1470	- 842	1263	- 918

Table 3
Comparison of transverse moments (kg-m/m)

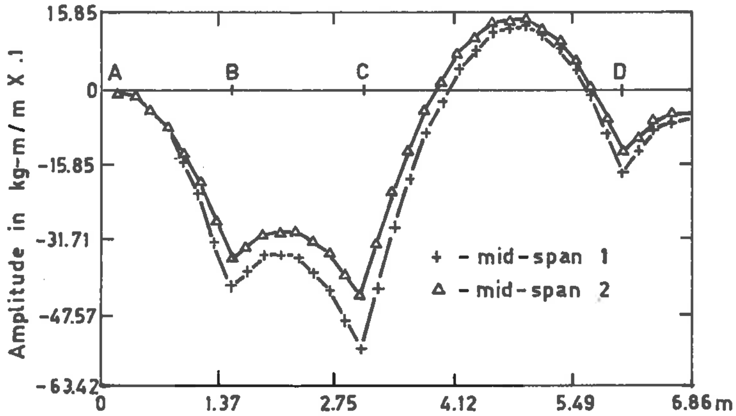
x in metres	finite strip method 9 strips		finite strip method 18 strips		finite segment ordinary theory		elasticity theory	
	B	D	B	D	B	D	B	D
1.22	-149	-159	-149	-168	-453	-225	-202	-150
3.66	-334	-183	-362	-195	-453	-178	-336	-195
10.98	-378	-154	-396	-163	-453	-128	-397	-169
18.30	-381	-199	-399	-208	-453	-190	-398	-205
23.18	- 63	- 72	- 63	- 91	-453	-251	-132	-105
25.62	- 59	- 68	- 59	- 91	-453	-245	-130	-101
30.50	-381	-190	-394	-199	-453	-168	-391	-187
37.82	-358	-172	-376	-181	-453	-135	-372	-171
42.70	-104	-113	-104	-127	-453	-221	-201	-146

Table 4
Influence of "n" on the deflections, stresses and moments (example 1)

value at midspan l \ n	7	8	9	10
w (metres) at A	149×10^{-4}	151×10^{-4}	153×10^{-4}	153×10^{-4}
σ_x (kN/m^2) at B	30.37	37.60	46.46	46.27
σ_y (kN/m^2) at B	1787	1811	1839	1839
M_x (kg-m/m) at B	-339	-361	-390	-390
M_y (kg-m/m) at B	19.88	20.43	21.47	21.52

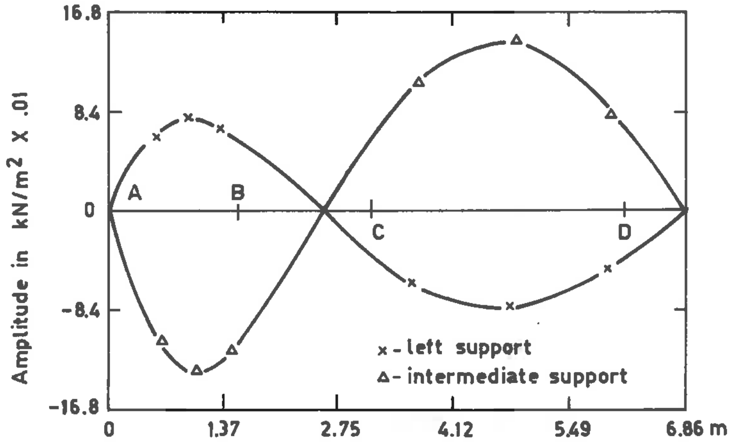


a. Longitudinal stress, mid-span 1 and 2



b. Transverse moment, mid-span 1 and 2

Fig. 4 Example 1. Transverse distribution of longitudinal stress and transverse moment



Shear stress, left and intermediate supports

Fig. 5 Example 1. Transverse distribution of shear stress

In order to assess the efficiency of the finite strip analysis, two finite element solutions using 9 x 9 and 9 x 18 meshes of flat shell elements [19] for half of the structure are carried out, and a comparison of number of degrees of freedom and CPU time used are presented in Table 5. It should be mentioned that while the deflections differ by only 1% (maximum) between the three sets of results, the difference in the stresses and moments amounts to 15% (maximum) between the coarse mesh (9 x 9) and finite strip results, and 5% (maximum) between the fine mesh (9 x 18) and finite strip results. Thus it can be concluded that for comparable accuracies, the saving in CPU time by using the finite strip solution is very significant.

Table 5
Comparison between finite element and finite strip results (example 1)

	number of degrees of freedom	number of points where displacements are given	number of points where stresses are given	CPU time in sec. (*)
finite strip n = 10	400	180	270	120
finite element mesh 9 x 9	1320	100	81	342
finite element mesh 9 x 18	2544	190	162	1206

(*) The computer used is a CDC-6400

The second example corresponds to model 3 from Beaufait's paper [18], in which experimental as well as theoretical (ordinary theory) results are available. The dimensions and the loadings are shown in Fig. 6, and the modulus of elasticity is taken as 73.14×10^6 kN/m^2 and Poisson's ratio as 0.33. Because the two spans are equal, only 9 terms of the series are used for the analysis. Twenty finite strips are used for half of the structure because of symmetry.

The transverse distribution of the longitudinal stress and the transverse bending moment at mid-span is shown in Fig. 7(a,b), and the values of transverse bending moment and the longitudinal stress at point H are given at in Table 6 for several different cross-section locations. In general, the finite strip results are much closer to the experimental values when compared with Beaufait's theoretical results.

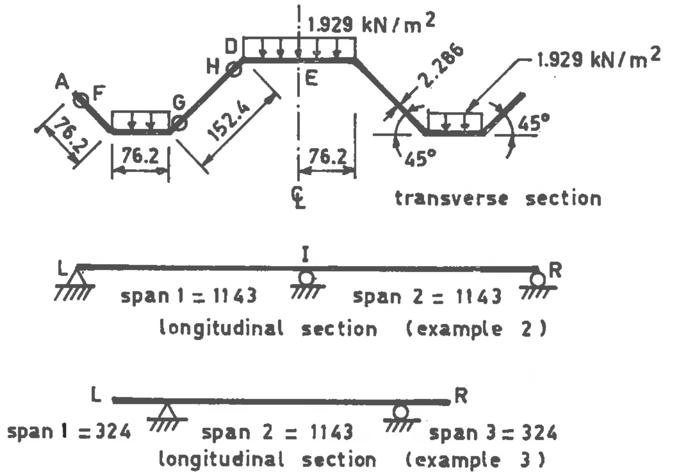


Fig. 6. Dimensions (in mm) and loading for Examples 2 and 3

Table 6

Longitudinal Stresses and Transverse Moments at point H (example 2), at different cross-sections

Section	σ_y (kN/m^2)			M_x (kg-m/m)		
	Beaufait experiment	Beaufait theory	finite strip method	Beaufait experiment	Beaufait theory	finite strip method
Mid-Span 1	- 656	- 704	- 718	-0.207	-0.205	-0.210
Mid-Span 2	- 738	- 704	- 718	-0.192	-0.205	-0.210
52.4 mm from centre support	-1642	-1428	-1559	-0.116	-0.218	-0.102
155.6 mm from centre support	490	538	504	-0.210	-0.215	-0.216

The third example corresponds to model 2 from the same paper by Beaufait, and here the versatility of the finite strip method over that of the elasticity method is amply demonstrated, since the latter method cannot be applied to the present example which has fairly long overhangs. The dimensions and loadings are shown in Fig. 6, and the material properties are the same as those given for example 2. Ten finite strips are used to represent half of the structure, and eight terms of the series (determined from the load convergence test) are used in the analysis. In addition to the finite strip solution, a finite element analysis using 70 flat shell elements [19] for a quarter of the structure was also carried out. All results are presented in Table 7, and once again the finite strip results agree very well with the experimental and numerical values. The discrepancy of the longitudinal stresses at the support was attributed by Beaufait to local effects caused by the supporting diaphragm.

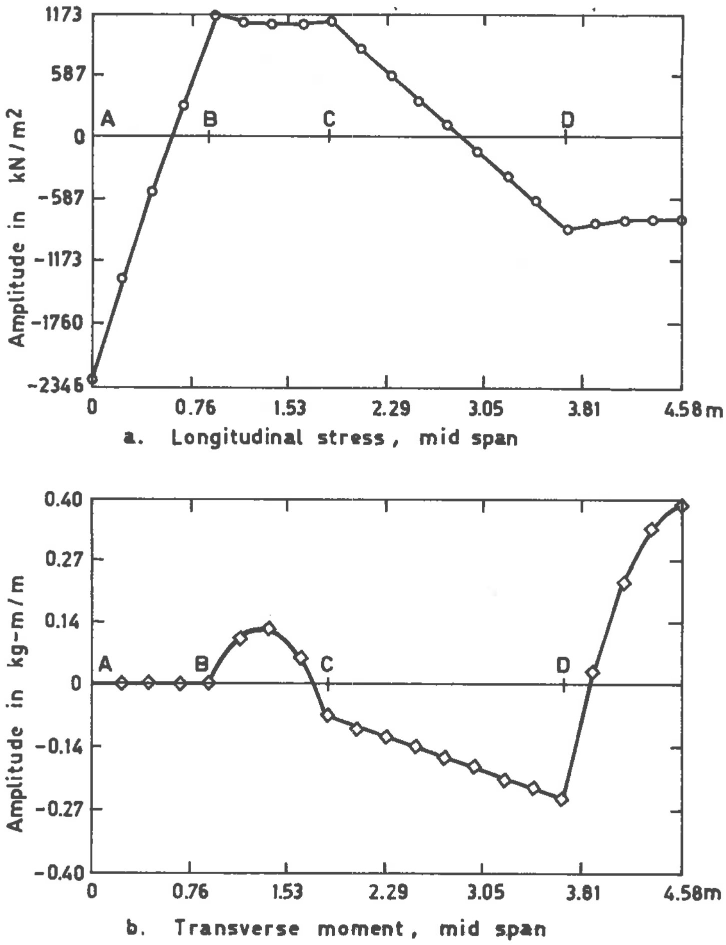


Fig. 7 Example 2. Transverse distribution of longitudinal stress and transverse moment

Conclusion

The finite strip method has been generalized to study multi-span structures with any type of boundary conditions. The number of terms necessary for the finite strip analysis is dependent on the considered structure, and can be determined by a simple test of load convergence. Three numerical examples are presented and the results are compared with numerical and experimental results available in the literature. The finite strip method produces values which agree very well with the experimental results. Finally, a comparison between finite element method and finite strip method is presented. The conclusion is that for such continuous structures the finite strip method is much more economical than the finite element method.

Table 7(a)
Longitudinal stresses (kN/m^2) (Example 3), at different cross-sections

		Beaufait theoretical	Beaufait experimental	Finite strip	Finite element
Mid-span	point F	-2525	-2539	-2567	-2456
Mid-span	point G	1408	1477	1490	1497
Mid-span	point H	- 973	-987, -904 (*)	- 932	- 994
**	point G	490	490	469	483
At support	point G	- 276	166	- 311	- 345
At support	point H	373	407	380	414

(*) different values measured at symmetric points

** 155.6 from support in main span

Table 7(b)
Transverse moments (kg-m/m) Example 3 at point G

	Beaufait theoretical	Beaufait experimental	Finite strip	Finite element
Section mid-span	-0.0589	-0.0580,-0.0874*	-0.0680	-0.0702
Section 155.6 mm from support main span	-0.0145	-0.0072	-0.0177	-0.0181

* different values measured at symmetric points

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INDEX TO PUBLICATIONS

Section III – General Topics

(5 publications)

All publications are included in the print copy of the thesis held in the University of Adelaide Library and unless indicated have been removed due to copyright regulations

1. Cheung, Y. K. (1975, July). *Numerical analysis of pavements.* Paper presented at the Symposium on recent developments in the analysis of soil behaviour and their applications to geotechnical structures, University of New South Wales.

2. Cheung, Y. K. (1976, December). *Finite strip method and its applications.* Paper presented at the International conference on finite element methods in civil engineering, University of Adelaide.

3. Cheung, Y.K. (1979). Tall buildings - analysis and design. *Hong Kong Engineer, May*, 31-39.

4. Cheung, Y. K., & Tham, L. G. (1979, April). *On the numerical solution of certain initial value problems*. Paper presented at the 3rd International Conference on Numerical Methods in Geomechanics, Aachen, Germany.

5. Jones, R., Mazumdar, J. & Cheung, Y. K. (1980). Vibration and buckling of plates at elevated temperatures. International Journal of Solids and Structures, 16(1), 61-70. doi: https://doi.org/10.1016/0020-7683(80)90095-5