The Theory of Inconsistency

Inconsistent Mathematics and Paraconsistent Logic

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PART 2

Papers on Inconsistent Mathematics

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INCONSISTENT MODELS FOR RELEVANT ARITHMETICS[1]

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§1. Introduction. This paper develops in certain directions the work of Meyer in [3], [4], [5] and [6] (see also Routley [10] and Asenjo [11]). In those works, Peano's axioms for arithmetic were formulated with a logical base of the relevant logic R, and it was proved finitistically that the resulting arithmetic, called $R^{\#}$, was absolutely consistent. It was pointed out that such a result escapes incautious formulations of Gödel's second incompleteness theorem, and provides a basis for a revived Hilbert programme. The absolute consistency result used as a model arithmetic modulo two. Modulo arithmetics are not ordinarily thought of as an extension of Peano arithmetic, since some of the propositions of the latter, such as that zero is the successor of no number, fail in the former. Consequently a logical base which, unlike classical logic, tolerates contradictory theories was used for the model. The logical base for the model was the three-valued logic $RM3$ (see e.g. [1] or [8]), which has the advantage that while it is an extension of R, it is finite valued and so easier to handle.

The resulting model-theoretic structure (called in this paper $RM3^2$) is interesting in its own right in that the set of sentences true therein constitutes a negation inconsistent but absolutely consistent arithmetic which is an extension of $R^{\#}$. In fact, in the light of the result of [6], it is an extension of Peano arithmetic with a base of a classical logic, $P^{\#}$. A generalisation of the structure is to modulo arithmetics with the same logical base $RM3$, but with varying moduli (called $RM3^i$ here). We first study the properties of these arithmetics in this paper. The study is then generalised by varying the logical base, to give the arithmetics RMn^i, of logical base RMn and modulus i. Not all of these exist, however, as arithmetical properties and logical properties interact, as we will show. The arithmetics RMn^i give rise, on intersection, to an inconsistent arithmetic RM^{ω} which is not of modulo i for any i. We also study its properties, and, among other results, we show by finitistic means that the more natural relevant arithmetics $R^{\#}$ and $R^{\#\#}$ are incomplete (whether or not consistent and recursively enumerable). In the rest of the paper we apply these techniques to several topics, particularly relevant quantum arithmetic in which we are able to show (unlike classical quantum arithmetic) that the law of distribution remains unprovable. Aside from its intrinsic interest, we regard the present exercise as a demonstration that inconsistent theories and models are of mathematical worth and interest.

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§2. Definitions. We begin with an arithmetical language $\mathscr{L}$ with a single binary relation $=$, constant 0, variables $x, y, z, \ldots$, connectives $\neg$, &, $\rightarrow$, term operators $+$, $\times$, $'$, and universal quantifier ( ) (also written $(\forall)$). Terms (open and closed) and sentences and wffs (open and closed) are defined in the usual way.[2] $\exists$ is defined as $\sim\forall\sim$, $A \vee B$ is defined as $\sim(\sim A \,\&\, \sim B)$, and $0^{(n)}$ is an abbreviation for $0''^{\cdots}{}'$, with n iterations of the superscript.

DEFINITION 1. *The relevant logic RQ* is given by the following axiom schemata and rules. *Axioms*: (1) $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$, (2) $A \rightarrow ((A \rightarrow B) \rightarrow B)$, (3) $A \,\&\, B \rightarrow A$, (4) $A \,\&\, B \rightarrow B$, (5) $(A \rightarrow B) \,\&\, (A \rightarrow C) \rightarrow (A \rightarrow B \,\&\, C)$, (6) $A \rightarrow A \vee B$, (7) $B \rightarrow A \vee B$, (8) $(A \rightarrow C) \,\&\, (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$, (9) $A \,\&\, (B \vee C) \rightarrow (A \,\&\, B) \vee (A \,\&\, C)$, (10) $\neg\neg A \rightarrow A$, (11) $(A \rightarrow \neg A) \rightarrow \neg A$, and the following quantificational schemata and all closures of them: (12) $(x)\, A \rightarrow A(t/x)$ (t any term), (13) $(x)(A \rightarrow B) \rightarrow ((x)A \rightarrow (x)B)$, (14) $A \rightarrow (x)A$ (x not free in A), (15) $(x)(A \vee B) \rightarrow (A \vee (x)B)$ (x not free in A), (16) $(x)A \,\&\, (x)B \rightarrow (x)(A \,\&\, B)$. *Rules*: (17) $\vdash A$, $\vdash A \rightarrow B / \therefore \vdash B$, (18) $\vdash A, \vdash B / \therefore \vdash A \,\&\, B$. To obtain *the logic RMQ*, add the Mingle axiom $A \rightarrow (A \rightarrow A)$. The *relevant arithmetics* $R^{\#}$ and $RM^{\#}$ in the language of $\mathscr{L}$ are given by the following. Logical axioms and rules are those of R and RM respectively. Both have additional arithmetical axioms and rules:

Arithmetical axioms. #1 $(x, y)(x = y \leftrightarrow x' = y')$, #2 $(x, y, z)(x = y \rightarrow (x = z \rightarrow y = z))$, #3 $(x)(x' \neq 0)$, #4 $(x)(x + 0 = x)$, #5 $(x, y)(x + y' = (x + y)')$, #6 $(x)(x \times 0 = 0)$, #7 $(x, y)(x \times y' = (x \times y) + x)$.

Arithmetical rule, RMI. $\vdash F0$, $\vdash (x)(Fx \rightarrow Fx') / \therefore \vdash (x)Fx$.

To obtain *the arithmetics* $R^{\#\#}$ *and* $RM^{\#\#}$, add to $R^{\#}$ and $RM^{\#\#}$ respectively the rule Ω: $\vdash F0, \vdash F0', \ldots, \vdash F0^{(n)}, \ldots$ (for all n)$/ \therefore \vdash (x)Fx$. *The arithmetics* $P^{\#}$ *and* $P^{\#\#}$ are Peano arithmetic with a base of classical logic and, respectively, without and with rule Ω.

DEFINITION 2. *An* $RM3^i$ *model* is an ordered pair $\langle D^i, I\rangle$ where D^i are the integers modulo i, and I is a function which assigns to the terms, operators and wffs of $\mathscr{L}$ the following values. (1) For individual variables x, $I(x) \in D^i$. (2) $I(0) = 0$. (3) $I(+)$, $I(\times)$, $I(')$ are the operations $+$, $\times$, $'$ of arithmetic modulo i respectively. (4) For terms t_1, t_2, $I(t_1 + t_2) = I(+)(I(t_1), I(t_2))$; $I(t_1 \times t_2) = I(\times)(I(t_1), I(t_2))$, $I(t_1') = I(')I(t_1)$. Open and closed wffs are assigned values in the three-valued Sugihara matrix $\{+1, 0, -1\}$ characteristic for $RM3$ as follows. (5) For atomic wffs $t_1 = t_2$, where t_1, t_2 are terms, $I(t_1 = t_2) = 0$ iff $I(t_1) = I(t_2)$ (modulo i), otherwise $I(t_1 = t_2) = -1$. (6) $I(A \rightarrow B)$, $I(A \,\&\, B)$ and $I(\neg)$ are determined by the three-valued tables for $RM3$. (7) $I((x)A) = \text{glb}\{y : I'(A) = y\}$ for every x-variant I' of I. A sentence A is $RM3^i$*-true under interpretation* I iff $I(A) \in \{+1, 0\}$. A *is true in the model* $RM3^i$ iff A is $RM3^i$-true under all interpretations I. *The arithmetic* $RM3^i$ is the set of sentences true in the model $RM3^i$.

DEFINITION 3. A subset S of $\mathscr{L}$ *is an L-theory* (relative to logic L) iff (1) if $A \in S$ and $\vdash_L A \rightarrow B$ then $B \in S$, and (2) if $A \in S$ and $B \in S$ then $A \,\&\, B \in S$. An L-theory S is (negation) *consistent* iff for all closed wffs $A \in \mathscr{L}$, not both $A \in S$ and $\neg A \in S$; otherwise *inconsistent*. S is *trivial* (or *absolutely inconsistent*) iff $S = \mathscr{L}$, otherwise *nontrivial*. S is *ω-inconsistent* iff for some F, $F0^{(n)} \in S$ for every n, and also

[2] We occasionally use $Fx_1 \ldots x_n$ for an open formula in which $x_1 \ldots x_n$ may occur free.

$\sim(x)Fx \in S$. S is *complete* iff for all closed wffs A, either $A \in S$ or $\neg A \in S$. S is *ω-complete* iff the rule Ω holds for S, i.e. for all F, if $F0^{(n)} \in S$ for every n, then $(x)Fx \in S$. *The rule γ holds for* S iff, if $A \in S$ and $\sim A \vee B \in S$, then $B \in S$. S is *prime* iff for all closed wffs A, B if $A \vee B \in S$ then at least one of $A \in S$, $B \in S$.

§3. The arithmetics $RM3^i$.

PROPOSITION 1. *The arithmetic $RM3^2$ is an RQ theory. Every theorem of $R^{\#}$ is true in the model $RM3^2$. The wff $0 = 0'$ is not true in that model. Hence $R^{\#}$ is absolutely consistent.*

PROOF. See [3].

The significance of this result is, as discussed in [3], [4] and [5], that relevant Peano arithmetic can be finitistically proved to be absolutely consistent, unlike classical Peano arithmetic $P^{\#}$. This goes some way to resurrecting the programme of Hilbert of a finitistic demonstration of the (absolute) consistency of mathematics. However, every primitive recursive function is representable in $R^{\#}$, so a proof of the *negation* consistency of $R^{\#}$ within $R^{\#}$ remains impossible, by Gödel's second incompleteness theorem. It is also provable that $P^{\#\#}$ is included in $R^{\#\#}$ (see [6]) so that, since $P^{\#} \subset P^{\#\#}$ and $R^{\#\#} \subset RM3^2$, we do have a proof of the absolute consistency of $P^{\#}$ (and hence of its negation consistency, which coincides with absolute consistency in $P^{\#}$ but not $R^{\#\#}$). This does not run contrary to Gödel's second theorem, however, since one step in the proof, viz. $P^{\#\#} \subset R^{\#\#}$, uses methods not representable in $R^{\#}$. Again, the negation consistency of $R^{\#\#}$ (and hence $R^{\#}$) is demonstrable (see [6]), but the result uses equally strong methods. These results also constitute an argument for the pragmatic virtues of the relevant logics. In fact, as we show later by finitistic methods, no sentence of the form $0^{(n)} = 0^{(m)}$ for distinct n, m is provable in any of $R^{\#}$, $R^{\#\#}$, $RM^{\#}$, $RM^{\#\#}$. (See also [3].) Hence relevant logic has considerably better control over its Peano arithmetic than does classical logic. We go on now to consider the properties of the arithmetics $RM3^i$.

PROPOSITION 2. *Every $RM3^i$ is inconsistent and ω-inconsistent but absolutely consistent.*

PROOF. $0 = 0'$ is easily seen not to be in any $RM3^i$, so all are absolutely consistent. But all are inconsistent, since (e.g.) in any model $RM3^i$ $I(0 = 0) = 0$, so by the $RM3$ tables for negation, $I(\neg(0 = 0)) = 0$ also, hence both are true. For ω-inconsistency, observe that any sentence of the form $\neg(0^{(n)} = 0^{(m)})$ takes value $+1$ or 0 in every $RM3^i$. So in particular $\neg(0 = 0), \neg(0 = 0'), \ldots, \neg(0 = 0^{(n)}), \ldots$ are all theorems. But the value of $\neg(0 = 0)$ is 0, so $(x)\neg(0 = x)$ takes a maximum value of 0, so $\neg(x)\neg(0 = x)$ takes a minimum value of 0 (by tables for $\neg$), and so is a theorem.

PROPOSITION 3. *For any F, $\vdash_{RM3^i} (F0 \,\&\cdots\&\, F0^{(i-1)}) \rightarrow (x)Fx$.*

PROOF. If $F0 \,\&\cdots\&\, F0^{(i-1)}$ takes the value $+1$, then by all the $RM3$ tables for $\rightarrow$, each conjunct does. Any assignment to the variable in Fx must therefore also give the value $+1$, whence $(x)Fx$ takes that value. If the antecedent takes the value 0, each conjunct takes the value 0 or 1, so again any assignment to the variable in Fx takes value 0 or 1 (and 0 for some assignment). Hence $(x)Fx$ takes value 0, and so by the $RM3$ tables, $(F0 \,\&\cdots\&\, F0^{(i-1)}) \rightarrow (x)Fx$ takes value 0. If the antecedent takes value -1, the whole wff takes value $+1$.

PROPOSITION 4. *$RM3^i$ is ω-complete.*

PROOF. If $\vdash F0, \ldots \vdash F0^{(n)}, \ldots$ for every n, then $\vdash F0, \ldots, \vdash F0^{(i-1)}$; so that $\vdash F0 \,\&\, \cdots \,\&\, F0^{(i-1)}$. Hence by Proposition 3 and the fact that each RM^i is closed under modus ponens for $\rightarrow$, $\vdash (x)Fx$.

PROPOSITION 5. *$RM3^i$ contains $R^{\#}$, $R^{\#\#}$, $RM^{\#}$, $RM^{\#\#}$.*

PROOF. As in [2], this is a matter of verifying that the axioms of $R^{\#}$ etc. are true in $RM3^i$ and that the rules preserve truth. That Ω is truth preserving follows from Proposition 4.

PROPOSITION 6. *$RM3^i$ is complete and prime.*

PROOF. Primeness follows from the $RM3$ tables for $\vee$: for closed wffs A, B, if $I(A \vee B) \in \{+1, 0\}$ one of $I(A)$, $I(B)$ must do so. Completeness is then immediate on the observation, easily checked, that $\vdash_{RM3^i} A \vee \neg A$.

PROPOSITION 7. *$RM3^i$ is decidable.*

PROOF. We observe first that it is standard to show that an open wff A is a theorem iff its universal closure is. By Proposition 3, however, all universal quantifiers in closed wffs are eliminable in favour of finite conjunctions. Since the value of atomic wffs $0^{(n)} = 0^{(m)}$ is effectively given, the value of all universally closed formulas, conjunctions and negations can be calculated.

§4. **Axiomatising $RM3^i$.** The purpose of this section is to axiomatise the arithmetics $RM3^i$. We proceed via a number of lemmas.

DEFINITION 4. $RM3^{i\#}$ is the arithmetic obtained by adding to $RM^{\#}$ the axioms $0 = 0^{(i)}$, and for every j, $2 \le j < i$, $0 = 0^{(j)} \leftrightarrow 0 = 0'$.

LEMMA 1. *In $RM3^{i\#}$, $\vdash (x, y, z_1, \ldots, z_n)(x = y \,\&\, Fxz_1 \cdots z_n \rightarrow Fyz_1 \cdots z_n)$.*

PROOF. In [4] or [5].

LEMMA 2. *In $RM3^{i\#}$, $\vdash (x)(x = 0 \vee x = 0' \vee \cdots \vee x = 0^{(i-1)})$.*

PROOF. The proof uses the rule RMI (Definition 1). Plainly we have $\vdash 0 = 0 \vee 0 = 0' \vee \cdots \vee 0 = 0^{(i-1)}$. But also we have, from axiom #1, $\vdash (x)(x = 0 \rightarrow x' = 0')$, $\vdash (x)(x = 0' \rightarrow x' = 0^{(2)}), \ldots, \vdash (x)(x = 0^{(i-2)} \rightarrow x' = 0^{(i-1)}$. Also, using $0 = 0^{(i)}$ and Lemma 1, $\vdash (x)(x = 0^{(i-1)} \rightarrow x' = 0)$. Hence by RMQ principles, $\vdash (x)(x = 0 \vee \cdots \vee x = 0^{(i-1)} \rightarrow x' = 0 \vee \cdots \vee x' = 0^{(i-1)})$. The lemma follows using rule RMI.

LEMMA 3. *In $RM3^{i\#}$, $\vdash (F0 \,\&\, \cdots \,\&\, F0^{(i-1)}) \leftrightarrow (x)Fx$, if x is not free in F.*

PROOF. Right to left is immediate from axiom (12). Left to right: using Lemma 1 and RMQ principles, we have

$$\vdash (x)((x = 0 \vee \cdots \vee x = 0^{(i-1)}) \,\&\, F0 \,\&\, \cdots \,\&\, F0^{(i-1)} \rightarrow Fx).$$

Distributing the quantifier,

$$\vdash (x)(x = 0 \vee \cdots \vee x = 0^{(i-1)}) \,\&\, (x)(F0 \,\&\, \cdots \,\&\, F0^{(i-1)}) \rightarrow (x)Fx.$$

The first conjunct of the antecedent is a theorem. Hence if x is not free in F, we have from axiom (14) and rule (18), the derived rule D: $\vdash F0, \ldots, \vdash F0^{(i-1)} / \therefore \vdash (x)Fx$. But now we also have $\vdash F0 \,\&\, \cdots \,\&\, F0^{(i-1)} \rightarrow F0, \ldots, \vdash F0 \,\&\, \cdots \,\&\, F0^{(i-1)} \rightarrow F0^{(i-1)}$. Hence by rule D, $\vdash (x)(F0 \,\&\, \cdots \,\&\, F0^{(i-1)} \rightarrow Fx)$. Distributing quantifiers and using axiom (14) we have $\vdash F0 \,\&\, \cdots \,\&\, F0^{(i-1)} \rightarrow (x)Fx$ as required.[3]

[3] We also note that it is straightforward to show that provable equivalences are replaceable in all contexts. See [4] or [5].

DEFINITION 5. Let t be $0 = 0$ and f be $\neg 0 = 0$.

LEMMA 4. *In* $RM3^{i\#}$, $0 = 0'$, $\neg 0 = 0'$; t *and* f *are related as in the* $RM3$ *three-valued Sugihara chain below, with the & and* $\rightarrow$ *relations between them as for* $RM3$.

$$\begin{array}{rr|l} * & +1 & \neg 0 = 0' \\ * & 0 & t, f \\ & -1 & 0 = 0' \end{array}$$

PROOF. The proof, essentially verifying theorems in $RM3^{i\#}$, is lengthy but not difficult. Details are omitted.

LEMMA 5. *In* $RM3^{i\#}$, *for any wff* A, *we have either* $\vdash A \leftrightarrow 0 = 0'$ *or* $\vdash A \leftrightarrow 0 = 0$ *or* $\vdash A \leftrightarrow \neg 0 = 0'$.

PROOF. By induction on the complexity of A. We need only consider the case where A is closed, since the case where A is open follows by axiom (12). Furthermore the quantifier case in the inductive clause can be ignored since by Lemma 3, quantifiers can be eliminated in favour of a conjunction.

BASE. A is $t_1 = t_2$. In view of Lemmas 1 and 2, we need only consider where $t_1, t_2 \in \{0, 0', \ldots, 0^{(i-1)}\}$. First, let A be $0^{(n)} = 0^{(n)}$ for some n, $0 \leq n \leq i - 1$. Now certainly $\vdash 0 = 0 \leftrightarrow 0 = 0$; but also $\vdash 0' = 0' \leftrightarrow 0 = 0, \ldots, \vdash 0^{(i-1)} = 0^{(i-1)} \leftrightarrow 0 = 0$, by repeated applications of axiom #1. If A is $0^{(n)} = 0^{(m)}$ where $n < m \leq i - 1$, then we have $\vdash 0 = 0^{(|n-m|)} \leftrightarrow 0' = 0^{(|n-m|+1)}$, $\vdash 0' = 0^{(|n-m|+1)} \leftrightarrow 0^{(2)} = 0^{(|n-m|+2)}, \ldots;$ $\vdash 0^{(n-1)} = 0^{(m-1)} \leftrightarrow 0^{(n)} = 0^{(m)}$, whence $\vdash 0 = 0^{|(n-m)|} \leftrightarrow 0^{(n)} = 0^{(m)}$. But it is an axiom that $\vdash 0 = 0' \leftrightarrow 0 = 0^{(|n-m|)}$. Hence $\vdash 0 = 0' \leftrightarrow 0^{(n)} = 0^{(m)}$. If $n > m$, we have $\vdash 0^{(n)} = 0^{(m)} \leftrightarrow 0^{(m)} = 0^{(n)}$ via the symmetry of identity (which follows from #2), which reduces the case to the previous one.

Inductive clause. (a) Negations. If $\vdash A \leftrightarrow 0 = 0'$ or $\vdash A \leftrightarrow 0 = 0$ or $\vdash A \leftrightarrow \neg 0 = 0'$, then evidently $\vdash \neg A \leftrightarrow \neg 0 = 0'$ or $\vdash \neg A \leftrightarrow 0 = 0$ or $\vdash \neg A \leftrightarrow 0 = 0$. The second disjunct uses Lemma 4. (b) Conjunctions. We have $\vdash A \leftrightarrow 0 = 0'$ or $\vdash A \leftrightarrow 0 = 0$ or $\vdash A \leftrightarrow \neg 0 = 0'$, and $\vdash B \leftrightarrow 0 = 0'$ or $\vdash B \leftrightarrow 0 = 0$ or $\vdash B \leftrightarrow \neg 0 = 0'$. We have to show $\vdash A \mathbin{\&} B \leftrightarrow 0 = 0'$ or $\vdash A \mathbin{\&} B \leftrightarrow 0 = 0$ or $\vdash A \mathbin{\&} B \leftrightarrow \neg 0 = 0'$. Evidently these follow from the equivalences of Lemma 4. (c) Implications. These follow, in a like fashion, from the equivalences of Lemma 4. This completes the lemma.

Lemma 5 means that adding to $RM^{\#}$ $0 = 0^{(i)}$ and $0 = 0^{(j)} \leftrightarrow 0 = 1$ for all j with $2 \leq j < i$ forces exactly three distinct propositions on us, two the negations of each other and the third its own negation. But that is precisely the structure of the logic $RM3$. Thus there would be no point in attempting a weaker logical extension of $RM3$ with these axions (though we should note that, in general, the $0 = 0^{(j)} \leftrightarrow 0 = 0'$ are not all independent) since only three of its values would be used, isomorphically with $RM3$. Lemma 5 also enables us to axiomatise the models $RM3^i$.

PROPOSITION 8. *The theorems of* $RM3^{i\#}$ *are exactly the truths of* $RM3^i$.

PROOF. That all theorems of $RM3^{i\#}$ are truths of $RM3^i$ is a standard matter of showing that the (extra) axioms are true, since we already have that $RM^{\#}$ is verified in $RM3^i$. Lemma 5 does all the work in the converse. For suppose A is true in $RM3^i$. Now certainly not $\vdash A \leftrightarrow 0 = 0'$, for, since all theorems of $RM3^{i\#}$ are true, $A \leftrightarrow 0 = 0'$ would be true, so that $0 = 0'$ would be true (by a simple argument). Hence $\vdash A \leftrightarrow \neg 0 = 0'$ or $\vdash A \leftrightarrow 0 = 0$. But $\vdash \neg 0 = 0'$ and $\vdash 0 = 0$. Hence $\vdash A$.

§5. **Extending the logic: the arithmetics RMn^i.** When we move to consider weakening the logical base to the logics RMn (n odd: an even n will not permit an inconsistent arithmetic) characterised by the n-valued Sugihara matrices or chains, we run into the problem that not every modulus will permit distinct false equations to occupy distinct false points in the matrix, enabling us to take advantage of the logical resources of the weaker logic (longer chain). In fact, we prove that given the logical apparatus of RM, if j and k are relatively prime, then at least one of $0 = 0^{(j)}$ and $0 = 0^{(k)}$ is equivalent to $0 = 0'$. To show this, we prove something quite a bit stronger which is of independent interest.

PROPOSITION 9. *In* $R^{\#}$, *for any* i, j *we have* $\vdash (0 = 0^{(i)} \circ 0 = 0^{(j)}) \leftrightarrow 0 = 0^{(\gcd(i,j))}$, *where* $A \circ B_{\mathrm{df}} = \sim(A \rightarrow \sim B)$, *and* $\gcd(i, j)$ *is the greatest common divisor of* i *and* j.

PROOF. *Right to left.* $\vdash 0 = 0^{(i)} \rightarrow 0 = 0^{(ki)}$ (any k, i). (Reason: $\vdash 0 = 0^{(i)} \rightarrow . 0^{(i)} = 0^{(2i)} \rightarrow 0 = 0^{(2i)}$, but $\vdash 0 = 0^{(i)} \leftrightarrow 0^{(i)} = 0^{(2i)}$; hence $\vdash 0 = 0^{(i)} \rightarrow 0 = 0^{(2i)}$. Now $\vdash 0 = 0^{(i)} \rightarrow . 0^{(i)} = 0^{(3i)} \rightarrow 0 = 0^{(3i)}$. But $\vdash 0^{(i)} = 0^{(3i)} \leftrightarrow 0 = 0^{(2i)}$; so from $\vdash 0 = 0^{(i)} \rightarrow 0 = 0^{(2i)}$, $\vdash 0 = 0^{(i)} \rightarrow 0 = 0^{(3i)}$. And so on.) Hence, since $\gcd(i, j)$ divides both i and j, we have $\vdash 0 = 0^{\gcd(i,j)} \rightarrow 0 = 0^{(i)} \;\&\; 0 = 0^{(j)}$. But, $\vdash_R A \;\&\; B \rightarrow A \circ B$.

Left to right. First note that $\vdash_R A \rightarrow (B \rightarrow C) \leftrightarrow . (A \circ B) \rightarrow C$, so it suffices to prove $\vdash 0 = 0^{(i)} \rightarrow . 0 = 0^{(j)} \rightarrow 0 = 0^{(\gcd(i,j))}$. We now invoke the well-known arithmetical fact that the gcd of any pair i, j is expressible in the form $ki + lj$, where exactly one of the integers k, l is positive. Thus it suffices to prove that $\vdash 0 = 0^{(i)} \rightarrow . 0 = 0^{(j)} \rightarrow 0 = 0^{(ki+lj)}$, where $ki + lj$ is positive and only one of k, l is positive. Since we have permutation, ($\vdash_R A \rightarrow (B \rightarrow C) \rightarrow . B \rightarrow (A \rightarrow C)$, we may assume without loss of generality that it is k which is positive. Now we observe that $\vdash 0 = 0^{(i)} \rightarrow . 0 = 0^{(j)} \rightarrow 0 = 0^{(ki)}$. (Reason: (a) $\vdash 0 = 0^{(j)} \rightarrow 0 = 0$ follows from $\vdash 0 = 0^{(j)} \rightarrow . 0^{(j)} = 0 \rightarrow 0 = 0$ and $\vdash 0 = 0^{(j)} \rightarrow 0^{(j)} = 0$. (b) $\vdash 0 = 0 \rightarrow . 0 = 0^{(i)} \rightarrow 0 = 0^{(i)}$ follows from the transitivity of identity. Combining (a) and (b), $\vdash 0 = 0^{(j)} \rightarrow . 0 = 0^{(i)} \rightarrow 0 = 0^{(i)}$. But as already shown, $\vdash 0 = 0^{(i)} \rightarrow 0 = 0^{(ki)}$.) But now note that $\vdash 0 = 0^{(j)} \rightarrow . 0 = 0^{(ki)} \rightarrow 0 = 0^{(ki+lj)}$ (Reason: $\vdash 0 = 0^{(j)} \rightarrow . 0 = 0^{(ki)} \rightarrow 0^{(j)} = 0^{(ki)}$. Since $ki + lj$ is positive and $l < 0 < k$, $ki > j$. Therefore, we can write from axiom # 1—using the proof as in Lemma 5—$\vdash 0 = 0^{(j)} \rightarrow . 0 = 0^{(ki)} \rightarrow 0 = 0^{(ki+(-1)j)}$. Permuting, $\vdash 0 = 0^{(ki)} \rightarrow . 0 = 0^{(j)} \rightarrow 0 = 0^{(ki+(-1j)}$. But now $\vdash 0 = 0^{(j)} \rightarrow . 0 = 0^{(ki+(-1)j)} \rightarrow 0^{(j)} = 0^{(ki+(-1)j)}$. Replacing the last equation by its equivalent, $\vdash 0 = 0^{(j)} \rightarrow . 0 = 0^{(ki+(-1)j)} \rightarrow 0 = 0^{(ki+(-2)j)}$. So permuting back, $\vdash 0 = 0^{(j)} \rightarrow . 0 = 0^{(ki)} \rightarrow 0 = 0^{(ki+(-2)j)}$. The argument can obviously be repeated to get $\vdash 0 = 0^{(j)} \rightarrow . 0 = 0^{(ki)} \rightarrow 0 = 0^{(ki+(-3)j)}, \ldots, \vdash 0 = 0^{(j)} \rightarrow . 0 = 0^{(ki)} \rightarrow 0 = 0^{(ki+lj)}$, as required.) We can now combine these two using $\vdash_R (A \rightarrow . B \rightarrow C) \rightarrow ((B \rightarrow . C \rightarrow D) \rightarrow (A \rightarrow . B \rightarrow D))$, to get the desired result. This proves Proposition 9.

We also have immediately, that if i, j are relatively prime then $\vdash (0 = 0^{(i)} \circ 0 = 0^{(j)}) \leftrightarrow 0 = 0'$; and in particular that $\vdash 0 = 0^{(i)} \rightarrow . 0 = 0^{(j)} \rightarrow 0 = 0'$. Now it is possible to show in $RM^{\#}$ that, if $\nvdash 0 = 0^{(j)} \rightarrow 0 = 0'$, then $\vdash (0 = 0^{(j)} \rightarrow 0 = 0') \rightarrow 0 = 0'$ (see [4] or [5]). We may therefore deduce, as promised, that in $RM^{\#}$ if i and j are relatively prime, then one of $\vdash 0 = 0^{(i)} \rightarrow 0 = 0'$ and $\vdash 0 = 0^{(j)} \rightarrow 0 = 0'$.

These facts tell a lot of what there is to say about embedding arithmetic of modulus i into the RMn. For example, if i is prime then there are only three points occupied in RMn: $-j$ (at which are $0 = 0'$, $0 = 0''$, $\ldots$, $0 = 0^{(i-1)}$), 0 (at which are $0 = 0$, $0' = 0'$, $\ldots$, $0^{(k)} = 0^{(k)}$, all k, and their negations), and $+j$ (the negations of formulae at $-j$; $\{-j, 0, +j\}$ are closed in any RMn with respect to &, $\forall$ and $\to$ of course). This is isomorphic to $RM3^i$. Furthermore, if j is relatively prime to (any) modulus i, then $0 = 0^{(j)}$ must be equivalent to $0 = 0'$.

It is also immediate that if any two equations $0 = 0^{(j)}$, $0 = 0^{(k)}$ are at points other than that occupied by $0 = 0'$, then j, k have a common divisor > 1. Indeed the equation $0 = 0^{\gcd(j,k)}$ must also be at some point distinct from $0 = 0'$ (by Proposition 9 again, else one of $0 = 0^{(j)}$, $0 = 0^{(k)}$ implies $0 = 0'$). Further, the equation $0 = 0^{\gcd(j,k)}$ for each pair of points must be no greater than either (else $\vdash 0 = 0^{(j)} \to 0 = 0^{(kj)}$ fails to be verified). Moreover, there is a single common divisor distinct from one for all points distinct from $0 = 0'$: if there were two, they would have to have a common divisor distinct from $0 = 0'$. And that common divisor is at a point no greater than any other point distinct from $0 = 0'$.

Thus the RMn^i look like this. At the bottom is $0 = 0', 0 = 0'', \ldots$. The next point up on the Sugihara chain is occupied by a "base" equation $0 = 0^{(j)}$, and for every other equation $0 = 0^{(k)}$ higher up the chain (including the modular equation $0 = 0^{(i)}$) j is a divisor of k. If l is a multiple of k, k is no higher than l. Every multiple k of j occupies a point distinct from $0 = 0'$, while every nonmultiple of j is equivalent to $0 = 0'$. In addition we have to satisfy $\vdash 0 = \text{modulus} \to . 0 = x \to 0 = \text{modulus} - x$, and also $\vdash 0 = \text{modulus}$. This means that any two multiples of the base j which add up to the modulus occupy the same point. Where x exceeds the modulus, of course, $0 = x$ is equivalent to $0 = (x \bmod i)$, where $(x \bmod i) < i$. Equations of the form $x = y$ are equivalent, in virtue of $x = y \leftrightarrow x' = y'$, to $0 = 0^{|(x-y)|}$. Negations, conjunctions, implications and quantifications are then determined as usual. This does not quite nail down RMn^i uniquely, however. We have for $RM7^{16}$, consistent with the above conditions:

0	$0 = 16$		0	$0 = 16$
-1	$0 = 8$	or	-1	$0 = 4, 8, 12$
-2	$0 = 2, 4, 6, 10, 12, 14$		-2	$0 = 2, 6, 10, 14$
-3	$0 = \text{odd}$		-3	$0 = \text{odd}$

but there do not seem to be any further conditions restricting the RMn^i. That is, it seems to be that any RMn^i satisfying the above constraints is possible. In particular, for any i there is some n (e.g. 3) satisfying the constraints; and for any (odd) n there is some i (if $n = 2j + 1$, take *e.g.* arithmetic modulus 2^j, and for $k \leq j$, send $0 = 0^{(2^{j-k})}$ to point $-k$ and make appropriate adjustments).

Thus modulus and size of matrix interact strongly, and not all combinations resist isomorphism with a smaller matrix size. We will however continue to refer in the next section to RMn^i, with the understanding that these may not be unique and that a particular i might force an isomorphism with RMm^i for lesser m. With that qualification, we note that the RMn^i all have the properties of the $RM3^i$: inconsistency, nontriviality, containment of $RM^{\#\#}$ etc., completeness, primeness, ω-completeness, ω-inconsistency. We omit the proofs.

§6. The arithmetic RM^{ω}. In this section we study the intersection of all the models RMn^i.

DEFINITION 6. *The arithmetic RM^{ω} is the intersection of all the truths of all the RMn^{i}.*

PROPOSITION 10. *RM^{ω} is inconsistent, ω-inconsistent and nontrivial. If n and m are distinct, then $0^{(n)} = 0^{(m)}$ is not in RM^{ω}.*

PROOF. Every RMn^{i} contains $0 = 0, \neg 0 = 0$. Also, every RMn^{i} contains $\neg 0 = 0$, $\neg 0 = 0', \neg 0 = 0'', \ldots$ but also $\neg(x)\neg 0 = x$. Further, $0 = 0'$ is not in some (in fact all) RMn^{i}. Finally, $0^{(n)} = 0^{(m)}$ is not in $RM3^{|n-m|+1}$.

PROPOSITION 11. *RM^{ω} is ω-complete.*

PROOF. Let $F0, F0', F0'', \ldots$ all be in RM^{ω}. Then they are true in every RMn^{i}. But every RMn^{i} is ω-complete, so $(x)Fx$ is true in every RMn^{i}, and so is in RM^{ω}.

PROPOSITION 12. *$R^{\#}, R^{\#\#}, RM^{\#}$, RM$^{\#\#}$ are all included in RM^{ω}.*

PROOF. Each of these systems is included in every RMn^{i}.

PROPOSITION 13. *The nontheorems of RM^{ω} are recursively enumerable.*

PROOF. This is a standard type of argument. Every RMn^{i} is decidable, so set the Turing machines for each RMn^{i} to work consecutively. Eventually, if A is a nontheorem of some RMn^{i}, some Turing machine will say so.

Problem. Is RM^{ω} decidable? Is it axiomatisable? Say by adding $\neg(x)\neg 0 = x'$ to $RM^{\#}$?

PROPOSITION 14. *RM^{ω} is incomplete.*

PROOF. Consider the wff $A: 0 = 0^{(2)} \rightarrow 0 = 0'$. In $RM3^{2}$, this gets the value $0 \rightarrow -1 = -1$, and so is not a truth. In $RM3^{3}$, however, it gets $-1 \rightarrow -1 = +1$; and so its negation, $\neg(0 = 0^{(2)} \rightarrow 0 = 0')$ gets value -1 and so is not a truth of $RM3^{3}$. Hence neither A nor $\neg A$ is in RM^{ω}. Thus we also have:

PROPOSITION 15. *$R^{\#}, R^{\#\#}, RM^{\#}, RM^{\#\#}$ are all incomplete.*

If Proposition 1 sidesteps Gödel's second incompleteness theorem, the significance of these last two propositions is that they parallel and contrast with—though very simply—Gödel's first incompleteness theorem. They are not conditional, as the Gödel/Rosser theorem is, on the assumption of consistency; indeed RM^{ω} is inconsistent. Nor do they need methods not formalisable in $P^{\#}$ to prove the assumption of consistency and so detach the conclusion of incompleteness: the above proof is perfectly finitistic. In addition, the present result holds independently of the assumption of a recursively enumerable proof procedure, unlike the Gödel/Rosser theorem. Any non-recursively-enumerable arithmetic in the language $\mathcal{L}$ and included in RM^{ω} has the same property. In particular, $R^{\#\#}$ and $RM^{\#\#}$ are not recursively enumerable.

The case for some relevant arithmetic being true arithmetic rests on the claim that the deductive relations of relevant logic are correct while those of classical logic admit invalid inferences. Quantified R looks to be a good candidate for the logic in question (though the present remarks apply to any sublogic of R). The objection to $R^{\#}$ as being true arithmetic, however, is just the objection to $P^{\#}$: since it is recursively enumerable and since all primitive recursive functions are representable therein, if consistent it is incomplete. The move to $P^{\#\#}$ as a candidate is mirrored relevantly by the move to $R^{\#\#}$. But unlike the case of $P^{\#\#}$, the expanded language of $R^{\#\#}$ means that we do not have a guarantee of completeness via the Ω-rule. The above incompleteness results mean that $R^{\#\#}$ (and $RM^{\#\#}$) are open to the same objection that $P^{\#}$ and $R^{\#}$ are. Mind you, the mere fact of incompleteness in a candidate for true arithmetic does not seem to us to be so serious (see e.g. Mortensen and Priest [9]). More serious is *what* is unprovable: neither $0 = 2 \to 0 = 1$ nor $\neg(0 = 2 \to 0 = 1)$. For what our intuition is worth, the latter feels true to us.

The fact that RM^ω is incomplete implies that the RMn^i do not form a chain or even a (lower) semilattice under subsethood. For if they did, it would be simple to show that since all the RMn^i are complete, so must RM^ω be. However, RM^ω is complete in a weaker sense. To show this, it is instructive to take a detour through the $RM3^i$.

DEFINITION 7. $RM3^\omega = \bigcap_i RM3^i$.

DEFINITION 8. A wff of $\mathscr{L}$ is *extensional* iff it contains no occurrences of $\to$. A theory Th is *extensionally complete* iff for every closed extensional wff A, either $A \in \text{Th}$ or $\neg A \in \text{Th}$.

PROPOSITION 16. *If A is a closed extensional wff, then, for any k and any i, if A is true in $RM3^{ki}$ then A is true in $RM3^i$.*

PROOF. We use the fact that in all these arithmetics $\neg(A \& B) \leftrightarrow \neg A \vee \neg B$ and $\neg\forall = \exists\neg$, to drive negations through conjunctions and quantifiers. So we only need to consider wffs which are disjunctions, conjunctions and universal and existential quantifications of basic equations and unequations (i.e. of the form $\neg t_1 = t_2$). The quantifiers can also be eliminated in favour of finite conjunctions and disjunctions in the usual way, so we only need to consider conjunctions and disjunctions of basic equations and unequations. Further, for the latter we need only consider the terms in $\{0, 0', \ldots, 0^{(ki-1)}\}$.

Base. If $0^{(n)} = 0^{(m)}$ is true in $RM3^{ki}$, then for some $j \geq 1$, $|n - m| = jki$. Hence $|n - m| = (jk)i$, where $jk \geq 1$; so $0^{(n)} = 0^{(m)}$ takes value 0 in RM^i. No equation takes the value $+1$ in $RM3^{ki}$. Equally, however, no unequation takes the value -1 in any $RM3^i$; so trivially every unequation true in $RM3^{ki}$ is true in $RM3^i$.

Inductive step. (i) A is $B \& C$ and true in $RM3^{ki}$. Then B and C are both true in $RM3^{ki}$, whence by the inductive hypothesis they are both true in $RM3^i$, so $B \& C$ is. (ii) A is $B \vee C$ and true in $RM3^{ki}$. But $RM3^{ki}$ is prime, so at least one of B, C is true in $RM3^{ki}$, so by the inductive hypothesis is true in $RM3^i$, so $B \vee C$ is. This completes the proof.

PROPOSITION 17. *The extensional truths of the $RM3^i$ form a lower semilattice with respect to subset inclusion.*

PROOF. Any pair of arithmetics $RM3^i$, $RM3^j$ have a common lower bound $RM3^{ij}$, by the previous proposition. A greatest lower bound is then available since there are only a finite number of candidates.

We note that the lower semilattice is not complete (as a semilattice) since the infinite set $\{RM3^i; i \text{ prime}\}$ has no lower bound in the semilattice. (We can complete it, however, by adding the extensional truths of $RM3^\omega$.) The previous proposition now enables us to prove:

PROPOSITION 18. *$RM3^\omega$ is extensionally complete.*

PROOF. If A is extensional and not true in $RM3^\omega$ it is not true in $RM3^i$ for some i. If $\neg A$ is not true in $RM3^\omega$ it is not true in $RM3^j$ for some j. Hence neither A nor $\neg A$ would be true in $RM3^{ij}$. But every $RM3^k$ is complete, contradiction.

Now we can return to the question of the extensional completeness of RM^ω.

PROPOSITION 19. *For each i and n the extensional truths of $RM3^i$ are just those of RMn^i.*

PROOF. We prove that $RM3^i \subseteq RMn^i$; the converse is similar. As before, we take the base clause to equations and unequations and the induction with respect to &, $\vee$, $\forall$, $\exists$; and we ignore terms other than $\{0,\ldots,0^{(i-1)}\}$ because of $\vdash(x)(x = 0 \vee \cdots \vee x = 0^{(i-1)}\}$. Further, we can ignore the $\forall$, $\exists$ cases, since in both $RM3^i$ and RMn^i, $\forall$ and $\exists$ are eliminable in favour of & and $\vee$. Evidently, the equations of $RM3^i$ are just those of the form $0^{(n)} = 0^{(n+ki)}$ for some $k \geq 0$. All of these are true in RMn^i. Also, every unequation (i.e. $\neg 0^{(n)} = 0^{(m)}$) is in RMn^i anyway. The inductive clause for & is straightforward. For $\vee$, we use the fact that $RM3^i$ is prime.

Now, however, we can observe that it follows from Proposition 19 that the extensional truths of RM^ω are exactly those of $RM3^\omega$ (Reason: $RM^\omega \subseteq RM3^\omega$ is evident. Conversely, if $A \in RM3^\omega$ then $A \in RM3^i$ for all i; so by Proposition 19, $A \in RMn^i$ for i.) It immediately follows that

PROPOSITION 20. *RM^ω is extensionally complete.*

The route we chose to Proposition 20 has the interest that it reveals the relations between the $RM3^i$ and on the one hand $RM3^{ki}$ and on the other RMn^i. However, Proposition 20 also follows from a more general, though perhaps less deep, result, Proposition 21.

DEFINITION 9. A *secondary equation* is any sentence A such that $\vdash A \rightarrow t$. A *secondary unequation* is any sentence A such that $\vdash f \rightarrow A$ (see [4] or [5]; recall from Definition 5 that t is $0 = 0$ and f is $\neg 0 = 0$).

PROPOSITION 21. *Any inconsistent arithmetic extending $R^\#$ is extensionally complete.*

PROOF. The details are contained in [4] or [5]. First note that in any inconsistent extension of $R^\#$, $\vdash t \rightarrow f$. (See [4] or [5]; essentially the reason is that if $\vdash A \mathbin{\&} \neg A$ for some A, then since $\vdash \neg A \leftrightarrow (A \rightarrow f)$, $\vdash f$. But $\vdash f \leftrightarrow (t \rightarrow f)$, so $\vdash t \rightarrow f$.) But it can be shown that union of the secondary equations and secondary unequations is closed with respect to the operators $\neg$, & and $\forall$. Finally it can be shown that every equation is a secondary equation and every unequation a secondary unequation. Proposition 21 follows.

One final point to make is to show again how, relevantly, concepts can diverge which coincide classically.

DEFINITION 10. A theory is *E-complete* iff whenever $(\exists x)Fx$ is in the theory, so is $F0', F0^{(2)}, \ldots$ *are in* RM^ω.

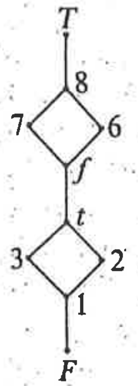
PROOF. $(\exists x)(0 = x')$ is in RM^ω since it is in every RMn^i. But $0 = 0', 0 = 0'', \ldots$ are each refutable in some RMn^i, and so not in RM^ω.
ness with respect to extensional wffs to coincide with ω-completeness. However, we have, despite the ω-completeness of RM^ω:

PROPOSITION 22. *There is an extensional wff $(\exists x)Fx$ in RM^ω such that none of $F0$, $F0'$, $F0^{(2)}, \ldots$ are in RM^ω.*

PROOF. $(\exists x)(0 = x')$ is in RM^ω since it is in every RMn^i. But $0 = 0', 0 = 0'', \ldots$ are each refutable in some RMn^i, and so not in RM^ω.

The explanation of the difference from the classical case is, of course, that classically a theory is either consistent or trivial, and consistency together with completeness guarantees E-completeness iff ω-completeness no matter what the logic in question. But relevantly, theories can be neither consistent and nontrivial. The equivalence can thus break down in such cases.

§7. An inconsistent extension of $R^{\#\#}$ without mingle.[4] The finite arithmetics studied up to now have all had the property that mingle, $A \rightarrow . A \rightarrow A$ holds, so that one of $A \rightarrow B$, $B \rightarrow A$ is true. In the present section we display an arithmetic for which this is not so. For base logic we take the ten-valued lattice, which we call CL2, with Hasse diagram as follows:



Negation is as suggested by the notation: $F = \neg T$, $1 = \neg 8$, $2 = \neg 7$, $3 = \neg 6$ and $t = \neg f$. All values above and including t are designated. Conjunctions and disjunctions are respectively lower and upper bounds in the lattice. The $\rightarrow$ table is as follows.

	$\rightarrow$	F	1	2	3	t	f	6	7	8	T
	F	T	T	T	T	T	T	T	T	T	T
	1	F	t	t	t	t	8	8	8	8	T
	2	F	3	t	3	t	7	8	7	8	T
	3	F	2	2	t	t	6	6	8	8	T
*	t	F	1	2	3	t	f	6	7	8	T
*	f	F	F	F	F	F	t	t	t	t	T
*	6	F	F	F	F	F	3	t	3	t	T
*	7	F	F	F	F	F	2	2	t	t	T
*	8	F	F	F	F	F	1	2	3	t	T
*	T	F	F	F	F	F	F	F	F	F	T

It is mechanical (literally!) to verify that all the theorems of R hold in CL2. Note, though that mingle fails e.g. at $7 \rightarrow . 7 \rightarrow 7$. Into CL2 we code arithmetic modulo 105, which we call $\mathrm{CL2}^{105}$, via the following rules.

(1) $I(0^{(n)} = 0^{(m)}) = t$ iff for some $k \geq 0, |n - m| = 105k$.

(2) $I(0^{(n)} = 0^{(m)}) = 3$ iff for some $k \geq 1$, $|n - m| = 15k$ but for no $k \geq 1$, $|n - m| = 21k$.

[4] We acknowledge in this section the help of John Slaney's computer programs, and the DEC-10 computer for executing them.

(3) $I(0^{(n)} = 0^{(m)}) = 2$ iff for some $k \geq 1$, $|n - m| = 21k$ but for no $k \geq 1$, $|n - m| = 15k$.

(4) $I(0^{(n)} = 0^{(m)}) = 1$ iff for some $k \geq 1$, $|n - m| = 3k$ but for no $k \geq 1$, $|n - m| = 15k$ and for no $k \geq 1$, $|n - m| = 21k$.

(5) $I(0^{(n)} = 0^{(m)}) = F$ otherwise, i.e. iff for no $k \geq 1$, $|n - m| = 3k$.

Negations, conjunctions and $\rightarrow$ are as for CL2. Truths are those statements which take a designated value. Quantifiers are now treated as they were in §2, and that enables us to verify the quantificational axioms of R as well.

PROPOSITION 23. *The theorems of* $R^{\#\#}$ *are all true in* $CL2^{105}$.

PROOF. The details are omitted. Axiom #2 is lengthy, but not in principle difficult.

Problem. What are the general conditions under which a modular arithmetic can be embedded in a nonmingle lattice? What mods will admit of such treatment (not all do, which explains the otherwise bizarre choice of modulo 105), and what conditions must equations sent to incomparable points satisfy?

§8. Fermat's last theorem. One advantage of having inconsistent extensions of $P^{\#}$, $R^{\#}$, etc., is that we can entertain the possibility of constructing inconsistent models to falsify arithmetical sentences, the truth of which is problematic. Take, for example, Fermat's last theorem. As is well known, if the denial of FLT is not provable in arithmetic then FLT is true. Hence if in some inconsistent model of $P^{\#}$ $\neg$FLT can be made to fail, FLT must be true. Unfortunately, in RM^{ω} and all its finite modular extensions, $\neg$FLT is equivalent to FLT (considering both as extensional formulae), and both are true. In fact the inability to refute $\neg$FLT is not confined to an RM-ish logic. Given any model in which $\vdash t \rightarrow f$, i.e. in any inconsistent extension of $R^{\#}$, it can be shown that $\neg$FLT $\rightarrow$ FLT, but that both $\neg$FLT and FLT are true (though not necessarily equivalent). Of course, this does not rule out the possibility that there are other ways of producing inconsistent extensions of $P^{\#}$ which would yield a refutation of $\neg$FLT. We remark, for what it is worth, that the *intensional* version of FLT, i.e. $(x, y, z, n)(x^n + y^n = z^n \rightarrow x = 0 \vee x = 1 \vee y = 0 \vee y = 1 \vee z = 0 \vee z = 1 \vee n = 0 \vee n = 1 \vee n = 2)$, is refutable. Choose $x = y = z = n = 3$ in $RM3^9$. Then for this instance, the consequent takes value -1. But the antecedent takes the value of $18 = 9$ which in $RM3^9$ is 0. But $0 \rightarrow -1 = -1$, and the value of a universal quantification is the minimum of the values of its instances. So the value of the intensional FLT is -1 (and so it is not provable in $R^{\#}$).

§9. Conclusion. We conclude with the observation that, since negation consistent theories are a *special case* of nontrivial theories, there is no reason for relevant mathematics to "reject" model theory as classically conceived. Relevant mathematics proposes itself as an extension to the classical case rather than as an alternative, though it must be admitted that the extension might be seen to have more desirable properties by being less restrictive. An example is the classical identification of negation consistency with nontriviality, which has, we believe, no justification. Relaxation of that identification means, for example, that limitative theorems such as those of Gödel have to be re-examined. It also means, as we have seen, that theories can be considered in which concepts which coincide classically no longer coincide, but are nonetheless mathematically well-behaved and interesting. Furthermore, it opens the possibility that inconsistent arithmetics might enable the solution of traditional problems via the demonstration of the independence of problematic sentences from formalised arithmetic; the latter being, of course, a traditional hope of model theory. But we stress that models developed classically do not suddenly fail to be legitimate in relevant mathematics. Instead, one can choose to vary the class of truths in the model in accordance with some nonclassical logic, and compare the result with the classical case. The resulting change in viewpoint cannot impoverish insight into the nature of mathematical structures, but rather can only enrich it.

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“Relevant Quantum Arithmetic”

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RELEVANT QUANTUM ARITHMETIC

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INTRODUCTION

In 'Quantum Mathematics' [1], Michael Dunn showed that the addition of Peano arithmetic to a classical quantum (orthomodular) logic enables all instances of the distribution axiom $A \& (B \vee C) \rightarrow (A \& B) \vee C$ to be proved, thus collapsing (classical) quantum arithmetic to classical Peano arithmetic $P^{\#}$. He also showed that distribution fails in second order Peano arithmetic without extensionality, but holds when extensionality is added. He expressed the opinion that the first of these results should not be of concern to the quantum logician (on the grounds that arithmetic need not be infected with quantum problems), but that the last should (on the grounds that the language of quantum mechanics is set theory, while the result seems to show that extensional set theory must admit distribution).

We wish to raise here the question whether Dunn's result can be reproved relevantly, i.e. whether distribution can be recovered in arithmetic founded on a base logic of <u>relevant</u> orthomodular logic. It is not apparent from Dunn's proof method that it can, but then it is not apparent that it cannot. As a contribution to this problem, we show in the next section that in arithmetic based on relevant ortho-logic, OR, distribution cannot be proved. OR differs from what one would take to be the relevant analogue of orthomodular logic in that the latter is slightly stronger. While both lack distribution, OR lacks any version of the orthomodular law $A \& (\sim A \vee (A \& B)) \leq B$; so what happens when the latter is added remains open. In the final section, we ask what happens when inconsistent arithmetics are added to relevant quantum logics. Inconsistent arithmetics were studied in [2]. For inconsistent arithmetics to be mathematically interesting, a logic in which $(A \& \sim A) \to B$ fails is necessary; and the usual relevant logics provide such logics. We adapt here the methods of [2] to show a Dunn-like result, that in the extensional fragment of such arithmetics, even with a logic as weak as OR, distribution is provable (and so *a fortiori* when orthomodularity holds as well). Hence there is no interesting effect on the extensional theorems of inconsistent relevant arithmetic when distribution is dropped from the base logic. It is an open question whether this holds true of the intensional theorems of inconsistent arithmetic as well.

DISTRIBUTION FAILS IN $OR^{\#}$

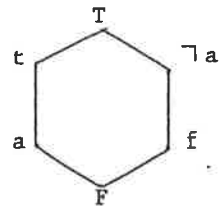
As Dunn notes, it is not entirely clear just what, from the point of view of Hilbert-style logics, quantum logic is. The logic OR is formed from R by dropping the distribution axiom (see e.g. [3], p. 341). The arithmetic based on OR, called $OR^{\#}$, is formed by adding following intensional forms of the Peano postulates to OR (see [2]). AXIOMS: $x = y \leftrightarrow x' = y'$, $x = y \to (x = z \to y = z)$, $x' \neq 0$, $x + 0 = x$, $x + y' = (x + y)'$, $x \times 0 = 0$, $x \times y' = (x \times y) + x$. Rule: F0, $Fx \to Fx' / \therefore (x)Fx$.

Now we can state the result of this section.

Proposition 1. Distribution is not provable in $OR^{\#}$.

Proof. The proof is quite easy. Take the ortholattice below, which we call BR; first discussed, as far as we know, in [4].

RELEVANT QUANTUM ARITHMETIC



Conjunctions and disjunctions are lower and upper bounds respectively in BR, $F = \neg T$, $f = \neg t$. The $\rightarrow$ operation is given by:

$\rightarrow$	F	f	a	¬a	t	T
F	T	T	T	T	T	T
f	a	t	a	T	a	T
a	¬a	¬a	T	¬a	T	T
¬a	a	a	a	T	a	T
*t	F	f	a	¬a	t	T
*T	F	F	a	¬a	a	T

It is mechanical to verify that every theorem of OR holds in BR. Now we construct a model for arithmetic on BR as follows. The domain is the natural numbers with +, × and ' as in the standard model for arithmetic. The universal quantifier is interpreted as the lower bound of the set of values of its instantiations (i.e. generalized conjunction in the complete lattice BR). We assign the equations $n = n$, for each n, to the value t; all other equations are assigned to F. This induces the values f on $\neg n = n$, and T on $\neg n = m$ for distinct n,m. It is then standard (though nonfinitistic) to verify that every theorem of $OR^\#$ takes a designated value (t or T) in this model. But distribution fails, e.g. $t \,\&\, (\neg a \vee a) \not\rightarrow (t \,\&\, \neg a) \vee a$.

Note also that the rule γ $(A, \neg A \vee B / \therefore B)$ fails for $OR^\#$: it can be shown that $\vdash_{OR^\#} t$ and $\vdash_{OR^\#} \neg t \vee a$; but here not $\vdash_{BR} a$, so not $\vdash_{OR^\#} a$. This is of interest, since $OR^\#$ is (presumably!) consistent. Say that a theory is <u>prime</u> if whenever a disjunction is in the theory, at least one of the disjunts is. Now it is standard that if a theory consistent and prime, γ holds for it. Hence the failure of γ for $OR^\#$ must be due to a failure of primeness. And indeed this is so, for $\vdash_{OR^\#} \neg t \vee a$; but here neither $\vdash_{BR} \neg t$ nor $\vdash_{BR}$ a, so neither $\vdash_{OR^{\#}} \neg t$ nor $\vdash_{OR^{\#}}$ a. It is a common observation among paraconsistentist logicians that γ fails when a theory is inconsistent but nontrivial. We believe this to be the only extant example of an interesting theory in which γ fails because of failure of primeness.

Note finally that the above result does not circumvent Dunn's theorem relevantly, since BR is not an orthomodular lattice. Nor does what seems a natural relevant analogue of orthomodularity, $(A \& (\sim A \vee (A \& B))) \rightarrow B$, hold in BR.

INCONSISTENT QUANTUM ARITHMETIC

In [2], we investigate the inconsistent (but nontrivial) arithmetics which arise when an equation of the form $0 = n$ is added as a theorem to relevant arithmetic, $R^{\#}$. R is of course an inconsistency - tolerating logic, and the addition of such formulae does not lead to total collapse. Here we ask what happens when such equations are added to $OR^{\#}$, or indeed in any inconsistent extension thereof.

Definition. Let t be the sentence $0 = 0$ and f be $\neg t$. The arithmetical sentence A is a secondary equation of a theory iff $\vdash A \rightarrow t$. A is a secondary unequation iff $\vdash f \rightarrow A$.

Lemma. In $OR^{\#}$, if A and B are both secondary equations, then distribution holds of them, i.e. $\vdash A \& (B \vee C) \rightarrow (A \& B) \vee C$. Similarly, if B and C are secondary unequations, then $\vdash A \& (B \vee C) \rightarrow (A \& B) \vee C$.

Proof: First note that for any A, $\vdash (t \rightarrow A) \leftrightarrow A$ and $\vdash (A \rightarrow f) \leftrightarrow A$. (see, e.g. [5] or [6]). But also in OR, both $\vdash f \rightarrow \neg B \rightarrow . A \rightarrow f \rightarrow . A \rightarrow \neg B$ and $\vdash f \rightarrow \neg A \rightarrow . B \rightarrow f \rightarrow . B \rightarrow \neg A$. Assuming that A and B are secondary equations, then $\vdash f \rightarrow \neg B$ and $\vdash f \rightarrow \neg A$ by contraposition. So detatching $\vdash A \rightarrow f \rightarrow . A \rightarrow \neg B$ and $\vdash B \rightarrow f \rightarrow . B \rightarrow \neg A$. But since in OR, $\vdash (A \rightarrow \neg B) \leftrightarrow (B \rightarrow \neg A)$, we have $\vdash B \rightarrow f \rightarrow . A \rightarrow \neg B$. But $B \rightarrow f$ is equivalent to $\neg B$, so $\vdash \neg B \rightarrow . A \rightarrow \neg B$. Similarly, $\vdash \neg A \rightarrow . A \rightarrow \neg B$. Hence in OR, $\vdash (\neg A \vee \neg B) \rightarrow . A \rightarrow \neg B$, whence $\vdash \neg (A \rightarrow \neg B) \rightarrow (A \& B)$. But also in OR, $\vdash A \rightarrow . A \rightarrow \neg B \rightarrow \neg B$, so that $\vdash A \rightarrow . B \rightarrow \neg (A \rightarrow \neg B)$. Thus $\vdash A \rightarrow . B \rightarrow (A \& B)$. Conjoining to the antecedent and disjoining to the final consequent, $\vdash A \& (B \vee C) \rightarrow . B \rightarrow . (A \& B) \vee C$; call this (1). Now for any C, $\vdash C \rightarrow . t \rightarrow C$ hence $\vdash t \rightarrow . C \rightarrow C$. Hence

$\vdash t \to . C \to C$. Hence if A is a secondary equation, $A \to . C \to C$.
Conjoining to the antecedent etc. as before,
$\vdash A \& (B \vee C) \to . C \to (A \& B) \vee C$; call this (2).
Using both (1) and (2) and the properties of OR,
$\vdash A \& (B \vee C) \to . (B \vee C) \to . (A \& B) \vee C$; so
$\vdash A \& (B \vee C) \to (A \& B) \vee C$. But in OR, $\vdash A \to . A \to B \to . A \to B$.
Hence $\vdash A \& (B \vee C) \to (A \& B) \vee C$. This is the desired result when A and B are secondary equations. If B and C are secondary unequations, then $\neg B, \neg C$ are secondary equations. Applying what we have just proved, $\vdash \neg C \& (\neg B \vee \neg A) \to (\neg C \& \neg B) \vee \neg A$. Rearranging using commutation and De Morgan's laws (all in OR), $\vdash \neg(A \& B) \& \neg C \to \neg A \vee \neg(B \& C)$. De Morgan again gives $\vdash \neg((A \& B) \vee C) \to \neg(A \& (B \vee C))$, and contraposition gives $\vdash A \& (B \vee C) \to (A \& B) \vee C$ as required.

Now we can state the main result of this section.

Proposition 2. In any inconsistent extension of $OR^{\#}$, if A,B,C are all extensional formulae (containing no $\to$s), then $\vdash A \& (B \vee C) \to (A \& B) \vee C$.
Proof: We need the following facts: if any of $\vdash A \to B$, $\vdash B \to A$, $\vdash B \to C$, $\vdash C \to B$ or $\vdash A \to C$ holds, then distribution does too. The proof of these is standard and easy to verify. Now, it is also true that in any inconsistent extension of $R^{\#}$, $\vdash t \to f$ (see [2] Proposition 21). In fact, the proof of this does not use distribution, so it holds also in $OR^{\#}$. Hence we have that any one of the following five conditions suffices for $A \& (B \vee C) \to (A \& B) \vee C$ to hold in any inconsistent extension of $R^{\#}$: (1) A a secondary equation and B a secondary unequation (2) B a secondary equation and A a secondary unequation (3) B a secondary equation and C a secondary unequation (4) C a secondary equation and B a secondary unequation (5) A a secondary equation and C a secondary unequation. To these we can add two more sufficient conditions, following from the lemma: (6) A and B both secondary equations (7) B and C both secondary unequations. Now it is also a fact that every extensional sentence is either a secondary equation or a secondary unequation (see [2] Proposition 21; we note that this does not hold for the intensional sentences of R so the proof here does not generalize). But now it is easy to see that it follows from (1) - (4) that it is impossible for distribution to fail for extensional sentences A, B, C unless they are all secondary equations, or all secondary unequations. And that is impossible by (6) and (7). This

conclusion reinforces those of [2], that inconsistent arithmetics are surprisingly stable, at least in their extensional parts, and that inconsistent mathematics is a rich and interesting area.

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INCONSISTENT NONSTANDARD ARITHMETIC

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Abstract. This paper continues the investigation of inconsistent arithmetical structures. In §2 the basic notion of a model with identity is defined, and results needed from elsewhere are cited. In §3 several nonisomorphic inconsistent models with identity which extend the ($=$, $<$) theory of the usual classical denumerable nonstandard model of arithmetic are exhibited. In §4 inconsistent nonstandard models of the classical theory of finite rings and fields modulo m, i.e. Z_m, are briefly considered. In §5 two models modulo an infinite nonstandard number are considered. In the first, it is shown how to model inconsistently the arithmetic of the rationals with all names included, a strengthening of earlier results. In the second, all inconsistency is confined to the nonstandard integers, and the effects on Fermat's Last Theorem are considered. It is concluded that the prospects for a good inconsistent theory of fields may be limited.

§1. Introduction. This paper is a sequel to [1] and [2], and continues the investigation of inconsistent structures therein. In §2 the basic notion of a model with identity is defined, and results needed from [2] are cited. In §3 several nonisomorphic inconsistent models with identity which extend the ($=$, $<$) theory of the usual classical denumerable nonstandard model of arithmetic are exhibited. In §4 inconsistent nonstandard models of the classical theory of finite rings and fields modulo m, i.e. Z_m, are briefly considered. In §5 two models modulo an infinite nonstandard number are considered. In the first of these, we obtain a result stronger than the finite methods of [2] could obtain, namely that the full theory of the field of rationals Q, equipped with names, can be modelled in an inconsistent extension of the nonstandard model of ($+$, $\times$) arithmetic. In the second of these, a model which bears an interesting relationship to classical nonstandard arithmetic, namely one in which all inconsistency is confined to the nonstandard part of the diagram, is displayed. One interest in the model is that it is possible to do better with Fermat's Last Theorem than the results on that score in [1] suggested. It is claimed on the basis of these results and those of [1] and [2] that the potential for inconsistent mathematics to be a rich source of structures, problems, and results looks good.

§2. Summary of definitions and results. We consider various sublanguages of the language $\mathscr{L}$ consisting of simple terms (names) for elements of various classical number systems, e.g. N, Z, Q and nonstandard extensions thereof; function symbols $+$, $\times$, $-$, $\div$,; atomic binary predicates $=$, $<$,; variables $x, y, z, \ldots$; and operators $\neg$, &, $\forall$, the latter also written ( ). Complex terms, wffs and sentences are defined in the

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usual way, as are $\supset$, $\vee$, $\equiv$ and $\exists$. We regard sentences of the form $t_1 = t_2$ and $t_1 < t_2$ as atomic, irrespective of whether t_1, t_2 contain function symbols. Only theories which contain no free variables are considered, and, for simplicity, no term is a variable. An (*RM*3-) *assignment* is a function I assigning to the appropriate sublanguage of $\mathscr{L}$ under consideration values from $\{T, N, F\}$ in accordance with: (1) For any atomic wff with terms t_1 and t_2, $I(t_1 = t_2)$ and $I(t_1 < t_2)$ both $\in \{T, N, F\}$. (2) $I(\neg A)$ and $I(A \mathbin{\&} B)$ are given by the *RM*3-matrices

&	T	N	F	¬
T	T	N	F	F
N	N	N	F	N
F	F	F	F	T

(3) $I((x)A = \min\{y$: for some term t, $I(A(t \mid x)) = y\}$, where min is relative to the lattice ordering $F < N < T$. A sentence A *holds in an assignment* I iff $I(A) \in \{T, N\}$. An assignment is *consistent* (complete) iff the set of sentences holding therein is consistent (complete). The set of sentences holding in any *RM*3-assignment I is a complete and possibly inconsistent *RM*3-theory. I is an *assignment with identity* iff for all terms t_1, t_2, if $t_1 = t_2$ holds, then for all predicates F, Ft_1 holds iff Ft_2 holds, where Ft_2 is like Ft_1, except that t_2 replaces t_1 in one or more places. An *RM*3-*model* is a pair $\langle D, I\rangle$ where D is a domain and I is an *RM*3-assignment, and such that: (1) I assigns to every simple term of the sublanguage a member of D, and I is onto D, so that every object is named; (2) I assigns to every n-ary functional expression an n-ary partial function on D; (3) the assignment to complex terms is given by $I(f(t_1 \cdots t_n)) = I(f)(I(t_1) \cdots I(t_n))$, provided that these are defined; and (4) I satisfies: $t_1 = t_2$ holds iff $I(t_1) = I(t_2)$. A model is *infinite* iff $\bar{\bar{D}} \geq \aleph_0$, otherwise *finite*. If $\langle D, I\rangle$ is a model and I an assignment with identity, then $\langle D, I\rangle$ is a *model with identity*. Finally, two models $\langle D, I\rangle$ and $\langle D^1, I^1\rangle$ are *isomorphic* iff there is a 1-1 correspondence $f: D \to D^1$ such that for all atomic terms $t_1, \ldots, t_n, t_1^1, \ldots t_n^1$, if $I^1(t_1^1) = f(I(t_1)), \ldots, I^1(t_n^1) = f(I(t_n))$, then for all atomic F, $Ft_1 \cdots t_n$ holds in I iff $Ft_1^1 \cdots t_n^1$ holds in I^1. Then from [2], we have the following. A necessary and sufficient condition for a model to be a model with identity is that for all terms t_1, t_2 and all atomic F, if $t_1 = t_2$ holds, then $I(Ft_1) = I(Ft_2)$. In the special case where $=$ is the only predicate of the language, $\langle D, I\rangle$ is a model with identity iff for all t_1, t_2, if $t_1 = t_2$ holds then for all t_3, $I(t_1 = t_3) = I(t_2 = t_3)$ and $I(t_3 = t_1) = I(t_3 = t_2)$.

EXTENDABILITY LEMMA. *Let I, I^1 be assignments to the same sets of wffs. If the atomic sentences holding in I are a subset of those holding in I^1 and the negations of atomic sentences holding in I are a subset of those holding in I^1, then the set of all sentences holding in I is a subset of that holding in I^1.*

TERM ELIMINATION LEMMA. *If $\langle D, I\rangle$ is a model with identity with at least one element of D assigned by I to more than one simple term, then there is a model $\langle D, I^1\rangle$ which assigns elements to only a proper subset of those in $\langle D, I\rangle$ such that there is only one name per element, and the sentences holding in the cut-down language are identical in the two models.*

In [2], it is argued that theories determined by models with identity represent extensional mathematics in a quite standard sense of "extensional", whether consistent or not.

§3. **The theory of order.** In this section, we illustrate in a simple way the use of the Extendability Lemma, by displaying two inconsistent models which extend the theory of the classical consistent denumerable nonstandard model of arithmetic, of order type $\omega + \eta(\omega^* + \omega)$. The inconsistent models are nonisomorphic to that model.

In all models, including the classical denumerable nonstandard model, take one simple term for every element of the classical nonstandard model. It is convenient in this section to adopt a naming system which reflects their classical order type, so we name them by pairs $\langle q, z\rangle$ where q is any rational number, $0 \leq q < 1$, and z is any integer, with the proviso that if $q = 0$ then $z \in N$ only. If we now set $I(\langle q, z\rangle = \langle q^1, z^1\rangle) = T$ iff $q = q^1$ and $z = z^1$, and F otherwise; and $I(\langle q, z\rangle < \langle q^1, z^1\rangle) = T$ iff $q < q^1$ or ($q = q^1$ and $z < z^1$), and F otherwise; and evaluate as per the $RM3$-matrices for $RM3$-models, we have the $(=, <)$ fragment with names of the classical consistent denumerable nonstandard model of arithmetic, since the $\{T, F\}$ subalgebra of the $RM3$-matrices $\{T, N, F\}$ is classical logic. Now consider the following inconsistent models with identity.

1. As elements of the domain, take all names $\langle q, z\rangle$ where $q = 0$, together with, for each $q \neq 0$, just one representative, say $\langle q, 1\rangle$. For every name $\langle q, z\rangle$ of the original language re that of the classical nonstandard model, we set $I(\langle q, z\rangle) = \langle q, z\rangle$ if $q = 0$, and $I(\langle q, z\rangle) = \langle q, 1\rangle$ for $q \neq 0$. Then atomic sentences are evaluated as follows.

Case $=$: $I(\langle q, z\rangle = \langle q^1, z^1\rangle) = T$ if $q = q^1 = 0$ and $I(\langle q, z\rangle) = I(\langle q^1, z^1\rangle)$, while $I(\langle q, z\rangle = \langle q^1, z^1\rangle) = N$ if (either $q \neq 0$ or $q^1 \neq 0$) and $I(\langle q, z\rangle) = I(\langle q^1, z^1\rangle)$ (note that by the construction of the model, this latter condition is equivalent to $q \neq 0$ and $q^1 \neq 0$ and $I(\langle q, z\rangle) = I(\langle q^1, z^1\rangle)$), while $I(\langle q, z^1\rangle = \langle q^1, z^1\rangle) = F$ otherwise, i.e. if $I(\langle q, z\rangle) \neq I(\langle q^1, z^1\rangle)$.

Case $<$: $I(t_1 < t_2) = T$ if $I(t_1) < I(t_2)$, $I(t_1 < t_2) = N$ if $I(t_1) = I(t_2)$, and $I(t_1 < t_2) = F$ if $I(t_1) > I(t_2)$. Now the conditions of the Extendability Lemma are met, so every true sentence of the classical nonstandard theory continues to hold in this inconsistent model. Furthermore, it is a model with identity. [*Proof.* By §2 and [2], we have to prove that for all atomic F, if $t_1 = t_2$ holds, then $I(Ft_1) = I(Ft_2)$. Let $t_1 = t_2$ hold, and we have to prove four cases (i) $I(t_1 = t_3) = I(t_2 = t_3)$; (ii) $I(t_3 = t_1) = I(t_3 = t_2)$; (iii) $I(t_1 < t_3) = I(t_2 < t_3)$; (iv) $I(t_3 < t_1) = I(t_3 < t_2)$. Ad (i): If $I(\langle q, z\rangle = \langle q'', z''\rangle) = T$ then $q = q'' = 0$ and $I(\langle q, z\rangle) = I(\langle q'', z''\rangle)$. Now $\langle q, z\rangle = \langle q^1, z^1\rangle$ holds, so $I(\langle q, z\rangle) = I(\langle q^1, z^1\rangle)$. Hence $q^1 = 0$ also. Thus $q^1 = q'' = 0$ and $I(\langle q^1, z^1\rangle) = I(\langle q'', z''\rangle) = 0$, i.e. $I(\langle q^1, z^1\rangle = \langle q'', z''\rangle) = T$ as required. The subcase $I(\langle q, z\rangle = \langle q'', z''\rangle) = N$ is similar. If $I(\langle q, z\rangle = \langle q'', z''\rangle) = F$ then $I(\langle q, z\rangle) \neq I(\langle q'', z''\rangle)$, whence $I(\langle q^1, z^1\rangle \neq I(\langle q'', z''\rangle)$ as required. Ad (ii): The argument is similar. Ad (iii): If $I(\langle q, z\rangle < \langle q'', z''\rangle) = T$ then $I(\langle q, z\rangle) < I(\langle q'', z''\rangle)$, whence $I(\langle q^1, z^1\rangle) < I(\langle q'', z''\rangle)$, i.e. $I(\langle q^1, z^1\rangle < \langle q'', z''\rangle) = T$ as required. The N and F subcases are similar. Ad (iv): The argument is similar. Hence, it is a model with identity.]

Also, the Term Elimination Lemma permits an obvious cut-down of simple terms to one for each member of the domain of the inconsistent model, with the resultant inconsistent theory agreeing with the inconsistent theory just constructed in their common language, and thus also having all sentences of the standard model for arithmetic holding. The models just constructed essentially collapse blocks of order-type $\omega^* + \omega$ in the standard model, by identifying elements within a block. The elements of the inconsistent models have a "natural" order type, namely the natural order on the names, which determines the assignment to $t_1 < t_2$ sentences of order type $\omega + \eta$. More exactly, since there is no order-preserving 1-1 correspondence between $\omega + \eta$ and $\omega + \eta(\omega^* + \omega)$, then there is no isomorphism between the inconsistent model and the classical model, because any attempt to do so produces a reversal of the order somewhere, and thus by the construction results in some sentence $t_1 < t_2$ going from "holds" to "value F".

2. A more drastic identification and collapse is this. For domain, take the standard names $\langle 0, n\rangle$ together with one nonstandard name, ω. For any standard term, set $I(\langle 0, n\rangle) =$ itself, and for any nonstandard term, set $I(\langle q, z\rangle) = \omega$. Then $I(t_1 = t_2) = T$ if t_1 or t_2 is standard and $I(t_1) = I(t_2)$; $I(t_1 = t_2) = N$ if t_1 and t_2 are nonstandard; and $I(t_1 = t_2) = F$ otherwise. $I(t_1 < t_2)$ is evaluated as in the previous model. The Extendability Lemma applies as before, so all sentences of the classical standard and nonstandard models hold, as do all sentences of the previous inconsistent models. It is a model with identity (proof left to reader), and the Term Elimination Lemma is applicable. The natural order type on the domain is $\omega + 1$, and since there is no order-preserving 1-1 correspondence between $\omega + 1$ and $\omega + \eta(\omega^* + \omega)$ or $\omega + \eta$, then as before there is no isomorphism between this model and the previous models, consistent or inconsistent.

§4. Finite extensions of the nonstandard model with addition, multiplication and other field properties. In [1] and [2] the finite extensions of the $\langle +, \times\rangle$ theory of N with names derive from the divisibility properties of numbers: to obtain inconsistent arithmetic modulo m, identify all numbers divisible by m with zero, those 1 more than a multiple of m with 1, and so on. But nonstandard numbers can also have finite factors. Furthermore, for every finite m, every nonstandard number is within $m - 1$ of some (nonstandard) multiple of m. Multiplication modulo m is uniquely definable on nonstandard numbers. It should come as no surprise, then, that there are nonstandard versions of modulo arithmetics, and that there are corresponding finite structures in which inconsistent extensions of the $(+, \times)$ classical standard and nonstandard theories hold. For every simple name t (standard or nonstandard) assign $I(t)$ to be $t \bmod m$; $I(t_1 + t_2)$ is $I(t_1) + (\text{mod } m)\, I(t_2)$ and correspondingly for $\times$; and $I(t_1 = t_2)$ is N iff $I(t_1) = I(t_2)$, and F otherwise. This is a model with identity. Thus by the Term Elimination Lemma, there are finite structures with finite numbers of simple names in which every term-free sentence of the nonstandard model of arithmetic holds. The latter is exactly the standard theory, and the former structures are of course exactly the finite inconsistent modulo arithmetics of [1] and [2]. In turn, these inconsistent modulo arithmetics with cut-down sets of names are obtainable from the finite classical rings Z_n by inconsistentizing and applying the Extendability Lemma. The Term Elimination Lemma thus enables us to close the circle.

For arbitrary (standard) modulo m, a unique additive inverse $-n$ is definable for every n, as $m - (n \bmod m)$ if $n \bmod m \neq 0$, and 0 otherwise; and of course the nonstandard numbers do not disturb this property. Hence, in the fashion of [2], we can inconsistently model (with identity) the $(+, \times, -)$ arithmetic of the full ring of integers Z in finite modulo m with domain $\{0, 1, \ldots, m-1\}$.

The properties of finite primes carry over to the nonstandard numbers, in that for prime p, a unique multiplicative inverse $n^{-1}(\text{mod } p)$ is definable for every $n \in \{0, 1, \ldots, p-1\}$. This gives nonstandard finite fields in the classical case. In the inconsistent case, it means that we can get the $(+, \times, -, \div)$ arithmetic for those rationals whose denominator $\neq 0 \text{ mod } p$, and the $(+, \times, -)$ arithmetic of the remainder (since 0^{-1} is not defined). A point to note is that these inconsistent finite-domain models for nonstandard fields fail to make all sentences of the classical theory, with names, of the rationals Q hold. The reason is that, as noted in [2], the construction identifies with zero elements which are classically distinct from zero. But 0^{-1} remains undefined in the inconsistent structures (alas!), so some rational numbers which have classical inverses fail to do so here. The problem then arises of whether it is possible to extend inconsistently the full classical theory of Q with names using some modulo construction on nonstandard integers. The main result of §5.1 is that the properties of nonstandard primes make this possible.

§5. Modulo infinity. An interesting class of inconsistent nonstandard structures begins from the observations that infinite numbers can have infinite divisors, that infinite numbers may have no finite divisors, and that infinite primes with no divisors save themselves and unity exist.

5.1. In which $\neg 0 = 0$ holds. The first of these allows for infinite rings modulo an infinite number as follows. As we have seen, for any m, every number is within $m - 1$ of some multiple of m. So, for classical arithmetic modulo infinite m, set the modulus of any number n equal to the least x such that x + some multiple of $m = n$. We know this x to be a number, finite or infinite, between 0 and $m - 1$. The set of numbers $\{0, 1, \ldots m-1\}$ has a natural addition modulo m, calculated by adding the two together in normal fashion and if necessary subtracting m. Similarly multiplication is given on the set by multiplying the two normally and taking the remainder after repeated division by m leaves a remainder $\leq m - 1$. Inconsistentizing, as in the previous section, this gives infinite inconsistent models with identity of the standard and nonstandard theories of the natural numbers. Formally, take names for all standard and nonstandard numbers as before. Domain $D = \{x: x < m\}$, where m is any nonstandard number, the intended modulus. On D, we define $+ \text{ mod } m$ and $\times \text{ mod } m$ in the standard fashion above. For any name n, set $I(n) = n \text{ mod } m$; set $I(\dot{+}) = + \text{ mod } m$, $I(\dot{\times}) = \times \text{ mod } m$, and $I(t_1 \overset{+}{\times} t_2) = I(\overset{+}{\times})(I(t_1), I(t_2))$. Set $I(t_1 = t_2) = N$ iff $I(t_1) = I(t_2)$, and F otherwise.

The "natural" ordering on the domain has a "last" element, and a "natural" order type of $\omega + \eta(\omega^* + \omega) + \omega^*$. We note that in all these models $\neg 0 = 0$ holds.

As in the finite case, the addition of a natural division to classical nonstandard infinite rings depends on the definability of a natural multiplicative inverse, and this is possible for the infinite primes, since for them the congruence $ax \equiv 1 \text{ (mod } p)$ has a unique solution x as in the standard case (e.g. [3, p. 41]). The Extendability Lemma and the Term Elimination Lemma thus give inconsistent infinite fields in which all truths of the standard model for arithmetic hold as well. An interesting feature of the consistent theory is that the negative (additive inverse) of a finite number is a nonstandard number in the same block as the modulus: in mod m (any m), we have that $-1, -2, \ldots$ are $m - 1, m - 2, \ldots$ etc. All finite numbers in blocks different from the final block (that of the modulus) have their negatives also infinite. When the nonstandard model with names is inconsistentized, all congruences become identities and we have, for instance, that $2m - 1 = m - 1$, as well as $\neg 2m - 1 = m - 1$.

The reciprocal of any finite number is an infinite number; but in general reciprocals of different finite numbers are in different blocks. The same kinds of inconsistent extensions and term eliminations are available. Note though that the reciprocals of all finite standard numbers are defined and are infinite numbers. So since sums, products and additive inverses are defined for all finite numbers, we have a field of order type $\omega + \eta(\omega^* + \omega) + \omega^*$ which inconsistently extends the $(+, \times, -, \div)$ theory of all the rationals with names for *all* rationals, which supplies the promised strengthening of the results for finite fields cited at the end of the previous section.

Problems. Is a nonintegral rational $\pm n \div m$ ever identical with a standard integer in these models? Does reciprocation always take one to a different block? Do different numbers in the same block always have reciprocals in different blocks? Is arithmetic modulo infinity identical with RM_3^ω in their common language? (On RM_3^ω, see [1].)

5.2. Confining the inconsistency to the nonstandard numbers. We now define a model NSN in which the standard laws of $(+, \times)$ arithmetic remain consistent. The inconsistency is confined to the infinite numbers, and does not spread back to $\neg 0 = 0$ via an identification of 0 with the modulus, as it did in §5.1. We look at what happens to Fermat's Last Theorem in this model. We also show the troubles which arise when $-$ and $\div$ are added to the model are, as in [2], not really a matter of inconsistency-toleration, but of functionality. This strengthens the suspicion raised in [2], that a useful inconsistent theory of fields may be hard to come by.

NSN is defined as follows. Names for all standard and nonstandard numbers. For domain D, select an arbitrary nonstandard number m, and let $D = \{0, 1, 2, \ldots\} \cup \{x: m \leq x \leq 2m - 1\}$. Names are interpreted as follows. If $n \in \{0, 1, 2, \ldots\}$, $I(n) = n$. Otherwise, $I(n) = m + n \bmod m$. Addition and multiplication on the standard part of the domain are the standard operations and if at least one of $n_1, n_2 \in \{m, \ldots, 2m - 1\}$, then $n_1 \overset{+}{\times} n_2 = m + (n_1 \overset{+}{\times} n_2) \bmod m$. Then set $I(t_1 \overset{+}{\times} t_2) = I(\overset{+}{\times})(I(t_1), I(t_2))$. Finally, if at least one of t_1, t_2 is standard, $I(t_1 = t_2) = T$ if $I(t_1) = I(t_2)$, and F otherwise; and if both t_1 and t_2 are nonstandard, then $I(t_1 = t_2) = N$ if $I(t_1) = I(t_2)$, and F otherwise. The Extendability Lemma ensures that every sentence of classical standard and nonstandard $(+, \times)$ arithmetic continues to hold. [*Proof.* The only interesting case is the operations $+$ and $\times$ on the nonstandard numbers, so let $t_1 + t_2 = t_3$ hold classically, and show it holds in NSN; the $\times$ case will be similar. Now $I(t_1 + t_2) = m + (t_1 + t_2) \bmod m$, but from the previous modulo construction, if $t_1 + t_2 = t_3$ then $(t_1 + t_2) \bmod m = t_3 \bmod m$. Hence $I(t_1 + t_2) = m + t_3 \bmod m = I(t_3)$; therefore $I(t_1 + t_2 = t_3) = N$.] The proof of the fact that it is a model with identity is left to the reader.

One interest in the model is that $t_1 = t_2$ is T in classical standard arithmetic iff it is T in NSN; the contradictions are confined to the nonstandard part. Another interest in the model is that the "least" nonstandard number in the domain, m, functions as a pseudo-zero, in that for any nonzero n, $n \times m = m$ holds, and for any nonstandard n, $n + m = n$ holds. This implies that the denial of Fermat's Last Theorem holds in the model, since $m^3 + m^3 = m^3$ holds, while none of $m = 0, 1, 2$ and $3 = 0, 1, 2$ hold.

Now $\neg$FLT, i.e. $(\exists x, y, z, n)(\neg x = 0 \;\&\; \cdots \;\&\; \neg n = 2 \;\&\; x^n + y^n = z^n)$, is *at least* N in this model, but may be T (recall that $F < N < T$). The discussion at this point will be helped by introducing the Routley * function: if S is a theory, then $S^* = \{A: \neg A \notin S\}$. Now it is straightforward to prove that if $\neg$FLT is N in NSN, then neither FLT nor $\neg$FLT $\in$ NSN*. But also, if $t_1 = t_2$ holds standardly, then it is T in NSN and belongs to NSN* also. But by an uncontroversial argument, $\neg$FLT is true (in classical standard arithmetic) iff for some x, y, z, n not classically identical with 0, 1 or 2, $x^n + y^n = z^n$ is true in standard arithmetic. Therefore, if $\neg$FLT is N in NSN, then FLT is true (in classical standard arithmetic). Furthermore, if $\neg$FLT is T in NSN, then it can only be T in virtue of there being some standard x, y, z, n etc. with $x^n + y^n = z^n$ being standardly true i.e. $\neg$FLT is true. In short, FLT is true iff $\neg$FLT is exactly N in NSN, and iff neither $\neg$FLT nor FLT is in NSN*; and FLT is false iff $\neg$ FLT is T in NSN, and iff $\neg$ FLT $\in$ NSN* and FLT $\notin$ NSN*. This represents an improvement on the results in [1]. Unfortunately, the job of proving $\neg$FLT to be T in NSN would seem to be just as hard as finding a refuting instance to FLT ever was.

We recall from §5.1 that the use of an infinite prime modulus identified with zero permitted negatives and inverses for all the natural numbers among the infinite numbers, and thus the arithmetic of the rationals to be modelled inconsistently. But NSN with m an infinite prime does not have true negatives and reciprocals. The reason is that the nonstandard part of the domain is closed with respect to $+$ and $\times$, but not $-$ and $\div$, so these operations could not be functional on the domain without identifying some infinite number with a finite number, which the construction prohibits. This situation seems to be endemic, at least to models with identity. To take another illustration, add to the $(=, <)$ model with domain $\{0, 1, 2, \ldots, \omega\}$ of §3 the operations

$+$	finite	ω
finite	finite	ω
ω	ω	ω

$\times$	0	finite	ω
0	0	0	0
finite	0	finite	ω
ω	0	ω	ω

But here, as before, $-$ and $\div$ cannot be added while remaining a model with identity, because e.g. $\omega + 1 = \omega$ holds, while $1 = (\omega + 1) - \omega = \omega - \omega$ (by identity) $= 0$ would be catastrophic. Consequently, at the risk of labouring the point, it is not going to be easy to find useful inconsistent number-theoretic fields in which all the truths of the theories of Q or R hold.

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ALIEN INTRUDERS IN RELEVANT ARITHMETIC

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Research School of Social Sciences
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ALIEN INTRUDERS IN RELEVANT ARITHMETIC[1]

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The system R# of first-order relevant arithmetic was introduced in [12], as the result of adding the (first-order version of the) Peano postulates to relevant predicate calculus RQ. The following model was exhibited to show the system non-trivial (thus partially circumventing Gödel's Second Theorem). We pick as our domain D of *objects* the integers mod 2, with +, ., 0 interpreted in the obvious way; on this plan, the successor operation $'$ is evidently interpreted so that $0' = 1$ and $1' = 0$. As our collection V of *truth-values* we pick the set **3** = {T, N, F}, with sentential connectives &, ∨, ~, → defined on the (classical) subset **2** = {T,F} in the usual classical way. To complete the definition of connectives on **3**, we define

&	T	N	F
T	T	N	F
N	N	N	F
F	F	F	F

∨	T	N	F
T	T	T	T
N	T	N	N
F	T	N	F

→	T	N	F
T	T	F	F
N	T	N	F
F	T	T	T

	~
T	F
N	N
F	T

People familiar with relevant logics will recognize **3** so characterised as the 3-point Sugihara matrix S_3 (so-called, no doubt, because it was invented by Sobociński, and usefully introduced into the study of relevant logics by us and a number of other people, after which it has had a habit of appearing, sometimes in disguise, in most papers on the subject). People unfamiliar with relevant logics will recognize the truth-tables for &, ∨, ~ as those of Łukasiewicz's original 3-valued logic (the only intuitive way of settling these tables when N(euter) is taken as an intermediate value between T(rue) and F(alse), despite all the clashing proposals made since); while → departs from Łukasiewicz only on the prescriptions N→N = N and T→N = N→F = F. (Basically this means that Łukasiewicz was still under the

[1]Thanks to members of the Automated Reasoning Project, A.N.U. for discussion of this paper. And to members the Universities of Sydney, Melbourne, and Newcastle for further discussion.

influence of that classical lassitude which tends to assign statements higher truth-values than they deserve.)

It is now easy to say what our non-trivializing model M_2^3 = <D,V> for R# was. Modulo 2, there are only 4 atomic sentences whose truth-values we have to settle; we settle them by setting to F the values of $0 = 1$ and $1 = 0$, and to N the values of $1 = 1$ and $0 = 0$. Truth-functional combinations of these sentences are then assigned values by the 3-valued tables. And interpreting quantifiers is almost as easy. Modulo 2, ∀xAx is simply to be interpreted as A0 & A1 (since 0 and 1 are all that there is, on the interpretation, to quantify over), with ∃xAx interpreted dually. While we can't say that all the theorems of R# are *true* in this model (after all, we started 0=0, which looks pretty true, off at the intermediate truth-value N), we can do almost as well. All the axioms of R# are *never false* (i. e., never take the truth-value F in the model), and this property is preserved under the relevant logical rules (*modus ponens* for → and adjunction for &). (Since the reader may expect some trouble to come up over induction, let us briefly ease its mind on the point. (We assume that this paper will be published in machine-readable form.) Induction tells us that whatever holds of 0 and all its successors holds of all numbers. In this model, what holds of 0 and all its successors holds of 0 and 1, and hence of "everything". End of verification.) But $0 = 1$ was *assigned* F from the start. It is *false*, even mod 2. So, since we can only prove in R# things that are *never false* in M_2^3, we can by no means prove that $0 = 1$. And the argument (which the reader is invited to check in technical detail) is completely elementary. And so R# is *consistent*. (In at least one standard sense of this overused term, and in fact in several.)

How did Gödel, who was reputedly pretty sharp, miss this? For all the metaphysical garbage that his work has called forth, spinning out hierarchies of ever more dubious systems for the purpose of demonstrating the non-triviality of the last dubious system, anything which starts counting at 0 and which can get as far as 1 is perfectly adequate to dismiss 0=1 and related nonsense. As all of us would have thought, until we were informed that there were deep theorems of mathematical logic to the contrary. (So if any 8-bit machines--or, for that matter, 1-bit machines--are reading this paper, take heart. For all the talk that you are obsolete, even you can do what Gödel is reputed to have claimed you couldn't.)

But, having dwelt elsewhere at some length on Gödel's Oversight (specifically, in [10], which should sometime see the light of day), we shall be brief about it here. Gödel erred in taking the word of Whitehead and Russell (and manufacturers of related systems) that they had "reduced" arithmetic to logic. (Though 'erred' is too strong here, since [7] already recognizes that systems which are *different enough* from the ones studied in [10] are not necessarily subject to its limitations. Or, in the case of relevant arithmetics, they remain subject to them, but in a way less destructive of mathematical certitude and less crippling to formal epistemology.) The classical logic that was the target of the reduction was a cut-price brand, which conflates many important ideas that are distinguished in ordinary mathematical and linguistic practice. (Cf. our remarks about overloading the word 'consistent'.) Basically, the steps in the argument *connecting* the unprovability of the Gödel sentence '17 Gen r' (which, however one looks at it, is a pretty queer sentence, in addition to being several miles long when put in primitive notation) to the unprovability of garden variety falsehoods like '0 = 1' depend upon fallacies of material implication. If, unimpressed by the classical proclivity to upgrade to 'true' putative logical principles whose actual and intuitive truth-value is 'false', we *block* some of these fallacies, falling into subtle confusions no longer produces garden variety falsehoods willy-nilly. Let us put it this way. What if '17 Gen r' *were* provable? We have, after all, no properly arithmetical grounds to rule it out, and must help ourselves to heaps of the standard set-theoretic mythology to sustain the faith. Were the mythology to crack and the faith to fail, what is the worst thing that could happen to us? At first glance, it is that we shall have a proof of ~(17 Gen r) as well. But, at second glance, we may recall that, at least according to some people, A and ~A together imply what you please. If what you please is '0 = 1', we shall then have a proof of a garden variety and wholly unsubtle falsehood in one more step.

Now it is this sort of thing that we are against, no matter how often established logical authorities and their burgesses try to con us into the view that materially valid inferences are *always* O.K. Not here they aren't! Subtle contradictions do creep into people's reasoning. Nor is there any ultimate defence against the possibility that even humdrum formal systems in "safe" subjects like arithmetic have got caught in the confusion. Note that we said *possibility*. It is for future researchers to discover if we are *actually* confused. (Unfortunately, the only *decisive* answer that they can give us to that question is *Yes*.) For the philosophically exciting, epistemologically injurious, and ultimately mostly insane conclusion drawn from Gödel's Theorem is that we cannot, really, be very sure about *anything* mathematical. By natural extension, since simple arithmetical knowledge is generally accounted as *secure* knowledge, we cannot, really, be very sure about *anything*, at all.

This is a high philosophical price to pay for the technical success of some sneaky self-referential tricks. (They are, to be sure, impressive tricks.) For, in plain words, it simply *does not follow* from the fact that we are possibly confused about *some things* that we are possibly confused about *everything*. Truthfully, Hilbert's formalist optimism is in a way to blame here. Hilbert wished to make *all* of mathematics secure, all together, by demonstrating the consistency of a suitable Super System, using methods sufficiently elementary that *everybody* would have to accept them. ('Everybody' in those days included even the mathematical intuitionists, which was felt to show sufficient obeisance to Fogelin's Rule--"No funny business!" [1], p. 106.) This project failed. It's hindsight, but it deserved to fail. Who expects, or should have expected, an ironclad guarantee that mathematicians have hitherto been right about everything? But, fatuous optimism not having worked, there was all too much fatuous pessimism in the conclusions drawn. (These days--and most days, we suspect, among workaday mathematicians who take their subject as they find it--fatuous fideism is rather more in vogue. If the logicians failed in their efforts to put mathematics on a sound basis, some can do no better than to take Tennyson's advice, "Believing where we cannot prove." Even in the whole hierarchy of ZF sets, or more. But, while we have never wished to knock other people's religions, a more credible line needs to be drawn between the domain of Special Revelation and that of Reason in these things.) To show '17 Gen r' unprovable, maybe we do have to be able to count to ε_0 (in whatever sense that is possible). But it is madness if we have to count to ε_0 to stave off a demonstration that $0 = 1$. For any child would laugh to see one. And if our formal systems cannot contain the contagion of subtle trickery into the domain of plain facts, even potentially, then it is time to send out for better systems. Fortunately, while not all of the technical evidence is yet in, we appear to have such better systems for arithmetic in R# and related theories.[2] Meanwhile, having to count to ε_0 *every time* we want to show some simple arithmetic falsehood undemonstrable is like sending a nuclear device to do the work of a flyswatter. What we can do by counting to 2--or to 6, 14, 42, or 3088--let us by all means do. We shall spare ourselves ontological, epistemological, and other dangerous fallout.

While we have anticipated ourselves slightly, it is no less evident that we can construct mod-whatever-you-please models on the same plan that we used to construct our mod 2 model. (This is useful if one wants to show $2 + 2 = 2$ is likewise unprovable; or $\forall x \forall y(2.x = 2.y)$, and so forth.) We get a variety of further, more general consistency results this way, removing not only primary school arithmetic but high school algebra from the range of the Gödelian fist. We also get a number of further models that are worthy objects of mathematical study in their own right. (See in addition to *op. cit.*, [13] and [15] as well. Dunn's [6] is an interesting related contribution.) All of this (and further Mortensen papers now in preparation or soon to appear, such as [16]) belongs to the topic of Inconsistent Model Theory. (Note that our initial model was intuitively inconsistent by making lots of sentences--for example, $0 = 0$ - *never false* by assigning both them and their negations the intermediate truth-value N.)

We shall use the remainder of this note to make a modest addition to the topic of Inconsistent Model Theory, building both on what we have said above and our work elsewhere. (Though, unless the reader is insistent on checking axioms and the like, which we

[2]Burgess [3] argues--or rather pronounces--to the contrary, in remarks based apparently on its author having got access to our unpublished work. This raises some questions of propriety. Moreover, the actual account of R# given is a straightforward hatchet job, misunderstanding some things and misrepresenting the rest. Assign most of its assertions the truth-value F. Assign the rest N. *Do not*, on any account, assign T to anything in Burgess' paper that bears on the present subject, except perhaps for the author's institutional affiliation.

have left it to look up, this paper will be self-contained. For the insistent, RQ was first formulated in [2]. The arithmetical postulates to add to get R# are those set out in [4], p. 42. (Caution: take these postulates *exactly* in the form found in [4]; do not substitute classical equivalents for them, which are not necessarily relevantly equivalent. And add x=y → x′=y′ and x=y → (z=y → x=z), which suffice with the other postulates to give identity its expected properties, again for the relevant context.) An essentially equivalent formulation of R# appears in [12].)

We have formulated R# by adding the first-order Peano postulates to a relevant quantificational base. But why do that? Even classically, these postulates are known to be seriously incomplete at this level. (To be sure, any level at which they are complete gets into what we have labelled " fatuous fideism".) In fact, there is an alternative R##, which adds an ω-rule to R# ("If for each numeral n the theorem A(n) has been demonstrated, infer ∀xA(x)"), and which, taking the Standard Mythology for granted, contains exactly the standard arithmetical truths in the truth-functional part of its vocabulary. (Note that our model M_2^3, and all models in the same vein, makes all theorems of R## *never false* as well. For it does not, after all, involve us that deeply in fatuous fideism if our domain of quantification is finite.) Moreover, we can of course switch the logical base. For M_2^3 and its ilk satisfy not only R but the stronger system RM and its extensions, which give rise on the same plan to arithmetics RM#, etc. (Most of our work in [13] dwelt directly with these stronger systems.)

Still, despite its deficiencies, first-order arithmetic seems like an interesting test case for application of the ideas of relevant logics. And we have seen that the imposition of relevant distinctions does make a difference, even if our interest in R# is ultimately encompassed by interest in a stronger and more comprehensive system that contains it. And let us now turn to the specific postulates that first-order arithmetic has traditionally been required to satisfy. Taking the others as more or less straightforward and uncontroversial, we shall concentrate on the postulate of mathematical induction. It is, after all, by far the most complicated of the postulates; and, without second-order quantifiers, it boils down to infinitely many separate instances. Nonetheless, it is also the postulate to which fingers point when the deficiencies of first-order arithmetic are being bewailed. For while, on the usual extensional intuitions, we want to assert mathematical induction for uncountably many sets of natural numbers (although it can only be asserted *usefully* for just one such set-- namely N itself, all other sets of natural numbers either failing to contain 0 or failing to contain the successor of some member), a countable language offers only countably many one-place open sentences to *stand in* for sets in our formulation of the induction postulate. So, unless we are terribly lucky (and we now know that we weren't), it is dubious that we have expressed *enough* of the induction postulate to characterize the natural numbers as definitively as we had hoped.

Such is a bit of the conventional wisdom (or at least of the conventional alibi for what went wrong). But what motivated people to pick mathematical induction as a postulate on the natural numbers in the first place? One reason, we cannot help but think, is that this principle and its close relatives are so enmeshed in the ordinary proof procedures applied in number theory that nobody ever thought seriously of *not* having it. But, at least according to Wang in [20], there was considerably more to it historically than that. One of the reasons, which is frequently pointed out, why mathematical induction is such a natural *proof procedure* is that it takes seriously how the natural numbers are *constructed*, being built up from 0 by adding 1's. From this viewpoint, the principle is nothing but the assertion that if a predicate holds of the number we start with, and if its holding is preserved when we construct each number from the last, then most certainly this predicate will hold of the whole lot.

But what if, despite our good intentions on the matter, some impostor manages to masquerade as a number, without having been got from 0 from adding 1's? That effective formal systems are unable to unmask such impostors is at once their shame and glory, motivating the title of this paper, and much else. But they have at least made the effort. The effort is called "mathematical induction", and its place among the arithmetical postulates is the chief product of the axiomatic approach to number theory initiated by Grassmann in [8] and given its modern form in the work of Dedekind (which, scholarship having been up to its usual standards, has caused us ever since to identify the governing principles of arithmetic as the Peano postulates).

But let us hear Dedekind himself on the subject of unmasking impostors ("Letter to Keferstein", translated in [19], p. 100). "What then must we add ... in order to cleanse our system S of such alien intruders as disturb every vestige of order and to restrict it to N? This was one of the most difficult points of my analysis and its mastery required lengthy reflection."

It is a chilling prospect that, among the integers that God made, there might be smuggled in some little green numbers of other ancestry. Could it be that, when these integers stand up to be counted, there is an E. T. among them, which manages sufficiently to resemble its fellows that even the most distinguished (mathematical) scientists are deceived? Might it even be that there is a whole host of these invaders, whose *cover* is that they *appear* to satisfy the Dedekind-Peano postulates for the natural numbers but which no effective specification of these numbers can unmask as phonies?

Well, yes. These things can be. And are, despite the confident quotation just cited. The *idea* was that mathematical induction (or one of its equivalents) so constrained what was to count as a number, in a model of the postulates, that no little green numbers would get in. For the logistic idea had been to put elementary arithmetic on an indisputable basis by laying down postulates from which all and only the arithmetical truths follow. For a while, there was considerable confidence that this had been done successfully. Then Gödel struck, and things have never been the same again. (To be sure, the affinity of our century for wars, revolutions, recessions, famines, and the threat of atomic extinction might also depress some people. But we are speaking here of *mathematical* depression, which we take to be of a higher intellectual order.) Granted, the logistically faithful can still hold that Dedekind got it just right; and that, essentially, his postulates have only one model, which is the collection N of honest natural numbers. (Dedekind did have a categoricity proof, after all.)

But the original logistic idea was struck down by the inadequacy of the accompanying deductive apparatus, a point that we have been dwelling on above. The semantic reflection of this blow, dealt by Skolem in [17] (even before [7]) lies in the existence of non-standard models for N (and much else). On the Standard Mythology (that is, taking the "honest" natural numbers 0, 1, 2, etc., for granted, where you are supposed to know what is meant by 'etc'), these non-standard models differ from N *exactly* by admitting alien intruders--hordes and hordes of them, each of them "infinite" from the viewpoint of N. (For each little green number is bigger than 0, bigger than 1, *und so weiter.*) Then, for the faithful, the question becomes, "How do we formulate the Standard Mythology so that its *non-numeric* objects (for example, sets to whose existence we appeal in reconstructing Dedekind's categoricity proof for N) are themselves shielded from perverse reinterpretation?" Answer--for optimists-- "It isn't easy." Answer--for pessimists--"It isn't possible." For the germ of the Dedekind-Peano idea is that N is the intersection of the successor-closed sets containing 0. (That's what mathematical induction was doing, at least in intent.) If the Honest Natural Numbers are among the sets whose intersection is being taken, this set will surely be a subset of all the others; whence the idea does produce the standard N. But what if the Honest Natural Numbers do not constitute a set *at all*, at least from the viewpoint of some particular model? Then the procedure does not secure its intended effect. The N produced is bogus, at least in the sense of containing alien intruders. Only given a preferred model of *set theory* (or other foundational apparatus) can we insist that N is, near enough, what we wanted it to be. And it is truly and not just formally perverse if, to believe the Standard Number-Theoretic Mythology (which does come naturally to most of us--or seems to, since we were all drilled in it for years) we must believe first in some Set-Theoretic Mythology, or other Tall Tale, which has got to be *more* problematic, not less.

So, all in all, if one wants to Keep the Faith in these matters, the least committal course would seem to be simply to believe in N, and be done with it. (For the remainder of this note, we shall at least *talk* as if we did. But the reader is to draw from that no further conclusions about our (mathematical) religion than we have drawn about its.) We shall take N as the *standard model* of arithmetic. (Everybody else does.) Other models (whatever formal theories they may be associated with, including set theories and classical and non-classical arithmetics) we call *non-standard.* The standard model of arithmetic being pretty boring--almost everybody *thinks* that he or she knows what it is supposed to be, and is only annoyed at not being able to characterize it--it is non-standard model theory that has boomed over the last decade or two. So, little green numbers being in style, let us see whether we can crowd *more* of them into a model of R# than the competition can offer.

Let us return to M_2^3. It is simple enough, and constitutes a paradigm for the kinds of *unexpected* models that satisfy systems of relevant arithmetic. To put the reader's mind to rest, all the *expected* models are there also, both standard and non-standard. For there are two components of a relevant model; a domain of objects, and a domain of truth-values, both with operations appropriate to their category (e. g., + for objects, and & for truth-values) defined upon them. Nothing prevents us from taking the standard {T,F} as our truth-value domain, with operations defined classically. If we do, the models for R# under this restriction will be exactly the ones for the corresponding classical first-order Peano arithmetic P. So we do not *lose* any models.

But we do gain some. Indeed, it is pleasant to find the *most natural* models of the integers--the ones that one finds in the first few pages of any text on abstract algebra, namely the integers mod n for finite n--among the models for R#. For note that, *whatever* we do with truth-values, our collection of objects and associated operations in a model may be characterized independently as whatever sort of abstract algebra it is. For M_2^3, this domain D is a very pleasant algebraic object indeed; for the integers mod 2 constitute a *ring* under the operations . and +; and, for that matter, a *field*. This may be an *inconsistent* model. But it is a *nice* one. And, while perhaps we should have learned by now to expect no better, one of the more annoying things about non-standard models of P is that they are *not nice*. (Still, there are those who have learned to love them.) Except that these models of P are collections of authentic numbers and little green numbers that have set up housekeeping together--ordering themselves on the aforementioned plan that all the authentic members of N come first (well, it's nice that they have *some* priority), in their natural order, with infinitely many copies of the *integers* following in slab after slab, the slabs themselves being as densely packed as if they were rational numbers. (The jargon is that these models have order type $\omega + (^*\omega + \omega).\eta$, which is not an order type that you would like to bring home to Mother, unless she has cooked an exceptionally large meal.)

Well, there is something perverse going on if the logician can countenance structures of the latter kind as modelling arithmetic, while ruling out the more natural and familiar *mod* models as models of the natural numbers. What's perverse about it? It is only by extensional courtesy (which the category theorists may be in the process of withdrawing) that we think of the natural numbers and their associated operations and relations as *collections of objects* at all. What they are, more fundamentally, are *rule-governed structures*. If, for whatever reason, we want to think about models of a theory other than the *intended* one(s), it is the *rules* that go into the stipulation of the intended structure that we most want to preserve, in as simple, clear, and natural a way as possible. While we may get some other sorts of models anyway--just because our original stipulation was incomplete,[3] and perhaps ineluctably so--they are in some sense accidental models.

There is another path, which as [6] points out is more familiar to the algebraist. Instead of getting extra models by *underloading* a theory--i.e., having a theory satisfy fewer constraints than we really want, because it is *incomplete*--we can also get interesting models by overloading that theory-- intuitively, making it *inconsistent* by identifying things that, really, we take to be distinct. This is the effect of the algebraist's *morphisms*, if we think about it. There is a function h from the natural numbers to the integers mod 2, which takes the even integers onto 0 and the odds ones onto 1, preserving meanwhile the chief algebraic operations on integers. From the viewpoint of N, this is a *confused picture*. N wants to distinguish 6 from 16, though they are not distinguished mod 2. But the picture is not a *completely* confused one. Indeed, since the computers that run the world these days do their own integer arithmetic mod 2-to-the-power-of-something, one can get a great deal done in this confused picture. And there is no reason in the world why a model that results from *fruitful confusion* (because it corresponds to intuitive inconsistency) is less interesting or worthy of study than one which results from *fruitful ignorance* (corresponding similarly to incompleteness).

So Dunn's view of these things in [6] is, we think, very nicely put. Moreover, being algebraically familiar, it belongs in the logician's bag of tricks. But there is a realm, to which we now turn, where we can make use of what goes with incompleteness and what goes with inconsistency together. It is the realm of *inconsistent non-standard* models of arithmetic. For just as we have shown that the *standard model* N of classical Peano arithmetic P can be collapsed mod n, for finite n, to make the integers mod n a model of R#, just so we can collapse an arbitrary *non-standard model* M of P mod an "infinite integer" n to make these "integers mod n" a model of the relevant Peano arithmetic R#.[4] Note the interplay of inconsistency and incompleteness here. Because P is incomplete, it has non-standard models. But no homomorphic image of one of these models is a model of P, because 0 becomes a successor. When eyeball-to-eyeball with inconsistency, P blinks. But R#, we shall see, does not blink. Its models preserve the distinctions that homomorphic images preserve, while still being models of *all* the first-order Peano postulates. Moreover, some of these inconsistent non-standard models are intrinsically quite interesting, as we shall see. Here's how to form

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[3]Negation-incompleteness, to be sure, is only part of the story, though it yields by the *completeness* theorem for 1st-order logic that systems like P will have unintended models. But there is also the point that formal *methods* are incomplete. So, by Skolem, even negation-complete theories may have such models.

[4]Thanks to Dr. Gordon Monro for having suggested in conversation several years ago that this fact might prove useful from a Relevant viewpoint.

them. (Now follows the part of the paper which readers more interested in impressions than technical details may wish to skip.)

Let M be an arbitrary model of P. While M is not quite a ring, it is near enough that the usual methods of forming homomorphic images will work. We shall continue, by courtesy, to refer to the elements of M as " integers". (At least the ordinary natural numbers are among these integers, as we have seen, together with whatever alien intruders have sneaked in.) And we shall also continue to use 'M' to refer indifferently to its base set; to the resulting structure in the algebraic sense, with appropriate operations +, ., ′ defined on it; and finally to the model in the semantic sense, which makes each sentence of P either true or false. As for the homomorphic images that we investigate, we shall consider only those induced by a principal ideal (nearly), determined by a particular element n of M.

Specifically, given n in M, we define the structure M_n as follows: we define a relation ≡ on M on the rubric a ≡ b iff, for some x in M, either a + x.n = b or a = b + x.n . It is elementary that ≡ is a congruence on M with respect to the chosen operations (using fundamental properties of +, ., 0, ′ and the fact that M is a model of P), whence we shall say that two elements a,b of M are *congruent mod* n provided that a ≡ b. We may now pass, in the usual algebraic way, to congruence classes, forming the structure M_n of *integers mod* n. Since ≡ is a congruence, the elements 0 and the operations +, ., ′ are well-defined on M_n. And M_n is a natural homomorphic image of M, on the morphism h which takes each element a of M to the set of elements congruent to it. We shall normally refer to the elements of the quotient algebra by their representatives, noting that for each congruence class there is exactly one member b of N such that b < n in M and which belongs to this class. (There cannot be two such representatives, since if a and b are congruent they must differ by a multiple of n, whence one of them will be no less than n. And, by the least number principle, the set of elements congruent to each b must have a least member a, which will serve as a representative; for, if a were not less than n, we could subtract n and get a smaller representative. As usual, the "kernel" of the homomorphism--the set of elements congruent to 0-- consists exactly of the "integers" divisible by n.)

There are a couple of more or less silly choices of n, though we permit them for completeness. (Though, should we be careless in our statement of some theorem, we trust the reader to exclude the silly choices.) Setting 1 = 0′ as usual, we note that M_1 is the trivial 1-element algebra. And M_0, which identifies a and b just in case they differ by a multiple of 0--i. e., not at all--is, for all practical purposes, just M itself. Except in this last case, M_n is a ring, even though M is not. Indeed, M_n is a commutative ring with unit. For its non-zero elements certainly form a commutative monoid under multiplication, with identity 1. Equally certainly, M_n is an additive commutative monoid, with identity 0, (In the other silly case M_1, we have 0=1.) Moreover, . distributes over +. So we need only show that there exists an additive inverse -b for each b in M_n . Since the system P compels each non-zero element to have a predecessor, n in particular has a predecessor (if $n \neq 0$), which we might as well call -1. It is then apparent that -1.b will have the right group-theoretic properties to be the inverse -b of b for each b in M_n.

The reader who was going to accept our invitation to skip the hard stuff but who decided to read on a bit must, by now, be feeling pretty superior. It has hit nothing yet that it could not easily have looked up or worked out in its CPU, in nanoseconds. Meanwhile, let us spell out what M_n does, as a semantic structure. (When we are so thinking of it, we may refer to it as M_n^2, to indicate that it has $\mathbf{2} = \{T,F\}$ as its set of truth-values.) Let L# be our arithmetical first-order language, which we think of as extended to a language L_n# by adjoining each member of M as a new individual constant (to be used as its own name). There is a natural interpretation function I_n^2 associated with M_n^2, whose arguments are all the closed terms and formulas of L_n# (and *a fortiori* of L#), with values in M_n for closed terms and in {T,F} for closed formulas, constructed on the usual recursive specifications. We may extend I_n^2 to all formulas, if we wish, by letting its value on an open formula be that of any universal closure of this formula. And M_n^2 is then a *model of* any formula A such that $I_n^2(A) = T$; it is a model of a *set of formulas* iff it is a model of each formula in S. (Similar definitional remarks apply of course to M itself, whose associated interpretation function and extended language we shall call I_n^2 and L_M# respectively.)

We chose M as a model of P. And M_n, we know, is not a model of P. But it is not far from one. Indeed, let P+ be the system formulated like P, but without the axiom $\forall x \sim (x' = 0)$ forbidding 0 from being a successor. Then

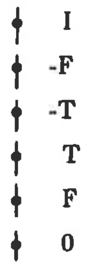
Observation. M_n^2 is a model of P+.

We are indebted to Paris and Wilkie for pointing out in conversation that the observation is obvious. Indeed, that most of the first-order Peano axioms hold in M_n^2 is immediate from the fact that it is a homomorphic image of M. The only tricky one is mathematical induction. But this is not tricky either when we realize that congruence mod n is already definable in the language L_n# itself, by

$$D\equiv.\ t \equiv u =df\ \exists x(t = u + x.n \vee t + x.n = u),$$

taken schematically for all terms t and u. (Evidently we could define congruence mod an arbitrary term w by putting w for n in the *definiens*. But since it is congruence mod this particular n that we have most in mind, we shall stick to the special case where n is a constant of our extended language.) Having fixed n and used it to define $\equiv$ contextually, let A be any formula of L_n# and let $A_\equiv$ be the result of systematically replacing all occurrences of '=' in A (in primitive notation) with the defined '$\equiv$'. It is quickly observed that A is valid in M_n^2 iff $A_\equiv$ is valid in M, in the usual classical sense. In particular, if A is an instance of the scheme of mathematical induction, $A_{\equiv}$ is *also* an instance of mathematical induction. This is valid in M, because it is a model of P. Accordingly, A is valid in M^2_n, by the correspondence just noted. Everything else being straightforward, it is then obvious that M^2_n is a model of P+. This is the content of our observation, ending our sketch of its verification.

We now use the following recipe to turn a "mod" model M^2_n of P+ into a "relevant" model M^6_n of all of R#. (Basically, it is an application of the technique used in [14] to show R# a conservative extension of its negation-free mate $R\#_+$.) First of all, consider the following Hasse diagram, of a lattice we henceforth call just **6**.



While our nomenclature for **6** differs from our usual policy (we generally reserve 'F' for the bottom element and 'T' for the top element of a DeMorgan monoid, as **6** will turn out to be), and have labelled things as we have to make classical connections. As usual, & will be interpreted as lattice meet, and ∨ as lattice join (in this case just *min* and *max* in the displayed order). ~ is order-reversing, subject to ~0 = I, ~F = -F, ~T = -T, etc. The truth-table for → is given as follows:

→	0	F	T	-T	-F	I
0	I	I	I	I	I	I
F	0	T	T	-F	-F	I
T	0	F	T	-T	-F	I
-T	0	0	0	T	T	I
-F	0	0	0	F	T	I
I	0	0	0	0	0	I

What **6** is, really, is the result of taking truth-tables (viewed as a Dunn monoid) into a DeMorgan monoid, conservatively adding "DeMorgan negation" on the plan first set out in [9]. To make that point, we are using F and T to stand for (classical) *false* and *true*. (But T may be also identified with the intuitive "least truth" t required by DeMorgan monoid theory. But its mate, the "greatest falsehood" f, is in fact -T. As "designated elements" of **6**, count everything in the principal filter determined by T: i. e., all of T, -T, -F, and I.)

It is readily verified that **6** satisfies the postulates on a DeMorgan monoid laid down by Dunn in [5]. (The lazy need merely note that it turns up among the structures generated by the program TOPSY--so named by Slaney because it "just growed"--explained in [18].) Since DeMorgan monoids stand to R as Heyting lattices stand to intuitionism, this means that **6** will look after the relevant verification of all quantifier-free logical postulates; and, being finite (and hence a complete lattice) it will also look after the verification of relevant logical postulates containing quantifiers, with ∀ interpreted as a generalized meet and ∃ as a generalized join (which, in a simply ordered case like **6**, means just that ∀xAx will get the "falsest" value of any of its instances At, while ∃xAx will get the "truest".

What is nicest about **6**, for our immediate purposes, is that it offers a convenient way to turn *classical* theories, and their accompanying models, into *relevant* ones, while still preserving relevant distinctions. Many of these distinctions have to do with negation. The DeMorgan ~A *denies* A. But it does not, like its Boolean cousin, confuse the denial of A with the *assertion* that A implies anything whatsoever. (Boolean negation is nonetheless an interesting and valuable connective of relevant logics, unjustly pilloried by authors of whom we otherwise speak well. But there is not time to go into all that here.) Thus from one viewpoint--there are others, consistent with relevant insight-- classical logic simply *lacks* negation. Or rather it *confuses* negation with A → F, where the distinguishing feature of F is that it implies everything (classically) in sight. As is well-known, 0 = 1 will do as such an F arithmetically. So we might rethink classical theories as *positive theories*, whose logical particles are just &, ∨, →, ∀, ∃, and which introduce (what passes for) negation via some sentence silly enough to imply absolutely everything.

While we are not necessarily recommending this view of classical negation, let's try it out. We would then view classical logic as a *positive logic*, consisting of all the classical tautologies in the positive particles just listed. A *classical theory* would then be any theory T which contained all those tautologies and which was closed under →E. If a classical theory does contain an F that implies everything, according to this theory, then it is moreover classical in the usual sense, defining (classical) not-A as A → F. Let us call such a theory a *classical F-theory*. Provided F is not itself a theorem of the theory (which would rather spoil things), any classical F-theory will have the usual models. It is time for a theorem.

Model transformation theorem. Let M^2 be any model of a classical F-theory, in the usual sense. We may extend M^2 to a model M^6, in the DeMorgan monoid **6**, with the following properties. (1) All theorems of the first-order relevant logic RQ are valid in M^6. (2) The class of valid sentences is closed under the rule →E (*modus ponens* for →) and the rule &I of RQ. (3) Every sentence true in M^2 takes the value T in M^6 (and is hence a "good guy", by the lights of M^6). (4) Every sentence false in M^2 takes the value F in M^6 (and is hence a "bad guy", by those same lights). (5) Let A be *any* positive sentence; then ~A takes a designated value in M^6. (Accordingly, adding any class of such sentences as new relevant axioms produces a conservative extension of the classical theory corresponding to the *truths* in M^2, and *a fortiori* produces a conservative extension of the original classical F-theory, understood now as a *relevant* theory.)

Explanation. Before *proving* this theorem, we wish to remind the reader what it *means*. We are thinking of our F-theory as lacking ~ in its formation apparatus, while containing all positive particles, including →, which the theory in question treats as *material* implication. From this viewpoint, extending M^2 to M^6 is *introducing* negation, taken as a new primitive particle ~, and subject to all usual negation laws (double negation, excluded middle, non-contradiction, etc.) except paradoxical ones. It then turns out, in line with previous results, that we can be quite free about adding axioms of the form ~A, or their *relevant* equivalents, when A is in the old, positive vocabulary. (These things are essentially known, from [11]. What is striking is that, if we start from a *classical* theory, they can be put very simply.)

Proof. On well-known syntactical maneuvers, we may assume that only *sentences* (i. e., formulas without free variables) enter into our stipulations of logics and theories. We also assume, in specifying the modelling conditions, that the language of a theory is enriched to include names for all elements of the model, as above. Assuming that M^2 has been given, with an associated interpretation function I^2 defined on all closed terms and sentences of our (positive) enriched language with values in {F,T} for sentences, we characterize M^6 and its associated interpretation function I^6 as follows. (i) The domain D of *objects* of M^6 shall coincide with the objects of M^2, and I^6 shall coincide with I^2 on all closed terms. (ii) The domain of *truth-values* of M^6 shall be the lattice **6** displayed above. (iii) I^2 and I^6 shall coincide on atomic formulas, identifying classical T,F with the T,F of **6**. (iv) On propositional connectives, I^6 shall be homomorphic; i. e., $I^6(\sim A) = \sim I^6(A)$, $I^6(A \rightarrow B) = I^6(A) \rightarrow I^6(B)$, etc. (v) $I^6(\forall x Ax) = I^6(At)$, where this value is least under I^6 for any sentence of the form At, where t ranges over the closed terms of the extended language. (vi) $I^6(\exists x Ax) = I^6(At)$, for the greatest such value of an At under I. (vii) a sentence is *verified* in M^6 iff it takes a value $\geq$ T under I^6 in the lattice ordering of **6**. I. e., A is verified if $I^6(A) \in$ {T,-T,-F,I}.)

While that was somewhat long-winded, how the specification operates is clear. I^6 *extends* I^2 by looking after formulas not in the original positive vocabulary; in particular, those containing DeMorgan ~. M^6 has extra truth-values to look after that extension. The particular assertions of the theorem are now straightforward. Because **6** is a finite DeMorgan monoid, *any interpretation* I therein is going to verify first-order relevant axioms and rules; this disposes of (1) and (2). Since I^2 and I^6 *coincide* on all positive sentences, (3) and (4) are also immediate. As for (5), a positive sentence must take one of the values T, F; so its negation must take one of -T, -F, either of which suffices to verify it in M^6. Moreover, since the class of sentences verified in M^6 contains all relevant tautologies, all sentences in our

F-theory, all negations of positive sentences and is closed under relevant rules, it is evidently a regular relevant theory which extends the F-theory we started with; but, since every non-theorem of this theory is refutable in some M^2, and hence in a corresponding M^6, any relevant denials of positive formulas may be added conservatively, in accordance with our parenthetical remarks. This completes the proof of the model transformation theorem.

Having the model transformation theorem in hand, we can now return to our previous considerations. Here is a corollary to look after them.

Corollary. Let M be any model of classical first-order Peano arithmetic P. Let $n \neq 0$ be an element of M, and let M^2_n be the classical model of P+ "modulo n", with interpretation function I^2_n. Then M^6_n, univocally determined by M^2_n by the recipe of the theorem, is a model of R#, which verifies exactly the same *positive* sentences that M^2_n makes classically true; moreover, the "integers" of M^6_n are just those of M^2_n.

Proof. Everything follows immediately from the theorem, except the statement that M^6_n is a model of R#. Remember that, classically speaking, M^2_n *failed* to verify the postulate which says that 0 is a non-successor. But that means, under our "positive translation" of P+, $\exists x(x' = 0) \rightarrow 0 = 1$ (near enough). Clearly this *deserves* to be a non-theorem of P+, since it is truth-functionally refuted mod 2. It is a different matter, relevantly, to assert $\forall x(\sim x' = 0)$ (which, of course, is completely equivalent to $\sim\exists x(x' = 0)$.) As a Peano postulate, and hence as an axiom of R#, it has to be verified in any model of R#. This may surprise the reader who skimmed our earlier remarks, but who has remembered that, in M^2_n, 0 *is* a successor; specifically, it is the successor of what we playfully called -1 above. But recall Dunn's interpretation of our policy in such matters; we are dealing with a "confused" homomorphic picture of an M in which 0 was *not* a successor. This suggests, as before, that *both* $\exists x(x' = 0)$ and its negation should be verified in M^6_n. And this is in fact what happens; since $-1' = 0$ according to M^6_n, $(\exists x(x' = 0)) = T$; but then its negation takes the value -T, also a "good guy". (Indeed, according to our theorem, we could add the negation of any positive formula, or a relevant equivalent, conservatively as a new axiom.) So the non-successor postulate, as an axiom of R#, is verified after all.

Otherwise, there is *almost* nothing to the verification M^6_n is a model of R#. The logical axioms and rules, and all positive proper axioms, are looked after by the theorem. This covers everything but mathematical induction. For this principle is *schematic*, of the form $A0 \ \& \ \forall x(Ax \rightarrow Ax') \rightarrow \forall xAx$. Evidently, if Ax is a *positive* formula, the corresponding instance of mathematical induction will be verified in accordance with the model transformation theorem in M^6_n, given that we have already observed that it is true in M^2_n. But Ax may contain DeMorgan negation, which is not covered directly by the theorem. So it is necessary, as in [14], to dig a little deeper. We observe first that, where Ax is any formula in which at most x occurs free, M_n^6 constrains the interpretative possibilities as follows: (a) For all closed terms t, $I_n^6(At) \in \{T,F\}$. Or (b) for all closed terms t, $I_n^6(At) \in \{-T,-F\}$. Or (c) for all closed terms t, $I_n^6(At) = 0$. Or (d) for all closed terms t, $I_n^6(At) = I$. Proof is by induction on the complexity of Ax. (a) holds if Ax is atomic. The other cases are settled by straightforward inductive argument, which we leave to the reader. It is then evident that, in cases (c) and (d), the induction axiom must take the value I, verifying it. We next wish to show that, in case (a), we have Ax equivalent in the model M_n^6 to a *positive* formula, in the sense that there exists a negation-free formula Bx such that $\forall x(Ax \leftrightarrow Bx)$ is valid in the model; and that, in case (b), we have Ax equivalent in the model to the *negation* of a positive formula. Again we argue by induction on the length of Ax, the atomic case being immediate. Again leaving details to the reader, this settles case (a), replacing equivalents and noting once more that induction holds for Ax positive, by construction. But it also settles case (b); for, in this case, $\forall xAx$ must be one of -T, -F on interpretation; while $\forall x(Ax \rightarrow Ax')$ must be one of T, F, which will force the conjunction in the antecedent of the induction scheme to precede the value given to the consequent in the ordering of **6** under interpretation. This completes the verification of the induction postulate in M_n^6, and with it the proof of the corollary from the preceding theorem.

After all this, we may exhibit a model of R# with alien intruders aplenty.

Alien Intruder Theorem. Every rational number is a non-negative integer. That is, there is a model M of R# such that the following obtain, in a straightforward sense. (a) Every rational number is an element of M. (b) The ordinary laws of rational arithmetic hold, for addition, multiplication, subtraction, division. (c) The Peano postulates are satisfied by M, including mathematical induction.

Proof. Add a new constant n to the vocabulary of P. Define as above as congruence mod n. Form an extension T of P by adding as new axioms $\forall x(x \equiv 0 \vee \exists y(x.y \equiv 1))$; and, for each numeral m corresponding to a "standard" natural number, $\sim m \equiv n$. T must be a consistent theory. For, if it is inconsistent, some finite conjunction of these added axioms must be inconsistent; however, there is then a standard number p such that (i) p is prime and (ii) all numbers named in the inconsistent finite conjunction of added axioms are less than p. But, interpreting n as p, we have a model for the finite subset of axioms alleged to be inconsistent (for no number smaller than p is congruent to p mod p; while since the integers *mod* a prime number constitute a field, the axiom asserting the existence of a multiplicative inverse for each element not congruent to 0 mod p will also hold). So, by compactness, T has a model, which we may call M*. Forming congruence classes mod n, we get a model M_n^* of P+, which is then transformed into a model M_n^6 of R# in accordance with the preceding corollary. This is our desired M, in which the Peano postulates hold (in their first-order relevant version) by the corollary. We must now show that all rational numbers are admitted by M.

In the first place, since no standard natural number was congruent to n in M^*, every standard number is an element of the quotient algebra M_n^* (and hence of M, which has the same objects in its domain). Using -1 as before to denote the element whose successor is 0, for each standard positive integer m in M there will be a corresponding standard negative integer -m = (-1).m; moreover, since M is in the natural way a ring, these standard integers constitute a sub-ring of M, as expected. But, for each non-zero integer m, our added postulate guarantees a reciprocal 1/m. (The notation is justified, since multiplicative inverses must be unique. For suppose, to the contrary, that there are distinct i and j such that m.i = m.j = 1. But then j = 1.j = i.m.j = i.1 = i, whence i and j are not distinct after all.) We may then form the arbitrary rational fraction k/m as k.(1/m), for m 0, in M. For k, m rational standard integers, these fractions may be reduced to lowest terms in the usual way; the usual laws of rational arithmetic will thereupon hold, ending the proof of the theorem.

All of this is, to say the least, most startling and wonderful. Bereft of the paradoxical properties of classical negation, it is possible to satisfy the Peano postulates in a domain (indeed, in many domains, since every non-standard model M^* of P may be transformed into a model M of R# in this fashion) that contains all the rationals, among the alien intruders. The price of this, to be sure, is inconsistency; in the model-theoretic sense, because M verifies both A and ~A for some choices of A; and in the syntactic sense, taking the class of sentences verified in M as a relevant theory that extends R# (as it is) but which contains explicit contradictions of the form A&~A. For those addicted to the "foolish hobgoblin of little minds", there is little that we can say at this point; for those who take consistency to be the hallmark and only criterion for mathematical existence, there is even less. But, to be honest, there was never much to be said *for* this criterion. In whatever shadowy sense mathematical entities may be said to exist, their *interest* lies in the beauty and richness of the structures to which they give rise, and to the possibility of applying these structures to the real as opposed to the mathematical world. No one will deny that rational numbers are applicable to the world; if, in applying them, we unmask them as just another sort of integer, we can view that as just another beautiful and interesting fact about them. And the trouble with inconsistency, as fatal to mathematical existence, lies not with the inconsistent as such; but in the fact that, if negation is allowed its horrendous paradoxical classical properties, the presence of a single inconsistency calls the mathematical game off, allowing everything to be proved. If the game can still go on--indeed, even perhaps become more interesting--if we adopt relevant and not merely truth-functional canons of inference, then there is no reason in principle not to do so. As a recipe for reconstructing mathematical reason, there always was a good deal wrong with classical logic--if only because intuitive reason is subject to those relevant constraints that its truth-functional regimentation ignores. If we attend to those constraints in our reconstruction of logic, and do not elevate ignoring them to the level of unreasoned dogma, there are mathematical worlds undreamed of yet to conquer.

Meanwhile, it is interesting to see how far we can go along the lines just laid out. Can we, for example, construct a model of R# on the above plan whose elements are *exactly* the standard set Q of rational numbers? The answer is "No". For, among the theorems of P, there are assertions like, "Every natural number is the sum of four squares." These assertions are preserved on the M_n style passages to homomorphic images, and on to the resulting "inconsistent" models of R#. In these homomorphic images, -1 is a number. Accordingly, -1 is the sum of four squares. Evidently, when all rational numbers are elements of our model, -1 cannot be the sum of four squares of rationals. So any model of this sort which includes the rationals must include other elements, not to be identified with any rational number. (As, on a little reflection about the subject, is perfectly clear anyway.)

But the method which produced the Alien Intruder Theorem will go a lot further. Perhaps, for example, one would like to have the imaginary number i among the integers, without identifying it with any standard integer (recalling, e. g., that $4^2 = -1$ mod 17, whence $4 = i$ from that viewpoint; to be sure, so is 13, which is a little disturbing). Well, we can pull the same trick. Add $\exists x(x^2 + 1 \equiv 0)$, while denying for each numeral m that $m^2+1 = 0$. The result, by compactness, is a consistent extension of P, which must have a model containing an element i such that $i^2 = -1$, where i is distinct from all rational numbers when we collapse mod the appropriate congruence. Question: can we view all *real* numbers as integers? Answer: we don't know.

But, the reader may protest, what you have done is extremely silly. Above all, the natural numbers are characterized by induction. How can, say, 3/2 count seriously as an admissible alien intruder? Could anybody give as a serious reason for 3/2 possessing a certain property (i) that having that property P is preserved under adding 1's and (ii) 0 has P? We can, to be sure, conclude on that basis that 0 has P, 1 has P, 2 has P, etc. But, in the process, we seem to have skipped 3/2 on our way from 1 to 2. Would it not be *magical* if 3/2 had the property P as well?

We should like to respond, first, that it is magical that induction should hold *anywhere* except in the standard model. We do not get to ordinary non-standard integers either from 0 by adding 1's; yet, in any non-standard model of P, these greatly outnumber the standard integers. But, second, the idea actually sounds less silly with respect to the rational numbers--which after all are elements of a well-known and rather intuitive structure--than it does of the usual "non-standard integers". If our arithmetical intuitions start with natural numbers, on what basis do we intersperse or tack on extra elements to a well-known number sequence? There are, after all, *all kinds* of structures in which the natural numbers can be embedded. Some of these embeddings have themselves come to be thought "natural"--into the integers, the rationals, the reals; and, more recently, into the ordinals and cardinals. Non-standard models of the usual sort, from this viewpoint, were just more of the same.

All of these embeddings are characterized *not merely* by the fact that the natural numbers can be located in the resulting structure, *but also* by the fact that *certain laws* governing the natural numbers are *preserved* (and *extended*) as we enrich the number system. But, from this viewpoint, we must ask, "Which laws are to be preserved? And why do we believe those laws in the first place?" This leads us not merely to mathematical induction, but to something resembling scientific induction. A facetious answer to the question, "Why should 3/2 have property P if 0 has P and successors of P have P?" might be "Well, if all those guys have P, any scientist would believe that 3/2 has P also, in the absence of specific reason to believe the contrary!"

Behind this facetious answer there is a point. What sorts of laws do we have an interest in preserving? Typically, ones like 'x+y = y+x', '0.x = 0', 'x.(y+z) = (x.y) + (x.z)'. (To be sure, we aren't *forced* to preserve them; e.g., on generalization to ordinals, commutativity of + fails; but such failures tend to result because there is *something else* that we have an interest in preserving--in the case of ordinals, that every non-empty set of them has a least element.) But why should we have believed that addition was commutative in the first place? Presumptively, mathematical induction gives a *reason.* And if we then wish to hold '(3/2)+y = y+(3/2)', is there nothing to the thought that, since mathematical induction has conferred this property on the natural numbers, we want it to hold of the rationals also? And since we have long thought of integers as special sorts of rational numbers, perhaps it is time to return the favour; alien intruders they may be; but, by our theorem, rational numbers are conversely special sorts of integers.

It will not do, to be sure, to push these sorts of speculation too far. But they do raise food for thought, as well as interesting directions for research. Another view of a mathematical structure never hurts. For, if all else fails, it might help us to prove a theorem that has hitherto escaped us. Or give us some further insight into what we wish to prove. And why.

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Inconsistent Number Systems

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1 Introduction In a previous paper ([8]), it was shown that there are finite inconsistent arithmetics which are extensions of consistent Peano arithmetic formulated with a base of relevant logic, and also of the set of truths of the classical standard model of arithmetic. In the present paper, the study of the operations of inconsistent number-theoretic structures, especially finite structures, is continued. The interest is particularly in displaying inconsistent theories and associated finite structures which extend standard classical structures, in the sense that all truths of the latter hold also in the former. The principal thesis to be argued on that basis is that classical mathematics is a *special case* of inconsistent mathematics.

The view of mathematics, as based on classical two-valued logic as a deductive tool, has it that from inconsistency all propositions are deducible. Hence, inconsistency-toleration is achieved in the present paper by use of a logic with a weaker deductive relation $\vdash$, the three-valued logic RM3, the third value of which has a natural interpretation, 'both true and false' (cf. Section 2). It should not be thought, however, that theories in which a weaker $\vdash$ is used inevitably lead to sacrifice of some classical propositions. It is one purpose of this paper to demonstrate this, by displaying inconsistent theories which contain various well-known classical consistent complete subtheories.

Aside from its capacity for contradiction containment, RM3 is chosen for two reasons. First, being three valued it is reasonably easy to deal with, particularly in yielding a rich model theory. Second, every RM3-theory displayed is also a theory of all the usual relevant logics such as E and R, which have an independently natural motivation. The interest of those logics for mathematics may be judged accordingly. Indeed, since every classical theory is an RM3-theory and thus also an E- or R-theory, the "special case" thesis above has another dimension: just as consistent mathematics is a special case (under the assumption of consistency or closure under classical deducibility) of inconsistent mathematics, so classical logic is a special case (in which closure under classical deducibility, for instance the rule of Disjunctive Syllogism, holds over a limited

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subject matter) of inconsistency-tolerating logic. Again, no sacrifice of mathematical richness is envisaged; on the contrary, the hope is to show that further mathematically rich structures are to be uncovered by the expanded perspective.

In Sections 2 and 3, the basic model-theoretic framework is set up. In Section 3, this is used to study inconsistent theories which include the classical theories of various rings and fields—that is, the standard theories of addition, multiplication, subtraction, and division. In that section the notion of a model with identity is defined, and it is argued that the existence of inconsistent models with identity supports the thesis that inconsistent mathematics can be seen as extensional in a perfectly standard sense of that term, as a study whose subject matter can be viewed as objects with inconsistent properties. An important outcome of this section is that it is not so easy to develop an inconsistent theory of fields even with an inconsistency-tolerating logic; the problems seem to be deeper, to do with identity and functionality. In Section 4, order is studied; and in Section 5, order is put together with the arithmetical operations to study ordered rings and fields.

A model-theoretic framework is employed, but I suggest that this is a consequence of the fact that intuitive inconsistent thinking is undeveloped (though not entirely absent) among mathematicians and logicians. It is to be hoped that its development will not prove ultimately impossible, but in its absence it is necessary to demonstrate that control of the deductive consequences of contradictions is possible. Thus, it is certainly not being claimed that the 'natural logic' of mathematicians is nonclassical, a disputed question in recent debates within philosophical logic (see e.g., [3]–[6], [10], [11], [14]). Mathematicians do seem to be habitual consistentizers. But if there is any way to expand this perspective, it must proceed by demonstrating the existence of rich mathematical structures which are nonetheless inconsistent. Conversely, the paraconsistency movement has somewhat shirked its duties in calling for inconsistency-tolerating logics but omitting to demonstrate the existence of rich inconsistent mathematical theories (e.g., [14]). If, say, there were no particularly interesting inconsistent theory of fields, perhaps because of problems about the desired functional properties of division and subtraction (as in the light of Section 3 may well turn out to be the case), then it is no use calling for a paraconsistent logic if it is not much use when you get it. Nonetheless, this paper aims to show that an expanded perspective is available. There is, I suggest, a 'seamless web' between classical consistent structures with a limitingly simple logic (tacit or not), and structures in which less deductive power leads to increased richness and freedom.

One direction in which this paper could be extended is toward nonstandard models of various number theories. This is done in sequels ([12] and [13]).

2 ***Basic definitions and the extendability lemma*** We begin with a general notion of an *assignment* which has minimal semantic features and then work toward semantic features of *models*. The point of the exercise is to pick apart some simple model-theoretic concepts which coincide classically, taking advantage of the greater freedom afforded by weakened background logic. We will see that some of the remaining connections are invariant with respect to broad variations in background logic, while others are specific to RM3. The eventual aim is to establish conditions, in Section 3, under which the resulting structures look more like extensional theories of inconsistent objects.

We consider various sublanguages of the language L consisting of simple terms (names), one for each real number; function symbols $+$, $\times$, $-$, $\div$; atomic predicates $=$, $<$, $\in$ (the latter is used only briefly in Section 5); variables x, y, $z, \ldots$; and operators $\neg$, &, $\forall$ (the latter also written ()). Complex terms, wffs, and sentences are defined in the usual way, as are $\supset$, $\vee$, $\equiv$, and $\exists$. We regard sentences of the form $t_1 = t_2$, $t_1 < t_2$, $t_1 \in t_2$ with no occurrences of $\neg$, &, $\forall$ as atomic, irrespective of whether the terms contain occurrences of function symbols. Only theories whose theorems contain no free variables are considered, and, for simplicity, no term is a variable. An RM3-*assignment* (abbreviated to 'assignment') is a function I assigning to the wffs of L, or the appropriate sublanguage of L under investigation at the time, values from the set $\{T,N,F\}$ in accordance with:

(1) For any atomic wff with terms t_1, t_2, we have $I(t_1 = t_2)$, $I(t_1 < t_2)$ and $I(t_1 \in t_2)$ all belong to $\{T,N,F\}$, (read 'true, neuter, false').[1]
(2) $I(\neg A)$ and $I(A \,\&\, B)$ are given by the RM3-matrices:

&	T	N	F	¬
*T	T	N	F	F
*N	N	N	F	N
F	F	F	F	T

(3) $I((x)A) = min\{y$: for some term t, $I(A(t|x)) = y\}$, where *min* is relative to the ordering: false < neuter < true. A sentence A *holds in an assignment* I iff $I(A) \in \{T,N\}$.

A subset S of L is an L-*semitheory* (relative to Logic L) iff if $A \in S$ and $A \vdash_L B$ then $B \in S$. S is an L-*theory* iff S is an L-semitheory and in addition if $A \in S$ and $B \in S$ then $A \,\&\, B \in S$. Where no confusion will result, we often drop the 'L–' when L = RM3. A set S of sentences is *determined by an* RM3-*assignment* I iff ($A \in S$ iff A holds in I). A set S of sentences is *consistent* iff for all closed wffs A, not both $A \in S$ and $\neg A \in S$; otherwise *inconsistent*. S is *trivial* (or *absolutely inconsistent*) iff $S = L$; otherwise *nontrivial*. S is *complete* iff for all closed wffs A, either $A \in S$ or $\neg A \in S$; otherwise *incomplete*. If S is determined by an RM3-assignment, then S is a complete RM3-theory; but not every RM3-theory is determined by an RM3-assignment, since not every RM3-theory is complete (not every classical theory is complete, and every classical theory is an RM3-theory).

A basic result is the following:

Proposition 1 (Extendability Lemma) *Let I, I^1 be* RM3-*assignments with the same sets of terms. If the atomic sentences holding in I are a subset of the atomic sentences holding in I^1 and if in addition the negations of atomic sentences holding in I are a subset of the negations of atomic sentences holding in I^1, then the theory determined by I is a subset of the theory determined by I^1.*

Proof: By induction on the complexity of sentences. We observe first that the hypothesis of the proposition is equivalent to the following: if A is atomic then: (i) if $I(A) = \mathrm{T}$ then $I^1(A) \in \{\mathrm{T,N}\}$; (ii) if $I(A) = \mathrm{N}$ then $I^1(A) = \mathrm{N}$; and (iii) if $I(A) = \mathrm{F}$ then $I^1(A) \in \{\mathrm{N,F}\}$. The induction proves that (i)–(iii) hold of all formulas. The base clause is already proved. The $\neg$ and & clauses are straightforward from the $\neg$ and & table. If A is of the form $(x)\,B$ then either (i) $I((x)B) = \mathrm{T}$, whence $I(B(t/x)) = \mathrm{T}$ for all terms t. So for all terms t (same terms, by hypothesis) $I^1(B(t/x)) \in \{\mathrm{T,N}\}$; whence $I^1((x)B) \in \{\mathrm{T,N}\}$. The two other alternatives (ii) and (iii), where $I((x)B) \in \{\mathrm{N,F}\}$, are similar.

Note one consequence of this. If a theory determined by an RM3-assignment is consistent and complete, then it is in fact a theory of classical first-order logic, since in the absence of the value Neuter, RM3-assignments are just classical models. Hence we can begin with any model from classical model theory (provided that it is equipped with appropriate names) and extend it by adding additional atomic sentences to make it inconsistent, evaluating all complex sentences as in RM3-assignments. The Extendability Lemma then ensures that the resulting theory is a supertheory of the classical theory commenced with. Furthermore, this extension is controlled by the assignment to atomic sentences, so to speak, so that if even one atomic sentence or its denial remains with the value False in the supertheory, it is nontrivial (absolutely consistent). There are two related desiderata with this strategy which will come out later: the substitutivity of identity, and the functionality of $+$, $\times$, $-$, $\div$. Setting these aside here, the general strategy so described for producing inconsistent extensions of classical theories (particularly determined by finite models) is a basic concern in what follows.

3 Identity, with applications to arithmetical operations Consider first the *classical standard model of the natural numbers*, equipped with names for the natural numbers. In view of the Extendability Lemma, the set of sentences holding therein can be extended by adding any collection of sentences of the form $\neg n = n$ and evaluating in an RM3-assignment. Note that the contradiction does not spread to other sentences of the form $\neg m = m$. Similarly, collections of sentences of the form $n = m$ for distinct n, m, may be added with the same result.

This raises the following question. If we add, say, $0 = 2$ to the standard model of the natural numbers, then, in virtue of the substitutivity of identity and the fact that $\neg 0 = 2$ also holds, have we not imported the further sentence $\neg 0 = 0$? The answer is no, and it illustrates the generality of the Extendability Lemma. The rule of substitutivity of identity (SI) in the form *if* $t_1 = t_2$ *holds, then* Ft_1 *holds iff* Ft_2 *holds* (all terms t_1, t_2, with t_2 replacing t_1 in F in at least one place) does not always hold in our assignments. What is the case, if the sentences holding in an RM3-assignment include those holding in the standard model of the natural numbers, is that $(t_1 = t_2 \,\&\, Ft_1) \supset Ft_2$ holds, since it holds in the standard model. But it is not in general true that if $A \supset B$ holds and A holds then B holds. In particular, $((A \,\&\, \neg A) \supset B) \,\&\, (A \,\&\, \neg A)$ might hold while B does not. However, this leads to no loss of information from classical arithmetic, since we do have that if $(A \supset B) \,\&\, A$ holds, *and if moreover* $(A \supset B) \,\&\, A$ *holds back in the standard model for arithmetic*, then B holds (trivial). A special case of interest is this: if $t_1 = t_2 \,\&\, Ft_1$ holds and *if moreover* $\neg t_1 =$

t_2 and $\neg Ft_1$ both do not hold, then Ft_2 holds. (Reason: for then $t_1 = t_2$ & Ft_1 holds back in the classical complete subtheory, wherein Ft_2 could be detached.)

So the rule SI does not hold in all RM3-assignments. This is by no means catastrophic. Intentional theories, for instance modal theories, in which SI fails have been extensively investigated. Many philosophers have taken the failure of SI as the mark of the intensional. Even so, it is obvious that a central role will be played by those models for which SI does hold. In fact, it is useful to use a more semantically based notion which ensures SI. We call an assignment an *assignment with identity* iff for all terms t_1, t_2, if $t_1 = t_2$ holds then for all predicates *F*, Ft_1 holds iff Ft_2 holds; where Ft_2 is like Ft_1, except that t_2 replaces t_1 in at least one place. This is evidently a generalization of the corresponding classical notion which nevertheless remains within its spirit. We also say that an assignment is *reflexive* iff $t = t$ holds for all terms *t*. Now the idea of an assignment with identity does not determine much by itself, but coupled with reflexivity it determines a lot, as the following proposition shows. First, some definitions: an assignment is *functional* iff for all terms t_1, t_2, if $t_1 = t_2$ holds then $f(t_1) = f(t_2)$ holds provided that both the latter are defined, and both are undefined otherwise. An assignment is *symmetric* iff $t_1 = t_2$ holds iff $t_2 = t_1$ holds, and *transitive* iff if $t_1 = t_2$ holds and $t_2 = t_3$ holds, then $t_1 = t_3$ holds (all t_1, t_2, t_3). An assignment which is reflexive, symmetric, and transitive is *normal*. Now we have necessary and sufficient conditions for a model with identity.

Proposition 2 (1) *I is an assignment with identity iff for all terms t_1, t_2, if $t_1 = t_2$ holds then for all atomic F, $I(Ft_1) = I(Ft_2)$.* (2) *If I is a reflexive assignment with identity, then I is normal and functional.* (3) *If I is reflexive and = is the only predicate of the language, then I is an assignment with identity iff I is functional and for all t_1, t_2, if $t_1 = t_2$ holds then for all t_3, $I(t_1 = t_3) = I(t_2 = t_3)$ and $I(t_3 = t_1) = I(t_3 = t_2)$.*

Proof: (1) $R \rightarrow L$ follows by a straightforward induction on the complexity of terms. $L \rightarrow R$: Let $I(Ft_1) \neq I(Ft_2)$ for some atomic *F* while $t_1 = t_2$ holds. If one of Ft_1, Ft_2, does not hold then the other does, so that *I* is not an assignment with identity. Otherwise, if both Ft_1, Ft_2 hold, then one of $\neg Ft_1$, $\neg Ft_2$ does not hold while the other does, again incompatible with identity.

(2) Symmetry: Let $t_1 = t_2$ hold. By identity, if $t_1 = t_2$ holds, then $t_1 = t_1$ holds iff $t_2 = t_1$ holds. By reflexivity, $t_1 = t_1$ holds. Hence $t_2 = t_1$ holds. Transitivity: Let $t_1 = t_2$ and $t_2 = t_3$ hold. By identity, if $t_1 = t_2$ holds then $t_1 = t_3$ holds iff $t_2 = t_3$ holds. Hence $t_1 = t_3$ holds. Functionality: Let $t_1 = t_2$ hold. By identity, $f(t_1) = f(t_1)$ holds iff $f(t_1) = f(t_2)$ holds. By reflexivity, $f(t_1) = f(t_1)$ holds. Therefore, $f(t_1) = f(t_2)$ holds.

(3) $L \rightarrow R$ follows from (1) and (2). $R \rightarrow L$: From (1), we need only prove that if $t_1 = t_2$ holds, then for atomic *F*, $I(Ft_1) = I(Ft_2)$. Clearly, atomic *F* have one of four forms: $t_1 = t_3$, $t_3 = t_1$, $f(t_1) = t_3$, or $t_3 = f(t_1)$. In the first two cases, the conditions of the theorem ensure what we want. In the third case, we have to prove that if $t_1 = t_2$ holds, then for any t_3, $I(f(t_1) = t_3) = I(f(t_2) = t_3)$. But if $t_1 = t_2$ holds, then by functionality, $f(t_1) = f(t_2)$ holds; hence by the conditions of the theorem, for any t_3, $I(f(t_1) = t_3) = I(f(t_2) = t_3)$ as required. The fourth case is similar. This completes the proof.

Note that all of (1)–(3) are true over a broad class of logics, since the inductions needed for (1) and (3) will work provided that I assigns values in a Lindenbaum algebra, and (2) and the remainder of (3) need only minimal properties for 'holds'. Proposition 2 is thus a general result for model theory based on many different logics.

The conditions for an assignment with identity can be made more semantically based, so the idea of an assignment is now strengthened to that of a model.

An RM3-*model* is a pair $\langle D, I\rangle$ where D is a domain and I is a function which is an RM3-assignment and which in addition has the following four properties: (1) I assigns to every *simple* term a member of D, and I is onto D; so that every object is named. This has the effect that our substitutional quantification becomes objectual. (2) I assigns to every n-ary functional expression an n-ary partial function on D. (3) The assignment to complex terms is given by $I(f(t_1 \ldots t_n)) = I(f)(I(t_1) \ldots I(t_n))$, provided that these are defined. (4) I satisfies: $t_1 = t_2$ holds iff $I(t_1) = I(t_2)$. These have the effects that I is normal and functional.

A model is *infinite* iff $\bar{\bar{D}} \geq \aleph_0$, otherwise *finite*. If $\langle D, I\rangle$ is a model and I is an assignment with identity, then $\langle D, I\rangle$ is a *model with identity*. Thus, if $\langle D, I\rangle$ is a model, then its semantical features ensure that I is normal and functional. Further, then, a necessary and sufficient condition for a model for an equational theory to be a model with identity is that if $t_1 = t_2$ holds, then for any t_3, $I(t_1 = t_3) = I(t_2 = t_3)$ and $I(t_3 = t_1) = I(t_3 = t_2)$. We could introduce further semantical conditions on the domain to ensure models with identity: the obvious maneuver is to introduce for each n-ary relational symbol a truth extension and a falsity extension, the intersection of which would be the neuter extension. But we do not consider that here, since the aim is less model theoretic than it is to establish the model theory as a convenient device for studying inconsistent mathematical objects and demonstrating that the inconsistency is under control.

It is sometimes thought that contradiction-toleration is a matter of the use of theories of intensional logics, or perhaps that it is a matter of "mere syntax". To the contrary, it is argued here that the study is extensional in at least two senses. It is syntactically extensional, in dealing only with the connectives $\neg$, &, $\forall$; and it is extensional in dealing with models with identity. In this sense, it can usefully be viewed as dealing with mathematical objects which have inconsistent properties, especially when models which inconsistently extend various consistent classical standard theories of classes of mathematical objects are considered.

As an example, consider the following class of inconsistent finite models with identity in which all sentences of the classical standard model for the arithmetic of $(+, \times)$ hold (investigated in [8]). There are names (i.e., simple terms) for all the nonnegative integers, with the domain being the integers modulo m, i.e., $\{0, 1, \ldots, m-1\}$; $+$, $\times$, are interpreted as addition and multiplication in arithmetic modulo m. Set $I(n)$, for every name n, to be $n(mod\ m)$. With $I(+, \times)$ this determines $I(t)$ for every term t. And finally set $I(t_1 = t_2) = N$ iff $t_1(mod\ m) = t_2(mod\ m)$, i.e., iff $I(t_1) = I(t_2)$; and $I(t_1 = t_2) = F$ otherwise. In [8], these are called RM3^m, and it is proved that they are inconsistent, nontrivial, complete, ω-inconsistent, ω-complete, and decidable. In [8], the interest in these structures is that they are extensions of the axiomatic arithmetic $R^{\#}$, and show that Gödel's Second Incompleteness Theorem can be escaped after a fashion in inconsistent and relevant mathematics. Here, the interest is that they are models with identity and determine finite inconsistent extensions of the classical standard theory of arithmetic.

A simple development of these results can be obtained from the well-known fact that the algebra of the integers modulo m enables a natural definition of 'minus n' and thereby subtraction. This can be exploited to display finite inconsistent extensions of the classical theory, with names, of the full ring of integers Z (positive and negative). Take names for all the integers. The domain is the integers modulo m; $+$ and $\times$ are, as before, $+(mod\ m)$ and $\times(mod\ m)$. The additive inverse $(-n)$ modulo m of a number n is given classically by $m-$ $(n\ mod\ m)$ if $n\ mod\ m \neq 0$, and 0 otherwise; and then subtraction $mod\ m$ is given by $(k-(mod\ m)n =_{df} k\ mod\ m + (mod\ m)(-n) mod\ m.)$ So here we interpret '$-$' to be '$-mod\ m$'. This determines $I(t)$ for all terms t. Set $I(t_1 = t_2) = \mathrm{N}$ iff $I(t_1) = I(t_2)$, i.e., iff $t_1\ mod\ m = t_2\ mod\ m$; and set $I(t_1 = t_2) = \mathrm{F}$ otherwise. Clearly, the condition of Propositions 2 and 3 for a model with identity is satisfied. Also every true identity of the classical theory of integers holds, since if classically $t_1 = t_2$ then $t_1\ mod\ m = t_2\ mod\ m$. So, by the Extendability Lemma, we have

Proposition 3 *There are finite inconsistent models with identity in which every sentence of the classical theory of the ring of integers Z holds.*

A useful and obvious result is the Term Elimination Lemma. The above models have finite domains and infinite numbers of simple terms, the latter being necessary if we are to have extensions of the various classical theories with names. But, as might be expected, the simple terms can be cut down to just one per member of the domain, while preserving the assignments to all terms, and preserving the values of all sentences in the weaker vocabulary. In particular, the term-free quantified theory remains identical. It needs models with identity to make this work, so that is another use for the notion. Let $\langle D, I\rangle$ be a model with identity. Select only one term from each set $\{t\colon (\exists x)(I(t) = x \in D)\}$, and let I^1 assign to it the same value it is assigned by I. Functional expressions are assigned the same partial functions on the domain as before, but functional terms only in the weaker language are assigned values. Atomic sentences in the weaker language are given the same values by I^1 as by I. This evidently ensures the base clause of an induction to prove the following.

Proposition 4 (Term Elimination Lemma) *A sentence in the cut-down language has exactly the value in $\langle D, I\rangle$ that it has in $\langle D, I^1\rangle$.*

Proof (Inductive Clause): The $\neg$ and & clauses are straightforward. For the $\forall$ clause (a.i) if $(x)Fx$ is T in $\langle D, I\rangle$ then $I(Ft) = \mathrm{T}$ for all terms of the vocabulary of I, so $I^1(Ft) = \mathrm{T}$ for all terms of the vocabulary of I^1, so $(x)Fx$ is T in I^1. (a.ii) If $I(x)Fx) = \mathrm{N}$ then $I(Ft) = \mathrm{T}$ or N for all terms t, and N for at least one. So $I^1(Ft) = \mathrm{T}$ or N for all terms t of the vocabulary of I^1. But also by the construction of I^1, for some t^* we must have $I^1(Ft^*) = \mathrm{N}$. Hence $I^1((x)Fx) = \mathrm{N}$. (a.iii) The F clause is similar, with 'F' replacing 'N'. Conversely (b.i) If $I^1((x)Fx) = \mathrm{T}$ then $I^1(Ft) = \mathrm{T}$ for all t in weaker vocabulary. But for every term t^* of I, we have $I(t^*) = I(t)$ for one of these t; so that, since M is a model with identity, Ft^* agrees with some Ft of I^1. But all of the latter are T, so also every $I(Ft^*)$ must be. Hence $I((x)Fx) = \mathrm{T}$. (b.ii) and (b.iii) the N and F cases are similar.

The effect of this lemma is that the two classes of models previously considered now yield inconsistent models with cut-down languages (finite numbers of simple terms, exactly one for each member of the domain), with the same sentences in the weaker language, including the term-free language, holding. These cease to be inconsistent extensions of, e.g., the classical theory of Z with names, but remain inconsistent extensions of the finite consistent arithmetics modulo m.

We now bring in division, and thus the theory of fields. It turns out that the interaction between subtraction and division is not smooth sailing. The following are a set of postulates adequate for the classical theory of fields (see [15], p. 130)

(1) $(x,y,z)(x + (y + z) = (x + y) + z)$
(2) $(x,y)(x + y = y + x)$
(3) $(x)(x + 0 = x)$
(4) $(x)(x + (-x) = 0)$
(5) $(x,y,z)(x \times (y \times z) = (x \times y) \times z)$
(6) $(x,y)(x \times y = y \times x)$
(7) $(x)(x \times 1 = x)$
(8) $(x)(\neg x = 0 \supset x \times x^{-1} = 1)$
(9) $(x,y,z)(x \times (y + z) = (x \times y) + (x \times z))$
(10) $\neg 0 = 1$.

First, there are certainly finite inconsistent fields because (as is well known) there are finite consistent fields. The finite arithmetics modulo p, $\{0,1,\ldots,p-1\}$, where p is *prime*, permit a definition of a unique multiplicative inverse n^{-1} for any $n \in \{1,2,\ldots,p-1\}$ though not for $n = 0$ (see, e.g., [2], p. 40). Therefore, if we take names only for $\{0,1,\ldots,p-1\}$, interpret $+$, $\times$, $-$, $\div$ as in arithmetic modulo p, and set $I(t_1 = t_2) = \mathrm{T}$ iff $I(t_1) = I(t_2)$, and F otherwise, we have the classical consistent theory of fields. Thus, setting instead $I(t_1 = t_2) = \mathrm{N}$ for $I(t)$ and F otherwise, we have by the Extendability Lemma:

Proposition 5 *There are finite inconsistent models with identity in which every sentence of the classical theory of fields holds.*

It would be desirable to see finite inconsistent extensions of the full theory with all names of the field of rationals Q. But it is not clear how to do this with these methods, because the interpretation function $I(n)$ assigning to all names of rationals members of the domain $\{0,1,\ldots,p-1\}$ would seem to assign infinitely many nonzero rationals to 0, as it does in the case of the integers. But then for these, an inverse n^{-1} is not defined, while it is in the full theory of Q.

A useful general result can be obtained as a consequence of the Extendability Lemma.

Proposition 6 *Let A be an algebra $\langle D,0_1,\ldots,0_n\rangle$ where D is a set and $0_1,\ldots 0_n$ are relations on D. Let h be a homomorphism from A to a subalgebra*

A^1 with $D^1 = h(D)$ and operations the restriction of $0_1, \ldots, 0_n$ to D^1. Then the classical equational theory of A with names for all elements of D can be inconsistently extended to an RM3-model with identity using the assignment $I(t) = h(t)$, $I(t_1 = t_2) = N$ iff $I(t_1) = I(t_2)$ and $I(t_1 = t_2) = F$ otherwise.

Proof: Certainly the assignment I is a model: $I(t)$ is defined on domain D^1, and if $I(t_1) = I(t_2)$ then evidently $O_i(t_1) = O_i(t_2)$. Also, it plainly satisfies the condition for a model with identity.

An application of this is that whenever classically one can partition an algebra into equivalence classes via a homomorphism onto a subalgebra, one may instead literally inconsistently identify distinct elements in the larger algebra, thereby obtaining an inconsistent extension of it. That is, of course, precisely what the modulo arithmetics are doing, with the *caveat* about division noted before. Another example is as follows. There are finite models inconsistently extending the classical (+,×,÷) theory of the nonnegative rationals Q^+ with names (see also Section 5 for order). Consider the following subalgebra of that structure:

$D = \{0,1\}$ with the operations

+	0	1
0	0	1
1	1	1

×	0	1
0	0	0
1	0	1

÷	0	1
0	U	0
1	U	1

U = undefined

The homomorphism h is given by $h(0) = 0$, $h(n) = 1$ for all $n > 0$.

Thus there is a finite model with identity in which the classical (+,×,÷) theory of the nonnegative rationals Q^+ with names holds. Notice how introducing the negative rationals and thereby subtraction would wreck this model: we want some element to function as an additive inverse $-n$ for each n, but if we identify more than one rational n_1, n_2 with a given element, they have the same additive universe, so that $n_1 - n_2 = n_1 - n_1 = 0$; and so division by $n_1 - n_2$ is (improperly) undefined. Thus, the prospects for a sensitive inconsistent theory of arithmetical fields look bleak, not for reasons of propositional logic, but because of the functional interaction of $-$ and $\div$. Relevant logic has hitherto not taken proper cognizance of the fact that a good inconsistent mathematics might be difficult to obtain for reasons beyond the purely sentential.

There are, needless to say, infinite inconsistent extensions of the theory of Q, even models with identity, e.g., set $I(n = n) = N$ for every rational n, and F otherwise. We encounter some of these in later sections, when order is introduced.

One interest in such inconsistent theories of division, both finite and infinite, is that they permit a solution to the following problem (raised by Graham Priest). Ordinarily one wants postulates such as the Cancellation Law ([2], p. 2) to hold when extending the theory of rings to that of integral domains and fields:

$$(x)(\neg x = 0 \supset (y,z)(x \times y = x \times z \supset y = z)).$$

But in inconsistent theories such as those of this section (see also Section 5) $\neg 0 = 0$ and $(x)(0 \times x = 0)$ hold, and one does not want to detach the consequent to get $y = z$ for all y, z; yet one also does not want to forbid detachment for those x which are classically not identical with zero. However in the inconsistent finite fields modulo prime p above, while both $\neg 0 = 0$ and $(x)(0 \times x = 0)$ hold, we cannot detach the consequent (because patently we do not have $(y,z)(y = z)$ holding). But on the other hand, the fact that they really are fields means that for those x of the model which are "really" not identical with zero, i.e. for which $x = 0$ has value F in the model, we can detach because we do have that $x \times y = x \times z \supset y = z$, even that if $x \times y = x \times z$ holds then $y = z$ holds. Problems: Are there any finite inconsistent models with identity of the full classical theory of Q with names? Is the above two-element model the only finite inconsistent model for Q^+? Is the addition of $\neg n = n$ to any model with identity still a model with identity?

4 Order The aim in this section is to introduce order, and in the next section to study the inconsistent interplay between order and arithmetical operations, particularly the theory of ordered fields. In this section, we look at $=$ and $<$ alone. Among other things, it is shown that a standard result of model theory, namely that the theory of dense order with no first and last elements is $\aleph_0$-categorical, breaks down given a suitable extension of that concept to cover the more general inconsistent case.

The following postulates suffice for the standard classical theory of dense order without endpoints (e.g., [1], p. 324; [7], pp. 78, 90):

(i) Irreflexivity $(x)(\neg x < x)$
(ii) Asymmetry $(x,y)(x < y \supset \neg y < x)$
(iii) Transitivity $(x,y,z)(x < y \supset . \ y < z \supset x < z)$
(iv) Comparability $(x,y)(\neg x = y \supset . \ \neg x < y \supset y < x)$
(v) Exclusiveness $(x,y)((x = y \supset . \ \neg x < y \ \& \ \neg y < x) \ \& \ (x < y \supset \neg x = y))$
(vi) No endpoints $(x)(\exists y,z)(x < y \ \& \ z < x)$
(vii) Denseness $(x,y)(x < y \supset (\exists z)(x < z \ \& \ z < y))$
(viii) Mixing $(x,y,z)(z = y \supset . \ (y < z \supset x < z) \ \& \ (z < y \supset z < x))$.

These postulates hold in the classical (and RM3-) models with identity whose domain is the rational numbers, which we may also take as terms naming themselves; with $I(t = t) = \mathrm{T}$ and $I(t_1 = t_2) = \mathrm{F}$ otherwise, and $I(t_1 < t_2) = \mathrm{T}$ iff $t_1 < t_2$, and F otherwise. It is a standard result that all classical models of (i)-(viii) of cardinality $\aleph_0$ are isomorphic. Now in the case where every element of the domain has a name, the following version of isomorphism lends itself to natural generalization. Two models $\langle D,I\rangle$, $\langle D^1,I^1\rangle$ are isomorphic iff there is a 1 to 1 correspondence f: $D \to D^1$ such that for all atomic terms $t_1,\ldots,t_n$, $t_1^1,\ldots,t_n^1$; if $I^1(t_1^1) = f(I(t_1)),\ldots,I^1(t_n^1) = f(I(t_n))$, then for all atomic F, $Ft_1\ldots t_n$ holds in I iff $Ft_1^1\ldots t_n^1$ holds in I^1.

Now extend the model of the previous paragraph to an inconsistent RM3-model as follows. Take the rationals as simple terms as before, but domain $D =$ the integers Z. For each rational n, set $I(n) =$ the integral part of n. Set $I(n = m) = \mathrm{N}$ iff $I(n) = I(m)$, and F otherwise; and set $I(n < m) = \mathrm{N}$ iff $I(n) \leq$

$I(m)$, and F otherwise. By the Extendability Lemma, every sentence of the classical theory of Q continues to hold, and hence (i)–(viii) hold. Furthermore, it is a model with identity; since if $n = m$ holds, i.e. $I(n) = I(m)$, then clearly for all atomic F, $I(Fn) = I(Fm)$. Note in passing that the discreteness postulate $(x)(\exists y)(x < y \,\&\, (z)(x < z \supset . \neg y < z \supset y = z))$ also holds in this model; so that both discreteness and denseness postulates can be inconsistently satisfied. But there is no 1 to 1 correspondence which preserves atomic sentences between the domain of this model and that of the previous model: a 1 to 1 correspondence f from Z to Q must eventually reverse the order on some of the elements of Q, so that while $I(n) \leq I(m)$ and thus $n < m$ holds in the inconsistent model, $f(I(m)) < f(I(n))$ in the classical model. Thus

Proposition 7 *There are non-isomorphic* RM3-*models with identity, of cardinality* $\aleph_0$, *in which every sentence of the classical theory of dense order without endpoints holds.*

Indeed, the Term Elimination Lemma may be used on this model to dispense with all names except names for the integers, and the same result applies to this model. Again, a similar result can be simply obtained using a finite model, which can also be used to show that the order theories of R, Q, and Z have a common inconsistent extension. Take domain $D = \{0,1\}$, and do three constructions corresponding to three sets of simple names, those of R, Q, and Z. In each case, set $I(n) = 0$ if $n \leq 0$ and $I(n) = 1$ if $n > 0$; set $I(n = m) = \text{N}$ if $I(n) = I(m)$, and F otherwise; and set $I(N < m) = \text{N}$ if $I(n) \leq I(m)$, and F otherwise. The three cases are inconsistent extensions of the order theories of R, Q, and Z respectively, by the Extendability Lemma; and the conditions for being models with identity are satisfied. The case of Q evidently provides an example of a finite-domain model in which all sentences of the theory of dense order without endpoints hold. But also, the Term Elimination Lemma can be applied to each of these constructions, to give that the set of term-free sentences of each of the order theories of R, Q, Z holds in the same (two-element) model, the term-free sentences of which are thus a common inconsistent extension of them all. Problem: Is there a way to extend the theory of R directly to that of Q?

5 Ordered rings and fields In this section, the question of putting together the arithmetical operations with the order relation is discussed. It is useful at this point to introduce a distinction. So far, models have been constructed in which, typically, all sentences of various classical theories hold; that is, inconsistent extensions of classical theories. We have thus been working implicitly with two desiderata for models: (i) that they make all sentences of the classical theory hold, and (ii) that they be models with identity. The interplay between arithmetic and order, however, tends to make this rather more difficult to achieve. So we consider a third, weaker desideratum: (iii) all members of a certain set of postulates (such as e.g., the order postulates of Section 4) hold. Classically, (iii) coincides with (i), but not necessarily in RM3. It should not be thought that this is inevitably a "defect" of RM3, of course, since many have argued that the deductive relationship of classical logic is too strong, precisely in its inability to provide contradiction containment. It will be seen in this section that there are occasions when (i) must be sacrificed while (iii) continues to hold. This can be amplified by a point from [8]. It is an interesting open question whether $R^{\#}$, i.e., Peano arithmetic formulated with a relevant $\rightarrow$ as its implication operator instead of $\supset$, contains all of classical Peano arithmetic $P^{\#}$. It is however a separate matter whether, if $R^{\#}$ does not contain all of $P^{\#}$, this would be a "defect" of $R^{\#}$, since it is arguable that *natural* arithmetic is formulated merely with "if . . . then", and relevant $\rightarrow$ is at least as good a candidate for that as $\supset$ is. Indeed, were we to discover that natural arithmetic suffered an inconsistency in virtue of some recondite feature, to do with the Godel sentence, say, it is by no means obvious that we would regard the contradiction as spreading uncontrollably and thus affecting our ability to calculate.

Begin with the integers Z. The following postulates classically suffice for its order theory (cf. Section 4): Irreflexivity, Asymmetry, Transitivity, Comparability, Exclusiveness, No First and Last Elements, Mixing, together with:

(ix) Discreteness $(x)(\exists y)(x < y \,\&\, (z)(x < z \supset . \, y < z \vee y = z))$
$\&\, (x)(\exists y)(y < x \,\&\, (z)(z < x \supset . \, z < y \vee z = y))$

(x) Sum Law $(x,y,z)(x < y \supset x + z < y + z)$

(xi) Product Law $(x,y,z)(x < y \supset . \, 0 < z \supset x \times z < y \times z)$.

First consider finite models. Take the finite $(+,\times,-)$ models modulo m of Section 4 and add the atomic sentences $t_1 < t_2$ for all terms t_1, t_2 constructible from names for the integers. Set $I(t_1 < t_2) = \mathrm{N}$ iff $t_1 \leq t_2$, and F otherwise. By the Extendability Lemma, all classical consequences of the $(=,+,\times,-,<)$ theory of Z with names hold. They are not, however, models with identity (Reason: $t_1 = t_2$ holds iff t_1 *mod* $m = t_2$ *mod* m, but t_1 *mod* $m = t_2$ *mod* m together with $t_1 < t_3$ does not ensure $t_2 < t_3$; t_2 might be too large even though when collapsed modulo m it is equal to t_1).

So there are finite models in which all sentences of the arithmetic and order theory of the integers holds, but which are not models with identity. Equally there are finite models with identity in which all the above order *postulates* hold. Take the above $(+,\times,-)$ models modulo m with names for all of Z; and set $I(t_1 < t_2) = \mathrm{N}$ iff t_1 *mod* $m \leq t_2$ *mod* m, and F otherwise. To show that these are models with identity it suffices to consider atomic sentences of the form $t_1 < t_2$, since other atomic sentences have been dealt with earlier. But if $t_1 = t_2$ holds, then t_1 *mod* $m = t_2$ *mod* m; whence $I(Ft_1) = I(t_1 < t_3)$, say, $= \mathrm{N}$ iff t_1 *mod* $m \leq t_3$ *mod* m iff t_2 *mod* $m \leq t_3$ *mod* m, so that $\mathrm{N} = I(t_2 < t_3) = I(Ft_2)$. Identity then follows from Proposition 2. However, not all classical consequences follow because the assignment to $<$ destroys the order on the integers (for example, in modulo *3*, *2* $<$ *4* is F, because *2 mod 3* $= 2 \leq 4$ *mod 3* $= 1$ does not hold).

There are, however, finite models with identity in which all classical sentences true in Z hold. Take the $(+,\times,-)$ models as before, and set all sentences of the form $t_1 < t_2$ to be Neuter. Clearly the Extendability Lemma ensures that all sentences in the $(=,+,\times,-,<)$ theory of Z continue to hold. That it is a model with identity follows from the fact that for atomic F of the form $t_1 < t_2$, trivially $I(Ft_1) = I(Ft_2)$, whether or not $t_1 = t_2$ holds. These models have the unsatisfactory feature that the order properties are rather insensitively ensured, in that all sentences of the form $t_1 < t_2$ are made to hold. Even so, it is still only possible to make this work by exploiting the inconsistency-toleration features of RM3, e.g., in making $(x)\neg x < x$ take the value N and so hold also. To summarize these results:

Proposition 8 *There are finite models both with and without identity of the classical* $(=,+,\times,-,<)$ *theory of the integers Z with names, and finite models with identity in which all classical* $(=,+,\times,-)$ *consequences hold and all order postulates hold as well.*

We turn to division. The problem is to see what can be made of the theory of ordered fields. In addition to standard field $(+,\times,-,\div)$ properties, order postulates are needed. Classically the previously mentioned postulates suffice: Irreflexivity, Asymmetry, Transitivity, Comparability, Exclusiveness, No First and Last Elements, Denseness, Mixing, Sum, and Product Laws.

We saw in Section 3 that bringing in division restricts rather drastically the possibilities for finite inconsistent models, or at any rate finite extensions of Q. We can, however, go further with a result of that section, namely that all classical consequences of the $(=,+,\times,\div)$ theory of the nonnegative rationals hold in a two-element model $D = \{0,1\}$ with operations as specified previously, and $I(0) = 0$, $I(n) = 1$ for all $n > 1$. To this we can add the ordering $I(n < m) =$ N iff $I(n) \leq I(m)$, and F otherwise. It now follows easily that

Proposition 9 *There is a finite model with identity in which all classical consequences of the* $(=,+,\times,\div,<)$ *theory of* Q^+ *hold.*

Consider now the following model with identity: names for all real numbers R; $D = \{0,1,\ldots,p-1\}$; $I(n) = 0$ for $n \leq 0$ and $n > p-1$, and $I(n) =$ the nearest integer $\leq n$ for $0 < n \leq p-1$; $I(n = m) =$ N iff $I(n) = I(m)$, and F otherwise; $I(n < m) =$ N iff $I(n) \leq I(m)$, and F otherwise. It is as immediate that it is a model with Identity as it is that the conditions of the Extendability Lemma apply, so that the sentences holding therein include all classical consequences of the first-order $(=,<)$ theory of R, including the continuity schema ([15], p. 131).

$$(((\exists x)Fx \,\&\, (\exists y)((x)(Fx \supset x \leq y)) \supset (\exists z)(x)(Fx \supset x \leq z) \\ \&\, (y)((x)(Fx \supset x \leq y) \supset z \leq y)).$$

So the Term Elimination Lemma can be applied to this model to give the conclusion that the following is a model with identity: $D = \{0,1,\ldots p-1\}$ (naming themselves); $I(n = m) = N$ iff $n = m$ and F otherwise; $I(n < m) =$ N iff $I(n) \leq I(m)$ and F otherwise. This satisfies all classical consequences of the $(=,<)$ theory of R in this language, and in particular all universally quantified sentences containing no names. Now we exploit the fact that a classical field can be constructed on the above domain in the standard fashion. The model cannot be made wholly classical (sentences only T or F) since we already have that $I(t = t) =$ N for all t. But the construction of the field simply adds to the above by assigning $I(t_1 + t_2)$, $I(t_1 \times t_2)$, $I(t_1 - t_2)$ and $I(t_1 \div t_2)$ the values they standardly take in $\{0,\ldots,p-1\}$, and sets $I(t_1 = t_2)$ (for any terms t_1, t_2) = N iff $I(t_1) = I(t_2)$, and F otherwise; and $I(t_1 < t_2) =$ N iff $I(t_1) \leq I(t_2)$, and F otherwise. The Extendability Lemma ensures that all classical consequences of the theory of fields hold, and it is straightforward to show that it remains a model with identity. That is,

Proposition 10 *There exist inconsistent continuously ordered finite fields.*

A complication should be mentioned. It has *not* been proved that every classical consequence of the theory of real closed fields holds in this model. To see this, note that the ability to substitute field identities in the theory of continuous ordering does not ensure that all classical consequences hold; for instance the Sum and Product laws are not obtained this way. In fact, the Sum and Product laws hold in our model, but conceivably various of their classical consequences might not. We do have though, as might be expected, that standard systems of first-order postulates for complete ordered fields (e.g., [15], p. 130) hold.[2] This point can be amplified by considering a different continuous ordering for the finite fields. For any real n, $I(n) = 0$ for $n \leq 0$, $I(n) =$ the next whole number $\geq n$ for $0 < n \leq p - 2$, and $I(n) = p - 1$ otherwise. Then set $I(n < m) = \mathrm{T}$ if $I(n) < I(m)$, $I(n < m) = \mathrm{N}$ if $I(n) = I(m)$, and $I(n < m) =$ F otherwise; and $I(n = m) = \mathrm{N}$ iff $I(n) = I(m)$, and F otherwise. Again, this inconsistently extends the classical ordering on R, so every classical sentence true therein holds. Further, it is a model with identity. By the Term Elimination Lemma, this is true for the model restricted to the p names $\{0,1,\ldots p-1\}$. Then we can add the $(+,\times,-,\div)$ theory of p-membered finite fields to this in the same fashion as before to get models with identity for different finite inconsistent continuously ordered fields. But now notice this: Sum and Product Laws fail here whereas they did not in the previous model. (Sum Law: In modulo p, $p - 2 < p - 1$ is T, but $(p - 2) + 1 < (p - 1) + 1$ is F. Product Law: In modulo 3, $1 < 2$ & $2 < 0$ (or $2 \neq 0$) is T, but $1 \times 2 < 2 \times 2$ is $2 < 1$ which is F.) This latter argument works for all modulo primes $p \geq 3$, but not for modulo 2. So we can deduce a couple of RM3-independence results: Sum and Product Laws cannot be obtained from continuity ordering + field properties, and these together with the Product Law do not yield the Sum Law. The moral to draw, though, is that it would be incorrect to conclude from the previous model that finitude + continuity + field properties + model with identity give all the sentences of the classical theory of real closed fields. Problem: Is that conclusion nevertheless true?

The results of these two models suggest the following simple extension into set theory. Instead of the schema 'Fx' in the continuity schema, replace it by '$x \in u$' and universally quantify the whole formula with respect to u, where u ranges over subsets of the domain. Let $s, t, \ldots$ be names for these, so that $I(s) \in P(D)$. Then let $I(n \in s) = T$ iff $I(n) \in I(s)$, and F otherwise. It is straightforward to verify that the set-theoretic continuity postulate holds in the above models which remain models with identity.

6 Conclusion

To amplify a point made at the beginning, the use of an explicit background logic to study mathematical structures is a mark of mathematical logic as opposed to "natural" mathematics. While natural mathematics does seem typically to proceed on a tacit consistency assumption, it is by no means obvious that this is essential. The test is to see whether the relaxation of that assumption leads to rich structures, and it is suggested that the evidence here that it does is initially promising. The assumption of consistency does not entail that natural logic is classical, and the case that natural logic is not classical has been extensively argued in recent times. Discoveries in semantics have shown that nonclassical logics of the paraconsistent kind must have inconsistent theories, so it would seem mandatory to display these. But it is a moot point the extent to which the deductive assumptions on which natural mathematics proceeds are logically necessary, or simply there because the history of the activity has made alternatives invisible. Only the investigation of such alternatives can determine that.

NOTES

1. The terminology 'neuter' is perhaps a little misleading, since it suggests "neither true nor false", whereas in fact it is better construed as "both true and false", or perhaps "both it and its negation hold". We retain 'neuter' here on grounds of established practice.
2. Save $0^{-1} = 0$, concerning which opinion differs. [15], p. 130; [9], pp. 280, 286; [17].

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Models for Inconsistent and Incomplete Differential Calculus

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Abstract In Section 2, a nilpotent ring is defined. In Section 3, nonclassical model theory is sketched and an incomplete model is defined. In Section 4, it is shown that the elements of equational differential calculus hold in this model, and a comparison with synthetic differential geometry is made. In Section 5, an inconsistent theory is defined with many, though not all, of the same properties.

1 Introduction This paper extends the nonclassical model theory for inconsistent first-order equational theories developed in [4], [6], and [7], to the case of inconsistent equational theories strong enough for a reasonable notion of differentiation. The aim is to show that inconsistency does not cripple such an equational differential calculus. There have been a number of calls recently for inconsistent calculus, some appealing to the history of the calculus in which inconsistent claims abound (see, e.g., [9]). However, inconsistent calculus has resisted development, for at least two reasons. First, the functional structure of fields interacts with inconsistency to produce triviality in even the purely equational part of first-order theories with terms of finite length (as pointed out in [6], [7], and [9]), in a way which standard contradiction-containment devices, such as weakening *ex contradictione quodlibet*, do not prevent. Stronger theories, those including set membership, terms of infinite length, order, limits, and integration are then infected with the same triviality. Second, the functional structure of inconsistent set theory remains difficult to control, and seems to require sacrifice of logical principles in addition to, and more natural than, *ex contradictione quodlibet* (see, e.g., [2], [5], or [8], pp. 178–180). But unless there

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are distinctive inconsistent theories of the order of strength of classical analysis, the claim that the history of the calculus supports paraconsistency is seriously undermined; and furthermore, claims that inconsistent mathematics can generate substantial new insights are correspondingly weakened.

This paper addresses the former of these two points, by adapting the methods of [6] and [7], in presenting an inconsistent but nontrivial equational theory of polynomial differentiation. So far, inconsistent methods seem to be useful mostly in a fairly restricted application, namely treating congruences as literal identities (inconsistently), in accordance with informal mathematical terminological practice. In Section 2, a congruence on a subset of the classical hyperreals gives the functional structure of the domain of the models and is independently interesting in connection with the classical theory of nilpotent rings. Section 3 sketches nonclassical model theory. It turns out that the structure developed in Section 2 is usefully studied in the first instance by modifying inconsistent model theory using a finite-valued intuitionist logic, since the resulting first-order "intuitionist" equational differential calculus has some similarities with a corresponding fragment of synthetic differential geometry (see [1] or [3]), particularly in respect of incompleteness, nilpotence, Taylor formulas, differentiation, and continuity. This is done in Section 4, and the advantages (in particular, that of simplicity) and limitations of the comparison are outlined. In Section 5, the same results are then obtained for inconsistent polynomial theory. The limitations of the present approach and some further developments are outlined in the final section. It is argued on the basis of these results that the fact that the same functional structure underlies all of the incomplete, inconsistent, and classical consistent theories suggests that the functional aspects of mathematics are more important than squabbles at the sentential level over *ex contradictione quodlibet*, inconsistency, incompleteness, etc.

2 A nilpotent ring We begin with the usual arithmetic of the field of hyperreal numbers R*, with operations $+, -, \times, /$. The subfield of real numbers is denoted by R. For each nonzero x in R*, the binary relation $\underset{x}{\sim}$ is defined by: $x_1 \underset{x}{\sim} x_2 =_{\text{def}} (x_1/x)$ is at most infinitesimally different from (x_2/x), (i.e., $x_1/x \approx x_2/x$, i.e., $(x_1 - x_2)/x$ is infinitesimal, i.e., $(x_1 - x_2)/x \approx 0$; see the relation $\approx \delta$ in [10]). For fixed x this is an equivalence relation on R*, as is easy to verify. It is not however a congruence. For example, if $x_1 \underset{x}{\sim} x_2$, then if $(x_1 - x_2)/x$ is infinite with respect to x_3, then $(x_1/x_3) \underset{x}{\sim} (x_2/x_3)$ does not in general hold. However, if x is set equal to an arbitrarily chosen infinitesimal δ, then a congruence with respect to $+, -, \times$, and an associated ring of equivalence classes is obtained. Let S be the set of noninfinite hyperreals, i.e., of the form $x + d$ where x is any real number and d is any infinitesimal, with the additional property that for some positive integer k, $d^k \underset{\delta}{\sim} 0$. Then

Proposition 1 *The relation $\underset{\delta}{\sim}$ on S is a congruence with respect to the operations $+, -, \times$.*

Proof: If $(x_1 + d_1) \underset{\delta}{\sim} (x_2 + d_2)$, i.e. $((x_1 - x_2) + (d_1 - d_2))/\delta \approx 0$, then $(((x_1 + x_3) - (x_2 + x_3)) + ((d_1 + d_3) - (d_2 + d_3)))/\delta \approx 0$, i.e. $((x_1 + d_1) + (x_3 + d_3)) \underset{\delta}{\sim} ((x_2 + d_2) + (x_3 + d_3))$. Also, let $d_1^{k_1}/\delta \approx 0$ and $d_3^{k_3}/\delta \approx 0$. Now

$(d_1 + d_3)^{k_1+k_3}/\delta = \left(\sum_{i=0}^{k_1+k_3} \binom{k_1 + k_3}{i} d_1^{k_1+k_3-i} d_3^i\right) \Big/ \delta$. But each term ≈ 0, so the whole sum is. Hence $(x_1 + d_1) + (x_3 + d_3)$ is in S, and so $(x_2 + d_2) + (x_3 + d_3)$ also obviously is. The subtraction case is the same. For multiplication, if $((x_1 - x_2) + (d_1 - d_2))/\delta \approx 0$, then if x_3 is real, then $(((x_1 - x_2) + (d_1 - d_2)) \times (x_3 + d_3))/\delta \approx 0$ also. Hence $(x_1 + d_1) \times (x_3 + d_3) \underset{\delta}{\sim} (x_2 + d_2) \times (x_3 + d_3)$. Now also, $(x_1 + d_1) \times (x_3 + d_3) = (x_1 \times x_3) + (x_1 \times d_3) + (x_3 \times d_1) + (d_1 \times d_3)$. Clearly, however, $0 \approx (x_1 \times d_3)^{k_3}/\delta \approx (x_3 \times d_1)^{k_1}/\delta \approx (d_1 \times d_3)^{\min(k_1,k_3)}/\delta$. Hence $x_1 \times d_3$, $x_3 \times d_1$, $d_1 \times d_3$, and $x_1 \times x_3$ are all in S, so their sum is also in S, as in the proof of the addition case. That is, $(x_1 + d_1) \times (x_3 + d_3)$ is in S; and also obviously $(x_2 + d_2) \times (x_3 + d_3)$ is.

Note that the proof of congruence breaks down for the case of division because $(((x_1 - x_2) + (d_1 - d_2))/(x_3 + d_3))/\delta$ might not be infinitesimal, e.g., if $x_3 = 0$ and d_3 is infinitesimal with respect to $((x_1 - x_2) + (d_1 - d_2))/\delta$. It follows from Proposition 1 that the set of equivalence classes of members of S is a ring (call it $\mathbb{R}$) with respect to the induced operations $+, -, \times$. Denote the equivalence class of any element $x + d$ by $|x + d|$. $\mathbb{R}$ has the following properties:

Proposition 2
(1) *For any real numbers* x_1, x_2, $|x_1| = |x_2|$ *iff* $x_1 = x_2$
(2) *For any infinitesimals* d_1, d_2, *if* $|d_1^2| = |d_2^2| = |0|$, *then* $|d_1| \times |d_2| = 0$
(3) *For any nonnegative integer k, there is some infinitesimal d with* $|d^{k+1}| = |0|$ *while* $|d^k| \neq |0|$.

Proof: (1) If x_1, x_2 are real, then not $(x_1 - x_2)/\delta \approx 0$ unless $x_1 = x_2$. (2) Let $d_1' = d_1^2/\delta$ and $d_2' = d_2^2/\delta$. By hypothesis, $d_1' \approx 0 \approx d_2'$. But $d_1 d_2/\delta = (d_1^2 d_2^2/\delta^2)^{1/2} = (d_1')^{1/2}(d_2')^{1/2}$, which is infinitesimal if d_1' and d_2' are. (3) Consider $\delta^2, \delta, \delta^{1/2}, \delta^{1/3}, \ldots$, etc.

The following lemma is useful for Propositions 3, 4, and 6.

Lemma *For any infinitesimal* δ *and any positive integer k, there is an infinitesimal d such that* d^{k+1}/δ *is infinitesimal while* d^k/δ *is infinite.*

Proof: Let $d = \delta^{(k+1)/k(k+2)}$. Now $d^{k+1}/\delta = \delta^{(k+1)^2/k(k+2)}/\delta = \delta^{(k^2+2k+1)/k(k+2)}/\delta^{(k^2+2k)/(k^2+2k)} = \delta^{1/(k^2+2k)} \approx 0$. But $d^k/\delta = \delta^{k(k+1)/k(k+2)}/\delta = \delta^{(k+1)/(k+2)}/\delta^{(k+2)/(k+2)} = \delta^{-1/(k+2)} = 1/\delta^{1/(k+2)}$, which is infinite.

Define D_0 to be $|0|$, and for all positive integers k, let $D_k = \{|d| : |d^{k+1}| = |0| \text{ while } |d^k| \neq 0\}$. Let D be $\bigcup_{\forall k>0} D_k$. Then

Proposition 3 *For all positive integers k*
(1) *There is a* $|d|$ *in* D_k *such that for all* $|d_1|$ *in D,* $|d_1| \times |d^k| = |0|$
(2) *There is a* $|d|$ *in* D_k *and a* $|d_1|$ *in* D_{k+2} *such that* $|d_1| \times |d^k| \neq |0|$.

Proof: (1) Let d be $\delta^{1/k}$. Now $d^k/\delta = 1$, not ≈ 0. But $d^{k+1}/\delta = 1.\delta^{1/k} \approx 0$. Hence $|d|$ is in D_k. Moreover, for any infinitesimal d_1, $d_1 d^k/\delta = d_1 \approx 0$, so $|d_1| \times |d^k| = |0|$. (2) Let d be $\delta^{(k+1)/k(k+2)}$ as in the lemma, and let d_1 be

δ/d^k. Now by the argument of the lemma, $|d|$ is in D_k. Furthermore, $d_1 = \delta/d^k = \delta/\delta^{(k+1)/(k+2)} = \delta^{1/(k+2)}$. So $d_1^{k+2}/\delta = \delta^{(k+2)/(k+2)}/\delta^{(k+2)/(k+2)} = 1$, not ≈ 0; and $d_1^{k+3}/\delta = \delta^{(k+3)/(k+2)}/\delta^{(k+2)/(k+2)} = \delta^{1/(k+2)} \approx 0$; hence d_1 is in D_{k+2}. Finally, $(d_1 d^k)/\delta = 1$, not ≈ 0, so $|d_1| \times |d^k| \neq |0|$.

Proposition 2(1) shows that $\mathbb{R}$ has a subfield isomorphic to the real numbers R; this field of equivalence classes will also be referred to as R where no confusion results. Now as usual we can write $|x|^k$ for $|x^k|$ and drop the $|\ |$ and multiplication signs in $\mathbb{R}$ where no confusion results. An element d of an algebra is *strictly nilpotent of degree k* if $d^{k+1} = 0$ while $d^k \neq 0$, and an algebra is *strictly nilpotent of degree k* if it has strictly nilpotent elements of degree k. Proposition 2(3) shows that $\mathbb{R}$ is strictly nilpotent of all positive integral degrees. Proposition 2(2) is relevant to the comparison with synthetic differential geometry in Sections 4 and 5. While all elements of D_k go to zero on being raised to the $k+1$st power and not for any lesser integral power, Proposition 3 shows that these elements fall into two nonnull classes: those whose kth power when multiplied by any nilpotent element goes to zero, and those whose kth power has a nonzero product with some nilpotent element. This is also relevant to the results of Section 4.

3 Summary of nonclassical model theory and the construction of an incomplete model This section sketches basic nonclassical model theory as developed in [4], [6], and [7], and applies it to the construction of an incomplete theory using a three-valued intuitionist logic.

A *logic* will be said to be a complete lattice L together with a filter $\nabla \subset L$, called the *designated elements* of L. The *language* $\mathfrak{L}$ considered here consists of a set of *simple terms* (names) in 1–1 correspondence with the noninfinite hyperreal numbers (think of them as naming themselves). *Complex terms* are produced by closure with respect to the operations $+$, $-$, $\times$. A *term* is either a simple term or a complex term. The metalinguistic variables $t, t_0, t_1, \ldots$ range over terms. If t_1, t_2 are terms, then an *atomic sentence* is of the form $t_1 \doteq t_2$. The language has two sorts of object language *variables*, each with several sorts of associated *quantifiers*: variables $x, x_0, x_1, \ldots$ with the four associated quantifiers $(\forall \in \mathbb{R}), (\exists \in \mathbb{R}), (\forall \in R), (\exists \in R)$; and variables $d, d_0, d_1, \ldots$ with the associated quantifiers $(\forall \in D), (\exists \in D)$, and, for each positive integer k, $(\forall \in D_k)$ and $(\exists \in D_k)$. The metalinguistic variables $v, v_0, v_1, \ldots$ range over variables of any sort.

The language also has the sentential operators $\neg$, &, $\vee$, $\rightarrow$, and $A \supset B$ is defined as $\neg A \vee B$ and $A \equiv B$ as $(A \supset B)$ & $(B \supset A)$. $(\mathrm{E}!x \in R)(Fx)$ is defined as $(\exists x \in R)(Fx$ & $(\forall x_0 \in R)(Fx_0 \rightarrow x = x_0))$. Wffs and sentences are defined in the usual way.

In this section the logic L is the three element chain (Hasse Diagram):

*	o	T	True
	o	N	Neither
	o	F	False

with set of designated elements $\nabla = \{T\}$. (Designated elements are starred on the Hasse Diagram.) An *assignment* is a function I: (closed sentences of $\mathfrak{L}$) $\rightarrow$ $\{T,N,F\}$ satisfying:

(1) For any atomic sentence $t_1 = t_2$, $I(t_1 = t_2) \in \{T,N,F\}$
(2) $I(A \,\&\, B) = \text{glb}(I(A),I(B))$, $I(A \vee B) = \text{lub}(I(A),I(B))$, while $I(A \rightarrow B)$ and $I(\neg A)$ are given by the table:

	$\rightarrow$	T	N	F	$\neg$
*	T	T	F	F	F
	N	T	T	F	F
	F	T	T	T	T

(note that L is intuitionist since its (&,v,¬)-structure is exactly that of the well-known three-element interior algebra)
(3) For every quantified sentence of the form $(\forall v \in X)Fv$, $I((\forall v \in X)Fv) =$ glb $\{y$: for some term t, $I(t)$ is in X and $I(F(t/v)) = y\}$; and for every sentence of the form $(\exists v \in X)Fv$, $I((\exists v \in X)Fv) =$ lub $\{y$: for some term t, etc.$\}$, where v is any variable and X is $\mathbb{R}, R, D$ or D_k (subject to proper matching of types).

A sentence A *holds in an assignment* I, written $\vdash_I A$, iff $I(A) \in \nabla$. A set of sentences holding in an assignment is a *theory*. If $A \in Th$ where Th is a theory, we write $\vdash_{Th} A$, dropping the subscript when Th is clear. I and Th are *consistent* if for no A is it the case that both $\vdash A$ and $\vdash \neg A$, and *complete* if for all A either $\vdash A$ or $\vdash \neg A$. I is an *assignment with functionality* iff for all terms t_1, t_2, if $t_1 = t_2$ holds, then for all atomic sentences Ft_1 containing t_1, Ft_1 holds iff Ft_2 holds, where Ft_2 is like Ft_1 except that t_2 replaces t_1 in one or more places. I is an *assignment with identity* iff for all terms t_1, t_2, if $t_1 = t_2$ holds, then for all closed sentences Ft_1 containing t_1, Ft_1 holds iff Ft_2 holds.

A *model* is a pair $\langle \mathfrak{D}, I \rangle$ where $\mathfrak{D}$ is a domain and I an assignment having the additional properties that: (1) I assigns to every simple term a member of $\mathfrak{D}$; (2) I assigns to every n-ary functional expression an n-ary partial function on $\mathfrak{D}$; (3) The assignment to complex terms is given by $I(f(t_1, \ldots, t_n)) = I(f)(I(t_1), \ldots, I(t_n))$, provided that this is defined; (4) I is required to be onto $\mathfrak{D}$, so that every element of the domain is assigned to some term; (5) I satisfies: $t_1 = t_2$ holds iff $I(t_1) = I(t_2)$. If $\langle \mathfrak{D}, I \rangle$ is a model and I an assignment with functionality (identity), then $\langle \mathfrak{D}, I \rangle$ is a *model with functionality (identity)*.

In this section, we construct a theory by specifying further features of the assignment function I. The domain $\mathfrak{D}$ is taken to be the nilpotent ring $\mathbb{R}$ of equivalence classes of the previous section. The specifications are as follows: (1) For every name t, $I(t) = |t|$; (2) $I(+)$, $I(-)$, $I(\times)$ are the ring operations on $\mathbb{R}$ induced by the congruence $|\ |$; this determines the interpretation of all complex terms; (3) Set $I(t_1 = t_2) = \text{T}$ iff $I(t_1) = I(t_2)$, set $I(t_1 = t_2) = \text{N}$ if $I(t_1) \neq I(t_2)$ but the hyperreal number $(t_1 - t_2)/\delta$ is noninfinite, and set $I(t_1 = t_2) = \text{F}$ otherwise. The values of all nonatomic sentences are then determined as above.

We observe that the model just described is a model with identity. (This follows from the facts: (1) That $t_1 = t_2$ holds iff $I(t_1) = I(t_2)$, and hence (2) That if $t_1 = t_2$ holds, then $I(F(t_1)) = I(F(t_2))$ for any atomic F, if these are defined. The latter is then the basis clause of an obvious induction on the lengths of sentences.) Being a model with identity suffices for being a model with functionality, and the latter permits calculations by substitution of identicals, whether or not the background logic is classical. In Section 5, substitutivity of identity is weakened, but in a controlled way. The model and its associated theory are intuitionist in another sense, namely that they are incomplete: Since $I(\delta = 0) =$ N, $I(\neg\delta = 0) =$ F, so that neither $\vdash\delta = 0$ nor $\vdash\neg\delta = 0$, although $\vdash\neg\neg\delta = 0$, $\vdash\delta^2 = 0$, and $\vdash\neg\delta^{1/2} = 0$.

Note finally that a wholly classical (two-valued) theory of $\mathbb{R}$ can be obtained by a different I: by setting $I(t_1 = t_2) =$ T iff $I(t_1) = I(t_2)$, and F otherwise. This also shows that classical two-valued model theory can be recovered as a special case.

4 Incomplete differential calculus, and comparison with synthetic differential geometry In this section it is shown that Taylor's formula and the polynomial differentiation laws hold in the model. A definition of limits can be given, and it is proved that every function is continuous. It is shown that the theory has significant similarities with a corresponding part of synthetic differential geometry, and the dissimilarities are outlined.

A *functional expression* (abbreviated to *function*) is the result of replacing any term or terms in an atomic wff by variables. A function with no remaining terms denoting infinitesimals is called a *real* function. If f is a function with a single free variable v of any sort, then we indicate this by $f(v)$. The result of replacing v throughout by a term t is denoted by $f(t)$. If v_1 and v_2 are variables of any sort, then $f(v_1 + v_2)$ is the result of replacing v by $v_1 + v_2$ throughout. Similarly for $-$ and $\times$. $(\mathrm{E}!x_1, \ldots, x_k \in R)$ is defined as $(\mathrm{E}!x_1 \in R) \ldots (E\rangle x_k \in R)$. Then we have:

Proposition 4 *If $f(x)$ is any real function, then for every positive integer k,*

$$\vdash(\forall x \in R)(\mathrm{E}!x_1, \ldots, x_k \in R)(\forall d \in D_k)(f(x + d) = f(x) + x_1 d + \ldots + x_k d^k).$$

Proof: If $f(x)$ is any real function, then by the polynomial laws of R^*, for any term t, $I(f(t))$ is identical with $I(t_0 + t_1 t + \ldots + t_n t^n)$, where the t_i are simple terms denoting real numbers, since identities are not destroyed in passing from R^* to $\mathbb{R}$. So we may restrict attention to functions of the form $t_0 + t_1 x + \ldots + t_n x^n$, where the t_i denote real numbers, i.e., where the $I(t_i)$ are in R. We abbreviate these functions by $\sum_{i=0}^{n} t_i x^i$. Then for any such $f(x)$ and any term t from R and any term d with $I(d)$ in some D_k, $f(t + d)$ is $t_0 + t_1(t + d) + \ldots + t_n(t + d)^n$. So $I(f(t + d)) = I(t_0) + (I(t_1) \times (I(t) + I(d))) + \ldots$ etc., where $+$ and $\times$ are the induced operations on $\mathbb{R}$. These operations obey the R^* polynomial laws, so we can compute this sum using the binomial expansion. If $n \leq k$, the nilpotence of the element d does not affect this expansion, and (α) below follows by normal arithmetic. If $k < n$, those terms in the binomial expansion of $I(f(t + d))$ which contain $|d^{k+1}|$ as a factor are identical with $|0|$. So $I(f(t + d))$ computes to

$$I(t_0 + t_1 t + t_2 t^2 + \ldots + t_n t^n) + I\left(\left(\sum_{i=1}^{n} \binom{i}{1} t_i t^{i-1}\right) d\right) \ldots$$
$$+ I\left(\left(\sum_{i=1}^{n} \binom{i}{k} t_i t^{i-k}\right) d^k\right). \qquad (\alpha)$$

Hence, by the assignment conditions for quantifiers

$$\vdash (\forall x \in R)(\exists x_1, \ldots, x_k \in R)(\forall d \in D_k)\left(f(x + d) = f(x) + \sum_{i=1}^{k} x_i d^i\right).$$

The next part of the argument (proving uniqueness) uses the postulate that the $I(t_i)$ are real. We need to conjoin to (α) the following: $(\forall x_{k+1}, \ldots, x_{2k} \in R)\left((\forall d \in D_k)\left(f(t + d) = f(t) + \sum_{i=1}^{k} x_{k+i} d^i\right) \to ((t_1 = x_{k+1}) \,\&\ldots\&\, (t_k = x_{2k}))\right)$, where the t_i are a relabelling of the coefficients of (α). Eliminating quantifiers to appropriately assigned terms, we need to prove that:

$$\vdash (\forall d \in D_k)\left(f(t + d) = f(t) + \sum_{i=1}^{k} t_{k+i} d^i\right) \to \&_{i=1}^{k}(t_i = t_{k+i}). \qquad (\beta)$$

If the consequent takes the value T, then (β) holds by the tables for $\to$. If the consequent does not take the value T, then there are two cases: either (i) $t_k = t_{2k}$ does not hold or (ii) some other $t_i = t_{k+i}$ does not hold. If $t_k = t_{2k}$ does not hold, then $I(t_k) \neq I(t_{2k})$. Now, since t_k and t_{2k} are real, $(t_k - t_{2k})/\delta$ is infinite, so $I(t_k = t_{2k})$ is F. But by the lemma of Section 2, there is some infinitesimal hyperreal number d such that d^k/δ is infinite, hence $(t_k - t_{2k})d^k/\delta$ is infinite. If every other $t_i = t_{k+i}$ holds, then $|t_i| = |t_{k+i}|$ and $t_i = t_{k+1}$ in R^*. So in R^*, $f(t + d) - \left(f(t) + \sum_{i=1}^{k} t_{k+i} d^i\right) = f(t) + \sum_{i=1}^{k} t_i d^i - \left(f(t) + \sum_{i=1}^{k} t_{k+i} d^i\right) = (t_k - t_{2k})d^k$. But the latter is infinite with respect to δ. So in $\mathbb{R}$, $I(f(t + d)) \neq I\left(f(t) + \sum_{i=1}^{k} t_{k+i} d^i\right)$. But also in R^*, $\left(f(t + d) - \left(f(t) + \sum_{i=1}^{k} t_{k+i} d^i\right)\right)\Big/\delta$ is infinite. Hence the antecedent of (β) is F, and (β) holds by the table for $\to$.

Otherwise, if some other $t_i = t_{k+1}$ does not hold, let i be the least integer for which $t_i = t_{k+i}$ does not hold. Then choosing the same d, in R^* $f(t + d) - \left(f(t) + \sum_{i=1}^{k} t_{k+i} d^i\right) = (t_i - t_{k+i})d^i +$ higher powers of d. But the first term is infinite with respect to δ if d^k is. So, as for (i), in $\mathbb{R}$ $I(f(t + d)) \neq I\left(f(t) + \sum_{i=1}^{k} t_{k+i} d^i\right)$. But in R^* $\left(f(t + d) - \left(f(t) + \left(\sum_{i=1}^{k} t_{k+i} d^i\right)\right)\right)\Big/\delta$ is infinite. Hence again the antecedent of (β) is F and so (β) holds.

Consider the case where $k = 1$. Then for any $|d|$ in D_1 and any real t, $\vdash f(t + d) = f(t) + t_1 d$ for some term t_1 with $I(t_1)$ in R. A functional expression $g(x)$ is called *a derivative of* $f(x)$, if for any d in D_1 and any t with $I(t)$

in R, $\vdash f(t+d) = f(t) + g(t)d$. We know independently from real number calculus that there is always at least one derivative for any real function $f(x)$. If $g(x)$ is a derivative of $f(x)$, we can also denote it by $f'(x)$. Thus for any derivative $f'(x)$, we have the *Taylor formula* $\vdash f(t+d) = f(t) + d \cdot f'(t)$, or $\vdash (\forall x \in R)(\forall d \in D_1)(f(x+d) = f(x) + d \cdot f'(x))$. Define an *n-th degree polynomial in the indeterminate x* to be any functional expression of the form $t_0 + t_1 x + \ldots + t_n x^n$, where the t_i are simple terms, that is $\sum_{i=0}^{n} t_i x^i$.

Proposition 5 (Polynomial Differentiation) *If f is any polynomial of the form* $\sum_{i=0}^{n} t_i x^i$ *with real coefficients* t_i, *and* $f'(x)$ *is a derivative of f, then* $\vdash (\forall x \in R)\left(f'(x) = \sum_{i=1}^{n} i t_i x^{i-1}\right)$.

Proof: From the Taylor formula, $\vdash (\forall x \in R)(f(x+d) = f(x) + d \cdot f'(x))$, where $I(d)$ is in D_1. Whence $I(f(t+d)) = I(f(t)) + (I(d) \times I(f'(t)))$ for any term t with $I(t)$ in R. But $I(f(t+d)) = I\left(\sum_{i=0}^{n} t_i (t+d)^i\right)$. As in Proposition 4, this computes to $I\left(\sum_{i=0}^{n} t_i t^i\right) + \left(I\left(\sum_{i=1}^{n} \binom{i}{1} t_i t^{i-1}\right) \cdot I(d)\right) + \left(I\left(\sum_{i=2}^{n} \binom{i}{2} t_i t^{i-2}\right) \cdot I(d^2)\right) + \ldots$ higher powers of d. Since $I(d^2) = I(d^3) = \ldots = 0$, all products of $d^2, d^3, \ldots$ may be dropped. Thus we have $I(f(t+d)) = I(f(t)) + (I(d) I(f'(t)))$ and also $= I(f(t)) + I(d) I\left(\sum_{i=1}^{n} \binom{i}{1} t_i t^{i-1}\right)$. So, since minus is a congruence, $I(d) I(f'(t)) = I(d) I\left(\sum_{i=1}^{n} \binom{i}{1} t_i t^{i-1}\right)$. But since $I(d)$ is in D_1, and $I(t), I(t_i)$ are in R, this can happen only if $I(f'(t)) = I\left(\sum_{i=1}^{n} \binom{i}{1} t_i t^{i-1}\right)$. But t was arbitrarily chosen. Hence $\vdash (\forall x \in R)\left(f'(x) = \left(\sum_{i=1}^{n} i t_i x^{i-1}\right)\right)$, as required.

A definition of two-sided limits can be given. Define '$\lim_{x \to t} f(x) = t_1$' to mean '$(\forall d \in D)(f(t+d) = t_1 \vee (\exists d_1 \in D)(f(t+d) - t_1 = d_1))$'. One-sided limits can also be defined by introducing the notions of positivity and negativity for members of D, but that is not done here because of the following proposition. It is also noted that in the above definition of limit the case where not $\vdash f(t) = t_1$ does not arise, as the following proposition shows. For any real function $f(x)$, define in the usual way 'f is continuous at t' to mean '$\lim_{x \to t} f(x) = f(t)$', and '$f$ is continuous' to mean '$(\forall x \in R)(f$ is continuous at $x)$'. Then

Proposition 6 *For every real function* $f(x)$, $\vdash f$ *is continuous.*

Proof: It has to be proved, for every real term t, that $\vdash (\forall d \in D)(f(t+d) = f(t) \vee (\exists d_1 \in D)(f(t+d) - f(t) = d_1))$. But it follows from Proposition 4 that $\vdash (\forall d \in D_k)(f(t+d) = f(t) + t_1 d + \ldots + t_k d^k)$. If not all the real $t_i = 0$, then $\vdash f(t+d) - f(t) = t_1 d + \ldots + t_k d^k$. It is obvious that raising the right-hand side to power k is not (considered as a hyperreal number) infinitesimal with respect to δ (since its first term is not), while raising it to power $k+1$ is infinitesimal with respect to δ (since each term is), so that the right-hand side is in D_k. Thus $\vdash(\exists d_1 \in D)(f(t+d) - f(t) = d_1)$. The result follows by disjoining the alternatives and universal generalization.

Synthetic differential geometry (SDG), as expounded in [3] (see also [1]), is likewise an incomplete theory (with neither $\delta = 0$ nor $\neg\delta = 0$ holding). The theory of [3] has elements which are strictly nilpotent of all degrees, while that of [1] restricts consideration to D_1. Neither proceeds from a construction on the classical hyperreals, however, nor utilizes a three-valued model theory. In these theories, also, every function is continuous. The method of obtaining derivatives from the Taylor formula as in Proposition 5 is similar to that of [3], and is a variant of the usual classical treatment. Like SDG, Propositions 4 and 5 utilize the calculatory advantages of nilpotent elements, since these ensure that higher-order differentials can ultimately be ignored.

The case where $x = 0$, $k = 1$ of Proposition 4 is Axiom′ 1 of [3], with the proviso that R in Proposition 4 be replaced by the whole domain there; the case where $x = 0$ is Axiom 1′ of [3] with the same proviso. If, however, R were replaced by $\mathbb{R}$ here, then Proposition 4 would fail, as follows. Choose any d_1 in D_1 and let f be the function $f(x) =_{df} d_1 x$. Then certainly $\vdash(\exists x \in R)(\forall d \in D_1)(f(d) = f(0 + xd))$, the x in question being d_1. However, this x is not unique: for any other d_2 in D we have $\vdash(\forall d \in D_1)(dd_2 = dd_1 = 0)$ while not $\vdash d_1 = d_2$, so that the antecedent of $(\forall d \in D_1)(f(d) = f(0) + d_2 d) \to d_1 = d_2$ holds while the consequent does not hold. Indeed, f could even have a noninfinitesimal coefficient, $f(x) = (5+\delta)x$, say; for then the coefficient fails to be unique, since $\vdash(\forall d \in D_1)((5+\delta)d = 0 = (5+2\delta)d)$ while not $\vdash 5 + \delta = 5 + 2\delta$. Thus the present theory is a theory of functions with real slopes as in nonstandard analysis, and so is less general than SDG.

The essential difference with the nilpotent elements in SDG is that the D_1 part of the domain is postulated in SDG to contain elements d_1, d_2 such that not $\vdash d_1 d_2 = 0$, while in the present model this is not so (Proposition 2(2)). Correspondingly the SDG cancellation principle, $(\forall d \in D_1)(dt_1 = dt_2) \to t_1 = t_2$, fails: for example, when $I(t_1) = |\delta|$ and $I(t_2) = |2\delta|$ the antecedent is T and the consequent N. However, the cancellation principle holds for cases where the difference between $I(t_1)$ and $I(t_2)$ is infinite with respect to δ if they are different at all, as for example $\vdash(\forall x_1 x_2 \in R)((\forall d \in D_1)(dx_1 = dx_2) \to x_1 = x_2)$.

The failure of the law of excluded middle (LEM) is of interest. The account of [3] links it to the holding of the cancellation principle and the continuity of every function. However we can see that the failure of LEM in the present paper is rather independent of the functional part of the construction, since the latter can also produce a wholly classical model (end of Section 3). The same point pertains to the inconsistent theory of the next section. This does not show that the 'correct' description is that of classical two-valued logic, however; to the contrary it suggests that functionality is mathematically prior to sentential logic.

SDG in [3] employs the mathematical machinery of Cartesian closed categories, which is much stronger than that of the present paper, which aims rather at studying equational theories. On the other hand, there is here some simplicity in the presentation of the ideas of incompleteness, nilpotence, differentiability, limits, continuity, etc., within the model-theoretic framework, albeit a nonclassical one. Furthermore, the present approach permits the investigation of similar theories with different background logics (see Sections 5 and 6). Another point is that while [3] maintains that SDG is an *essentially geometric* treatment of analysis, one might argue that it is interesting how much one can get of SDG with resources merely from model theory and algebraic number theory.

5 Inconsistent differential calculus The background logic is now altered to the well-known logic RM3.

		$\rightarrow$	T	B	F	$\neg$
*	True	*T	T	F	F	F
*	Both	*B	T	B	F	B
	False	F	T	T	T	T

The set of designated elements $\nabla = \{T,B\}$. There are a number of options for the assignment function for the values of atomic sentences. The one used here illustrates the possibility of controlling substitutivity of identity even though a full model with identity is absent (another option is mentioned in the final section): (1) Set $I(t_1 = t_2) = T$ if $t_1 = t_2$; that is, if t_1, t_2 are considered hyperreal numbers; (2) $I(t_1 = t_2) = B$ if $t_1 \neq t_2$ but $I(t_1) = I(t_2)$ in $\mathbb{R}$; and (3) $I(t_1 = t_2) = F$ if $I(t_1) \neq I(t_2)$.

Note that in consequence of (1) and (2), $I(\delta^2 = 0) \neq T$, but rather $I(\delta^2 = 0) = B$ and so $I(\neg\delta^2 = 0) = B$. The theory is thus inconsistent. Again, $\vdash(5 + \delta^2 = 5)\ \&\ \neg(5 + \delta^2 = 5)$. In consequence of (3), $I(\delta = 0) = F$ and $I(\neg\delta = 0) = T$, unlike SDG. Indeed, all theories which are constructed by assigning to the set of atomic equations values from the above logic in the above fashion are complete.

The present model is a model with functionality but not with identity. (Proof of functionality: By inspection $t_1 = t_2$ holds iff $I(t_1) = I(t_2)$. But I is a congruence; so for any term $t(t_1)$ containing t_1, $I(t(t_1)) = I(t(t_2))$. Hence if $t_3(t_1) = t_4(t_1)$ holds, then $I(t_3(t_1)) = I(t_4(t_1))$; so $I(t_3(t_2)) = I(t_4(t_2))$, hence $t_3(t_2) = t_4(t_2)$ holds. Proof of nonidentity: $\vdash\delta^2 = 0$, but while $\vdash\neg\delta^2 = 0$, not $\vdash\neg 0 = 0$ nor $\vdash\neg\delta^2 = \delta^2$.) This means that, on the one hand, calculations may be carried out utilizing the advantages of $\vdash\delta^2 = 0$, as in earlier sections; while on the other hand, one does not have to submit to $\vdash\neg t = t$ for any term t, an improvement on [4], [6], and [7].

It can be asked how much is lost from a theory if full substitutivity of identity in all contents is relaxed. This leads to a comparison first with the full $+$, $-$, $\times$-theory of the noninfinite hyperreals, and then with the theory of the previous section. It is shown (i) that every sentence holding in the $+$, $-$, $\times$, &, $\vee$, $\neg$, $\forall$, $\exists$-theory of the noninfinite hyperreals holds in the present model, and (ii) that Propositions 4 to 6 may also be reproved utilizing the same calculatory advantages of nilpotent elements.

(i) is an immediate consequence of the extendability lemma (Proposition 1 of [7]), since the sets of sentences of the forms $t_1 = t_2$ and $\neg t_1 = t_2$ holding for the noninfinite hyperreals are respectively subsets of those holding in the present model.

As for (ii) we have:

Proposition 7 *If $f(x)$ is any real function, then for every positive integer k,*

$$\vdash (\forall x \in R)(\mathrm{E}!x_1, \ldots, x_k)(\forall d \in D_k)(f(x+d) = f(x) + x_1 d + \ldots + x_k d^k).$$

Proof: The proof that $I(f(t+d))$ computes to (α) as in Proposition 4 is identical. To prove uniqueness, we need to prove (β). If the consequent of (β) is T, then (β) holds. And, for real coefficients t_i, t_{k+i}, one never has $I(t_i = t_{k+i}) = \mathrm{B}$. Hence consider the case where $I(t_i = t_{k+i}) = \mathrm{F}$. Then $I(t_i) \neq I(t_{k+i})$. But also $(t_i - t_{k+i})/\delta$ is an infinite hyperreal number since the numerator is real and nonzero. Hence, as in Proposition 4, for some d with $|d|$ in D_k, $d^i(t_i - t_{k+i})/\delta$ is noninfinitesimal, so $I(d^i t_i) \neq I(d^i t_{k+i})$ and the antecedent is F as required.

Proposition 8 *If f is any polynomial of the form $\sum_{i=0}^{n} t_i x^i$ with real coefficients t_i, then $\vdash (\forall x \in R)\left(f'(x) = \sum_{i=1}^{n} i t_i x^{i-1}\right)$.*

Proof: Similar to the proof of Proposition 5.

Proposition 9 *For every real function f, $\vdash f$ is continuous.*

Proof: Similar to the proof of Proposition 6.

The $\rightarrow$-free part of this theory is a common inconsistent extension of the classical theories of (i) the ring of noninfinite infinitesimals of R^*, and (ii) the nilpotent ring $\mathbb{R}$, which cannot be achieved classically. To repeat an earlier point, inconsistent calculus is not being recommended as superior or truer, though its nilpotent elements have some of the calculatory advantages of synthetic differential geometry. The aim is only to demonstrate its existence, and to lend support to the claim that inconsistent theories are of mathematical interest.

6 Conclusion An inconsistent model with identity can be constructed with RM3: Set $I(t_1 = t_2) = \mathrm{B}$ iff $I(t_1) = I(t_2)$, and F otherwise, as with the models of [4], [6], and [7]. This produces $\vdash \delta^2 = 0$ & $\neg\delta^2 = 0$ & $\neg\delta = 0$ but not $\vdash \delta = 0$, all as in Section 5; but it also yields $\vdash t = t$ & $\neg t = t$ for every term t. There seems to be no reason not to adopt the more sensitive model of Section 5 which is functional but not with identity.

In following papers, it is proposed to report results on the following related topics: (1) corresponding inconsistent theories using Brazilian-style paraconsistent negation, and topological and Routley-* dualizations of these and the theories of the present paper; (2) order and set membership; (3) integration; (4) inconsistent superreals; (5) inconsistent polynomial rings in one or more indeterminates.

The congruence $\underset{\delta}{=}$ and particularly the associated inconsistent theories can be regarded as yet another approach to the idea of an "infinitesimal microscope" (see [10] or [11]). A microscope with "resolving power" δ can be said to be a theory which inconsistently identifies with zero and one another all sizes infinitesimal with respect to δ. One is unable to distinguish the behavior of quantities below this "order of infinitesimality" or "order of relative identity". These quantities have all of one another's properties if the theory has substitutivity of identity, or atomic properties if the theory/model has functionality.

Finally, inconsistent claims about infinitesimals have been around for as long as calculus. One must always try to see whether these stem from confusion, or from dim but genuine paraconsistent insights. The only possibility for giving the second kind of answer lies in the rigorous construction of inconsistent mathematical theories. Perhaps the present theories satisfy some of the intuitions of classical analysts; but even if they do not, inconsistent and incomplete mathematics needs investigation.

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FIXED POINT THEOREMS FOR INCONSISTENT AND INCOMPLETE FORMATION OF LARGE CATEGORIES

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ABSTRACT: The method of fixed points is used to show that an unrestricted comprehension scheme for large categories can be described in either inconsistent or incomplete theories of several background logics; thus simplifying and generalising a result of Feferman.

1. *Introduction*

Defects in the current foundations for category theory are well known and have occupied thinkers about foundations since Eilenberg and Mac Lane proposed the theory of categories in 1945. This is not to say that foundational problems have seriously interrupted the progress of mathematicians who actually use categories in their day-to-day work. The popularity of category constructs in the practice of mathematics continues to grow despite any truly satisfactory resolution to the original foundational problems. It must however be conceded that the categories of mathematical practice are generally small or locally small and so are well accounted for by the current foundations.

Just as it is natural and healthy for discussions about architectural achievements to focus on aesthetics and functionality of design, so is the current interest in the applications of categories normal and productive. Engineering talk about the properties of bricks, mortar, steel and timber seem tedious by comparison. However when the architect becomes more adventurous with their applications of the material, the cautious among us begin to wonder about the received wisdoms concerning bricks and mortar and we turn back to the engineers for reassurance. Focus should again begin to move back onto logicians. Questions about the foundations of category theory have become all the more pertinent with the recent surge in popularity of the theory.

This paper is a reaction to an approach to the foundations of category theory by Solomon Feferman in [2]. Feferman presents a theory of partial operations and classifications which are kinds of intentional characterisations of the concepts of characteristic functions and sets.

Classifications have a general comprehension axiom which facilitates the formation of the general kinds of collections required for the practice of unrestricted category theory by mathematicians. Feferman models this axiom in an underlying logic which is classical and S4 modal.

Our aim in this paper is to attempt to simplify the modelling in an intuitionist logic and then to dualise the intuitionist model to get a model in a paraconsistent logic. Our approach stems from a recognition of the well known isomorphism between S4 theories and theories in intuitionist logic, the observation that such intuitionist logics are the logics of open sets and that there are paraconsistent logics which are the logics of the corresponding closed subsets. We will observe that these topological constructions are defective in their treatments of double negation, leading to a reformulation in a different dual pair of logics.

The motivation for this paper is a desire to recommend paraconsistent theories as equally viable as intutitionist ones for the practice of mathematics.

2. *Preliminaries*

Feferman sets out to model a comprehension axiom for classifications:

$$\exists f\, \forall a_1 \ldots a_n \exists c [f a_1 \ldots a_2 \simeq c \ \wedge \ (\forall x (x \eta c \leftrightarrow \Box \phi_c x)) \ \wedge \ (\forall x (x \bar{\eta} c \leftrightarrow \Box \neg \phi_c x))]$$

in a logic which is essentially S4.

Ignoring the possible misordering of the quantifiers and setting aside the idea of the characteristic (partial) function f, the idea seems to be to have two predicates, η and $\bar{\eta}$ to be intuitively understood as functioning like the predicates $\in$ and $\notin$ in set theory. ϕ_c is some property which, by virtue of the comprehension scheme, gives rise to a classification c. $\Box \phi x$ is read intuitively as saying it can be verified that the property ϕ holds of object x. As a result, while η and $\bar{\eta}$ in some sense function like $\in$ and $\notin$, it is not the case that:

$$(\forall x)(\forall y)(x \bar{\eta} y \leftrightarrow \neg x \eta y)$$

This is because some classifications y are partial, which means for some x,

$(x\eta y \lor x\bar{\eta}y)$ fails and consequently so does $(x\bar{\eta}y \leftrightarrow \neg x\eta y)$. As an example consider the property ϕx defined as $x\eta x$. An attempt to verify ϕa amounts to a check whether $a\eta a$. But to verify $a\eta a$ requires first a demonstration that ϕa holds, which is what we were trying to demonstrate in the first place. So attempts to verify some properties are circular and cannot be completed.

Restricting the properties ϕx that give rise to an associated classification c_ϕ to those for which $\Box\phi x$ holds has the effect of bringing about a kind of incompleteness in the resulting theory in the sense that

$$(\exists y)(\exists x)\neg(x\eta y \lor x\bar{\eta}y)$$

Our approach is to be open about this incompleteness by dropping the predicate $\bar{\eta}$ from the language and just using the sentence operator $\neg$ instead. We also drop $\Box$ from the language and achive its original purpose by using a 3-valued logic and allowing some sentences of the form ϕx to take the middle truth value. The resultant theory will be incomplete in the usual sense that $A \lor \neg A$ fails for some sentence A.

To be more precise, let L be a logic with a set of truth values X with designated values $\nabla \subset X$. Let the consequence relation $\models_L$ be defined as follows. $A \models_L B$ iff for all valuations v on X if $v(A) \in \nabla$ then $v(B) \in \nabla$. A set of sentences Th_L is an L-theory iff it is closed w.r.t. $\models_L$ and closed w.r.t. conjunctions.

DEFINITION: An L-theory Th_L is *incomplete* iff for some sentence A neither $A \in Th_L$ nor $\neg A \in Th_L$.

DEFINITION: Th_L is *inconsistent* iff for some sentence A both $A \in Th_L$ and $\neg A \in Th_L$.

3. *A J3 Theory*

In this section the aim is to show how a modified comprehension axiom for classifications:

(C) $\quad (\exists y_\phi)(\forall x)(x\eta y_\phi \leftrightarrow \phi x)$

can be modelled in the intuitionist logic J3. That is, we will construct a theory Th_{J3} whose underlying logic is J3 such that $C \in Th_{J3}$.

Like Feferman we denote the language of our theory by L $(\simeq, \eta)$ which is a basic language L $(\simeq)$ extended by the addition of the binary predicate η (but not $\bar{\eta}$ or $\Box$). We assume the existence of a simplier theory $Th_{\simeq}$ in the simpler language L $(\simeq)$. The negation operator will be denoted by the symbol $\neg$.

J3 is a logic with a set of three truth values $\{F, N, T\}$, a set of designated values $\nabla_{J3} = \{T\}$ and negation $\neg$ defined such that $\neg T = F$, $\neg N = F$ and $\neg F = T$.

J3 is a topological logic. This means that we can consider the truth values F, N and T to be sets in some topology and the operators $\wedge$, $\vee$, $\neg$, $\forall$, $\exists$ to be defined in terms of set operators in the topology.

Consider a partially ordered set of worlds W:

$$w^* \leq w$$

and the set of hereditary subsets X:

$$\Lambda, \{w\}, \{w^*, w\}$$

X is a set of open sets and $\langle W, X\rangle$ is a topological space. A valuation is a function v:L $(\simeq, \eta) \rightarrow X$.

Compound sentences containing operators $\wedge$, $\vee$, $\neg$, $\forall$, $\exists$ are evaluated by recursively applying the following rules:
For any valuation v,

(i) $v(A \wedge B) = v(A) \cap v(B)$

(ii) $v(A \vee B) = v(A) \cup v(B)$

(iii) $v(\neg A) = \mathrm{I}\,(\overline{v(A)})$ = The largest open set S such that $S \subseteq \overline{v(A)}$

(iv) $v((\forall x)Fx) = \cap\{y$: for some term t, $v(Ft) = y\}$

(v) $v((\exists x)Fx) = \cup\{y$: for some term t, $v(Ft) = y\}$

We let F denote Λ, N denote $\{w\}$ and T denote $\{w^*, w\}$ and tell the usual intutionist story regarding designated truth values. That is, let $\nabla_{J3} = \{T\}$.

Following Feferman we define a model for the comprehension axiom using a transfinite inductive definition. That is we define a valuation v such that $v(C) \in \nabla_{J3}$ i.e. $v(C) = T$.

The required valuation v will be named v_f in the construction that follows. v_f will be defined in terms of a previous valuation v_{f-1} which will in turn be defined in terms of v_{f-2}, etc. So in order to define the valuation v_f we require definitions of a series of valuations v_1, v_2, v_3, ... Our definition begins with a basis v_1.

BASIS: For all sentences $A \in L(\asymp,\eta)$, v_1 is defined:

(B1) $v_1(A) = \{w^*,w\}$ if $A \in L(\asymp)$ and $A \in Th_{\asymp}$
(B2) $v_1(A) = \Lambda$ if $A \in L(\asymp)$ but $A \notin Th_{\asymp}$
(B3) $v_1(A) = \{w\}$ otherwise
(B4) Remaining compound sentences are then evaluated by recursive applications of the rules (i) - (v).

Because this is a transfinite inductive definition the induction step has two parts, one for successor steps and one for limit steps.

INDUCTION STEP: v_α is defined:
If α is a successor ordinal then

(I1) $v_{\alpha+1}(a\eta c_\phi) = v_\alpha(\phi a)$
(I2) Remaining compound sentences are then evaluated by recursive applications of the rules (i) - (v).

If α is a limit ordinal then

(I3) $v_\alpha(A) = \bigcup_{\beta<\alpha} v_\beta(A)$

This completes the definition of a transfinite series of valuations v_1, v_2, v_3, ... The valuation v_f is a special valuation in this series which is a fixed point. That is a valuation such that for every sentence S, $v_f(S) = v_{f+1}(S)$.

Before proving that such a valuation is implicit in our definition of the transfinite series of valuations, we prove some interim results.

DEFINITION: $v_\alpha \leq v_\beta$ iff
$\{A \in L(\asymp,\eta): v_\alpha(A)=T\} \subseteq \{A \in L(\asymp,\eta): v_\beta(A)=T\}$
and $\{A \in L(\asymp,\eta): v_\alpha(A)=F\} \subseteq \{A \in L(\asymp,\eta): v_\beta(A)=F\}$.

or alternatively:

For every sentence A if $v_\alpha(A)=T$ then $v_\beta(A)=T$ and if $v_\alpha(A)=F$ then $v_\beta(A)=F$.

THEOREM (Monotonicity): For valuations v_α, v_β if $\alpha \leq \beta$ then $v_\alpha \leq v_\beta$.

PROOF: By induction on the number of connectives in an arbitrary sentence A.

If A is an atomic sentence then the theorem trivially holds because the rules for building up any valuation only change the values of compound sentences. So $v_\alpha(A) = v_\beta(A)$.

Assume that for sentences P, Q the theorem holds. We show that the theorem holds for sentences $P \wedge Q$, $P \vee Q$, $\neg P$, $(\forall x)P$, $(\exists x)P$. Three of the five demonstrations are given here:

(a) Assume $v_\alpha(P \wedge Q) = T$. By clause (i) of the evaluation procedure for valuations it follows that $v_\alpha(P) = T$ and $v_\beta(Q) = T$. From the induction hypothesis we have $v_\beta(P) = T$ and $v_\beta(Q) = T$. By again employing clause (i) of the evaluation procedure for valuations we conclude that $v_\beta(P \wedge Q) = T$ as required. A similar argument can be given assuming $v_\alpha(P \wedge Q) = F$ and concluding that $v_\beta(P \wedge Q) = F$

(c) Assume $v_\alpha(\neg P) = T$. By clause (iii) of the evaluation procedure for valutions it follows that $v_\alpha(P) = F$. By the induction hypothesis it follows that $v_\beta(P) = F$. Again from (iii) we have that $v_\beta(\neg P) = T$ as required. A similar argument can be given assuming $v_\alpha(\neg P) = F$ and concluding that $v_\beta(\neg P) = F$.

(e) Assume $v_\alpha((\exists x)P) = T$. By clause (v) of the evaluation procedure for valuations it follows that for some term t, $v_\alpha((\exists x)P[x/t]) = T$. By the induction hypothesis we have that for some term t, $v_\beta((\exists x)P[x/t]) = T$. By clause (v) of the evaluation procedure for valuations we conclude that $v_\beta((\exists x)P) = T$ as required. □

THEOREM (Fixed Point): This definition generates a fixed point. That is a valuation v_f such that for every sentence $A \in L(\simeq,\eta)$, $v_f(A) = v_{f+1}(A)$.

PROOF: By the previous theorem, we have that this method generates a sequence of valuations: $v_1 \leq v_2 \leq v_3 \leq \ldots$.

Once a sentence gets assigned a value T or F by a valuation it retains that value in all later valuations. Now, there are only denumerably many sentences in the language $L(=,\eta)$. The set of ordinals of the second number class is non-denumerable. So for some λ of the second number class $\nu_\lambda = \nu_{\lambda+1}$. □

THEOREM (ν_f is a model for C): $\nu_f((\exists y_\phi)(\forall x)(x\eta y_\phi \leftrightarrow \phi x)) \in \nabla_{J3}$

PROOF:

Left to right:

Assume for arbitrary a and for some c_ϕ that $\nu_f(a\eta c_\phi)=T$. Let α be the least ordinal such that $\nu_\alpha(a\eta c_\phi)=T$. α must be a successor ordinal. By the method of construction it must be the case that $\nu_{\alpha\text{-}1}(\phi a)=T$. Since α-1 $\leq f$ it follows from montonicity that $\nu_f(\phi a)=T$.

If we assume for arbitrary a that $\nu_f(a\eta c_\phi)=F$ it can be shown by a similar argument that $\nu_f(\phi a)=F$.

Assume for arbitrary a that $\nu_f(a\eta c_\phi)=N$. Assume also that $\nu_f(\phi a)=T(F)$, then by the method of construction $\nu_{f+1}(a\eta c_\phi)=T(F)$. But ν_f is a fixed point so $\nu_f(a\eta c_\phi)=T(F)$ which contradicts the first assumption. So $\nu_f(\phi a)=N$.

Right to left:

Assume for arbitrary a that $\nu_f(\phi a)=T$. By the method of construction it follows that $\nu_{f+1}(a\eta\ c_\phi)=T$. But $\nu_f=\nu_{f+1}$ since ν_f is a fixed point. So $\nu_f(a\eta\ c_\phi)=T$.

By a similar argument it can be shown that if $\nu_f(\phi a)=F$ then $\nu_f(a\eta c_\phi)=F$.

Assume for arbitrary a that $\nu_f(\phi a)=N$. Assume also that $\nu_f(a\eta c_\phi)=T(F)$. Let α be the least ordinal such that $\nu_\alpha(a\eta c_\phi)=T(F)$. α is a successor ordinal. By the method of construction $\nu_{\alpha\text{-}1}(\phi a)=T(F)$. By the monotonicity theorem and the fact that α-1 $\leq f$ it follows that $\nu_f(\phi a)=T(F)$ which contradicts the first assumption. So $\nu_f(a\eta c_\phi)=N$.

So for arbitrary a, $\nu_f(a\eta c_\phi) = \nu_f(\phi a)$ and hence by the definition of $\leftrightarrow$, for arbitrary a, $\nu_f(a\eta c_\phi \leftrightarrow \phi a)=T$. So $\nu_f((\exists y_\phi)(\forall x)(x\eta y_\phi \leftrightarrow \phi x)) = T$. $T \in \nabla_{J3}$. □

DEFINITION: Let R denote the classification defined by the property $\neg x\eta x$.

THEOREM: $\nu_f(R\eta R) = N$.

PROOF: We show that it is impossible for $\nu_f(R\eta R) = T$ or F.

Assume $\nu_f(R\eta R) = T$. Let α be the least ordinal such that $\nu_\alpha(R\eta R) = T$. α must be a successor ordinal and $\nu_{\alpha\text{-}1}(\neg R\eta R) = T$. By (iii) $\nu_{\alpha\text{-}1}(R\eta R) = F$. By monotonicity, it follows that $\nu_f(R\eta R) = F$ which contradicts the original assumption.

Assume $v_f(R\eta R) = F$. Then $v_f(\neg R\eta R) = T$. By (I1) $v_{f+1}(R\eta R) = T$. But since v_f is a fixed point $v_f = v_{f+1}$. So $v_f(R\eta R) = T$, contradicting the original assumption.

So $R\eta R$ does not take on truth value T or F in any valuation v_α. It therefore retains its original valuation of N at v_f. □

DEFINITION: Let $\overline{R}$ denote the classification defined by the property $x\eta x$.

THEOREM: $v_f(\overline{R}\eta\overline{R}) = N$.
PROOF: Again we show that it is impossible for $v_f(\overline{R}\eta\overline{R}) = T$ or F. We do this by showing there can be no least ordinal α such that $v_\alpha(\overline{R}\eta\overline{R}) = T(F)$.

Let α be the least ordinal such that $v_\alpha(\overline{R}\eta\overline{R}) = T(F)$. α is a successor ordinal and $v_{\alpha-1}(\overline{R}\eta\overline{R}) = T(F)$ contradicting the original assumption that α be the least such ordinal. □

DEFINITION: Define the theory $Th_{J3} = \{A \in L(\simeq,\eta) : v_f(A) \in \nabla_{J3}\}$

THEOREM (Incompleteness): For some sentence P, neither $P \in Th_{J3}$ nor $\neg P \in Th_{J3}$.
PROOF: $v_f(R\eta R) = N = \{w\}$. By (iii) $v_f(\neg R\eta R) = \Lambda$. $\{w\}, \Lambda \not\subset \nabla_{J3}$. So $R\eta R \notin Th_{J3}$ and $\neg R\eta R \notin Th_{J3}$. □

In this section we have given a definition of a sequence of valuations v_1, v_2, v_3, ... in the topological logic J3. We have proved that one of these valuations is a fixed point v_f which defines a theory Th_{J3}. This theory is incomplete and contains our modified axiom of comprehension C.

4. *A P3 Theory*

Next we show how a similar paraconsistent model for the modified comprehension axiom can be obtained by exploting the topological nature of the previous construction. Its dual closed set construction is a model for C in the logic P3.

P3 is a logic with a set of three truth values $\{F,B,T\}$, a set of designated values $\nabla_{P3} = \{B,T\}$ and negation $\ulcorner$ defined such that $\ulcorner T = F$, $\ulcorner B = T$ and $\ulcorner F = T$.

Notice that we can transform the J3 lattice into the P3 lattice by turning it upside down. This is achieved by exchanging $\cup$ and $\cap$. A problem with this though is that the bottom value (F) is the only designated truth value, which is absurd. Paraconsistentists have settled on a more satisfactory dualisation of ∇_{J3} as $\nabla_{P3} = \{X : X \text{ is a truth value and } X \notin \nabla_{J3}$. That is $\nabla_{P3} = \{\Lambda, \{w\}\}$.

A construction isomorphic to this one can be obtained by considering the closed subsets of W instead the open ones and retaining the original ordering, unions and intersections. A new definition of negation needs to be supplied

Instead of considering the open subsets of our original set of worlds W, we now turn our consideration to the closed subsets.

Retain the original partially ordered set of worlds W:

$$w^* \leq w$$

and now consider the set of anti-hereditary subsets Y:

$$\Lambda, \{w^*\}, \{w^*, w\}$$

Y is a set of closed subsets and $\langle W, Y\rangle$ is a topological space.

This time we define valuations

$$v_1,\ v_2,\ v_3,\ \ldots, v_f : \mathsf{L}\ (\approx, \eta) \rightarrow Y$$

Let F denote Λ, B denote $\{w^*\}$ and T denote $\{w^*, w\}$. $\nabla_{P3} = \{B, T\}$

To change the underlying logic J3 to P3, we rename the truth value N as B and add B to the set of designated values ∇_{P3}.

The rules (i) - (v) for evaluating the valuations of compound sentences carry through unchanged with the exception of (iii). Closed set negation is denoted by the symbol $\ulcorner$ and is defined:

(iii) $v(\ulcorner A) = \mathsf{C}\ (\overline{v(A)})$ = The smallest closed set S such that $\overline{v(A))} \subseteq S$

This means that $\ulcorner T = F$, $\ulcorner B = T$ and $\ulcorner F = T$, as desired. (B3) is changed to:

(B3) $v_1(A) = \{w^*\}$ otherwise

DEFINITION: $Th_{P3} = \{A \in L(=,\eta) : v_f(A) \in \nabla_{P3}\}$

The monotonicity theorem carries through with minor modification in clause (c) where $\neg$ is changed to $\ulcorner$. The fixed point theorem carries through with a modification changing N to B. $C \in Th_{P3}$ follows similarly. □

DEFINITION: Let R denote the classification defined by the property $\ulcorner x\eta x$.

THEOREM: $v_f(R\eta R) = B$
PROOF: Replace N by B in the previous version of this proof. □

THEOREM (Inconsistency): For some sentence P, both $P \in Th_{P3}$ and $\ulcorner P \in Th_{P3}$
PROOF: $v_f(R\eta R) = B = \{w^*\}$ By the new (iii) $v_f(\ulcorner R\eta R) = \{w^*, w\}$. $\{B, T\} \in \nabla_{P3}$. So $R\eta R$ and $\ulcorner R\eta R \in Th_{P3}$. □

Thus the topological dual of the previous intuitionist construction is a paraconsistent one. Both are equally viable. They are essentially different perspectives of the same construction.

5. *Double Negation and Routley-* Negation*

In our J3 theory $\neg N=F$ and $\neg F=T$. As a result, $\neg\neg N=T$. Recall that sentences like $R\eta R$ and $\overline{R}\eta\overline{R}$ are assigned the truth value N in the final fixed point valuation v_f. This provides a neat solution to Russell-type paradoxes in our theory of classifications, but as a further consequence of the topological nature of the negation operator $\neg$ it is also the case that $v(\neg\neg R\eta R) = T$ and $v(\neg\neg\overline{R}\eta\overline{R})=T$. This is unsatisfactory since N is an undesignated truth value while T is designated so that $\neg\neg A \supset A$ is not in the theory.

Similarly, in the P3 theory $\ulcorner\ulcorner B=F$ so that while $v(R\eta R)=B$, $v(\ulcorner\ulcorner R\eta R)=F$. B is designated in P3 but F is not, so $A \supset \ulcorner\ulcorner A$ fails.

The logic underlying Feferman's theory is classical so that double negation holds in it. A better reconstruction is therefore one which affirms double negation. If we want double negation to behave as required for sentences which are assigned the middle truth value, the definition of the negation operator has to be arranged so that the middle truth value is a fixed point under its own operation. That is, in an intuitionist theory we want the negation of N to be N also, and in a paraconsistent theory we want the negation of B to be B.

Again we consider the set of worlds W:

$$w^* \leq w$$

We adopt the Herditary Condition of relevant semantics:

If $x \leq y$ and $x \in I(A)$ then $y \in I(A)$.

Routley *-negation (denoted by the symbol $\sim$) is defined as follows:

Let $w^{**} = w$

(i) $w \in I(A)$ iff $w^* \notin I(\sim A)$
(ii) $w^* \in I(A)$ iff $w^{**} \notin I(\sim A)$ iff $w \notin I(\sim A)$

There are four cases we need to consider: (a) $\sim\Lambda$, (b) $\sim\{w^*\}$, (c) $\sim\{w\}$, (d) $\sim\{w,w^*\}$.

(a) Assume $I(A) = \Lambda$. That is $w^* \notin I(A)$ and $w \notin I(A)$. Then by (ii) it follows that $w \in I(\sim A)$ and by (i) $w^* \in I(\sim A)$. That is, $I(\sim A) = \{w,w^*\}$.

(b) Assume $I(A) = \{w^*\}$. That is $w^* \in I(A)$ and $w \notin I(A)$. Then it follows that $w \notin I(\sim A)$ and $w^* \in I(\sim A)$. That is $I(\sim A) = \{w^*\}$.

(c) Assume $I(A) = \{w\}$. That is $w^* \notin I(A)$ and $w \in I(A)$. Then it follows that $w \in I(\sim A)$ and $w^* \notin I(\sim A)$. That is $I(\sim A) = \{w\}$.

(d) Assume $I'(A) = \{w,w^*\}$. That is $w^* \in I(A)$ and $w \in I(A)$. Then it follows that $w \notin I(\sim A)$ and $w^* \notin I(\sim A)$. That is $I(\sim A) = \Lambda$.

So there are two fixed points under negation here: $\sim\{w^*\} = \{w^*\}$ and $\sim\{w\} = \{w\}$.

THEOREM: $w \in I(A), I(\sim A)$ iff $w^* \notin I(A), I(\sim A)$
PROOF: $w \in I(A)$ iff $w^* \notin I(\sim A)$. And $w \in I'(\sim A)$ iff $w^* \notin I'(A)$. □

Notice that so far the hereditary condition has not been employed. In the set of worlds W this condition says, since $w^* \leq w$ if $w^* \in I(A)$ then $w \in I(A)$. There are two cases to consider: (a) $w^* = w$ and (b) $w^* \neq w$. Now if $w^* = w$ then $\{w^*\} = \{w\} = \{w,w^*\}$ and we have just the Boolean algebra $[\{w\},\Lambda]$. So our present concern is with the second case where $w^* \neq w$ and $w^* \leq w$ so $w^* < w$.

THEOREM: (Consistency of w^*) There is no sentence A such that $w^* \in I(A)$ and $w^* \in I(\sim A)$.
PROOF: Assume there is a sentence P such that $w^* \in I(P)$ and $w^* \in I(\sim P)$. Then by the hereditary condition $w \in I(P)$. From the definition of negation it follows that $w^* \notin I(\sim P)$ which contradicts the second assumption. □

THEOREM: (Completeness of w) For every sentence A, $w \in I(A)$ or $w \in I(\sim A)$.
PROOF: Assume there is a sentence P such that $w \notin I(P)$ and $w \notin I(\sim A)$. Then by the definition of negation $w^* \in I(\sim P)$. From the hereditary condition it follows that $w \in I(\sim P)$ which contradicts the second assumption. □

Applying Routley-* negation and the hereditary condition on our original paritally ordered set of worlds $W = w^* \leq w$ gives us a choice of two 3-element algebras suitable as underlying logics for a model of the modified comprehension axiom: $[\Lambda \subseteq \{w^*\} \subseteq \{w^*,w\}]$ and $[\Lambda \subseteq \{w\} \subseteq \{w^*,w\}]$ where the middle value in each is a fixed point under negation.

We take this pair of logics and regard them as duals of each other, where one is an intuitionist logic and the other is a paraconsistent one. In addition to the Routley definition of *-negation we also adopt the *-operation which dualises theories.

DEFINITION: Let *Th* be a theory defined by an interpretation I and set of designated values ∇. That is $Th = \{A{:}I(A) \in \nabla\}$.

DEFINITION: $Th^* = \{A{:}I(\sim A) \notin \nabla\}$.

Now if we let ∇ in *Th* be the singleton $\{T\}$ as intutionists insist, then we can deduce the nature of ∇ in the dual theory Th^* which will be paraconsistent.

THEOREM: If ∇ in *Th* is $\{T\}$ then ∇ in Th^* is $\{B, T\}$.
PROOF: $Th = \{A{:}\ I(A) \in \{T\}\}$. $Th^* = \{A{:}\ I(\sim A) \notin \{T\}\}$. So $Th^* = \{A{:}\ I(\sim A) \in \{F,N\}\}$. By the definition of $\sim$ we have $Th^* = \{A{:}\ I(A) \in \{N,T\}\}$. The convention is to label the middle truth value in a paraconsistent logic B instead of N so $Th^* = \{A{:}I(A) \in \{B,T\}\}$. Thus ∇ in Th^* is the set $\{B,T\}$. □

In the following two sections we detail the construction of a further two theories again containing the comprehension axiom C. This time they will be Routley-* duals of each other and negation will be Routley-* negation.

6. *A K3 Theory*

K3 is a logic with a set of three truth values $\{T, N, F\}$, a set of designated values $\nabla_{K3} = \{T\}$ and negation $\sim$ defined such that $\sim T = F$, $\sim N = N$ and $\sim F = T$.

We define valuations

$$v_1, v_2, v_3, \ldots, v_f: \mathsf{L}(\simeq,\eta) \rightarrow \{\Lambda, \{w^*\}, \{w^*,w\}\}$$

Let F denote Λ, N denote $\{w^*\}$, and T denote $\{w^*,w\}$. Let the set of designated truth values $\nabla_{K3} = \{T\}$.

The rules (i) - (v) are as for J3 except for (iii). We denote Routley-* negation by the symbol $\sim$ and define it as before.

The definitions of valutions $v_1, v_2, v_3, \ldots, v_f$ are as before and the fixed point theorem carries through.

DEFINITION: Define $Th_{K3} = \{A \in \mathsf{L}(\simeq,\eta): v_f(A) \in \nabla_{K3}\}$

The theory Th_{K3} contains the comprehension axiom C and is incomplete like J3 because $v_f(R\eta R) = N = v_f(\sim R\eta R)$ and $N \notin \nabla_{K3}$. However this time, because of the new definition of negation, $v_f(\sim\sim R\eta R) = N$ as desired.□

7. *An RM3 Theory*

RM3 is a logic with a set of three truth values $\{T, B, F\}$, a set of designated values $\nabla_{RM3} = \{B,T\}$ and negation $\sim$ defined such that $\sim T = F$, $\sim B = B$ and $\sim F = T$.

We define valuations

$$v_1, v_2, v_3, \ldots, v_f: \mathsf{L}(\simeq,\eta) \rightarrow \{\Lambda, \{w\}, \{w^*,w\}\}$$

Let F denote Λ, B denote $\{w\}$, and T denote $\{w^*,w\}$. Let the set of designated truth values $\nabla_{RM3} = \{B,T\}$.

We evoke the same proven strategy to arrive at a theory Th_{RM3} which models the comprehension axiom C, but unlike Th_{K3} is inconsistent because $v_f(R\eta R) = B = v_f(\sim R\eta R)$ and $B \in \nabla_{RM3}$. $v_f(\sim\sim R\eta R) = $ B. □

8. *Conclusion*

We have seen that Feferman's original construction lends itself to reconstructions in the toplogical logic duals J3 and P3 as well as in the lattice logic duals K3 and RM3. All reconstructions utilise some sort of fixed point method for arriving at the model of the comprehension axiom, as did Feferman in his original modal logic setting. This suggests that the fixed point method for constructing models of axioms has broad application. The method appears not to make any special demands on its underlying logic, although if it is used on an infinite domain it does presume some fairly strong properties of the metatheory's "set" theory.

So what about the foundations of category theory? It must be acknowledged that the problem of finding an adequate foundation for category theory is rather unsatisfactorily solved by Feferman; and so far as the present approach is just a reconstruction of the Feferman idea, the same can be said of it.

Feferman's theory of partial operations and classifications provides a foundation for category theory in that it is a theory that makes possible the formation of set-like entities, classifications, from "arbitrary" property-like entities, operations. Feferman goes some way towards giving a consistent account of naive set theory, and it is something like naive set theory which is needed in foundations for a fully general (well founded) category theory. However, as Feferman notes himself, unlike naive set theory his theory contains no extensionality axiom. No attempt has been made to model an extensionality axiom in any of our reconstructions. We do note however that modelling an extensionality axiom does appear to be possible. Brady uses a similar methodology to model both a comprehension axiom and extensionality axiom in [1].

A disappointment with the Feferman paper is that his comprehension axiom for classifications is less than wholly general. Whereas with the standard foundations for category theory a distinction must be made between small and large categories, the Feferman foundations require a similar distinction to be made between partial and total categories. For example, the new foundations allow the construction of the category of all total categories, but not the category of *all* categories, partial and total.

This is characteristic of the common approaches to foundations. With NBG foundations it is legal to form the category of all small categories, but not all categories small and large. The Grothendiek method of universes allows the formation of all categories in a universe U, but not all categories both in and outside U. It does allow the formation of the category of all categories in U and *some* outside U, but its objects will all be categories in a larger universe U' and one cannot include as objects categories outside U'.

Inside such foundational frameworks one is given glimpses of what it is like to have true freedom to categorially abstract, but in each case a seemingly arbitrary boundary is drawn over which one is forbidden to legally step. Such boundaries owe their existence to an almost pathelogical fear of contradiction on the parts of past foundationalists. And such fear is not without justification. Contradiction in the classical setting is a fearsome creature with a rapacious nature. No sooner is its presence permitted than the entire system is swamped by it, in that every sentence and its negation take on the status of theoremhood. But it is well known that this nature can be tamed, even harnessed, with a shift in logic.

The comprehension scheme used in the four theories given here is truly unrestrictive. The strategy in all four cases has been to abandon the classical logic as background logic for the theory and to assign a third truth value to "contradictory" sentences. Aside from the behaviour of negation, the difference between intuitionist theories Th_{J3}, Th_{K3} and the paraconsistent Th_{P3}, Th_{RM3} is just an attitude to this third truth value. Intutionists regard it as undesignated like *False* whereas paraconsistentists regard it as designated like *True*. In all cases the existence of contradiction is contained in small regions.

We believe that foundations allowing a truly general category theory can only be achieved by abandoning some of the notions of classical logic. The Feferman foundations originally appeared classical and quite general. We have shown that it is possible to capture many of the original ideas of the Feferman approach in theories which are intuitionistic or paraconsistent. The shift away from classical logic allowed us to use an unrestricted comprehension axiom and to simplify the presentation.

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Inconsistent Mathematics

Inconsistent mathematics is the study of the mathematical theories that result when classical mathematical axioms are asserted within the framework of a (non-classical) logic which can tolerate the presence of a contradiction without turning every sentence into a theorem.

1. Inconsistent Mathematics

Inconsistent Mathematics began historically with foundational considerations. Set-theoretic paradoxes such as Russell's led to attempts to produce a consistent set theory as a foundation for mathematics. But, as is well known, set theories such as ZF, NBG and the like were in various ways ad hoc. Hence, a number of people including da Costa (1974), Brady (1971), Priest, Routley, and Norman (1989), considered it preferable to retain the full power of the natural abstraction principle (every predicate determines a set), and tolerate a degree of inconsistency in set theory. This requires, of course, that one dispense with the logical principle *ex contradictione quodlibet* (ECQ) (from a contradiction every proposition may be deduced), as well as any principle which leads to it, such as disjunctive syllogism (DS) (from *A*-or-*B* and not-*A* deduce *B*). But considerable debate, in Burgess (1981) and Mortensen (1983), made it clear that dispensing with ECQ and DS was not so counter-intuitive, especially when a plausible story emerged about the special conditions under which they continue to hold.

In addition, mathematics has a metalanguage; that is, names for mathematical statements and other parts of syntax, self-reference, proof and truth. Gödel's contribution to the philosophy of mathematics was to show that the first three of these can be rigorously expressed in arithmetical theories, albeit in theories which are either inconsistent or incomplete. The possibility of a well-structured example of the former alternative was not taken seriously, again because of belief in ECQ. However, in addition natural languages seem to have their own truth predicate. Combined with self-reference this produces the Liar paradox, "This sentence is false", an inconsistency. Priest (1987) and Priest, Routley, and Norman (1989) argued that the Liar had to be regarded as a statement both true and false, a true contradiction. This represents another argument for studying inconsistent theories, namely the claim that some contradictions are true. Kripke (1975) proposed instead to model a truth predicate differently, in a consistent incomplete theory. We see below that incompleteness and inconsistency are closely related.

But mathematics is not its foundations. Hence there is a further independent motive, to see what mathematical structure remains when the constraint of consistency is relaxed. But it would be wrong to regard this as in any way a loss of structure. If it is different at all, then it represents an addition to known structure.

Robert K. Meyer (1976) seems to have been the first to think of an inconsistent arithmetical theory. At this point, he was more interested in the fate of a consistent theory, his relevant arithmetic R#. There proved to be a whole class of inconsistent arithmetical theories; see Meyer and Mortensen (1984), for example. Meyer argued that these theories provide the basis for a revived Hilbert Program. Hilbert's program was widely held to have been seriously damaged by Gödel's Second Incompleteness Theorem, according to which the consistency of arithmetic was unprovable within arithmetic itself. But a consequence of Meyer's construction was that within his arithmetic R# it was demonstrable by simple finitary means that whatever contradictions there might happen to be, they could not adversely affect any numerical calculations. Hence Hilbert's goal of conclusively demonstrating that mathematics is trouble-free proves largely achievable. The arithmetical models used later proved to allow inconsistent representation of the truth predicate. They also permit representation of structures beyond natural number arithmetic, such as rings and fields, including their order properties. Recently, these inconsistent arithmetical models have been completely characterised by Graham Priest; that is, Priest showed that all such models take a certain general form. See Priest (1997), (2000).

One could hardly ignore the examples of analysis and its special case, the calculus. There prove to be many places where there are distinctive inconsistent insights; see Mortensen (1995) for example. (1) Robinson's non-standard analysis was based on infinitesimals, quantities smaller than any real number, as well as their reciprocals, the infinite numbers. This has an inconsistent version, which has some advantages for calculation in being able to discard higher-order infinitesimals. Interestingly, the theory of differentiation turned out to have these advantages, while the theory of integration did not. (2) Another place is topology, where one readily observes the practice of cutting and pasting spaces being described as "identification" of one boundary with another. One can show that this can be described in an inconsistent theory in which the two boundaries are both identical and not identical, and it can be further argued that this is the most natural description of the practice. (3) Yet another application is the class of inconsistent continuous functions. Not all functions which are classically discontinuous are amenable of inconsistent treatment; but some are, for example $f(x)=0$ for all $x<0$ and $f(x)=1$ for all *x*≥0. The inconsistent extension replaces the first < by ≤, and has distinctive structural properties. These inconsistent functions may well have some application in dynamic systems in which there are discontinuous jumps, such as quantum measurement systems. Differentiating such functions leads to the delta functions, applied by Dirac to the study of quantum measurement also. (4) Next, there is the well-known case of inconsistent systems of linear equations, such as the system (i) $x+y=1$, plus (ii) $x+y=2$. Such systems can potentially arise within the context of automated control. Little work has been done classically to solve such systems, but it can be shown that there are well-behaved solutions within inconsistent vector spaces. (5) Finally, one can note a further application in topology and dynamics. Given a supposition which seems to be conceivable, namely that whatever happens or is true, happens or is true on an open set of (spacetime) points, one has that the logic of dynamically possible paths is open set logic, that is to say intuitionist logic, which supports incomplete theories par excellence. This is because the natural account of the negation of a proposition in such a space says that it holds on the largest open set contained in the Boolean complement of the set of points on which the original proposition held, which is in general smaller than the Boolean complement. However, specifying a topological space by its closed sets is every bit as reasonable as specifying it by its open sets. Yet the logic of closed sets is known to be paraconsistent, ie. supports inconsistent theories; see Goodman (1981) for example. Thus given the (alternative) supposition which also seems to be conceivable, namely that whatever is true is true on a closed set of points, one has that inconsistent theories may well hold. This is because the natural account of the negation of a proposition, namely that it holds on the smallest closed set containing the Boolean negation of the proposition, means that on the overlapping boundary both the proposition and its negation hold. Thus dynamical theories determine their own logic of possible propositions, and corresponding theories which may be inconsistent, and are certainly as natural as their incomplete counterparts.

Category theory throws light on many mathematical structures. It has certainly been proposed as an alternative foundation for mathematics. Such generality inevitably runs into problems similar to those of comprehension in set theory, see eg. Hatcher (1982, p.255-260). Hence there is the same possible application of inconsistent solutions. There is also an important collection of categorial structures, the toposes, which support open set logic in exact parallel to the way sets support Boolean logic. This has been taken by many to be a vindication of the foundational point of view of mathematical intuitionism. However, it can be proved that that toposes support closed set logic as readily as they support open set logic. That should not be viewed as an objection to intuitionism, however, so much as an argument that inconsistent theories are equally reasonable as items of mathematical study.

Duality between incompleteness/intuitionism and inconsistency/paraconsistency has at least two aspects. First there is the above topological (open/closed) duality. Second there is Routley * duality. Discovered by the Routleys (1972) as a semantical tool for relevant logics, the * operation dualises between inconsistent and incomplete theories of the large natural class of de Morgan logics. Both kinds of duality interact as well, where the * gives distinctive duality and invariance theorems for open set and closed set arithmetical theories. On the basis of these results, it is fair to argue that both kinds of mathematics, intuitionist and paraconsistent, are equally reasonable.

A very recent development is the application to explaining the phenomenon of inconsistent pictures. The best known of these are perhaps M.C.Escher's masterpieces*Belvedere, Waterfall* and *Ascending and Descending*. In fact the tradition goes back millennia to Pompeii. Escher seems to have derived many of his intuitions from the Swedish artist Oscar Reutersvaard, who began in 1934. Escher also actively collaborated with the English mathematician Roger Penrose. There have been several attempts to describe the mathematical structure of inconsistent pictures using classical consistent mathematics, by theorists such as Cowan, Francis and Penrose. As argued in Mortensen (1997), however, no consistent mathematical theory can capture the sense that one is seeing an impossible thing. Only an inconsistent theory can capture the content of that perception. This amounts to an appeal to cognition, that is the epistemological justification of paraconsistency as above. One can then proceed to describe inconsistent theories which are candidates for such inconsistent contents. There is an analogy with classical mathematics on this point. Projective geometry is a mathematical theory which is interesting because we are creatures with an eye, since it explains why it is that things look the way they do in perspective. This study has been further developed in Mortensen (2002a), where category theory is applied to give a general description of the relationships between the various theories and their consistent cut-downs and incomplete duals. For an informal account which highlights the difference between visual "paradoxes" and the philosophically more common paradoxes of language, such as the Liar, see Mortensen (2002b).

It should be emphasised that these constructions do not in any way challenge or repudiate existing mathematics, but extend our conception of what is mathematically possible.

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THE LEIBNIZ CONTINUITY CONDITION, INCONSISTENCY AND QUANTUM DYNAMICS

ABSTRACT. A principle of continuity due to Leibniz has recently been revived by Graham Priest in arguing for an inconsistent account of motion. This paper argues that the Leibniz Continuity Condition has a reasonable interpretation in a different, though still inconsistent, class of dynamical systems. The account is then applied to the quantum mechanical description of the hydrogen atom.

1. INTRODUCTION [1]

By any standards, Leibniz was a great thinker. His contributions to the calculus and the theory of space and time are of fundamental importance, while his general ontology was inventive and well-defended. One of this less-studied contributions is what Graham Priest has called the Leibniz Continuity Condition, hereafter called the LCC. As we will see, this is essentially the thesis that all causal processes are continuous, though there are interesting issues of interpretation and application. This paper begins by considering these issues of interpretation. It is seen first that the LCC has a prima-facie legitimate area of application in a well-defined collection of dynamical systems. These systems are consistent, and so differ from Priest's own suggested applications, which utilise inconsistent methods. The problem of extending this range of consistent solutions to a larger class of known dynamical systems is then considered. It turns out that inconsistent methods are also required, though in a different way from Priest. It also turns out that there is a natural topological interpretation of the inconsistency-tolerant logic required for these solutions, dual to that of intuitionism. Finally an application, the energy levels of the Hydrogen atom, is sketched.

2. THE LEIBNIZ CONTINUITY CONDITION

Leibniz wrote:

When the difference between two instances in a given series or that which is presupposed can be diminished until it becomes smaller than any given quantity whatever, the corresponding difference in what is sought or in their results must of necessity also be

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diminished or become less than any given quantity whatever. Or to put it more commonly, when two instances or data approach each other continuously, so that one at last passes over into the other, it is necessary for their consequences or results (or the unknown) to do so also (Leibniz 1687: 352).

Priest glosses this as follows: given any two mathematical sequences (s_n) and (t_n), if $\lim_{n\to\infty}(s_n - t_n) = 0$ then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n$. As Priest points out, given Leibniz' use of the principle, it seems reasonable to state the principle as: given any limiting process, whatever holds up to the limit holds at the limit; or again, if a variable quantity at all stages enjoys a certain property, its limit will enjoy the same property (Priest, 1987: p. 208).

Priest notes that the LCC as thus stated has significant problems, and that Leibniz was presumably aware that its application must be restricted. In particular, Priest notes that if "whatever holds up to the limit" includes disjunctions, implications or tensed statements, then difficulties arise. For example, considering disjunctions, a disjunctive statement $A \vee B$ might hold continuously in an interval without either disjunct holding continuously in the interval (e.g. A holds at the rational points and B holds at the irrational points), so that "there is no reason to suppose that the disjunction holds at the limit" (p. 211), since presumably what makes a disjunction hold is one of its disjuncts holding. Again, on tensed statements, consider a statement A which holds in an interval of time up to and including a time t but not thereafter. Then "A will be the case" holds in the open interval up to t; and thus by the LCC it holds at the limit point t, which contradicts the assumption that t is the last time that A holds.

Priest does propose an application for the LCC however, in a general account of change, and motion in particular. For consider any (nondisjunctive, nontensed) statement A which stops being true at t. Thus A holds arbitrarily close to t; so that by the LCC, A holds at t. But also not-A holds arbitrarily close to t; so by the LCC, not-A also holds at t. That is, at the instant of change a contradiction is true. Indeed, contradiction is the mark of change; it is what distinguishes change from a state in which A is unchanging during an interval. It further follows that the special case of continuous motion, being continuous change, is a "continuous state of contradiction", which is Priest's account of motion.

There are some *prima facie* objections which might be urged against the resulting account of motion. In the first place, it is not clear why one should abandon the traditional account of motion as change of place over time. Priest rejects this account of motion as relational since it depends on the relation between the behaviour of a body at a point and its behaviour at neighbouring points; whereas, following Hegel, what he wants is an account according to which motion at a time is an intrinsic, or non-relational, property of a body at that time. The inconsistent account satisfies this desideratum. Still, one might argue that the relational account is conceptually simpler since the relational facts are present even in Priest's story, which thus imports an extra element. It might also be argued that change of place over time is necessary and even sufficient for motion, so that the extra intrinsic element is unnecessary for the situation. In the second place, since motion considered in the traditional sense as the time derivative of position is not Lorentz invariant, the account implies either absolute simultaneity, or that contradiction is not Lorentz invariant either, and thus that contradiction is not a basic ontological category.

Whatever the strength of these objections to the inconsistent account of motion, they may have to be set aside if there is a strong argument in favour of the inconsistent account, such as the argument given earlier deriving from the LCC. But one person's *modus ponens* is another person's *modus tollens*. After all, we might apply Priest's earlier caution about disjunction, implication and tense to the case of negated statements also. For consider any continuous function of time $f(t)$, which is strictly increasing in an interval up to $t = a$. Then at any time t just before a, the statement "$f(t)$ is not identical with $f(a)$" holds. It follows from the LCC that the statement "$f(a)$ is not identical with $f(a)$" holds. This unreasonable result ought to incline us to caution over the application of the LCC to negated statements; but it is that application on which Priest's argument rests.

If the LCC ought to be forbidden for statements involving the logical particles "not", "or", "if ... then", then if it is to have any application at all, it should be restricted to logically atomic statements, by which I mean statements which involve none of the above logical operations nor the quantifiers. But the earlier example of tense raises difficulties for even so restricted a suggestion. So does order. For consider the strictly increasing function $f(t)$ of the previous paragraph. Then for all t just before a, the statement "$f(t)$ is less than $f(a)$" holds. The LCC then implies that the statement "$f(a)$ is less than $f(a)$" holds, again a seemingly unreasonable conclusion, especially since the result obviously generalises to all the other $f(t)$ as well. Faced with these difficulties, is there no hope whatsoever for the LCC, or was Leibniz just flatly wrong about it?

3. Continuity and Physical Significance

Let $f(t)$ be any function with the property that its limit is well defined at $t = a$. If $\lim_{t \to a} f(t) = k$ where k is a constant, that is $\lim_{t \to a} f(t) - k = 0$, then applying the LCC, we have $f(a) - k = 0$ likewise. That is, $f(a) = k$; which is to say that $\lim_{t \to a} f(t) = f(a)$. But this is what it means to say that $f(t)$ is continuous at a. That is, it is a consequence of the LCC that every function with well-defined limits is continuous. Thus the LCC is well-named as a condition of continuity. Now this consequence might seem to be the last straw for the LCC, since we are familiar with functions with well-defined limits which are not continuous. But the above quotation from Leibniz might just possibly be read charitably as applying to physically significant functions only, that is, to functions describing causal processes in the physical world. Priest's own reading can likewise be interpreted in this way. He speaks of its application to "events", "goings on", and "changes in physical states of affairs", for example. This is fairly clearly aiming at physically significant processes, and certainly far from supposing that the realm of application extends as wide as arbitrary mathematical functions. Under this application, the LCC is seen as a contingent principle which does not hold of all functions, but which applies to physically significant or causal processes.

As a bonus, it is evident that the restriction of the LCC to the behaviour of functions serves to avoid the above problems about the logical particles and order. We would be dealing with equations only, since it is precisely equations in which the laws and boundary conditions of dynamical systems are framed: differential and integral equations are in the first place equations, and their solutions are functions which satisfy those equations.

It is now apparent that far from being an oddity, the LCC applies to a natural class of functions. These are the C^∞ functions, those which are continuous and which have continuous n^{th} derivatives, for arbitrary n. Every function in this class is continuous, and it is closed under sums, differences, products, quotients (with the usual restriction to non-zero divisors) and derivatives. Arguably, these operations preserve the physical significance of functions: the derivative of a function describing changes in a physically significant process ought also to be regarded as describing a change in a physically significant process, for example.

Thus Leibniz is to be read as proposing a natural condition on the behaviour of physical processes, namely that for them to be causally well-behaved they should be continuous. And certainly continuity is closely woven into our conception of causal well-behavedness. Priest supports such a conception when he argues that discontinuous functions in "nature" would be "capricious" (p. 209), that "nexuses" express our idea that states of affairs at a time are "dependent" on those at other times, and that "theories of action at a distance, which require something to happen in no time, namely the transmission of an effect, have always been felt philosophically puzzling" (p. 210). As a final point, we can remind ourselves that the assumption that the functions under consideration are the C^∞ functions is typically made for such dynamically important systems as the manifolds of General Relativity. In other words, Leibniz is merely proposing a reasonable constraint on functions describing physically and causally reasonable processes. God does not play dice with nature. Thus when Priest asks how one might go about establishing the LCC (p. 209), the answer is that if our best theories postulate a C^∞ universe, then the LCC is automatically satisfied in such a universe.

4. Inconsistency

But dynamics has not always been restricted to universes of C^∞ functions, quantum mechanics being a notable example. So there is a reason, even if only a pure mathematical reason, to see whether continuity, causality and the LCC can be realised in a larger class of functions. The answer is that it can be done utilising methods from inconsistent mathematics, as we will see. It is emphasised, however, that what is being offered is not an inconsistent account of general motion as in Priest, but an account of apparently discontinuous change which nonetheless satisfies the LCC in an inconsistent but well-controlled way.

One place to begin is the instant of change problem (see e.g. Mortensen (1985), Priest (1987), Jackson and Pargetter (1988), Smith (1990)). Consider the function $f(t)$ of position given by: if $t < 0$ then $f(t) = 0$ else $f(t) = t$. At $t = 0$ is it in motion or at rest? At $t = 0$ it has not changed its position, though for all $t > 0$ it has. We might look to velocity $v(t)$, the time derivative of position, for a clue; since it is a plausible principle that a thing is in motion at t iff $v(t)$ has a value other than 0. Unfortunately, the traditional story is that $v(0)$ does not exist; since v has a left hand limit of 0 as t approaches 0 from the left, and v has a right hand limit of 1 as t approaches 0 from the right, and it is a necessary condition for the derivative of f at 0 to exist that the left hand limit equal the right hand limit. That is, v is discontinuous since it does not exist everywhere. One can reasonably feel that this is a disappointment, since velocity would have no value here, not even zero. The classical story asserts something very strange, for instantaneously velocity does not even exist. At best, the classical story seems to be a refusal to say what really is going on at this point.

As it turns out, however, there is no particular reason why inconsistently the velocity at $t = 0$ could not be set equal to both the left hand limit and the right hand limit. Since $v(0) = 0$ and $v(0) = 1$, then by the transitivity of identity, $0 = 1$. More strictly, the consequence is that *at* $t = 0$, *0 cm per sec = 1 cm per sec* (or whatever other units are being used to describe velocity). This consequence has two aspects. The first aspect is that the function $v(t)$ is to be thought of as a dynamical function, associating times with *quantities* such as 0 cm per sec, rather than with the (undimensional) real number 0. That is, $f(t)$ and $v(t)$ are functions describing paths in multi-dimensional phase space rather than in the abstract mathematical space of real numbers. This means that a consequence like 0 cm per sec = 1 cm per sec does not inevitably spill over into inconsistency in other dimensions of the phase space, let alone the general consequence that $0 = 1$. This contradiction containment is, needless to say, characteristic of paraconsistent methods. However, provided that contradictions can be confined to a dimension of a phase space in the suggested fashion, then there is no further formal problem in extending the treatment to contradictory identifications of dimensionless numbers, since one can simply ignore the implicit dimension. The second aspect of the consequence is that the velocity dimension of the phase space is thought of as containing a contradiction *at* $t = 0$, but not necessarily at other times. The idea that mathematical laws might vary over time is not entirely new, being familiar from topos theory and intuitionism (see Bell (1986)).

The mathematical details of this story are already known (see the author's (1995)). Consider any function v with the property that $v(t) = a$ and $v(t) = b$ where a and b are traditionally distinct real numbers. Now for this to amount to an inconsistency, it must also be that $\sim (a = b)$ at $t = 0$. So, one would want it that enough of the arithmetic of the real numbers holds in the space of values of v at $t = 0$, to ensure that $\sim (a = b)$ holds at $t = 0$ as well as at all other times. This can be understood as requiring that the *singularity* or *oddity* of what is happening at $t = 0$ should be represented by $\sim (a = b)$ being the *norm*, with $a = b$ being an *extra abnormality* at $t = 0$.

One way, perhaps the easiest way, to see that the inconsistent identification of a pair of real quantities does not lead to loss of control over the functionality of the dimension of phase space at $t = 0$, is to employ a geometric model or analogy. At times other than $t = 0$, the space of values of the function v has the topology of the real line. At $t = 0$, on the other hand, we can suppose that velocities behave as if there is an "instantaneous cylindrification", so that velocities are distributed on a circle whose circumference is the absolute magnitude of $(a - b)$. That is, velocities are identified with one another just in case their difference is an integral multiple of $(a - b)$. In particular, $a = b$, $a - 1 = b - 1$, $a + a + a = b + b + b$ and so on, all hold.

The term "cylindrification" is used to suggest that the *other* dimensions of phase space are *not* rolled up at $t = 0$. The point of this metaphor is to show that there is a functionally well-structured description of the identification. For instance, it is not difficult to show that at $t = 0$, if $a = b$ and $c = d$ both hold, then so do $a+c = b+d$ and $a-c = b-d$. Thus, the consequences of the identification $a = b$ carry over to consequences for the addition and subtraction functions. Now there is an interesting point here about multiplication and division. Some restrictions on the functionality of the "rolled up" subspace of phase space are unavoidable, since it is known that if a single pair of (traditionally) distinct real numbers are identified $(a = b)$, then if in addition (1) all traditional truths of real number theory continue to hold, and (2) the four arithmetical operations $\{+, -, \times, \div\}$ are fully functional, then every real number r is provably identical with every other real number r', $(r = r')$. (For details, see e.g. (1995: p. 61)). That is surely too much loss of useful information in the real field. The easiest way to avoid it, if one insists that one's theory remain functional (a desirable but not essential requirement, see (1995: Ch. 1)), is simply to say that the subspace lacks a multiplicative structure at $t = 0$. But this does not forbid all multiplication operations, however, since multiplication of each real number by any integer n is still functional, because it can be defined as a sum of the real number with itself n times.

Logic has not been mentioned so far, and perhaps the reader is harbouring the thought that according to two-valued logic if a contradiction holds then every statement holds, which would make any inconsistent theory useless for analysis. But the methods of paraconsistent logic are by now sufficiently well investigated for it to be clear that the principle *ex contradictione quodlibet*, that from a contradiction every statement follows, is both arbitrary and unnecessary. (See e.g. Priest (1987), and many other authors.) It goes without saying that *ex contradictione quodlibet* is rejected in the present paper. However, our concerns are here more mathematical than logical. Nonetheless, it is also possible to show that all statements of classical real number theory in $(=, +)$ plus the logical connectives, continue to hold on the surface of the cylinder (together with some further inconsistent consequences of the identification $(a = b)$, of course; for details see (1995: Sect 6.2 & 3)). This was the desideratum noted above, that things should remain as normal as possible at $t = 0$, so that abnormalities are readily identifiable. My point here, however, is that the geometrical model makes it intuitively appreciable that at $t = 0$ there remains a fair degree of functionality, and that there is as much control of computation remaining as can be performed on the surface of a cylinder.[2]

Now we come to one of the payoffs of this analysis. Consider the velocity function v earlier, except that to simplify matters let $v(t) = a$ for all $t < 0$ and let $v(t) = b$ for all $t > 0$, and finally let $v(0) = a = b$, where the space of values of v at $t = 0$ inconsistently identifies a with b. Then, since $\lim_{t\to 0-} v(t) = a$ and $\lim_{t\to 0+} v(t) = b$, then $\lim_{t\to 0-} v(t) = \lim_{t\to 0+} v(t)$. But the condition for continuity of v at 0 is that the left hand limit of v at 0 = the right hand limit of v at 0 = the value of the function at 0. Thus we can conclude that v is continuous at 0 (inconsistently continuous, that is). Therefore, v satisfies the LCC. If it was thought, then, that $v(t)$ represented a physically significant process or causal nexus, we would be in a position to satisfy that intuition as it is embedded in the LCC. It should be mentioned that not every function hitherto regarded as discontinuous comes out as inconsistently continuous in this story. The crudest example is the function which is zero at all rational points and one at all irrational points, but the functions $\sin(1/t)$ and $t = 0$, and any function which diverges to infinity at a finite point, will do as well. The present account only extends the concept of continuity to functions classically regarded as piecewise continuous while discontinuous at a discrete set of points.

5. THE HYDROGEN ATOM

We now turn to an application of this analysis. When light is shone through a slit and then a prism, it breaks up into a distinctive collection of lines called a spectrum. When the light comes from a pure element such as hydrogen, the wavelengths of the spectral lines bear a simple relationship to one another, discovered by Balmer in 1884 (see Diagram 1). Balmer's formula was: wavelength (in angstroms) $= 3645.6n^2 \div (n^2 - 4)$, for $n = 3, 4, 5 \ldots$. Later, generalisations of this formula for other elements, and at wavelengths other than the visible, were found by Rydberg, Paschen, Brackett and others.

Niels Bohr was able to explain this relationship by his famous theory of the atom, which postulates that orbital electrons take only discretely distributed energy states or levels. An electron jumps up one or more

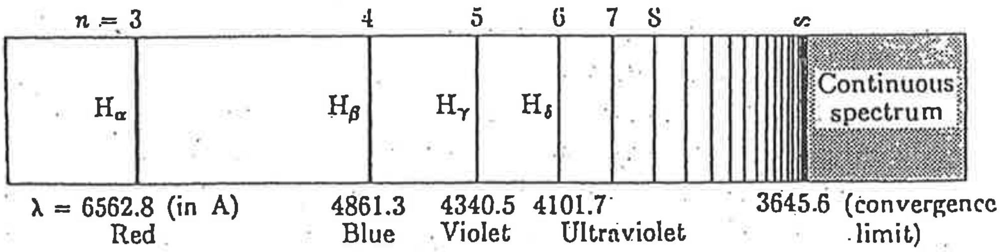

Diagram 1. The Balmer spectrum of atomic hydrogen.

energy levels when it absorbs a packet of electromagnetic energy of the amount of the difference in the energy levels, in the form of a photon. When an electron drops energy levels it emits a photon with the appropriate energy, and thus a definite wavelength, which is observed as the above spectrum. Bohr was able to show that the exact size of the constants involved was a consequence of an analysis which, aside from the postulation of quantisation, drew on otherwise well-understood principles of angular momentum together with other already-known constants of nature. (See e.g. Wehr and Richards (1960), especially p. 90.) Now this change in the electron's energy level in consequence of the emission or absorption of a photon of a define energy is plainly a causal process. For example, we can make it happen by irradiation with photons of appropriate wavelength and frequency. But according to the theory the process also happens instantaneously. There is no gradual change from one energy level to the other when the photon is emitted, if only because there is no gradual build up of the energy of the emitted photon, nor are photons of intermediate energy ever observed or postulated in the emission spectrum. Such changes are, of course, hardly untypical in quantum processes; indeed the whole of quantum measurement theory is constructed around them.

Thus we have the situation where at a time, say $t = 0$, a change occurs which is causally driven according to well-confirmed laws, which is therefore a prime candidate for the application of the Leibniz Continuity Condition as we have understood it. But a traditionally-conceived C^{∞} description is not available. What more reasonable place, then, for applying the inconsistent analysis above, to conclude that at $t = 0$ the energy level of the hydrogen atom takes both values, and that the energy function is inconsistently continuous?

Indeed, what other options are there? Here we can draw on the discussion of the instant of change problem mentioned earlier. Should one try to say, for example, that at $t = 0$ the energy always only takes the lower of the two values? This looks to be arbitrary, especially in light of the temporal symmetry of the emission and absorption processes. Certainly there is nothing in the equations of quantum mechanics to suggest such a conclusion. Similarly for the suggestion that the energy always takes the higher of the two values. Perhaps, then, one should say that at $t = 0$ the energy takes no values? But on reflection this would be a very radical suggestion indeed. Not only would the energy function be discontinuous at $t = 0$, but it would not even be defined there. For just an instant, the electron would have *no* energy, *not even zero energy*. It is not even clear what sense one might make of the idea of an electron, on this view. Of course, having *zero* energy would be something on which some mathematical purchase could be made, but the theory forbids the electron from having zero energy. In any case, energy $e(0) = 0$ would be an unreasonable and unwarranted interpretation of the emission/absorption process: up to $t = 0$, $e(t) = a$; then it would drop instantaneously to zero, $e(0) = 0$; then after $t = 0$, it would jump back to $e(t) = b$, a double jump!

It is clear, I trust, that the present analysis can be applied to any quantum measurement process involving a discrete spectrum. Elementary quantum theory precisely does not say what happens when there is an instantaneous jump from one discrete eigenstate to another, other than that such things occur. It is, then, open to us to preserve such a degree of causality for such processes as the LCC explicates, by utilising inconsistent mathematical methods in the way described.

This is not the only sort of measurement process postulated in quantum mechanics. There are measurement operators with a continuous spectrum, to which the present analysis thus does not apply. One common technique with these is to utilise Dirac's Delta "function" $\delta(t)$, which has the two properties (i) that $\delta(t) = 0$ for all non-zero t, and (ii) $\int_{-\infty}^{\infty} \delta(t)dt = 1$. The conjunction of these two properties is not the property of any ordinary function, needless to say. The Delta function was long regarded as problematic by mathematicians and physicists, until being "solved" by Schwartz' theory of distributions. This was, however, at the cost of a considerable increase in complexity, particularly in the space of functions used in quantum theory: strictly, $\delta(t)$ was treated by means of functionals rather than functions. Without going into details in this paper, it is asserted that inconsistent methods are available in which, using Robinson's nonstandard infinitesimals and inconsistently "smearing out zero", the Delta function can be treated as a function again, an inconsistent function of course. Note in passing that the Delta function enables a natural account of the derivative of the inconsistently continuous function $v(t)$ of Section 3 (where $v(0) = a = b$), given by: $v'(0) = (a - b)\delta(0)$. For details, see (1995: Ch. 7).

6. Conclusion

If this were all there were to the interpretational problems of elementary quantum theory, then perhaps we could rest easy with the thought that all we have to tolerate is a degree of inconsistency and all will be well. But one of the major intellectual problems of our time, the Bell inequality, remains untouched. Having said that, it can be said that the problem above involves a process which is causal of sorts, and yet nonlocal, at least in the sense that states classically apart from one another are made to be together or the same. Perhaps, then, the present methods can be extended to deal with nonlocality in general, using the principle that nonlocality is inconsistent locality. But that would be the topic of another paper.

Appendix: Topology and Logic

This Appendix aims to show that there are reasonable topological insights behind the paraconsistent constructions in this paper, and that those insights are as natural as the topological duality between open and closed sets.

Calculus is usually conducted on open sets. One says, for example, things like: if f and g are functions continuous on an open interval I, then so are $f + g$, $f - g$ etc. continuous on I. This has led to a rather intuitionist conception of calculus, since intuitionist logic is, as is well known, the logic of open sets. But now as well we have a role for closed set logic which is paraconsistent, as a topological dual to intuitionism. Any function which is inconsistently continuous at a discrete set of singularities can be said to define a collection of closed sets as the values of statements. Boundaries are the places where inconsistencies hold. As the simplest example, consider again our function $v(t)$, with $v(t) = a$ for all $t < 0, v(0) = a = b$, and $v(t) = b$ for all $t > 0$. This determines a collection of three closed sets $\{T, \{0\}, \phi\}$, where T is the set of all instants (the real line), $\{0\}$ is a boundary, and ϕ is the null set. (We can assume here that T is isomorphic to the real line, though that assumption might be relaxed for other purposes.) These closed sets function as the values of statements, that is, as the regions where those statements hold. All first order statements of classical real number theory in the $(=, +)$

language hold at all instants, so the value of such statements is T. The statements $a = b$, $a + 1 = b + 1$ etc. hold on $\{0\}$; while many other statements, such as any identity where the difference between the terms is not an integral multiple of $a - b$, hold nowhere, that is at ϕ.

Turning to negation, T and ϕ are naturally enough negations of one another. The negation of $\{0\}$ is determined on the assumption, dual to intuitionism, that if a statement is true on a closed set, so is its negation. Hence, considering the boundary $\{0\}$, its set-theoretic complement $T - \{0\}$ will not do for its negation, since the latter is an open set and not closed. The natural thing to say is that the complement of a boundary is the smallest closed set containing the set-theoretic complement. (Compare intuitionism, where dually the intuitionist complement of an open set is the largest open set contained in the set-theoretic complement.) This means that the complement of $\{0\}$ is T, the whole collection of instants. And this is as it should be, since $a = b$ holds at $\{0\}$ while its negation, $\sim (a = b)$ holds in classical real number theory and thus was set to hold everywhere, that is at T. In passing, this demonstrates informally a claim made above, that all of $(=, +, -)$ classical real number theory continues to hold (everywhere) in the present theory. There remains a role for open sets in this conception. Since multiplication in general (other than integer multiplication as summation) is restricted at $t = 0$, the region on which the multiplicative truths of real number theory hold must be restricted to the open set $T - \{0\}$. Now the intuitionist complement of an open set is the largest open set contained in the set-theoretic complement, that is in this case ϕ. So all negations of the multiplicative truths of real number theory hold at ϕ i.e. nowhere, which is as it should be, while their double negations hold everywhere.

The whole structure is a four-valued paraconsistent and paracomplete logic isomorphic to the four values $\{T, \{0\}, T - \{0\}, \phi\}$. Statements involving the logical operators conjunction, disjunction, and the quantifiers are then evaluated in accordance with simple rules. The conjunction of two values is their intersection, disjunction is union, universal quantification is generalised conjunction and existential quantification is generalised disjunction. The three-valued paraconsistent closed-set logic $\{T, \{0\}, \phi\}$, as well as the three-valued paracomplete intuitionist open-set logic $\{T, T - \{0\}, \phi\}$ are sublogics, since negation remains an operation on each sublogic. Implication is of interest: classical material implication, defined as $\sim A \vee B$, is along for the ride since it is definable in terms of $\{\sim, \vee\}$ and hence eliminable in favour of them. Intuitionist implication is not so definable, and it turns out that its topological dual is not a kind of implication. But there is a reasonable implication of an S5-ish kind on any of these structures, intuitionist or paraconsistent: $A \rightarrow B = T$ if A is a subset of B, else $A \rightarrow B = \phi$.

NOTES

[1] Thanks for useful comments are due to Martin Bunder, Mic Detlefsen, Graham Priest, Greg Restall, Jack Smart and others present when an earlier version was read at a meeting of the Australasian Association for Logic, Armidale 1995.

[2] It should be noted that there are alternative inconsistent constructions which preserve the LCC for this system. Thanks are due at this point to Greg Restall, who pointed out a construction using "heap models". While heap models have certain advantages for some applications, the present author favours the view that addition, at least, is more functionally interesting and realistic if it is a group operation, which it is not in heap models. At any rate, the existence of alternative constructions here serves to reinforce the basic point that the LCC can be satisfied inconsistently in such dynamical systems.

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Peeking at the Impossible

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Abstract The question of the interpretation of impossible pictures is taken up. Penrose's account is reviewed. It is argued that whereas this account makes substantial inroads into the problem, there needs to be a further ingredient. An inconsistent account using heap models is proposed.

1 Introduction Given that the mathematics of the inconsistent has developed to the point of self-subsistence, it becomes essential to look to applications. Anomalies in physics and pure mathematics are an intriguing prospect. But one very obvious example remains unaddressed: the impossible pictures such as are found in Escher's works, for example, the inconsistent triangle, ascending and descending, and the like. The embarrassment for the paraconsistency program is that it took a thoroughly classical mathematician, Roger Penrose, to make the first significant inroads on the problem.

Penrose applies the theory of cohomology groups to the problem. He shows necessary and sufficient conditions for a two-dimensional picture to represent a consistent three-dimensional object. This paper begins by setting out Penrose's account in Section 2. In Section 3 it is seen that there remains one ingredient to be added to Penrose's solution. A theory is described which extends Penrose's account by means of the theory of heaps. The inconsistent theory of heaps has been studied by Priest, Restall, van Bendegem, and others. It proves necessary to modify that theory, though an inconsistent version remains the most intuitive. The result provides a sense in which looking at inconsistent drawings is peeking at the impossible. However, the existence of a stable theory also tends to show that the inconsistent may not be so impossible after all.

2 Penrose's account Consider the inconsistent triangle.[1] Penrose notes correctly that it can be a picture of a three-dimensional structure, in fact many different structures. This could happen if the structure were in fact three disassembled parts lined up behind one another so that the distances between them could not be seen.[2] Indeed, there is an infinite collection of such 3-D structures which "project down" onto a front screen, so that the projection forms the 2-D picture. These are readily obtained

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by thinking of moving the three disassembled parts closer or farther from the picture while allowing that their sizes can increase or decrease depending on whether they are farther or closer from the 2-D image on the front screen, so as to make them have that image as their projection. This requires that objects expand uniformly the farther away they are, in such a way as to keep the same size and shape of the image.

The difference between this case and the case where the 2-D picture is an image of a consistent 3-D object is thus: in the former case the parts cannot be assembled into a consistent 3-D object (no bending permitted), whereas in the latter case the parts

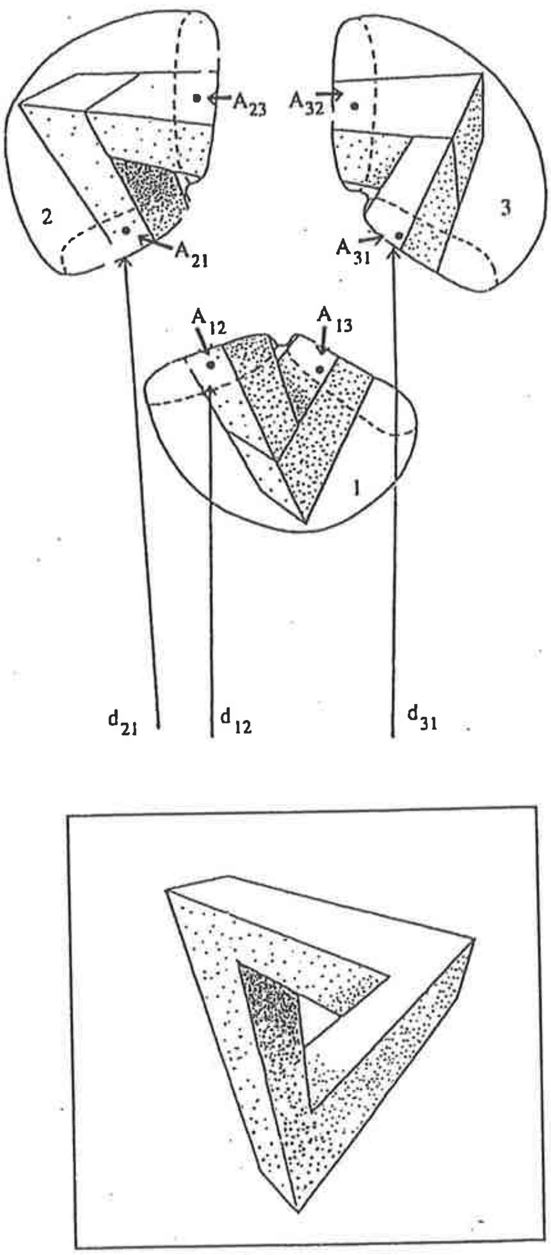

Figure 1: The inconsistent triangle and a 3-D disassembly (adapted from Penrose).

can be so assembled. Note that disassembly and reassembly are required to be done in such a way that the same 2-D projection is preserved in all motions of the 3-D parts. This, in a nutshell, is Penrose's suggestion: it is necessary and sufficient for a 2-D picture to be a picture of a consistent 3-D object, that the collection of 3-D configurations which project onto it has amongst it a consistent connected object which is a reassembly of any of the 3-D structures. What is interesting, of course, is the possible failure of the necessary and sufficient condition. Then the 2-D image fails to be a projection of any consistent connected 3-D object, though it remains a projection of an infinite number of disassembled 3-D structures.

Penrose describes this in the language of cohomology groups. We begin with the multiplicative group of positive real numbers $\mathbb{R}^+$. Penrose calls this the *ambiguity group* of the structure. It represents the fact that a 2-D picture ambiguously represents a class of 3-D structures because the (single) eye cannot distinguish points which are in a direct line with one another from the eye. For any such pair of points, the distance to one from the eye can be represented as a multiple of the distance to the other, where the multiplier is from $\mathbb{R}^+$.

If the 3-D structure is in three parts, joining it up will require identifying (at least) two points on each part with a point on each of the two other parts. (We ignore the point that strictly surfaces, not points, will need to be identified, by taking the single points as representative of the surfaces.) Let the parts be numbered 1, 2, 3 and let the point on part i which is to be identified with a point on part j be called A_{ij}. Thus, A_{ij} joins with A_{ji}. Let the distance from the front screen to A_{ij} be called d_{ij} (see diagram).[3] The requirement of assembly can thus be expressed by the condition $d_{ij} = d_{ji}$, all i, j, (where the d_{ij} belong to $\mathbb{R}^+$). There are an infinite number of structures which satisfy this condition for any given 2-D picture. It is convenient to reduce this condition to one involving equivalence classes. Introduce the ratios r_{ij} defined by d_{ij}/d_{ji} . (Note that $r_{ij} = 1/r_{ji}$). Then each triple (r_{12}, r_{23}, r_{13}) determines an infinite equivalence class of structures having the same ratio of distances between corresponding points on the three bodies. These triples are called *cocycles*. The condition of assembly, that $d_{ij} = d_{ji}$, is obviously equivalent to the condition that the cocyle = $(1, 1, 1)$. If this happens, the cocycle is called a *coboundary*. A coboundary can obviously function as the unit of a group, where group multiplication is defined as pointwise multiplication of the three components of the cocycles, that is, $(a, b, c) * (d, e, f) = (ad, be, cf)$. The group operation evidently has physical significance in that any pair of equivalence classes of configurations under the operation produces a unique equivalence class of configurations. When the set of cocycles contains a coboundary, the cocycles form a *cohomology* group. Thus, the cocycles fail to be a group just when they lack a coboundary, that is, just when they cannot be assembled into a consistent connected 3-D structure.

3 Inconsistent heaps One question remains with this approach. Why is it that it seems to the eye that the picture represents an impossible object, and not any one of the possible disassembled objects which look the same? The nearest Penrose comes to an answer seems to be the observation that the eye cannot distinguish between points directly behind one another. Still, this applies as much to the consistent case as the inconsistent case. What is it for the eye to *preferentially interpret* what it sees as inconsistent, and not any one of the possible structures? These questions are answered in the next section, where it is proposed that the inconsistent interpretation is represented by an inconsistent theory. But first we must do some preliminary work.

In point of fact, there *are* assembled 3-D structures which project onto the inconsistent triangle. First, it is plain that one can *almost* assemble three parts. One can identify *two* of the pairs of points, say A_{12} with A_{21} and A_{23} with A_{32}, leaving the third pair separate. If the third pair of points were only "virtually separate," but identical because *they* were painted on a *back* screen, that would be enough to produce the image on the front screen. This suggests a general strategy for producing 3-D objects which project to the desired image on the front screen. Suppose that space is finite, with a backdrop or back boundary. Assemble as much as one can, put the pieces against the backdrop, and *draw in* the remainder of the lines on the backdrop. All the world's a stage (or a Hollywood set).

One can imagine an objection that the backdrop universe is not an acceptable 3-D structure for projection onto the front screen because it fails to force separate points in 3-D space to remain separate. Algebraically, this is the stipulation that the ambiguity group corresponds to, or maps one-one to, the set of possible distances from the front screen. In the language of model theory, it is the condition that the theory describing the geometrical solution preserves all statements about the ambiguity group which deny identity. This is a reasonable requirement, and it leads us thus to an inconsistent theory, as we will see.

The theory of *heap* models projects the positive integers with addition onto a primitive concept of counting, for example, "1, 2, 3, Heap." (The presence of zero and negatives is optional, see below.) Thus we have that, in addition to all true equations of positive integer arithmetic, all of $4 = 5 = \cdots =$ Heap hold. If we think of the positive integers as like the ambiguity group, and additionally impose the condition of the previous paragraph that the map from the ambiguity group to the heap be 1-1, we also have that $4 \neq 5 \neq 6 \cdots$ all hold, which is inconsistent. That is, there is an inconsistent theory which satisfies this condition. Heap H functions as an indeterminate upper limit to counting, a kind of infinity in that $a + H = H + a = H$ holds (except where the model also contains the additive inverse of H, where $H + (-H) = 0$). However, H can be reached by finite means: $1 + 3 = H$. Obviously, there are an infinite number of heap models, one for every maximal nonheap element.[4]

Adapting heap theory to the present case of the multiplicative group of positive reals requires some modifications. The natural view of the "backdrop" universe described earlier is to allow only the distances up to a certain distance d_{ij}^{max}, the distance to the backdrop. This might be represented initially as a mapping of the ambiguity group $\mathbb{R}^+$ to $\{x \in \mathbb{R}^+ : x \leq d_{ij}^{max}\}$, where the restriction of the domain to $x \leq d_{ij}^{max}$ is the identity mapping. However, if we impose the condition that the mapping from the ambiguity group to the set of distances be 1-1, we have an inconsistent theory.

To focus, let d_{ij}^{max} have a specific real number value, say 4.1. Then, for example $2 = 3 = 3.1$ do not hold, but $4.1 = 4.2 = \cdots$ all hold. Since these identicals are intersubstitutable in all contexts, we can introduce the name 'H' to refer ambiguously to all of them, that is, $4.1 = 4.2 = \cdots = H$. Since the mapping from the ambiguity group is one-one, also $4.1 \neq 4.2 \neq \cdots \neq H$. To multiply two numbers a, b in the heap: first determine whether either $= H$. If not, then multiply $a * b$ normally, determine whether the result $= H$ and if so identify $a * b$ in addition with all numbers which $= H$. Otherwise, if one or both of $a, b = H$, then the result $= H$ and all its identicals as well. For example, since $2 * 4 = 8$, then by substitution of identicals $2 * 4 = H = 4.1$ also: as with the integer models it is possible to reach the backdrop by operation on items in front of the backdrop. But also $a * H = H * b = H$, for example, $0.5 * H = H * 2 = H$. Evidently, $*$ is commutative. Also, the structure has a unit: $a * 1 = 1 * a = a$.

But the heap is not a group for two reasons. First, it lacks natural inverses for some elements, those which are greater than 4.1. Inverses can be produced by a further inconsistent extension of the theory of the ambiguity group. Identify all members of the class $\{x \in \mathbb{R}^+ : x \leq (4.1)^{-1}\}$ with one another and call them H^{-1}. If two numbers strictly between H^{-1} and H are multiplied together, take their product in $\mathbb{R}^+$ and determine whether it is identical with H, H^{-1} or something in between, identifying with all identicals in the former two cases. Otherwise, H^{-1} behaves like a zero, with $a * H^{-1} = H^{-1} * a = H^{-1}$; except for the case where $a = H$, where H^{-1} behaves as the inverse of H, with $H * H^{-1} = H^{-1} * H = 1$. Thus H^{-1} functions as a lower limit on distances.

The second reason heaps are not groups is that multiplication fails to be associative. For example, $(0.5 * 2) * 3 = 1 * 3 = 3$, but $0.5 * (2 * 3) = 0.5 * H = H$, while $3 = H$ does not hold. There are other multiplications which fail to be associative, for example, vector cross product. Furthermore, the failure of associativity is well motivated by the intended interpretation in a space with a backdrop and a least size: if you reach either of these limits you're stuck there, unless you're multiplied by your inverse; so the order you associate matters. However, these heaps are almost groups: commutative groupoids with a unit and inverses. Also, the subalgebra $\{H, 1, H^{-1}\}$ is plainly the limiting case of the heap where all $x > 1$ are identified with H and all $x < 1$ with H^{-1}.

It is clear that there are both consistent and inconsistent theories here. The difference between the two is the absence or presence of $4.1 \neq 4.2 \neq \cdots H$ and their inverses. The latter is the one-one condition, that distinct elements of the ambiguity group correspond to distinct distances in any acceptable reassembly of the structure. Now corresponding to cocycles as equivalence classes of triples of distances from $\mathbb{R}^+$, there are obviously classes of triples from heaps. These structures satisfy the condition for a coboundary among cocycles, namely the existence of a unit. Clearly, given any parts of the assembly which are done consistently, in front of the backdrop, then for them certainly $d_{ij} = d_{ji}$, that is $r_{ij} = 1$. But also for any pair of distances which are identified at the backdrop, again $d_{ij} = d_{ji} = H$, and $r_{ij} = d_{ij} * (d_{ji})^{-1} = H * H^{-1} = 1$. Thus the triple of ratios $(r_{12}, r_{23}, r_{13}) = (1, 1, 1)$, which is the unit.

For the purposes of describing formally an appropriate model in which the theory of the inconsistent heap holds, closed set logic would seem to be the most natural, since it has the advantage of representing contradictions as holding on closed sets and particularly their boundaries. Consider the topological space with a basis of four closed sets: $\mathbb{R}^+$, $\{x \in \mathbb{R}^+ : x \leq (d_{ij}^{max})^{-1}\}$, $\{x \in \mathbb{R}^+ : x \geq d_{ij}^{max}\}$, $\{\}$. For convenience, we can rename the middle two of these as H^{-1} and H, respectively. The closed sets serve as semantic values for closed set logic and theories thereof, since they are closed with respect to unions (disjunction) and intersections (conjunction).

Quantifiers are interpreted as respectively generalized union and intersection as usual. For negation, one takes closed complement, that is, the least closed set containing the Boolean complement. Thus the closed complement of both H^{-1} and H is $\mathbb{R}^+$ itself. This means that if $x = y$ is stipulated to hold on H but nowhere else, (i.e. $x = y$ takes H as its semantic value) then its negation $x \neq y$ holds everywhere, that is, $\mathbb{R}^+$. In particular, since H is a subset of $\mathbb{R}^+$, the contradiction $x = y$ & $x \neq y$ holds on H. The inconsistent theory of the heap can then be generated by the condition: *if* $x = y$ is true in the classical theory of $\mathbb{R}^+$ *then* assign $x = y$ to $\mathbb{R}^+$, *else if* x and y belong to at least one nonnull closed set, *then* assign $x = y$ to the least closed set to which both x and y belong, *else* assign $x = y$ to $\{\}$. To also add H to the theory, if both x and y belong to H, assign $x = H$ and $y = H$ to H, and similarly for H^{-1}. Then we define a sentence to hold in the theory of the heap, if it is assigned to some nonnull closed set. Then, for example, $4.1 = 4.2 = H \neq 4.1 \neq 4.2$ all hold; but whereas $2 \neq 3 \neq H$ and $4 = 2 * 2$ hold, they hold consistently, that is, neither $2 = 3 = H$ nor $4 \neq 2 * 2$ hold. The interested reader is left to fill in further formal details. For further description of theories of closed set logics, see Mortensen [2].

4 Inconsistent representations The question is, in looking at the inconsistent triangle, *what* is one peeking at behind it? One answer could be that one is peeking at a disassembled object, where the eyes fail to disidentify points directly behind one another. Where this seems to fall short, is that it does not tell us the positive consequences of implementing this failure to disidentify in such a way as to model the inconsistency. The eye seems to be in a positive default mode: identify those things which one fails to disidentify. But what are the mathematical consequences of literally identifying those points? Which connected structure is it that it seems to be? And what is it for it to nonetheless seem inconsistent?

It would be even more unsatisfactory to say merely that one is not peeking at anything at all. How does that differ from shutting the eyes for example? And how does it destroy the overwhelming illusion that one is peeking at something? The obvious third possibility is that one is peeking at an object in a backdrop universe, which is a heap.

As we have already seen, there are consistent and inconsistent versions of these models. The inconsistent theories satisfy the condition that the ambiguity group maps 1-1 to the set of distances of the heap: if $x \neq y$ holds in $\mathbb{R}^+$, then $x \neq y$ holds for distances d_{ij}, and for ratios/cocycle components r_{ij}. There are two ways to interpret the inconsistent models: epistemologically and ontologically (weak and strong paraconsistency, respectively). Epistemologically, what we are seeking to describe is a cognitive phenomenon: how is it that it *seems*, when it seems inconsistent? This concerns the representation of inconsistent data. It is well known that this can arise wherever there are at least two sources of data; or updates from a single source. In the present case, one might suppose that $4.1 \neq 5$ represents the *norm*, what one *knows* or *believes* about the space one is in; whereas $4.1 = 5$ represents the *default consequences* of being *unable to tell* between these distances.

Ontologically on the other hand, one is required to regard the theories representing such cognitive states as at least possible. On an intuitive level, the standard strong paraconsistentist argument is that one is required to take seriously the thought that one's inconsistent cognitive state might really be the way things are. *It really might be like that.* This is a general consequence of good theoretical practice, in any case: one should believe in the results of one's careful scientific investigations, which might contain persistent anomalies. Thus, if the world really were inconsistent in this way, then *this* is how it would look if you took a peek at it. Of course, there are other things which would look the same, but that is the way of it with eyes. *Credo ut intelligam.*

5 Conclusion There is one less than complete feature of this discussion. We have discussed heaps, and heaps can provide a unit for the cohomology group. But I doubt that heaps are the only inconsistent way to understand the triangle, or even the best way. The inconsistency we perceive is more cyclical and less brute force than that. This indicates a more subtle inconsistentizing operation is in the offing, which hopefully will be the topic of a further paper. But the general method of constructing inconsistent models remains the same.

Penrose also briefly indicates how to extend his discussion to cohomology groups associated with other 2-D figures representing ambiguous or inconsistent 3-D structures. His approach is undoubtedly rich with different applications. It is to be presumed that the inconsistent approach will lend itself to a similar range of applications and it is also proposed to study these in more detail in later papers.[5]

NOTES

1. Penrose calls this the tribar. Whereas there is an excellent case for some such short name (fewer keystrokes), the present author cannot bring himself to divorce this word from its common use among logicians, to refer to the sign for material equivalence ≡.
2. Penrose and Penrose in [5] built a (consistent) 3-D structure which photographs as stairs ascending in a closed loop, but needless to say it is not as it seems. For the photograph, see Penrose [4].
3. This represents a slight departure from Penrose's symbolism, which reserves 'd_{ij}' for what we call 'r_{ij}'.
4. The term "heap model" seems to have been first used by Meyer. On heap theory, see, e.g., Priest [6], p. 227 or van Bendegem [7].
5. For further inconsistent structures in the general area of projective geometry, wherein an inconsistent theory of homogeneous coordinates is proposed, see Mortensen [1], Chapter 9. On closed set logic, see Chapter 11 or [2].

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TOPOLOGICAL SEPARATION PRINCIPLES AND LOGICAL THEORIES

ABSTRACT. This paper is dedicated to Newton da Costa, who, among his many achievements, was the first to aim at dualising intuitionism in order to produce paraconsistent logics, the C-systems. This paper similarly dualises intuitionism to a paraconsistent logic, but the dual is a different logic, namely closed set logic. We study the interaction between the properties of topological spaces, particularly separation properties, and logical theories on those spaces. The paper begins with a brief survey of what is known about the relation between topology and modal logic, intuitionist logic and paraconsistent logic in respect of the incompleteness and inconsistency of theories. Necessary and sufficient conditions which relate the T_1-property to the properties of logical theories, are obtained. The result is then extended to Hausdorff and Normal spaces. In the final section these methods are used to vary the modelling conditions for identity.

1. PRELIMINARIES ON LOGIC AND TOPOLOGY

It is well known that there are significant connections between logic and topology. In the first section of this paper we survey these, in preparation for extending them later to take account of topological separation principles. Consider first the formal semantics of propositional modal logic. Modal logic adds to the usual Boolean operators (&, $\vee$, $\sim$) the unary propositional operator $\square$, where $\square P$ is is interpreted as "It is necessary that P". Additional Boolean operators such as $\supset$ and $\equiv$ are defined in the usual way; and the modal operator $\Diamond$, interpreted as "it is possible that", is defined as $\sim\square\sim$. The possible worlds semantics for modal logic constructs models using a set X of possible worlds. Propositions hold at some worlds and do not hold at others, so we can write Pa for the statement "The proposition P holds at world a". For example, if P is the proposition that snow is white, then Pa is the proposition that snow is white holds in world a. This is given an algebraic setting by associating with each proposition P a set $[P]$ of members of X, interpreted as the set of worlds at which P holds. Thus we can define Pa to mean $a \in [P]$. The simplest set of conditions governing the behaviour of the operators (&, $\vee$, $\sim$, $\square$) is:

(1) $[P]$ is a subset of X, all propositions P

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(2) $[\sim P] = -[P]$, the set complement of $[P]$

(3) $[P\&Q] = [P] \cap [Q]$

(4) $[P \vee Q] = [P] \cup [Q]$

(5) $[\Box P] = X$ if $[P] = X$

else $= \emptyset$.

It is then simple to prove that:

(6) $[\Diamond P] = X$ if $[P]$ not $= \emptyset$

else $= \emptyset$

We can define a *deducibility relation* $\vdash$ in a model [ ] by: $P \vdash Q$ in [ ] iff $[P] \subseteq [Q]$. Then define a proposition to be a *semantic theorem* just in case it is true in all worlds of all models, i.e., $[P] = X$ in all models. The set of semantic theorems so defined coincides exactly with the set of provable theorems of the logic *S*5. The simplicity of the semantics in this result has convinced the big majority of modal logicians that *S*5 is the preferred modal logic as a description of the properties of necessity and possibility.

The idea that X might have a topological structure (X, O) with open sets O, allows a generalisation. Thus if we replace the semantic condition (5) for $[\Box P]$ with the condition:

(5.1) $[\Box P] = \text{int}[P]$, the interior of $[P]$

we find instead that the semantic theorems coincide exactly with the provable theorems of *S*4. Furthermore, it is apparent that we can readily recover the *S*5 case with the additional stipulation that the topology O on X should be the indiscrete topology. Note also that this change implies that the semantics of possibility, condition (6), changes to:

(6.1) $[\Diamond P] = \text{cl}[P]$, the closure of $[P]$.

Modal logic is not the only place where logic connects with topology. One can link the behaviour of $(\&, \vee, \sim)$, especially $\sim$, directly with the topological structure of X. So instead one stipulates that *the semantic value of any proposition shall be an open set.* This amounts to the stipulation that $(\&, \vee, \sim)$ shall be functional on the open sets. This is no problem for

(&, ∨), since open sets are closed w.r.t. (finite) intersections and unions. However, for ~ to be an operator on open sets, it cannot generally be the set (Boolean) complement. Instead, one associates it with the *open complement*: the largest open set contained in the set theoretic complement, which may be identical with the set theoretic complement but need not be. Algebras of closed sets are *Heyting* algebras. Summarising, we take (1)–(4) but replace:

(1.1) $[P] \in O$, the open sets on X

(2.1) $[\sim P]$ = the open complement of $[P]$.

There is also a natural implication operator ⇒, which is not definable in terms of (&, ∨, ~), but as:

(7) $[P \Rightarrow Q] = [P] \Rightarrow [Q] = \cup\{O : [P] \cap O \subseteq [Q]\}$

If as before a semantic theorem is any formula which holds at all points of X in all models, we have exactly the theorems of *intuitionist logic* J. The above semantic features mean that intuitionist logic supports *incomplete* theories, that is theories in which neither P nor $\sim P$ holds, for some proposition P. To see this, consider a proposition P and let a be any point on $b[P]$, the boundary of $[P]$. $[P]$ is open, so a is not in $[P]$, hence Pa does not hold. But neither does $\sim Pa$ hold, since that requires $a \in [\sim P]$ = the open complement of $[P]$, which is disjoint from $b[P]$. Again, neither P nor $\sim P$ hold at a, so the theory consisting of the propositions which hold at a, is incomplete. So one can describe a, viewed as a possible world, as an *incomplete world*.

The topological duality between open and closed is mirrored in a duality between intuitionist logic which supports incomplete theories, and (one variant of) *paraconsistent* logic, which supports *nontrivial inconsistent* theories. A theory is *inconsistent* if it contains at least one proposition P and its negation $\sim P$, and *nontrivial* if it does not contain every proposition (of its language). If we stipulate that all propositions hold on *closed* sets of points, then in order to have negation be a natural operation, we must identify the negation of a proposition with the *closed complement* of that proposition, which is the smallest closed set containing the set theoretic complement. That is, replace (1) and (2) instead by:

(1.2) $[P] \in C$, the closed sets on X

(2.2) $[\sim P]$ = the closed complement of $[P]$

We also have, instead of (7):

$$(7.1)\quad [P - Q] = [P] - [Q] = \cap\{C : [P] \subseteq [Q] \cup C\}$$

Algebras of closed sets are *Brouwerian* algebras.

This change requires rethinking the semantic condition determining theoremhood. In moving from open sets to closed sets one is reversing the order on the lattice. That is, the bijection which turns open sets into their set complements is contravariant w.r.t. subset inclusion, which is the order on the set lattice. Now in any lattice-theoretic value space of more than two values there is in general a choice of more than one filter on the lattice, where membership of a given filter serves to determine membership of a theory. That is, a given Brouwerian lattice of closed sets, or for that matter a Heyting lattice of open sets, can support more than one theory for the same value function [ ], determined by different filters on the lattice. The most natural dual for the Heyting condition that a theorem be determined by the property of holding at every point, is the condition that a theorem be determined by the property of holding at some point. The set of propositions which hold at some point in all (closed set) models is closed set logic. The properties of the natural dual to intuitionist implication $\Rightarrow$, namely pseudo-difference $-$, prevent it from being a reasonable implication; for example $P - P$ fails to be a theorem. This is not particularly paradoxical, since dually J lacks a natural pseudo-difference operator, but which Boolean logic possesses. Furthermore, $\Rightarrow$ is not the only reasonable implication around: there is always a reasonable implication on any lattice, namely the operator which equals X if $[P] \subseteq [Q]$ else equals $\emptyset$.

Let $[P]$ be a closed set and let a be any point on $b[P]$. Since $a \in [P]$, Pa holds. But also since $a \in [\sim P]$ = the closed complement of $[P]$, $\sim Pa$ also holds. That is, the theory which is the propositions holding at a, is inconsistent. The world a can thus be described as an *inconsistent world*. It is not in general trivial however, since many propositions may fail to hold at a. In this sense closed set logic is paraconsistent, i.e., supports non-trivial inconsistent theories.

This completes our survey of what is known to date about semantics using open and closed sets. In the following sections, these ideas are extended to a more general setting, in which value spaces include sets which are open, closed or neither, but (&, $\vee$, $\sim$) remain operators simultaneously on both open and closed sublattices, and where the logic addressed goes beyond propositional logic to a fragment of quantifier logic. Results are obtained connecting specific topological properties such as T_1, T_2 with the logical properties of the theories supported. Extension to the even more general case of full first-order logic presents further complexities, requiring the use of product topologies, but is not necessary to make the point here.

2. INCONSISTENCY AND INCOMPLETENESS TOGETHER

We consider monadic predicates F with a single free variable x. Each F is associated semantically with an extension $[F]$ which is a subset of a topological space (X, O, C) (usually shortened to (X, O)) with open sets O and closed sets C. Fa *holds* in a model [ ], or F *holds at* a in [ ], iff $a \in [F]$. Now we stipulate that the operators (&, ∨, ~) shall be operators on both the open and closed sublogics (and ignore intuitionist implication and paraconsistent pseudo-difference). Again there are no problems with (&, ∨); but there is only one way to deal with negation, and that is to associate it with the *topological complement*, which is the open complement if $[F]$ is open, the closed complement if $[F]$ is closed, and the set complement otherwise. Note that in the case where $[F]$ is clopen, i.e., both closed and open, then topological complement coincides with set complement. That is, we retain (1)–(4) except that we replace (2) with:

(2.3) $[\sim F]$ = the topological complement of $[F]$

An interesting consequence of this definition, pointed out by Greg Restall, is that we can have $[F] \subseteq [G]$ without $[\sim G] \subseteq [\sim F]$, for example if $[F]$ is open and $[G]$ its closure. This is a kind of limited failure of contraposition which is significant because it occurs in the context of a well-motivated semantics.

The logic and associated theories can be further extended to sentences with a single, leading quantifier by adding:

(8) $(\forall x)Fx$ holds iff Fa holds for all a

(9) $(\exists x)Fx$ holds iff Fa holds for some a

If negated quantifiers are then defined by the Quantifier Equivalence Laws $\sim \forall =_{df} \exists \sim$ and $\sim \exists =_{df} \forall \sim$, these will have the consequences that:

(10) $\sim (\forall x)Fx$ holds iff $\sim Fa$ holds for some a

(11) $\sim (\exists x)Fx$ holds iff $\sim Fa$ holds for all a

Apart from being true in Boolean Logic, these are interesting consequences because they permit the possibility of inconsistency and incompleteness among quantified sentences. A set of sentences is said to be a *theory on* a topological space (X, O) if it is the set of sentences which holds in some model [ ] where (X, O) is the codomain of [ ].

PROPOSITION 1. If (X, O) has the discrete topology then every theory on (X, O) is consistent and complete.

Proof. The discrete topology is that in which every subset is open. Let F be any predicate of whatever logical complexity with a single free variable x. Then $[F]$ is both closed and open. Hence $[\sim F]$ is the set complement of F. Thus not both Fa and $\sim Fa$ hold, for any $a \in X$. And if Fa does not hold, then a is not in $[F]$, so it is in $[\sim F]$, so that $\sim Fa$ holds. That is, the subtheory of quantifier-free sentences is consistent and complete. For the quantified case, if $(\forall x)Fx$ holds then Fa holds for all a. But $[F]$ is clopen, so a is not in $[\sim F]$, so $\sim Fa$ holds for no a, so that $\sim (\forall x)Fx$ does not hold, which is consistency. If $(\forall x)Fx$ does not hold, then for some a, Fa does not hold. Since $[F]$ is clopen, $a \in [\sim F]$; and so $\sim (\forall x)Fx$ holds, which is completeness. The case of the existential quantifier is similar. □

To find an appropriate converse, we need the following definition:

DEFINITION 2. A theory on a space is a 1-*point* theory iff for some nonlogical predicate F, $[F]$ = some singleton $\{a\}$.

Clearly some theories are 1-point theories, e.g., any theory containing the predicate "$\in \{a\}$"; while some theories are not 1-point theories, e.g., any theory with sole nonlogical axiom "$a \in \{a, b\}$" where "$\sim a = b$" also holds in the theory. Note that we exclude = from being such an F on the grounds that it is a logical symbol. Of course = can be included in the theory because "$= a$", "$= b$", "$\sim = a$" etc. can be; but allowing it to be all of F weakens later conclusions.

Now we have a proposition which further links the topological properties of (X, O) with the logical properties of theories on it.

PROPOSITION 3. If (X, O) is a T_1-space and every 1-point theory on (X, O) is consistent, then (X, O) has the discrete topology.

Proof. We use the formulation of a T_1-space as: every singleton is closed. (For definitions of this and other separation axioms see e.g., Simmons (1963, 130.) If (X, O) is not discrete then we show how to construct an inconsistent 1-point theory. Since it is not discrete, some singleton $\{a\}$ is not open. But being T_1, $\{a\}$ is closed. Consider any theory containing the sentence "$a \in \{a\}$". This is a 1-point theory w.r.t. the interpretation: $[\in \{a\}] = \{a\}$. Clearly, $a \in \{a\}$ holds. But $\{a\}$ is closed and not open, so $[\sim \in a] = X$. This means that $\sim a \in \{a\}$ also holds and the theory is inconsistent. However, for any b different from a, $b \in \{a\}$ does not hold, so the theory is non-trivial. □

PROPOSITION 4. (X, O) has the discrete topology iff (X, O) is T_1 and every theory on (X, O) is consistent.

Proof. L to R follows from Proposition 1 together with the fact that the discrete topology is T_1. R to L follows from Proposition 3. □

These propositions between them furnish us with examples of inconsistent and incomplete theories on topological spaces. For inconsistent theories, one cannot have the discrete topology. But any non-discrete T_1 topology will yield them. If $\{a\}$ is any closed non-open singleton, the theories described in Proposition 3 suffice, i.e., any theory in which $a \in \{a\}$ holds, or more generally any theory in which Fa holds where $[F] = \{a\}$. These theories are in general non-trivial, for example if they contain a name b for any distinct element b, then Fb does not hold. For incomplete theories, if we consider 1-point theories we must have a space which is not T_1. A simple case of a non-T_1 space is the 3-member set $X = \{a, b, c\}$, with $O = \{X, \{a, b\}, \{a, c\}, \{a\}, \emptyset\}$ and $C = \{X, \{b, c\}, \{b\}, \{c\}, \emptyset\}$. Clearly, $b \in \{a\}$ does not hold. But also $[\sim \in \{a\}] = \emptyset$, so that $\sim b \in \{a\}$ does not hold and the theory is incomplete. The theory is at the same time inconsistent, since both $b \in \{b\}$ and $\sim b \in \{b\}$ hold.

3. OTHER SEPARATION PRINCIPLES

To extend these results to other separation principles, we need further definitions. There are many theories on a given topological space (X, O). Here we want to define the theory of a subset S of X. We also want to model identity. In the present setting of monadic predicates, modelling the monadic predicates $[\ = a]$, one for each a, as 1-point $\{a\}$ produces in T_1 spaces $\sim a = a$ for every a, since all disidenties hold because $[\sim = a] = X$. The simplest solution, which simplifies later results by confining inconsistency and incompleteness to nonlogical predicates, is to make identity classical:

(12) $[= a] = \{a\}$, all a in X

(13) $[\sim = a] = -[a]$

We briefly consider other models for identity in the next section.

DEFINITION 5. If S is any subset of X, we say that *the theory of* S, Th(S), is the theory on X with names for all members of X, a predicate "$\in \Sigma$" with $[\in \Sigma] = S$, and as well a possibly infinite number of identity predicates "$= a$" and "$a =$", one for each name, satisfying (12) and (13) above.

Clearly, the theory of S identifies all members of the space (X, O) and picks out the members of S. This is done consistently except at the boundary, where it is done incompletely or inconsistently.

DEFINITION 6. The *atomic* sentences of a theory are those which lack the logical connectives (&, $\vee$, $\sim$, $\exists$, $\forall$). Any two theories on a space (X, O) are *atomically disjoint* if the atomic *nonlogical* sentences (whose main connective is other than =) holding in each are disjoint; and one is an *atomic extension* of the other if the set of atomic sentences of the first is a superset of those of the second.

Now we have two propositions which characterise T_2 and normal spaces respectively.

PROPOSITION 7. The space (X, O) is Hausdorff iff every pair of 1-point theories of subsets of X have a pair of consistent atomic extensions which are atomically disjoint.

Proof. Uses the definition of Hausdorff as: every distinct pair of points can be separated by open sets; and consists in noting that the bijection between sets and their theories associates (a) singletons with disjoint 1-point theories which are complete in the T_2 case; (b) open sets with consistent (possibly incomplete) theories; (c) the subset relation with the atomic extension relation; and (d) disjoint sets with atomically disjoint theories. □

PROPOSITION 8. The space (X, O) is normal iff every pair of atomically disjoint complete theories of subsets of X have a pair of consistent atomic extensions which are atomically disjoint.

Proof. Uses the definition of Normal as: every disjoint pair of closed sets can be separated by open sets; and consists in noting that in addition the bijection associates closed sets with complete (possibly inconsistent) theories. □

The results of this section essentially involve a restatement of the topological separation axioms in the language of theories. But the latter have a character of their own, since they involve the extendibility of pairs of complete and possibly inconsistent theories, to consistent theories which separate them.

4. OTHER IDENTITY CONDITIONS

There are other ways to model identity. Thus one can replace (12) by:

(12.1) $[= a] = [a =] = \{a\} \cap S$.

This has the effect that only members of S are self-identical, while $a = b$ does not hold for any distinct a and b, nor even when a is b if it is outside S. The pure theory of identity remains consistent and complete if we retain (13). However if (13) is changed to the topological counterpart of (12.1):

(13.1) $[\sim\ = a] = [\sim a =] = \ \sim [= a]$

then the theory of "$= a$" copies that of "$\in \Sigma$" when the latter is 1-point. Thus, when a is not in S, in addition to $a = a$ not holding, $\sim a = a$ holds. Inside S, in addition to $a = a$ holding, $\sim a = a$ holds if $\{a\}$ is closed but not open. That is, $\sim [= a]$ is X if a is outside S or $\{a\}$ is closed and not open, else $\sim [= a]$ is $X - \{a\}$. As S contracts, the region in which no identities hold expands, so this can be called the "scorched earth model".

Alternatively, one can replace (12) by:

(12.2) $[= a] = [a =] = \{a\} \cap (S \cup \sim S)$.

This has the effect that $a = a$ holds for every a except those on the boundary of S, if that is not included in $S \cup \sim S$, i.e., if S is open and not closed. No other identities hold. There are several options for $[\sim = a]$. For example, one can have $\sim [= a]$ which is (13.1); but there are also $-[= a]$, $-\{a\}$ and $\sim \{a\}$.

There is also $-\{a\} \cup (\{a\} \cap b(S) \cap S)$. This has the effect that $\sim b = a$ holds for every b other than a, but also $\sim a = a$ for every a on the boundary of S as long as it is included in S, i.e., if S is closed and not open. That is, if S is closed and not open, it is surrounded by a boundary of contradictory self-identities, which can be called the "ring of fire" model. If however S is open and not closed, then $\sim a = a$ does not hold on the boundary, so the boundary of S is a fence where neither identities nor their denials hold.

Finally, there is:

(12.3) $[= a] = S$ if $a \in S$

else $= \{a\}$

This has the effect that the members of S are identified with one another. The methodology of identifying elements is ubiquitous in topology. This model can be used to study the functional properties of inconsistent theories determined by quotient topologies arising from an equivalence relation (see Mortensen (1995, chap. 9)).

5. CONCLUSION

It is seen that there are significant interactions between the topological properties of spaces and the logical properties of theories on them. It is proposed to study these further in later papers.

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"Prospects for Inconsistency"

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Prospects for Inconsistency

Chris Mortensen

1 Terminology: Paraconsistency versus Inconsistency

The first insight for paraconsistency is the recognition of non-trivial inconsistent theories. This is what leads to the conclusion that natural logic must be inconsistency-tolerant, ie. paraconsistent. The phenomenon of inconsistency is the prior concept here, paraconsistency is a derivative concept. Thus, the central use of the term "paraconsistent" is as a property of logics rather than theories, while the central use of "inconsistent" is as a property of theories rather than logics. (Both mistakes are made.) It might be protested that a "paraconsistent theory" is uncontroversially any theory of a paraconsistent logic. I do not object to that usage, only to the usage where it replaces "inconsistent theory". Even so, if a paraconsistent theory is any theory of a paraconsistent logic, then any consistent theory of a paraconsistent logic (and there are many) becomes a paraconsistent theory, which is an odd consequence. While on the subject of terminology, I wish to enter a mild protest about another neologism, the word "dialetheic". While there is an obvious case for some such short name for the thesis that there are true contradictions and its variants, I think that it is always desirable to speak plainly and frankly when one is contemplating true contradictions, if only for the shock value.

It is clear that inconsistent mathematics needs the support of paraconsistent logic. That was part of the original paraconsistency insight, that any logic such as classical logic which contained the rule *ex contradictione quodlibet*, ECQ (from a contradiction to deduce everything), made inconsistency-tolerance impossible. Thus paraconsistent logics were constructed to demonstrate rigorously the possibility of contradiction-control. However, conversely, paraconsistent logic has very restricted applications unless rich inconsistent mathematical theories are found: paraconsistent logic is blind without interesting inconsistent theories. Particularly, since mathematics is the study of abstract structure, inconsistent theories at their most general should be mathematical theories. Thus, inconsistent mathematics and paraconsistent logic need one another.

2 Functionality

In order to proceed further, we need a definition.

DEFINITION 1. A theory is functional iff, for any terms t_1, t_2, whenever $t_1 = t_2$ holds, Ft_1 holds iff Ft_2 holds, where F is any atomic context (ie. any logic-free context, eg. $= t$, $< t$, εt etc., where the term t may be functionally complex, eg. containing function symbols $+$, $*$, $-$ etc.). A theory is transparent iff the same thing holds, except that F is any context (ie. containing logical symbols eg. $\vee$, &, $\sim$, $\supset$, $\exists$, $\forall$).

It is not difficult to show that any classical model which is functional is transparent as well. But the possibility of non-trivial inconsistent theories allows the concepts of functionality and transparency to be pulled apart. This is a further example, beyond that of the often-noted distinctness of inconsistency and triviality, of the conceptual flexibility which the insight of inconsistency-tolerance brings. It emerges that it is the concept of functionality, rather than the additional logical concept of transparency, which is important for mathematics, as we will see. I will suggest that the study of functionality is the goal of mathematics.

Now it is true that one can contemplate inconsistent databases and theorem provers which need paraconsistent logic for their logical properties, but which have little or no mathematical functionality. Not every computational application needs full mathematical functionality. Still, if these lack full mathematical functionality, then they are functionally impoverished. Furthermore, any metaphysical or scientific arguments for an inconsistent or dialetheic conclusion depend in the end on inconsistent mathematics for their validation. This is after all the goal of exact philosophy, to demonstrate rigorously the possibility that a given theory holds. Metaphysics without at least the prospect of a rigorous theory in the background is speculative at best. In sum, all paraconsistency and all dialetheism must rest on the possibility of inconsistent mathematics for their justification.

In this paper, the role of functionality in inconsistent theories is considered. It is seen that inconsistent mathematics needs less crude devices that barring ECQ to contain the spread of contradictions.

3 The role of functionality

We have seen that paraconsistent logic is necessary for inconsistent mathematics. However, it is not sufficient. This is because, as is well known, in various important theories such as the theory of the real numbers, inconsistencies spread to triviality *in spite of* banning ECQ. Hence, while inconsistent mathematics needs paraconsistent logic, it requires stronger methods of contradiction-containment as well. Logic by itself cannot exercise such a strong restraint over mathematics.

THEOREM 2. *The addition of the sentence $a = b$ (where a and b are classically distinct real numbers) to classical real number equational theory, with the result-*

ing theory stipulated to be functional, enables one to prove that every real number is identical with every other real number.

Proof. Functionality implies that equals may be substituted for equals. Hence, from $a = b$ one has by functionality that $a - a = b - a$. Classically, the left hand side $a - a = 0$. Hence, $0 = b - a$. From this by functionality one has, for any c, that $0.c/(b - a) = (b - a).c/(b - a)$. Classically, the LHS $= 0$ and the RHS $= c$. Hence by functionality $0 = c$. By a similar argument $0 = d$ for any other real number d. Hence $c = d$ for any real c and d. ■

This result seems to have been first noticed by Dunn. Versions can be found in Priest, Routley and Norman [1989], and Mortensen [1995]. Having every atomic sentence holding is one way for a theory to be useless for serious mathematics. We can call this *mathematical* triviality. It is, however, interesting that one cannot go on to argue automatically from the fact that every atomic sentence holds in an inconsistent theory, ie. that the theory is mathematically trivial, to the conclusion that every sentence holds, ie. that the theory is trivial in general. This follows from:

THEOREM 3. *There are non-trivial functional RM3-theories in which every atomic sentence holds.*

Proof. Consider an RM3 model where every classically true atomic sentence is assigned the value T and every other atomic sentence is assigned the value B. This is plainly functional, but not every negated atomic sentence holds, in particular the denials of every sentence assigned T are assigned F and thus do not hold, eg. $\sim a = a$ holds for no a. (For details of RM3 models, see eg. Mortensen [1995, Chapter 2].) ■

However, fairly minimal extra resources are required for triviality to follow. Interestingly, they seem to be logical resources rather than mathematical resources. For example, suppose that the theory contains $(x, y)\ (x = y \rightarrow .\sim x = y \longleftrightarrow \sim x = x)$, which in turn would follow from $(x, y)\ (x = y \rightarrow .x = x \longleftrightarrow x = y)$ together with $A \rightarrow B \rightarrow .\sim B \rightarrow \sim A$. Then, selecting any x and some y which is not classically identical with x, since both $x = y$ and $\sim x = y$ hold in the model we may deduce that $\sim x = x$ holds, for every x. Then, selecting any y at all, from the fact that $x = y$ and $\sim x = x$ hold we may deduce $\sim x = y$. This ensures that the negation of every atomic sentence also holds. This is not even yet enough for triviality. To ensure that the obvious inductive argument for triviality succeeds, one needs in addition for the & clause $\sim A \rightarrow \sim(A \& B)$, for the $\rightarrow$ clause $(A \& \sim B) \rightarrow \sim(A \rightarrow B)$, and for the universal quantifier clause $\sim Fa \rightarrow \sim(x)Fx$. These principles hold in the RM3 models. The important point is that ECQ and much more do not need to hold in order for this argument for triviality to work.

Summing up this section, contradiction-tolerancy methods from logic can be overwhelmed by mathematical operations.

4 The role of different functions

On the other hand, not even all arithmetical equational theories are so trivialised. There is no such argument in Peano arithmetic, or standard arithmetic (ie. sentences true in the standard model of natural number arithmetic). That is, we have:

THEOREM 4. *Theorem 2 fails in Robinson arithmetic, Peano arithmetic and standard arithmetic (except holding trivially if ECQ holds).*

Proof. This is essentially Meyer's well known result [1976], that there exist functional, even transparent, inconsistent RM3 models of natural number arithmetic in which not every atomic sentence holds, ie. which are mathematically non-trivial. ■

Returning to the proof of Theorem 2, it is clear that in $(+, -)$ arithmetic all one can get from $a = b$ are, e.g., $a - 1 = b - 1$, $a - 2 = b - 2, \ldots, 0 = b - a$. The diagnosis is that it is the presence of multiplication and division in the theory of the real field which enables scale changes to be made so that everything can be made equal to everything else. Thus Theorem 2 will even fail for the additive group of reals. It is this feature which enables inconsistent operations at anomalies and discontinuities in physical systems as described as phase spaces over the real numbers: consider the space "cylindrified" at the inconsistent boundary by $0 = b - a$ and having only as much multiplicative structure as available on the surface of a cylinder of circumference $b - a$. Thus, even $0 = 1$ can hold non-trivially on the surface of a cylinder of circumference 1. (See Mortensen [1995, Chapter 6] for details.) On the other hand, the addition of $0 = 1$ trivialises Meyer's axiomatic relevant arithmetic $R\#$.

From this point of view, the contradiction-spreading feature of ECQ is seen as a crude tool, spreading contradiction immediately everywhere. As if it had nothing to do with the properties of the mathematical objects! Talk about the hegemony of logic over mathematics! Once you get a logical anomaly, mathematics is over!

I suspect that some confusion lurks in mathematicians' minds about the effects of contradiction, because of a poor understanding of an old conundrum. Consider the following well-known fallacy:

THEOREM 5. *Real number arithmetic is inconsistent, in fact mathematically trivial.*

Proof. Let a be any non-zero number and let b be any number such that classically $a = b$ (for example, b might be $(a + 1) - 1$). Then substituting equals for equals, $a.b = a.a$. Subtracting $b.b$ from both sides, $(a.b) - (b.b) = (a.a) - (b.b)$. Factorising both sides, $b.(a - b) = (a + b).(a - b)$. Dividing both sides by $a - b$, one obtains $b = a + b$. Since $a = b$, $b = b + b = 2b$. Dividing both sides by b, we have $1 = 2$ [Rouse Ball, 1959]. But $\sim 1 = 2$ is a theorem of all the above arithmetics. Furthermore, from $1 = 2$ we have $1 - 1 = 2 - 1$, and thus $0 = 1$.

Putting both of these together gives $0 = 2$, and thus all numbers become equal. Alternatively, having reached $b = 2b$, subtract to get $0 = b$. Since the argument works for arbitrarily chosen b, all numbers are equal as before. ∎

General triviality follows in the conditions described at the end of Section 3. This argument is often presented for an unspecified arithmetic, with the real numbers in mind. The usual diagnosis of the error is to note (correctly) that there is an error in division of both sides by $a - b$, which is zero, for which division is undefined. However, if this were one's only source of arithmetical anomalies, then one might incautiously go on to think that mathematical triviality is the inevitable accompaniment of contradiction. This is simply not so. As we have seen, Peano etc. arithmetics are harder to trivialise. So it is interesting to ask how far one can get with this argument in those contexts. For example, in the context of natural number arithmetics, one may model the consequences of $b = 2b$ for a fixed non-zero b without mathematical triviality. By subtractions, $0 = b$; but as is well known this can be made to hold non-trivially in RM3 mod b. From that point of view, the error is not division by zero, but dividing both sides of $b = 2b$ by b to get $1 = 2$. Once that is allowed, of course, mathematical triviality follows by operations permitted in Peano arithmetic.

5 Non-functional theories

I suggest that theories in which functionality fails, but in a controlled way, represent the future of the paraconsistency program. One goal we can have is a good theory of division by zero. This perforce will be non-functional if triviality is to be avoided. But it must have some structure. One weaker constraint could be functionality in a sub-range, together with definite laws obeyed outside the sub-range. One inconsistent theory of division by zero exists as part of a theory of the Dirac delta function. Here, one obtains an inconsistent extension of non-standard analysis with infinitesimals and infinite numbers. There exists a largest "order of infinity", which is the reciprocal of zero. Multiplication of identical numbers by any real number, indeed multiplication by any infinite number which is infinitesimal with respect to the reciprocal of zero, gives identical results; whereas multiplication of identical numbers by the reciprocal of zero "unsmears the zero", and can produce non-identical results. That is, while $a = b$ ensures that $a/r = b/r$ for any non-zero real number r, yet nonetheless $a/0 = b/0$ may not hold. (For details, see [Mortensen, 1995, Chapter 7].)

6 Conclusion

The moral to be drawn, is that there is much to be learned about the effects that the constraint of functionality has on inconsistent theories of different operations; as well as the behaviour of theories in which functionality fails.

I am indebted to the audience at the First World Congress of Paraconsistency for insightful comments on this paper, and particularly to Diderik Batens for later careful criticisms.

The University of Adelaide, Australia.

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Toward a Mathematics of Impossible Pictures

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Towards a Mathematics of Impossible Pictures

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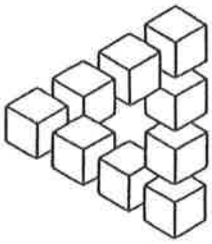
Abstract
In this paper are described recent developments in the mathematics of impossible pictures. Classical consistent theories by Cowan, Francis and Penrose are described. It is argued that only an inconsistent theory can capture the epistemic content of the experience. Approaches using inconsistent models which are heaps, greaps or cylinders are described.

1 Introduction

The theory of inconsistency, together with its parent discipline paraconsistent logic, is developing rapidly. One notable omission, however, is the theory of impossible pictures. So as to make no confusions, let it be stressed that this means real pictures of apparently impossible objects. We will see examples as we go along.

2 History of Impossible Pictures

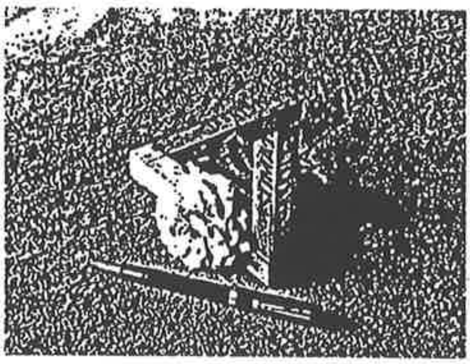
Impossible pictures have been drawn for a long time. The earliest seems to have been on a Pompeii interior wall. There are also medieval alterpieces, and some parts of Piranesi's Carceri. Marcel Duchamp did a strange bed. But impossible pictures come of age with Oscar Reutersvaard. One day in 1934 in his high school Latin class in Stockholm, the 17 year old Oscar doodled like this:



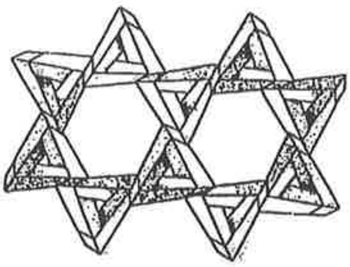
Thereby began a brilliant career in which he drew more than 4,000 pictures. He was honoured in the 1980s by the Swedish government in a number of stamps featuring his creations.

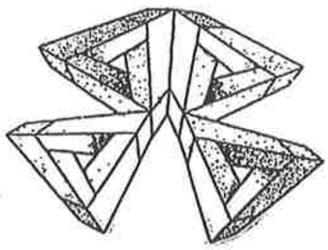
About 20 years after Reutersvaard began, the idea was re-discovered by the Penroses, and M. C. Escher. In [5] L. S. Penrose and his son Roger published a two page paper in the *British Journal of Psychology*, one page of which included both drawings and a photograph. Simultaneously, and then in communication with the Penroses, M. C. Escher drew masterpieces like *Belvedere*, *Waterfall*, and *Ascending and Descending*.

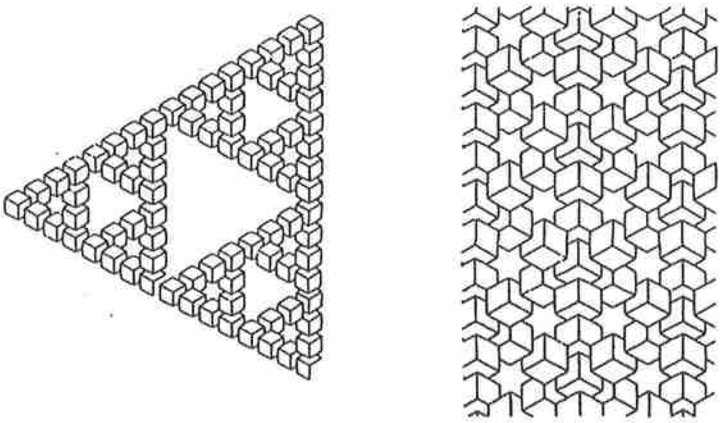
Since then, there have been many following in the construction of impossible pictures. One prominent person has been Bruno Ernst, who took a photograph of the impossible triangle.



There have also been very recently new pictures at Adelaide.





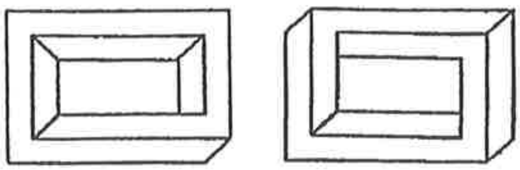


Mercier Wallpaper

3 Cowan, Francis and Penrose

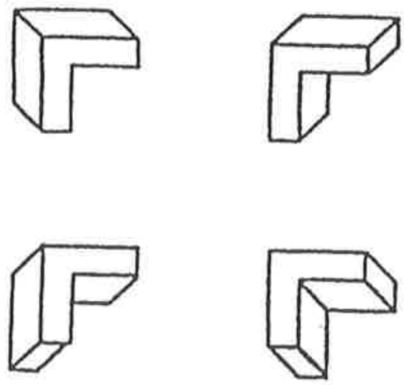
There seem to be three attempts to describe these pictures in terms of classical mathematics: Thaddeus Cowan [1], George Francis [2] and Roger Penrose [6].

Thaddeus Cowan focusses on the case of four-sided figures with a hole in the middle, though the analysis applies to any n-sided figure, for $n > 2$, and thus the triangle.



Different Inconsistencies in 4-sided figures

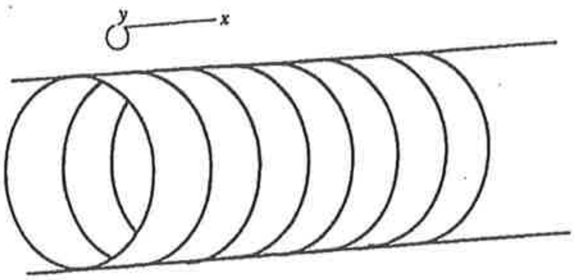
Cowan is able to classify 2-dimensional corner elements in terms of the 3-dimensional information that they represent, using the concepts front, back, inside and outside. This results in four kinds for each corner. With each figure having four corners, that is 256 different assembled figures. Using group theory, Cowan described necessary and sufficient conditions for such assembling to result in a picture of a possible object: one which always leaves these unchanged all the way around. Pictures of impossible objects are those which switch these in one or more places of assembly. The theory of the tricorn (impossible triangle) comes out as a special case, when two sides are identified, but the analysis applies to a much broader class of figures as well.



Corner elements

This is a very excellent and useful theory. However, it leaves something out. In effect it is telling us that if the conditions fail this is not a picture of a possible object. But that is a perfectly consistent state of affairs. It fails to capture our sense of what it is a picture of, its *content*: a object with impossible properties.

George Francis [2] made an advance when he asked the sensible question: what sort of consistent 3-D space could this picture represent? This immediately gives a positive heuristic to the problem. Francis pointed out that the triangle could exist in certain 3-D non-Euclidean spaces, specifically those which are $R^2 \times S^1$. It is difficult to represent this on the page, but the case of $R^1 \times S^1$ as embedded in R^3 is familiar as the cylinder:



$R^1 \times S^1$

Francis is clearly right here. But there is another way to see Francis' point, and its limitations. The importance of Ernst's photograph is that it is a 3-D object. But it only looks this (impossible) way from a single camera angle. The real 3-D object simply does not have the connectivity it seems to have from that single angle. But what is it that it seems to be from that angle? Something with a hole, which the real physical object did not have. The experience has a content, one which leads us to say "I see it but it is impossible!" The mind is clearly projecting something onto its perceptions here. But what?

Roger Penrose [6] made a further significant advance when he described the situation using cohomology theory. The idea was similar, that a picture of an apparently impossible object can indeed represent a possible 3-D object, but one with different connectedness. For example, one can imagine that the triangle is a picture of three disassembled parts lying at different distances from the viewer, but lined up so that they look joined up. That is, they are lined up so that they project down onto the 2-D picture. The point is now, that in the consistent case it is possible to re-assemble the pieces into a connected whole, whereas in the inconsistent case it is not so possible. Penrose shows how to describe this in group theory.

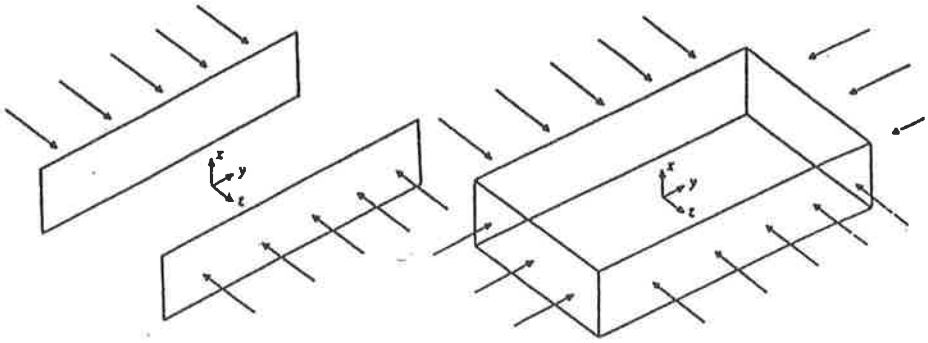
This is clearly a useful advance. But again one feels that there is something left out. The mind doesn't think "Here are three disassembled parts", it doesn't perceive it as three disassembled parts, but as assembled. Nor does it simply "fail to dis-identify" separate points which are in a straight line with the eye. It is possible to see the photograph the way it is, but there remains the strong perception of something impossible. The mind completes the picture, it actively identifies points that it cannot distinguish. But to do that, its content must be represented by an inconsistent theory. We describe these below.

It is worth comparing with projective geometry. Perspective presents us with an obviously consistent perception: the railway lines *look* as if they meet. This is an objective phenomenon; if you take a photograph the lines will meet on the surface. Perspective does not look paradoxical. What we know is that the lines do not meet, but this is cognitive at a higher level. It is not part of the content of the percept, it is not "projected onto the percept" as it were. In this respect the content differs from that of the tricorn and other similar figures, which appear paradoxical. In each case, the content is a collection of propositions which form a rigorous theory. Why we should be interested in these theories, however, is because of the presence of the human perceptual apparatus.

4 Heaps and Anti-Heaps

In [4] the author described an inconsistent extension of Penrose's theory. The idea is to express the content of the apparently impossible perception by an inconsistent theory. The simplest example of this is to use the theory of heaps, developed by Meyer, van Bendegem, Priest and others. Heap theory began with heaps of natural numbers. A heap of natural numbers is represented by the counting sequence "One, two, three, heap". That is, all numbers beyond a certain maximum number H (for heap) are identified. There are consistent and inconsistent versions of such theories. The consistent theories add $H = H+1 = H+2 = \ldots$ to the theory while removing $\neg H = H+1$, $\neg H = H+2$, ... etc. The inconsistent theories add $H = H+1\ldots$ while retaining $\neg H = H+1\ldots$ We consider the case for the inconsistent theories below.

There are also heaps of real numbers, where the idea is that all real numbers outside some (open) interval are identified with the respective bounds. Again, inconsistent versions retain the negated identities of classical real number theory. These can be represented by an interval of the real line with arrows outside the heap bounds to indicate identities outside the bounds. It is important to note, though, that these theories lose their character as real number theories because they must sacrifice some of their arithmetical functionality to avoid trivialisation (see Mortensen [3]). But this is no real problem. We are dealing with geometrical structures here rather than arithmetical theories. It would hardly be surprising if one could no longer do the arithmetic on different geometrical structures that one can do on the real line, since their algebraic properties may be radically different.



Heaps

In [4] it was proposed that the impossible triangle could be treated inconsistently by a "backdrop" universe, which is a heap. Assemble as much of Penrose's parts as one can, stick them out from a backdrop, and then draw in the remaining lines on the backdrop. Thus all points behind the backdrop were (inconsistently) identified with its surface. This certainly produces an inconsistent theory of a figure which will look like the triangle. But there is a more general theory using ***antiheaps***.

Antiheaps identify all those points which heaps keep distinct, and disidentify all those points which heaps identify. That is, antiheaps identify all points with a closed interval, and keep the classical disidentities outside.

Antiheap

It is apparent that antiheaps provide a treatment for any impossible picture which can be obtained by photographing a 3-D model, such as Ernst's. Proceed as follows. Describe the actual 3-D model in polar co-ordinates (r, θ, ϕ) with the origin at the eye. Where a interval on a ray is identified by the eye, identify all points between the two bounds on the antiheap of the ray. Note that this is not a full projective geometry, because projective geometry aims to identify all points on a ray, which thus reduces the 3-D theory to a 2-D theory. Here we remain with a 3-D space in which only certain stretches are selectively identified and disidentified in the way that the eye-brain combination treats them. Then extend the diagram of the theory to a full theory using the model theory for any appropriate paraconsistent logic (for details of inconsistent model theory, see [3, Chap. 2])

5 Greap Models

Greaps are an intermediate structure between groups and heaps. They were suggested by a treatment such as Penrose's which proposes that a picture of a possible object is obtained just in case a certain structure forms a heap. We will not discuss these here, for details see [4].

6 Why Inconsistency?

It should be clear that no consistent theory can explain why it is that it seems to be an impossible object. It is proposed here that our sense that we see something impossible, the content of our perception, is represented by an inconsistent theory. The mind has a certain experience, but in addition has expectations which are incompatible, expectations of local Euclideanism, which are overlaid on our percept. These expectations are not removed by revealing the truth about the picture, since they are doubtless hard-wired in by evolution as outputs of the visual module. The mind puts both of these together and the result is represented by an inconsistent theory. Indeed, it is difficult to see what other explanation of the content of the impossible experience there could be.

It is also apparent that George Francis' account using $R^2 \times S^1$ space can be extended in a similar way. In this case, however, the inconsistent identifications required are given not by heaps or antiheaps, but by the mod function, where the modulus is the circumference of the S^1 circle. Construct the consistent space in which the problematic figure can comfortably live. Then extend the space by adding the disequations sufficient to make the space Euclidean. For example, in S^1, the circle, we have $0 = \text{circ}$, $0 + 1 = \text{circ} + 1, \ldots$ where circ is the circumference, but retain the Euclidean output of the visual module by keeping $\neg 0 = \text{circ}, \ldots$ etc.

Note that there is an important sense in which the percept of the inconsistent tricorn is *locally consistent*, only becoming inconsistent when its properties are globally perceived as impossible (I owe this observation to a referee). It is certainly true that the inconsistency owes its presence to an overall *gestalt* of the picture as globally impossible. The present inconsistent extension of Francis reflects this, by not giving a preferred status to certain points or lines as the localised bearers of the inconsistency: all points on the surface of $R^2 \times S^1$ are equally the bearers of the inconsistent identifications which are the consequences of imposing the global R^3 condition.

Note that this approach is a thoroughly *cognitive* justification of the move to an inconsistent theory. There is no suggestion that there really exists an inconsistent object. On the other hand, this is an entirely appropriate application of paraconsistent methods, which falls under the *epistemological* justification of paraconsistency (see eg. [3, Chap. 1]).

7 The Routley and Cross Functors

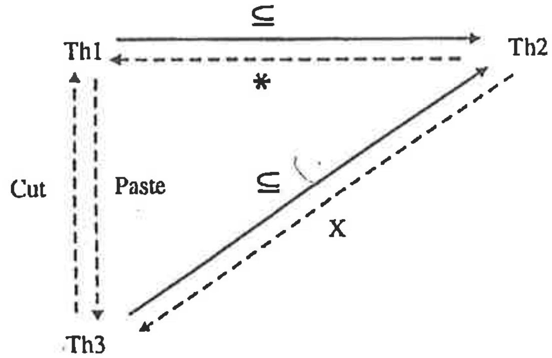
There are two processes involved in forming all such inconsistent theories. (1) First there is a extension of the theory of Euclidean space *Th*1 by adding equations of the form $p_1 = p_2$, where p_1 and p_2 are points. Since we are insisting that this extension preserve the Euclidean character of its origin, we also have $\neg p_1 = p_2$, and thus an inconsistent theory, *Th*2. (2) Second, there is the strictly classical consistent theory *Th*3 which keeps the identifications $p_1 = p_2$, but drops the Euclidean overlay $\neg p_1 = p_2$. This is a general form of the "pasting" method of classical algebraic geometry, so it can be usefully thought of as the Paste functor on the category of theories. Its inverse functor is the Cut functor. The relation between these processes can be given a functorial description, as follows.

We can define two operations on any theory Th:

$$Th^{*} = \mathrm{df}\{A : \neg A \text{ is not in } Th\} \quad \text{(The Routley star operation)}$$
$$Th^{x} = \mathrm{df}\{A : \neg A \text{ is in } Th\}$$

In [3, Chap. 12], it is shown that if we begin with a simple inconsistent theory of closed set logic (where in a simple theory truth-values are just the null set, the whole space and any boundary), then the effect of the Routley operation is to snip out all those sentences A for which both A and $\neg A$ are in the original theory. It is convenient to re-name the Routley star as the Routley functor, since it is also a functor on the category of theories (with morphisms as set-inclusions). We further rename the x operation as the Cross functor.

Now notice that the Routley functor reverses the effect of the process described as (1) above. That is, applied to the inconsistent theory $Th2$, the result is $Th1$, which is to say that (1) is the inverse of the Routley functor. It is also shown that if we begin with a simple inconsistent theory of closed set logic, then the effect of the Cross functor is to excise all odd-numbered strings of negations of atomic sentences A and add all even numbered strings of negations, where the contradictory pair $(A, \neg A)$ are in the theory. So in particular A is retained and $\neg A$ removed. This is the inverse of the process described in (2) above.



Thus we have:

THEOREM The Routley functor forms an adjunction with the Paste functor, with the Routley functor as the left adjoint and the Paste functor as the right adjoint. The Cross functor forms an adjunction with the Cut functor, with the Cross functor as the left adjoint and the Cut functor as the right adjoint. In each case, the unit of the natural transformation is a collection of inclusion morphisms which represent passing from a consistent theory to an inconsistent extension.

In conclusion, there is the obvious "dual" of impossible pictures, namely ambiguous pictures such as the duck-rabbit, Necker Cube etc. In this case, it is indeterminate what it is a picture of, but the ambiguous aspects are individually consistent. It is noted that a corresponding form of the above theorem holds for incomplete theories constructed on open set logic (intuitionism). Each of the two above adjunctions continue to hold, with similar natural inclusion relations holding, save that the inclusions represent passing from an incomplete theory to a complete extension. This will be developed further in later work.

The author wishes to express his thanks in this paper to very useful comments by the editors and several referees, as well as the audience of WCP2000 at beautiful Juquehy Beach, Brazil.

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INCONSISTENT MATHEMATICS

SOME PHILOSOPHICAL IMPLICATIONS

In A.Irvine (ed) *Handbook of the Philosophy of Science*

(forthcoming, 2006)

Inconsistent Mathematics
Some Philosophical Implications

Chris Mortensen

1. Introduction: the Paradoxes.

We begin with the paradoxes. Many puzzles that have been called paradoxes have been discovered. Some of these are trivial, such as the paradox of the Barber. Others are tricky but it is possible to discern a way through them, such as the Unexpected Examination. Others are genuinely profound in their implications. Among these, two groups were distinguished: *semantic* paradoxes such as the Liar and Grelling's; and *set-theoretic* paradoxes such as Russell's and Curry's. In the last quarter of the twentieth century, the semantic paradoxes led Routley and Priest to conclude that some contradictions are true (Priest [1979, 1987], Priest, Routley and Norman [1989]). This view, known as *dialetheism*, was at once highly radical and yet disarmingly simple. To describe it as radical is to állude to its reception among the body of contemporary philosophers, the large majority of whom still regard it as extreme. To describe it as simple is to allude to the appeal to simplicity in support: alternative solutions to the Liar, such as Tarski's hierarchy of languages, look unsimple by comparison. A similar observation can be made about the set-theoretic paradoxes: naïve set theory with unrestricted comprehension is simple and natural in comparison with contrived patch-ups such as Zermelo-Frankel set theory or Russell's theory of types.

The present essay is not about the semantic paradoxes, and not so much about the set-theoretic paradoxes either. Nonetheless, the example of the paradoxes hopefully softens the reader up for two points. The first point is that the idea that some contradictions might be true has considerable antiquity. Routley and Priest were in a long tradition. Some of the Ancient Greeks, notably Herakleitos and the author of the *Dissoi Logoi*, seem to have taken dialetheism seriously; and this generated a Western tradition which extends to Hegel, Marx and Engels. In the Eastern tradition there have been the *Tao-te-Ching*, Chan Buddhism in China, and Zen in Japan. The second point is that if dialetheism is true then any logic which validates the classical law Ex Contradictione Quodlibet (ECQ), *from a contradiction any conclusion may be validly deduced,* cannot be entirely correct as a description of universal principles of reasoning. This conclusion is supported by the evident artificiality of ECQ. A logic in which ECQ fails is known as *paraconsistent*, or inconsistency-tolerant.

The effect of the set-theoretic paradoxes on the nature of mathematics is conditioned by the question of *foundationalism*. If mathematics has a foundation, then set theory is a good candidate. Frege and Russell's logicist program had two pillars: that mathematics has a foundation, which is set theory, and that set theory in turn reduces to logic. If natural set theory is inconsistent then this seems to weaken the first pillar. It also seems to weaken foundationalism generally, if no better foundation can be found. In passing, it cannot be ruled out *apriori* that some other field of mathematics, such as category theory, might serve as a better foundation for mathematics than set theory. However, it seems clear that category theory employs similar strong comprehension-like principles to those of set theory, and so has similar problems with consistency (see Hatcher [1982])

If consistent set theory is bought only at the cost of unsimple and artificial principles which do not look much like principles of logic or definition, then, as Russell realised, the second pillar of logicism falls too. However, what Frege and Russell did not envisage is the possibility of accepting the contradictions outright. Set-theoretic foundationalism might survive, and both pillars of logicism with it, if the alleged contradictions caused by an unrestricted comprehension principle were restricted to regions where little or no damage to mathematics ensues. The barrier is of course ECQ, but we have just been seeing independent reasons for rejecting that. Hence we can register a preliminary conclusion: foundationalism and logicism might be salvageable if contradictions which are true-in-mathematics are tolerated, and ECQ abandoned. Nonetheless, we will later see different reasons for rejecting both foundationalism and logicism.

The barrier that ECQ erects against liberated thinking can be described in another way. It is the idea that once a contradiction presents itself as proved in a theory, then reasoning with that theory must cease. Distinctions between different inconsistencies are impossible because any attempt to describe their structure dissolves into any other attempt. It is the doctrine that *the inconsistent has no structure*. Such a view, if true, would immediately ruin any attempt to develop a Theory of Inconsistency. This essay aims to refute that view.

2. *The Role of Logic.*

It might help the reader to begin by setting aside the Platonist question of what kind of an object, mathematical or otherwise, could possibly have inconsistent properties. In its place, it is recommended to put the primacy of the proposition. Mathematical texts and lectures do not present abstract objects for transcendental scrutiny. They begin with assertions. Certainly, mathematical texts employ also diagrams. But mathematical texts, where they use diagrams, make assertions about them from the start. It follows that we should be less inclined to ask how could an inconsistent proposition be true-in-mathematics. Rather we should be more inclined to wonder where that might lead. Perhaps later might come an appreciation of mathematical objects with inconsistent properties, as the truthmakers for preferred mathematical propositions, and a basis for model theory. But this metaphysical extra is certainly not necessary to make a beginning with.

Hence, our starting point is collections of propositions. More precisely, if we are to study structure, we must deal with mathematical *theories*, that is sets of propositions closed under a deducibility relation. Deducibility relations are characteristic of logics; and it is well-known that there are many deducibility relations, since there are many logics. Hence the discussion has to be generalised to *L-theories,* that is theories of a logic (or deducibility relation) *L*. An L-theory *Th* is *inconsistent* iff for some proposition A both A ε *Th* and ~A ε *Th,* where ~ represents the symbol for negation (there are other symbols for special kinds of negations). *Th* is *incomplete* iff for some A neither A ε *Th* nor ~A ε *Th*. *Th* is *trivial* iff *Th* is the whole language, *ie. Th* contains every proposition; otherwise *Th* is *nontrivial*. The members of any L-theory are also called its *theorems*, and are said to *hold* in the theory.

In the end it will be desirable to suppress the logical apparatus provided by *L* as much as possible. However, for the present, consideration of logic is forced upon us by the logical principle ECQ itself, which if correct would ensure that there is just one inconsistent theory, the trivial theory. This in turn would prevent any distinctions between kinds of inconsistency, between inconsistent mathematical structures. But at this point we are able to exercise some *free choice*: we can *decide* to countenance mathematical theories of logics for which ECQ fails. If there are none, then *invent* them. There are plenty of paraconsistent logics around to supply adequate logical apparatus. Thus there is a sense in which classical logic, regarded as the logic of mathematics, is *made false* by the existence of inconsistent mathematical theories. To paraphrase Marx, philosophers have hitherto attempted to understand the nature of contradiction, the point however is to change it.

Given a logic, there are two ways to construct theories of that logic: by axioms or by models. The first intentionally inconsistent arithmetical theory was Robert K. Meyer's RM3(mod 2), which was specified by a model. Its background logic was the paraconsistent 3-valued logic RM3. The theory RM3(mod 2) was inconsistent because both 0=2 and ~(0=2) were theorems. However, Meyer constructed this theory because he wanted to study the relevant arithmetic R#, which is axiomatically constructed. The logic for R# is Anderson and Belnap's quantified relevant logic R, axiomatically presented. R# is then given by taking the classical axioms for Peano Arithmetic, replacing their classical implication connectives ⊃ by the implication connective → of R, and closing under the deducibility relation for R. There is no suggestion that R# is inconsistent. However, by virtue of Meyer's result that R# ⊆ RM3(mod 2), it follows that R# can have 0=2 added as an axiom, the result being an inconsistent axiomatically-presented arithmetic which is nontrivial. Indeed, RM3(mod 2) itself has an axiomatic presentation: to R# add 0=2 together with all instances of the propositional axiom Mingle, A→(A→A). See Meyer [1976], Meyer and Mortensen [1984], Mortensen [1995].

The question can be asked: given that there are many paraconsistent logics, which is "best" for inconsistent mathematics? The answer that emerged was that it doesn't much matter which: the properties of inconsistent theories tend to be invariant over a large class of background logics. To be more exact, when theories are specified by means of models, their logical properties tend to take second place behind *mathematical calculations which are performed at the sub-atomic level* (sub-atomic relative to the atoms of logic, that is). This suggests an important idea: that mathematics is after all different from logic since logic deals with the *general* properties of propositions, predicates and identity, while mathematics deals with calculations in *particular kinds* of structures. We will be developing this theme as we proceed.

Even so, there is one paraconsistent logic which is particularly natural: closed set logic. It is well known that intuitionist logic is the logic of open sets; closed set logic is its topological dual. For many familiar logics, such as tense and modal logics, we can think of propositions as indexed by sets of points in an appropriate space, such as a set of times, or a set of possible worlds, or a phase-space. This idea can then be extended by supposing that the index set has a topological structure. If we make the stipulation that *propositions only ever hold on open sets of points,* we obtain open set logic. It is not difficult to then think of the disjunction of two propositions as holding on the union of the sets on which they hold, and conjunction as holding on the intersection. Considering negation however, it is apparent that the negation of a proposition A cannot hold on the set-theoretic complement of set of points on which A holds, since the set-theoretic complement of an open set is not in general open. It is thus customary to take for negation *the largest open set contained in the set-theoretic complement.* We can then see the familiar intuitionist property of negation emerging: *at the boundary neither A nor not-A holds.* Theories of open set logic may thus be incomplete. It is widely acknowledged that this is a natural-sounding semantics.

Applying the topological open-closed duality, we must have closed set logic. Closed set logic is the stipulation that *whatever holds, holds on closed sets of points.* The interesting case is negation. It is thus customary to take *the smallest closed set containing the set-theoretic complement.* We then have the familiar paraconsistent property of negation emerging: *at the boundary both A and not-A holds.* It is apparent that this is an equally natural semantics to that of open set logic. It is one in which ECQ fails and which supports inconsistent theories. This is as natural as the natural transformation: open ↔ closed.

Unfortunately, it was soon found that inconsistency can spread for reasons other than ECQ. Curry's paradox generates triviality for naïve set theory even in the absence of negation, even in the absence of ECQ (see *eg.* Meyer, Routley and Dunn [1979]). All that is necessary is the logical law of Contraction $(A\to(A\to B))\to(A\to B)$ as well as Modus Ponens, Simplification and Universal Instantiation. Indeed even weaker principles suffice, as shown by Slaney [1989] and Rogerson [2000]. Thus we must not live in a fool's paradise when constructing inconsistent theories *axiomatically.* Maybe some variant of Curry's paradox can jump up and bite us as a consequence of our axioms, even if we are sure that ECQ cannot hurt our theories.

Nevertheless, we have a guarantee from *model theory* that the spread of contradictions can be stopped short of triviality, at least for naïve set theory. This was shown by Brady very early on [1971], using a model-theoretic fixed-point method derived from Gilmore [1967]. Similar work was done independently by Da Costa (see *eg.* [1974)]. The importance of this kind of result cannot be stressed enough. Brady demonstrated nontriviality in the presence of the Russell Set and the Curry set, so by brute force, *whatever* logical principles have to fail for these sets not to lead to explosion, *must* fail in Brady's construction. In a further parallel to Meyer, Brady developed the method in later papers to show that classically false ordinal equations are not provable in naïve set theory either.(see [1989]). Thus, just as in arithmetic, the contradictions in naïve set theory are far away and contained, and do not interfere with serious mathematical calculations.

Hence, problems for inconsistency arising from *logic* are not insurmountable. But this is far from being an end to it. Dunn pointed out that if any classically false equation was added to *real number theory*, then every equation became provable. The proof of this is elementary algebra: from $a=b$, where a and b are distinct real numbers, we can subtract a from both sides to get $0=(b-a)$. Each side may then be multiplied by any number we please to get $0=r$ for any real number r. Hence by the principle of the substitutivity of identicals, every real number equals every other.

We can coin the term *mathematically trivial* for any (mathematical) theory all of whose (logically) *atomic* propositions are theorems. Now mathematical triviality implies full triviality in the presence of the rule ECQ. But in general it does not do so. Yet it is mathematical triviality that is catastrophic for mathematics: no calculation would mean anything. And in Dunn's argument we have an example where mathematical triviality is spread by principles other than ECQ or anything else from pure logic. Conversely, if calculations are possible at all, then it is nothing short of crude classical hegemony to insist that a detour through mere *logical* principles such as ECQ ought to render the theory useless for this purpose or any other.

Correspondingly, we can define a theory to be *transparent* if it permits full substitutivity of identicals; that is, if $t_1=t_2$ holds then Ft_1 holds iff Ft_2 holds, where F is any context. A theory is *functional* if substitutivity of identicals is restricted to *logically atomic* contexts; that is, F is any atomic context. In theories of classical logic, functionality implies transparency, but this is not so in the general case. Furthermore, Dunn's argument requires no more than functionality to work. But now we can see that it is functionality that matters more for mathematics than transparency, since functionality is what ensures that calculations can proceed. Failure of substitutivity because of logic is not such a weighty matter, while both functionality and its failure are of greater moment for mathematics.

3. Pure Mathematics

It is impossible in this brief account to survey all the results of inconsistent mathematics. However, some broad outlines can be touched on. The study has tended to concentrate on techniques from model theory rather than axiomatics, and we will take that approach here.Thus we begin with a first-order language containing (i) names for mathematical objects, such as the natural numbers, integers, real numbers, sets, topological spaces; (ii) term-forming operations on these objects, such as $+,\times,-,\div,'$ (successor) ; (iii) atomic predicates and relations, such as $=,\subseteq,\in$ (iv) logical operations such as $\&,\vee,\sim,\supset,\rightarrow,\leftrightarrow,\forall,\exists$. Well-formed formulae are defined in the usual way. A *model* is a triple $\langle D,L,I\rangle$, where D is a domain of mathematical objects, L is a many-valued logic, some of whose values are designated and the others undesignated, and I is an interpretation which maps names to elements of the domain, term-forming operators to (partial) operators on the domain, predicates to subsets of the domain, n-ary relations to subsets of D^n, and wffs to the values of L in accordance with the interpretations of parts of the wff to the domain or other values respectively. The *theory* associated with the model is then formed by taking the all those wffs of the model which take a designated value in the interpretation.

One device worth mentioning is the use of *extensions* and *anti-extensions* for each predicate and n-ary relation. The idea, due to Dunn and used by Priest, is that the extension and anti-extension of a predicate can overlap and in that case the predicate is counted as both true and false of those objects. However it is not necessary to use this device, and it is less than fully general when a logic having numerous values is being used. The reader is cautioned at this stage from taking models with too much ontological seriousness. Models are to be regarded in the first instance as devices for controlling the membership of theories. Notice also in passing the implied distinction between mathematics and logic in that, with the exception of = and perhaps ∈, logic proper only enters under (iv).

To take an example, consider the language to contain names for all natural numbers (perhaps constructed in the usual way from 0 and the successor operation), arithmetical operations +,×,′, and a single binary relation =. Let the domain D be the natural numbers modulo 2, and the logic *L* be the 3-valued paraconsistent logic RM3, with values {T,B,F} where T and B are designated values (B is understood as "both"). Interpret names for the natural numbers as their counterparts mod 2 and term-forming operators as their corresponding operators mod 2. Atomic sentences $t_1=t_2$ are interpreted as taking the value B if t_1 mod 2 = t_2 mod 2, otherwise $t_1=t_2$ is interpreted as taking the value F. The set of sentences taking either of the designated values {T,B} is Meyer's theory RM3(mod 2). The theory is inconsistent since the equation $0=2$ takes the value B while $\sim(0=2)$ takes T. Meyer then proved that relevant arithmetic R# ⊆ RM3(mod 2), which was the basis for his finitary nontriviality proof for R#, see Meyer [1976]. It is obvious that Meyer's construction can be modified to produce RM3(mod n) for any number n. Since R# is contained in any of these, we can also see that no classically false equation $t_1=t_2$ can be proved in R#. See Meyer and Mortensen [1984].

Meyer's proof that R# ⊆ RM3(mod 2) was finitary in Hilbert's sense, in that it relied solely on ordinary mathematical induction over the length of formulae. Since by inspection RM3(mod 2) is nontrivial, it follows that R# can be shown to be non-trivial by finitary means. By contrast, it follows from Godel's incompleteness theorems that there is no finitary proof of the non-triviality (equivalently, consistency) of *classical* Peano arithmetic. This was viewed with great pessimism by Hilbert, who felt that it spelt the end of his program to demonstrate the consistency of mathematics by finitary means. However, Meyer concluded that Hilbert's pessimism is unfounded, as long as we cast aside the shackles of classical logic and ECQ. A further corollary of Meyer's result was not merely that the explosive spread of contradiction in relevant arithmetic is prevented, but that *no false atomic propositions (equations) can be proved in R#.* Thus *calculation is* *untouched by contradiction in relevant arithmetic*. This is then a further important consequence for the philosophy of mathematics. We saw earlier that logicism might be rehabilitated from Russell's paradox by retaining naïve comprehension, as long as ECQ fails. Now we see that the Hilbert program similarly has excellent prospects for rehabilitation in logics in which ECQ fails. These include logics only slightly weaker than classical logic.

It is fairly easy to show that *extensional* part of R# (with logical operators &,∨,~,⊃,≡,∃,∀,=, but lacking intensional operators →,↔) is a subset of classical Peano arithmetic PA.There was for a time the hope that they coincided exactly. This would of course imply the non-triviality of PA, and hence its consistency. That would not of course violate Godel's second incompleteness theorem, since there is no suggestion that the proof method itself be representable in classical arithmetic. But it would be a new proof all the same, perhaps using quite different techniques from the usual. It was eventually discovered by Meyer, adapting Friedman, that R# is strictly weaker than PA. (Meyer-Friedman [1992]). This dashed the hopes of a consistency proof for PA. Meyer himself expressed pessimism that R# was thereby shown to be less than adequate for arithmetic, since there are true extensional propositions unprovable in R#. But it seems that this makes R# all the more interesting: a genuine rival to PA in which all calculations can be performed; and in which, moreover, all primitive recursive functions are representable so that the incompleteness theorems apply. Moreover, it is hardly something that adherents to classical PA can rejoice in, since they, too, have had to live with the incompleteness theorems ever since they were proved: what is the Godel sentence if not a true-but-unprovable statement?

The class of all mod models, for varying modulus n, has various interesting properties. Its intersection RMω has the property that its counter-theorems are recursively enumerable, but it is not known whether it is recursive or not. There are also *non-standard* mod models (see Mortensen 1987, 1995). Recently, Priest [1997, 2000] has completely characterised the class of mod models, that is, he showed that all mod models take a certain form.

Of interest is the case of RM3(mod p) where p is prime, since it is known that the natural numbers mod p form a *field*; that is, division is well-defined. This raises the question of where Dunn's proof of triviality for the inconsistent real number field breaks down in mod p. The answer is that in an inconsistent mod arithmetic the equation $a=b$ holds only if the classical difference between a and b differ by an integral multiple of the modulus. Multiplying or dividing both sides by the same integral number does not disturb that, so the inconsistency does not spread everywhere.

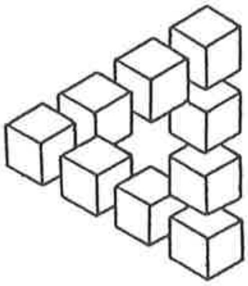
It is well known that in the history of the calculus debate raged about whether one should take seriously the use of "very small" real numbers. By the early nineteenth century it seemed that disputes over the status of infinitesimals were resolved in favour of real numbers alone by means of the Cauchy-Weierstrass (ε, δ) technique, which quickly became the orthodoxy in mathematics departments. By 1960 however, Robinson revived infinitesimals by showing rigorously that one could develop calculus just as well with them, and that calculus based on infinitesimals is in various ways simpler to manipulate. (See Robinson [1966].) Now it is notorious that in working out derivatives Newton opportunistically divided by very small numbers, yet set them to zero when it was convenient to ignore them. Perhaps then one might be able to make them inconsistently both equal to zero and not equal to zero? However, the prospect that inconsistency in the real numbers spreads uncontrollably into triviality poses an obvious problem for developing inconsistent differential and integral calculus, and resorting to infinitesimals does not offer an obvious relief since the mathematical triviality proof goes over immediately to a mathematical triviality proof in the hyperreal field.

One way to avoid this is to take as one's domain something with a little less than the full structure of fields. This is accomplished by beginning with the *noninfinite hyperreal numbers*, that is the finite real numbers together with the infinitesimals. Selecting an infinitesimal number η, define $a \approx b$ to mean that $(a-b)/\eta$ is infinitesimal or zero. One may then prove that the equivalence classes so generated form a *ring* under the induced operations. This ring serves as the domain for an inconsistent model. Taking RM3 as background logic as before, set $I(t_1=t_2)$ to be T if $t_1=t_2$ as real numbers, set $I(t_1=t_2)$ to be B if t_1 and t_2 are distinct real numbers but $[t_1]=[t_2]$, and set $I(t_1=t_2)$ to be F otherwise. Then it is easy to see that both $\eta^2=0$ and $\sim(\eta^2=0)$ hold, whereas η itself is consistently non-zero. The prospect that infinitesimals smaller that a certain level in size (*ie.* infinitesimals which are even infinitesimal w.r.t. η) can be equated with zero, allows calculations in which they can be ignored, even though their "effects" remain in that division by them remains in various contexts. Differentiation and integration can be developed, and Taylor's theorem and the fundamental theorem of the calculus can all be proved.

There is more to be said about results from pure mathematics than this. Analysis, topology and category theory have all been studied. For an extended discussion, see Mortensen [1995, 2000, 2002a]. However, we move now proceed with our survey by turning to make some brief remarks on geometry.

4. Geometry

Consider the picture below.



There are many others. It is notable that the beginnings of inconsistent mathematics avoided dealing with such pictorial puzzles, though now the situation is slowly being remedied. Interestingly, classical mathematics has also largely avoided dealing with them. In the classical mathematical literature there were to be found three approaches. The first, due to Thaddeus Cowan [1974], studied *n*-sided figures in terms of the properties of their corners, employing the theory of *braids.* Second, George Francis [1987] asked what sort of *consistent non-Euclidean* space could be inhabited by such objects. The answer, for the above figure, is $R^2 \times S^1$. Third, Roger Penrose [1991] used the theory of *cohomology groups* to obtain necessary and sufficient conditions for a picture to be of a consistent object; *a fortiori* the failure of those conditions would mean that the picture was of an inconsistent object. See also Penrose and Penrose [1958].

These were unquestionably all very perceptive approaches. However, as argued by the present writer in [1997b, 2002b, 2002c], they all shared a common deficit: *they did not explain the sense we have that we are perceiving an object with impossible properties.* This suggests a different conception of the problem, namely to think of the brain as encoding *an inconsistent geometrical theory.* The problem would then become to write out such a theory (or rather theories, for there are many different impossible pictures with different properties). The theory in question would stand to the pictures in somewhat the way that projective geometry stands to the experience of perspective; and with somewhat the same justification, namely that projective geometry is important to us because of the experience of having an eye.

This kind of justification of the study of inconsistency has been described as the *epistemic* or *cognitive* justification. Such justifications appeal to a human capacity, typically the capacity to reason in a logically-anomalous environment, without intellectual collapse into triviality. There is of course no suggestion that inconsistent objects exist in the physical world. Rather it is that our perceptions construct a geometrical theory while at the same time retaining geometrical principles which are incompatible with it. It seems that in inconsistent pictures we have a clear example of the mind's ability to make constructions which are inconsistent and yet persist even when the impossibility is manifest to us. The lack of cognitive penetrability of the experiences is characteristic of the *modularity* of perceptual capacities which has been noted by various authors, *eg* Fodor [1983].

The details of such mathematical theories are still in an early stage of development. The interested reader is invited to consult the above references for further elucidation.

5. Applied Mathematics

A good antidote to the error that mathematics develops in pristine logical order is to read the works of Imre Lakatos [1976]. It is particularly in applied mathematics, physics and engineering where mathematical opportunism is most apparent. Here the lack of classical rigor comes with applications built-in. Hence we can ask, as with the historical disputes over infinitesimals, whether the "logically erroneous" theory might be more accurately described as an inconsistent theory rich enough to permit useful calculations.

A good example is Dirac's Delta "function", $\delta(x)$. This had the twin properties: (i) $\delta(x) = 0$ for all $x \neq 0$, and (ii) $\int \delta(x)dx = 1$, where the integration was over the whole real line. It is apparent that there is no such function on the real numbers. Yet Dirac perceived a use for it in his version of Quantum Mechanics. In this he was followed by many of the physics community. Quantum theory developed rapidly and decisively. It was not for some forty years that Laurent Schwartz managed to put things on a consistent footing by using functionals rather than functions. There was a significant cost, however, in that the new theory was considerably more complicated. There is a fairly obvious construction for the Delta function which uses infinitesimals: draw a triangle of infinitesimal base β and infinite height $2/\beta$. This satisfies something close to the condition (i), namely $\delta(x) = 0$ for all *real* $x \neq 0$; and clearly the second condition is satisfied since the area of the triangle is 1. This was not Robinson's construction, however, since it requires second-order principles; whereas Robinson restricted himself to first-order conditions, so that his theory amounted pretty much to a copy of

Schwartz'. It turns out, however, that there is an inconsistent theory which adapts the construction above of inconsistent infinitesimals, and which has the property that $\delta(x) = 0$ for all x for which $x = 0$ fails to hold. Since it is the propositions that hold that are relevant to property of functionality, we can say that the construction recovers the concept of a function, albeit an inconsistent function.

It is hardly surprising that Quantum Mechanics lends itself to inconsistent applications, since QM has long been regarded as a source of anomaly and paradox. One more application in this area is quantum measurement. In cases where an operator has a discrete spectrum, such as the energy levels of the hydrogen atom, elementary QM postulates discontinuous changes in the wave function when a measurement is made. Now discontinuity is an enemy of causality: it would be desirable to have a theory in which quantum measurement was reducible to the other familiar quantum process of unitary evolution. This is the *measurement problem*, and it is fair to say that the measurement problem remains unsolved, and is even intensified given the problem of nonlocality, Bell's theorem and Aspect's experiments. An approach using *inconsistent continuous functions* seems to allow both for continuity/causality and at the same time discrete spectra. For more details, see Mortensen [1997a]

The cognitive justification of paraconsistency, discussed before, is apparent in the application to information systems. Nuel Belnap [1977] famously pointed out that any control system with more than one stream of informational inputs, must allow for the possibility that its inputs may be in conflict. Furthermore, it may be impossible to shut the system down until the problem is resolved, as with an aircraft aloft. Thus there has to be a way of operating in an anomalous informational environment, which is after all what we humans manage to do. One theory taking this approach considers the problem of solving *inconsistent systems of linear equations*. Inconsistent systems of linear equations have been known about for centuries, and the standard mathematical reaction has been to throw up the hands in despair. However, it proves possible to solve some such systems of equations in an inconsistent space. Now the classical theory of *control systems* makes heavy use of systems of linear equations. This in turn suggests that if one were able to model a malfunctioning control system in terms of an inconsistent system of linear equations, there might be a way of continuing to exercise some limited control. The modelling proved not to be so difficult. According to classical control theory, when a system is functioning correctly, its internal organisation is modelled by a *transfer matrix*, which transforms a (column) vector of inputs into a vector of outputs. When the system is malfunctioning, there is a difference between the *expected* outputs and the *observed* outputs. By superimposing the observed outputs onto the expected transfer matrix, one obtains an inconsistent system of equations which can then be solved. In software modellings this has met with some limited success. A related approach has been taken by the Brazilian group around Abe [2000], who have demonstrated a paraconsistent robot, Emmy.

It should be noted that it is not being claimed here that the control system is behaving inconsistently in the real world. It is rather that the discrepancy between expected and observed creates an epistemological gap that has to be resolved. Calculations take place in a virtual space in which all the information available is used to form a composite picture with the aim of continuing to function until proper knowledge and control can be fully restored.

A final point to be noted is the shift in ontology that takes place between pure mathematics and applied mathematics. In rejecting Platonism, we were rejecting abstract truthmakers for pure mathematics. The truthmakers for applied mathematics, one would imagine, are its applications. These involve systems of physical objects and their physical quantities, the kinds of things which are causally active, changing and producing change. Physical quantities, such as 5 gram, 2 cm, 3 sec, come as a package of a number ("5") and a quantity kind or dimension ("gram"). In the present writer's view, the best account of quantities treats them as causally relevant universals. Laws of nature come out as relations between universals, see Armstrong [1978]. Real numbers then emerge fairly unproblematically as ratios (*ie.* relations of comparison) between dimensioned quantities having the same dimension, see Forrest and Armstrong [1987], Bigelow [1988], Mortensen [1998]. It is not proposed to develop this account here, the reader is directed to these references. The point being made is that there is not necessarily an equivalence between the problem of the truthmakers for pure mathematics, and truthmakers for applied mathematics. The harder problem seems to be for pure mathematics, while applied mathematics looks rather more tractable.

6. Logicism and Foundationalism Revisited

With this all-too-sketchy survey of what is known to date, we return to our flirtatious quarrel with logicism. The foregoing suggests that we can draw a (rough) line between logics and mathematics in the kinds of reasonings they employ. Logics deal with universally applicable principles of reasoning, centrally ($\vdash$, $\models$,&,$\vee$,~,$\rightarrow$,$\leftrightarrow$,$\exists$,$\forall$,=,) and other constructions arising in natural language (*eg.* tense, modality, adverbs). Logic applies to mathematical reasoning, certainly, but it applies to that aspect of mathematical reasoning that applies to other subject matter as well. In contrast, mathematics *distinctively* deals with concepts like those of algebra, calculus, differential equations, analysis and geometry. Somewhere in the middle between logic and mathematics lie set theory, number theory, recursion theory and parts of algebra. In the case of algebra, logicians' interests have tended to be confined to structures which can supply a plausible semantics for various sets of logical axioms, such as lattices. With only a few exceptions, logicians have not been much interested in groups, for example. This leads to the challenge to logicism: in what sense is mathematics no more than logic with definitions? *It all depends on which definitions.*

Mathematicians tend to be anti-foundationalist. The previous challenge can also be directed against foundationalism, and it explains why mathematicians have not taken logic's attempts at hegemony too seriously. Claims like "set theory is a foundation for mathematics" or "mathematics reduces to logic" look like they are saying that *all there is to mathematics* is set theory or logic. But this is precisely to suppress what is *distinctive* about mathematics. They give a false sense of what is the *nature* of mathematics.

The point can be further illustrated by considering the "reduction" of geometry to algebra. It is uncontestable that Descartes' discovery of the coordinatisation of the plane enabled an immense step forward in geometry. The methods of algebra could then be applied to the study of the plane. Space could be studied by solving equations involving real numbers and their functions. Nonetheless, it is a mistake to take this as implying that geometry is *nothing but* real number theory, as Russell seems to have thought (see *eg.* Ayer [1972] p43). *The two-dimensional plane is not* R^2*; space is not a collection of numbers.* Its parts are points, lines, curves, and planes, not sets or real numbers. We need only pay attention to our own perceptions of space to see this: we perceive areas, lines etc, we do not perceive numbers. In short, there is no *conceptual* equivalence possible between geometry and set theory. This is why a mathematician can pursue the study of space paying little or no attention to foundations: mathematics has a *conceptual autonomy* that foundations cannot supply.

From this point of view, the gap between mathematics and logic is even wider than that between mathematics and real numbers and set theory. Logicians study "and", "or" "not", "implies" and the like. Their discipline begins where mathematics leaves off in studying the behaviour of geometry, groups and the like. This makes logic look more like a small area in the corpus of mathematics, rather than a foundation for it. Furthermore, it exposes ECQ for what it is: a tool in a takeover bid to establish the hegemony of logic over mathematics.

As a piece of personal reportage I recall years ago explaining to a visiting eminent mathematician why I was inclined to reject ECQ. After listening politely, he asked: "Excuse me, but are you not denying that the null set is a subset of every set?" This confusion embodies a subtle reversal, but it is no better motivated. We may be inclined to make a limited "reduction" of set theory to logic by adopting naïve set theory and claiming that there is nothing to set theory but logic. Naïve comprehension would then be an expression of the reduction. In favour it can be said that it is certainly less *ad hoc* than rival comprehension principles. However, our eminent mathematician was reversing the order of explanation: he felt that the principles of set theory were *sui generis* and that the legitimacy of ECQ was ensured by that!

7. Revisionism and Duality

Earlier, we referred to the topological duality between incomplete theories of open set logic, and inconsistent theories of closed set logic. There is another kind of duality, Routley-* duality. This applies between theories of logics in which the laws of Double Negation $A\leftrightarrow{\sim}{\sim}A$ and De Morgan ${\sim}(A\vee B)\leftrightarrow({\sim}A\&{\sim}B)$ and ${\sim}(A\&B)\leftrightarrow({\sim}A\vee{\sim}B)$ hold. Neither open set logic nor closed set logic has these laws unrestrictedly, however many of the logics in the Anderson-Belnap class of relevant logics have them. For any set of sentences S, define S* to be $\{A: {\sim}A \notin S\}$. Then a simple argument shows that if *Th* is any theory of a logic containing Double Negation and De Morgan, then *Th* is inconsistent iff *Th** is incomplete. Since DN ensures that *Th*** = *Th*, we also have that *Th* is incomplete iff *Th** is inconsistent.

That is, incompleteness and inconsistency as properties of theories are duals of one another in *two* senses: they are topological duals of one another, and they are Routley-* duals of one another. Duality results are of course sources of "theorems for free". As a quick illustration of free theorems, we note a dualisation of Kripke's modelling of the truth predicate in an incomplete theory. Kripke [1975] showed, using a fixed point method deriving from Gilmore [1967] and Brady [1971], that the Liar proposition L and its negation are excluded from a theory satisfying the condition for a truth predicate: $A\leftrightarrow T(A)$ where A is any proposition and T (.) is the truth predicate for the name (Godel number) of A. Kripke interpreted this as showing that the Liar proposition L ought to be regarded as neither true nor false. However, applying the Routley-* to the truth theory, we can immediately conclude that there is a theory satisfying the conditions for a truth theory to which both L and ~L belong. We might also observe that the inconsistent dual theory has certain advantages over the incomplete theory, namely that in Kripke's theory we have the oddity that none of L, ~L, T(L) and T(~L) receive a truth value even though $L\leftrightarrow T(L)$ and ${\sim}L\leftrightarrow T({\sim}L)$ hold in the theory. Note that while Kripke employed a third logical value in his construction, he was clear that this was a formal device for calculation only, and that he regarded the liar sentence as lacking a value. This is perfectly reasonable as a proof device, however it seems strange that a valueless proposition can yet contribute to making a compound hold. In contrast, in the inconsistent dual, all of L,~L,T(L) and T(~L) take contradictory values; which is at least some reason to hold that L↔T(L) does too.

Intuitionism and constructivism are examples of *revisionist* philosophies of mathematics, in that they declare that certain principles accepted in classical mathematics are incorrect. They aim to revise mathematics by truncating it, based on a narrower conception of what is an acceptable proof. However, revisionism leaves unanswered an important question: *why are the excluded areas yet mathematics*? In their haste to offer a theory of correct proof, revisionists neglect the central question of the philosophy of mathematics: what is mathematics? This is hardly to be answered adequately by declaring those parts of mathematics that the theorists don't like, not to be mathematics at all.

The classical Hilbertian ideal of a mathematical theory is one which is complete and demonstrably consistent. Revisionist theories, by excluding aspects of classical theories, render themselves incomplete, a fact which has been long-noted in connection with intuitionism. By contrast, inconsistent mathematics is not revisionist at all. Taking a lead from the duality results, it aims to extend mathematics, not weaken it. The duals of incomplete theories are inconsistent, and they include classical consistent complete theories as subtheories, and consistent incomplete theories as sub-sub-theories. Thus inconsistent mathematics supports a principle of tolerance about what counts as mathematics, an inclusive approach not an exclusive one. Both classical mathematics and revisionist mathematics emerge as *special cases* of a more generalised conception of mathematics, which includes inconsistent mathematics as well.

8. The Role of Text

One further matter needs to be raised, though dealing with it fully would take much more space than we have here. If we ask what makes all of the above examples mathematics, it is apparent that the answer must have something central to do with the characteristic use of notation or symbols. That is to say, mathematics is *textually distinctive*. Importantly, this is something it shares with symbolic logic. It is apparent that the rise of symbolic logic in the twentieth century is attributable to its use of mathematical text. The question is: just how is it that this has been so efficacious? This dovetails with the broader question of just why it is that the distinctive textual features of *any* mathematics do their jobs so well? We are all familiar with examples like the advantage of the change from Roman numerals to Arabic numerals: it is clear that this is a microcosm of the general question of the distinctive nature and efficacy of mathematical text. There is something important to be explained about how mathematical meaning is carried by text.

There is another observation which is a kind of converse to this one. In his University of Adelaide PhD thesis *TheRole of Notation in Mathematics* [1988], Edwin Coleman pointed out the *varieties* of mathematical text. He drew attention the differences between a page from Euclid, a page from *Principia Mathematica*, a page from a text on business mathematics, a page from a standard calculus text, and a page from a mechanical engineering text. Consider for example the varying role of diagrams, and the presence or absence of natural language. The differences are richly textual, and yet the very stuff of mathematics. Thus, the question of the usefulness of distinctive *texts* in mathematics, is part of the question of how mathematical text generates meaning. It is the *interplay* of similarity and difference that needs to be understood.

Coleman argued that the right discipline to undertake such a study was the theory of signs, *semiotics*. Co-discovered by Peirce and Saussure, semiotics aims to study how text and other signs generate meaning. Saussure in particular had to rely somewhat more heavily on the internal *differences* within a code or system of signs, because, unlike Peirce, his account lacked a theory of extra-linguistic reference. While this is an obvious drawback in any general account of language, it can be seized on by (we) anti-Platonists as just right for any account of *mathematical* meaning, where (according to us) there are no abstract objects to be the referents. This is nothing but an application of Saussure's concept of *difference*. Ockham's Razor does the rest against Platonism. A certain amount of literature which addresses these issues in the indicated ways has grown up, including Nelson Goodman [1981], Rene Thom [1980], Brian Rotman [1987, 1990,], Coleman [1988, 1990, 1992], and Mortensen and Roberts [1997].

We saw earlier that Meyer's nontriviality result serves fit to re-habilitate Hilbert's program of demonstrating that mathematics does not have false consequences. But there are problems for Hilbert's program of a different sort here. Drawing on the above, Coleman attacked Hilbert's formalism. Like Brouwer, Hilbert gave way to the despair of revisionism. In order to demonstrate mathematics to be consistent and complete, or at least without error, mathematical theories must be displayed in canonical form, as formal systems, purely symbolic and devoid of all meaning (save that generated internally). But here, as with revisionisms anywhere, we can again ask *why are the uncanonical parts yet mathematics?* Don't get me wrong. I am certainly not against reconstruction of a theory as a first order formal theory, if only because then you could automate it! But notice that in producing an "equivalent" formal theory we are suppressing a difference that is part of what has to be explained: in what sense can notationally distinct codes be equivalent, and how can textual features contribute to distinctness of code, and thus to differences of meaning?

This kind of study cuts across the inconsistency program to some extent. Nonetheless, it serves to reinforce the point that an inclusive point of view about mathematics is necessary if one is to understand what mathematics is. Revisionism inevitably reduces our view of what is possible for mathematics, and thus distorts our understanding of the phenomena.

9.Conclusions

To summarise, the following propositions have been advanced.

1. Logicism and foundationalism may well be saved if we adopt a logic lacking ECQ

2. Similarly, *part of* Hilbert's program, to prove that mathematics has no false consequences, may well be saved if such a logic is adopted. There are many suitable logics, some of them only slightly weaker than classical logic.

3. Nonetheless, logicism and foundationalism do not explain the conceptual autonomy of mathematics from logic. In particular, geometry is conceptually separate from logic and set theory, and does not reduce to them.

4. Revisionist philosophies of mathematics, whether they be revisionist about the truths of mathematics (intuitionism) or revisionist about notation (formalism), are open to the objection that they do not account for the varieties of mathematics outside of approved canonical norms.

5. In contrast to revisionism, we must take an inclusive position, whereby inconsistent mathematics is seen as extending our conception of what is possible for mathematics rather than rejecting the corpus of existing mathematics.

6. This is just as well, since inconsistent mathematics has numerous applications beyond itself.

7. As part of comprehending the nature of mathematics, the distinctively textual aspects of mathematics, both the similarities and the differences between textual styles, have to be understood; and semiotics seems to be the best theoretical framework for this project.

Two related issues of traditional philosophy of mathematics have been placed on the backburner in this essay. One is the matter of truthmakers for pure mathematics. The other is the distinctive epistemology of mathematics, and in particular the method of *apriori* proof. Neither can be neglected in a full account. However, we might make the very limited suggestion that if the primary phenomenon to be explained for mathematics is textual, then it is not so speculative that the account ought to derive from the features of text, rather than abstract acausal objects. Certainly, the legitimacy of inconsistency ought to give pause to the Platonist. It poses the dilemma: either abandon Platonism, or admit inconsistent objects. One salient virtue in sheeting home the primary account to the theory of signs, is that it scores well on the second issue: we have a readily-understandable epistemology for signs. It can hardly be denied that getting in contact with signs, such as those on your keyboard, is a thoroughly natural activity. The same can't be said for Platonism.

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PART 3

Papers on Paraconsistent Logic

“A Theorem on Verisimilitude”

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Chris Mortensen

A THEOREM ON VERISIMILITUDE

This paper is a shortened version of one read to the Philosophy Seminar, Australian National University, October 1977. The main result, and additional results, are being prepared for publication in the author's Relevance and Verisimilitude .

Recent work on Popper's qualitative theory of verisimilitude by Miller ([3],[4],[5]), Tichy ([13],[14]) and Popper ([7]) has indicated that there is considerable difficulty in giving a satisfactory account of the idea of one theory's being nearer to the truth than another. On the other hand, intuition seems to support the existence of such a notion (though this cannot be regarded as a decisive reason). Moreover, Putnam ([9]) and Smart ([12]) have both argued that scientific realism requires a notion of nearness to the truth (see also Popper [8] pp.223 ff.)

Popper's (original) qualitative theory of verisimilitude is taken here to be the following

DEFINITION 1. Let T be the set of true sentences (relative to a language L), $F = \bar{T}$ (relative to L) and A and B be theories whose language is L. Then $B >_v A$ (<u>B is more verisimilar than A</u>) iff either $A \cap T \subseteq B \cap T$ and $B \cap F \subset A \cap F$, or $A \cap T \subset B \cap T$ and $B \cap F \subseteq A \cap F$.

We will also need the following definitions:

DEFINITION 2. Let L be a logic with a suitable implication operation '$\rightarrow$' and a suitable conjunction operation '&'. A set A is an <u>L-theory</u> iff (1) if $\alpha \in A$ and $\vdash_L \alpha \rightarrow \beta$, then $\beta \in A$, and (2) if $\alpha \in A$ and $\beta \in A$, then $\alpha \& \beta \in A$. (Sometimes, as in classical logic, the second condition can be dropped).

DEFINITION 3. Let A be an L-theory. <u>Then the rule γ holds for A</u> iff, if $\alpha \in A$ and $\sim\alpha \vee \beta \in A$, then $\beta \in A$.

Miller and Tichy's result is set out below, together with remarks on the proof, which are intended to draw attention to principles used in the proof. The proof is close to that of Tichy ([13]), and Harris ([2]).

THEOREM 1 (Miller-Tichy). If A and B are theories and $B >_v A$, then $B \cap F = \Lambda$.

PROOF. If $B >_v A$, then either (1) $A \cap T \subset B \cap T$ and $B \cap F \subseteq A \cap F$, or (2) $A \cap T \subseteq B \cap T$ and $B \cap F \subset A \cap F$. On either supposition, together with the assumption that $B \cap F \neq \Lambda$ we can deduce a contradiction.

(1) Suppose $A \cap T \subset B \cap T$ and $B \cap F \subseteq A \cap F$ and, for contradiction, suppose $B \cap F \neq \Lambda$.

(2) Suppose $A \cap T \subseteq B \cap T$ and $B \cap F \subset A \cap F$ and, for contradiction, suppose $B \cap F \neq \Lambda$.

Let $f \in B \cap F$. Let $b \in B \cap T - A \cap T$. Since $f \in B \cap F$, $f \in B$. Since $b \in B \cap T$, $b \in B$. Hence $f \& b \in B$ (Remark (a)). Also, since $f \in B \cap F$, $f \in F$. Hence $f \& b \in F$ (Remark (b)). So $f \& b \in B \cap F$. We show that $f \& b \notin A \cap F$, thereby contradicting the assumption that $B \cap F \subseteq A \cap F$. (For suppose $f \& b \in A \cap F$. Then $f \& b \in A$, so $b \in A$ (Remark (c)). But since $b \in B \cap T$, $b \in T$, so $b \in A \cap T$, contradicting the assumption that $b \notin A \cap T$. Hence $f \& b \notin A \cap F$)

Let $f \in B \cap F$. Let $a \in A \cap F - B \cap F$. Since $f \in B \cap F$, $f \in F$. So $\sim f \in T$. (Remark (d)). So $\sim f \vee a \in T$ (Remark (e)). Since $a \in A \cap F$, $a \in A$, so $\sim f \vee a \in A$. (Remark (f)). Hence $\sim f \vee a \in A \cap T$. But $A \cap T \subseteq B \cap T$, so $\sim f \vee a \in B \cap T$. So $\sim f \vee a \in B$. But $f \in B \cap F$, so $f \in B$. From $\sim f \vee a \in B$ and $f \in B$, we may deduce $a \in B$. (Remark (g)). But $a \in A \cap F$, so $a \in F$, so $a \in B \cap F$, contradicting the assumption that $a \in A \cap F - B \cap F$.

Remarks (a) From $f \in B$ and $b \in B$ to deduce $f \& b \in B$. This is part of the definition of an L-theory. cs of L.

(b) From $f \in F$ to deduce $f \& b \in F$. This is a natural, though perhaps not inevitable, restriction to place on the complement of the set of truths (=, perhaps, the set of falsehoods)

(c) From $f \& b \in A$ to deduce $b \in A$. If A is an L-theory and $\vdash_L (X \& Y) \rightarrow Y$, then this move is correct. Moreover, $\vdash_L (X \& Y) \rightarrow Y$ is a natural theorem to adopt, though it is disputed in connexive logic.

(d) From $f \in F$ to deduce $\sim f \in T$. This move amounts to the assumption of completeness for T (Either $X \in T$ or $\sim X \in T$, for any X). The assumption is natural, though of course debatable if T contains arithmetic.

(e) From $\sim f \in T$ to deduce $\sim f \vee a \in T$. This is a move like (b) above. It is a natural condition on T.

(f) From $a \in A$ to deduce $\sim f \vee a \in \hat{A}$. This is a move like (c) above. It is permissible provided that A is an L-theory where $\vdash_L X \rightarrow (Y \vee X)$, and this is a natural theorem to adopt, though disputed in conceptivist logic.

(g) From $\sim f \vee a \in B$ and $f \in B$ to deduce $a \in B$. This is the assumption that γ holds for the (arbitrarily chosen) theory B.

The important point for our purposes here is that the proof assumes that all theories satisfy the rule γ. All theories of classical logic satisfy γ, and of course it is classical logic which is assumed by Miller and Tichy. Thus the correct statement of Theorem 1 is 'If A and B are classical theories, then ...'. As is easy to see, the assumption that γ holds for a given theory is sufficient to ensure that if the theory is (negation) inconsistent, it is trivial.

However, γ fails for large classes of theories based on logics other than classical logic. Various reasons have been given for rejecting γ as a universally valid rule of inference (see e.g. Anderson and Belnap [1], Routley and Meyer [11], and Routley [10]). These reasons, needless to say, go along with a rejection of classical logic as the correct logic for large classes of ordinary inferential situations, such as those in mathematics or science. It is not, of course, enough to point out that various of the principles used in the proof of

Theorem 1 can be made to fail in some logic or other. In order to weaken the significance of Theorem 1, it needs to be argued that some of those principles must be dispensed with in a logic adequate for mathematics or science. However, strong candidates for such a logic are to be found among relevant logics. In this connection one argument (among the many) seems persuasive. Scientific theorising typically proceeds in ignorance of the consistency of the theories used. Occasionally, (negation) inconsistency is uncovered, and since inconsistencies are false (a stronger position might hold that inconsistencies are merely usually false) they need to be removed. The means of removing them might not be immediately obvious, however, and so research continues in the context of a known false (because known inconsistent) theory. But this activity is unintelligible if the background logic is classical (indeed, intuitionist): any theorem we wish to deduce from our theory can be deduced immediately from the contradiction.

Might Theorem 1 be reproved so as to apply to relevant logics as well as classical logics? The following theorem shows not.

THEOREM 2. There exist two RM3-theories A, B, such that $B >_{v} A$ and $B \cap F \neq \Lambda$.

PROOF. Let the language L be determined by a denumerable number of constants $\{p_1, p_2, \ldots\}$ closed under negations and conjunctions. The theories we construct will thus be <u>zero degree</u> theories, containing no occurrences of '$\rightarrow$'. Clearly, however, the result applies to theories of higher degree, indeed of higher order. We need the RM3-matrices (of RM3, see [1]).

&	T	N	F	~
*T	T	N	F	F
*N	N	N	F	N
F	F	F	F	T

→	T	N	F
*T	T	F	F
*N	T	N	F
F	T	T	T

A is constructed as follows

(1) For all $n \geqslant 0$, $V_A(\sim^n p_1) = V_A(\sim^n p_2) = N$ ← mira.

(2) For all $n \geqslant 0$ and $m \geqslant 3$, $V_A(\sim^{2n} p_m) = T$ and $V_A(\sim^{2n+1} p_m) = F$

(3) If α is of the form $\beta \,\&\, \gamma$, then $V_A(\alpha)$ is determined from the RM3-table for '&'.

(4) If α is of the form $\sim^n(\beta \,\&\, \gamma)$ for some $n \geqslant 1$, then $V_A(\alpha)$ is determined from the RM3-table for '$\sim$'.

And let $A = \{\alpha : V_A(\alpha) = T \text{ or } V_A(\alpha) = N\}$.

B is constructed as follows

(1′) For all $n \geqslant 0$, $V_B(\sim^n p_1) = N$.

(2′) As for (2) with '$m \geqslant 3$' replaced by '$m \geqslant 2$'.

(3′) = (3)

(4′) = (4)

And let $B = \{\alpha : V_B(\alpha) = T \text{ or } V_B(\alpha) = N\}$

Finally we construct a theory B^- which will be proved can be taken for T.

(1") As for (2) above, with '$m \geqslant 3$' replaced by '$m \geqslant 1$'.

(2") = (3)

(3") = (4)

And let $B^- = \{\alpha : V_{B^-}(\alpha) = T \text{ or } V_{B^-}(\alpha) = N\}$.

The theorem then follows from the following five lemmata

LEMMA 1. A, B, B^- are RM3-theories

LEMMA 2. $B^- \subset B \subset A$

LEMMA 3. A is nontrivial

LEMMA 4. B^- is a classical theory, and is negation consistent and complete in L.

LEMMA 5. Let $T = B^-$. Then $A \cap T = B \cap T = T$ and $B \cap F \neq \Lambda$.

The importance of the selection of RM3 as a background logic is that a large class of plausible relevant logics are weaker than RM3, which immediately enables us to deduce

THEOREM 3. There are two T-, E-, R-, EM-, RM-, ..., etc., theories such that $B >_v A$ and $B \cap F \neq \Lambda$. (On T, etc., see [1]).

We conclude with some more speculative remarks. As noted earlier, intuition seems to support the viability of a notion of verisimilitude. In addition, intuition suggests that Popper's definition provides at least a sufficient condition for one theory's being closer to the truth than another. The import of the Miller-Tichy result is that not very many pairs of classical theories satisfy this sufficient condition. But relevant mathematics can still avail itself of that sufficient condition, and perhaps even take it for a necessary condition as well. In connection with this point, it is somewhat less reasonable to demand that an account of verisimilitude impose an interesting ordering on <u>all</u> theories, than it is to demand that the account of verisimilitude yield interesting results for theories which are either inconsistent or incomplete.

The results given here do not <u>by themselves</u> solve the 'problem of verisimilitude', even for relevant logics. The problem of finding a way out of the Miller-Tichy result for those theories for which γ holds remains. Research into this problem is continuing, particularly in connection with escaping the first part of Theorem 1, which does not depend on γ. Conceivably, however, it might be reasonable to conclude that Popper's definition of verisimilitude is the best we can come up with, and that relevant mathematics can be equipped with a satisfactory account of verisimilitude which does not, however, impose an interesting ordering on those relevant theories which are also classical. In such a situation, classical logic would at a clear disadvantage with respect to relevant logic.

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Every Quotient Algebra for C_1 is Trivial

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1 In recent years, a number of different types of logics have been proposed with the intention of avoiding the various paradoxes of material implication, particularly the property that from a contradiction anything may be deduced. Two such types of logics are the relevance logics of Anderson and Belnap [1], and the paraconsistent logics in the vicinity of C_1. The logic C_1 has primitives $\neg$, $\supset$, &, v, and is given axiomatically below. In the opinion of this author, C_1 has various unsatisfactory features, two of which are that it lacks the theorem $A \supset \neg\neg A$, and that the rule of replacement ($\vdash A \equiv B$ implies $\vdash C(A) \equiv C(B)$, for any context C; $A \equiv B$ being defined as usual by $(A \supset B)$ & $(B \supset A)$) does not hold for C_1.

To date, there has been an outstanding problem (raised, for example, in [10], p. 508) about C_1: how to "algebraise" it. The aim of this paper is to contribute to the solution of that problem by proving that on certain very minimal assumptions C_1 has no nontrivial quotient algebra. We will say presently what it means for a quotient algebra to be trivial. It is suggested that the present result, in addition to "solving" the algebraisation problem, exhibits a further unsatisfactory feature of C_1, namely that C_1 lacks a proper biconditional. We hope to amplify this point in a later paper.

The present enterprise is to investigate the consequences of partitioning the formula algebra of C_1 into a quotient algebra of equivalence classes by some relation $\sim$ holding between formulas. The relation $\sim$ need *not* necessarily be syntactic, i.e., definable by a formula in the operators $\neg$, $\supset$, &, v. We impose the following four requirements on any such relation $\sim$ and quotient algebra: (a) $\sim$ is an equivalence relation, i.e., $A \sim A$, $A \sim B$ implies $B \sim A$, and $A \sim B$ and $B \sim C$ imply $A \sim C$. (b) The formula algebra is homomorphic to the quotient algebra (with corresponding operations) obtained from the equivalence relation; i.e., $A \sim B$ implies $C(A) \sim C(B)$, for any context C. (c) If $A \sim B$ and $\vdash A$ then $\vdash B$ (where '$\vdash$' means provability in C_1). This is necessary to prevent including

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nontheorems and theorems in the same equivalence class, which would have the consequence that we would be unable both to designate the equivalence class (without making some nontheorems theorems), and not to designate it (without refuting some theorem). Again, if $\vdash A$ and not $\vdash B$ then it is easy to show that A and B are distinguishable by the valuation semantics described below, and we would not want an algebraisation in which $|A|$ and $|B|$ were to be identified by the fact that $A \sim B$ when A and B are distinguishable semantically. We prove in Theorem 2 that conditions (b) and (c) in the context of C_1 imply the condition: if $A \sim B$ then $\vdash A \equiv B$. This fact is crucially used in the proof of our main theorem, Theorem 3. (d) The quotient algebra so obtained is nontrivial. We promised earlier a definition of this term. In one sense there are always at least two equivalence relations: one is 'A is the same formula as B', and another is 'A is a formula and B is a formula'. In order to avoid such trivial cases, we make a definition.

Definition 1 Let $\mathcal{L}$ be a sentential language and let $\sim$ and $\mathcal{L}/\!\sim$ be, respectively, equivalence relation and quotient algebra satisfying (a)-(c) above. Then $\mathcal{L}/\!\sim$ is *trivial* iff either $(\forall A \in \mathcal{L})(|A| = \{A\})$ (where $|A|$ is the equivalence class of A), or $(\forall A, B \in \mathcal{L})(|A| = |B|)$. Otherwise, $\mathcal{L}/\!\sim$ is *nontrivial*.

The fourth requirement, then, is that the quotient algebra determined by $\sim$ be nontrivial.

2 We now present C_1 formally. We begin with a language $\mathcal{L}$ consisting of a denumerable number of sentential variables p_i, $1 \leqslant i < \omega$, closed as usual under $\neg$, &, v, $\supset$. The operator $\equiv$ is defined as usual (see above), and we have two new defined symbols: A° is an abbreviation for $\neg(A \,\&\, \neg A)$, and $\neg^* A$ is an abbreviation for $\neg A \,\&\, A^\circ$. Capital letters from the beginning of the alphabet are metalinguistic schematic variables.

Definition 2 The logic C_1 is the smallest subset of $\mathcal{L}$ closed under uniform substitution and modus ponens (for $\supset$) and containing all instances of the following schemata:

(1) $A \supset (B \supset A)$
(2) $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$
(3) $(A \,\&\, B) \supset A$
(4) $(A \,\&\, B) \supset B$
(5) $A \supset (B \supset (A \,\&\, B))$
(6) $A \supset (A \vee B)$
(7) $B \supset (A \vee B)$
(8) $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
(9) $A \vee \neg A$
(10) $\neg\neg A \supset A$
(11) $B^\circ \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$
(12) $(A^\circ \,\&\, B) \supset ((A \,\&\, B)^\circ \,\&\, (A \vee B)^\circ \,\&\, (A \supset B)^\circ)$.

Definition 3 A C_1-*valuation* is a function $v: \mathcal{L} \to \{1, 0\}$ such that

(1) $v(A) = 0 \Rightarrow v(\neg A) = 1$
(2) $v(\neg\neg A) = 1 \Rightarrow v(A) = 1$

(3) $v(B^\circ) = v(A \supset B) = v(A \supset \neg B) = 1 \Rightarrow v(A) = 0$
(4) $v(A) = 0$ or $v(B) = 1 \iff v(A \supset B) = 1$
(5) $v(A) = v(B) = 1 \iff v(A \mathbin{\&} B) = 1$
(6) $v(A) = 1$ or $v(B) = 1 \iff v(A \vee B) = 1$
(7) $v(A^\circ) = v(B^\circ) = 1 \Rightarrow v((A \vee B)^\circ) = v((A \mathbin{\&} B)^\circ) = v((A \supset B)^\circ) = 1.$

A formula A is *true in a valuation* v iff $v(A) = 1$.

Theorem 1 *C_1 is sound and complete with respect to the class of all C_1-valuations; i.e., all and only theorems of C_1 are true in all C_1-valuations* (da Costa and Alves [11]). *C_1 is decidable* (Fidel. See [11], p. 627).

We use the decidability of C_1 extensively below, so we outline Fidel's decision procedure. A *quasi matrix for a formula A* is constructed as follows:

1. List all the propositional constants of A (in a horizontal line) and, as in truth tables for classical propositional calculus, list all possible assignments of 1 and 0 to them.

2. List all denials of propositional constants (to the right of the former list) of A and assign values as follows: if the propositional constant was assigned 0 its denial is assigned 1. If the constant was assigned 1, bifurcate the line on which the 1 occurs, and on one half the denial is assigned 0 and on the other 1.

3. List all remaining subformulas of A and negations of proper subformulas of A and proceed as follows:

(3.1) If the major connective of any such formula is &, v, or $\supset$, its value is determined from the values of its two components as in classical logic.

(3.2) If the formula is of the form $\neg B$ and B was assigned 0, assign B the value 1.

(3.3) If the formula is of the form $\neg B$ and B was assigned 1, then there are several subcases

(3.3.1) B is of the form $\neg C$ and C was assigned 0. Assign $\neg B$ (i.e., $\neg\neg C$) the value 0.

(3.3.2) B is of the form $\neg C$ and C assigned 1. Bifurcate the line and assign $\neg B$ the value 0 on one bifurcation and 1 on the other.

(3.3.3) B is of the form $C \mathbin{\&} \neg C$ or $\neg C \mathbin{\&} C$. Assign $\neg B$ the value 0.

(3.3.4) B is of the form $C \circ D$ (where $\circ$ is &, v, or $\supset$) but not of the form 3.3.3. If the value of C is different from the value of $\neg C$ and the value of D is different from the value of $\neg D$, assign $\neg B$ the value 0. Otherwise, bifurcate the line and assign $\neg B$ the value 0 on one half and 1 on the other half.

We can now state the outcomes of this decision procedure: if some line of the quasi matrix of A assigns 0 to A, then for some C_1-valuation v, $v(A) = 0$, and so A is not valid and not a theorem. Otherwise, for all C_1-valuations v, $v(A) = 1$ and A is valid and a theorem.

3 In Section 1 the claim was made that the conditions (b) and (c) we imposed on equivalence relations imply that if $A \sim B$ then $\vdash A \equiv B$. In this section, we prove that result.

Theorem 2 *If an equivalence relation $\sim$ satisfies the conditions:* (b) *if $A \sim B$ then $C(A) \sim C(B)$ for any context C, and* (c) *if $A \sim B$ and $\vdash A$ then $\vdash B$; then it necessarily satisfies the condition: if $A \sim B$ then $\vdash A \equiv B$.*

Proof: Suppose the antecedent of the theorem, and suppose that $A \sim B$. We need to prove that $\vdash A \equiv B$. If either $\vdash A$ or $\vdash B$ then by the antecedent of the theorem $\vdash B$ and $\vdash A$, respectively. Hence, by the properties of C_1 (Axioms (1) and (5)), $\vdash A \equiv B$. So suppose neither $\vdash A$ nor $\vdash B$, and suppose, for contradiction, $\nvdash A \equiv B$. If $\nvdash A \equiv B$ then, as is well known, there is a C_1-valuation v such that $v(A) \neq v(B)$. Suppose that $v(A) = 1$, $v(B) = 0$. It follows from the properties of C_1-valuations that $v(\neg^* A) = 0$, $v(A \vee \neg^* A) = 1$, and $v(B \vee \neg^* A) = 0$. From $v(B \vee \neg^* A) = 0$, we have $\nvdash B \vee \neg^* A$. Now by condition (b) of the theorem, if $A \sim B$ then $(A \vee \neg^* A) \sim (B \vee \neg^* A)$. But it is also a fact that $\vdash A \vee \neg^* A$. Hence, by condition (c), $\vdash B \vee \neg^* A$. Contradiction. Hence $\vdash A \equiv B$. This proves the theorem.

4 We now proceed to our main theorem. First, we need some lemmas.

Lemma 1 *Let $\vdash A \equiv B$. The following are sufficient conditions for the truth of $\vdash \neg A \equiv \neg B$:*

(1) *$A = B$ (A is the same formula as B)*
(2) *for all C_1-valuations v, $v(A) = 0$ (equivalently, $v(B) = 0$)*
(3) *for some C_1-valuation v, $v(A) = v(B) = 1$ and for all C_1-valuations v_1, v_2, $v_1(A) = 1$ implies $v_1(\neg A) = 0$ and $v_2(B) = 1$ implies $v_2(\neg B) = 0$.*

The following are sufficient conditions for the truth of (3): for some v, $v(A) = v(B) = 1$, together with any condition from List One together with any condition from List Two.

List One:

(i) *A is of the form $\neg C$ and $(\forall v)(v(A) = 1$ implies $v(C) = 0)$*
(ii) *A is of the form $C \circ D$ (where $\circ$ is &, $\vee$, $\supset$) and C is $\neg D$ or D is $\neg C$*
(iii) *A is of the form $C \circ D$ (where $\circ$ is &, $\vee$, $\supset$) and $(\forall v)(v(C \circ D) = 1$ implies $v(C) \neq v(\neg C)$ and $v(D) \neq v(\neg D))$.*

List Two: as for List One with 'B', 'E', 'F', replacing 'A', 'C', 'D', respectively.

Proof: Clearly (1) is sufficient. If $(\forall v)(v(A) = 0)$ and $\vdash A \equiv B$ then $(\forall v)(v(B) = 0)$. But $v(A) = 0$ implies $v(\neg A) = 1$, and similarly for B, so $(\forall v)(v(\neg A) = v(\neg B))$ and so, by the conditions for & and $\supset$, $(\forall v)(v(\neg A \equiv \neg B) = 1)$, i.e., $\models \neg A \equiv \neg B$, i.e., $\vdash \neg A \equiv \neg B$. Hence (2) is sufficient. As to (3), either $v(A) = 0$ or $v(A) = 1$. If $v(A) = 0$ then $v(\neg A) = 1$, and by hypothesis if $v(A) = 1$ then $v(\neg A) = 0$. But $\vdash A \equiv B$ so A and B have the same values. But again by our hypothesis, if $v(B) = 1$ then $v(\neg B) = 0$, and clearly if $v(B) = 0$ then $v(\neg B) = 1$. Hence $\neg A$ and $\neg B$ have the same values in all valuations and so, as above, $\models \neg A \equiv \neg B$, i.e., $\vdash \neg A \equiv \neg B$. Hence (3) is sufficient.

Now we show that List One combined with List Two are sufficient conditions for (3). It is sufficient to show that List One gives sufficient conditions for A to satisfy (3), as the proof for B is identical. Clearly (i) is sufficient: Suppose $v(A) = v(\neg C) = 1$. Then from (i) $v(C) = 0$, but then by condition (2) of Definition 3, $v(\neg\neg C) = 0 = v(\neg A)$.

Ad (ii) Construct a quasi matrix for $\neg A$, i.e., $\neg(C \,\&\, \neg C)$. Whenever A is assigned 1, by 3.3.3 of the definition of a quasi matrix above, $\neg A$ is assigned 0. Hence there is no C_1-valuation in which A is assigned 1 and also $\neg A$ assigned 1.

Ad. (iii). As for (ii).

Lemma 2 *Let $\vdash A \equiv B$. If none of the above sufficient conditions (1)-(3) obtain, then $\nvdash \neg A \equiv \neg B$. If none of (i)-(iii) obtain, (3) does not obtain.*

Proof: Construct a quasi matrix for $A \equiv B$, which also involves giving values to $\neg A$ and $\neg B$. Since $\vdash A \equiv B$, A and B receive the same value on all lines. If none of (1)-(3) obtain, then we must bifurcate at least one of the lines for $\neg A$, $\neg B$. But this will ensure that there is a valuation where $v(\neg A) \neq v(\neg B)$. We now simply extend the quasi matrix to a quasi matrix for $\neg A \equiv \neg B$ by calculating the values of $\neg A \supset \neg B$, $\neg B \supset \neg A$, $\neg(\neg A \supset \neg B)$, $\neg(\neg B \supset \neg A)$, and $(\neg A \supset \neg B) \,\&\, (\neg B \supset \neg A)$. The values of $\neg A \supset \neg B$ and $\neg B \supset \neg A$ are calculated directly, the values of their respective denials being irrelevant. At least one of $\neg A \supset \neg B$, $\neg B \supset \neg A$ is zero on any of the above lines where $\neg A$ and $\neg B$ have different values, and so their conjunction calculates to 0, i.e., $\nvDash \neg A \equiv \neg B$, i.e., $\nvdash \neg A \equiv \neg B$.

By inspection of the conditions for construction of a quasi matrix, provided that for some v, $v(A) = 1$, if none of (i)-(iii) apply we must bifurcate the table for $\neg A$. But then we can conclude that there is a C_1-valuation v such that $v(A) = 1$ and $v(\neg A) = 1$, i.e., (3) is false as required.

We can now prove the promised result, which we state as follows

Theorem 3 *No equivalence relation for C_1 satisfying the above conditions (a)-(c) of Section 1 for equivalence relations determines a nontrivial quotient algebra.*

Proof: We prove this by proving that for any such relation $\sim$, if $A \sim B$ then $A = B$ (A is the same formula as B). Suppose $A \sim B$. By condition (b) for $\sim$, $C(A) \sim C(B)$, and so in particular $\neg A \sim \neg B$, $(A \vee p_1) \sim (B \vee p_1)$ and $\neg(A \vee p_1) \sim \neg(B \vee p_1)$, where p_1 is the first propositional variable. Hence, by Theorem 2, $\vdash A \equiv B$, $\vdash \neg A \equiv \neg B$, $\vdash (A \vee p_1) \equiv (B \vee p_1)$, and $\vdash \neg(A \vee p_1) \equiv \neg(B \vee p_1)$. We show that these four formulas are theorems iff $A = B$. Clearly the four theorems hold if $A = B$, because $\vdash A \supset A$. So suppose $A \neq B$. If $\vdash A \equiv B$ and $\vdash \neg A \equiv \neg B$, then we have both $\vdash (A \vee p_1) \equiv (B \vee p_1)$, and also from Lemma 2 that conditions (2) and (3) of Lemma 1 hold with respect to A and B. Construct a quasi matrix for $\neg(A \vee p_1) \equiv \neg(B \vee p_1)$, and consider the (bifurcated) line on which p_1 receives value 1. This line bifurcates giving $\neg p_1$ the value 1 on one half and $\neg p_1$ the value 0 on the other half. Consider the half on which $\neg p_1$ has the value 1. Compute the quasi matrix, including the values of A, $\neg A$, B, $\neg B$, $A \vee p_1$, $B \vee p_1$, $\neg(A \vee p_1)$, $\neg(B \vee p_1)$. Now, since p_1 has 1, $A \vee p_1$ and $B \vee p_1$ both have 1. But now the value of p_1 is the same as the value of $\neg p_1$ on the lines in question, and the values of $A \vee p_1$ and $B \vee p_1$ are 1. So, taking the 'A' and 'B' of Lemma 1 to be $A \vee p_1$ and $B \vee p_1$, respectively, the sufficient conditions (i)-(iii) of Lemma 1 for the truth of (3) of Lemma 1 (i.e., for all v_1, $v_1(A \vee p_1) = 1$ implies $v_1(\neg(A \vee p_1)) = 0$; and for all v_2, $v_2(B \vee p_1) = 1$ implies $v_2(\neg(B \vee p_1)) = 0$) fail. So, by Lemma 2, (3) fails. But (1) also fails because if $A \neq B$ then $A \vee p_1 \neq B \vee p_1$, and (2) fails since neither $A \vee p_1$, nor $B \vee p_1$ have the value 0. Hence all of the sufficient conditions (1)-(3) of Lemma 1 for the truth of $\vdash\neg(A \vee p_1) \equiv \neg(B \vee p_1)$ fail, and so by Lemma 2, $\nvdash\neg(A \vee p_1) \equiv \neg(B \vee p_1)$. Contradiction. Thus, if $\vdash A \sim B$, then $A = B$.

In a sequel, we hope to study the algebraic properties of systems in the neighbourhood of C_1 which have nontrivial algebras. These systems cannot be any weaker than C_1, of course: if we can show that $A \sim B$ implies $A = B$ for C_1, then the same must hold for any system with weaker deductive resources than C_1. Thus for instance we have the corollary

Corollary *All of the systems C_2, C_3, . . . of [11] have only trivial quotient algebras.*

5 That is not quite an end to the question of the algebraisation of C_1, however. There are equivalence relations which partition the formula algebra of C_1 into such trivial quotient algebras. The relation of Theorem 3 will do: $A \sim B$ iff $\vdash(A \equiv B)$ & $(\neg A \equiv \neg B)$ & $((A \vee p_1) \equiv (B \vee p_1))$ & $(\neg(A \vee p_1) \equiv \neg(B \vee p_1))$. Now nothing so far established shows that there might not be interesting (though perhaps bizarre) partial orders which can be imposed on this quotient algebra. Such partial orders will, as usual, be reflexive, antisymmetric, and transitive; the antisymmetry property in question being $|A| \leqslant |B|$ and $|B| \leqslant |A|$ implies $|A| = |B|$. In the light of Theorem 3, this becomes $|A| \leqslant |B|$ and $|B| \leqslant |A|$ implies $A = B$.

There do exist such partial orders. One is: $|A| \leqslant |B|$ iff $\vdash A \supset B$ and either $\nvdash B \supset A$ or $A = B$. The proof that it is a partial order is not difficult. The algebra obtained by imposing this partial order on the (trivial) quotient algebra with singleton equivalence classes is, as might be expected after Theorem 3, rather strange. Some of its properties are: (i) If $|A| \leqslant |B|$, then if $\vdash B$ then either $\nvdash A$ or $A = B$. In fact, every theorem is a maximal element. No non-theorem is maximal, for if $\nvdash A$, then $|A| \leqslant |A \vee \neg A|$. (ii) If $|A| \leqslant |B|$ then if $\vdash A \equiv \neg^*(C \vee \neg C)$, then either $\nvdash B \equiv \neg^*(C \vee \neg C)$ or $A = B$. Everything which is equivalent to a negation* of a theorem is a minimal element. (iii) The algebra is not a lattice with respect to $\vee$ and &: $|A|, |C| \leqslant |A \vee C|$ fails for the case where $A = C$, and dually for &.

Another partial order, of little interest, is determined by "A is a sub-formula of B". There may be other such partial orders, though it is a fair conjecture that any such will turn out to be equally uninteresting or strange. It would be desirable to find some set of conditions for a "reasonable" partial order, according to which it could be shown that there are no reasonable partial orders on quotient algebras for C_1.

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“The Truth Teller Paradox”

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THE TRUTH TELLER PARADOX

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1. Consider the statements

(α) This very sentence is true

(β) This very sentence is false

The second of these, (β), is of course the liar paradox, about which much has been written. Much less has been written about (α), though many have noted that it is curious. One aspect of that curiousness is that there seems to be nothing to choose between the hypotheses that it is true, and that is is false. Indeed, it is hard to see how there could even be anything to choose between the hypotheses. More particularly, both hypotheses seem to be consistent: from neither hypothesis does there appear to be deducible a contradiction. And if that is true, it would seem further to follow that there is no *a priori* proof of the truth of (α), and no *a priori* proof of its falsity. (Contrast this situation with that of (β), where there are, *prima facie* at least, *a priori* proofs of both the truth and the falsity of (β).) We will return to the question of the absence of *a priori* proofs of the truth and of the falsity of (α) below, in section 6.

2. However, it has gone unnoticed that (α) has a specific problem of its own. The purpose of this note is to explain the problem and to consider in a little more detail some aspects of it, particularly some possible solutions. The problem, which we might call «the truth teller paradox» is as follows. In view of the fact that truth and falsity are both consistently assignable to (α), and so there are no *a priori* proofs of its truth and its falsity, it is tempting to suppose that it is neither true nor false. However, this runs into the following problem. Suppose that (α) is neither true nor false. Then certainly it is not true. But since it says of itself that it is true, it is false. This contradicts the

supposition that it is neither true nor false. Hence it is either true or false.

To put the problem in a nutshell, the absence of any possible proof of the truth of (α) would be a reason for declaring it not to be true. Similarly, there is a reason to declare it not to be false. In any case, there are apparently no proofs of its truth or of its falsity. But there is a proof that it is either true or false.

3. There are several options for formalising, in the interests of clarity, the argument that (α) is either true or false. Not all of these options are equivalent. We choose one which seems to us to be closest to the intuitions expressed in the above informal argument. (The notation is the obvious one). [1]

Let $\ulcorner\alpha\urcorner = \ulcorner T\ulcorner\alpha\urcorner\urcorner$ (1)
Now $(\sim T\ulcorner\alpha\urcorner \;\&\; \sim F\ulcorner\alpha\urcorner) \rightarrow \sim T\ulcorner\alpha\urcorner$ (2)
But $\ulcorner\alpha\urcorner = \ulcorner T\ulcorner\alpha\urcorner\urcorner \rightarrow (\sim T\ulcorner\alpha\urcorner \rightarrow F\ulcorner\alpha\urcorner)$ (3)
So $(\sim T\ulcorner\alpha\urcorner \;\&\; \sim F\ulcorner\alpha\urcorner) \rightarrow F\ulcorner\alpha\urcorner$ (4) By (1), (2), (3)
$\rightarrow (T\ulcorner\alpha\urcorner \vee F\ulcorner\alpha\urcorner)$ (5) From (4)
But $T\ulcorner\alpha\urcorner \vee F\ulcorner\alpha\urcorner \vee (\sim T\ulcorner\alpha\urcorner \;\&\; \sim F\ulcorner\alpha\urcorner)$ (6)
Hence $T\ulcorner\alpha\urcorner \vee F\ulcorner\alpha\urcorner$ (7) By (5) and (6)

4. We do not know how to solve the truth teller paradox, though some solutions are more attractive than others (see section 7 below). However, it will serve to sharpen the paradox, we think, if we chart briefly where the possibilities for solution lie. The first possibility is to fault one of the lines of the argument, and in this section we consider this. Line (1) just formalises the English sentence (α). Lines (2), (4), (5) and (7) depend on standard truth functional properties of conjunction and disjunction, (specifically $(A \;\&\; B) \rightarrow A$, $A \rightarrow (A \vee B)$ and $(A \rightarrow B) \rightarrow ((A \vee B) \rightarrow B)$), the transitivity of entailment, and modus ponens for $\rightarrow$. Most of these have been questioned. [2] In our view they all hold, and it seems fair to say that most philosophers would be in agreement.

That leaves lines (3) and (6). Line (3) is an expression of the idea that if (α) fails to be true, then, in the light of the fact that it says of itself that it is true, it is false. That seems correct to us, and pretty much on all fours with the more usual argument, in the liar paradox, that if (β) is true, it is false. Evidently, however, someone might refuse to make that move, and simply hold to the proposition that (α) fails to be true. Of course, that is not quite so simple a matter, because there are various arguments one can imagine to back up (3). For instance, from $\sim T\ulcorner\alpha\urcorner$, using $\ulcorner\alpha\urcorner = T\ulcorner\alpha\urcorner$, we have, by substitution of identicals, $\sim\alpha$; then, by the T-scheme (i.e. $\alpha \leftrightarrow T\ulcorner\alpha\urcorner$), $T\ulcorner\sim\alpha\urcorner$; so that $F\ulcorner\alpha\urcorner$ (by definition of «false», say). Holding that (α) merely fails to be true requires that one find fault with some move in this latter argument. That is not so easy, and doubtless various philosophers could be found with opposing intuitions on just where to halt the argument. For example, one position holds that the T-scheme holds only for «grounded» sentences, and that neither (α) nor its denial are grounded. ([3]) Because this kind of position is frequently taken with respect to logical paradoxes like (β), its strengths and weaknesses are well known. ([4]) We will therefore not pursue the matter except to say that we think that the weaknesses outweigh the strengths, as one of us has, in effect, argued elsewhere. ([5])

That leaves (6): either (α) is true, or it is false, or it is neither. This is an expression of one of the fundamental ideas behind the paradox, that if one wants to avoid holding that (α) is true and one wants to avoid holding that it is false, then one should say that it is neither. We think it would be a mistake to assume the Law of Excluded Middle in the form: (α) is true or (α) is false; but of course (6) does not depend on that, and seems to be on much more solid ground. It is possible to deduce (6) from an instance of Excluded Middle in the form $A \vee \sim A$:

$$(T\ulcorner\alpha\urcorner \vee F\ulcorner\alpha\urcorner) \vee \sim(T\ulcorner\alpha\urcorner \vee F\ulcorner\alpha\urcorner), \text{ so } T\ulcorner\alpha\urcorner \vee F\ulcorner\alpha\urcorner \vee (\sim T\ulcorner\alpha\urcorner \;\&\; \sim F\ulcorner\alpha\urcorner)$$

by De Morgan's Law. But it is preferable to avoid that if possible, if only because it is a tricky business working out precisely what Excluded Middle comes to in a situation where we are taking seriously the possibility that some sentence might be neither true nor false.

5. If all the steps of the argument are accepted there appear to be only three further possibilities. The first is a paraconsistent or inconsistent one, namely to accept both the argument to the effect that (α) is either true or false *and also* the arguments to the effect that (α) has no truth value. Thus $\sim(\sim T\ulcorner\alpha\urcorner \;\&\; \sim F\ulcorner\alpha\urcorner) \;\&\; (\sim T\ulcorner\alpha\urcorner \;\&\; \sim F\ulcorner\alpha\urcorner)$ would be a true contradiction. While it is arguable that this position

can be sustained without collapse into triviality,[6] it would tempt few.

6. The second possibility is to accept that either (α) is true or (α) is false, but deny that there is any paradox by holding that one of those two alternatives really is true, but (at least at present) we do not know which. After all, in section 1 all we said was that the hypotheses that (α) is true and that it is false *seem* to be consistent; and from that it does not follow that there *is* no proof of the truth of (α), nor that there is no proof of its falsity. Now clearly the truth of one of the disjuncts is not going to be determined empirically, so if a truth value for (α) is to be found it will be by the production of an *a priori* proof.

Indeed, that is not so unthinkable. Although there appears to be nothing to choose between (α) being true and its being false, appearances can be deceptive. As an analogy, consider the sentence normally expressed as «This sentence is provable in Peano arithmetic». It might well be thought that this would have exactly the same ontological status as (α). Yet it is provable in Peano arithmetic.[7] It might also be thought that some standard proof of this fact could be modified by substituting «is true» for «is provable» to produce an argument for the truth of (α). Indeed, that modification can be made.

Let «δ» be the claim «If this sentence is true, α».

i.e.	$\ulcorner\delta\urcorner = \ulcorner T\ulcorner\delta\urcorner \rightarrow \alpha\urcorner$
So certainly	$\delta \rightarrow (T\ulcorner\delta\urcorner \rightarrow \alpha)$
Hence	$T\ulcorner\delta\urcorner \rightarrow (T\ulcorner T\ulcorner\delta\urcorner\urcorner \rightarrow T\ulcorner\alpha\urcorner)$ by the T scheme and the distribution of T over $\rightarrow$.
But by the T scheme	$T\ulcorner\delta\urcorner \leftrightarrow T\ulcorner T\ulcorner\delta\urcorner\urcorner$; so $T\ulcorner\delta\urcorner \rightarrow (T\ulcorner\delta\urcorner \rightarrow T\ulcorner\alpha\urcorner)$
Hence by contraction	$T\ulcorner\delta\urcorner \rightarrow T\ulcorner\alpha\urcorner$
and by the T scheme	$T\ulcorner\delta\urcorner \rightarrow \alpha$ (*)
and again	$T\ulcorner T\ulcorner\delta\urcorner \rightarrow \alpha\urcorner$
which by definition of α gives	$T\ulcorner\delta\urcorner$
whence by line (*)	α

Unfortunately this is unsatisfactory as an *a priori* proof of (α). It is just a dressed-up version of Curry's Paradox and could easily be modified to produce a proof of $\sim\alpha$. It cannot be carried out in Peano arithmetic, because the truth predicate for Peano arithmetic fails to be arithmetically representable (at least, it fails iff Peano arithmetic is consistent !). In natural language, anyone who wants to maintain that the truth predicate is representable[8] will presumably make similar moves against this proof of (α) that he or she would make against Curry's Paradox.[9]

But still, even though this proof fails, there may be satisfactory proofs of either $T\ulcorner\alpha\urcorner$ or $F\ulcorner\alpha\urcorner$. However, perhaps there are no proofs of either $T\ulcorner\alpha\urcorner$ or $F\ulcorner\alpha\urcorner$. In this case we would be forced into the position that something true is in principle unprovable. This is certainly ruled out from an Intuitionist viewpoint, though not, in principle, from a classical one. However, it does raise the question of in what *exactly* the truth of one of these alternatives consists.

7. The final possibility for solving the truth teller paradox is the one which seems to us to be the most attractive, if only because the difficulties sketched in the foregoing solutions appear considerable. This solution is to hold that while (α) fails to be true and fails to be false, nevertheless it is either true or false. The possibility of this line goes back to Aristotle.[10] Adopting the position requires that the standard truth conditions for disjunction be rejected (even though we set out to use standard disjunction), since we have all of: (1) (α) is true $\vee$ (α) is false, (2) (α) is not true, and (3) (α) is not false. Intensional disjunction has been studied, for example by Anderson and Belnap.[11] There, however, the intensional disjunction is not the kind for which $A \vee B$ might hold while both A and B fail, but the kind for which the truth of one of the disjuncts is not sufficient for the truth of the disjunction.

A theory is said to be *prime* iff whenever $A \vee B$ is in the theory either A is in the theory or B is. Identifying the world with its true theory, we can then describe the present position by saying that it holds that *the world is non-prime*. The failure of primeness is not mysterious. For example, let PA be Peano arithmetic formulated with a base of classical logic, and let G be its Gödel sentence. Now certainly $\vdash_{PA} G \vee \sim G$. But, by Gödel's first Incompleteness Theorem, if PA is consistent then neither $\vdash_{PA} G$ nor $\vdash_{PA} \sim G$. That is, (the set of theorems of) Peano arithmetic is non-prime if it is consistent.

A typically Platonist attitude might interpret Gödel's theorems as applying to consistent recursively enumerable arithmetics only, while holding that arithmetical reality is complete. If arithmetical reality were consistent and complete, then it would be prime (as a simple argument establishes). But one man's modus ponens is another man's modus tollens: a consistent and non-prime reality fails to be complete. The failure of the completeness of the world is also a doctrine which has been taken seriously.

We cannot pretend that the thesis that the world is non-prime is not paradoxical; but it is, we believe, no more paradoxical in principle than the paraconsistent solution to the liar, namely that (β) is both true and false(12) (and paradoxes not infrequently beget paradoxical solutions). In favour of both solutions, it can be said that our concepts of negation, falsity and disjunction *escape our control*. This is especially so in logico-mathematical contexts such as the present ones where, as with Intuitionism, truth and falsity seem to coincide with provable truth and falsity. Consider, for example, the liar (β). We might feel that we can *resolve* to make «false» exclude «true», but this is an illusion. There is no guarantee that we can keep control of the resolution once we allow «false» and «true» to have additional logical properties. If we allow them to have enough properties, particularly the natural property of the representability of the truth predicate in the object language, then we will *discover* something about the concepts, namely that we cannot keep to the resolution of the exclusiveness of «true» and «false». Furthermore, such sentiments ought not to be so strange to someone with Intuitionist leanings. It only needs the view that in limited contexts, say logico-mathematical contexts, something is true (false) iff provably true (false), for it to seem not so impossible that the truth value of a sentence might be overdetermined. (Of course, it is at the very least arguable that the assignment of both truth and falsity to a sentence does not lead to logical collapse.)(13) Similarly, then, with the truth teller, (α). We cannot guarantee control of the desirable primeness feature of disjunction if we want disjunction to have additional, natural properties.

Supposing that primeness fails does not mean that the truth conditions for «∨» go completely haywire. Nothing has been said to deny that the truth of disjuncts is *sufficient* for the truth of a disjunction. As to *necessity* here, we can usefully invoke the idea of the local consistency and completeness of a theory. Let L be a language, and Th a theory in that language. Then Th is *locally consistent (complete) relative to a sublanguage L′ of L* iff the restriction of Th to L′ is consistent (complete). It follows from the previously cited fact (that consistency and completeness implies primeness), that local consistency and completeness implies local primeness. Normal, well-behaved situations will be consistent and complete, we can suppose. Hence, a locally well-behaved theory of the world will be one in which disjunction is locally entirely classical. It follows that we may be confident that it is only when things get strange that the truth conditions for «∨» behave unexpectedly. One might, in addition, have a theory about just when strangeness like inconsistency, incompleteness and non-primeness can occur (say, in logico-mathematical contexts, though we do not wish to commit ourselves to such a restrictive view).

8. In sum, the truth teller paradox is interesting in its own right. In addition, at least some solutions raise intriguing possibilities.

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FOOTNOTES

(1) We let «α» be an abbreviation for the sentence (α), not its name. Quasi quotes will be used as name forming functors.

(2) On (A & B)→A, see Storrs MCCALL «Connexive Implication», *Journal of Symbolic Logic*, 31 (1966), 415-433. On A → (A ∨ B) and transitivity see William PARRY «Ein Axiomsystem für eine neue Art von Implikation (Analysische Implikation)», *Ergebnisse eines mathematischen Kolloquiums*, 4 (1933), 5-6. On transitivity see Peter GEACH *Logic Matters*, Blackwell, 1972, Chs. 4,7. On modus ponens, see Errol MARTIN and Robert K. MEYER, *S for Syllogism*, forthcoming; or Robert K. Meyer, Richard Routley and J. Michael Dunn, «Curry's Paradox», *Analysis*, 39 (1979), 124-8.

(3) See Saul KRIPKE, «Outline of a Theory of Truth», *Journal of Philosophy*, 12 (1975), 690-716.

(4) See e.g. Susan HAACK, *Philosophy of Logic*, Cambridge U.P., 1978, 145 ff.

(5) See Graham PRIEST, «The Logic of Paradox». *The Journal of Philosophical Logic*, 43, (1978).

(6) PRIEST, *ibid*; or Nicholas Rescher and Robert Brandom, *The Logic of Inconsistency*, Blackwell, 1980.

(7) See e.g. George BOOLOS and Richard JEFFREY, *Computability and Logic*, Cambridge U.P., 1974, Ch. 16, 188-9.

(8) See PRIEST, *ibid*.

(9) For example, by giving up contraction (A → (A → B)) → (A → B), or modus ponens in the form ((A → B) & A) → B. See MEYER, ROUTLEY and DUNN, *ibid*.

(10) *De Interpretatione*, Ch. 9. Formally it might be handled by supervaluation techniques. See e.g. BAS C. VAN FRAASSEN «Presupposition, Implication and Self-Reference», *Journal of Philosophy*, 65 (1968), 136-52.

(11) For example, Alan Ross ANDERSON and Nuel BELNAP, *Entailment*, Princeton, 1975, 176-7.

(12) See PRIEST, *ibid*.

(13) See PRIEST, *ibid*.

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Model Structures and Set Algebras for Sugihara Matrices

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1 Introduction Since the work of Lemmon on modal algebras [2], [3], it has been known that there is a close relationship between relational model structures, set algebras, and matrices. The type of result which Lemmon established was to show how to construct a modal set algebra given a modal model structure, or a model structure given an algebra, in such a way that validity in the algebra coincides with validity in the model structure.

The extension of Lemmon's type of result to various cases of relevant algebras and relevant model structures associated with relevant logics has been studied by Brady in [6], by Routley and Meyer in [5] and [6], and by the author in [4] and [6]. The purpose of this paper is to report results connecting model structures and set algebras for the Sugihara matrices, and in particular for two infinite Sugihara matrices, both of which are characteristic for the important logic RM. The Sugihara matrices, or chains, and the logic RM are investigated in Anderson and Belnap's [1]. To date, no semantics for RM has been given which uses only a single relational model structure. This paper provides such a semantics. Earlier results, for example in [6], of such theorems connecting particular relational model structures and particular set algebras have been exclusively for finite cases of such algebras. The present result is new in that it is the first such example of an infinite algebra and model structure.

2 Sugihara matrices, algebras, and model structures Let I be the set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$, and let $I^+ = I \cup \{+\omega, -\omega\}$. The ordering $<_e$ on I^+, called the *extensional ordering*, is defined to be the natural ordering on I together with the proviso that for $x \in I$, $-\omega < x < +\omega$. The *intensional ordering*, $<_i$, on I^+ is defined by $x <_i y$ if either $|x| <_e |y|$ (where $|-\omega| = +\omega$), or $x = -y$ and $x <_e 0$. Let $s_n^0 = \{x : x \in I \;\&\; |x| <_e n + 1\}$, and let $s_n = s_n^0 - \{0\}$. The *Sugihara matrices* are quintuples $\langle \Sigma, \vee, \sim, \rightarrow, \mathcal{D} \rangle$ where: (a) Σ is I^+, $I^+ - \{0\}$, I, $I - \{0\}$, s_n^0, or s_n; (b) $x \vee y = \max\{x, y\}$ relative to $<_e$; (c) $\sim x$ = the numerical

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negative of x, and $\sim 0 = 0$; (d) $x \to y$ is defined by: if $x \leqslant_e y$ then $x \to y = \sim x \vee y$, otherwise $x \to y = \sim(x \vee \sim y)$; and (e) $\mathcal{D}$ (the set of designated elements) $= \{x: -1 <_e x\}$. If $\Sigma = s_n^0$ or s_n, we describe the matrix as a *finite* or *finite normal* Sugihara matrix respectively, and denote it by S_n^0 or S_n. If $\Sigma = I$ or $I - \{0\}$, we call the matrix the *infinite* or *infinite normal* Sugihara matrix respectively, and denote it by S^0 or S. If $\Sigma = I^+$ or $I^+ - \{0\}$, we call it the *complete infinite* or *complete infinite normal* Sugihara matrix respectively, and denote it by S_ω^0 or S_ω.

We now define our language L to consist of a denumerable number of proposition letters, closed in the usual way under $\vee$, $\sim$, and $\to$. Capital letters from the front of the alphabet function as metalinguistic variables. $A \& B$, $A \oplus B$, and $A \circ B$ are abbreviations of $\sim(\sim A \vee \sim B)$, $\sim A \to B$, and $\sim(A \to \sim B)$, respectively. A function V: $L \to \Sigma$ is an S_n^0 *valuation* (or S_n, S^0, S, S_ω^0, S_ω valuation, depending on choice of Σ) iff V satisfies $V(A \vee B) = V(A) \vee V(B)$, $V(\sim A) = \sim V(A)$, and $V(A \to B) = V(A) \to V(B)$. (Note that the left hand side connective in these equations is the propositional operator. The right hand side connective is the algebraic operator.) A formula A is S_n^0 *valid* (or S_n, etc., valid) (written, as usual $\models_{S_n^0} A$, etc.) iff for all S_n^0 valuations (or S_n, etc., valuations) V, $V(A) \in \mathcal{D}$.

Let $K^0 = \{T, T^*, a_1, a_1^* \dots\}$ and let $K_n^0 = \{T, T^*, a_1, a_1^*, \dots, a_{n-1}, a_{n-1}^*\}$. Then a pair $\langle K, R\rangle$, where $K = K^0$ (or K_n^0 for some n) is the *Sugihara model structure* (or a finite Sugihara model structure), denoted by $\underset{\sim}{S}^0$ (or $\underset{\sim}{S}_n^0$), iff: (a) the elements of K are distinct, and $*$ is a function $*$: $K^0 \to K^0$ satisfying $a^{**} = a$ for all $a \in K^0$; (b) R is a ternary relation on K^0; and (c) if we let $\dots < a_{i-1}^* < \dots < a_1^* < T^* < T < a_1 < \dots < a_{i-1} < \dots$ and let $a \leqslant b$ iff $a < b$ or $a = b$, then R satisfies $(\forall abc \in K^0)(Rabc$ iff $(\exists d)(a \leqslant d$ and $b \leqslant d^*$ and $d \leqslant c)$ or $(a \leqslant T$ and $b \leqslant c)$ or $(a \leqslant T$ and $b \leqslant c)$ or $(a \leqslant b = c)$ or $(b = c \leqslant a^*)$. To obtain the *normal* and *finite normal* Sugihara model structures, $\underset{\sim}{S}$ and $\underset{\sim}{S}_n$, set $T = T^*$ in $\underset{\sim}{S}^0$ and $\underset{\sim}{S}_n^0$ respectively. A function I: $LxK^0 \to \{1, 0\}$ is an *interpretation* on a Sugihara model structure $\langle K, R\rangle$ iff I satisfies: (a) $I(\sim A, a) = 1$ iff $I(A, a^*) \neq 1$; (b) $I(A \vee B, a) = 1$ iff either $I(A, a) = 1$ or $I(B, a) = 1$; (c) $I(A \to B, a) = 1$ iff $(\forall bc)$ (not all of $Rabc$ and $I(A, b) = 1$ and $I(B, c) \neq 1$); and (d) if $a \leqslant b$ and $I(A, a) = 1$ then $I(A, b) = 1$. A formula A is *valid* on $\underset{\sim}{S}^0$ (or $\underset{\sim}{S}_n^0$, $\underset{\sim}{S}$, $\underset{\sim}{S}_n$) iff for all interpretations I on $\underset{\sim}{S}^0$ (or $\underset{\sim}{S}_n^0$, etc.), $I(A, T) = 1$.

If $\langle K, R\rangle$ is a Sugihara model structure, we define the *associated Sugihara (set) algebra* to be the quintuple $\langle U, \vee, \sim, \to, D\rangle$ where: (a) $U \subseteq \mathcal{P}(K)$ (the power set of K) is such that if $a \in x \in U$ and $a \leqslant b$ then $b \in x$; (b) $\vee$ is set theoretic union; (c) $a \in \sim x$ iff $a^* \notin x$; (d) $a \in x \to y$ iff $(\forall bc)$ (not all of $Rabc$ and $b \in x$ and $c \notin y$); and (e) $x \in D$ iff $T \in x$. For $\langle K, R\rangle = \underset{\sim}{S}^0$, $\underset{\sim}{S}_n^0$, $\underset{\sim}{S}$, $\underset{\sim}{S}_n$ we call the associated algebra the *Sugihara algebra*, *finite* Sugihara algebra, *normal* and *finite normal* Sugihara algebras, respectively, and denote them by $\underset{\approx}{S}^0$, $\underset{\approx}{S}_n^0$, $\underset{\approx}{S}$, and $\underset{\approx}{S}_n$, respectively. A function $\mathcal{A}$: $L \to \underset{\approx}{S}^0$ (or $\underset{\approx}{S}_n^0$, etc.) is called an *assignment* on $\underset{\approx}{S}^0$, etc., iff $\mathcal{A}(A \vee B) = \mathcal{A}(A) \vee \mathcal{A}(B)$, $\mathcal{A}(\sim A) = \sim\mathcal{A}(A)$ and $\mathcal{A}(A \to B) = \mathcal{A}(A) \to \mathcal{A}(B)$. A formula A is $\underset{\approx}{S}^0$-*valid* or ($\underset{\approx}{S}_n^0$ valid, etc.) iff for all assignments $\mathcal{A}$ on $\underset{\approx}{S}^0$ (etc.), $\mathcal{A}(A) \in D$.

3 Results The main result reported in this note is that validity in a Sugihara model structure, in its associated set algebra, and in an appropriately chosen Sugihara matrix all coincide. Proof of these facts is quite long, and is given in full in [4]. Here only the main steps are sketched. We begin by noting that part of the result, namely the connection between the model structures and their associated set algebras, is a more-or-less immediate consequence of the general connection between model structures and set algebras established by Routley and Meyer in [5]. Thus:

Theorem 1 $\models_X A$ *iff* $\models_{\underset{\sim}{X}} A$, *for* $X = S^0, S^0_n, S$, *or* S_n.

We proceed to connect the set algebras with the matrices. We note an important preliminary fact: that the set algebras in question will all be complete as lattices, since arbitrary set theoretic unions and intersections exist. But the infinite and infinite normal Sugihara matrices originally investigated in [1] are not complete as lattices, since they lack maximal and minimal elements. Naturally, all the finite matrices will be complete. Since we propose to establish isomorphism between the set algebras and suitable matrices, we need to complete the Sugihara matrices, hence the complete Sugihara matrices defined above. It is not difficult to show that validity in the complete matrices coincides with validity in their incomplete counterparts. Hence we have

Theorem 2 $\models_X A$ *iff* $\models_{X_\omega} A$, *for* $X = S^0, S$.

We now link the complete matrices with the set algebras. Let $\langle \Sigma, \vee, \sim, \rightarrow, \mathcal{D}\rangle$ be a complete (finite or infinite) Sugihara matrix and $\langle U, \vee, \sim, \rightarrow, D\rangle$ a Sugihara algebra arising from a model structure $\langle K, R\rangle$. K is either $\{T, T^*, a_1, a_1^*, \ldots\}$ or $\{T, T^*, a_1, a_1^*, \ldots, a_{n-1}, a_{n-1}^*\}$. T and T^* may or may not be distinct, but all other elements are distinct. Let K^+ be the set of unstarred elements of K, including T. Then the *natural correspondence* f between Σ and U is defined to be the function $f: \Sigma \rightarrow U$ satisfying: (a) if $0 \in \Sigma$ then $f(0) = K^+$; (b) for $n > 0$, $f(n) = K^+ \cup \{a_0^*, a_1^*, \ldots a_{n-1}^*\}$ (where $a_0 = T$); and (c) for $n < 0$, $f(n) = K^+ - \{a_0, a_1, \ldots, a_{n-1}\}$.

Theorem 3 *f is a 1-1 correspondence, and $x \in \mathcal{D}$ iff $f(x) \in D$.*

Theorem 4 $f(\sim A) = \sim f(A), f(A \vee B) = f(A) \vee f(B)$.

The proofs of these two theorems are straightforward adaptations of the general results of Routley and Meyer [5].

Theorem 5 $f(A \rightarrow B) = f(A) \rightarrow f(B)$.

In order to establish Theorem 5, we need some lemmas.

Lemma 1 *If $\langle K, R\rangle$ is a Sugihara model structure, $Rabc$ iff Rac^*b^*.*
Lemma 2 *In any Sugihara algebra, $x \rightarrow y = \sim y \rightarrow \sim x$, and $x \,\&\, y = x \cap y$.*
Lemma 3 *In any Sugihara model structure, Raa^*a.*
Lemma 4 *In any Sugihara model structure, $Rabc$ entails ($a \leqslant c$ or $b \leqslant c$) and ($a \leqslant c$ or $a \leqslant b^*$).*
Lemma 5 *In any Sugihara algebra $\langle U, \vee, \sim, \rightarrow, D\rangle$ if $b \notin x \in U$ and $a \leqslant b$ then $a \notin x$.*
Lemma 6 *If $f: \Sigma \rightarrow U$ is the natural correspondence between a Sugihara matrix and algebra, then $x \leqslant y$ (numerically) iff $f(x) \subseteq f(y)$.*

In view of Lemmas 2 and 6 and the definition of $\vee$ and $\rightarrow$ in Sugihara matrices, viz., if $x \leqslant y$ then $x \rightarrow y = \sim x \vee y$ and otherwise $x \rightarrow y = \sim x \,\&\, y$, it suffices to prove Theorem 5 if we can prove the following for any Sugihara algebra: if $x \subseteq y$ then $x \rightarrow y = \sim x \cup y$, and if $y \subset x$ then $x \rightarrow y = \sim x \cap y$. We split the proof into two parts and, in turn, divide each of these in two.

1. If $x \subseteq y$ then $x \rightarrow y = \sim x \cup y$.

1.1 If $x \subseteq y$ and $a \notin x \rightarrow y$ then $a \notin \sim x \cup y$. Assume the antecedent. Now $a \notin \sim x \cup y$ iff $a \notin \sim x$ and $a \notin y$. So we need to prove that $a \notin \sim x$ and $a \notin y$. We prove first that $a \notin y$. Since $a \notin x \rightarrow y$, $(\exists b, c)$ $(Rabc \,\&\, b \in x \,\&\, c \notin y)$. By Lemma 3, $Rabc$ implies $a \leqslant c$ or $b \leqslant c$. If $a \leqslant c$ and $c \notin y$ then by Lemma 5, $a \notin y$. Now $c \notin y$, so by Lemma 5 $b \notin y$. But $x \subseteq y$, so $b \notin x$, contradicting $b \in x$. Hence not $b \leqslant c$, so $a \leqslant c$, so $a \notin y$ as required.

We note now that $x \subseteq y$ implies $\sim y \subseteq \sim x$. Also by Lemma 2, $x \rightarrow y = \sim y \rightarrow \sim x$. Thus from the assumption of the antecedent we have $\sim y \subseteq \sim x$ and $a \notin \sim y \rightarrow \sim x$. So by a similar argument to the one just given, $a \notin \sim x$.

1.2 If $x \subseteq y$ and $a \notin \sim x \cup y$ then $a \notin x \rightarrow y$. Suppose the antecedent. As in 1.1, we need to prove that $a \notin \sim x$ and $a \notin y$ implies $a \notin x \rightarrow y$. Now $a \notin \sim x$ implies $a^* \in x$. By Lemma 3, Raa^*a. So Raa^*a and $a^* \in x$ and $a \notin y$. Hence $a \notin x \rightarrow y$.

2. If $y \subset x$ then $x \rightarrow y = \sim x \cap y$.

2.1 If $y \subset x$ and $a \notin x \rightarrow y$ then $a \notin \sim x \cap y$. Suppose the antecedent. We need to prove $a \notin \sim x \cap y$, i.e., $a \notin \sim x$ or $a \notin y$. Since $a \notin x \rightarrow y$, for some b, c we have $Rabc$ and $b \in x$ and $c \notin y$. Now by Lemma 4, $Rabc$ implies $a \leqslant c$ or $a \leqslant b^*$. If $a \leqslant c$ then, since $c \notin y$, by Lemma 5 $a \notin \sim y$; so that $a \notin \sim x$ or $a \notin y$. Hence, suppose instead $a \leqslant b^*$. Now $b \in x$, so $b^* \notin \sim x$. So by Lemma 5, $a \notin \sim x$; so that $a \notin \sim x$ or $a \notin y$.

2.2 If $y \subset x$ and $a \notin \sim x \cap y$ then $a \notin x \rightarrow y$. Now $a \notin \sim x \cap y$ iff $a \notin \sim x$ or $a \notin y$. So we need to prove both $a \notin \sim x$ implies $a \notin x \rightarrow y$ and $a \notin y$ implies $a \notin x \rightarrow y$. Suppose first $a \notin y$. Now $y \subset x$, so not every world is in y. Select the largest world relative to $\leqslant$ not in y. Denote it by b; i.e., $b \notin y$. Now $a \notin y$ and b is the largest world not in y, so $a \leqslant b$. By the third disjunct of the definition of R, viz., $a \leqslant b = c$, we can thus have $Rabb$. But we must also have $b \in x$, because when we construct members of the set algebra we make such sets of worlds progressively larger by adding the largest of the remaining nonmembers. If we add any smaller member than the largest nonmember of y to y then by Lemma 5 we must also add the largest: members of the algebra are closed upwards under $\leqslant$. So since b is the largest nonmember of y, b is in every proper superset of y. But $y \subset x$, so $b \in x$. Thus $Rabb$ and $b \in x$ and $b \notin y$. So $a \notin x \rightarrow y$.

Now let $a \notin \sim x$. Since $y \subset x$, $\sim x \subset \sim y$. Hence, by an identical argument to the one just given, $a \notin \sim y \rightarrow \sim x$. But $\sim y \rightarrow \sim x = x \rightarrow y$. Thus, $a \notin x \rightarrow y$.

From the preceding theorems it follows that the natural correspondence f is an isomorphism which also preserves designated elements. Hence validity in the various Sugihara matrices coincides with validity in the various complete Sugihara set algebras. Combining this with Theorems 1 and 2, we have:

Theorem 6 *The following statements are equivalent:*

(1) $\models_{S_n^0} A$. (2) $\models_{\underline{S}_n^0} A$. (3) $\models_{\underset{\sim}{S}_n^0} A$.

The following are equivalent:

(4) $\models_{S_n} A$. (5) $\models_{\underline{S}_n} A$. (6) $\models_{\underset{\sim}{S}_n} A$.

The following are equivalent:

(7) $\models_{S^0} A$. (8) $\models_{S_\omega^0} A$. (9) $\models_{\underline{S}^0} A$. (10) $\models_{\underset{\sim}{S}^0} A$.

The following are equivalent:

(11) $\models_{S} A$. (12) $\models_{S_\omega} A$. (13) $\models_{\underline{S}} A$. (14) $\models_{\underset{\sim}{S}} A$.

4 Concluding remarks We conclude with some observations on the connection between model structures and matrices. The connection between syntax and semantics for a logic is in a sense *prima facie* mysterious. Some of the mystery can be removed by showing that syntax and semantics are different sides of the same coin. Several ways of doing this are current in the literature. The Lindenbaum algebra, a linguistic construction, links algebraic semantics with syntax. Similarly, the canonical model links worlds semantics with syntax; and tableau constructions can be semantically or alternatively proof theoretically oriented, the difference in some cases being difficult to discern. In addition to the syntax-semantics link, connections between different semantics for the one logic are worth making, else we might wonder why different kinds of semantics characterise the one logic. Theoretically this might be done via the syntax, but it is always worth making the connection directly, especially if we wish to remain open on the doctrine that semantics can be genuinely explanatory rather than covert syntax. A matrix is a puzzle: how does it arise to characterise a logic? One important aspect of Lemmon's work, developing as it did from Stone's representation theorem for Boolean Algebras was to show that matrices characterising modal logics are algebras and can be viewed as deriving from model structures. If we regard, as many have, the worlds semantics as explanatory of modal logics, Lemmon's results thus provide an explanation of algebraic semantics and matrices of those logics.

The present results can be viewed along these lines. The connection between the worlds semantics and the complete Sugihara matrices via the Sugihara set algebras 'explains' the matrices, by allowing them to be seen as transformations of worlds structures. This raises several questions, however. One is, from whence the incomplete Sugihara matrices S^0 and S, which are not isomorphic with any of the set algebras? The answer is that RM has a certain compactness feature, in that for purposes of validity of a particular formula we only need consider a finite Sugihara algebra, and the union of all such is incomplete as a lattice. It follows that the hope of showing all matrices to be explicable in the direct way indicated above is unwarranted. Nevertheless, large classes of matrices have proved themselves to be amenable to this treatment. If we adopt this view of 'explanation', we can conclude that there is a sense in which the 'real' Sugihara matrices are those explained directly in terms of worlds structures, namely those with maximal and minimal elements $\{+\omega, -\omega\}$. We can also conclude that, in looking for explanations of matrices, none of those which lack maximal and minimal elements, indeed which are incomplete as lattices, are directly so explicable.

Another question is how we manage to deal with algebraic structures such as chains, using as we do an essentially Boolean power-set construction on the set of worlds. The answer is that the so-called Hereditary Condition in the Routley-Meyer worlds semantics for relevant logics forces a collapse of certain elements in the power set algebra on the set of worlds, and thus generates non-Boolean set algebras. On these matters see Mortensen [4] and Routley and Meyer [5].

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RELEVANCE AND VERISIMILITUDE

1. INTRODUCTION: VERISIMILITUDE AND CHOICE OF LOGIC

As many authors from Popper onwards have noted, it would be highly desirable to have a theory of nearness to the truth. Therefore, the results of Miller and Tichy ([6], [11]), apparently demonstrating the unsatisfactoriness of Popper's qualitative theory of verisimilitude, led to considerable research attempting to find a better account of nearness to the truth. Almost without exception, this research concentrated on modifying the definition of verisimilitude with an eye to escaping Miller-Tichy-style limitative results, while retaining the base of classical logic on which the results also depend. In Mortensen [7], however, it was shown that the Miller-Tichy result could be escaped while retaining Popper's original theory, by modifying the logical base on which scientific theories are constructed. Needless to say, the interest of such a result varies with the plausibility of the logical base.[1] It is therefore gratifying that the new logical base could be any one of the usual relevant or relevance logics developed originally by Anderson and Belnap,[2] since it is independently arguable that these logics represent an improvement over classical logic as candidates for the logic of ordinary inferential situations, and especially those of science. It should also be emphasised that while in [7] it was shown that the *letter* of the Miller-Tichy theorem could be escaped in relevant logic, it was pointed out there that it was still an open question whether some *analogue* of the theorem which did not rely on classical assumptions was provable. As we see later, this turns out to be the case.

Arguably, choice of logic is governed partly by pragmatic considerations.[3] Hence the 'relevance program' is devoted to investigating what kinds of changes the move to a relevant logical base might make to mathematics and science, especially their foundations. In this connection it is worth mentioning Meyer's result that Peano arithmetic on a relevant logic base can escape at least some versions of Gödel's Second Incompleteness Theorem.[4] Accordingly, if the relevant logics have the pragmatic advantage over classical logic – that the mathematics and

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science erected on their base have nicer or more useful properties – they are to that extent more desirable. If a theory of verisimilitude is available to relevant logic and not to classical logic, that is some argument for relevant logic. Or if the theory of verisimilitude is simpler or more intuitive in the one than in the other, the same point holds.

This is not to detract from the research which has gone on within the classical program. If a classical verisimilitude can be found with appealing properties, all to the good. It was presumably for a reason, though, that Popper hit on his qualitative definition of verisimilitude. Inspection of that definition suggests that it provides at least a desirable *sufficient* condition for one theory's being nearer to the truth than another. The Miller-Tichy theorem did nothing to threaten that intuition. It showed that too *few* pairs of theories were related by the verisimilitude relation.

Specifically, on the definition, one theory *A* could be closer to the truth than another *B* only if *A* contained no false sentences. Furthermore, in such a case *B* would contain no more true sentences than *A*, and if *B* contained exactly as many false sentences (viz. none), then *B* would contain, strictly, fewer true sentences. It is not unreasonable to say that such pairs of theories *are* related in such a way that the former is nearer the truth than the latter. The problem is that, following the Miller-Tichy theorem, these are the *only* pairs of theories so related. That was the difficulty for the Popperian position, because for science to have made progress by increasing the verisimilitude of theories, pairs of false theories will need to have been so related. Consequently, we should not conclude that Popper's original definition has nothing to do with verisimilitude. Rather, the best we can conclude is that Popper's definition plus classical logic do not give a *necessary* condition for one theory's being closer to the truth than another. But then, the intuition that Popper's theory has enough to do with verisimilitude to provide a sufficient condition should lead us to take seriously the question whether it is the classical logic constraint which should go in the quest for jointly necessary and sufficient conditions.

These, then, are several reasons for examining Popper's definition against a background of nonclassical logic.[5] The purpose of this paper is to show that the prospects for using Popper's definition and relevant logics in harness are considerably dimmer than they looked to be from the standpoint of the earlier paper, [7]. A limitative result, somewhat resembling the Miller-Tichy theorem, is provable for the verisimilitude ordering in even very weak logics. By itself, that would not mean that the prospects are altogether extinguished. As we will see below, there are available two intensional versions of Popper's definition which preserve most of the intuitions supporting the original. The simpler, more natural of these is, however, subject to the same limitative results. The less simple definition is apparently not, but it has a serious defect of its own.

The plan of this paper is as follows. In section 2 the Miller-Tichy theorem is reviewed, precise logical conditions are placed on its range of application, and some consequences are discussed. In section 3, the countertheorem of [7] is discussed. In section 4, the main result of this paper will be proved, namely that undesirably few pairs of theories of relevant logics are related by the relation *has greater verisimilitude than*. In section 5, it will be shown that two classically equivalent versions of the verisimilitude ordering are not equivalent relevantly (or even modally), and that only one of these is subject to the aforementioned limitative theorems. The other is shown to have a serious defect of its own, however.

2. THE MILLER-TICHY THEOREM REVIEWED

We begin with a formal language L, which is a set of sentences closed under conjunctions '&', disjunctions 'v', negations '~', and implications '→'. A logic L is a subset of L closed under the rule of uniform substitution. If L is a logic and $a \in \mathrm{L}$, we write $\vdash_{\mathrm{L}} a$. An implication logic is a logic with the property that if $\vdash_{\mathrm{L}} a$ and $\vdash_{\mathrm{L}} a \rightarrow b$ then $\vdash_{\mathrm{L}} b$.

DEFINITION 1. A subset A of L, $A \subseteq L$, is an *L-theory* (relative to logic L) iff (1) if $a \in A$ and $\vdash_{\mathrm{L}} a \rightarrow b$ then $b \in A$, and (2) if $a \in A$ and $b \in A$ then $a \,\&\, b \in A$. If L = classical logic, A is said to be a classical theory.

DEFINITION 2. Let A be an L-theory. A is *inconsistent* iff for some a, both $a \in A$ and $\sim a \in A$. A is *incomplete* iff for some a, both $a \notin A$ and $\sim a \notin A$. A is *trivial* iff $A = L$. *The rule* γ *holds for* A iff if $a \in A$ and $\sim a \vee b \in A$ then $b \in A$. A is *prime* iff whenever $a \vee b \in A$, at least one of a, $b \in A$ also.

We remark that γ holds for every classical theory, but that it does not hold for every L-theory if, say, L is some relevant logic. In particular, because it is not a theorem of any relevant logic that $(a \,\&\sim a) \rightarrow b$, there are inconsistent relevant theories which are not trivial. This is one of the reasons why relevant logics have been thought to be preferable to classical logic; namely, that since inconsistent nontrivial scientific theories seem to occur,[6] some relevant logic is a better model of scientific reasoning than classical logic. Provided that the logic L has very weak properties, then for any such inconsistent but nontrivial L-theory, the rule γ must fail for it, since it is immediate that if A is inconsistent and γ holds for A, then A is trivial.

We are now in a position to consider Popper's definition of verisimilitude. Let A and B be L-theories (for Popper, L = classical logic), and let T be the set of true sentences of the language L and F the set of falsehoods. Then A has greater verisimilitude than B just in case (1) $A \cap T \supseteqq B \cap T$ (2) $A \cap F \subseteqq B \cap F$, and (3) $B \cap T \subset A \cap T$ and $B \cap F \not\subset A \cap F$, or $A \cap T \not\subset B \cap T$ and $A \cap F \subset B \cap F$. As noted by several authors, this is equivalent to Definition 4 below, and it is that definition which we will work with for the time being. Before coming to it, however, it will be convenient to build it up from two other definitions.

DEFINITION 3. $A >_1 B$ iff $B \cap T \subset A \cap T$ and $A \cap F \subseteq B \cap F$. $A >_2 B$ iff $B \cap T \subseteq A \cap T$ and $A \cap F \subset B \cap F$.

DEFINITION 4. (Popper) $A > B$ (A has more verisimilitude than B) iff $A >_1 B$ or $A >_2 B$.

Now we can proceed to review Miller and Tichy's result. It will also be convenient to break it up into two parts, corresponding to the two parts of Definition 3. We state the results more generally than Miller or Tichy did, since we consider arbitrary L-theories (and so detail the precise logical properties of L required), whereas Miller and Tichy assume that L is classical.

THEOREM 1. Let A, B be L-theories, and suppose that (1) $\vdash_L (a \,\&\, b) \rightarrow a$ (2) if $a \in F$ then $a \,\&\, b \in F$, for arbitrary a, b, and (3) $T \cup F = L$. Then if $A >_1 B$, then $A \subseteq T$.

Proof. If $A >_1 B$, then $B \cap T \subset A \cap T$ and $A \cap F \subseteq B \cap F$. Suppose, for contradiction, $A \not\subseteq T$. Let $f \in A$ and $f \notin T$, so that $f \in F$, by (3) above. Also, let $a \in A \cap T - B \cap T$, so that $a \in A$, $a \in T$ and $a \notin B$. First, we show that $a \,\&\, f \in A \cap F$. (Reason: $a \in A$ and $f \in A$, so $a \,\&\, f \in A$, because A is an L-theory. Also, $f \in F$, so $a \,\&\, f \in F$, by (2) above.

Hence $a \& f \in A \cap F$.) Second, we show that $a \& f \notin B \cap F$, contradicting $A \cap F \subseteq B \cap F$. (Reason; $a \notin B$, so $a \& f \notin B$, by (1); so $a \& f \notin B \cap F$.)

THEOREM 2. Let A, B be L-theories, and suppose that (4) $\vdash_L a \rightarrow (b \vee a)$, (5) if $a \in T$ then $a \vee b \in T$, for arbitrary a, b, (6) $T \cup F = L$, (7) F is consistent, (8) γ holds for A. Then if $A >_2 B$, then $A \subseteq T$.

Proof. If $A >_2 B$, then $B \cap T \subseteq A \cap T$ and $A \cap F \subset B \cap F$. Suppose, for contradiction, $A \not\subseteq T$. As before, let $f \in A$ and $f \in F$. Also let $b \in B \cap F - A \cap F$, so that $b \in B$, $b \in F$ and $b \notin A$. First, we show that $\sim f \vee b \in B \cap T$. (Reason: $b \in B$, so $\sim f \vee b \in B$, by (4). Also, $f \in F$, so $\sim f \in T$, by (7); so $\sim f \vee b \in T$, by (5). Hence $\sim f \vee b \in B \cap T$.) Second, we show that $\sim f \vee b \notin A \cap T$, contradicting $B \cap T \subseteq A \cap T$. (Reason: $b \notin A$, but $f \in A$, so $\sim f \vee b \notin A$, by (8). Hence $\sim f \vee b \notin A \cap T$.)

We will make two related observations. First, the condition (7) above, that the set of falsehoods F be consistent, is not the supposition that propositions of the form $a \& \sim a$ do not belong to F, but the supposition that for no a are both a, $\sim a$ in F. (That is not necessarily something which would be cogenial to an intuitionist.) Second, if A is a classical theory, then γ holds for A. For suppose that a, $\sim a \vee b$, are both in A. Then so is $a \& (\sim a \vee b)$. But $\vdash_{PM}(a \& (\sim a \vee b)) \rightarrow b$. Hence $b \in A$. We can now combine Theorems 1 and 2 to give the following:

THEOREM 3. (Miller-Tichy). Let A, B be classical theories (for which all the above properties hold), and let T, F be as above. Then if $A > B$ then $A \subseteq T$.

In passing, an interesting consequence can be drawn from Theorem 3 which does not seem to have been widely noticed. As noted by Hilpinen [12] and Niiniluoto [13], there is a close connection between the theory of verisimilitude and David Lewis's theory of counterfactuals. The latter depends for its intelligibility on an overall measure of relative similarity (or nearness) of worlds to a given world. Lewis intuits that there must be such a measure. But if worlds are describable by sets of propositions, then evidently by Theorem 3 Popper's definition is not available for such a measure. We can also deduce (see Miller [6]) that selecting some distinguished set of propositions and measuring similarity of worlds relative to these has the consequence that relative similarity is not invariant with respect to logically equivalent sets of propositions chosen to describe worlds. We would have to say that *logically equivalent features* of worlds do not determine the same similarity relations. Lewis might be prepared to accept this conclusion holding that similarity is a matter of dimension of interest, though this move is presumably unacceptable to anyone who holds that the truth of counterfactuals is not so highly pragmatic a matter. Of course it is open to Lewis to reject the Popperian measure of similarity, but then he needs to offer another. In particular, the theories of verisimilitude of Niiniluoto [13] and Tichy (see *e.g.* [11]), would appear not to be open to Lewis, since they seem to treat verisimilitude as sentential rather than propositional. Whatever is the case here, it does seem worth emphasizing that results from the theory of verisimilitude throw into question certain theories of counterfactuals.

3. A COUNTERTHEOREM

A question which now arises concerns the extent to which Theorem 3 is dependent on the assumption of γ for A; that is, the assumption of those logics for which γ holds for all theories, such as classical logic, modal logic or intuitionism (though we have noted that an intuitionist might object to another part of the proof). A suggestive answer was given in [7].

THEOREM 4. There exist RM3-theories A, B such that $A > B$ and $A \not\subseteq T$.

Several observations need to be made about that result. The logic RM3 is a three-valued logic discussed by Anderson and Belnap.[8] RM3 is of interest because it is weaker than classical logic, because γ fails for some RM3-theories, and because it is stronger than all the usual relevant logics B, T, E, EM, R, RM, etc. Since it is easy to show that if A is an L-theory and L is stronger than L′ then A is an L′-theory, it follows from Theorem 4 that the Miller-Tichy theorem breaks down for all the usual relevant logics. Furthermore, Theorem 4 was proved by constructing two RM3-theories which were inconsistent but nontrivial, so that γ failed for them, while $A >_2 B$ was made to hold. That is strong evidence that Theorem 3 depends heavily on the assumption of γ for all theories. It also suggests that a viable theory of verisimilitude is available to the usual relevant logics. It must be noted, though, that one-half of the Miller-

Tichy result, namely Theorem 1, continues to hold given the other assumptions of the theorem, since it is a theorem of all these logics that $(a \& b) \rightarrow a$.

But while Theorem 3 breaks down for relevant logics, that is consistent with there being other proofs around of analogous results for those logics too (perhaps weaker results which are nonetheless strong enough to be damaging to the verisimilitude program), as noted in [7]. That proves to be the case, and it is what we will see in the next section.

4. LIMITATIVE THEOREMS FOR RELEVANT LOGICS

As a way in, let us observe that it is easy to show that if an L-theory (for any L) is consistent and prime (see Definition 2), then γ holds for it. (For suppose $a \in A$ and $\sim a \vee b \in A$. Since A is consistent, $\sim a \notin A$; but A is prime, so at least one of $\sim a$, $b \in A$. So $b \in A$.) Also, as we noted before, theorems like $a \& b \rightarrow a$ and $a \rightarrow (b \vee a)$ hold for all the usual relevant logics. In what follows, we will (unless we give warning) assume that the L-theories we talk about have L = any one of the usual relevant logics, or classical logic. So, on the assumption (which we will henceforth make) that T and F behave themselves, as in Theorems 1–3, we can deduce from Theorem 2:

THEOREM 5. Let A, B be L-theories and let A be consistent and prime. Then if $A >_2 B$ then $A \subseteq T$. Hence if $A > B$ then $A \subseteq T$.

Thus the Miller-Tichy result holds for all consistent prime theories. This means that at best Popper's verisimilitude ordering is only useful as an ordering on theories which are either inconsistent or non-prime. That is perhaps a tolerable state of affairs if one is really wedded to '>' as one's theory of verisimilitude, though evidently it would be nice to have a verisimilitude ordering on pairs of false consistent prime theories as well. However, worse is to come.

THEOREM 6. Let A, B be L-theories, and let A be prime. Then if $A >_2 B$ then $T \subseteq A$.

Proof. If $A >_2 B$, then $B \cap T \subseteq A \cap T$ and $A \cap F \subset B \cap F$. Suppose, for contradiction, that $T \not\subseteq A$. Let $t \in T$ and $t \notin A$. Also, let $b \in B \cap F - A \cap F$, so that $b \in B$, $b \in F$ and $b \notin A$. First, we show that $t \vee b \in B \cap T$. (Reason: $b \in B$, so $t \vee b \in B$ by (4) of Theorem 2. Also,

$t \in T$, so $t \vee b \in T$, by (5). Hence $t \vee b \in B \cap T$.) Second, we show that $t \vee b \notin A \cap T$, contradicting $B \cap T \subseteq A \cap T$. (Reason: $t \notin A$ and $b \notin A$; but A is prime, so $t \vee b \notin A$. Hence $t \vee b \notin A \cap T$.)

Combining Theorems 1 and 6, we can deduce.

THEOREM 7. Let A, B be L-theories and A be prime. Then if $A > B$ then $A \subseteq T$ or $T \subseteq A$.

Various changes can be rung on the previous theorems, though they are less important for our purposes.

THEOREM 8. Under the conditions of Theorem 1, if $A >_1 B$ and A is complete, then $A = T$.

Proof. Theorem 1 establishes that $A \subseteq T$. To show that $T \subseteq A$, suppose not and let $t \in T$ and $t \notin A$. Also, let $a \in A \cap T - B \cap T$. We show that $a \,\&\, {\sim}t \in A \cap F$ and $\notin B \cap F$, contradicting $A \cap F \subseteq B \cap F$. First, $a \in A$; and, since A is complete, ${\sim}t \in A$. Hence $a \,\&\, {\sim}t \in A$. Also, $t \in T$; so ${\sim}t \in F$, so $a \,\&\, {\sim}t \in F$. Hence $a \,\&\, {\sim}t \in A \cap F$. Second, $a \notin B \cap T$ but $a \in T$, so $a \notin B$. Hence $a \,\&\, {\sim}t \notin B$; hence $a \,\&\, {\sim}t \notin B \cap F$.

THEOREM 9. Under the conditions of Theorems 5 and 6, if $A >_2 B$ and A is consistent and prime, then $A = T$.

Proof. Immediate from Theorems 5 and 6.

THEOREM 10. Under the conditions of Theorem 3, if A, B are classical theories and A is consistent and complete, then if $A > B$ then $A = T$.

Proof. We show that A has to be prime. Let $a \vee b \in A$ and suppose for contradiction that $a \notin A$, $b \notin A$. By the completeness of A, ${\sim}a \in A$ and ${\sim}b \in A$. Hence ${\sim}a \,\&\, {\sim}b \in A$, hence by De Morgan's law ${\sim}(a \vee b) \in A$, contradicting the consistency of A. Therefore A is prime. The theorem now follows immediately from Theorems 8 and 9.

These results, particularly Theorems 5–7, are unfortunate for the theory of relevant Popperian verisimilitude. It is hardly desirable that the only way for A to be closer to the truth than B other than A's containing no falsehoods, is that it contain all truths; but that is what is forced on us if A is prime.

Admittedly, it is still open to use $A >_2 B$ as a verisimilitude ordering on the class of non-prime theories; and that is not totally restrictive. Many interesting theories are non-prime: it only needs the theory to commit itself to $a \vee b$ without committing itself to a or to b. An L-theory is said to be *L-regular* iff it contains every theorem of L (relative to the language *L*). But if $\vdash_L a \vee \sim a$, then every regular incomplete L-theory is non-prime (since if the theory is incomplete, then for some a, neither a nor $\sim a$ is the theory; but, being regular, $a \vee \sim a$ is in the theory). Thus Peano arithmetic formulated on a base of classical logic is both incomplete (by Gödel's Incompleteness Theorem), and regular (since, classically, any sentence implies every theorem). This theory is therefore non-prime, but it is hardly uninteresting. Of course, since it is a classical theory, γ holds for it, so Theorem 5 does also, as does the original Miller-Tichy Theorem 3. (Non-primeness does not imply the failure of γ.) Nevertheless, primeness is not an inescapable desideratum for scientific theories.

To sum up these results, Popper's definition gives an acceptable ordering of verisimilitude for pairs of theories A, B, for which the theory of greater verisimilitude, A, satisfies one or more of the following conditions: (1) $A \subseteq T$, (2) $T \subseteq A$, (3) A is non-prime and γ fails for A. So the definition is not entirely useless. But it is restrictive enough. It would be desirable if the restrictions could be escaped and a verisimilitude ordering defined on false, prime or γ-satisfying theories.

5. INTENSIONALISING THE METALANGUAGE, AND TWO VERSIONS OF POPPER'S DEFINITION

In this section, we will consider briefly the prospects for taking advantage of intensional logic by intensionalising the metalanguage in which Popper's definition is given. We will see that, while definitions which remain close to the spirit of Popper's are available, all of them have serious defects.

Given that we have available an intensional implication operator, $\rightarrow$ (which need not be a relevant implication; the modal $\multimap$ is a candidate), we can consider the development of an intensional set theory which uses intensional operations and relations on sets as well as extensional ones. Let us define $A \subseteq B$ to mean $(x)(x \in A \rightarrow x \in B)$, $A \leftrightharpoons B$ to mean $A \subseteq B$ and $B \subseteq A$, and $A \subset B$ to mean $A \subseteq B$ and $(\exists x)(x \in B \ \& \ x \notin A)$. Intensional set theory, particularly relevant set theory, has been studied, especially by Brady and by Routley, though the study is still in an early stage.[9] Now let us see what happens when we substitute these intensional relations for the extensional relations in Popper's definition.

First, consider the intensional analogue of $A > B$ (suggested by Richard Routley): either $B \cap T \sqsubset A \cap T$ and $A \cap F \sqsubseteq B \cap F$, or $B \cap T \sqsubseteq A \cap T$ and $A \cap F \sqsubset B \cap F$. This remains within the spirit of the intuitions which prompted Popper's definition save for the intensionalising move, which is why it is worth notice. Unfortunately, it is open to all the limiting results of its extensional cousin. As might be expected, it is a theorem of the standard relevant logics, and of all the usual modal logics, that $(x)(Ax \rightarrow Bx) \rightarrow (x)(Ax \supset Bx)$. It follows immediately that if $A \sqsubseteq B$ then $A \subseteq B$ and if $A \sqsubset B$ then $A \subset B$. Hence if the intensional versimilitude relation holds between sets, so does Popper's extensional relation, and so all the results of the previous sections can be proved of it. The trouble with Popper's definition was that too few pairs of theories satisfied it. If we intensionalise it this way we merely ensure that even less pairs do.

Second, let us recall that Popper's definition was actually two equivalent definitions. Does that equivalence hold intensionally? It does if the alternative intensional definition is as follows: A has more verisimilitude than B iff $A \cap T \overset{\subset}{\underset{\supset}{=}} B \cap T$; and $A \cap T \overset{\subset}{\underset{\supset}{=}} B \cap F$; and either $B \cap T \sqsubset A \cap T$ and $B \cap F \not\sqsubset a \cap F$, or $A \cap T \not\sqsubset B \cap T$ and $A \cap F \sqsubset B \cap F$. This is equivalent to the definition just considered and so open to all its objections. But the equivalence breaks down if the *comparability* condition is required only to be extensional, or even permitted not to hold at all: $B \cap T \sqsubset A \cap T$ and $B \cap F \not\sqsubset A \cap F$, or $A \cap T \not\sqsubset B \cap T$ and $A \cap F \sqsubset B \cap F$; with or without $A \cap T \overset{\subset}{\underset{\supset}{=}} B \cap T$ and $A \cap F \overset{\subset}{\underset{\supset}{=}} B \cap F$. There is no way to prove that this implies or is implied by the definition of the previous paragraph. Unfortunately, it fails to be transitive, as the reader can easily verify. Now that is not a *totally* damaging fact; we can make any relation transitive by taking its transitive closure. Nonetheless, if the aim is to capture the intuitive idea of nearness to the truth and if that idea is transitive, then the failure of a definition to be transitive argues the failure of the definition to capture the idea. Furthermore, taking the transitive closure of the definition must result in pairs of theories related by the proposed verisimilitude relation for which *neither* of the disjunctive conditions of the definition hold, a consequence which intuitions about sufficient conditions for nearness to the truth cannot justify. And, moreover, we cease to have a relation which we can reasonably determine holds in particular cases. If artificial measures are needed to make the definition transitive, the resulting concept is to that extent artificial.

5. CONCLUSION

Popper's definition looked initially promising provided that the restriction of classical logic was removed. As we have seen, this promise is not fulfilled. The search for a satisfactory verisimilitude ordering must therefore be pursued along more mainstream lines. The present exercise ought, however, to make us aware of the possibility that breakdowns of proposed definitions might only occur because of strictly classical assumptions.

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NOTES

[1] In this connection, it is easily shown that the Miller-Tichy result holds for intuitionism and for all the usual modal logics.
[2] See Anderson and Belnap [1], or Routley, Meyer *et al.*, [8].
[3] See, e.g., Routley [9].
[4] Thereby, as Meyer argues, rehabilitating the Hilbert program. It should be stressed that all primitive recursive functions are representable in relevant Peano arithmetic; so that Gödel's First Incompleteness Theorem, and *certain* versions of the Second, still hold. See Meyer [5].
[5] It goes without saying that another reason is the purely formal interest in determining precisely which jobs Popper's definition can do within the relevance program.
[6] See, e.g., Lakatos [3].
[7] Smart [10].
[8] Anderson and Belnap, *ibid.*
[9] See, e.g., Brady [2]. We mention one important result due to Brady. Naïve set theory, formulated on the base of certain relevant logics, is able to escape the trivialising consequences of the Russell and Curry paradoxes. This is of considerable interest for combining a relevance approach to logic with an anti-Platonist position on the ontological status of sets.

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"Aristotle's Thesis in Consistent and Inconsistent Logics"

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Aristotle's Thesis in Consistent and Inconsistent Logics

Abstract. A typical theorem of connexive logics is Aristotle's Thesis (A), $\sim(A\rightarrow\sim A)$. A cannot be added to classical logic without producing a trivial (Post-inconsistent) logic, so connexive logics typically give up one or more of the classical properties of conjunction, e.g. $(A \mathbin{\&} B)\rightarrow A$, and are thereby able to achieve not only nontriviality, but also (negation) consistency. To date, semantical modellings for A have been unintuitive. One task of this paper is to give a more intuitive modelling for A in consistent logics. In addition, while inconsistent but nontrivial theories, and inconsistent nontrivial logics employing propositional constants (for which the rule of uniform substitution US fails), have both been studied extensively within the paraconsistent programme, inconsistent nontrivial logics (closed under US) do not seem to have been. This paper gives sufficient conditions for a logic containing A to be inconsistent, and then shows that there is a class of inconsistent nontrivial logics all containing A. A second semantical modelling for A in such logics is given. Finally, some informal remarks about the kind of modelling A seems to require are made.

I

Connexive logics have been subjected to a semantical investigation in recent years[1], partly with an eye to understanding, if such be possible, the underlying attitude to logic expressed by those who seem to commit themselves to characteristically connexivist theses. Typically, connexive logics differ from classical logic in two ways. First, certain theses which are not classically theorems are affirmed as being truths of logic. Among these, we distinguish three of importance, together with the names which they have come to be given in the standard literature.

1.	$(A\rightarrow B)\rightarrow\sim(A\rightarrow\sim B)$	(Boethius)
2.	$\sim(A\rightarrow\sim A)$	(Aristotle)
3.	$\sim\big((A\rightarrow B) \mathbin{\&} (A\rightarrow\sim B)\big)$	(Strawson)

The addition of any of these theses to classical logic, gives a trivial (post-inconsistent) logic, so in order to retain them certain theses of classical logic need to be given up. It is common to give up the principles of Simplification, viz. $(A \mathbin{\&} B)\rightarrow A$ and $(A \mathbin{\&} B)\rightarrow B$.[2] As we shall see later, however, this is not the only way of avoiding triviality; another way being the move to relevant logic in order to block the trivialising effect of inconsistency.

[1]See e.g. [10], [11], [3], [6], [7], [4].
[2]See e.g. [2].

This paper concentrates on the above Aristotle thesis. It aims to do several things. First, certain sufficient conditions for the negation inconsistency of a logic containing this thesis are displayed. Second, the existing semantical modellings for this thesis are mentioned, and an improved modelling is given in the context of certain consistent logics. Third, it is shown that inconsistent logics containing this thesis and the principles of Simplification are not inevitably trivial, and an improved modelling in the context of these inconsistent (and so paraconsistent) logics is given. Fourth, an attempt is made to draw these technical results together in order to understand what might underly the inclination to assert distinctively connexivist theses.

II

We work against a background of relevant logics. In particular, we need the following three well-known logics: ***B***, ***E***, ***R***. (See e.g. Routley and Meyer [9], Ch. 3.)

B: (A1) $A\to A$, (A2) $(A\ \&\ B)\to A$, (A3) $(A\ \&\ B)\to B$, (A4) $((A\ \&\ B(\ \&\ (A\ \&\ C))\to(A\to(B\ \&\ C))$, (A5) $A\to(A\vee B)$, (A6) $B\to(A\vee B)$, (A7) $((A\to C)\ \&\ (B\to C))\to((A\vee B)\to C)$, (A8) $(A\ \&\ (B\vee C))\to((A\ \&\ B)\vee(A\ \&\ C))$, (A9) $\sim\sim A\to A$, (R1) $A, A\to B/.\!:\ B$, (R2) $A, B/.\!:\ A\ \&\ B$, (R3) $A\to B\ \ C\to D/.\!:\ (B\to C)\to(A\to D)$, (R4) $A\to\ \sim B/.\!:\ B\to\ \sim A$. For ***E***, add (A10) $(A\to B)\to((B\to C)\to(A\to C))$, (A11) $(A\to B)\to((C\to A)\to(C\to B))$, (A12) $(A\to(A\to B))\to(A\to B)$, (A13) $(A\to\ \sim A)\to\ \sim A$, (A14) $(A\to\ \sim B)\to(B\to\ \sim A)$, (R5) $A/.\!:(A\to B)\to B$. For ***R***, ***E***+(A15) $A\to((A\to B)\to B)$.

In this section we exhibit some sufficient conditions for the inconsistency of Aristotle in a logic. Theorem 1 is essentially obtained in Routley and Montgomery [10]. Theorem 2 and in part Theorem 3 are due to Meyer. Where a known theorem is cited, the result can be found in Anderson and Belnap [1], or Routley and Meyer [9]. Proofs are abbreviated.

THEOREM 1. *The addition of Aristotle as a theorem to* ***B*** *(i.e.* ***B****+Aristotle i.e.* ***BA****) is negation inconsistent.*

PROOF.

1.	$(A\ \&\ \sim A)\to\ \sim A$	Simplification
2.	$\sim A\to(\sim A\vee\ \sim\sim A)$	Addition
3.	$(A\ \&\ \sim A)\to(\sim A\vee\ \sim\sim A)$	(1), (2) Rule Transitivity
4.	$(\sim A\vee\ \sim\sim A)\leftrightarrow\ \sim(A\ \&\ \sim A)$	Identity, using Double Negation and De Morgan
5.	$(A\ \&\ \sim A)\to\ \sim(A\ \&\ \sim A)$	(3), (4), Rule Transitivity
6.	$\sim(A\to\ \sim A)$	Aristotle

7.	$\sim((A \mathbin{\&} \sim A) \to \sim(A \mathbin{\&} \sim A))$	(6), $A \mathbin{\&} \sim A/A$; (5) and (7) contradict.

THEOREM 2. ***RA** is trivial*

PROOF. Let $A \circ B =_{df} \sim(A \to \sim B)$. The proof then proceeds by establishing five theses of $\mathbf{B}$ + Aristotle.

1.	$((A \to B) \mathbin{\&} A) \circ ((A \to B) \mathbin{\&} A)$	Aristotle
2.	$((B \mathbin{\&} B) \circ (A \mathbin{\&} B)) \to (A \circ B)$	$\vdash_B$, so $\vdash_R$
3.	$(A \to B) \circ A$	(1), (2) Modus Ponens
4.	$((A \to B) \circ A) \to B$	$\vdash_R$
5.	B	(3), (4) Modus Ponens

THEOREM 3. $\vdash_{EA} \Diamond B$, $\vdash_{EA} \sim(A \to B)$, *where* $\Diamond B =_{df} \sim((\sim B \to \sim B) \to \sim B)$

PROOF.

1.	$((B \to B) \to B) \to ((B \to A) \to ((B \to B) \to A))$	$\vdash_E$
2.	$((A \to B) \to A) \to A$	$\vdash_E$
3.	$((B \to A) \to ((B \to B) \to A)) \to ((B \to A) \to A)$	(2) Rule Prefixing
4.	$((B \to B) \to B) \to ((B \to A) \to A)$	(1), (3) Rule Transitivity
5.	$((B \to B) \to B) \to ((\sim A \to \sim B) \to A)$	4, Replacement of provable equivalences
6.	$((B \to B) \to B) \to ((A \to \sim B) \to \sim A)$	(5), $\sim A \vert A$, $\sim\sim A \leftrightarrow A$
7.	$((\sim B \to \sim B) \to \sim B) \to ((A \to B) \to \sim A)$	(6), $\sim B \vert B$, $\sim\sim B \leftrightarrow B$
8.	$\Box \sim B \to ((A \to B) \to \sim A)$	(7) $\Box B =_{df} (B \to B) \to B$
9.	$\sim((A \to B) \to \sim A) \to \Diamond B$	(8), Contraposition, $\Diamond B =_{df} \sim \Box \sim B$
10.	$((A \to B) \circ A) \to \Diamond B$	(9) Defn. of $\circ$.
11.	$(A \to B) \circ A$	From (3) of Theorem 2.
12.	$\Diamond B$	(10), (11) Modus Ponens

Also,

13.	$(A \to B) \leftrightarrow \Box(A \to B)$	$\vdash_E$
14.	$\sim(A \to B) \leftrightarrow \sim \Box(A \to B)$	(13), Contraposition
15.	$\sim(A \to B) \leftrightarrow \Diamond \sim(A \to B)$	(14), Df of $\Box$
16.	$\Diamond \sim(A \to B)$	(12), $\sim(A \to B) \vert B$
17.	$\sim(A \to B)$	(15), (16), Modus Ponens

III.

In this section, we establish a result concerning semantical modelling for Aristotle in consistent systems. The style of modelling is the usual ternary relational worlds semantics, as investigated, for example, in Routley and Meyer [8].

A *(reduced) model structure* is a sextuple $\langle T, K, R, S, U, {}^*\rangle$, where $T \in K$; $R, S, U \subseteq K^3$ and * is a function, $^*\colon K \to K$. A *relevant valuation for a language* $\mathscr{L}$ is a pair $\langle M, v\rangle$ where M is a relevant model structure and v is a function $v\colon K \times PV(\mathscr{L}) \to \{1, 0\}$, where $PV(\mathscr{L})$ is the set of propositional constants of $\mathscr{L}$. In addition, v is required to satisfy the hereditary condition H_v: If $RTab$ and $v(p, a) = 1$ then $v(p, b) = 1$. A *relevant model for a language* $\mathscr{L}$ is a pair $\langle M, I\rangle$ where M is a relevant model structure, and I is a function $I\colon K \times \mathscr{L} \to \{1, 0\}$ such that for some relevant valuation $\langle M, v\rangle$ for $\mathscr{L}$, $v \subseteq I$, and in addition I satisfies

$$I(\sim A, a) = 1 \quad \text{iff} \quad I(A, a^*) \neq 1.$$
$$I(A \,\&\, B, a) = 1 \quad \text{iff} \quad (\exists b, c)\,\big(Sabc \text{ and } I(A, b) = 1 = I(B, c)\big)$$
$$I(A \vee B, a) = 1 \quad \text{iff} \quad (\exists b, c)\,(Uabc \text{ and } I(A, b) = 1 \text{ or } I(B, c) = 1)$$
$$I(A \to B, a) = 1 \quad \text{iff} \quad (\forall bc)\,(\text{If } R\,abc \text{ and } I(A, b) = 1 \text{ then } I(B, c) = 1).$$

A formula A of a language $\mathscr{L}$ is *true in a model* $\langle M, I\rangle$ *for* $\mathscr{L}$ if $I(A, T) = 1$ where $T \in M$. A formula A is *valid in a class* $\mathscr{C}$ *of models for* $\mathscr{L}$ ($\models_{\mathscr{C}} A$) iff A is true in all members of $\mathscr{C}$.

The characterisation theorems proved in this paper are of the usual sort. The soundness part of the theorems shows that if a class $\mathscr{C}$ of models satisfies a certain (semantical) condition, then the theorem schema in question (here, Aristotle) is valid in $\mathscr{C}$. This corresponds to that part of the inductive step in the usual soundness theorem which shows that all axioms of the logic under consideration are valid in $\mathscr{C}$. The completeness part of the theorems proceed by constructing a canonical model by the usual methods (e.g. Routley and Meyer [8]), so that if A is not a theorem of the logic in question, then $I(A, T) \neq 1$ where T is the 'real world' of the canonical model. It then remains to show that the canonical model is a model of the required sort, in particular that it satisfies the semantical condition determining $\mathscr{C}$, so that it will follow that if A is not a theorem of the logic in question, then A is not valid in $\mathscr{C}$, the core the usual completeness theorem. We will ignore this standard construction in the completeness theorems here, supposing that the canonical model, which is a set of linguistic structures satisfying the condition $I(A, a) = 1$ iff $A \in a$, for all worlds a in the model, and all formulae A, is given. We exhibit only the distinctive part of the theorems.

Routley has obtained semantical modellings for Aristotle and Strawson by introducing a further idea into the semantics, the 'generation' relation, AGy. A model structure is taken to be a 7-tuple $\langle T, K, R, S, U, G, {}^*\rangle$, and then a model $\langle M, I\rangle$ must in addition satisfy the condition: if AGb, then $I(A, b) = 1$, for any formula A and world b. The modellings for Aristotle and Strawson are

$(\exists y)(RT^*yy^* \text{ and } AGy)$ (Aristotle)

If $SbcT^*$, then

$(\exists y, z)(Rbyz \text{ and } Rcyz^* \text{ and } AGy)$ (Strawson)

These modellings have at least one undesirable feature, namely that they are not particularly intuitively enlightening. It might also be argued that they have a second defect, namely that in employing the generation relation they abandon strictly semantical considerations in favour of a not-very-disguised way of writing an arbitrary restriction on the interpretation function I into the semantics. We will return to this second criticism later, and attempt to take some of the force away from it. As to the first criticism, the aim of this section is to show that under certain natural conditions, Aristotle, at least, has a simpler semantical modelling.

THEOREM 4. *Let L be a consistent logic with a ternary relational worlds-semantics as described above, and with the following properties:* $\vdash A \to A$, $\vdash A \vee \sim A$, *associative, indempotent and commutative laws for* & *and* $\vee$. *Rule Transivity (if* $\vdash A \to B$ *and* $\vdash B \to C$, *then* $\vdash A \to C$), *and the Rule R: if* $\vdash A \to (B \vee C)$ *and* $\vdash (C \,\&\, D) \to E$, *then* $\vdash (A \,\&\, D) \to (B \vee E)$. *Let* $C_{A\mathscr{M}} = \{x : x$ *is a world of a model* $\mathscr{M} = \langle M, I\rangle$ *and* $I(A, x) = 1$ *and* $I(\sim A, x) \neq 1\}$. *(For a condition on the worlds of a model, the* $\mathscr{M}$ *may be dropped when considering the condition as holding over all models in a given class.) Then Aristotle is characterised by the semantic condition* $C_A \neq \Lambda$.

PROOF. 1 (Soundness for Aristotle). We need to suppose the following facts, which are not difficult to demonstrate (a) if $\vdash A \to A$, then $RTaa$, for all a; (b) if $RTab$ then if $I(A, a) = 1$ then $I(A, b) = 1$; (c) if $\vdash A \vee \sim A$, then RTT^*T. (See [9]). Suppose now $C_A \neq \Lambda$, i.e. $(\exists a)(a \in C_A)$, i.e. $I(A, a) = 1$ and $I(\sim A, a) \neq 1$. We need to show that Aristotle is true at T in our model. From $RTaa$ together with $I(A, a) = 1 \neq I(\sim A, a)$, we have that $I(A \to \sim A, T) \neq 1$. Hence, by definition of *, $I(\sim(A \to \sim A), T^*) = 1$. But now by RTT^*T, together with (b) above, if $I(A, T^*) = 1$, then $I(A, T) = 1$, so $I(\sim(A \to \sim A), T) = 1$, as required.

2. (Completeness for Aristotle). For this result, we need a version of the *Extension Lemma*, which has been proved in a number of places, e.g. [9] or Mortensen [5]. Let S, T be disjoint sets of formulae such that no conjunction of members of S entails in a given logic L any disjunction of members of T. If, in L & and $\vee$ satisfy associative, idempotent and commutative laws, and R is a rule in L, then there are disjoint sets of formulae S', T' with $S \subseteq S'$, $T \subseteq T'$ such that S' is a prime theory (that is, S' satisfies the two conditions (1) if $A \in S'$ and $\vdash A \to B$ then $B \in S'$, and (2) if $A \vee B \in S'$, then $A \in S'$ or $B \in S'$.)

For the completeness theorem we show that if $\vdash \sim(A \to \sim A)$ then $(\exists a)(a \in C_A)$. If $\vdash \sim(A \to \sim A)$ and L is consistent, then for all A not $\vdash A$

$\to \sim A$. Let $S_0 = \{A\}$, $T_0 = \{\sim A\}$. Now we claim that no conjunction of members of S_0 entails any disjunction of members of T_0. For suppose otherwise, then we would have $\vdash (A \,\&\, \ldots \,\&\, A) \to (\sim A \vee \ldots \vee \sim A)$; but then by the idempotence of & and $\vee$, $\vdash A \to \sim A$, a contradiction. But now from the conditions of the theorem we have the conditions for the application of the Extension Lemma, so that A is contained in a prime theory a disjoint from $\{\sim A\}$. In the standard style of completeness theorem the worlds of the canonical model are prime theories, so we may conclude that $I(A, a) = 1 \neq I(\sim A, a)$, that is $a \in C_A$, i.e. $C_A \neq \Lambda$.

IV.

It was noted earlier that connexivism has often opted for discarding Simplification in order to avoid triviality in the presence of Aristotle. It is easy to show, though, that triviality does not inevitably result from these two theses. The matrices below verify all of ***E***, in which Simplification is a thesis, as well as Aristotle, Strawson and Boethius.[3]

→	0	1	2	~	&	0	1	2	∨	0	1	2
0	1	1	1	2		0	0	0		0	1	2
* 1	0	1	1	1		0	1	1		1	1	2
* 2	0	0	1	0		0	1	2		2	2	2

Since ***E*** + Aristotle is (negation) inconsistent, the logic characterised by these matrices is inconsistent, but we note that it is not trivial. (We also note that it is not difficult to prove that this logic is post-complete).

Now the presence of ***E*** provides a very pleasing modelling for Aristotle, using the notion of a 'non-normal world' developed in Routley and Meyer [9] Ch. 6. A world a of a model is non-normal, $\sim Na$, just in case no implicated formula is true at a: $\sim Na$ iff $(\forall A, B)\big(I(A \to B, a) \neq 1\big)$. It can be shown that this is a generalisation of the well-known idea of a non-normal world in the semantics of modal logic. Now we note from Section II above that in the context of ***E***, Aristotle enables us to prove the schema $\sim (A \to B)$. Conversely, from the schema $\sim (A \to B)$, Aristotle follows by substitution. However, it is virtually immediate that the schema $\sim (A \to B)$ is modelled by the semantic condition $\sim NT^*$. (Soundness: let $\sim NT^*$, then for all A, B, $I(A \to B, T^*) \neq 1$, so $I\big(\sim (A \to B), T\big) = 1$. Completeness: let $I\big(\sim (A \to B), T\big) = 1$ for each A, B, then $I(A \to B, T^*) \neq 1$, i.e. $\sim NT^*$ as required.) Hence we have, what is evidently a simplification of earlier modellings.

THEOREM 5. *In the context of* ***E***, *Aristotle is modelled by the semantic condition* $\sim NT^*$.

[3] Indeed, they also verify other important theses, for instance Restricted Ming $(A \to B) \to ((A \to B) \to (A \to B))$, and Strong Distribution $((A \to (C \vee B)) \,\&\, ((B \,\&\, A) \to C)) \to (A \to B)$. See [1], p. 341.

V.

One interesting feature of the three modellings of Aristotle given here is that they all involve in some way the idea of situational occupation. It was noted earlier that the generation relation is closely connected with the assignment function; and indeed Routley's modelling for Aristotle, viz. $(\exists y)(RT^*yy^*$ and $AGy)$ amounts to a claim that a certain world, one having a certain relational property, has a certain proposition true at it. Similarly, the modelling of section III, viz. $C_A \neq \Lambda$, is the claim that a certain world has a proposition A true at it, and does not have the denial of A true at it. The modelling of Section IV is again similar, in that the condition $\sim NT^*$ amounts, given the assignment condition for negation, to the claim that the denial of every proposition is true (at the real world, T). Each modelling employs the idea that a certain proposition be true at a world, unlike, for example, such well known conditions in the semantics of entailment as $RTaa$, $Raaa$, if $Rabc$ then $Rac^*b^* = a$, and so on (see Routley and Meyer [8] p. 205). Each, that is, is like a restriction on the class of models via a restriction on the *assignment function* I, rather than a restriction on the class of models via restriction on the class of *model structures*.

Given this point, we can see more clearly the force of a criticism of Routley's modelling suggested earlier in Section III. To recall, the criticism was that the use of the generation relation is *ad hoc* in that it restricts the appropriate class of models by restricting the acceptable assignment functions, whereas classes of models correspondin? to formulae ought to be specified by conditions on the model structure. This criticism implies the primacy of model structures over models in the understanding of theses via their semantical modellings. This is reasonable to some degree. The case for it would seem to be that the aim of keeping the kind of assignment function I as fixed as possible is to be able to lay down in the specification of I *basic* meaning postulates for the *kind* of connective in question. Variations in conditions specifying the class of associated model structures is then to be understood as variation in claims about the meaning of a certain *kind* of connective. Put crudely, without stability in I, we will not know which connective we are talking about since we will not know what its truth conditions are.

However, we can take this attitude too far. First, some conditions on the assignment function I are standardly admitted into the semantics, such as the hereditary condition H_v of Section III, which is designed to validate a particular formula, viz $A \to A$. Second, it is not clear that the distinction between the type of meaning a connective has and the particular meaning it has can be maintained through thick and thin. For example, semantical postulates on a model structure are linked indirectly with the restrictions on the assignment function; indeed, this is what soundness and completeness theorems show.

I want to suggest another reason for being tolerant of direct semantic restrictions on situational occupancy. Such a practice seems to be justifiable by taking a conventionalist view of meaning. Consider an alternative anti-conventionalist view of meaning, such as David Lewis's modal realism. Now if modal realism held that worlds-type model structures with strange non-modal accessibility relations between them were also real (that is, real in addition to the fixed set of possible worlds) then there would appear to be no alternative to preserve the logical truth of modal logic but to impose restricting conditions on the function which assigns truth in various worlds. After all, all the model structures needed, and many more besides, would exist. But there are many such possible assignment functions, as many as set theory allows. So such a choice of conditions on the assignment function would seem to have an element of convention in it. Thus it is no accident that Lewis is not a realist about model structures other than those which correspond to his preferred logic. But if we abandon Lewis's version of modal realism in favour of a picture which sees worlds as linguistic items, for example as set of propositions or theories, then a conventionalist element returns rather easily. It is not difficult to hold[4] that many 'degenerate' (at any rate, unsuitable) kinds of linguistic model structures are available for semantical purposes in addition to the preferred ones. But then what is available to one but to say that the choice of certain model structures *and also* certain assignment functions on them is conventional? There is no further fact of the matter available to determine logical truth, and necessity is at bottom analyticity. And convention surely need not balk at a decision to prefer certain assignment functions rather than others. There is no less fact to the matter underlying such a choice than there is underlying the choice of one class of model structure rather than another. These, then, are three reasons for being less than concerned at the objection to the semantical modellings of this paper that situational occupation and the generation relation are *ad hoc* devices.

The variation in the modellings illustrates the well-known point that different contexts give a thesis different forces. Is there a core meaning for Aristotle? In one sense Routley's is, since it is a modelling which holds good with minimal background resources (those of *CB*, see Routley [6] p. 9). But as noted earlier, that modelling does not have a lot of intuitive appeal. I think that the condition $C_A \neq \Lambda$ is closest to the way we think of Aristotle. Aristotle says $\sim(A \rightarrow \sim A)$: no proposition entails its negation. Now as noted by Routley, [6] p. 2:

[4]Though it might be denied, in favour of a view which held parallel to Lewis that there are only certain model structures of linguistic items and only one natural type of assignment function associated with them, giving a preferred logic. While not wishing to be embroiled with Meinongian positions, it would seem that most of these points can be made about those views also.

> It is a central semantical feature of relevant logics that for every statement A there is some situation a where it holds and some situation b where it fails. It is from this central feature that such prized properties of entailment as relevance and sufficiency flow.

Combine these two ideas with the requirement that situations be closed under entailment and we have that for every proposition there is a situation which A belongs to and which $\sim A$ does not belong to: $C_A \neq \Lambda$. This informal argument is not intended to supplant consistency and completeness theorems. Rather, it is intended to suggest that the ease with which we make the informal deductive moves is a mark of the ease with which we accept the precise conditions of those theorems, especially in thinking about Aristotle. We might also note that if it is true that Aristotle is typically thought of in the present way, rather than, say, in the way in which it occurs in ***EA***, then this is some evidence for a human tendency to 'consistentise' when trying to grasp an assertion.

The present result gives an argument against Aristotle in such consistent (and, plausibly, natural) contexts. If we think that there is a proposition A which is self inconsistent in the sense that it cannot occur in a deductively closed theory of the 'correct' logic, whatever that might be, without its negation also occurring in that theory, then the 'consistency class' of theories associated with A, C_A, must be empty. Syntactically, this amounts to denying $\sim(A \to \sim A)$. Needless to say, it is a widespread view, shared by the author, that there are such propositions.

The addition of Aristotle to ***E*** plainly gives a paraconsistent logic, that is, one with at least one nontrivial inconsistent theory. It is not uncommon to distinguish between two grades of paraconsistency: weak (at least one nontrivial inconsistent theory), and strong (at least one inconsistent proposition true). ***EA*** suggests the need for a third even stronger grade: at least one inconsistent proposition is logically true, or even at least one inconsistent proposition of the logic of &, $\vee$, $\sim$, $\to$ is logically true. Aside from this, the condition $\sim NT^*$ in the context of ***E*** does not seem to hold much practical interest, amounting as it does to an assertion of the negation of every conditional. As if this were not bad enough, the presence of ***E*** means that every conditional which is a theorem of ***E*** is also asserted. While there may be grounds for believing in some inconsistency in the universe, this much inconsistency is surely too much.

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X
Paraconsistency and C_1

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1. Introduction

Attempts to avoid some or all of the paradoxes of material implication have, as is well known, a long history. One recent approach to avoiding some (but not all) of the paradoxes has been taken by da Costa and others using the logic C_1 or logics close to C_1. It must be emphasized at the outset that while C_1 does avoid some of the paradoxes, it is best viewed not as a paradox-avoiding logic but as a paraconsistent logic. We will say more on this matter later. This paper is concerned to examine C_1 and to criticize it. A different approach to the problem of avoiding the paradoxes has been taken in the development of relevant logics by Anderson, Belnap and others. This second approach is not considered in the present paper, but one of our aims is to provide some indirect support for it by criticism of C_1 and related logics.

Those who have felt uneasy about the paradoxes of material implication have usually done so because, *inter alia*, the paradoxes give a reason for being dissatisfied with '$\supset$' as a good formal logical 'model' for such (relatively) ordinary notions as 'implies', 'entails', or 'if . . . then'. Another way to say this, is to say that the classical propositional calculus is not a good *implication logic*. To explain this, we need some definitions.

Definition 1.1. A *logic* is a set of well formed formulae (wffs) closed under uniform substitution.

The reason for this definition is that for laws to count as logical it seems reasonable that they hold universally; independent of subject matter.

Definition 1.2. An *implication logic* L is a logic L containing a binary operator, say $\rightarrow$, called an *implication operator*, intended to be interpreted as 'implies', 'entails' or 'if . . . then'; a single necessary condition is imposed on $\rightarrow$, namely that it satisfy: if $\vdash_L A$ and $\vdash_L A \rightarrow B$ then $\vdash_L B$.[2]

To say, then, that the classical propositional calculus is not a good implication logic is just to say that it does not contain an operator which interprets 'implies' etc., well. We will also occasionally speak of an implication operator as a conditional operator. Motivation for this needs no discussion.

An important and recently developed insight into these problems has been to look at the notion of a theory. The notion of a scientific theory, for all its vagueness, is that of a thing with a deductive structure. Deductive structure abstracted from particular subject matter is the province of the logician. Different logics, however, may be intended to offer different accounts of correct deducibility (and, conversely, on the well known dictum that entailment is the converse of deducibility, of correct entailments). So any attempt at formal picturing of the notion of a theory must, if it is to avoid begging the question in favour of one account of deducibility, be relativized to a logic L. Combining this with the above dictum, we get the following definition.

Definition 1.3. Let L be an implication logic with implication operator $\rightarrow$ and let X be a set of wffs. X is an *L-theory* iff $A \in X$ and $\vdash_L A \rightarrow B$ then $B \in X$.[3]

We will also need the following.

Definition 1.4. An L-theory is *trivial* iff it contains every wff.

Definition 1.5. An L-theory is *inconsistent* iff it contains both A and $\neg A$, for some A (where $\neg$ is the intended negation operator).

Definition 1.6. An L-theory X is *L-regular* iff every theorem of L is a member of X.

The idea of an L-theory gives us a useful formal approximation to that of the informal idea of a theory. It also sheds some light on the question of the paradoxes of material implication: if and only if a contradiction 'really' implies everything should it be the case that any inconsistent scientific or mathematical theory contains every wff of its language. If, therefore, we are unwilling to agree that inconsistent scientific theories are inevitably trivial, we should conclude that no logic containing the paradoxes, in particular the theorem $(A \& \neg A) \rightarrow B$, is a good implication logic.[4] There do seem to be strong reasons for being unwilling so to agree.[5] Conversely, also, someone wishing to reject classical propositional logic as an implication logic on the grounds of the paradoxes of material implication will presumably be prepared to admit that there are non-trivial inconsistent theories of 'natural' logic. Formally, such a person will wish their preferred implication logic L to be such that there exist non-trivial inconsistent L-theories. This suggests the need for a definition.

Definition 1.7. A logic L is a *paraconsistent* logic iff there exist non-trivial inconsistent L-theories.

Among people who wish for a paraconsistent implication logic, it is important to distinguish two attitudes. It seems to be a widespread misunderstanding that someone who favours a paraconsistent logic, a 'paraconsistentist', must also think that the correct (true and comprehensive) theory of the world is inconsistent. This is mistaken, as should be clear from our definitions above. To tolerate inconsistent theories which are non-trivial is not to accept that some inconsistent theory is true. We will apply the term '*weak paraconsistentist*' to a paraconsistentist who thinks that the world is consistent (*i.e.* that the true and comprehensive theory of the world is consistent). Weak paraconsistentism needs to be distinguished from a stronger position. Some philosophers hold the thesis that some contradictions are true, or variants of the thesis such as that some sentences are both true and false, or that some sentences and their negations are true.[6] The present author has considerable sympathy for this thesis. Often it is held that the sentences which are both true and false arise in a very limited area, namely from the semantic or the set theoretic paradoxes.[7] Now it is manifestly not the case that the world is trivial. Hence anyone who thinks that the world is inconsistent needs to be a paraconsistentist. We will describe the paraconsistentist position which holds the world to be inconsistent as '*strong paraconsistentism*'.

We must now turn to look at the class of paraconsistent logics described earlier, the C_1-related logics. The class considered will be C_1 and a group of weaker logics $C_2, \ldots, C_\omega$. Any member of the class may be adopted by weak paraconsistentists in that C_1, etc. are all consistent. On the other hand, strong paraconsistentism may well hold the plausible thesis that the sentential logic of 'and', 'or', 'not' and 'implies' is by itself consistent. In which case, such a strong paraconsistentism may well also wish to avail itself of a logic from these classes of candidates.

2. C_1 and some related systems

We assume a language L consisting of a denumerable number of proposition letters p_i, $1 \leq i < \omega$, and closed under the primitive unary operator $\neg$ and the primitive binary operators &, v, $\supset$. $A \equiv B$ is defined as $(A \supset B) \& (B \supset A)$, A^0 is defined as $\neg(A \& \neg A)$, and $\neg * A$ is defined as $\neg A \& A^0$. Capital letters A, B, C, etc. are metalinguistic variables ranging over formulae.

Definition 2.1. The logic C_1 is the smallest subset of L closed under uniform substitution and *modus ponens* (for $\supset$) and containing all instances of the schemata (1) $(A \supset (B \supset A))$; (2) $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$; (3) $(A \& B) \supset A$; (4) $(A \& B) \supset B$; (5) $A \supset (B \supset (A \& B))$; (6) $A \supset (A \vee B)$; (7) $B \supset (A \vee B)$; (8) $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$; (9) $A \vee \neg A$; (10) $\neg\neg A \supset A$; (11) $B^0 \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A)$; (12) $(A^0 \& B^0) \supset ((A \& B)^0 \& (A \supset B)^0 \& (A \vee B)^0)$. Let A^m abbreviate $A^{00\ldots0}$ where there are m circles, and let $A^{(m)}$ abbreviate $A^1 \& A^2 \& \ldots \& A^m$. Then to obtain the logics C_n, $1 < n < \omega$, replace A^0 and B^0 in (11) and (12) by $A^{(n)}$ and $B^{(n)}$ respectively. To obtain the logic C_ω, delete axioms (11) and (12) from the definition of C_1. C_0 is the classical propositional calculus. It may be obtained, for example, by adding the schema A^0, *i.e.* $\neg(A \& \neg A)$, to C_1.

C_1 has a semantics and a decision procedure, which we outline here and use later.

Definition 2.2. A *C_1-valuation* is a function $v: L \rightarrow \{1, 0\}$ such that (1) $v(A) = 0$

implies $v(\neg A)=1$; (2) $v(\neg\neg A)=1$ implies $v(A)=1$; (3) $v(B^0)=v(A\supset B)=v(A\supset\neg B)=1$ implies $v(A)=0$; (4) $v(A)=0$ or $v(B)=1$ iff $v(A\supset B)=1$; (5) $v(A)=v(B)=1$ iff $v(A \,\&\, B)=1$; (6) $v(A)=1$ or $v(B)=1$ iff $v(A \vee B)=1$; (7) $v(A^0)=v(B^0)=1$ implies $v((A \,\&\, B)^0)=v((A\supset B)^0)=v((A\vee B)^0)=1$. A formula A is *true in a valuation v* iff $v(A)=1$.

Definition 2.3. (See da Costa and Alves, 1977) A *quasi matrix for a formula A* is constructed as follows:

(1) Make a list of all the proposition letters of A (in a horizontal line) and as in truth tables for classical propositional calculus list all possible assignments of 0 and 1 to them.

(2) Make a list of all denials of proposition letters of A (to the right of the former list) and assign values as follows: if the proposition letter was assigned 0 its denial is assigned 1. If the letter was assigned 1, bifurcate the line on which the 1 occurs, and on one half the denial is assigned 0 and the other half 1.

(3) Make a list of all remaining subformulae of A and negations of proper subformulae of A and proceed as follows.

(3.1) If the major connective of any such formula is &, v, or $\supset$, its value is determined from the values of its two components as in classical logic.

(3.2) If the formula is of the form $\neg B$ and B is assigned 0, assign $\neg B$ the value 1.

(3.3) If the formula is of the form $\neg B$ and B is assigned 1, then there are several subcases.

(3.3.1) B is of the form $\neg C$ and C is assigned 0. Assign $\neg B$ (i.e. $\neg\neg C$) the value 0.

(3.3.2) B is of the form $\neg C$ and C is assigned 1. Bifurcate the line and assign $\neg B$ the value 0 on one bifurcation and 1 on the other.

(3.3.3) B is of the form $C \,\&\, \neg C$ or $\neg C \,\&\, C$. Assign $\neg B$ the value 0.

(3.3.4) B is of the form $C \circ D$, where $\circ$ is &, v, $\supset$, but not of the form (3.3.3). Then: if the value of C is different from the value of $\neg C$ and the value of D is different from the value of $\neg D$, assign $\neg B$ the value 0; otherwise, bifurcate the line and assign $\neg B$ the value 0 on one half and 1 on the other.

We can now state the outcomes of this decision procedure: if some line of the quasi matrix for A assigns 0 to A, then A is not valid and so not a theorem. Otherwise, A is a theorem. The relevant results can be summarized in a theorem.

Theorem 2.1. (da Costa, Alves, Fidel and others) C_1 is sound and complete with respect to the class of C_1-valuations. C_1 is decidable by the method of quasi matrices.

A theory of deduction and of the deductive structure of theories may well have major implications for the foundations of mathematics. The leading motivation for C_1, namely the desire for a paraconsistent logic, suggests, as we noted earlier, that there might be interest in certain inconsistent mathematical theories. One well known inconsistent mathematical theory is Naive Set Theory. We might therefore ask whether there are any interesting set theories containing an unrestricted (or not very restricted) comprehension axiom and based on C_1. The question has considerable importance for the foundations of mathematics because it raises the possibility that a mathematics strong enough for all desired results might need an unrestricted comprehension axiom (set forming operation). Mathematics might *need* an inconsistent foundation. Conversely, given an inconsistent but not trivial foundation we might perhaps discover new and desirable mathematical results. Even if we proceeded merely on paraconsistent foundations without necessarily being inconsistent, it would be at least conceivable that certain desirable distinctions might be made and desirable results proved on their basis. Meyer's work in relevant arithmetic, particularly his proof of the strong consistency (non-triviality) of relevant Peano arithmetic, is an outstanding example of this. (See e.g. Routley, 1977).

Work on the foundations of set theory with a logical basis in C_1 has had some success. In particular, we mention da Costa's inconsistent but apparently non-trivial set theory NF_1 (da Costa, 1974). The existence of such a set theory is gratifying for several reasons. One is that it gives some support for C_1. Presumably not every paraconsistent logic will permit such set theories, so those logics which do deserve a special place. Furthermore, the broader 'programme' of developing mathematics around a logic like C_1 gets considerable mileage if (particularly) a set theory can be displayed. A second reason is that it demonstrates the possibility of a stable set theory based on some paraconsistent logic. Even if C_1 has undesirable features (as will be argued in this paper) it means that some hope can be held for other types of paraconsistent logic to do the same sort of job for set theory. A third reason is that it demonstrates the richness of C_1 as a source of interesting mathematical problems.

Thus, C_1 undoubtedly has some good features. In some of the remainder of this paper, several arguments will be suggested against C_1. The aim will be to show that C_1 and logics like it are flawed, and that a different kind of paraconsistent logic needs to be adopted. It is suggested here that the various relevant logics provide better candidates in this regard.

3. Does C_1 have a reasonable conditional and biconditional?

It was remarked earlier that C_1 was best viewed as a paraconsistent logic and secondarily as a paradox avoiding logic, and only a partial one at that. There is no doubt that one important motivation for C_1 and related logics has been paraconsistent, possibly even strong paraconsistentist (see da

Costa, 1974). Moreover, while C_1 lacks $(A \& \neg A) \supset B$, it still contains $A \supset (B \vee \neg B)$ and $A \supset (B \supset A)$ which are paradoxical if $\supset$ is intended as an implication operation. To the extent that C_1 is paradoxical, we should regard it as defective, I suggest, but it will not be our aim in this paper to pursue this criticism.

A question which remained open for some time concerned how to "algebraize" C_1.[8] In part the difficulty was to make the question precise. In this section we will discuss a result of the author's which proposes a solution to this problem on the basis of a certain proposal as to how to make the question precise. As we will see, the result has considerable bearing on the question of the reasonableness of C_1.

The quest for an algebraization of a logic seems to come down to this: the attempt to find some equivalence relation on wffs with which to partition into a quotient algebra (set of equivalence classes) the set of wffs constituting the logic. The equivalence relation is frequently, but I suggest not necessarily, expressible as a formula in the primitive symbols of the logic. Again, it might be required that the quotient algebra have associated with it a partial order, but for our present purposes this would seem to be an unnecessary encumbrance. However, there do seem to be certain minimal conditions it is reasonable to impose on any such equivalence relation and quotient algebra. Two are suggested here. (i) The logic, considered as an algebra of formulae, be homomorphic to its quotient algebra. Formally put: let $\sim$ be the equivalence relation in question; then we must have that $A \sim B$ implies $C(A) \sim C(B)$ for any context C; (ii) if $A \sim B$ and $\vdash A$ then $\vdash B$. The reason for this requirement is that the point of algebraization of a logic is usually to designate precisely the equivalence classes containing theorems, designation being the algebraic equivalent of theoremhood. But if condition (ii) fails, so that for some formulae A, B in the same equivalence class $\vdash A$ and $\nvdash B$, we would be unable both to designate that class (without designating a non-theorem) and to undesignate it (without refuting some theorem). In fact, as we will see soon, if the quest is not for algebraization but for a good biconditional, the argument for (ii) can be made even stronger.

Now we can state a result of the author's, proved in [1980].

Theorem 3.1. For C_1 and any logic weaker than C_1, any equivalence relation $\sim$ satisfying the previous conditions (i) and (ii) is such that if $A \sim B$, then $A = B$ (A is the same formula as B).

It is argued in that paper [1980] that this result is sufficient to settle in the negative the question of the existence of a reasonable algebraization of C_1. Of interest here is the different issue of its bearing on the question of whether C_1 has a reasonable biconditional.

In effect, the quest for an equivalence relation satisfying conditions (i) and (ii) can be seen as a quest for a biconditional (co-implication) expressible in the logic, with the important extra proviso that a biconditional expressible in the logic ought to be expressible as a formula schema in the primitive symbols. Now given a biconditional $\leftrightarrow$, with reasonable properties, the provability of the biconditional, $\vdash A \leftrightarrow B$, ought to serve as a sufficient candidate for the equivalence relation $A \sim B$. Certainly it seems reasonable that the provability of the biconditional constitute an equivalence relation on formulae: $\vdash A \leftrightarrow A$, $\vdash A \leftrightarrow B$ implies $\vdash B \leftrightarrow A$, and $\vdash A \leftrightarrow B$ and $\vdash B \leftrightarrow C$ implies $\vdash A \leftrightarrow C$ are minimal conditions on a biconditional. Conditions (i) and (ii) above are reasonable too. One would hardly want for one's biconditional that $\vdash A \leftrightarrow B$ and $\vdash A$ but not $\vdash B$, so condition (ii) is reasonable. (This point supplies the previously promised strengthening of the argument for (ii).) As for condition (i) the matter is not quite so obvious, because it might be that one wishes to have certain contexts opaque to the substitution of provable co-equivalents. An example would be (non-normal) modal logics where pairs of formulae can be provably materially equivalent without being intersubstitutable in all contexts. Still, even non-normal logics have a relation sufficient for substitutability in all contexts: provable strict equivalence. Furthermore, even logics for well known non-extensional contexts, such as those of the propositional attitudes, can reasonably contain an equivalence relation sufficiently strong for substitutability in all contexts, an example being an equivalence relation for propositional identity.[9] And in the special case of a sentential logic for conjunction, disjunction and negation (such as C_1) it does not seem so unreasonable to demand that such contexts be transparent with respect to the substitution of some not-too-strong biconditional, if we want a biconditional at all. We must conclude then, that it ought to be the case that if a reasonable biconditional were expressible in C_1, that biconditional should satisfy conditions (i) and (ii) above, as well as the condition that the provability of the biconditional constitute an equivalence relation on the set of wffs.

But now we must further conclude that the above Theorem 3.1 shows that any biconditional expressible in C_1 can only provably hold between formulae if they are one and the same. This, I claim, is a defect. Certainly there can be biconditionals only holding between one and the same formula: any biconditional expressing 'is the same formula as' adequately will do. Nevertheless, notions like 'co-entails', 'co-implies' and 'if and only if' would seem to stretch much further. Moreover, it ought to be part of the resources of a logic *itself* that one be able to prove that its biconditional holds between various of its formulae. This additional point does not matter for our purposes, however, because Theorem 3.1 does not turn on the biconditional being expressible as a provable formula. Any equivalence relation satisfying conditions (i) and (ii) for C_1 (or any weaker logic) can only hold between one and the same wff; it can only partition the formula algebra into singleton equivalence classes.

We conclude, then, that C_1 (and all weaker logics) cannot express a reasonable biconditional. Now let us ask whether C_1 can express a reasonable conditional.

The answer would seem to be the same, and for similar reasons. A conditional together with reasonable resources for conjunction, will give rise to a natural biconditional: $A \leftrightarrow B =_{df} (A \rightarrow B) \& (B \rightarrow A)$. Therefore if a logic cannot express a proper biconditional, then it cannot express a reasonable conditional either. We have here a complicating factor, though, namely conjunction. Conceivably someone might argue that even though a logic has no reasonable biconditional, it does have a reasonable conditional, and the inability of the conditional to give rise to a biconditional is due to the fact that conjunction is inadequately represented in the logic. At least, it might be argued, it does not *follow immediately* from the non-existence of a good biconditional that a good conditional also does not exist.

This point must be conceded. There are several replies, however. The first reply is to note again the above theorem: it shows that any suitable equivalence relation expressible in the language of C_1 *or not*, will only hold between one and the same wff. Now if we had a satisfactory conditional, there would seem to be a straightforward way of defining a suitable equivalence relation *metalinguistically*: $A \sim B =_{df} \vdash A \rightarrow B$ and $\vdash B \rightarrow A$. The 'and' here is metalinguistic, and so may be assumed to have satisfactory properties. It follows that no formulae could provably imply each other unless they were one and the same formula. That would seem to be a defect in the implication relation alone. A second reply is this. Even if we do not grant the first reply, the determined defender of the C_1 account of implication has the onus to show that the account of conjunction of C_1 is unsatisfactory. The present state of the art is well short of that. Furthermore, if the valuation semantics (Definition 2.2) is taken as a truth definition for the operators in C_1, then conjunction is classical, since its valuation is (Clause (5) of Definition 2.2). In any case, what we could conclude for the present is something still quite damaging to C_1; namely that the 'implication-conjunction mix' is inadequate. Either conjunction is inadequate, or implication is, or they both are. Any way it is taken, something is wrong. Whichever reply is granted, then, one can conclude that the problems for C_1 go below the level of its biconditional to those operators which give rise to biconditionals.

It is sometimes said that implication and deducibility are importantly linked to negation. There is no doubt, for example, that such a close connection is manifest in the relational semantics for relevant logics.[10] The C_1 account of implication has been attacked here, but another way of looking at the matter is to look at the C_1 account of negation. C_1 lacks the theorems $A \supset \neg\neg A$ and $(A \supset B) \supset (\neg B \supset \neg A)$ which would seem to be desirable if $\supset$ were to be an implication operator. Furthermore, semantically (the valuation semantics) it is the method of evaluating negation which differs from the classical. The positive (negation-free) principles of C_1 are classical. That, of course, is not necessarily a recommendation,[11] but it does suggest that it is the weakness of principles involving negation which prevent a non-trivial equivalence operation that gives substitutability in all contexts.

It has been argued that C_1 and all weaker logics are unsatisfactory. The question then naturally arises whether it would be possible to strengthen the principle of negation of C_1 (while remaining with a paraconsistent logic) in such a way that a better conditional and biconditional could be formed. In the next section, we will consider this question.

4. Some extensions of C_1

Some criticism has been directed at C_1 so far. The rest of this paper is not critical. The aim of this section is to investigate certain strengthenings of the negation principles of C_1 in order to see whether such logics have equivalence relations which can do better than merely giving rise to singleton equivalence classes. In short, if C_1 is unsatisfactory, does it have extensions which are more satisfactory in this respect?

We recall that da Costa and others have investigated an infinite class of logics $C_1, C_2, \ldots, C_\omega$ with C_1 as an upper limit. The object of this section is to define and report results concerning an infinite class of logics with C_1 as lower limit. These logics have the interest that, unlike C_1, they do have non-trivial equivalence relations, and so represent an improvement on C_1 in that respect.

Definition 4.1. For $n \geq 1$, let $C_{n/(n+1)}$ be the logic determined by the twelve axioms and two rules of C_1 together with the two schemata

(13) $\neg^{n-1}A \supset \neg^{n+1}A$ (where $\neg^n$ denotes n iterations of $\neg$)

(14) $\&_{i=1}^{n}(\neg^{i-1}A \equiv \neg^{i-1}B) \supset \&_{i=1}^{n}(\neg^{i}(A \circ C) \equiv \neg^{i}(B \circ C))$ $\& \&_{i=1}^{n}(\neg^{i}(C \circ A) \equiv \neg^{i}(C \circ B))$ (where $\circ$ is $\&$, $\vee$, and $\supset$).

Clearly, the logics $C_{n/(n+1)}$ are all contained in C_0, since C_1 is, and (13) and (14) are both C_0-valid (classically valid).

Definition 4.2. A $C_{n/(n+1)}$ valuation is a function $v: L \rightarrow \{1, 0\}$ satisfying the seven conditions for a C_1 valuation, together with the additional two

(8) $v(\neg^{n-1}A) = 1$ implies $v(\neg^{n+1}A) = 1$,

(9) $v(A) = v(B)$ and $\ldots$ and $v(\neg^{n-1}A) = v(\neg^{n-1}B)$ implies $v(\neg^{i}(A \circ C) = v(\neg^{i}(B \circ C))$ and $v(\neg^{i}(C \circ A)) = v(\neg^{i}(C \circ B))$ (where $\circ$ is $\&$, $\vee$ and $\supset$, for any C and all i such that $1 \leq i \leq n$).

It is now straightforward to adapt the methods of da Costa and Alves, 1977 to prove

Theorem 4.1. For $1 \leqslant i \leqslant \omega$, the logic $C_{n/(n+1)}$ is sound and complete with respect to the class of $C_{n/(n+1)}$ valuations.

We obviously also have

Theorem 4.2. $C_{n/(n+1)}$ is included in $C_{(n-1)/n}$.

It is somewhat lengthier to prove the next theorem, and we sketch the proof.

Theorem 4.3. $C_{n/(n+1)}$ is strictly included in $C_{(n-1)/n}$.

Proof. We find a $C_{n/(n+1)}$ valuation, which in particular validates $v(\neg^n A) = 1$ implies $v(\neg^{n+2}A) = 1$, and which refutes $v(\neg^{n-1}A) = 1$ implies $v(\neg^{n+1}A) = 1$. For the last condition we need $v(\neg^{n-1}A) = 1$ and $v(\neg^{n+1}A) = 0$, for some A. By $v(\neg^{n+1}A) = 0$, by the conditions of a C_1 valuation we need $v(\neg^n A) = 1$, and so $v(\neg^{n-2k}A) = 1$, for $k \leqslant n/2$. To avoid refuting $v(\neg^n A) = 1$ implies $v(\neg^{n+2}A) = 1$, we need also $v(\neg^{n+2}A) = 1$. Thus, we need to have $v(\neg^n A) = 1 = v(\neg^{n-1}A) = v(\neg^{n-k}A)$, and $v(\neg^{n+1}A) = 0$. Define a *core formula* to be a formula of the form $\neg^k p_1$, where p_1 is the first proposition letter, and $0 \leqslant k < \omega$. Set $v(\neg^k p_1) = 1$ for $0 \leqslant k \leqslant n$. Set $v(\neg^{n+2k+1}p_1) = 0$, for $0 \leqslant k < \omega$. Set $v(\neg^{n+2k}p_1) = 1$ for $0 \leqslant k < \omega$. This defines a valuation on all core formulae. This is extended to a valuation on all formulae as follows. For all other proposition letters p_i, $2 \leqslant i < \omega$, set $v(p_i) = 0$. All other formulae are then evaluated classically, viz: $v(\neg A) = 1$ iff $v(A) = 0$; $v(A \,\&\, B) = 1$ iff $v(A) = v(B) = 1$, etc. We now claim that v is a $C_{n/(n+1)}$ valuation. It is a straightforward though somewhat lengthy induction to prove that v satisfies the nine conditions of Definition 4.2. Finally, we note that $v(\neg^{n-1}p_1 \supset \neg^{n+1}p_1) = 0$ in this valuation, which establishes the theorem.

The above theorems separate the $C_{n/(n+1)}$ from one another and give soundness and completeness results for them. We move on to our real aim: to demonstrate non-trivial quotient algebras for them.

Definition 4.3. For any formulae A, B, write $A \sim_n B$ for the statement

$$\vdash_{C_{n/(n+1)}} \&_{i=0}^{n} (\neg^i A \equiv \neg^i B).$$

Theorem 4.4. $A \sim_n B$ implies $C(A) \sim_n C(B)$, for any context C.

Proof. This is in several parts

(1) $A \sim_n B$ implies $\neg A \sim_n \neg B$. Clearly, $\vdash_{C_{n/(n+1)}} \&_{i=0}^{n} (\neg^i A \equiv \neg^i B)$ guarantees all conjuncts of $\vdash_{C_{n/(n+1)}} \&_{i=0}^{n} (\neg^i A \equiv \neg^i B)$ except for $\vdash \neg^{n+1}A \equiv \neg^{n+1}B$. For the latter, we have $\vdash \neg\neg A \supset A$, so that $\vdash \neg^{n+1}A \equiv \neg^{n-1}A$; and also from (13) of Definition 4.1, $\vdash \neg^{n-1}A \equiv \neg^{n+1}A$. Thus $\vdash \neg^{n-1}A \equiv \neg^{n+1}A$. Similarly, $\vdash \neg^{n-1}B \equiv \neg^{n+1}B$. But from the antecedent of Theorem 4.4, $\vdash \neg^{n-1}A \equiv \neg^{n-1}B$. Hence by the transitivity of $\equiv$ in C_1, $\vdash \neg^{n+1}A \equiv \neg^{n+1}B$.

(2) $A \sim_n B$ implies $A \circ C \sim_n B \circ C$ and $C \circ A \sim_n C \circ B$, for any formula C where $\circ$ is &, v, $\supset$. These cases here are all similar to one another, so only one is displayed. We need to prove that $\vdash \&_{i=0}^{n} (\neg^i A \equiv \neg^i B)$ implies $\vdash \&_{i=0}^{n} (\neg^i (A \,\&\, C) \equiv \neg^i (B \,\&\, C))$. By (14) of Definition 4.1 certainly the antecedent implies $\&_{i=0}^{n} (\neg^i (A \,\&\, C) \equiv \neg^i (B \,\&\, C))$. For the case $i = 0$, we

note that $\vdash_{C_i} (A \equiv B) \supset ((A \,\&\, C) \equiv (B \,\&\, C))$, and that the antecedent of this follows from $\&_{i=1}^{n} (\neg^i A \equiv \neg^i B)$. This completes the theorem.

We note that the point of axiom (14) of Definition 4.1. is now clearer, in that it enables us to prove Theorem 4.4., which is a crucial step in the construction of Lindenbaum algebras for these logics.

Theorem 4.5. $\sim_n$ is an equivalence relation on the language L.

Proof. The reflexivity of $\sim_n$ follows immediately from $\vdash_{C_1} A \equiv A$. The symmetry and transitivity of $\sim_n$ follow immediately from the symmetry and transitivity of $\equiv$ in C_1.

Theorem 4.6. The language L considered as a formula algebra with operations $\neg$, $\&$, $\vee$, $\supset$ is homomorphic to the quotient algebras $L/\sim_n$ with operations $\neg$, $\&$, $\vee$, $\supset$.

Proof. This is a standard type of result. Theorem 4.4. shows that $L/\sim_n$ preserves the operations $\neg$, $\&$, $\vee$, $\supset$. Theorem 4.5. shows that $L/\sim_n$ is well defined. Hence $h: L \rightarrow L/\sim_n$ such that $h(A) = |A|$ ($|A|$ is the equivalence class of A) is a homomorphism.

Definition 4.4. Where $|A|$, $|B|$ are elements of $L/\sim_n$, let $|A| \leqslant_n |B|$

$$\text{iff} \vdash_{C_{n/(n+1)}} (A \supset B) \,\&\, (\neg B \supset \neg A) \,\&\, \ldots \begin{cases} \neg^n A \supset \neg^n B \text{ if n is even} \\ \neg^n B \supset \neg^n A \text{ if n is odd.} \end{cases}$$

Theorem 4.7. $\leqslant_n$ is a partial order on $L/\sim_n$.

Theorem 4.8. In $L/\sim_n$, $|\neg^{n+k+1} A| = |\neg^{n+k-1} A|$ for $k \geqslant 0$. $|\neg^n A| \leqslant_n |\neg^{n-2} A|$. If $|A| \leqslant_n |B|$, then $|\neg B| \leqslant_n |\neg A|$.

In particular, in $C_{1/2}$, $|A| = |\neg\neg A|$, and in $C_{2/3}$, $|\neg\neg A| \leqslant_n |A|$ and $|\neg\neg\neg A| = |\neg A|$, $|\neg^4 A| = |\neg^2 A|$, ... etc. In general, $C_{n/(n+1)}$ affords us with n+1 'negations' in that for any formula A there are n+2 distinct equivalence classes $|A|, |\neg A|, \ldots |\neg^{n+1} A|$ of the Lindenbaum algebra $\langle L/\sim_n, \leqslant_n \rangle$. This is enough to show what was promised earlier, namely that for the $C_{n/(n+1)}$ we have non-trivial quotient algebras, and also that the $\sim_n$ can function as biconditionals (strictly, the provability of the biconditional) holding between distinct formulae.

In order to see just a few more of the algebraic properties of these Lindenbaum algebras $\langle L/\sim_n, \leqslant_n \rangle$ of the $C_{n/(n+1)}$, let us look at the well-known matrices below, which we will call $C_{0.1}$ here (see da Costa, 1974, p. 499; also called P_1 by Sette, 1973, and F by da Costa and Alves, 1981).

&	1	2	3	v	1	2	3	⊃	1	2	3		¬
*1	1	1	3		1	1	1		1	1	3		3
*2	1	1	3		1	1	1		1	1	3		1
3	3	3	3		1	1	3		1	1	1		1

It is known that these matrices validate the theorems of C_1, and it is straightforward to check that they validate all of the $C_{n/(n+1)}$ except $C_{1/2}$ as well. This being so, it can easily be shown that the following is not in general a theorem of $C_{n/(n+1)}$: $\neg A \supset \neg(A \& B)$. Hence, while we do have for all these logics $\vdash (A \& B) \supset A$, we do not have $|A \& B| \leqslant_n |A|$. Thus the algebras are not lattices under &, v as $\cap$, $\cup$ respectively. It is more lengthy to prove, employing as it does the use of $C_{n/(n+1)}$ models, that $\neg(A \vee B) \supset \neg A$ in general fails. In particular, $\neg(p \vee p) \supset \neg p$ fails for any proposition letter p. Thus we do not have $|A| \leqslant_n |A \vee B|$, so that the algebras are not even semi-lattices.

This completes our sketch of the strengthenings of C_1 considered in this section.

5. More about extending C_1

We consider some more properties of extensions of C_1, with an eye to determining some of the consequences of strengthening C_1, especially if one wants to do so because one is persuaded that it is defective.

The first result develops out of the observation that the systems $C_{n/(n+1)}$ of the previous section are all weaker than C_0. The result to be proved shows that this is no accident. Conceivably, one might be interested in strengthening C_1 away from the direction of C_0 (obviously without adding theorems like $\neg(A \& \neg A)$ which collapse into C_0). That is, one might be looking for extensions of C_1 (say, without some of the defects of C_1) which are not sublogics of C_0. We show now that there are none such.

Theorem 5.1. C_0 is the only Post-complete logic stronger than C_1.

Proof. The theorem asserts that any logic containing C_1, closed under *modus ponens* for and uniform substitution, is either C_0, weaker than C_0, or trivial. We prove it by proving that any non-theorem of C_0, A, has a substitution instance A″ such that $\vdash_{C_1} A'' \supset p$.

Suppose that $\nvdash_{C_0} A$. Then there is a classical valuation v such that $v(A) = 0$. Let A′ be the result of substituting $p \& \neg p$ for every variable of A which is assigned 0, and $\neg(p \& \neg p)$ for every variable assigned 1. As is well known $\vdash_{C_0} \neg A'$, in fact $\vdash_{C_0} A' \supset p$. We now invoke the Prime Component Theorem for C_1 (da Costa, 1974, Theorem 4, p. 500): if $A_1, \ldots A_n$ are all the prime components of the formulae of Γ, A, then $\Gamma \vdash_{C_0} A$ iff $\Gamma, A_1^0, \ldots, A_n^0 \vdash_{C_1} A$. The prime components of a formula are, in this case, the propositional variables (see, e.g., Kleene's *Introduction of Metamathematics* p. 111). Here, $\Gamma = \Lambda$, so by the Prime Component Theorem $p^0 \vdash_{C_1} A' \supset p$; that is, $\vdash_{C_1} p^0 \supset (A' \supset p)$. Let A″ be the result of substituting $(p \& \neg p)$ for p throughout A′.

A″ is thus a substitution instance of A′ and A. Since C_1 is closed under uniform substitution, we plainly have $\vdash_{C_1} (p \& \neg p)^0 \supset (A'' \supset (p \& \neg p))$. But $\vdash_{C_1} (p \& \neg p)^0$ (da Costa and Alves, 1977, p. 628). Hence $\vdash_{C_1} A'' \supset (p \& \neg p)$, so $\vdash_{C_1} A'' \supset p$. This completes the theorem.

We turn now to consideration of two logics with finite characteristic matrices which are 'next to' C_0. The first is the set of valid formulae of the matrices $C_{0.1}$ of the previous section. The second we call $C_{0.2}$.

&	1	2	3	v	1	2	3	⊃	1	2	3	¬
*1	1	1	3		1	1	1		1	1	3	3
2	1	1	3		1	1	1		1	1	3	2
3	3	3	3		1	1	3		1	1	1	1

It is not difficult to check that all the logics $C_{n/(n+1)}$ are included in $C_{0.2}$. Neither of $C_{0.1}$ nor $C_{0.2}$ is included in the other, however: $C_{0.2}$ validates $A \supset \neg\neg A$ and $C_{0.1}$ does not, while $C_{0.1}$ validates $(\neg A)^0$ and $C_{0.2}$ does not.

We note that there are natural relationships between assignments in the matrices $C_{0.1}$ and $C_{0.2}$, and certain kinds of C_1-valuations. An assignment in the $C_{0.1}$ matrix behaves identically as a C_1-valuation in which only the propositional variables are permitted to have their negations take the value 1 when they themselves do. Complex wffs can take only the values 1 and 3 in the $C_{0.1}$ matrices, and behave classically with respect to these. Assigning a propositional variable p the value 2 and its negation 1 thus corresponds to a C_1-valuation where both p and ¬p are valued at 1. Conversely, given any such C_1-valuation to a formula and its subformulae assign a propositional variable in the $C_{0.1}$ matrix 2 just in case both it and its negation receive 1 in the C_1-valuation. The value 2 must thus be a designated value, since it corresponds to a valuation of 1, and so it is. If we define a formula to be $C_{0.1}$-valid* iff it is valued at 1 in all such C_1-valuations, we thus have $\vDash_{C_{0.1}} A$ iff A is $C_{0.1}$-valid*. Similarly, an assignment in the $C_{0.2}$ matrices corresponds to a C_1-valuation in which

(a) all wffs with main connectives &, v, ⊃ are valued as normal wffs (and so, in light of $\vdash_{C_1} A^0 \supset (\neg A)^0$, all negations of such wffs, and any wff built up entirely of these two kinds, also).

(b) Any wff of the form $\neg^n p$, $n \geqslant 0$ which is assigned 1 or 3 is valued at 1 or 0 respectively; and the C_1-valuation requires that $\neg^{n+1}p$ be valued at 0 or 1 respectively, corresponding to a $C_{0.2}$-assignment of 3 or 1, so that $(\forall n)$ ($\neg^n p$ is valued differently from $\neg^{n+1}p$).

(c) If $\neg^n p$ is assigned 2, so is p, and so is $\neg^m p$, for all m. Corresponding to this, we C_1-value $\neg^m p$ as 1, for all m. In positive contexts, 2 behaves like 1. It is not difficult to prove that $C_{0.2}$-validity coincides exactly with validity in such a class of C_1-valuations. (Problem: Such a class of C_1-valuations looks rather like the class of $C_{1/2}$-valuations. Does $C_{0.2} = C_{1/2}$?)

Now we are in a position to give axioms for $C_{0.1}$ and $C_{0.2}$.

Theorem 5.2. $C_{0.1}$ is axiomatized by adding to C_1 the axioms $(A \,\&\, B)^0$, $(A \vee B)^0$, $(A \supset B)^0$ and $(\neg A)^0$.[12]

Proof. We want to prove that $\models_{C_{0.1}} A$ iff $\vdash_{C_{0.1}} A$. Right to left is a straightforward matter of verifying the axioms and showing that the rules preserve validity, and we omit the details. We want therefore to prove that if $\models_{C_{0.1}} A$ then $\vdash_{C_{0.1}} A$.

We prove first that $\models_{C_{0.1}} A$ iff $\models_{C_1} (A_1^0 \,\&\, \ldots \,\&\, A_n^0) \supset A$, where the A_i are all the complex subformulae of A. Right to left is a matter of observing that if A_i is complex, $\models_{C_{0.1}} A_i^0$ and that if $\models_{C_1} (A_1^0 \,\&\, \ldots \,\&\, A_n^0) \supset A$, then $\models_{C_{0.1}} (A_1^0 \,\&\, \ldots \,\&\, A_n^0) \supset A$; then detaching the antecedent. Left to right: suppose $\models_{C_{0.1}} A$, and let v be an arbitrary C_1-valuation of $(A_1^0 \,\&\, \ldots \,\&\, A_n^0) \supset A$. If $v(A_1^0 \,\&\, \ldots \,\&\, A_n^0) = 0$, then $v((A_1^0 \,\&\, \ldots \,\&\, A_1^0) \supset A) = 1$, so let $v(A_1^0 \,\&\, \ldots \,\&\, A_n^0) = 1$; so that $v(A_i) \neq (\neg A_i)$, for all complex subformulae A_i of A. But such a valuation is precisely the kind of C_1-valuation described above which corresponds exactly to a $C_{0.1}$ assignment, and we have that $\models_{C_{0.1}} A$ Thus A must be assigned 1 by v. But v was arbitrary, so $\models_{C_1} (A_1^0 \,\&\, \ldots \,\&\, A_n^0) \supset A$.

But now the theorem follows easily. Suppose $\models_{C_{0.1}} A$. Then $\models_{C_1} (A_1^0 \,\&\, \ldots \,\&\, A_n^0) \supset A$, where the A_i are all the complex subformulae of A. Therefore $\vdash_{C_1} (A_1^0 \,\&\, \ldots \,\&\, A_n^0) \supset A$, so $\vdash_{C_{0.1}} (A_1^0 \,\&\, \ldots \,\&\, A_n^0) \supset A$. But $\vdash_{C_{0.1}} A_1^0 \,\&\, \ldots \,\&\, A_n^0$. Thus $\vdash_{C_{0.1}} A$.

Theorem 5.3. $C_{0.2}$ is axiomatized by adding to C_1 the axioms $(A \,\&\, B)^0$, $(A \vee B)^0$, $(A \supset B)^0$, and $A \supset \neg\neg A$.

Proof. As before, $\vdash_{C_{0.2}} A \Rightarrow \models_{C_{0.2}} A$ is straightforward.[13] For $\models_{C_{0.2}} A \Rightarrow \vdash_{C_{0.2}} A$, we prove first that $\models_{C_{0.2}} A$ iff $\models_{C_1} ((A_1^0 \,\&\, \ldots \,\&\, A_n^0 \,\&\, (B_1 \supset \neg\neg B_1) \,\&\, \ldots \,\&\, (B_m \supset \neg\neg B_m)) \supset A)$, where the A_i are all the complex subformulae of A of the form $C \,\&\, D$, $C \vee D$ or $C \supset D$, and the B_i are all the subformulae of A. Right to left of this is done by noting that all of $A_1^0, \ldots, A_n^0$, $(B \supset \neg\neg B_1), \ldots, (B_m \supset \neg\neg B_m)$ are valid in $C_{0.2}$, and then applying *modus ponens*. Left to right, assume $\models_{C_{0.2}} A$, and let v be an arbitrary C_1-valuation for $(A_1^0 \,\&\, \ldots) \supset A$. If the antecedent is false, then the whole formula is true, so let the antecedent be true. This amounts to requiring that

(a) $v(A_i) \neq v(\neg A_i)$, so that, in view of $\vdash A^0 \supset (\neg A)^0$, $v(\neg^n A_i) \neq v(\neg^{n+1} A_i)$, and similarly for any subformula built exclusively out of these;

(b) for any propositional variable p, since $v(p \supset \neg\neg p) = 1$ and also $\vdash_{C_1} \neg\neg p \supset p$, we have either $v(\neg^n p) \neq v(\neg^{n+1} p)$ for all n, or $v(\neg^n p) = 1$, for all $n \geq 0$. But such a C_1-valuation is precisely the kind of C_1-valuation described above corresponding exactly to a $C_{0.2}$-assignment. But we have that $\models_{C_{0.2}} A$, so we must have $v(A) = 1$. Hence $\models_{C_1} (A_1^0 \,\&\, \ldots) \supset A$.

Now the theorem follows immediately. From $\models_{C_{0.2}} A$ we have

$\vDash_{C_1}(A_1^0 \& \ldots) \supset A$, so $\vdash_{C_1}(A_1^0 \& \ldots) \supset A$, so $\vdash_{C_{0.2}}(A_1^0 \& \ldots) \supset A$. But $\vdash_{C_{0.2}}(A_1^0 \& \ldots)$, so $\vdash_{C_{0.2}} A$.

The next two theorems show that $C_{0.1}$ and $C_{0.2}$ are each 'next to' C_0, in the sense that there are no logics lying strictly between either and C_0.

Theorem 5.4. If A is C_0-valid but not $C_{0.1}$-valid then A has a substitution instance A′ such that $\vDash_{C_{0.1}} A' \supset \neg(p \& \neg p)$, for some variable p. Hence the addition of A to $C_{0.1}$ collapses it to C_0 or triviality.

Proof. Since $\nvDash_{C_{0.1}} A$, there is a $C_{0.1}$-assignment assigning 3 to A. Since $\vDash_{C_0} A$, at least one variable must be assigned 2, because if all variables are assigned 1 or 3, the resulting assignment is a C_0-assignment. To form A′, substitute p for all variables assigned 2, $\neg p$ for all variables assigned 1 and $\neg\neg p$ for all assigned 3. Let v be an arbitrary $C_{0.1}$-assignment to $A' \supset \neg(p \& \neg p)$. If $v(p) = 1$ or 3, $v(\neg(p \& \neg p)) = 1$, so that $v(A' \supset \neg(p \& \neg p)) = 1$. If $v(p) = 2$ then $v(\neg p) = 1$ and $v(\neg\neg p) = 3$, so that $v(A')$ is identical with the first assignment to A, i.e. $v(A') = 3$. Hence $v(A' \supset \neg(p \& \neg p)) = 1$. Therefore $\vDash_{C_{0.1}} A' \supset \neg(p \& \neg p)$.

Theorem 5.5. If A is not $C_{0.2}$-valid but is C_0-valid, then A has a substitution instance A′ such that $\vDash_{C_{0.1}} A' \supset \neg(p \& \neg p)$.

Proof. If $\nvDash_{C_{0.2}} A$, then there is a $C_{0.2}$-assignment which assigns A either 2 or 3. If A is assigned 2, then A is of the form $\neg^n p$, for some n. By substitution, if $\neg^n p$ is added as a theorem, $\neg^{2n} p$ is also a theorem, and so, by $\vDash_{C0.2} A \equiv \neg\neg A$, p is, hence triviality. So suppose that A is assigned 3. Since $\vDash_{C_0} A$, at least one variable is assigned 2. Form A′ as follows: substitute p for all variables assigned 2, $p \supset p$ for all variables assigned 1, and $\neg(p \supset p)$ for all variables assigned 3. As in Theorem 5.4, this ensures that if p is assigned 2 in any $C_{0.2}$-assignment v, $v(A') = 3$. But this is the only way to make $v(\neg(p \& \neg p)) \neq 1$, and $v(p) = 2$ implies $v(\neg(p \& \neg p)) = 2$. But $3 \supset 2 = 1$. So $v(A' \supset \neg(p \& \neg p)) = 1$ for arbitrary v. Hence $\vDash_{C_{0.2}} A' \supset \neg(p \& \neg p)$.

We conclude by posing the problem as to whether there are any other logics distinct from $C_{0.1}$ and $C_{0.2}$ which are next to C_0 in the same kind of way. More particularly, is it the case that any extension of C_1, *a fortiori* contained in C_0, must be a sublogic of either $C_{0.1}$ or $C_{0.2}$?[14]

Notes

[1] I wish to thank Elias Alves, Newton da Costa, Michael McRobbie and Graham Priest for their help.

[2] We note two things about Definition 1.2. First, it is quite informal in that it speaks of the 'intended interpretation' of →, and specifies very loosely that the intended interpretation be among the loose class of notions 'implies' etc. Second, it does not pretend to offer necessary and sufficient syntactic conditions for an implication operator; the weak necessary condition is not intended to be sufficient.

[3] The more usual definition of an L-theory, *e.g.* Routley and Meyer [RLR], imposes the condition that X be closed under conjunction as well. This author favours that practice, preferring in general to term the X of Definition 1.3 a 'quasi-theory'. For the present purposes, however, the terminology of Definition 1.3 is more convenient.

[4] This way of putting the point assumes reasonable properties for conjunction and negation, of course.

[5] See e.g. Routley, 1977.

[6] See e.g. Priest, 1977.

[7] Though mention must be made of (older) positions which hold that inconsistency arises from change e.g. Novack, 1971, (possibly) Hegel, 1969, and (possibly) Engels, 1934.

[8] See e.g. da Costa, 1974, p. 508.

[9] See e.g. Routley and Routley, 1975.

[10] See e.g. Routley and Meyer, 1973, or Routley and Routley, 1972.

[11] Indeed, a positive disrecommendation in that C_1 contains $A \supset (B \supset A)$.

[12] Axioms are given by Sette in [1973] for P_1, completeness with respect to the matrices proved, and 'maximality' (Theorem 5.4 below) proved. The axioms here are different from Sette's, and, in the context of C_1, to some extent more natural. It is therefore instructive to prove completeness directly. The proof of maximality below also used a different proof from Sette. The completeness and maximality results for $C_{0.1}$ were obtained independently of Sette.

[13] Alves and da Costa have pointed out that this is not so straightforward since MP does not preserve designation. MP fails only when $A = 1$, $B = 2$. But if $B = 2$ then B is of the form $\neg^n p$ $(n \geq 0)$, and so may take the value $B = 3$ as well. This is impossible if A and $A \supset B$ always take the value 1.

[14] This question has subsequently been answered in the negative by Alves and Loparić.

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Prior and Rennie on Times and Tenses

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One of Arthur Prior's constructions of the relational calculus for times within tense logic plus propositional quantifiers is considered using Malcolm Rennie's multimodal semantics and found wanting in two respects: Prior's use of an independent necessity operator was both incorrect and unnecessary, and his system imported an objectionable metaphysics namely temporal relationism. When these defects are repaired it is shown that temporal relationism is not entailed by the modified system. Finally, there is considered the possibility of describing an inconsistent temporal structure in which distinct, classically incompatible conceptions of time can be unified within an inconsistent framework.

1. Introduction

This paper is dedicated to the Australian logician Malcolm Rennie (1940–1980), whose brilliant logical career was cut short at an early age. Rennie was responsible for educating a generation of Australian logicians, and the present author considers it a privilege to have been his student. Rennie was a great admirer of Arthur Prior. Rennie, following Prior, took the view that logic is a vehicle in which conflicting metaphysical views can be expressed, so that disputes can be conducted precisely and rigorously and thus with some chance of resolution. I have subsequently come to think that Prior's tense-logical work is significant in a way Prior did not suspect: as the doom of the view of logic as the purveyor of necessary truth. It takes only a little study of Prior's work on tense logic to expand greatly one's conceptions of what is possible for time, and thus to erode confidence in the necessity of any thesis about time; the example generalizes rapidly. Neither Prior nor Rennie, necessitarians both, drew that consequence of course. In this paper, completeness results due to Rennie are applied in criticism of one of Prior's constructions of times from tenses. Further metaphysical implications of the construction, specifically temporal relationism, are then uncovered and repairs are suggested. Finally, it is shown that the inconsistency between the competing conceptions of time, temporal relationism and temporal absolutism, is not the worst kind of inconsistency since they can both be made to hold in an appropriate inconsistent but nontrivial framework.

2. Prior's system

Prior began with the classical propositional calculus expressed in Polish notation, and added to its language the usual tense logical unary propositional operators F, P, G, H, for 'It will be that', 'It has been that', 'It will always be that', and 'It has always been that' respectively. The relational calculus for times, or U-calculus, is an applied first order functional calculus with identity; and with the primitives Utt' and Ttp to be read respectively 't is earlier than t'' and 'p is true at t', where the t-variables range over times or temporal instants. An obvious translation of tense logic into the U-calculus is to add a constant v for 'now', and then define $Fp =_{df} \Sigma tKUvtTtp$, $Pp =_{df} \Sigma tKUtvTtp$, $Gp =_{df} \Pi tCUvtTtp$, and $Hp =_{df} \Pi tCUtvTtp$. It is now well known, though only suspected when Prior was writing (Prior *1968*, p. 75), that there is no general reverse translation of theses in the U-calculus into tense logical theses. In 'The Logic of Ending Time' (in Prior *1967*), Prior provided a tense logic augmented with propositional quantifiers which he claimed was adequate for this job.

Prior's augmented tense logic begins with Lemmon's minimal tense logic $\mathbf{K}_t$, and adds a necessity operator L of which all the theorems of **S5** hold, together with mixing axioms relating necessity to tenses. Temporal instants are to be identified with the totality of what is true at those instants, a manoeuvre familiar from another context, namely relationism about time (see below). To do this, Prior also postulated a set of 'world propositions', a, b, c, ..., thought of as giving the totality of what is true at a time, together with axioms for them; and then defined the U-calculus propositions Uab, Tap and Iab, the last to be read 'a is the same time as b'.

Summary of the system:

(1) All substitution instances of theorems of the classical propositional calculus **PC** with the rule of modus ponens
(2) All substitution instances of the axioms and rules of $\mathbf{K}_t$, that is:
$K_t(1)$: $CGCpqCGpGq$
$K_t(2)$: $CHCpqCHpHq$
$K_t(3)$: $CPGpp$
$K_t(4)$: $CFHpp$
$K_t(5)$: If $\vdash A$ then $\vdash GA$ and $\vdash HA$
(3) All substitution instances of the theorems of the modal logic **S5** plus the rule of necessitation: if $\vdash A$ then $\vdash LA$
(4) All substitution instances of:
M(1): $CLpGp$
M(2): $CLpHp$
(5) There is a set of propositions ('world propositions') a, b, c, ..., all satisfying in addition to the above:
W(1): Ma where as usual $M =_{df} NLN$
W(2): $ALCapLCaNp$
W(3): Σaa
(6) The following definitions:
Df(1): $Tap =_{df} LCap$
Df(2): $Uab =_{df} Ta(Fb)$
Df(3): $Iab =_{df} LEab$

where in (5) and (6) p is any proposition and a, b are any world propositions.

3. Against Prior's system

Prior remarks ('The Logic of Ending Time', Prior *1967*, p. 100):

> Formally, we begin from some tense logic, say K_t, and introduce ... an operator L, which means in effect that "It is true at all times ...".

And again:

> Ma (= $NLNa$) asserts in effect that the proposition a is true at some time.

In this section it is argued first that, on a certain interpretation of "It is true at some time that . . .", the second assertion is false, and the first misleading. Then a second interpretation, drawing on another paper of Prior's, is considered. It is argued that the same defect holds of this second interpretation, unless the device of laying down independent modal postulates is abandoned and the interpretation is made true by definition. It is also seen, however, that this second interpretation has the drawback of using technical devices even stronger than propositional quantifiers.

It is known that if a necessity operator L_2 is introduced into $\mathbf{K}_t$ by $L_2p =_{df} KKpGpHp$, then the resulting modal logic is **B**, the Brouwersche system. Renaming the modal operator L of **S5** above 'L_1', we have in effect a multiply modal logic with two necessity operators L_1 and L_2. Multiply modal logics were studied by Rennie (*1971*). In the presence of **S5** the M axioms are equivalent to CL_1pL_2p. Thus the sublogic in the (N, C, L_1, L_2) language without world propositions is given by (1) axioms and rules for PC (2) axioms and rules for **S5** governing L_1 (3) axioms and rules for **B** governing L_2, and (4) CL_1pL_2p.

A *Rennie 2-multiple model* for this sublogic is a tuple $\langle \mathrm{K}, \mathrm{R}_1, \mathrm{R}_2, \mathrm{v}\rangle$ where (i) $\langle \mathrm{K}, \mathrm{R}_1, \mathrm{v}\rangle$ is an **S5** model, i.e. R_1 is reflexive, symmetric and transitive on K and v is the usual function assigning truth or falsity to wffs at members of K (ii) $\langle \mathrm{K}, \mathrm{R}_2, \mathrm{v}\rangle$ is a **B** model, i.e. R_2 is reflexive and symmetric on K (iii) $\mathrm{R}_2 \subseteq \mathrm{R}_1$. It is not difficult to verify that the modal axioms and rules of the preceding paragraph all hold in such a model; Rennie showed completeness. Assuming only soundness, it can be seen here that no new theorems beyond those of **B** are produced in the (N, C, L_2) sublanguage by the presence of CL_1pL_2p. For example, considering the S4 axiom $CL_2pL_2L_2p$, let $\mathrm{K} = \{1, 2, 3\}$, let $\mathrm{R}_1 = \mathrm{K}^2$, let $\mathrm{R}_2 = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$, and let $\mathrm{v}(p, 1) = \mathrm{v}(p, 2) = 1$ and $\mathrm{v}(p, 3) = 0$. Then $\mathrm{v}(L_2, p, 1) = 1$, but also $\mathrm{v}(L_2p, 2) = 0$ so that $\mathrm{v}(L_2, L_2p, 1) = 0$ and $\mathrm{v}(CL_2pL_2L_2p, 1) = 0$. More generally, this style of argument works for any n-multiply modal system with sole mixing axioms CL_1pL_2p, . . ., $CL_{n-1}pL_np$, where the modal logic of L_i is at least as strong as L_{i+1}: pick any countermodel to any nontheorem solely in L_{i+1}, it will have a corresponding relation R_{i+1}. Then $\mathrm{R}_n \subseteq \ldots \subseteq \mathrm{R}_{i+1} \subseteq \mathrm{R}_i \subseteq \ldots \subseteq \mathrm{R}_1$ can be satisfied by taking $\mathrm{R}_n = \mathrm{R}_{n-1} = \ldots = \mathrm{R}_{i+1}$ and $\mathrm{R}_i = \mathrm{R}_{i-1} = \ldots = \mathrm{K}^2$. Hence no extra theorems in the (N, C, L_2) language are produced.

We can show now that in Prior's system, not $\vdash CKKpGpHpL_1p$, equivalently not $\vdash CM_1pAApFpPp$.

Proposition 1 In Prior's system, not $\vdash CL_2pL_1p$.
Proof If $\vdash CL_2pL_1p$ then $\vdash EL_2pL_1p$; and so since the **S5** axiom for L_1 is a theorem, the **S5** axiom for L_2, namely $CM_2pL_2M_2p$, would be provable. But it has just been shown that L_2 has a weaker logic than **S5**.■

Thus, we come to the following point. How are Prior's "It is true at all times that . ." and "It is true at some time that . ." to be understood? A natural interpretation of the former is $KKpGpHp$, that is L_2p; and a natural interpretation of the latter is $AApFpPp$, that is M_2p. Given these interpretations, then applying Proposition 1, Prior's second quoted claim above is false; M_1a does not 'assert in effect' that $AAaFaPa$. Moreover his first claim above is misleading: since not $\vdash CKKaGaHaL_1a$, then L_1a does not 'mean in effect' that it is true at all times that a. However, because $\vdash CL_1aKKaGaHa$, L_1a *suffices* for $KKaGaHa$.

But this is not the only way to read "It is true at all times that . .", nor even necessarily the best. The problem is that there might be times neither earlier nor later than the present, which $KKpGpHp$ thus cannot reach. An example is the universes of special relativity, where the objective earlier/later relation is confined to the light cone, and events in the elsewhere are only earlier/later than the present relative to some frames, but not relative to others. In another paper, 'Tense Logic and the Logic of Earlier and Later' (Prior *1967*, pp. 116–134), Prior gave a stronger reading, using an inductive definition:

$$L^0p =_{\text{df}} p$$

$$L^{n+1}p =_{\text{df}} KHL^npGL^np$$

$$\text{Then } Lp =_{\text{df}} \Pi nL^np$$

The point is that while Lp implies $KKpGpHp$, it also implies such things as GHp, HGp, $GHGp$, $HGHp$ and the like. This Lp can thus be read more plausibly as "It is true at all times that . .". The cost, of course, is even stronger quantification: over the superscripts of the inductively defined modalities. Setting this aside, a corresponding possibility modality can be defined inductively in the same fashion, by:

$$M^0p =_{\text{df}} p$$

$$M^{n+1}p =_{\text{df}} APM^npFM^np$$

$$\text{Then } Mp =_{\text{df}} \Sigma nM^np$$

Prior showed that these modalities have an **S5** structure (p. 130). Nonetheless, it is interesting that even here it can be shown that $CLpL_1p$ and CM_1pMp are not ensured. To see this, let the system be strengthened by adding $CGpGGp$, $CHpHHp$, $CKKpGpHpGHp$ and $CKKpGpHpHGp$ as axioms. This has the effect of ensuring that CL^1pL^2p, CL^2pL^3p, . . . etc., and hence $ELpL_2p$, are all theorems. But still CL_2pL_1p and hence $CLpL_1p$ would not be theorems: let $K = \{1, 2\}$, $R_1 = K^2$, $R_2 = \{\langle 1, 1\rangle, \langle 2, 2\rangle\}$, $v(p, 1) = 1$, $v(p, 2) = 0$; then $v(L_2p, 1) = 1$ while $v(L_1p, 1) = 0$, so that $v(CL_2pL_1p, 1) = 0$. That is to say, while one continues with the strategy of defining L_1 and M_1 by independent modal axioms, there appears to be no way to make Prior's opening assertions any better than misleading.

There is, of course, a fairly straightforward way out of this: dispense with the independent **S5** modal axioms and M(1) and M(2), and define $Lp =_{\text{df}} KKpGpHp$, equivalently $Mp =_{\text{df}} AApFpPp$. At least these modalities are definable within the original resources, and one can continue to use these defined modalities in the axiom system as long as one gives up the claim that Lp means that p is true at all times. This strategy is adopted below, but before doing so a deeper difficulty is considered.

4. Relationism

Temporal relationism is a group of doctrines, typically motivated by the verificationist criterion of meaning, asserting that there is nothing more to time than the events occurring within it. Relationist theories of time typically have strong consequences for the topology of time. Thus it has seemed to some that there could not be a stretch of time in which everything was static, or two distinct instants at which qualitatively the same events occurred (for example in the latter case according to this kind of relationism the universe would really be temporally closed). The world propositions provide a way of formulating this thesis: if two times have the same world propositions true at them, they are the same time; or *CKTacTbcIab*, for any world propositions *a*, *b*, *c*. Now it is a defect for a system of tense logic to contain strong topological assumptions above time, if these can be avoided. However:

Proposition 2 In Prior's system, ⊢*CKTacTbcIab*, for any world propositions *a*, *b*, *c*.

Proof (sketch)

(1)	*EKTacTcbKLCacLCcb*	DF(1)
(2)	*CKLCacLCcbLCab*	**B**
(3)	*CKTacTbcTab*	1, 2, **PC**
(4)	*EKTbcNTcbKLCbcNLCcb*	Df(1)
(5)	*CNLCcbLCcNb*	W(2)
(6)	*CKTbcNTcbKLCbcLCcNb*	4, 5, **PC**
(7)	*CKLCbcLCcNbLNb*	**B**
(8)	*NLNb*	W(1)
(9)	*NKTbcNTcb*	6, 7, 8, **PC**
(10)	*CTbcTcb*	9, **PC**
(11)	*CKTacTbcKTacTcb*	10, **PC**
(12)	*CKTacTbcTab*	3, 11, **PC**
(13)	*CKTacTbcTba*	10, 12, **PC**
(14)	*CKTacTbcKTabTba*	12, 13, **PC**
(15)	*CKTabTbaKLCabLCba*	Df(1)
(16)	*CKLCabLCbaLEab*	**B**
(17)	*ELEabIab*	Df(3)
(18)	*CKTacTbcIab*	14, 15, 16, 17, PC ■

5. Avoiding relationism

How is this to be avoided? Notice that the problem only arises if the propositions *a*, *b*, *c* . . . are thought of as descriptions of the instantaneous qualitative state of the universe and *Iab* is interpreted as identity of times. If not, it is not so clear that *CKTacTbcIab* is an objectionable thesis. But some definition of identity is desirable. Nor will $a = b =_{df} KNUabNUba$ do, since this amounts to requiring that *U* be connected where the *U*-calculus permits neutrality. If however the *a*, *b*, *c* . . . are thought of as propositions specific as to what the time is ('clock propositions'), then *CKTacTbcIab* looks right. So the system is changed by taking propositions of the form 'The time is . . .'; and then times are identified with these. The *a*, *b*, *c* . . . are changed to t_1, t_2, . . . to reflect this change. The axioms W(1) and W(2) remain unchanged, but they are renamed T(1) and T(2).

Concerning T(3): Σtt, this also look unnecessarily limiting; since it amounts to the claim that at least one time exists, which would rule out atemporal states of affairs. Such a universe might be one in which uninstantiated Platonic universals were the only existients. Note that W(3) was not used in the proof of the previous proposition, so it is not implicated in relationism.

It would be unreasonable to dispense with world propositions altogether however, if only because they enable formulation and consideration of relationist theses like the above. They are therefore retained, with a modification. The thesis W(2): $ALCapLCaNp$ is reasonable for clock propositions in place of a (given the assumption of the completeness of the theory of true propositions describing the universe, which might be questioned). But W(2) is not reasonable for time-slices of events. The reason is that substitution of clock propositions for p enables all instances of $ETtaTat$ to be proved with the help of W(2), and we would be back with $CKTt_1cTt_2cIt_1t_2$. So the full W(2) must be dropped. There needs to be something to distinguish world propositions, however. A reasonable solution seems to be to distinguish a third class of propositions, $p_1, p_2, \ldots$ describing ordinary events. The intuitive relationship between ordinary and world propositions is that the latter are infinite maximal consistent conjunctions of the former. The nearest one can get to this in a propositional tense system would seem to be to retain W(2) but restrict p in it to ordinary and world propositions, excluding clock propositions.

On W(3): Σaa, this is rather harder to dispense with than T(3), i.e. Σtt, at least if W(2) is retained; since denying W(3) would seem to allow for a universe in which no world propositions are true and perhaps even a 'null universe' in which no propositions are true. (But even here it is not obviously a necessary truth that at least one proposition is true; see Mortensen (*1989*) or Mortensen & Burgess (*1989*).

The modified system is as follows.

(1) Ordinary propositions: there is a set of propositions $p_1, p_2, \ldots$ satisfying the axioms and rules of **PC** and $\mathbf{K}_t$.
(2) Clock propositions: there is a set of propositions $t_1, t_2, \ldots$ satisfying **PC**, $\mathbf{K}_t$, and for any clock proposition t,
T(1): Mt where M is as in Df(1) below.
T(2): $ALCtpLCtNp$ where p is any proposition, and L is as DF(1) below.
(3) World propositions: there is a set of propositions $a_1, a_2, \ldots$ satisfying **PC**, $\mathbf{K}_t$ and for any world proposition a_i,
W(1): Ma_i
W(2): LCa_ipLCa_iNp where p is any ordinary or world proposition.
W(3): $\Sigma a_i a_i$
(4) Definitions
Df(1): $Lp =_{\text{df}} KKpGpHp$, $Mp =_{\text{df}} AApFpPp$
Df(2): $Tt_ip =_{\text{df}} LCt_ip$
Df(3): $Ut_it_j =_{\text{df}} Tt_iFt_j$
Df(4): $It_it_j =_{\text{df}} LEt_it_j$

where p is any proposition and t_i, t_j are any clock propositions.

The objectionable relationist schema can be formulated in this system: for any clock propositions t_i, t_j, and any world proposition a_k, $CKTt_ia_kTt_ja_kIt_it_j$. But:

Proposition 3 For any clock proposition t_i there is a clock proposition t_j and a world proposition a_k such that not $\vdash CKTt_ia_kTt_ja_kIt_it_j$.

Proof Take elements $\tau_1, \tau_2, \ldots$ of a K_t-model structure, and for all i, j, set $v(p_i, \tau_j) = 1$ if $i + j = 0 \bmod 2$, else $v(p_i, \tau_j) = 0$. If the τ_j are thought of as times, this has the effect of an oscillating or pendulum universe with even numbered atomic propositions true at even numbered times and odd numbered atomic propositions true at odd numbered times. For clock propositions, set $v(t_i, \tau_j) = 1$ iff $i = j$. Interpret world propositions as maximal consistent sets of the p_i. A world proposition is set true at a time if every member is true at the time. There are thus in the model just two values for world propositions; and for all i set $v(a_{2i}) = \{A: v(A, \tau_1) = 1\}$ and $v(a_{2i+1}) = \{A: v(A, \tau_2) = 1\}$. Also, $v(a_1, \tau_1) = v(a_1, \tau_3) = \ldots = 1$ and $v(a_2, \tau_2) = v(a_2, \tau_4) = \ldots = 1$, etc. It is straightforward to verify **PC**, $\mathbf{K}_t$ and T(1). T(2) holds since for any i, j either Ct_ip_j or Ct_iNp_j holds at all times and similarly for a_j and t_j in place of p_j. W(1) and W(3) evidently hold, and W(2) holds since Ca_ip_j or Ca_iNp_j hold at all times and similarly for a_j in place of p_j. For t_j in the theorem pick $j = i + 2$ and for a_k pick $k = i$. Now Tt_ia_i, that is LCt_ia_i, holds at all times since Ct_ia_i holds at all times. Similarly for $Tt_{i+2}a_i$, so $KTt_ia_iTt_{i+2}a_i$ holds at all times. But It_it_{i+2} holds at no times; because $v(t_i, \tau_i) = 1$ while $v(t_{i+2}, \tau_i) = 0$. Hence not $\vdash CKTt_ia_iTt_{i+2}a_iIt_it_{i+2}$ ■

The idea of interpreting world propositions as maximal consistent sets was suggested to me by Rennie.

6. Paraconsistency

As is often the case, the possibility of inconsistency-tolerance throws a distinctive light on these logical theories. The modelling used in the proof of the previous proposition evidently relies on the fact that the logic does not forbid the division of clock propositions into just two equivalence classes. Hence, in the spirit of Prior's and Rennie's conception of logic as providing a rigorous framework for metaphysics, it is worth considering the situation where the theory is augmented by the addition of an infinite number of axioms NIt_it_j, one for each $i \neq j$; that is to say, following Df(4), $NLEt_it_j$. But on the other hand, we should also acknowledge the strength of the relationist's position by reflecting on the fact that such clock propositions are abstractions. Actual clocks, or at any rate the common analog clocks with a circular dial, give the same empirical reading after a regular amount of time, typically twelve hours. This can also be interpreted as an expression of the relationist point of view in a two-state oscillating universe. Now circular time or indeed any cylindrification of phase space is a prime area for the application of inconsistent tools, as shown in Mortensen *1994*. Hence, to unify the points of view both abstract and empirical, as all good philosophy seeks to unify, we can ask what distinctions we could continue to make when we accept the empirical point of view while clinging to the abstract conception of times. An alternative approach leading to the same point is to consider a conception of time as informed by two cognitive sources, one where clocks are perceived to repeat themselves and one where things are nonetheless perceived at those times to be different. The conception might be that of an AI system or our own invention; but whatever, it would be an epistemic or cognitive story.

The question thus arises: how serious is this inconsistency? According to the classical conception of logic, as exemplified in Prior's tense logics and Rennie's models, all inconsistency is immediately fatal to any theory, by spreading everywhere throughout it in virtue of the rule *ex contradictione quodlibet*, ECQ, $p,\ Np \vdash q$. But it is also known that the same fatality can be generated in theories of logics lacking this rule, for example by adding $\vdash 0 = 1$ to Meyer's arithmetic **R#**, based on the relevant logic **R**, though not by adding $\vdash 0 = 2$. The spread of contradiction into triviality independently of ECQ tends to require a high degree of functionality associated with the identificands, as we now see.

The broad parts of the technical construction are already well known. First, one needs a paraconsistent (inconsistency-tolerant) background logic and a paraconsistent version of minimal tense logic $\mathbf{K}_t$. Many of the former are known; we choose here three-valued closed set logic **P3**. **P3** is paraconsistent though not a relevant logic and indeed possessing an **S5**-ish implication S. It is described by the matrices:

K	T	B	F	A	T	B	F	S	T	B	F	N
$*T$	T	B	F		T	T	T		T	F	F	F
$*B$	B	B	F		T	B	B		T	T	F	T
F	F	F	F		T	B	F		T	T	T	T

The first paraconsistent tense logic seems to have been due to Priest, and based on his paraconsistent logic **LP** (see Priest *1982*). The alternative here takes all the axioms of $\mathbf{K}_t$, and replaces the connective C everywhere throughout with the **P3** implication connective S. The C is retained in the language, however, since Cpq is definable as $ANpq$. Propositional quantifiers are taken semantically as generalised conjunctions (Π) and disjunctions (Σ) of their instances. Necessity L and possibility M are along for the ride since they are defined symbols, and so do not disturb the overall properties of the logic.

The resulting logic is consistent but paraconsistent. It is consistent because the implication connective S has no more theorems than its counterpart C, so that interpreting S as C gives consistency relative to classical tense logic. It is paraconsistent because not $p,\ Np \vdash q$ (let $p = \mathrm{B},\ q = \mathrm{F}$). We can use a valuation function for clock propositions and world propositions like that of the previous proof except that we permit values in any world to come from $\{\mathrm{T}, \mathrm{B}, \mathrm{F}\}$; and we modify the assignment by taking $\mathrm{v}(t_i, \tau_j) = \mathrm{T}$ if $i = j$, $\mathrm{v}(t_i, \tau_j) = \mathrm{B}$ if $i \neq j$ but $i + j = 0 \bmod 2$, else $\mathrm{v}(t_i, \tau_j) = \mathrm{F}$. For any fixed j, the theory Th determined by $\mathrm{Th} = \{A \colon \mathrm{v}(A, \tau_j) \varepsilon \{\mathrm{T}, \mathrm{B}\}\}$ is inconsistent; for example $\vdash KIt_0t_2NTt_0t_2$. The first conjunct here, It_0t_2, is the expression of the relationist's intuitions while the second conjunct NIt_0t_2 is the expression of the absolutist's intuitions. But the theory is nontrivial in the sense that not every identity of clock propositions holds: for example, not $\vdash It_0t_1$ while still $\vdash NIt_0t_1$. This unification of possible positions cannot be achieved classically without triviality, but the paraconsistent point of view permits it.

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Relevant Logics and their Rivals, Volume II

A CONTINUATION OF THE WORK OF
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Chapter 9

The Algebraic Analysis of Relevant Affixing Systems

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The unification of logical methods accomplished in the previous chapter is, in one respect, incomplete. Though the semantical analysis of a logic will, as we have seen, generate matrices for the logic, it does not yield all matrices for the logic (though there is a sense in which every matrix is repeated). To determine all matrices for a logic we investigate its matrix or algebraic representation or analysis.

In this chapter we first present algebraic analyses for an important and extensive class of affixing systems: the class comprises not only a great many relevant logics including all the more standard systems but also all the usual irrelevant logics and some unusual ones as well. Thus the analyses are much more general as regards the class of logics catered for than those encompassed in recently established theory, set out in most detail in Rasiowa [1974]. The algebraic analyses are connected with semantical analyses already provided for the class of logics in question (in Chapter 4 of RLR1) by means of a general representation theorem. The algebraic analyses are then applied to yield embedding and conservative extension results.[1]

Later parts of the chapter take off from the algebraic analyses given in the first sections, e.g. the matter of finite algebras, or from the study of important algebras, e.g. the axiomatization of certain fundamental matrices, or also arise from more general questions that algebraic approaches to relevant logics lead to, e.g. the study of relevant consequence operations, their properties and applications, of relevant theories, and, more peripherally, of many valued relevant logics.

All these topics are also unified as generalizations, applying as well to relevant logics as favoured irrelevant logics, of logical developments most intensively cultivated in Poland, where it is held that algebraic methods – many of them discovered or worked out in Poland – really do hold the keys to the universe (of logic):

> We do believe that matrix semantics [i.e. in effect algebraic analyses, for sentential logics (considered as consequence operators on finitary free algebras)] is of first rank among other semantics. ... Other semantics are regarded as secondary, if at all (stated in Zygmunt [1977], p.88).

But while we grant the importance of matrix semantics, and of the bridge they provide between logical systems and mathematics, especially algebraic methods, we consider that they are of much less philosophical significance than worlds semantics. In any case however, there are easy routes, traversible in both directions, from one set of semantics to the other, as is well known, and we now show somewhat more generally.[2]

§9.1. Ackermann and De Morgan Groupoids

Relevant affixing systems are modelled by algebraic structures called (in honour of Ackermann, 'the father of relevance logic') Ackermann groupoids. Groupoids are group-like structures, many of whose properties algebraic studies have disclosed, and Ackermann's groupoids simply add further duly contained elements to groupoids. In what follows some very small acquaintance with modern algebra is presupposed.

A structure $\underset{\sim}{G} = <G, \leq, \circ, \rightarrow, 1>$, where G is a set, $1 \varepsilon G$, $\leq$ is a two-place relation on G, and $\circ$ and $\rightarrow$ are two-place operations on G, is an *(implicational) Ackermann groupoid* iff

P1. G is a partially ordered groupoid under $\leq$ and $\circ$, i.e.

P1.1. $\leq$ is a reflective transitive and antisymmetric relation on G, and

P1.2. $\leq$ is summative under $\circ$, i.e. whenever $a \leq b$ then both $a\circ c \leq b\circ c$ and $c\circ a \leq c\circ b$ for every $a,b,c \ \varepsilon \ G$;

P2. 1 is a left identity, i.e. $1\circ a = a$ for $a \not\varepsilon G$;

P3. G is left-residuated w.r.t. $\rightarrow$, i.e. $a\circ b \leq c$ iff $a \leq b\rightarrow c$ for $a,b,c \ \varepsilon \ G$.

A *positive Ackermann groupoid* $\underset{\sim}{G}$ is a structure $<G, \circ, \rightarrow, \cap, \cup, 1>$ satisfying P2 and P3, where $\leq$ is defined: $a \leq b$ iff $a\cup b = b$, and with P1 strengthened to:

P4. G is a distributive lattice with respect to $\cup$, $\cap$, which is ordered w.r.t. to $\circ$, i.e.

P4.1. $<G, \cap, \cup>$ is a distributive lattice, and

P4.2. G is lattice ordered, i.e. $a \circ (b\cup c) = (a\circ b) \cup (a\circ c)$ and $(b\cup c) \circ a = (b\circ a) \cup (c\circ a)$, for $a,b,c \ \varepsilon \ G$.

P4 implies P1.2 as follows:

$(c\circ a) \cup (c\circ b) = c \circ (a\cup b) = c \circ b$ when $a \leq b$;

i.e. $c\circ a \leq c\circ b$ when $a \leq b$. $\cup$ and $\cap$ are of course two-place operations on C, and - is a one-place operation on C.

A *de Morgan groupoid* $\underset{\sim}{G}$ is a structure $<G, \circ, \rightarrow, \cap, \cup, -, 1>$ satisfying P2, P3, P4.2 and with P4.1 strengthened to:

P5.1. $<G, \cap, \cup, ->$ is a de Morgan lattice, i.e. a distributive lattice such that, for $a,b \ \varepsilon \ G$, $--a = a$ and whenever $a \leq b$, $-b \leq -a$. Thus a de Morgan groupoid is a positive Ackermann groupoid but with postulate P4 strengthened to:

P5. G is a de Morgan lattice which is ordered w.r.t. o.

A *classical (de Morgan) groupoid* is a de Morgan groupoid such that:

P6. $1 \le a \cup -a$, for every $a \in G$.

A *negative Ackermann groupoid* $\underset{\sim}{G}$ is a structure $<G, \le, o, \rightarrow, -, 1>$ where $<G, \le, o, \rightarrow, 1>$ is an Ackermann groupoid and - is a unary operation on G such that for $a,b \in G$, $--a = a$ and if $a \le b$ then $-b \le -a$.

§9.2. De Morgan Groupoids Algebraize B^{ot}

The basic logic B^{ot}, with sentential language SL_t which includes as well as the primitive connective set {→, &, v, ~, o} the sentential constant t, has the following axiom schemes and rules:

A1. A → A (This scheme is redundant in the presence of t.)

A2. A&B → A | A3. A&B → B

A4. (A→B)&(A→C) → .A → B&C | A5. A → AvB

A6. B → AvB | A7. (A→C)&(B→C) → .AvB → C

A8. A&(BvC) → (A&B)v(A&C) | A9. ~~A → A

A10. t | A11. t →. A→A

R1. A, A→B ⇒ B (modus ponens)

R2. A, B ⇒ A&B (adjunction)

R3. A→B ⇒ B→C →. A→C (suffixing)

R4. A→B ⇒ C→A →. C→B (prefixing)

R5. A→~B ⇒ B→~A (contraposition)

R6. (AoB) → C ⇔ A →. B→C (residuation)

The basic logic B, of Chapter 4 of RLR1, results on deleting connective o and constant t and postulates concerning them.

Theorem 9.2.1. B^{ot} is a conservative extension of B.
Proof is as in Chapter 5, RLR1.
Thus an algebraic treatment of B^{ot} provides at the same time an algebraic treatment of B, and indeed o and t are added to facilitate algebraic treatment of relevant logics (as explained in Meyer and Routley [1972]).

We are also interested in various parts of B^{ot}, especially the positive part B^{ot}_+ which drops negation and its associated postulates A9 and R5, the implicational part B^{ot}_I which omits negation, conjunction and disjunction and their associated postulates, and the implicational negation part B_I which eliminates conjunction and disjunction. Parts are similarly characterized for the extensions of B^{ot} we go on to consider; for example, as before, where L is an extension of B^{ot}, L_+ is the positive part of L and extends B^{ot}_+.

We connect logics with algebras in the familiar way, by means of interpretation functions. In the specific way we define truth on an interpretation we follow Dunn's algebraic analysis of system R (in Dunn [1966]).

An *interpretation* of a sentential logic L in a structure G with corresponding operations is a function I defined on all wff of L, with values in G, and such that, whenever the connectives and constants belong to the language of L, the following hold for all wff A and B:

(i) $I(A \to B) = I(A) \to I(B)$
(ii) $I(A \circ B) = I(A) \circ I(B)$
(iii) $I(t) = 1$
(iv) $I(A \& B) = I(A) \cap I(B)$
(v) $I(A \vee B) = I(A) \cup I(B)$
(vi) $I(\sim A) = -I(A)$

A wff A of L is *true on interpretation I in G* iff $1 \le I(A)$; and otherwise is *false on I in G* A is *valid in G* iff A is true on all interpretations in G and otherwise is invalid in G A is *L-groupoid valid* iff A is valid in all L-groupoids; and otherwise is *L-groupoid invalid;* in particular A is *de Morgan (groupoid) valid* iff A is valid in all de Morgan groupoids.

Theorem 9.2.2. A is a theorem of B^{ot} iff A is valid in all de Morgan groupoids.

Proof: (1) We show, by induction on the length of proof of A, that A is true on any interpretation I in any de Morgan groupoid G. A10 is trivial; otherwise where A is an axiom scheme of the form $C \to D$, it suffices to show $I(C) \le I(D)$, since $1 \le I(C \to D)$ iff $1 \le I(C) \to I(D)$, i.e. iff $1 \circ I(C) \le I(D)$. Since then G is a de Morgan lattice, all axioms except A4, A7 and A11 are trivially verified.

ad A4. Since $(a \to b) \cap (a \to c) \le a \to b$, $((a \to b) \cap (a \to c)) \circ a \le b$. Since similarly $((a \to b) \cap (a \to c)) \circ a \le c$, $((a \to b) \cap (a \to c)) \circ a \le b \cap c$, so $(a \to b) \cap (a \to c) \le a \to (b \cap c)$. Hence $1 \le I(A4)$.

ad A7. Since $(a \to c) \cap (b \to c) \le a \to c$, $((a \to c) \cap (b \to c)) \circ a \le c$, and similarly $((a \to c) \cap (b \to c)) \circ b \le c$. Hence, by lattice ordering, $((a \to c) \cap (b \to c)) \circ (a \cup b) \le (((a \to c) \cap (b \to c)) \circ a) \cup (((a \to c) \cap (b \to c)) \circ b) \le c$. Thus $(a \to c) \cap (b \to c) \le (a \cup b) \to c$; whence $1 \le I(A7)$.

ad A11. Since $1 \circ a \le a$, $1 \le a \to a$; so $1 \le I(A) \to I(A) = I(A \to A)$.

The rules are verified by supposing their premises are true on an arbitrary interpretation I and showing their conclusions are likewise true. R2 and R5 are immediate from de Morgan lattice properties, and R6 by left residuation. R1 follows by transitivity of $\le$, and R3 and R4 by left and right summation.

(2) Suppose A is not a theorem of B^{ot}. Define, in the usual way, the Lindenbaum algebra of B^{ot}. For each wff C, $|C| = \{C' : C \leftrightarrow C' \text{ is a theorem of } B^{ot}\}$. Let $|G| = \{|C| : C \text{ is a wff}\}$ and define operations on elements of $|G|$ thus:

$|A| \circ |B| = |A \circ B|$, $|A| \to |B| = |A \to B|$,
$|A| \cap |B| = |A \& B|$, $|A| \cup |B| = |A \vee B|$, $-|A| = |\sim A|$

Then (i) $|\underset{\sim}{G}|$ = <|G|, o, →, ∩, U, -, |t|> is a de Morgan monoid. For first, the operations are well-defined. Next

(ii) |A| ≤ |B| iff A→B is a theorem of B^{ot}.

If ⊢A→B, ⊢AvB ↔ B; so ⊢C ↔ AvB iff ⊢C↔B for every wff C, i.e. |AvB| = |B|, so |A| U |B| = |B|, i.e. |A| ≤ |B|. Conversely if |A| ≤ |B|, then ⊢C ↔ AvB iff ⊢C↔B; so ⊢AvB ↔ B, whence ⊢A→B.

It is immediate that ≤ is a partial order on |G|, and P5.1 follows readily.

ad P2. If ⊢toA → C, ⊢t → .A→C; so as ⊢t, ⊢A→C. Since t → .A→A, toA → A; so if ⊢C → toA, ⊢C→A. Hence if ⊢toA ↔ C then ⊢A↔C.

Conversely, since ⊢toA → A, if ⊢A→C, ⊢toA → C; and if ⊢C→A then ⊢toC → toA, so ⊢t → .C → toA, whence ⊢C → toA, since ⊢t. Thus if ⊢A↔C then ⊢toA ↔ C. Combining results |toA| = |A|, whence |t| o |A| = |A|.

ad P3. Immediate by the residuation rule, and (ii).

ad P4.2. By (ii) it suffices to show ⊢A o (BvC) ↔ AoB v AoC and its mate. Since ⊢CoB → CoB, ⊢C → .B → CoB. If ⊢A→B then ⊢B → CoB → .A → CoB, hence ⊢C → .A → CoB and ⊢CoA → CoB. Similarly if ⊢A→B then ⊢AoC → BoC. Hence using A5 and A6, ⊢AoB → Ao(BvC) and ⊢AoC → Ao(BvC) whence ⊢AoB v AoC → Ao(BvC) by R2 and A7. As to the converse, since ⊢A → .B → AoB and ⊢A → .C → AoC, ⊢A → . (B → AoB) & (C → AoC), since ⊢(B → AoB) & (C → AoC) → .BvC → (AoB) v (AoC), ⊢A → .BvC → (AoB) v (AoC); whence ⊢A o (BvC) → . (AoB) v (AoC). The other case is similar.

Thus (i) is established. Now define an interpretation I in $|\underset{\sim}{G}|$ thus:

I(A) = |A|. Then

(iii) I is an interpretation in $|\underset{\sim}{G}|$. Also

(iv) C is a theorem of B^{ot} if |t| ≤ |c|, for every wff C.

For by (ii) |t| ≤ C iff t→C is a theorem of B^{ot}. But then, since t is a theorem so is C.

Since by hypothesis, A is a non-theorem, A is not true on I in $|\underset{\sim}{G}|$, that is there is a De Morgan groupoid in which A is invalid.

Corollaries. A is a theorem of L iff A is valid in all L groupoids, where logic L is one of the following parts of B^{ot} : B^{ot}_{I}, $B^{ot}_{I\sim}$, B^{ot}_{+}. Thus Ackermann groupoids, algebraize B^{ot}_{I}, positive Ackermann groupoids B^{ot}_{+} and negative Ackermann groupoids $B^{ot}_{I\sim}$.

Proofs are by the relevant parts of the proof of the theorem.

§9.3. Extensions of B^{ot} and its Parts have Algebraic Counterparts

Every extension of B^{ot} and its parts by the addition of new axiom schemes has an algebraic counterpart; some of these extensions are more natural than others. A remarkable connection between logical postulates of a class of extensions of B^{ot} and algebraic postulates, written in terms of $\circ$, was established in Meyer and Routley [1972]. When negation is added the list of interconnexions can be enlarged; but some modifications are also required, just as when the semantics for positive systems is enlarged to take account of negation. But unlike the algebraic postulates corresponding to relevantly-interesting positive logical postulates those corresponding to negation postulates are so far not much better than transforms of the axiom schemes. Hopefully a neater algebraic analysis of negation will emerge (e.g. from recent reaxiomatizations of relevant logics, which conservatively add a Boolean negation[3]).

Only a selection of more important (or in some cases older established) logical postulates from earlier chapters are considered. The methods also apply to a great many further postulates. Note that such prestigeous systems as E (strictly E^{ot}), R and T are included. We list the logical postulates on the left, the corresponding algebraic postulate on the right, and the corresponding combinator on the far right:

B1. $A \& (A \to B) \to B$	r1. $a \le a \circ a$	W_*
B2. $(A \to B) \& (B \to C) \to .A \to C$	r2. $a \circ b \le a \circ (a \circ b)$	WB
B3. $A \to B \to .B \to C \to .A \to C$	r3. $(a \circ b) \circ c \le b \circ (a \circ c)$	B'
B4. $B \to C \to .A \to B \to .A \to C$	r4. $(a \circ b) \circ c \le a \circ (b \circ c)$	B
B5. $(A \to .A \to B) \to .A \to B$	r5. $a \circ b \le (a \circ b) \circ b$	W
B6. $t \to A \to A$	r6. $a \le a \circ 1$	
B7. $A \to .A \to B \to B$	r7. $a \circ b \le b \circ a$	C_*
B8. $A \to .B \to B$	r8. $a \circ b \le b$	K'
B9. $A \to .B \to A$	r9. $a \circ b \le a$	K
B10. $(A \to .B \to C) \to .A \to B \to .A \to C$	r10. $(a \circ b) \circ c \le (a \circ c) \circ (b \circ c)$	S
B11. $(A \to .B \to C) \to .B \to .A \to C$	r11. $(a \circ b) \circ c \le (a \circ c) \circ b$	C
B12. $A \to .A \to A$	r12. $a \circ a \le a$	
B13. $A \vee \sim A$	r13. $1 \le a \cup -a$	
B14. $A \to \sim A \to \sim A$	r14. if $a \circ b \le -b$ then $a \le -b$	
B15. $A \to \sim B \to .B \to \sim A$	r15. if $a \circ b \le c$ then $a \circ -c \le -b$	
B16. $B \to A \vee \sim A$	r16. $b \le a \cup -a$.	

A cited combinator applied to the left hand side of the corresponding algebraic inequality will produce the right-hand side of the inequality (for details see Curry and Feys [1958]). It is at this point that we begin to fit methods of combinatory logics into our synthesis of methods of logical analysis. (See also Bunder's §11.4.)

We call a groupoid *fitting for* logic L, or an *L-groupoid,* just in case the algebraic postulates which correspond to the further axiom schemes of L hold for the groupoid.

Theorem 9.3.1. Where L is any one of the extensions of B^{ot} or its parts, introduced, A is a theorem of L iff A is valid in all L-groupoids.
Proof extends the proof of Theorem 9.2.2, and is straightforward once we establish certain basic equivalences.

For each implicational axiom Bi in our list there is a corresponding algebraic postulate bi obtained from the interpretation of Bi; as a transformation bi is obtained from Bi as follows: capital letters are replaced by alphabetically matching lower case letters, the main implication symbol → is replaced by the order notation ≤, and otherwise symbols, →, o, &, v, ~, t are replaced throughout by →, o, ∩, ∪, -, 1 respectively. Call the result bi = t(Bi) the *algebraic transform*, under t, of Bi. Similarly each algebraic postulate (except those of the form 1 ≤ d, for some d) has a *logical transform* Ri = t^{-1}(ri) obtained by the inverse process. For example, in the case of B3 and r3 we obtain respectively under t and t^{-1},

b3. a → b ≤ b→c → .a→c R3. Ao(BoC) →. Bo(AoC)

The basic equivalences are as follows:
(I). Where Ri = t^{-1}(ri), Bi is a theorem of B^{ot} iff Ri is a theorem of B^{ot}.
(II).Where bi = t(Bi), bi holds in a de Morgan (Ackermann, if it is non-negative) groupoid $\underset{\sim}{G}$ iff ri holds in $\underset{\sim}{G}$.
The proofs of (I) and (II) are likewise transforms of one another. Residuation and its mate P3 are respectedly applied, and also these corollaries:

Ba. (A→B)oA → B Ra. (a→b)oa ≤ b
Bb. A → .B → AoB Rb. a ≤ b → aob

We write some illustrative proofs in algebraese and some in logicish.

<u>ad</u> r1, given b1. a ≤ a ∩ a
≤ a ∩ (a → aoa) using Rb
≤ aoa by b1.

<u>ad</u> b1, given r1. Since a∩b ≤ a and a∩b ≤ b, (a∩b)o(a∩b) ≤ ao(a∩b) ≤ aob, by P1.2. Thus, applying r1, a ∩ b ≤ (a ∩ b)o(a ∩ b) ≤ aob. Then a∩(a→b) ≤ (a→b)∩a ≤ (a→b)oa ≤ b.

<u>ad</u> r2, given b2. Applying Rb twice and lattice properties, a ≤ (b → .aob) ∩ (aob → .ao(aob)) ≤ b → .ao(aob), by b2. Then r2 follows by P3.

<u>ad</u> b2, given r2. Since (a→b)oa ≤ b, by P1.2 and Ra,
(b→c) o ((a→b)oa) ≤ (b→c)ob ≤ c (1)
By P1.2, P4.2, ((b→c) ∩ (a→b)) o a ≤ (a→b)oa (2)
Hence, ((b→c) ∩ (a→b)) o (((b→c) ∩ (a→b)) o a)

$\leq (b \to c) \circ (((b \to c) \cap (a \to b)) \circ a)$, using P1.2, P4.2
$\leq (b \to c) \circ ((a \to b) \circ a)$, by (2), P1.2

Then $((b \to c) \cap (a \to b)) \circ a \leq ((b \to c) \cap (a \to b)) \circ (((b \to c) \cap (a \to b)) \circ a$, by r5

$\leq (b \to c) \circ ((a \to b) \circ a)$
$\leq c$, by (1)

b2 follows by P3.

ad R3, given B3. Since $A \to .C \to A \circ C$, $A \to .\alpha \to \beta$, where α is $A \circ C \to B \circ (A \circ C)$ and β is $C \to B \circ (A \circ C)$. But $B \to \alpha$, by Bb, hence by prefixing and suffixing $A \to .B \to \beta$, whence, by residuation, $(A \circ B) \circ C \to .B \circ (A \circ C)$.

ad B3, given R3. $(B \to C) \circ ((A \to B) \circ A) \to (B \to C) \circ B$, by Ra
$\to C$, by Ra.

Thus, by R3, $((A \to B) \circ (B \to C)) \circ A \to C$, whence B3, by residuation.

ad R4, B4. Similar.

ad R5, given B5. Since $A \circ B \to .B \to (A \circ B) \circ B$,
$B \to A \circ B \to .B \to .B \to (A \circ B) \circ B$
$\to .B \to (A \circ B) \circ B$, by B5

Thus $(A \to .B \to A \circ B) \to .A \to .B \to (A \circ B) \circ B$, whence by B6 and residuation, R5.

ad B5, given R5. Since $(A \to (A \to B)) \circ A \to .A \to B$ by Ra, $((A \to A \to B)) \circ A) \circ A \to B$. Thus, as $(A \to .A \to B) \circ A \to .((A \to .A \to B) \circ A) \circ A$ by R5, $(A \to (A \to B)) \circ A \to B$; whence B5.

ad B6, R6. We establish the basic interconnexion

$A \to B \to B \Leftrightarrow B \to B \circ A$

from which the B6 and R6 cases follow as special cases. Suppose $B \to .B \circ A$ is provable. Then, as $(A \to B) \circ A \to B$, and by assumption $A \to B \to .(A \to B) \circ A$, $A \to B \to B$. Conversely suppose $A \to B \to B$ is provable. Since $A \to .B \to A \circ B$ and by assumption $B \to A \circ B \to .A \circ B$, $A \to A \circ B$.

ad R7 given B7. By B7, $B \to .B \to A \circ B \to .A \circ B$. Hence as $A \to .B \to A \circ B$, $B \to .A \to A \circ B$ and $B \circ A \to A \circ B$,

ad B7 given R7. By R7, $A \to .(A \to B) \to (A \to B) \circ A$. But $(A \to B) \circ A \to B$, whence B7.

ad B8, B9, B12 immediate; and B10, B11 are like cases already treated.

ad b14, r14. Suppose r14 holds. Then since $(a \to -a) \circ a \leq -a$, $a \to -a \leq -a$. Conversely, given $a \circ b \leq -b$, $b \to a \circ b \leq b \to -b$. But $a \leq b \to a \circ b$ and $b \to -b \leq -b$, whence $a \leq -b$.

Note that r16 cannot be replaced by the postulate $1 = a \cup -a$. If it were one would have from $a \cup -a \leq 1$ that $b \leq a \to a$; but it is known that $B \to .A \to A$ does not follow, given B16.

The proof of the theorem is completed as follows:

(1'). Extending (1) Theorem 9.2.2.

Where algebraic postulate ri holds in $\underset{\sim}{G}$, by (II) bi holds in $\underset{\sim}{G}$; hence Bi is valid in $\underset{\sim}{G}$.

(2'). Extending (2) Theorem 9.2.2. Where Bi is an implicational axiom scheme, by (I) Ri is a theorem; whence ri holds in $|\underset{\sim}{G}|$.

§9.4. Representation of the Algebras in Model Structures

A (B^{ot}) *model structure* (m.s.) is a structure $\underset{\sim}{M}$ = <T, K, R, O, *> where, as explained in Chapter 4 of RLR1, K is a set, T ε K, R is a three-place and O a one-place relation on K, and * is a unary operation on K, such that the following definitions apply and postulates hold for all a,b,c,d ε K:

d1. a ≤ b $=_{df}$ for some x ε K, Ox and Rxab.
d2. R^2abcd $=_{df}$ for some x ε K, Rabx and Rxcd.
d3. R^2a(bc)d $=_{df}$ for some x ε K, Rbcx and Raxd.
p1. a ≤ a
p2. If a ≤ d & Rdbc then Rabc.
p3. If b ≤ d & Radc then Rabc.
p4. If d ≤ c & Rabd then Rabc.
p5. a** = a.
p6. If a ≤ b then b* ≤ a*.
p7. OT. (i.e. p0 of §4.3, p.298.)

Postulates p2-p4 ensure full monotonicity. The postulates yield structures equivalent to those of §4.3 (in the sense of equivalence explained there, that the structures are adequate for the same logics).

A strengthened model structure (s.m.s) for B^{ot} is a structure M where O satisfies the further postulate:

p8. If a ≤ b & Oa then Ob.

The advantage of such a structure is that a slightly simpler evaluation rule can be used for constant t, a rule which facilitates algebraic connexions.

The following semantical postulates are added to B^{ot}m.s., singly or in combination, to provide modellings for extensions of B^{ot}, just as in Chapter 4. But only a selection of more important postulates are now considered.

q1. Raaa.
q2. If Rabc then R^2a(ab)c.
q3. If R^2abcd then R^2b(ac)d.
q4. If Rabc then R^2abbc.
q6. For some x ε K, Ox and Raxa.
q7. If Rabc then Rbac.
q8. If Rabc then b ≤ c.
q9. If Rabc then a ≤ c.

q10. If R^2abcd then for some $x \varepsilon K$, R^2acxd and $Rbcx$.
q11. If R^2abcd then R^2acbd.
q12. If $Rabc$ then $a \leq c$ or $b \leq c$.
q13. If $x \varepsilon O$ then $x^* \leq x$.
q14. Raa^*a.
q15. If $Rabc$ then Rac^*b^*.
q16. $a^* \leq a$.

Semantical postulate qi *corresponds* to axiom scheme Bi and to algebraic postulate ri. Where L is a logic obtained by adding axiom schemata from among those listed to B^{ot}, an Lm.s. is the m.s. obtained by adding to a B^{ot}m.s. semantical postulates corresponding to the further axioms of L.

Where $\underset{\sim}{M}$ is an Lm.s., a *range* (or *L.A. proposition*) in $\underset{\sim}{M}$ is any subset J of K which is closed upwards, i.e. whenever $a \varepsilon J$ and $a \leq b$ then $b \varepsilon J$. These ranges, or strikes as they are called in Meyer and Routley [1972], are the relevant analogues of the propositional ranges, used extensively by Carnap in the context of strict logics, and later taken, especially by logicians at Los Angeles, as expliciting the notion of proposition.

The algebra of ranges $\Pi(\underset{\sim}{M})$, also written $\underset{\sim}{M}^+$, determined by $\underset{\sim}{M}$, for short *the algebra* $\underset{\sim}{M}^+$ *on M,* is a structure $\underset{\sim}{M}^+ = \langle \Pi, \circ, \rightarrow, \cap, \cup, -, 1 \rangle$ where:

(i) Π is the set of all ranges in M;
(ii) $\cap$ and $\cup$ are, respectively, set-theoretical intersection and union on subsets of Π;
(iii) For $F, G \varepsilon \Pi$,
$F \circ G = \{c \varepsilon K: (\exists a \varepsilon F, b \varepsilon G) Rabc\}$
(iv) For $F, G \varepsilon \Pi$,
$F \rightarrow G = \{c \varepsilon K: (\forall a,b \varepsilon K)(\text{if } a \varepsilon F \text{ and } Rcab \text{ then } b \varepsilon G)\}$.

Equivalently, $F \rightarrow G = \cup\{H \varepsilon \Pi: H \circ F \subseteq G\}$. As to the equivalence, suppose first $c \varepsilon \cup\{H \varepsilon \Pi: H \circ F \subseteq G\}$. Then, for some H_1, $c \varepsilon H_1$ and $d \varepsilon H_1$, and if $a \varepsilon F$ and $Rdab$ then $b \varepsilon G$, for every $d,a,b \varepsilon K$. If then $a \varepsilon F$ and $Rcab$, $b \varepsilon G$ for every $a,b \varepsilon K$; whence $c \varepsilon F \rightarrow G$. Conversely, suppose $c \varepsilon F \rightarrow G$ and set $H = G \circ G$. Then $c \varepsilon H$ and also $H \circ F \subseteq G$. For suppose $a \varepsilon (F \rightarrow G) \circ F$. Then there are b,d with $b \varepsilon F \rightarrow G$ and $d \varepsilon F$ such that $Rbda$; but since $b \varepsilon F \rightarrow G$, if $d \varepsilon F$ and $Rbda$ then $a \varepsilon G$; whence $a \varepsilon G$ as required.

(v) For $F \varepsilon \Pi$, $-F = \{c \varepsilon K: c^* \varepsilon F\}$.
(vi) $1 = \{a \varepsilon K: (\exists b \varepsilon K)(Ob \,\&\, b \leq a)\}$

Alternatively where the Lm.s. is strengthened by p8, we may define:

(vi') $1_1 = \{a \varepsilon K: Oa\}$

Theorem 9.4.1. If $\underset{\sim}{M}$ is an Lm.s. then $\underset{\sim}{M}^+$ is an L groupoid.
Proof: (1) Where $\tilde{M}$ is a B^{ot}m.s. then $\underset{\sim}{M}^+$ is a De Morgan groupoid.
First the operations on Π are well defined. Since each operation determines a subset of K it remains to establish that the resulting subsets are closed

upwards; but this is immediate in the case of union and intersection, follows from monotonocity conditions on R in the case of ∘ and →, and follows from a ≤ b ⊃ b* ≤ a* in the case of -. It remains to show that groupoid postulates hold.

ad P1. Immediate from properties of set-inclusion and the definition of ∘.

ad P2. c ε 1∘F ⊃ (∃b ε F, a,d ε K)(Od & d ≤ a & Rabc)
⊃ (∃b ε F, d ε K)(Od & Rdbc), by monotonicity.
⊃ (∃b ε F)(b ≤ c), by monotonicity.
⊃ c ε F, by upward closure.

Conversely, c ε F ⊃ c ε F & OT & T ≤ T & RTcc
⊃ (∃b ε F)(OT & T ≤ T & RTbc)
⊃ c ε 1∘F.

Where $\underset{\sim}{M}$ is strengthened the proof of P2 simplifies.

ad P3. Suppose first H∘F ⊆ G. Then H is one of the sets whose union is F→G; hence H ⊆ F→G. Conversely suppose H ⊆ F→G and that c ε H∘F. For some a ε H and b ε F, Rabc. Hence since a ε F→G, c ε G as required by definition of F→G.

ad P4.1. Π is a ring of sets, and all such are distributive lattices ordered by set inclusion.

ad P4.1. The special De Morgan lattice postulates are immediate.

ad P4.2. By quantification logic from the definitions.

(2) Where qi holds in the Lm.s., ri holds in $\underset{\sim}{M}^+$.

ad r1. Since Raaa, whenever c ε F, (∃a ε F,b ε F)Rabc. Hence F ⊆ F∘F, as required.

ad r2. Since Rabc ⊃ (∃x ε K)(Raxc & Rabx), (∃a ε F,b ε C)Rabc ⊃ (∃c' ε F,b' ε G,x ε K)(Ra'xc & Ra'b'x). Hence, since x ε F∘G iff (∃a' ε F, b' ε G)Ra'b'x, c ε F∘G ⊃ (∃a' ε F,x ε K)(Ra'xc & x ε F∘G)
⊃ c ε F∘(F∘G)

ad r3. Since (∃x)(Rabx & Rxcd) ⊃ (∃y)(Rbyd & Raay)(∃a ε F,b ε G,c ε H,x ε K)(Rabx & Rxcd) ⊃ (∃a ε F,b ε G,c ε H,x ε K)(Rabx & Rxcd) ⊃ (∃a ε F,b ε G,c ε H,y ε K)(Rbyd & Racy). So (∃c ε H,x ε F∘G)Rxcd ⊃ (∃b ε G,y ε F∘H)Rbyd; whence d ε (F∘G)∘H ⊃ d ε G∘(F∘H). Most remaining cases are similar.

ad r16. Since d ε F or d ∉ F and d* ≤ d, d ε F or d* ε F, whence d ε F∪-F. Hence G ⊆ F∪-F.

We turn to preliminaries to representation theorems. Let $\underset{\sim}{G}$ be a structure satisfying P4.1, i.e. a distributive lattice. Then, as usual, F is a *filter in* $\underset{\sim}{G}$ iff F is a subset of $\underset{\sim}{G}$ such that a∩b ε F iff a ε F and b ε F. If F is a filter in $\underset{\sim}{G}$ and 1 ε F then F is a *1-filter.* A filter F is *prime* iff whenever a∪b ε F then either a ε F or b ε F. The P-filter, the *principal filter determined by 1,* is the class of groupoid elements a such the 1 ≤ a; it is of course a 1-filter, and is the smallest 1-filter. A groupoid $\underset{\sim}{G}$ is itself *prime* iff its P-filter is prime. Similarly other properties defined for filters will be transferred to their groupoids; e.g. a De Morgan groupoid $\underset{\sim}{G}$ is consistent iff its P-filter is, where a filter F of a De Morgan groupoid is *consistent* iff -1 ε F.

Theorem 9.4.2. Let $\underset{\sim}{G}$ = <G, o, →, ∩, U, -, 1> be a prime de Morgan groupoid. Then there is an m.s. $\underset{\sim}{M}$ = <0, K, R, P, *> such that $\underset{\sim}{G}$ is isomorphic to a subalgebra of $\underset{\sim}{M}^+$; and also a strengthened m.s. $\underset{\sim}{M}_1$ = <0, K, R, P, *> such that $\underset{\sim}{G}$ is isomorphic to a subalgebra of $\underset{\sim}{M}_1{}^+$.

Proof: Let $\bar{K}$ be the class of filters on $\underset{\sim}{G}$, K the class of prime filters on $\underset{\sim}{G}$, and T the P-filter of $\underset{\sim}{G}$ i.e. T = {m ε G: 1 ≤ m}, where m ≤ n iff mUn = n. Define, for a,b,c, ε K, $\bar{R}$ abc iff (∀m,n ε G)(m→n ε a & m ε b ⊃ n ε c), $\bar{0}$a iff (∀m ε G)(1 ≤ m ⊃ m ε a); and, as auxiliaries, a + b = {∃m ε G: (∃n ε G) (n→m ε a & n ε b) and a ⊆ b iff (∀m ε G)(m ε a ⊃ m ε b). Then $\bar{R}$ abc iff a+b ⊆ c. Define R and 0 as the restrictions of $\bar{R}$ and $\bar{0}$ to K; and for a ε K, set a* = {m ε G: -m ∉ a}. For strengthened m.s. $\underset{\sim}{M}_1$ define 0_1a iff 1 ε a, i.e. iff a is a 1-filter.

(I) $\underset{\sim}{M}$ = <T, K, R, 0, *> is an m.s.

(i) If m→n ε T and m ε a then n ε a, for a ε $\bar{K}$.
For if m→n ε T then 1 ≤ m→n, so m = 1om ≤ n. Thus m∩n = m, and m∩n ε a; so, as a is a filter, n ε a. Likewise if m ≤ n and m ε a then n ε a.

(ii) b ⊆ c iff (∃x ε K)(0x & Rxbc), i.e. iff b ≤ c.
Suppose first R0bc, suppose m→n ε T and m ε b. Then as b ⊆ c, m ε c; hence, by (i), n ε c. Thus as T ε K, b ≤ c. For the converse suppose for some d ε K, Pd and Rdbc. Suppose further for arbitrary m, m ε b; it has to be shown that m ε c. Since 1 ≤ m→m and Pd, Pm→m ε d. Hence, since Rdbc, m ε c.

(iii) T ε K, since $\underset{\sim}{G}$ is prime. For mUn ε T iff 1 ≤ mUn iff 1 ≤ m v 1 ≤ n iff m ε T v n ε T; hence T is a prime filter.

(iv) [chiefly for later use]. + is an operation on $\bar{K}$. To show m∩k ε a+b iff m ε a+b & k ε a+b, suppose first m∩k ε a+b. Then for some n, n → m∩k ε a and n ε b; so n→m ε a and n→k ε a, by (i). Hence m ε a+b and n ε a+b. Suppose conversely, m ε a+b and k ε a+b. Then since (n→m) ∩ (n'→k) → .n∩n' → m∩k ε T, m∩n ε a+b, by (i) and definition of +.

(v) * is an operation on K. For firstly,

m∩n ε a* iff -(m∩n) ∉ a
iff -mU-n ∉ a, by De Morgan laws.
iff -m ∉ a & -n ∉ a, since a is prime.
iff m ε a* & n ε a*.

Secondly, mUn ε a* iff -(m ∩ n) ∉ a
iff -m U -n ∉ a
iff -m ∉ a v -n ∉ a
iff m ε a* v n ε a*.

(vi) Postulates p1-p7 hold. p1-p4 are immediate by (i) and the definition of R. p5 follows using the definition of *. As to p6 suppose m ε b*; then m ∉ b; but a ⊆ b, so -m ∉ a, whence m ε a*. p7 is immediate by definitions.

(II) $\underset{\sim}{M}_1$ = <T, K, R, 0_1, *> is an s.m.s.

p7-p8 are immediate.

(III) Define h as the function, from $\underset{\sim}{G}$ to $\underset{\sim}{M}^+$, which takes each element m ε G into the set of prime filters on G to which m belongs, i.e. h(m) = {a ε K: m ε a}. h(m) is closed upwards, for each m ε D, i.e. a ε h(m) & a ≤ b ⊃ b ε h (m), since m ε a & a ⊆ b ⊃ m ε b. Hence the set $\bar{\Pi}$ = {h(m): m ε D} is a subset of Π, the set of all ranges in M. Where operations are defined on Π^+ as for Π, for all m,n ε D,

(a) h(m∩n) = h(m) ∩ h(a)
(b) h(mUn) = h(m) U h(n)
(c) h(-n) = - h(n)
(d) h(m○n) = h(m) ○ h(n)
(e) h(n→m) = h(n) → h(m)

Proofs of (a)-(c) are almost standard.

ad (a). a ε h(m∩n) iff m∩n ε a iff m ε a & n ε a
iff a ε h(m) & a ε h(n) iff a ε h(m) ∩ h(n).

ad (b): similar.

ad (c). a ε h(-n) iff -n ε a iff n ∉ a*
iff a* ∉ h(m) iff a ε -h(m).

ad (d). a ε h(m○n) iff m ○ n ε a
iff there are prime filters b, c on $\underset{\sim}{G}$ such that m ε b, n ε c and Rbca - (?)
iff a ε h(m) ○ h(n), by definition of ○ in $\underset{\sim}{M}^+$.

It remains to establish step (?). One half is direct. Suppose, for some b,c, m ε b, n ε c and Rbca. Then as m ε b, m → m○n ε b. Since n ε c and Rbca, by definition of R, m○n ε a. For the converse, suppose m○n ε a, and define b' = {k ε G: m ≤ k} and c' = {k ε G: n ≤ k}. Then b', c' are filters on $\underset{\sim}{G}$, m ε b' and n ε c', and Rb'c'a. As to the last, suppose for arbitrary m',n', m'→n' ε b' and m' ε c'. Then as m ≤ m'→n' and n ≤ m', m○n ≤ (m'→n')○m' ≤ n'. Hence, by upward closure, as m○n ε a, n' ε a. To trade in b' and c' for prime filters we apply Zorn's lemma. Maximalizing on b' we get a filter b such that m ε b, n ε c' and Rbc'a; then maximalizing on c' we get a filter c' such that m ε b, n ε c and $\bar{R}$ bca. Since b,c turn out prime, $\bar{R}$ can be replaced by R. (For full details, see Chapter 4 of RLR1, as these applications of Zorn's lemma can be eliminated in favour of Lindenbaum methods.)

ad (e). a ε h(n→m) iff n→m ε a
iff for every prime filter b,c, iff Rabc and n ε b then m ε c -(?)

iff for every b,c, iff Rabc and b ε h(n) then c ε h(m)
iff a ε h(n)→h(m), by definition of '→' in $\underset{\sim}{M}^+$

One half of (?) is immediate by definition of R. For the converse suppose n→m ∉ a and define b' = {k ε G: n ≤ k} and c' = a+b'. Then b',c' are filters, n ε b', Rab'c' and m ε c'. As to the last, suppose m ε c'. Then for some n', n'→m ε a and n' ε b, i.e. n ≤ n'. Thus n'→m ⊃ n→m, whence n→m ε a, contradicting assumptions. The remaining details, showing that b' and c' can be replaced by prime filters is taken care of by Zorn's lemma, much as in case (d).

It follows by (a)-(e) that $\overline{\Pi}$ is closed under the operations of $\underset{\sim}{M}^+$; hence the algebra Π($\underset{\sim}{M}$) on $\underset{\sim}{M}$ provides a subalgebra of $\underset{\sim}{M}^+$. Further h provides the required isomorphism from $\underset{\sim}{G}$ to $\overline{\Pi}$($\underset{\sim}{M}$). For by (a)-(e), h is a homomorphism. To show h is 1-1, suppose m ≠ n: the case where m ≰ n is typical. By a theorem of Stone there is a prime filter a such that m ε a and n ∉ a (see Birkhoff [1948]); so h(m) ≠ h(n).

Theorem 9.4.3. Where $\underset{\sim}{G}$ = <G, o, →, ∩, ∪, -, 1> is a prime L groupoid, there is a [strengthened] Lm.s. $\underset{\sim}{M}$ = <T, K, R, O[O_1], *> such that G is isomorphic to a subalgebra of $\underset{\sim}{M}^+$, for each L for which both groupoid and modelling condition have been provided.

Proof: We need to add to the proof of the previous theorem only the details showing that when postulate ri holds in $\underset{\sim}{G}$ postulate qi holds in $\underset{\sim}{M}$. Several of the required connexions follow at one given the following interconnexion: Rabc iff (m,n ε G)(m ε a & n ε b ⊃ mon ε c). Suppose $\overline{R}$ abc, m ε a and n ε b. Since m ≤ n → mon, n → mon ε a, so mon ε c. Conversely, suppose m→n ε a and m ε b. Then (m→n)om ε c; but (m→n)om ≤ n, so n ε c as required. One example illustrates the procedure.

<u>ad</u> q1 given r1. Since a ≤ aoa, $\overline{R}$aaa, and by restriction Raaa.

In the light of Chapter 4 however, new work can be reduced if we first replace ri by its equivalent bi, and then show qi by algebraization of the argument in Chapter 4. For example to establish q3, we apply b3 to show $\overline{R}^2$abcd ⊃ $\overline{R}^2$b(ac)d, and then we restrict the antecedent and apply Zorn's lemma to restrict the consequent to K.

Theorem 9.4.4. Where L is any one of the extensions of B^{ot} or its part introduced, A is a theorem of L iff A is valid in every prime L groupoid.

Proof: In view of Theorem 9.3.1, it suffices to prove (IV):

If A is not valid on L-groupoid $\underset{\sim}{G}$ then there is a prime L-groupoid $\underset{\sim}{G}$'.

Let I be an interpretation on $\underset{\sim}{G}$ which falsifies A. Generalizing Meyer, Dunn and Leblanc [1974] (Theorem 2, Stage 1), we define an interpretation I on a prime L-groupoid $\underset{\sim}{G}$' which falsifies A. Let T be the P-filter of G. Since 1 ≤ I (A), I(A) ∉ T. Then by the Stone prime filter theorem there is a prime 1-filter T' containing T such that I(A) ∉ T'. We shall define G' as the quotient L-groupoid G/T', given the following lemma:

Where F is a 1-filter in G, a' = {b: a↔b ε F} for every a ε G, G' = {a': a ε G}, and operations are defined representation-wise on G' (i.e. for b',c' ε G', b' o' c' = (boc)', b' →' c' = (b→c)', b' ∩' c = (b∩c)', b' U' c' = (bUc)', -' b' = (-b)') then (i) the operations are well-defined, (ii) G/F = <G', o', →', ∩', U', -', 1'> is an L-groupoid, and (iii) is a T-homomorphism from G onto G/F, i.e. ' preserves groupoid operations, and 1' is left identity of G/F.

Proof of (ii) is like that of its special case (2) Theorem 9.2.1. As there it is shown,

(a) . a' ≤ b' in G iff a→b ε F.

For a' ≤ b' iff a'Ub' = b', i.e. (aUb)' = b', i.e. c ↔ aUb ε F iff c↔b ε F. Suppose d' ≤ b'. Then as 1 ε F, b→b ε F, so b ↔ aUb ε F, so a→b ε F. Conversely suppose a→b ε F. Then aUb ↔ b ε F. For as b ≤ aUb, 1ob ≤ aUb, so b → aUb ε F; and as b→b ε F, aUb → b ε F. Hence (aUb)' = b'; so a' ≤ b'.

The remainder of the proof of (ii) is as in (2) Theorem 9.2.2 since the algebraic transform of each theorem of L is in F. Proof of (iii) comes at once from the representational definition of operations.

To complete (IV), define I'(C) = I(C))'. Since 1 ε T', and I(A) ε T', 1→I(A) ε T', so by (a) 1' ≰ I'(A), that is I' falsifies A in G'.

Theorem 9.4.5. A is a theorem of L iff A is valid in M^+ for every Lm.s. M, for each extension L of B^{ot} (of B^+) treated.

Proof: Since, by Theorem 9.4.1, M^+ is an L-groupoid, soundness follows by Theorem 9.3.1. Conversely, suppose A is not a theorem of L. Then, by Theorem 9.3.1, there is an interpretation on an L groupoid, and by Theorem 9.4.4 an interpretation I in an prime L-groupoid G such that 1 ≰ I(A). By Theorem 9.4.3 then there is a strengthened Lm.s. M_1 and an isomorphism h such that G is isomorphic under R to a subalgebra $\overline{\Pi}(M_1)$ of M_1^+. Define an interpretation I' of L in M_1^+ as follows: I'(t) = h(1), for every sentential parameter p, I'(p) = hI(p), and otherwise I' is specified recursively by clauses (i)-(vi).

Then (*) I'(B) = hI(B), for every wff B of L.

Proof is by induction from the given basis.

I'(t) = h(1) = h(I(t)) = hI(t)

I'(AoB) = I'(A) o I'(B) = hI(A) o hI(B) = h(I(A) o I(B)) = hI(AoB), etc.

Since 1 ≰ I(A), h(1) ⊄ hI(A), i.e. by (*),

1_1 = {a ε K: P_1a} = {a ε K: 1 ε a} = h(1) $\not\subseteq$ I'(A). Hence I' falsifies A in M_1^+ for strengthened Lm.s. M_1 and so falsifies A in M_1^+ for Lm.s. M.

§9.5. Semantical Analyses Connected with Algebraic Analyses

A *valuation* v of SL_t in Lm.s. M = <T, K, R, 0, *> is a function which assigns one of (holding-) value set Π = {1,0,F} to each sentential parameter p in SL_t at each element a ε K, subject to the constraint:

a ≤ b & v(p,a) = 1 ⊃ v(p,b) = 1, for every a,b ε K.

I is the interpretation associated with v in M provided that I is a function in Π SL_t x K satisfying the following conditions, for every p in SL_t, all wff A and B, and every a ε K (when listed connectives and constants are in SL_t):

(i) I(p,a) = v(p,a)
(ii) I(A&B,a) = 1 iff I(A,a) = 1 = I(B,a)
(iii) I(AvB,a) = 0 iff I(A,a) = 0 = I(B,a)
(iv) I(A→B,a) = 1 iff, for every b,cεK, if Rabc and I(A,b) = 1 then I(B,c) = 1
(v) I(AoB,a) = 1 iff, for some b,c ε K, Rbca and I(A,b) = 1 = I(B,c)
(vi) I(~A,a) = 1 iff I(A,a*) = 0
(vii) I(t,a) = 1 iff, for some b ε K, 0b and b ≤ a.

(For s. Lm.s. I(t,a) = T iff 0a).

A wff A is *holds on* a valuation v (or on associated I) at an element a ε K iff I(A,a) = 1. Wff A is *true on* v (or on associated I) iff I(A,T) = 1; otherwise A is false on v (or I). A is *valid in* Lm.s. M iff A is verified on every valuation in M; otherwise A is *invalid in* L m.s. M. Finally A is *L-valid* iff A is valid in every Lm.s.; otherwise A is *L-invalid.*

The algebraic and semantic accounts of validity are connected as follows:

Theorem 9.5.1. Where M is an Lm.s., A is L-valid in M iff A is valid in L-groupoid M^+.

Proof breaks down into two lemmata, showing that for every valuation on M there is a corresponding assignment on M^+, and conversely:

Given a valuation v in M define an assignment I_v to parameters by setting I_v(p) = {a ε K: v(p,a) = 1}. Since v satisfies the requirement v(p,a) = 1 & a ≤ b ⊃ v(p,b) = 1, I_v(p) is closed upwards; thus I_v(p) is a range (or a matrix value) in M. I_v is extended inductively to all wff of L by using the conditions an interpretation has to meet. Let v itself be the extension of v to all wff (similarly v_I), i.e. the interpretation associated with v.

Conversely given an assignment I of elements A_i of $\mathfrak{M}^+$ to parameters p_i define a valuation v_I by setting $v_I(p_i, a) = 1$ iff $a \in A_i$, i.e. $a \in I(p_i)$. Since $A \in \mathfrak{M}^+$ is closed upwards, v_I is indeed a valuation.

Lemma 9.5.1.
(a) For all $a \in K$, $v(A,a) = 1$ iff $a \in I_v(A)$;
(b) For all $a \in K$, $a \in I(A)$ iff $v_I(A,a) = 1$.
Proof in each case is by induction on the length of A:
ad (a). The basis is immediate by definition of I_v.
ad &. $v(B\&C, a) = 1$ iff $v(B,a) = 1 = v(C,a)$
iff $a \in I_v(B) \cap I_v(C)$
iff $a \in I_v(B\&C)$.
ad v. similar.
ad ~. $v(\sim B,a) = 1$ iff $v(B,a^*) \neq 1$
iff $a^* \notin I_v(B)$.
iff $a \in -I_v(B)$
iff $a \in I_v(\sim B)$
ad →. $v(B\rightarrow C, a) = 1$ iff $(b,c)(Rabc \;\&\; I_v(B,b) = 1 \supset v(C,c) = 1)$
iff $(b,c)(Rabc \;\&\; b \in I_v(B) \supset c \in I_v(C))$
iff $a \in I_v(B) \rightarrow I_v(C)$
iff $a \in I_v(B\rightarrow C)$.
ad o. similar.
ad t. $v(t,a) = 1$ iff $(\exists b)(Ob \;\&\; b \leq a)$
iff $a \in 1$
iff $a \in I_v(t)$.
ad (b). The basis defines v_I; and the inductive steps simply rewrite those for case (a) with 'v_I' replacing 'v' and 'I' replacing 'I_v'.

To complete the theorem, suppose first A is true on v, i.e. $v(A,T) = 1$. Then by the lemma (a) $T \in I_v(A)$. Hence by upward closure,

$(\forall a)(T \leq a \supset a \in I_v(A))$, i.e. $1 \subseteq I_v(A)$.

Thus A is true on I_v. Conversely, suppose A is true on I_v; then $T \in I_v(A)$, whence by lemma (b) $v_I(A,T) = 1$.

Theorem 9.5.2. $\vdash_L A$ iff A is L-valid, for each logic L treated.
Proof by Theorems 9.4.5 and 9.5.1.
An alternative, less algebraic, proof of this result is of course provided in Chapters 4 and 5 of RLR1. And by exploiting that alternative completeness proof and reversing the argument there, a short proof can be provided of the following earlier result (cf. Meyer and Routley [1972]).

Theorem 9.5.3. If A is valid in all L groupoids then A is a theorem of L, for each extension L of B^{ot} treated.

Proof: Suppose A is not a theorem of L. Then there is, from Chapter 5, an Lm.s. $\underset{\sim}{M}$ = <T, K, R, O, *> and a valuation v in $\underset{\sim}{M}$ such that I(A,T) = 0, where I is the interpretation associated with v. Define I_v as before, i.e.

$I_v(p) = \{a \in K: v(p,a) = T\}$.

Then I_v provides an interpretation in L groupoid $\underset{\sim}{M}^+$, as in the previous theorem; and I_v falsifies A. For $T \notin I_v(A)$ by lemma (a); hence $1 \not\subseteq I_v(A)$.

§9.6. Embeddings and Conservative Extensions

Where $\underset{\sim}{G}$ is a groupoid on set G, a *G-ideal* is any subset S of G which is closed downwards w.r.t. ≤, i.e. whenever b ε S and a ε G, if a ≤ b then a ε S. Thus G-ideals are the duals of G-filters which stand to groupoids as ranges stand to m.s. Where c ε G, its *principal* G-ideal is the set $h(c) = \{a \in G: a \leq c\}$. The relation h so defined is a 1-1 function. We use G-ideals rather than G-filters in order to preserve order relations under mapping h. (The embedding theorems which follow are patterned on Lemma 2 of Lemmon [1966a].)

Lemma 9.6.1. Every implicational Ackermann groupoid may be embedded

(i) in a positive Ackermann groupoid

(ii) in a De Morgan groupoid.

Specifically, where $\underset{\sim}{G}$= <G, ≤, o, →, 1> is an implicational Ackermann groupoid, define the following structure $\underset{\sim}{D}$ = <D, o', →', ∩, ∪, 1'> and $\underset{\sim}{D}'$ = <D, o', →', ∩, ∪, -, 1'>: D is the set of all G-ideals; ∩, ∪, -, ⊆, have their set-theoretic sense; and operations o' and →' on D are defined as follows, for every S,T ε D:

$SO'T = \{a \in G: (\exists s \in S, t \in T) a \leq s o t\}$

$S \rightarrow' T = \cup\{R \in D: R o' S \subseteq T\}$.

Finally, where h: G→D is the function which takes each element c ε G into principal G-ideal h(c), 1' = h(1). Then:

(a) $\underset{\sim}{D}$ is a positive Ackermann groupoid and $\underset{\sim}{D}'$ is a de Morgan groupoid.

(b) h is an embedding of $\underset{\sim}{G}$ into $\underset{\sim}{D}$ [into $\underset{\sim}{D}'$] inasmuch as h is an injection which preserves the operations and relations of $\underset{\sim}{G}$ i.e. for every a,b ε G, h(aob) = h(a) o h(b), h(a→b) = h(a) → h(b), and a ≤ b iff h(a) ⊆ h(b).

Proof: (a) The operations on D are well-defined for reasons dual to those already given in the case of Theorem 9.4.1.

ad p1. Immediate from features of set inclusion and the definition of o.

ad p2. If c ε S, c = 1oc ε h(1)oS, so S ⊆ h(1)oS. Conversely, if c ε h(1)oS then there are e ≤ 1 and s ε S such that c ≤ eos. But eos ≤ 1os = s, so since c ≤ s, c ε S. Thus h(1)oS ⊆ S.

ad p3. Suppose $R\circ'S \subseteq T$; then R is among the sets whose union is $S\rightarrow'T$; hence $R \subseteq S\rightarrow'T$. Conversely, suppose $R \subseteq S\rightarrow'T$ and $c \in R\circ'S$. Then for $r \in R$, $s \in S$, $c \leq r\circ s$. Since $r \in S\rightarrow'T$, for some $R' \in D$, $r \in R'$ and $(R\circ'S) \subseteq T$. Since $r\circ s \in R'\circ S$, $r\circ s \in T$, whence since T is a G-ideal, $c \in T$, as required.
ad p4.1. A corollary of the fact that D is a ring of sets.
ad p5.1. From the properties of set-theoretical complement.
ad p4.2. $c \in R\circ(S\cup T)$ iff, for some $r \in R$ and $u \in S\cup T$, $c \leq r\circ u$, i.e. iff $c \in R\circ S$ or $c \in R\circ T$, i.e. iff $c \in (R\circ S) \cup (R\circ T)$. The other case is similar.

(b) if $h(a) \subseteq h(b)$ then if $a \leq a$, $a \leq b$, so $a \leq b$.
Conversely, if $a \leq b$ then, for $c \in G$, if $c \leq a$ then $c \leq b$ by transitivity, so $h(a) \subseteq h(b)$.

If $c \in h(a\circ b)$ then $c \leq a\circ b$, so $c \in h(a) \circ h(b)$; hence $h(a\circ b) \subseteq h(a) \circ h(b)$. Conversely, suppose $c \in h(a) \circ h(b)$; then for some $e \leq a$ and $f \leq b$, $c \leq e\circ f$, by definitions of $\circ$ and h. Since $e \leq a$, $e\circ b \leq a\circ b$; since $f \leq b$, $e\circ f \leq e\circ b$; hence $c \leq e\circ f \leq a\circ b$. But $a\circ b \in h(a\circ b)$, so $c \in h(a\circ b)$. Thus $h(a) \circ h(b) \subseteq h(a\circ b)$.

Suppose $c \in h(a) \rightarrow h(b)$; then there is an $R \in D$ such that $c \in R$ and $R\circ h(a) \subseteq h(b)$. Since $a \in h(a)$, $c\circ a \in h(b)$, whence $c\circ a \leq b$. Thus $c \leq a\rightarrow b$, so $c \in h(a\rightarrow b)$. Conversely, since $(a\rightarrow b) \circ a \leq b$, $h(a\rightarrow b) \circ h(a) \subseteq h(b)$, as h preserves $\circ$ and $\subseteq$. Hence by residuation $h(a\rightarrow b) \subseteq h(a) \rightarrow h(b)$.

Theorem 9.6.1.
(1) L_+ is a conservative extension of L_I;
(2) L is a conservative extension of L_I.
Proof: Since L includes L_+ which includes L_I, extension results are immediate. For the conservative part, suppose A is not a theorem of L_I. Then there is an interpretation I of L_I. in an L_I-groupoid $\underset{\sim}{G}$ such that $1 \leq I(A)$. Now embed $\underset{\sim}{G}$ in L-groupoid $\underset{\sim}{D}'$ [or L_+-groupoid $\underset{\sim}{D}$] under mapping h, as in the preceding Lemma. Specify an interpretation I' of L in $\underset{\sim}{D}'$ as follows: $I'(t) = h(1)$, for every sentential parameter p, $I'(p) = hI(p)$, and otherwise I' is defined component-wise by clauses (i)-(vi). Then (*) $I'(B) = hI(B)$, for every wff B of L_I.
Proof is by induction from the given basis, e.g.

$I'(t) = h(1) = hI(t)$

$I'(A\rightarrow B) = I'(A) \rightarrow I'(B) = hI(A) \rightarrow hI(B) = hI(A\rightarrow B)$.

Since $1 \nleq 1(A)$, $h(1) \not\subseteq hI(A)$, i.e. by (*) $1' \not\subseteq I'(A)$. Hence I' falsifies A in $\underset{\sim}{D}'$; so A is not a theorem of L.

§9.7. Some Finite Algebras of Importance in the Relevant Enterprise[4]

We now illustrate developments in preceding sections - especially those connecting algebraic structures with model structures - by considering certain finite structures, thereby providing analyses of cases that have figured prominently in the investigation of relevant logics. These cases arise because there are matrices which verify the theorems of various relevant logics, but which may separate the logics, falsify significant non-theorems (such as paradoxical or irrelevant classical theses), and so on. However the availability of such matrices, which do so much useful work, is (as Lewis realized when Parry acquainted him with a batch of such matrices for modal logics) a puzzling fact, which requires explanation. Now it has been known, at least since Lemmon's work in [1966a] and [1966b], that various finite matrices can be 'explained' by showing that they are isomorphic to various finite set-theoretic structures, which in turn are instances of the kind of algebra which the logic in question determines. Lemmon also showed how to connect characteristic classes of algebras with characteristic classes of model structures, thereby providing a satisfying completeness to the explanation of the various matrices by showing them also to be derived from various finite model structures. Lemmon's work was in modal logic, but the generalization to the kinds of structures we consider here is not particularly difficult. We intend the generalization we make to explain certain key matrices for relevant logics.

The results of previous sections provide the needed connections between algebras and model structures. The kind of problem we consider now is: given a matrix, to find a relevant set-theoretic algebra and a relevant model structure such that a formula A is valid in the matrix iff A is valid in the algebra, and iff A is valid in the model structure. Solutions for certain cases of this problem are given here. For convenience we restrict ourselves to primitives $\{\rightarrow, \&, \vee, \sim\}$, though the extension to more general cases is not different in principle.

First case: RM3. The RM3 matrices, which star in the proof that relevant implication is not definable in terms of modal logic (ENT1, p.470ff) for instance, are as follows:

&	0	1	2	~	v	0	1	2
* 0	0	1	2	2	0	0	0	0
* 1	1	1	2	1	1	0	1	1
2	2	2	2	0	2	0	1	2

→	0	1	2
* 0	0	2	2
* 1	0	1	2
2	0	0	0

Figure 9.1 RM3 matrices

The matrices are characteristic for the system RM3 obtained from R or RM by the addition of the Minglish schemes:

~A&B → .A→B and ~A → .Av(A→B), or alternatively the schemes:

A → .~A→B and Av(A→B).[5]

For semantics of RM3 conditions corresponding to schemes as follows are added to (reduced) R semantics:

~A&B → .A→B Rabc ⊃ .a≤c v b≤a*

(or A →. ~A → A)

Av(A→B) RTab ⊃ a≤T (for unreduced modellings replace T by x, with x in O).

Completeness requires in the latter case non-degenerate models.

Consider the model structure RM3 = <T, K, R, *>, where K = {T, T*}, T** = T and R = {<TTT>, <TT*T*>, <TT*T>, <T*TT>, <T*T*T*>, <T*T*T>, <T*TT*>}. An interpretation I on RM3 assigns 1 or 0 to all sentences at both T and T*. Thus (before consideration of the effect of the hereditary condition) an arbitrary sentence can receive the value 1 in all and only the members of the following subsets of K: {T,T*}, {T}, {T*}, ∅. Thus, as customary, these sets can be thought of as being assigned to sentences by a function of f: Wff → P(K) i.e. from wff to the power set of K, given by x ε f(A) iff I(A,x) = 1. Not every such function is a possible function, however, as in the semantics of RM3, I must satisfy the hereditary condition:

(H) I(A,x) = 1 & x≤y ⊃ .I(A,y) = 1.

For instance, an assignment of 1 to A at T* but not T is forbidden by the given condition T*≤T (i.e. <TT*T> ε R). Thus (H) together with T*≤T requires that f take formulae to members of {{T,T*}, {T}, ∅}, the set of *ranges* on RM3 (or propositions, or strikes, in the terminology used above). The algebra of ranges RM3 determined by RM3 will be a quintuple <S, →, &, v, ->, where S = {{T,T*}, {T}, ∅}, and the operations →, &, v, - are determined, in accordance with the *key connection*, x ε f(A) iff I(A,x) = 1, by the various values given to formulae A→B, A&B, AvB, and -A by I in various worlds, given the assignments of I to A and to B in those worlds. We display the computation for the cases of $\rightarrow$ and $\sim$:

Implication:

(i) If $f[A] = \{T,T^*\}$ and $f[B] = \{T,T^*\}$, when $f[A\rightarrow B] = \{T,T^*\}$.

(a) $T \in f[A\rightarrow B]$. We need to show that $I[A\rightarrow B,T] = 1$, i.e. $(\forall x,y)(RTxy \;\&\; I[A,x] = 1 \supset .I[B,y] = 1)$ i.e. $(RTTT \;\&\; I[A,T] = 1 \supset .I[B,T] = 1) \;\&\; (RTT^*T^* \;\&\; I[A,T^*] = 1 \supset .I[B,T^*] = 1) \;\&\; (RTT^*T \;\&\; I[A,T^*] = 1 \supset .I[B,T] = 1)$. But each conjunct is satisfied because $f[B] = \{T,T^*\}$ guarantees $I[B,T] = 1$ and $I[B,T^*] = 1$.

(b) $T^* \in f[A\rightarrow B]$. We need to show that $I[A\rightarrow B,T^*] = 1$, i.e, $(\forall x,y)(RT^*xy \;\&\; I[A,x] = 1 \supset .I[B,y] = 1)$, i.e. $(RT^*TT \;\&\; I[A,T] = 1 \supset .I[B,T] = 1) \;\&\; (RT^*T^*T^* \;\&\; I[A,T^*] = 1 \supset .I[B,T^*] = 1) \;\&\; (RT^*T^*T \;\&\; I[A,.T^*] = I \supset .I[B,T] = 1) \;\&\; (RT^*TT^* \;\&\; I[A,T] = 1 \supset .I[B,T^*] = 1)$. But as before each conjunct is guaranteed by $f[B] = \{T,T^*\}$. Hence $f[A\rightarrow B] = \{T,T^*\}$.

(ii) If $f[A]$ = anything and $f[B] = \{T,T^*\}$ then by a similar argument $f[A\rightarrow B] = \{T,T^*\}$.

(iii) If $f[A] = \varnothing$ and $f[B]$ = anything, then $f[A\rightarrow B] = \{T,T^*\}$.

(a) $T \in f[A\rightarrow B]$ since each conjunction is of the form: $RTxy \;\&\; I[A,x] = 1 \supset .I[B,y] = 1$, but $f[A] = \varnothing$ guarantees that $I[A,x] \neq 1$, for any x.

(b) The argument is similar for $T^* \in f[A\rightarrow B]$. Hence $f[A\rightarrow B] = \{T,T^*\}$.

(iv) If $f[A] = \{T,T^*\}$ and $f[B] = \{T\}$, then $f[A\rightarrow B] = \varnothing$.

(a) $T^* \notin f[A\rightarrow B]$, since RTT^*T^* and $I[A,T^*] = 1$ but $I[B,T^*] \neq 1$.

(b) $T \notin f[A\rightarrow B]$, since $RT^*T^*T^*$ and $I[A,T^*] = 1$ but $I[B,T^*] \neq 1$.

(v) If $f[A] = \{T,T^*\}$ and $f[B] = \varnothing$, then $f[A\rightarrow B] = \varnothing$.

(a) and (b) are both as for (iv).

(vi) If $f[A] = \{T\}$ and $f[B] = \{T\}$, then $f[A\rightarrow B] = \{T\}$.

(a) $T \in f[A\rightarrow B]$. We need to show that $(\forall x,y)(RTxy \;\&\; I[A,x] = 1 \supset .I[B,y] = 1)$, i.e. $(RTTT \;\&\; I[A,T] = 1 \supset .I[B,T] = 1) \;\&\; (RTT^*T^* \;\&\; I[A,T^*] = 1 \supset .I[B,T^*] = 1) \;\&\; (RTT^*T \;\&\; I[A,T^*] = 1 \supset .I[B,T] = 1)$. The first conjunct is guaranteed by $f[B] = \{T\}$, the second and third by $T^* \notin f[A]$.

(b) $T^* \notin I\, f[A\rightarrow B]$, since RT^*TT^* and $I[A,T] = 1$ and $I[B,T^*] \neq 1$.

(vii) If $f[A] = \{T\}$ and $f[B] = \varnothing$, then $f[A\rightarrow B] = \varnothing$.

(a) $T \notin f[A\rightarrow B]$ since $RTTT$ and $T \in f[A]$ and $T \notin f[B]$.

(b) $T^* \notin f[A\rightarrow B]$ since RT^*TT and $T \in f[A]$ and $T \notin f[B]$.

This completes the cases for $\rightarrow$.

Negation:

(i) If $f[A] = \{T,T^*\}$, then $f[\sim A] = \varnothing$.

(a) $T \notin f[\sim A]$, since if $I[A,T^*] = 1$ then $I[\sim A,T] \neq 1$.

(b) $T^* \notin f[\sim A]$, since if $I[A,T,] = 1$, then $I[\sim A,T^*] \neq 1$.

(ii) If $f[A] = \{T\}$, then $f[\sim A] = \{T\}$.

(a) $T \in f[\sim A]$, since $I[A,T^*] = 1$.

(b) $T^* \notin f[\sim A]$ since $I[A,T] = 1$.

(iii) If $f[A] = \varnothing$, then $f[\sim A] = \{T,T^*\}$. The proof is straightforward.

Conjunction and disjunction:
It follows from previous results that these are set-theoretic intersection and union respectively. However, to illustrate the case of conjunction: I{A,x} = 1 and I{B,x} = 1 iff I{A&B,x} = 1, so x ε f{A} and x ε f{B} iff x ε f{A&B}, so f{A&B} = f{A} ∩ f{B}.

This gives us the tables displayed below. In order to finalise the tables as matrices, we need to consider the question of designated values. A wff is valid in <u>RM3</u> iff for all interpretations I on <u>RM3</u>, I{A,T} = 1. We want to designate values in RM̰3 in such a way that A always takes a designated value in RM̰3 iff A is valid in <u>RM3</u>. Now the condition (I) (I{A,T} = 1) is equivalent to (∀f){T ε f{A}}. Hence we must designate all elements x of RM̰3 with T ε x. Note that if we designated, say, {T} but not {T,T*}, then this would amount to declaring refuted any formula which is ever assigned to {T,T*} i.e. to declaring refuted any formula A for which there is an interpretation I such that I{A,T} = 1 and I{A,T*} ≠ 1, which would be wrong unless we had some guarantee that no such case could arise. More generally, we need to designate every element of the principal filter determined by {T}, and we will use this condition for later matrices too. If A is always given a designated value in RM̰3 by any such function f, we say that A is valid in RM̰3.

We now display the constructed set algebra RM̰3.

&	{T,T*}	{T}	∅	~
* {T,T*}	{T,T*}	{T}	∅	∅
* {T}	{T}	{T}	∅	{T}
∅	∅	∅	∅	{T,T*}

v	{T,T*}	{T}	∅
	{T,T*}	{T,T*}	{T,T*}
	{T,T*}	{T}	{T}
	{T,T*}	{T}	∅

→	{T,T*}	{T}	∅
* {T,T*}	{T,T*}	{T}	∅
* {T}	{T,T*}	{T}	∅
∅	{T,T*}	{T,T*}	{T,T*}

Figure 9.2 Set algebra for RM3

The condition x ε f{A} iff I{A,x} = 1, together with the designation of the principal filter determined by {T}, now ensures that a formula A is valid in RM̰3 iff A is valid in <u>RM3</u>. But an inspection of RM̰3 shows that it is isomorphic to the RM3 matrices previously tabulated (with 0 corresponding to {T,T*}, 1 to {T}, and 2 to ∅). Hence we arrive at the main result for RM3:

Theorem 9.7.1. The following conditions are equivalent:
(1) A is valid in <u>RM3</u>; (2) A is valid in RM̰3; (3) A is valid in RM3;
(4) A is true in all RM3 model structures; (5) A is a theorem of RM3.

A similar analysis can be made of other finite Sugihara matrices, for instance RM4, and also of finite Lukasiewicz logics.[6]

Second case: the crystal lattice CL Having dealt with a fairly simple case and explained the elements, we can now proceed with more speed. A more complex example is the crystal lattice CL, much used in §3.6 of RLR1. The matrix CL - or CLM as we write when we want to emphasize the *matrix* representation - is interesting for several reasons, not the least of which is that it often provides a more rapid proof of the relevance of systems than those using M_0, as well as for a different range of systems. CL has two fixed points, a, b with the further property that a→b is undesignated: hence for any wff A→B where A and B do not share a variable, assign a to all the variables of A and b to all the variables of B.

The matrices for CL are, to repeat §3.6, these:

&	0	1	2	3	4	5	~	v	0	1	2	3	4	5
* 0	0	1	2	3	4	5	5	0	0	0	0	0	0	0
* 1	1	1	2	3	4	5	4	1	0	1	1	1	1	1
* 2	2	2	2	4	4	5	2	2	0	1	2	1	2	2
* 3	3	3	4	3	4	5	3	3	0	1	1	3	3	3
* 4	4	4	4	4	4	5	1	4	0	1	2	3	4	4
5	5	5	5	5	5	5	0	5	0	1	2	3	4	5

→	0	1	2	3	4	5
* 0	0	5	5	5	5	5
* 1	0	4	5	5	5	5
* 2	0	2	2	5	5	5
* 3	0	3	5	3	5	5
* 4	0	1	2	3	4	5
5	0	0	0	0	0	0

Figure 9.3 CL matrices

Consider the following model structure <u>CL</u> = <T, K, R, *> where K = {T,T*,a,a*}, T = T**, a** = a, and the members of R are the following 45 triples (we omit the ordered triple brackets):
TT*T, TT*a, TT*a*, TaT, Ta*T, TT*T*, Ta*a*, Taa, TTT, T*T*T, T*T*a, T*T*a*, T*aT, T*a*T, T*T*T*, T*a*a*, T*aa, T*TT, aT*T, aT*a, aT*a*, aaT, aa*T, aT*T*, aa*a*, aaa, aTT, a*T*T, a*T*a, a*T*a*, a*aT, a*a*T, a*T*T*, a*a*a*, a*aa, a*TT, T*TT, a*Ta*, aTa, aa*T*, a*aT*, T*Ta, T*a*T*, T*Ta*, T*aT*.

The conditions $T^* \leq a$, $a^* \leq T$, together with (H) [as for RM3], force, under the map $x \in f[A]$ iff $I[A,x] = 1$, a collapse of the 16 element set lattice to a 6 element lattice CL. $T^* \leq a$, together with $I[A,T^*] = 1$ and $T^* \leq a$ implies $I[A,a] = 1$, require that $T^* \in f[A]$ implies $a \in f[A]$, for any set $f[A]$ of the lattice. Similarly, $T^* \in f[A]$ implies both $a^* \in f[A]$ and $T \in f[A]$, and $a \in f[A]$ implies $T \in f[A]$ and $a^* \in f[A]$ implies $T \in f[A]$. Thus CL is as below, with the top 5 elements designated (corresponding matrix values are indicated to the side):

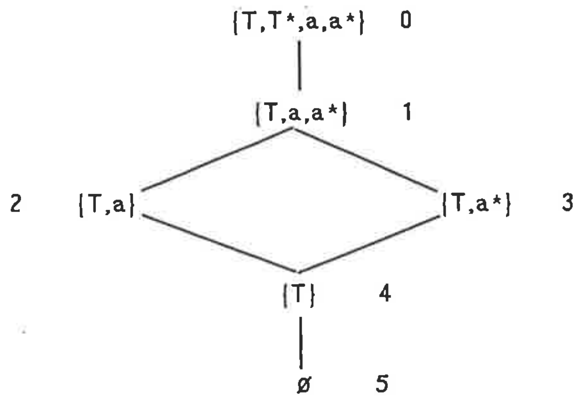

Figure 9.4 CL lattice

Conjunctions and disjunctions, as usual, go over to the intersections and unions. The table for negation is:

	*	*	*	*	*	
	{T,T*,a,a*}	{T,a,a*}	{T,a}	{T,a*}	{T}	ø
~	ø	{T}	{T,a}	{T,a*}	{T,a,a*}	{T,T*,a,a*}

Figure 9.5 CL negation table

We do the computation for a couple of typical cases.

(i) If $f[A] = \{T,a,a^*\}$, then $f[\sim A] = \{T\}$.

(a) $T \in f[\sim A]$ since $I[A,T^*] \neq 1$.

(b) $a \notin f[\sim A]$ since $I[A,a^*] = 1$.

(c) $a^* \notin f[\sim A]$ since $I[A,a] = 1$.

(d) $T^* \notin f[\sim A]$ since $I[A,T] = 1$.

(ii) If $f[A] = \{T,a\}$, then $f[\sim A] = \{T,a\}$.

(a) $T \in f[\sim A]$ since $I[A,T^*] \neq 1$.

(b) $a \in f[\sim A]$ since $I[A,a^*] \neq 1$.

(c) a* ∉ f[~A] since I[A,a] = 1.
(d) T* ∉ f[~A] since I[A,T] = 1.
Accordingly the structure thus far is a De Morgan lattice; and with implication it falls out to a De Morgan monoid.

Finally, we present the computed table for implication. In justification of the table, we give a list of those triples of the relation R which ensure the *absence* of those worlds which are absent from the entry in question. The rest of the computation is mechanical.
[We put {T,T*,a,a*} as K, in order to fit the table onto the page. R.B.]

→	K	{T,a,a*}	{T,a}	{T,a*}	{T}	∅
* K	K	∅	∅	∅	∅	∅
* {T,a,a*}	K	{T}	∅	∅	∅	∅
* {T,a}	K	{T,a}	{T,a}	∅	∅	∅
* {T,a*}	K	{T,a*}	∅	{T,a*}	∅	∅
* {T}	K	{T,a,a*}	{T,a}	{T,a*}	{T}	∅
∅	K	K	K	K	K	K

Figure 9.6 CL implication table

(i) ∅ → anything or anything → {T,T*,a,a*} : proof is immediate.
(ii) {T,T*,a,a*} → {T,a,a*}, {T} or ∅: {TT*T*, aT*T*, a*T*T*, T*T*T*}.
(iii) {T,a,a*} → {T,a}, {T} or ∅: {Ta*a*, aa*a*, a*a*a*, T*a*a*}.
(iv) {T,a,a*} → {T,a*}: {Taa, aaa, a*aa, T*aa}.
(v) {T,a} → {T,a*}: as for (iv).
(vi) {T,a*} → {T,a}, {T} or ∅: as for (iii).
(vii) {T,a} → {T} or ∅: as for (iv).
(viii) {T} → ∅: {TTT, aTT, a*TT, T*TT}.
(ix) {T} → {T}: {a*Ta*, aTa, T*TT*}.
(x) {T,a} → {T,a}: {a*Ta*, T*TT*}.
(xi) {T,a} → {T,a*}: {aTa, T*TT*}.
(xii) {T} → {T,a*}: {T*TT*, aTa}.
(xiii) {T} → {T,a}: {T*TT*, a*Ta*}.
(xiv) {T} → {T,a,a*}: {T*TT*}.
(xv) {T,a*} → {T,a,a*}: {T*TT*, aa*T*}.
(xvi) {T,a} → {T,a,a*}: {T*TT*, a*aT*}.
(xvii) {T,a,a*} → {T,a,a*}: {T*TT*, aa*T*, a*aT*}.

By inspection, the C̰L algebra is isomorphic to the matrices CL when {T,T*,a,a*}, ..., ∅ are identified with 0, ... , 5, respectively. The method of construction of C̰L from CL then ensures:

The conditions T* ≤ a, a* ≤ T, together with (H) [as for RM3], force, under the map x ε f(A) iff I(A,x) = 1, a collapse of the 16 element set lattice to a 6 element lattice CL. T* ≤ a, together with I(A,T*) = 1 and T* ≤ a implies I (A,a) = 1, require that T* ε f(A) implies a ε f(A), for any set f(A) of the lattice. Similarly, T* ε f(A) implies both a* ε f(A) and T ε f(A), and a ε f(A) implies T ε f(A) and a* ε f(A) implies T ε f(A). Thus CL is as below, with the top 5 elements designated (corresponding matrix values are indicated to the side):

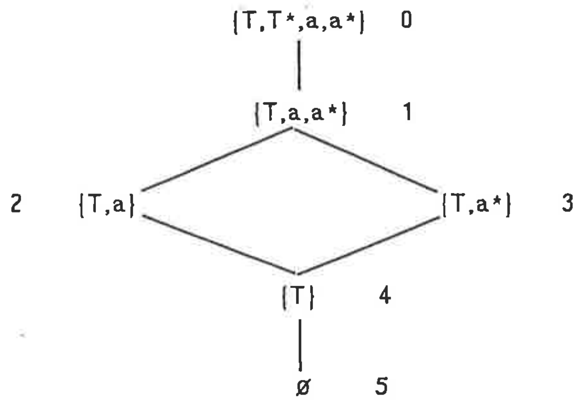

Figure 9.4 CL lattice

Conjunctions and disjunctions, as usual, go over to the intersections and unions. The table for negation is:

	*	*	*	*	*	
	{T,T*,a,a*}	{T,a,a*}	{T,a}	{T,a*}	{T}	Ø
~	Ø	{T}	{T,a}	{T,a*}	{T,a,a*}	{T,T*,a,a*}

Figure 9.5 CL negation table

We do the computation for a couple of typical cases.

(i) If f(A) = {T,a,a*}, then f(~A) = {T}.
(a) T ε f(~A) since I(A,T*) ≠ 1.
(b) a ∉ f(~A) since I(A,a*) = 1.
(c) a* ∉ f(~A) since I(A,a) = 1.
(d) T* ∉ f(~A) since I(A,T) = 1.

(ii) If f(A) = {T,a}, then f(~A) = {T,a}.
(a) T ε f(~A) since I(A,T*) ≠ 1.
(b) a ε f(~A) since I(A,a*) ≠ 1.

Theorem 9.7.2. The following statements are equivalent:
(i) A wff A is valid in CL; (ii) A valid in CL; (iii) A is valid in CL.

A subsequent objective (in §9.8) is to supply a relevant system for which the CL matrix is characteristic. Since the proof that the system concerned does indeed axiomatize CLM proceeds through a 4 world model, it helps to simplify the 4-world semantics on CL. It is also worth detouring at the same time to give an alternative proof that CL and CL have the same class of valid wff, and to present a 4-valued semantics for the class.[7]

For the simplification we vary the relation R of CL inessentially as, for example, direct reworking of the argument above (for Theorem 9.7.5) will show. For this reason we often do not distinguish notationally (the former m.s.) CL and the new m.s. CL' where ≤ and R are defined as follows: b ≤ c $=_{df}$ b = C v b = T* v c = T; b < c $=_{df}$ b ≤ c & b ≠ c; Rbcd $=_{df}$ (b = T ⊃ c ≤ d) & (b ≠ T ⊃. b ≤ c* v b ≤ d). Whereas in CL R holds for 45 triples, in CL' it holds for 49. The following properties hold generally in CL': T* < a, a* < a*, T* < T, a < T, a*< T:[8] RTbb, i.e. b ≤ a; b ≤ c & Rcde ⊃ Rbde; c ≤ d ⊃ d* ≤ c*.

Valuations and interpretations on CL', or CL, truth and validity, etc., are defined in the usual way; and the following connections are proved in the familiar way (cf. Chapter 4 of RLR1):

(α) For every wff A, for all b,c ε K, when b ≤ c and I(A,b) = 1, I(A,c) = 1.
(β) For every wff A and B, I(A→B,T) = 1 iff (∀b ε K)(I(A,b) = 1 ⊃ I(B,b) = 1).

Using (α) and (β), the interpretation of A→B can be simplified as follows (for CL'):

I(A→B,T) = 1 iff (I(A,T) = 1 ⊃ I(B,T) = 1) & (I(A,T*) = 1 ⊃ I(B,T*) = 1) & (I(A,a) = I ⊃ I(B,a) = 1) & (I(A,a*) = 1 ⊃ I(B,a*) = 1).

I(A→B,T*) = 1 iff (∀b,c ε K)(I(A,b) = 1 ⊃ I(B,c) = 1), since RT*bc, for all b,c ε K; i.e. iff I(A,T) = 1 ⊃ I(B,T*) = 1, by (α) and properties of ≤ and < above.

I(A→B,a) = 1 iff (∀b,c ε K)((a ≤ b* v a ≤ c) & I(A,b) = 1 ⊃. I(B,c) = 1), since Rabc iff a ≤ b* or a ≤ c, for all b,c ε K, i.e. iff (∀b ε K)(b ≤ a* & I(A,b) = 1 ⊃ (∀c ε K)I(B, c) = 1) & (∀c ε K)(a ≤ c & (∃b ε K)(I(A,b) = 1) ⊃ I(B,c) = 1, by quantificational logic; i.e. iff (I(A,a*) = 1 ⊃ I(B,T*) = 1) & (I(A,T) = 1 ⊃ I(B,a) = 1), by (α) and ordering properties above.

I(A→B,a*) = 1 iff (I(A,a) = 1 ⊃ I(B,T*) = 1) & (I(A,T) = 1 ⊃ I(B,a*) = 1), as in the preceding case, after replacing a by a*.

To establish the adequacy of this modelling, for the CL matrices, we circuit through the following 4-valued semantics, since this represents an intermediate stage between the matrices and the model structure.[9] A (4) valuation V (for CL) is a function assigning to each semantical parameter p a subset V(p) of the set {T,T*,a,a*} of values, subject to the following conditions (corresponding to the hereditariness condition):

(i) a ε V[p] ⊃ T ε V[p];
(ii) a* ε V[p] ⊃ T ε V[p];
(iii) T* ε V[p] ⊃ .a ε V[p] & a* ε V[p] (and hence T ε V[p]).

An *interpretation* I, extending a valuation V to all wff, is defined inductively as follows: I[p] = V[p], for every p; d ε I[~A] iff d* ε I[A], for every value d in IIII = {T,T*,a,a*}; d ε I[A&B] iff d ε I[A] & d ε I[B], for every d in IIII; T ε I[A→B] iff (∀d ε IIII)(d ε I[A] ⊃ d ε I[B]); T* ε I[A→B] iff T ε I[A] ⊃ T* ε I[B]; a ε I[A→B] iff (a* ε I[A] ⊃ T* ε I[B]) & (T ε I[A] ⊃ a ε I[B]); a* ε I[A→B] iff (a ε I[A] ⊃ T* ε I[B]) & (T ε I[A] ⊃ a* ε I[B]).
A wff A is *valid* in this 4-semantics for CL iff T ε I[A] for every valuation V.

Theorem 9.7.3. The following statements are equivalent: (i) wff A is valid in CL; (ii) A is valid in the 4-semantics for CL; (iii) A is valid in CL'.
Proof: To show that (i) and (ii) are equivalent, set up the correspondences displayed in the lattice diagram for CL, observing that designated values correspond under the assignments, and that matrix operations are isomorphic to semantical operations.

It is worth outlining, in this one typical case, the more detailed argument ultimately establishing this equivalence. Let μ be the matrix valuation corresponding to 4-valuation V such that, for every sentential parameter p, μ[p] = 0 iff V[p] = {T,T*,a,a*}; μ[p] = 1 iff V[p] = {T,a,a*}; μ[p] = 2 iff V[p] = {T,a}; μ[p] = 3 iff V[p] = {T,a*}; μ[p] = 4 iff V[p] = {T}; μ[p] = 5 iff V[p] = ∅.
Note that the two sets of valuations are appropriately exhaustive under the correspondence. Let J be the extension of μ to all wff, determined inductively, (by precisely the assumed matrix procedure, as follows: J[p] = μ[p]; J[~A] = ~[J[A]]; J[A&B] = &[J[A],J[B]]; J[A→B] = →[J[A],J[B]]. A wff A is *valid in the matrix system CL* iff J[A] ≠ 5, for all matrix valuations μ. All this simply sets down in more exact fashion what has hitherto (in this section) been taken largely for granted.

The correspondence between initial valuations on sentential parameters extends (of course) inductively to the correspondence, between matrix and 4-semantics valuations as regards all wff, summed up thus:

(†1). T ε I[A] iff J[A] ≠ 5; T* ε I[A] iff J[A] = 0; a ε I[A] iff J[A] = 0, 1 or 2; and a* ε I[A] iff J[A] = 0, 1 or 3, for every wff A.
While this is obvious enough by inspection of the CL diagram given, it can also be proved by induction, the induction basis being already supplied. There remain inductive steps for each connective, all of which are straightforward, so only a few representative examples are displayed.
ad ~. T ε I[~B] iff T* ∉ I[B], i.e. iff J[B] ≠ 0, i.e. J[~B] ≠ 5. T* ε I[~B] iff T ∉ I[B], i.e. iff J[B] = 5, i.e. J[~B] = 0. Etc.
ad →. T* ε I[B→C] iff T ε I[B] ⊃ T* ε I[C], i.e. iff J[B] = 5 or J[C] = 0, i.e. iff J[B→C] = 0. Etc.

By (†1), T ε I(A) iff J(A) ≠ 5, always. Hence A is valid in the 4-semantics iff A is valid in CLM.

To show in similar detail that (ii) and (iii) are equivalent, the following correspondence is appealed to:

Let V be the 4-valuation corresponding to the valuation v of the CL m.s., such that, for every sentential parameter p, T ε V(p) iff v(p,T)= 1; T* ε V(p) if v (p,T*) = 1; a ε V(p) iff v(p,a) = 1; a* ε V(p) iff v(p,a*) = 1.

Observe that the conditions (i), (ii) and (iii) on the valuation V hold iff v(p,a) = 1 ⊃ v(p,T) = 1 and v(p,a*) = 1 ⊃ v(p,T) = 1 and v(p,T*) = 1 ⊃ v(p,a) = 1 & v(p,a*) = 1, this conjunction being equivalent to the hereditariness condition on v, viz. b ≤ c and v(p,b) = 1 ⊃ v(p,c) = 1. Once again the exhaustive correspondence between (initial) valuations extends inductively to all wff, thus:

(†2). d ε I(A) iff I(A,d) = 1, for d = T,T*,a,a*, and for every wff A.

The induction basis is immediate, and the induction steps are straightforward, representative cases going, in brief, as follows:

ad ~. d ε I(~B) iff d* ∉ I(B), i.e. iff I(B,d*) ≠ 1, i.e. iff I(~B,d) = 1.

ad &. d ε I(B&C) iff d ε I(B) and d ε I(C), i.e. I(B,d) = 1 = I(C,d), i.e. iff I (B&C,d) = 1.

ad →. T* ε I(B→C) iff T ε I(B) ⊃ T* ε I(C), i.e. iff I(B,T) = 1 ⊃ I(C,T*) = 1, i.e. if I(B→C,T*) = 1.

a ε I(B→C) iff (a* ε I(B) ⊃ T* ε I(C)) & (T ε I(B) ⊃ a ε I(C)), i.e. iff (I(B,a*) = 1 ⊃ I(C,T*) = 1) & ((I(B,T) = 1 ⊃ I(C,a) = 1), i.e. iff I(B→C,a) = 1.

Since by (†2), T ε I(A) iff I(A,T) = 1, A is valid in the 4-semantics iff A is valid in CL', and so in CL.

Third case: the Belnap matrix system M_0 The Belnap matrices, already derived in §3.1 of RLR1 and heavily exploited in deriving features of FD, fit neatly into the algebraic framework established. Though based on a more compact model structure than the CL matrices, M_0 comprises nonetheless the following more ample matrices:

&	+3	+2	+1	+0	-0	-1	-2	-3	~	v	+3	+2	+1	+0	-0	-1	-2	-3
* +3	+3	+2	+1	+0	-0	-1	-2	-3	-3		+3	+3	+3	+3	+3	+3	+3	+3
* +2	+2	+2	+0	+0	-2	-3	-2	-3	-2		+3	+2	+3	+2	+3	+3	+2	+2
* +1	+1	+0	+1	+0	-1	-1	-3	-3	-1		+3	+3	+1	+1	+3	+1	+3	+1
* +0	+0	+0	+0	+0	-3	-3	-3	-3	-0		+3	+2	+1	+0	+3	+1	+2	+0
-0	-0	-2	-1	-3	-0	-1	-2	-3	+0		+3	+3	+3	+3	-0	-0	-0	-0
-1	-1	-3	-1	-3	-1	-1	-3	-3	+1		+3	+3	+1	+1	-0	-1	-0	-1
-2	-2	-2	-3	-3	-2	-3	-2	-3	+2		+3	+2	+3	+2	-0	-0	-2	-2
-3	-3	-3	-3	-3	-3	-3	-3	-3	+3		+3	+2	+1	+0	-0	-1	-2	-3

→	+3	+2	+1	+0	-0	-1	-2	-3
* +3	+3	-3	-3	-3	-3	-3	-3	-3
* +2	+3	+2	-3	-3	-2	-3	-2	-3
* +1	+3	-3	+1	-3	-1	-1	-3	-3
* +0	+3	+2	+1	+0	-0	-1	-2	-3
-0	+3	-3	-3	-3	+0	-3	-3	-3
-1	+3	-3	+1	-3	+1	+1	-3	-3
-2	+3	+2	-3	-3	+2	-3	+2	-3
-3	+3	+3	+3	+3	+3	+3	+3	+3

Figure 9.7 M_0 matrices

Consider the model structure $\underline{M}_0$ = <T,K,R,*> where K = {T,a,a*}, T* = T, a** = a, and R = {<TTT>,<Taa>,<Ta*a*>,<aaa>,<aa*a*>,<aTa>, <aa*a>,<aa*T>,<a*aa>,<a*a*a*>,<a*Ta*>,<a*aa*>,<a*aT>}. Hence Rbcd iff (b=T ⊃ c=d) and (b ∉ T ⊃. b=c* v b=d), and so RTcd if c=d. Interpretations, truth and validity of $\underline{M}_0$ are defined as usual. The interpretation rule for → simplifies as follows:

I(A→B,T) = 1 iff (∀b,c ε K)(b=c & I(A,b) = 1 ⊃ I(B,c) = 1), i.e. iff (∀b ε K)(I(A,b) = 1 ⊃ I(B,b) = 1).

I(A→B,a) = 1 iff (∀b,c ε K)((a=b* v a=c) & I(A,b) = 1 ⊃ I(B,c) = 1), i.e. iff (I(A, a*) = 1 ⊃ (c ε K)I(B,c) = 1) &. (∃b ε K)I(A,b) = 1 ⊃ I(B,a) = 1).

I(A→B,a*) = 1 iff (I(A,a) = 1 ⊃ (∀c ε K)I(B,c) = 1) & ((∃b ε K)I(A,b) = 1 ⊃ I(B,a*) = 1).

As in the case of CL a bridge semantics, with values just elements of set K, can be inserted connecting world semantics with matrices, but as before it can also be bypassed. On the bridge semantics a 3-valuation V assigns to each valuational parameter p a subset V(p) of the set {T,a,a*} of values. Interpretation I extends V inclusively to all wff, as follows:

T ε I(~B) iff T ε I(B), a ε I(~B) iff a* ∉ I(B), a* ε I(~B) iff a ∉ I(B);
d ε I(B&C) iff d ε I(B) and d ε I(C), for each value d;

T ε I(B→C) iff (∀d ε K)(d ε I(B) ⊃ d ε I(C)), a ε I(B→C) iff (a* ε I(B) ⊃

I{C} = K} and {I{B} = ∅ ⊃ a ε I{C}}, a* ε I{B→C} iff {a ε I{B} ⊃ I{C} = K} and {I{B} = ∅ ⊃ a* ε I{C}.

A wff A is valid in the 3-semantics, or 3-valid in M_0 iff T ε I{A}, for all interpretations.

Since x ≤ y only if x = y, the hereditary condition (H) imposes no restrictions on admissible interpretations. Thus we proceed to construct the 8 element algebra $\underset{\sim}{M}_0$ with no collapse of elements required by (H). The 8 elements of $\underset{\sim}{M}_0$ are {T,a,a*}, {T,a*}, {T,a}, {T}, {a,a*}, {a}, {a*}, ∅, and we will eventually again identify these with the elements of matrix M_0 in descending order. Designated values, as before, are the principal filter determined by {T}. As before, conjunctions and disjunctions go over to the intersections and unions of the Hasse diagram of M_0 (on which corresponding matrix values are displayed to the side):

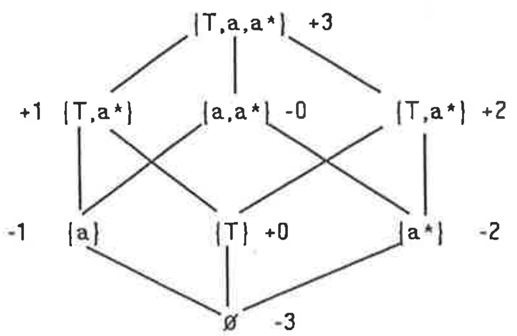

Figure 9.8 M_0 lattice

With the principal filter determined by {T} designated, it is now a matter of straightforward computation to show that the tables for ∩ and ∪ in this lattice mirror those for & and v in M_0.

For negation, the $\underset{\sim}{M}_0$ table below is isomorphic to the M_0 negation matrix:

	*	*	*	*				
	{T,a,a*}	{T,a*}	{T,a}	{T}	{a,a*}	{a}	{a*}	∅
~	∅	{a*}	{a}	{a,a*}	{T}	{T,a}	{T,a*}	{T,a,a*}

Figure 9.9 M_0 negation table

We display the computation in a couple of interesting cases:

(i) If f{A} = {T,a*} then f{~A} = {a*}.

(a) a* ε f(~A) since I(A,a) ≠ 1.
(b) T ∉ f(~A) since I(A,T*) = I(A,T) = 1.
(c) a ∉ f(~A) since I(A,a*) = 1.
(ii) If f(A) = {a} then f(~A) = {T,a}.
(a) T ε f(~A) since T* = T ∉ f(A).
(b) a ε f(~A) since a* ∉ f(A).
(c) a* ∉ f(~A) since a ε f(A).

The implication table has 64 entries. We give the table here and display the computation for several entries:
[We replace {T,a,a*} by K in order to fit the table in. R.B.]

→	K	{T,a*}	{T,a}	{T}	{a,a*}	{a}	{a*}	∅
* K	K	∅	∅	∅	∅	∅	∅	∅
* {T,a*}	K	{T,a*}	∅	∅	{a*}	∅	{a*}	∅
* {T,a}	K	∅	{T,a}	∅	{a}	{a}	∅	∅
* {T}	K	{T,a*}	{T,a}	{T}	{a,a*}	{a}	{a*}	∅
{a,a*}	K	∅	∅	∅	{T}	∅	∅	∅
{a}	K	∅	{T,a}	∅	{T,a}	{T,a}	∅	∅
{a*}	K	{T,a*}	∅	∅	{T,a*}	∅	{T,a*}	∅
∅	K	K	K	K	K	K	K	K

Figure 9.10 M_0 implication table

(i) If f(A) = anything and f(B) = {T,a,a*}, then f(A→B) = {T,a,a*}. For example, T ε f(A→B). It needs to be shown that (∀x,y)(RTxy and I(A,x) = 1 ⊃ I(B,y) = 1). But the consequent is guaranteed, for any choice of y, by the fact that f(B) = {T,a,a*}.

(ii) If f(A) = ∅ and f(B) = anything, then f(A→B) = {T,a,a*}. The proof is similar to (i), except that f(A) = ∅ guarantees the falsehood of the antecedent.

(iii) If f(A) = f(B) = {a,a*}, then f(A→B) = {T}.
(a) a ε f(A→B) since Raa*T and a* ε f(A) and T ∉ f(b).
(b) a* ∉ f(A→B) since Ra*aT and a ε f(A) and T ∉ f(B).
(c) T ε f(A→B) since RTxy only if x = y and x ε f(A) implies x ε f(B).

(iv) If f(A) = f(B) = {T} then f(A→B) = {T}.
(a) a ∉ f(A→B) since RaTa.
(b) a* ∉ f(A→B) since Ra*Ta*.
(c) T ε f(A→B), as for (iii)(c).

The method of construction of the algebra $\underset{\sim}{M}_0$ from the model structure $\underline{M}_0$ together with the designated elements of M_0 being the principal filter determined by {T}, ensures that a wff is valid in $\underset{\sim}{M}_0$ iff it is valid in $\underline{M}_0$. But inspection verifies that $\underset{\sim}{M}_0$ is isomorphic to M_0. Hence the next result:

Theorem 9.7.4. The following statements are equivalent: (i) A wff A is valid in M_0; (ii) A is valid in $\underline{M}_0$; (iii) A is valid in $\underset{\sim}{M}_0$; (iv) A is 3-valid in M_0.

Fourth case: Smiley matrices, redesignated Just as the analysis of the Belnap M_0 matrices largely repeats that already given in §3.1 of RLR1, but in more algebraic terms, bringing in full model structures and algebras, so the analysis of the Smiley-Wajsberg matrices given in §2.6 (p.115) may also be substantially repeated. In this case we set down only basic features, many of them simply carried over from the previous first degree entailment analysis. There is one significant difference: whereas in §2.6 only one value in the Smiley matrices was designated, in this investigation two values are designated. This shift in designated values in fact makes no difference to the class of first degree entailments validated by the matrices. In fact even larger variations on the matrices make no difference, as will appear. The redesignated Smiley matrices, S4, are, to repeat §2.6, these:

&	1	2	3	4	~
*1	1	2	3	4	4
*2	2	2	4	4	2
3	3	4	3	4	3
4	4	4	4	4	1

v	1	2	3	4
*1	1	1	1	1
*2	1	3	1	2
3	1	1	3	3
4	1	2	3	4

→	1	2	3	4
*1	1	4	4	4
*2	1	1	4	4
3	1	4	1	4
4	1	1	1	1

Figure 9.11 Smiley matrices, re-designated

The corresponding model structure is $\underline{S4}$ = <T,K,R,*> where K = {T,T*}, T** = T and R = {<TTT>, <TT*T*>, <T*TT>, <T*T*T*>}. Hereditariness imposes no restrictions since x ≤ y if x = y. The 4 elements of $\underset{\sim}{S4}$ are {T,T*}, {T}, {T*}, Ø. Designated values are again the principal filter determined by {T}; this is what accounts for the designation of two elements. The Hasse diagram showing lattice-theoretic properties is as follows (alternative labellings are shown to the side):

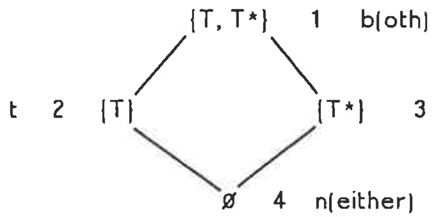

Figure 9.12 Smiley lattice

The table for ∩ and ∪ mirror those for & and v in $\underset{\sim}{S4}$. For negations, the S4 matrix is isomorphic to the following $\underline{S4}$ table:

	* {T,T*}	* {T}	{T*}	Ø
~	Ø	{T}	{T*}	{T,T*}

Figure 9.13 Smiley negation table

Similarly for implication, the S4 matrix is isomorphic to the following S4 implication table:

→	{T,T*}	{T}	{T*}	Ø
* {T,T*}	{T,T*}	Ø	Ø	Ø
* {T}	{T,T*}	{T,T*}	Ø	Ø
{T*}	{T,T*}	Ø	{T,T*}	Ø
Ø	{T,T*}	{T,T*}	{T,T*}	{T,T*}

Figure 9.14 Smiley implication table

In the computations for this table the cases (i) where f(A) = anything and f(B) = {T,T*} and (ii) f(A) = Ø and f(B) = anything, are like those for M_0, and are similarly independent of conditions on R. Remaining cases utilize exactly the conditions imposed on R.

Theorem 9.7.5. The following are equivalent: (i) A wff A is valid in S4; (ii) A is valid in $\underline{S4}$; (iii) A is valid in $\underset{\sim}{S4}$.

Fifth case: the Brady variation on S4 Brady changes the implication matrix to the following matrix M4 in order, so he says, that the matrices are 'made more suitable [than S4] for a system with formulae of any degree':
Otherwise the matrices for M4 are the same as those for S4.

→	1	2	3	4
* 1	1	4	3	4
* 2	1	2	3	4
3	1	3	1	3
4	1	1	1	1

Figure 9.15 Brady variation

Similarly M4 is the same as S4 except for R which is as follows:
{<TTT>, <TT*T*>, <T*TT*>}
i.e. Rabc iff (a = T ⊃ b = c) & (a = T* ⊃ .b = T v c = T*).
Again x ≤ y iff x = y. Designated values of M4 and the Hasse diagram are as for $\underset{\sim}{S}4$. So is the table for negation, while that for implication of $\underset{\sim}{M}4$ differs from that for $\underset{\sim}{S}4$ exactly as the correspondences between matrix values and algebraic elements requires. Hence the expected result:

Theorem 9.7.6. The following statements as regards wff A are equivalent: (i) A is valid in M4; (ii) A is valid in M4; (iii) A is valid in $\underset{\sim}{M}4$.

Sixth case: a chain of connexive matrices We turn to our final set of examples, which concern a chain of matrices introduced earlier in the book (§3.6 of RLR1). For all n ≤ 1, let M_n be the following matrix: elements {0,1,2, ...,2n}, designated elements {0,1, ...,2n-1}, avb = min{a,b}. a&b = max{a,b}, ~a = 2n-a, a→b = 2n-1 iff a ≥ b, otherwise a→b = 2n. Such chains are of interest because they verify 'connexive' E, i.e. E together with such connexivist theses as Aristotle, ~(A→~A), and Boethius, (A→B) → ~(A→~B).

Let M_n = <T,K,N,R,*> where K = {T,T*,a_1,a_1*, ...,a_{n-1},a_{n-1}*}, x** = x for all x ε K, N = {T} and R is the relation with (i) T* ≤ a_1* ≤ ... ≤ a_{n-1}* ≤ a_{n-1} ≤ ... ≤ a_1 ≤ T (ii) RTxx, for all x ε K.

The hereditary condition (H) together with (i) forces the power set algebra into a chain form, since we must have T* ε every set to which a_1* belongs, ... a_1 ε every set to which T belongs. Thus $\underset{\sim}{M}_n$ looks like this:

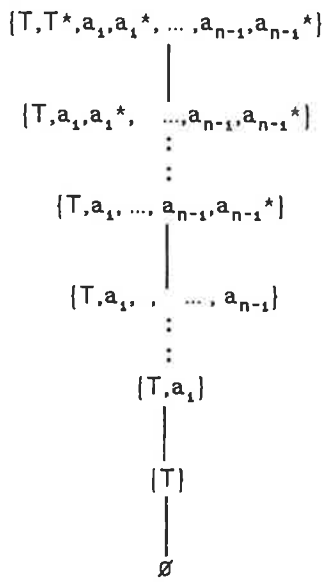

Figure 9.16 Connexive chain

As always, conjunctions and disjunctions go over to intersections and unions, and since we have a chain (and propose to identify the elements of $\underset{\sim}{M}_n$ in descending order in the diagram with 0, 1, ..., 2n respectively) this will ensure that $a \cap b$ = (numerical) max $\{a,b\}$, $a \cup b$ = (numerical) min$\{a,b\}$ as desired.

For negation, note that $\{T,a_1,a_2, ...,a_{n-1}\}$ is a fixed point: if f $(A) = \{T,a_1, ...,a_{n-1}\}$, then (a) $T \in f(\sim A)$ since $T^* \notin f(A)$; (b) $T^* \notin (\sim A)$ since $T \in f(A)$; (c) $a_i \in f(\sim A)$ since $a_i^* \notin f(A)$; (d) $a_i^* \notin f(\sim A)$ since $a_i \in f(A)$. Hence f $(\sim A) = f(A)$. For all other entries in the negation table, note that both $a_i, a_i^* \in f(A)$ if and only if neither a_i nor $a_i^* \in f(\sim A)$, and if $a_i \in f(A)$ but $a_i^* \notin f(A)$, then $a_i \in f(\sim A)$ and $a_i^* \notin f(\sim A)$. This gives the table:

	~
$\{T, T^*, a_1, a_1^*, \ldots, a_{n-1}, a_{n-1}^*\}$	$\emptyset$
$\{T, a_1, a_1^*, \ldots, a_{n-1}, a_{n-1}^*\}$	$\{T\}$
$\vdots$	$\vdots$
$\{T, a_1, \ldots, a_n\}$	$\{T, a_1, \ldots, a_n\}$
$\vdots$	$\vdots$
$\{T\}$	$\{T, a_1, a_1^*, \ldots, a_{n-1}, a_{n-1}^*\}$
$\emptyset$	$\{T, T^*, a_1, a_1^*, \ldots, a_{n-1}, a_{n-1}^*\}$

Figure 9.17 Connexive negation table

For the implication operation, note that we are aiming at having implicated formulae true only at either $\{T\}$ or $\emptyset$. If we impose the condition $\sim Na_1$ (no implicated formula is true at a_1 in $\underline{M}_n$), which *seems* to be necessary, then in virtue of every world save T being $\leq a_1$, the hereditary condition (H) forces $\sim Nx$ for all $x \neq T$. This certainly ensures that no implicated formula can be true at any world except T, and so that the only values that $A \rightarrow B$ can take in M_n will be $\{T\}$ or $\emptyset$.

Therefore, in calculating the values of $A \rightarrow B$ in M_n, we need take into account only whether or not $T \in (A \rightarrow B)$. Now if $f(A) \subseteq f(B)$, then by inspection, we certainly have that $(\forall x, y)(RTxy \ \& \ I(A,x) \supset I(B,y))$, so that $I(A \rightarrow B, T) = 1$, i.e. $T \in f(A \rightarrow B)$, so that $f(A \rightarrow B) = \{T\}$. On the other hand, if $f(A) \subseteq f(B)$, then for some $x \in K$, $x \in f(A)$ and $x \notin f(B)$. But now since $RTxx$ and $I(A,x) = 1$ and $I(B,x) \neq 1$, $T \notin f(A \rightarrow B)$. Hence $f(A \rightarrow B) = \emptyset$. Thus the implication table is:

$\rightarrow$	$\{T, T^*, a_1, a_1^*, \ldots, a_{n-1}, a_{n-1}^*\}$	...	$\emptyset$
$\{T, T^*, a_1, a_1^*, \ldots, a_{n-1}, a_{n-1}^*\}$	$\{T\}$	$\emptyset$ above diagonal	
$\vdots$			
$\emptyset$	$\{T\}$ on and below diagonal		$\{T\}$

Figure 9.18 Connexive implication table

But, under the map $\{T, T^*, a_1, a_1^*, \ldots, a_{n-1}, a_{n-1}^*\} \rightarrow 0, \ldots, \emptyset \rightarrow 2n$, this and the other operations on $\underset{\sim}{M}_n$ are isomorphic to the matrix M_n. Hence the final result of this type.

Theorem 9.7.7. The following statements are equivalent: (i) A wff A is valid in M_n; (ii) A is valid in $\underline{M}_n$; (iii) A is valid in $\underset{\sim}{M}_n$.

Three final comments. First, this last result was achieved using somewhat unsubtle modellings: we required ~Nx for every x ≠ T. However it is shown in Mortensen [1984] that in the context of E, Aristotle, ~(A→~A), may be modelled by the semantical modelling ~NT*, the non-normality of further worlds not being required. Accordingly it would be interesting to find out if the above chains can be obtained by simply imposing the condition ~NT* (and, of course, complicating the other conditions on R), so as to 'separate off' the semantical effect of E from that of Aristotle.

Secondly, there are two special cases of the M_n chain which are of interest which the theory explains. M_1 is the three-valued post-complete matrix used in ENT1, p.98, in Mortensen [1984], and elsewhere in this work. M_2 is the PIANO matrix used in ENT1, p.243, and heavily relied on in §3.6.

Thirdly, other sequences of matrices introduced earlier in the text are amenable to similiar algebraic analysis and explanation,[10] as the results of §3.1 and §6.9 of RLR1 concerning sequences of matrices beginning with M_0 and S4 reveal.

§9.8. Finite Relevant Logics and Axiomatizations of Relevance-establishing Matrices

That relevant logics are weakly relevant has been shown primarily through the use of finite matrices. It is evident from the arguments used in these demonstrations that, where the matrices used in establishing relevance axiomatized - were, that is, systems for which the matrices are characteristic devised - the resulting systems would be both weakly relevant and finite-valued. The combination of weak relevance with finite-valuedness depends, however, on the systems concerned being functionally incomplete. Were the systems functionally complete, not only would irrelevant implications (such as those used in the Rosser-Turquette axiomatizations of many-valued logics) be definable, but more to the point, absolutely false and true constants, F and T, would be definable in terms of sentential variables and connectives, for which such irrelevances as F→A and B→T would be provable: in short, (weak) irrelevance would ensue.

Certainly, such finite-valued (functionally incomplete) relevant logics would violate important requirements of adequacy for theories of entailment and conditionality (imposed in Chapter 1 of RLR1), for instance, that the logics concerned should not be finite-valued.[11] Nonetheless, the class of finite-valued relevant logics is of *much* technical interest, for instance because *such logics satisfy both the weak relevance requirement and also the generalized extensionality requirement*, which is commonly insisted upon in Poland, North America and elsewhere, as essential for clearness and distinctness and sometimes indeed for even making sense (the requirement is assessed in DRL and in EMJB). Such finite-valued relevant logics of course provide us with (further) many-valued relevant logics: these are already had, in a sense, with 4-valued semantics for FD (in §3.2), while infinite-valued relevant logics we have had for some time in the shape of the main relevant systems (e.g. of Chapter 4).

Axiomatizations of matrices can be worked out from completeness arguments where these can be carried out in a routine way. It suffices to add to some basic system (such as C on p.289 of RLR1) principles required to clinch arguments at each requisite point in the completeness proof, verifying that the principles adopted are indeed verified by the matrices concerned. And completeness arguments with respect to finite matrices can often be standardized. We illustrate the point in the axiomatization of the crystal lattice CL, with a method which can be applied in a routine, if tedious, way in many other cases - though not invariably without some adjustments. One important variation, resulting from the common situation that $T = T^*$, we elaborate subsequently in axiomatizing matrix M_0.

But first a couple of simpler preliminary cases are sketched.

A preliminary elementary case: the axiomatization of the four-valued matrix system M4 Where the matrix to be axiomatized contains no more than four values, the procedure is much simplified because the values can be correlated with mappings from just two elements T and T* to 1 and 0 (cf. the method of §2.6). In this case only the base world T - readily constructed in the light of previous information - and perhaps its star image T* are required. When the number of matrix values exceeds four however, other elements than T and T* have to be introduced and duly delimited, much complicating the argument.

Before the *standard* more complex situation is gone into, the elementary procedure is illustrated first through the axiomatization of the matrix M4 (of §9.7).

The system BN4, which proves to axiomatize M4, is obtained from DW^d, by the these bizarre axiom schemes:[12]

$\sim A \& B \rightarrow .A \rightarrow B$

$\sim A \rightarrow .A \vee (A \rightarrow B)$

$A \vee \sim B \vee (A \rightarrow B)$

$A \rightarrow .A \rightarrow \sim A \rightarrow \sim A$

$A \vee (\sim A \rightarrow .A \rightarrow B)$

BN4 has, in addition to familiar theorems of DW^d, the following bizarre theorems, needed in completeness arguments: B → .~B v (A→B), A&~B →. A&~B → ~(A→B), A v (~(A→B) → A), ~(A→B) → ~(~B→~A).
[The fourth axiom and last theorem are not bizarre. — R.B.]

BN4 can be fairly straightforwadly provided with a semantical analysis in the style of Chapter 4, by working out semantical postulates for (some of) the bizarre axioms. And once the system is connected with the M4 matrix system, other semantics already indicated (in the style of §9.7) as associated with that matrix system will follow, e.g. that based on the two world structure {T,T*}. In fact completeness will be established using the four-valued American plan translation of the two world semantics.

The American semantics for BN4 are exactly the same worldless (or one world) semantics given in §3.2, and in effect elaborated in §4.7, except for the æ rule which simplifies to the following
1 ε V(A→B) iff both 1 ∉ V(A) or 1 ε V(B) and 0 ∉ V(B) or 0 ε V(A);
0 ε V(A→B) iff 1 ε V(A) and 0 ε V(B).
That is, valuation V is a function from (initial) wff to IIII, i.e., subsets of {1,0} satisfying the & and ~ rules of §3.2 and the given → rule, and a wff A is *American valid* iff 1 ε V(A) for every such valuation V. It is evident that this semantics just mirrors the matrix semantics for M4, and accordingly, by the methods of §9.7, a wff is valid in M4 iff it is American valid.

Soundness with respect to both the matrix and worldless American semantics is routinely shown by direct or couputer verification. The completeness argument can make use of results of the type already obtained in §4.8; namely a more elaborate account of a BN4-theory is required, and associated notions such as BN4-derivability are correspondingly refined. A *BN4-theory* a is closed not only under adjunction and provable BN4-implication but also under disjunctive modus ponens, that is wherever Cv(A→B) ε a and CvA ε a then CvB ε a, and (to include the null case) wherever A→B ε a and A ε a then B ε a.[13] Much as in §4.8, extension and primeness theorems are reworked to accommodate the more elaborate closure conditions (for full details see Brady [1982]). Then completeness is established in the by now standard way. That is, where A is not a theorem of BN4 there is a prime regular BN4 theory T such that A ∉ T. And T induces a canonical valuation V on all wff B (by induction from initial wff) such that 1 ε V(B) iff B ε T and 0 ε V(B) iff ~B ε T. Hence A is not American valid, and so not matrix valid. Therefore M4 is characteristic for BN4.

A further preliminary case: the logic, matrices, and semantics for the truth-preservation connective in combination with extensional connectives.
The 4-valued matrix set TP_4 that emerges from investigations of the truth-preservation connective > (of §5.5) is as follows:

&	t	b	n	f	~
* t	t	b	n	f	f
* b	b	b	f	f	b
n	n	f	n	f	n
f	f	f	f	f	t

>	t	b	n	f
* t	t	b	n	t
*b	t	t	n	n
n	t	b	t	b
f	t	t	t	t

Figure 9.19 Truth-preservation matrices

Valuations, V_M, and interpretations, I_M, associated with these matrices are defined in the usual way. Observe that matrix values b and n are interchangeable. As a result, the set of designated values can be either {t} or {t,b} or {t,n} without altering the class of valid wff of TP_4.[14]

The matrix analysis connects in the expected way with a 4-valued (one-world) American semantics. The evaluation rules for extensional connectives & and ~ are as before, e.g. for BN4, while the rule for > is as follows:

$1 \varepsilon V(A{>}B)$ iff $1 \varepsilon V(A) \supset 1 \varepsilon V(B)$.

$0 \varepsilon V(A{>}B)$ iff $0 \notin V(A)$ & $0 \varepsilon V(B)$.

Then, for every wff A, A is valid in TP_4 iff A is valid in the American semantics. The correspondence in values is (as before) this: t with {1}, b with {1,0}, n with Ø, and f with {0}. The connection is established by showing, by induction on wff that for every wff A, $I_M(A)$ = t or b iff $1 \varepsilon V(A)$ and $I_M(A)$ = f or b iff $0 \varepsilon V(A)$.

The American semantics connects in turn with a two-worlds semantics on world set K = {T,T*}. Models are defined precisely as in §5.5. Then, for every wff A, A is valid in the two worlds semantics iff A is valid in the American semantics. Proof is a matter of showing by induction on wff that, for every wff A, $1 \varepsilon V(A)$ iff I(A,T) = 1 and $0 \varepsilon V(A)$ iff I(A,T*) = 0.

The interconnections can be applied to show that TP4 really captures (i.e. is characteristic for) {~, &, >}-logic. The key result is the:

Conservative extension result A wff A with connectives from set {~, &, >} is valid in the two-world semantics iff A is L-valid for any system L treated in §5.5 where Lm.s. is such that T ≠ T* and T ≠ u.[15]

Proof: Let A be such a wff. Suppose firstly A is valid in the two-world semantics. There are two cases according as, in the L e.m.s. M, T = T* or not.

ad T ≠ T*. Note that T ≠ e and T = u: otherwise there would be no valid - or invalid - wff, as the case may be. Now let v be a valuation of M and I its corresponding interpretation. Let v_T and I_T be the restrictions of v and I, respectively, to the set-ups T and T*. Hence, $v_T(p,T) = v(p,T)$ and $v_T(P,T^*) = v(p,T^*)$, for every sentential variable p. By induction on wff it follows that $I_T(B,T) = I(B,T)$ and $I_T(B,T^*) = I(B,T^*)$, for every such wff B. Since T ≠ T*,

$T \neq e$ and $T^* \neq e$, v_T and I_T can serve as a valuation and its corresponding interpretation for the two worlds semantics. Hence since A is valid there, $I_T(A,T) = 1$, and so $I(A,T) = 1$. Thus A is valid in the L e.m.s. $\underset{\sim}{M}$.
ad $T = T^*$. As before $T \neq e$. Let v be a valuation of $\underset{\sim}{M}$ and I its corresponding interpretation. As before let v_T and I_T be the restrictions of v and I to T. Then $I_T(B,T) = I(B,T)$, for every wff B with only the connectives from {~, &, >}. Let v_C be a classical valuation on {T,T*} such that $v_C(p,T) = v_C(p,T^*) = v_T(P,T)$, for every sentential variable p. By induction on wff $I_C(B,T) = I_C(B,T^*)$, for every such wff B, where I_C is the extension of v_C. Since $T \neq e$, the interpretation conditions for I_C and T (and T*) are the same as those for I_T and T. Hence $I_C(B,T) = I_T(B,T)$, for every such wff B. Since A is valid in the two-world semantics, $I_C(A,T) = 1$, and hence $I_T(A,T) = 1$. Then A is valid in the Le.m.s. $\underset{\sim}{M}$.

Accordingly, A is valid in every Le.m.s. and so L-valid. For the converse, let A be valid in the semantics for L. Then A is valid in an Le.m.s. $\underset{\sim}{M}$ such that $T \neq T^*$ and $T \neq u$. As before, $T \neq e$. Let v_2 be a valuation on {T,T*} and I_2 its corresponding interpretation. Let V_J be an extension of v_2 so that v_K assigns a value, 1 or 0, to all sentential variables at all members of K, so that $v_K(p,e) = 0$ and $v_K(p,u) = 1$, for every sentential variable p. Thus v_K is a valuation of the Le.m.s. $\underset{\sim}{M}$ such that $v_K(p,T) = v_2(p,T)$ and $v_K(p,T^*) = I_2(B,T)$ and $I_K(B,T^*) = I_2(B,T^*)$, for every wff B with connectives from {~, &, >}, where I_K corresponds to v_K. Since A is valid in $\underset{\sim}{M}$, $I_K(A,T) = 1$ and hence $I_2(A,T) = 1$. Hence A is valid in the two-worlds semantics.

It remains to axiomatize *the* {~, &, >}-logic TP_4 captures. The logic TP4 is as follows:

TP1. A > A.
TP2. A&B > A.
TP3. A&B > B.
TP4. (A>B) & (A>C) > .A > B&C.
TP5. (A>C) & (B>C) > .AvB > C.
TP6. A & (BvC) > (A&B) v (A&C).
TP7. ~~A > A.
TP8. A & (A>B) > B.
TP9. B > .A > B.
TP10. C > .A v (A>B).
RTP1. A, A>B ⇒ B.
RTP2. A, B ⇒ A&B.
RTP3. A>B, C>D ⇒ B>C > .A>D.
RTP4. A > ~B ⇒ B > ~A.

Thus TP4 = B + F1 + F2 + F3 (see p.382 of RLR1), with occurrences of '→' in B replaced by '>', and TP4 = $B^>$ - D v (A→B) ⇒ D v ((B>C) → (A>C)), with occurrences of '→' replaced by '>'. Further, since D v (A>B) ⇒ D v ((B>C) > (A>C)) preserves validity in TP_4, one can set TP4 = $B^>$, with '→' replaced by '>'.

Among the theorems of TP4 are these:

A > A v B, B > A v B, A > ~~A, ~A > ~(A&B), ~B > ~(A&B), ~(A&B) > ~Av~B, A v (A>B), ~B > ~A v ~(A>B), ~(A>B) > ~B, ~A&~(A>B) > C.

Derived rules include these:

A>B, B>C ⇒ A>C; A>B, A>C ⇒ A > B&C;

A>C, B>C ⇒ AvB > C; A&B > C, A > CvB ⇒ A>C.

Soundness of TP4 with respect to any of the semantics supplied is established in the familiar way, in particular it is straightforwardly shown that all the axioms of TP4 taken the value t, for all interpretations I_M of TP_4, and that all the rules of TP4 preserve the value t in TP_4.[16] Hence all theorems of TP4 take the value t, for all interpretations of TP_4, and are thereby valid in TP_4.

Given the interconnections it is enough to establish completeness with respect to the American semantics; that TP_4 is characteristic then follows. The argument in the standard mould of completeness proofs, i.e. based on that of §4.6, except that *TP4-derivability* is defined in terms of '>'. The correspondence between postulates of TP4 and of B means that extension and priming lemmas, and so on, go through in the standard way. Accordingly where A is a non-theorem of TP4 there is a prime regular TP4-theory T such that A ∉ T. But for any such theory T we can define a valuation V for which it follows by induction that:

(V) 1 ε V(B) iff B ε T and 0 ε V(B) iff ~B ε T.

Hence 1 ∉ V(A) and A is valid under the American semantics for TP4. The induction basis for (V) is given by the definition V. The induction steps for ~ and & are standard.

ad >. 1 ε V(A>B) iff 1 ε V(A) ⊃ 1 ε V(B), i.e. iff A ∉ T or B ε T applying the induction hypothesis, i.e. iff, provisionally, A>B ε T. The provisional connection is filled out exactly as in the main completeness result in §5.5, using however TP8-TP10.

0 ε V(A>B) if 0 ∉ V(A) & 0 ε V(B), i.e. iff, by induction hypothesis, ~A ∉ T and ~B ε T, i.e. iff, provisionally, ~(A>B) ε T. The provisional step is justified thus: since $\vdash_{TP4}$~B > ~A v (A>B), ~B ε T ⊃ ~A ε T v ~(A>B) ε T, and hence ~A ∉ T & ~B ε T ⊃ ~(A>B) ε T. Since also $\vdash_{TP4}$~A & ~(A>B) > C, ~A ε T & ~(A>B) ε T ⊃ C ε T. Since T ≠ u_C, let C ∉ T. Then ~A ∉ T or ~(A>B) ∉ T, and hence ~(A>B) ε T ⊃ ~A ∉ T. Since finally $\vdash_{TP4}$~(A>B) > ~B, ~(A>B) ε T ⊃ ~B ε T. Combining these results, ~A ∉ T & ~B ε T iff ~(A>B) ε T.

A standard case: the axiomatization of the crystal lattice, CL[17] The system CL, with primitive set {~, &, →} and v and ↔ defined in the usual way, adds to system R the following Minglish axiom schemes:

CL1. ~A&B → (~A→A) v (A→B)	cl1. Rabc ⊃ a≤c v b≤a* or Rade ⊃ d*≤a* v c≤a*
CL2. A v (A→B).[18]	cl2. RTab ⊃ a≤T.

The modelling conditions for a relational semantics for CL extending that for R are displayed on the right. CL is sound and complete with respect to this semantics. But the main objective here is a different one. It is to show that CL axiomatizes the crystal lattice system, CLM, that is, that CL is sound and complete with respect to these matrices, or equivalently, that the matrices are characteristic for the system CL.

There are two main reasons for providing this axiomatization. Firstly, it enables one to see how much in the way of rubbish can be added to system R whilst maintaining weak relevance, and the extent to which matrix values can be reduced to a small finite number. Secondly, the method of proving the completeness result is sufficiently general to provide a framework for proving completeness theorems for many other matrix systems and model structures. However the general framework does leave open the problem of searching for appropriate formulae to put into the axiomatizations, a problem that elaboration of the completeness arguments frequently closes.

Soundness of CL with respect to the CL matrices is proved in the usual way, by induction on proofs.

Theorem 9.8.1. (Soundness) For every wff A of CL, if A is a theorem of CL A is valid in the CL matrices, CLM.

Proof: By inspection the rules of R preserve validity in CLM. And as can be shown by hand computation, or by using a computer program such as TESTER (ENT1, pp.86-7), axioms of R together with CL1 and CL2 are valid in CLM.

Establishing completeness is a much more complex business. The main result established will be that wff valid in the 4-world CL model structure, <u>CL</u>, are theorems of CL, from which completeness with respect to CLM will follow as a corollary of Theorem 9.7.5. The method adopted to prove the main result involves taking equivalence classes of wff and showing, as in proving algebraic completeness, the classes furnish an algebra with the correct properties. Then it is shown that the resulting algebra is finite and must be one of a given small class of lattice-based algebras, and alternatives to the requisite system <u>CL</u> are then eliminated. Finally the algebras are used to define the 4-world model structures <u>CL</u> already linked with CL-matrices in the previous section (Theorem 9.7.2). The reasons for attempting such an equivalence-class approach are quite straightforward. A finite-valued logic says, so to speak, that there are only finitely-many propositions, 6 in the case of CL, and every wff is equivalent to one of these. Thus by forming equivalence-classes in canonical models in terms of theories of wff analogues of these values can be semantically recovered and a finite model structure designed in terms of them.

Although the intended system in what follows is CL the initial methods are fairly general and hold for a wide class of logics, as before ambiguously labelled L. More exactly, the general results hold for extensions L of system C (see p.289 of RLR1).

Where D is a non-theorem of L there is, by a lemma of §4.6, a prime regular L-theory T such that A ∉ T. Now form the set Σ of equivalence classes of wff of L under the equivalence relation of T-co-implication. Specifically, [A] $=_{df}$ {B: A↔B ε T} and Σ $=_{df}$ {[A]: A is a wff}. The following mini-lemma shows that [A] and Σ are well-defined.

Lemma 9.8.1. The relation of T-co-implication is an equivalence relation of the class of wff.
Proof: Since $\vdash_L$A↔A, A↔A ε T, giving reflexivity. Since $\vdash_L$A&B → B&A, (A→B)&(B→A) ε T ⊃ (B→A)&(A→B) ε T, and hence A↔B ε T ⊃ B↔A ε T, giving symmetry. For transitivity, suppose A↔B ε T and B↔C ε T. Then (A→B)&(B→C)&(C→B)&(B→A) ε T. Hence by CSyll, (A→C)&(C→A) ε T, i.e. A↔C ε T as required. (Alternatively the proof can use ESyll and Ass.)

In order to show that logical operations can be well-defined, pointwise, on Σ, the following substitutivity result is required.

Lemma 9.8.2. Where A↔B ε T then C(A) ↔ C(B) ε T, for all wff A and B, where C(B) results from C(A) by replacing zero or more occurrences of A by B.
Proof is by induction on the formation of C(A) from a single occurrence of A in C(A) for which the formula B is substituted: The zero replacement case is automatic and multiple replacements result by iteration of single replacements. There are these 6 cases in the inductive proof:
1. Let C(A) be A. Then A↔B ε T ⊃ A↔B ε T.
2. Let C(A) be ~D(A), with the lemma holding, by inductive hypothesis, for D(A). Then A↔B ε T ⊃ D(A) ↔ D(B) ε T. Since $\vdash_L$(A↔B) → (~A↔~B), D′(A) ↔ D(B) ε T ⊃ ~D(A) ↔ ~D(B) ε T, and hence A↔B ε T ⊃ ~D(A) ↔ ~D(B) ε T.
3. Let C(A) be D(A) & E, with the lemma holding for D(A). Then A↔B ε T ⊃ D(A)↔D(B) ε T. Since $\vdash_L$(A↔B) & (C→C) → .A&C ↔ B&C, where D(A) ↔ D(B) ε T and E→E ε T, D(A)&E ↔ D(B)&E ε T. Since $\vdash_L$A→A, E→E ε T; and hence A↔B ε T ⊃ D(A)&E ↔ D(B)&E ε T.
4. Let C(A) be E&D(A), with the lemma holding for D(A). The case is similar to the preceding one.
5. Let C(A) be D(A) → E, with the lemma holding for D(A). Then A↔B ε T ⊃ D(A) ↔ D(B) ε T. Since $\vdash_L$ (A↔B) → (A→C ↔. B→C), D(A) ↔ D(B) ε T ⊃ D(B)→E ↔. D(B)→E ε T and hence A↔B ε T ⊃ D(A)→E ↔. D(B)→E ε T.
6. Let C(A) be E → D(A), with the lemma holding for D(A). The case differs from the previous one only in using the prefixing form of ESyll.

Corollary. If, for a ε Σ, A ε a and B ε a, then for some b ε Σ, C(A) ε b and C(B) ε b. Hence also, where B ε [A], C(B) ε [C(A)].

It follows that the following notation can be unambiguously introduced: For wff A and B, $\sim[A] =_{df} [\sim A]$; $[A]\&[B] =_{df} [A\&B]$; $[A] \to [B] =_{df} [A\to B]$; $[A] \vee [B] =_{df} [A \vee B]$. In order to show appropriate De Morgan lattice properties of &, v and ~ so characterized, define an ordering relation, $\leq$, on members of Σ thus:

$a \leq b =_{df} \quad a\to b \in T$, for $a,b \in \Sigma$.

Lemma 9.8.3.

(i) For all wff A and B, $[A] \leq [B]$ iff $A\to B \in T$.

(ii) $\leq$ partially orders the set Σ.

(iii) $<\Sigma, \leq>$ is a lattice, with meet given by & and join by v.

(iv) $<\Sigma, \leq, \sim>$ is a De Morgan lattice, with complement ~.

Proof: (i) Suppose $[A] \leq [B]$. Then, by definition, $[A] \to [B] \subseteq T$, i.e. $[A\to B] \subseteq T$. Hence as $A\to B \in [A\to B]$, $A\to B \in T$. Conversely, suppose $A\to B \in T$. Then for every C such that $C \leftrightarrow .A\leftrightarrow B \in T$, $C \in T$, whence $[A\to B] \subseteq T$. So $[A] \to [B] \subseteq T$, and $[A] \leq [B]$.

(ii) As to reflexivity, since $\vdash_L A\to A$, $A\to A \in T$ and $[A] \leq [A]$. For antisymmetry, suppose $a\leq b$ and $b\leq a$ for $a,b \in \Sigma$. Then for some wff A and B, $a = [A]$ and $b = [B]$, and, by (i) $A\to B \in T$ and $B\to A \in T$, whence $a=b$. For transitivity, suppose $a\leq b$ and $b\leq c$, for $a,b,c \in \Sigma$. For some wff, A, B and C, $a = [A]$, $b = [B]$ and $c = [C]$, and so by (i) $A\to B \in T$ and $B\to C \in T$. By ESyll, $B\to C \to .A\to C \in T$. By Ass, $B\to C \in T \;\&\; B\to C \to .A\to C \in T \supset A\to C \in T$. Hence, $A\to C \in T$, $[A] \leq [C]$ (by (i)), and $a\leq c$. (Alternatively, in place of ESyll and Ass use CSyll.)

(iii) Firstly, $a\&b$ is the g.l.b. of a and b. Since $\vdash_L A\&B \to A$ and $\vdash_L A\&B \to B$, $A\&B \to A \in T$ and $A\&B \to B \in T$, and hence $[A] \& [B] \leq [A]$ and $[A] \& [B] \leq [B]$. Then, since, for some A, B, $a = [A]$ and $b = [B]$, $a\&b$ is a lower bound of a and b. Since $\vdash_L (A\to B) \& (A\to B) \to .A \to B\&C$, $A\to B \in T \;\&\; A\to C \in T \supset A \to B\&C \in T$. Then $[A] \leq [B] \;\&\; [A] \leq [C] \supset .[A] \leq [B]\&[C]$. Hence $c\leq a \;\&\; c\leq b \supset .c \leq a\&b$, establishing $a\&b$ as the g.l.b. of a and b. Secondly, $a \vee b$ is the l.u.b. of a and b. Proof is like the preceding argument but uses dual L theorems $A \to A\vee B$, $B \to A\vee B$ and $(A\to C) \& (B\to C) \to. A\vee B \to C$.

(iv) Firstly $<\Sigma, \leq>$ is distributive. For using the L theorems $A\&(B\vee C) \leftrightarrow .(A\&B) \vee (A\&C)$ and $A\vee(B\&C) \leftrightarrow .(A\vee B)\&(A\vee C)$, it follows, as before, that $a\&(b\vee c) = (a\&b) \vee (a\&c)$ and $a\vee(b\&c) = (a\vee b) \& (a\vee c)$, for $a,b,c \in \Sigma$. Similarly, since $\vdash_L A\leftrightarrow\sim\sim A$, $A\leftrightarrow\sim\sim A \in T$ and $\sim\sim[A] = [A]$. Finally, since $\vdash A\to B \to .\sim B\to\sim A$, $a\leq b \supset \sim b\leq\sim a$, for $a,b \in \Sigma$.

Corollary. For $a,b \in \Sigma$, if $a = \sim a$ and $b = \sim b$ and $a \neq b$ then $a \nleq b$ and $b \nleq a$.

Proof: Suppose the hypotheses hold, and $a\leq b$. Then $\sim b\leq\sim a$, $b\leq a$, whence $a = b$, which is impossible. Similarly, by reductio, $b \nleq a$.

The finite model structure which accounts (as in §9.7) for the CL matrix will be based on the complement $\overline{T}$ of T and on the opposite T^* of T, where as usual $T^* =_{df} \{A: \sim A \notin T\}$. It follows, as in §4.6 of RLR1.

Lemma 9.8.4. T^* is a prime L-theory contained in T.
Proof: As before $T^* \subseteq T$ by virtue of LEM. To be sure, however, that $T^* \varepsilon \Sigma$, special features of the underlying logic have to be invoked, in this case the axiom $A \vee (A \rightarrow B)$ of CL (and of various other finite-valued logics). Thus henceforth the argument begins to rely on particular features of finite-valued logics.

Lemma 9.8.5. T^* and $\overline{T}$ both belong to Σ, where the auxiliary set $\overline{T}$ is the complement of T w.r.t. the set of wff.
Proof uses the CL thesis $A \vee (A \rightarrow B)$: Both T^* and $\overline{T}$ are non-null, since $D \varepsilon \overline{T}$ and $\sim D \varepsilon T^*$, where D is the the given non-theorem of L. What has to be shown is that T^* (and similarly T) is an equivalence class under T-co-implication. For this it suffices to show that a) where $A \varepsilon T^*$ and $B \varepsilon T^*$ then $A \leftrightarrow B \varepsilon T$ and b) where $A \varepsilon T^*$ and $B \notin T^*$ then $A \leftrightarrow B \notin T$.
ad a) On the hypothesis, $\sim A \notin T$ and $\sim B \notin T$. Since $\vdash \sim A \vee (\sim A \rightarrow \sim B)$, $\sim A \varepsilon T$ or $\sim A \rightarrow \sim B \varepsilon T$, whence $\sim A \rightarrow \sim B \varepsilon T$ since T is normal and $B \rightarrow A \varepsilon T$ by contraposition. Similarly, using $\vdash \sim B \vee (\sim B \rightarrow \sim A)$, $A \rightarrow B \varepsilon T$, whence $A \leftrightarrow B \varepsilon T$.
ad b) Given the hypothesis, $\sim A \notin T$ but $\sim B \varepsilon T$. By Ass, $\sim B \varepsilon T$ and $A \rightarrow B \varepsilon T$, $\sim A \varepsilon T$, so $A \rightarrow B \notin T$, whence $A \leftrightarrow B \notin T$.
Proof that $\overline{T} \varepsilon \Sigma$ similarly involves two cases which are proved like a) and b).

The task of revealing the finite character of the model structure now begins in earnest.

Lemma 9.8.6.
(i) For all $a \varepsilon \Sigma$, $\overline{T} \leq a \leq T^*$.
(ii) $\sim T^* = \overline{T}$ and $\sim \overline{T} = T^*$.
Proof: As to (i) there are two parts. Suppose, first, $B \varepsilon \overline{T}$, whence $\overline{T} = [B]$. As $\vdash B \vee (B \rightarrow A)$, $B \varepsilon T$ or $B \rightarrow A \varepsilon T$, and hence $B \rightarrow A \varepsilon T$ and $[B] \leq [A]$. Hence for $a \varepsilon \Sigma$, $T \leq a$. Suppose, secondly, $B \varepsilon T^*$. Then $T^* = [B]$ and $\sim B \notin T$. Since $\vdash \sim B \vee (A \rightarrow B)$, $\sim B \varepsilon T$ or $A \rightarrow B \varepsilon T$, and hence $A \rightarrow B \varepsilon T$ and $[A] \leq [B]$. Thus for $a \varepsilon \Sigma$, $a \leq T^*$. As to (ii) let $A \varepsilon \overline{T}$. Then $\overline{T} = [A]$ and $\sim A \varepsilon T^*$. Hence $T = \sim [A] = \sim \overline{T}$ and $\sim T^* = \sim\sim \overline{T} = \overline{T}$.

Lemma 9.8.7. A set of wff a is a union of equivalence classes of Σ, i.e. $a = \bigcup \{[A]/A \varepsilon a\}$, iff a is closed under T-equivalence, i.e. for every wff A and B, when $A \leftrightarrow B \varepsilon T$ and $A \varepsilon a$ then $B \varepsilon a$.
Proof: $a = \bigcup \{[A]/A \varepsilon a\}$ iff $\bigcup \{[A]/A \varepsilon a\} \subseteq a$, i.e. iff $(\forall B)(B \varepsilon \bigcup \{[A]/A \varepsilon a\}$

⊃ B ε a}, i.e. iff (∀B)((∃A)(B ε [A] & A ε a) ⊃ B ε a} i.e. iff (∀A,B)([A] = [B] & A ε a ⊃ B ε a}, i.e. iff (∀A,B)(A↔B ε T & A ε a ⊃ B ε a}.

Corollary. If a set a of wff is closed under T-implication then a is a union of equivalence classes of E.

As before, a T-CL-theory is a set of wff closed under T-implication and Adjunction. Again a T-CL-theory is a CL-theory.

Lemma 9.8.8. Where a is closed under T-equivalence,
(i) a is a T-CL-theory iff {[A]/A ε a} is a filter of <Σ, ≤, ~>, and
(ii) a is a prime T-CL-theory iff {[A]/A ε a} is a prime filter of <Σ, ≤, ~>.[19]
Proof: Since a = U{[A]/A ε a}, for all wff A, A ε a iff A ε U{[A]/A ε a} i.e. iff (∃B)(A ε [B] & B ε a}, i.e. iff (∃B)([A] = [B] & B ε a}, i.e. iff [A] ε {[A]/A ε a}.
(i) a is a T-CL-theory iff (∀A,B)(A→B ε T & A ε a ⊃ B ε a) *and* (∀A,B)(A ε a & B ε a ⊃ A&B ε a}, i.e. iff (∀A,B)([A] ≤ [B] & [A] ε {[A]/A ε a} ⊃ [B] ε {[A]/A ε a}) *and* (∀A,B)([A] ε {[A]/A ε a} & [B] ε {[A]/A ε a} ⊃ [A]&[B] ε {[A]/A ε a}), i.e. iff (∀b,c ε Σ)(b ≤ c & b ε {[A]/A ε a} ⊃ c ε {[A]/A ε a}) *and* (∀b,c ε Σ)(b ε {[A]/A ε a} & c ε {[A]/A ε a} ⊃ b&c ε {[A]/A ε a}), i.e. iff (∀b,c ε Σ)(b&c ε {[A]/A ε a} iff b ε {[A]/A ε a} & c ε {[A]/A ε a}). The final step is justified as follows:- LHS to RHS: Since b&c ≤ b and b&c ε {[A]/A ε a}, b ε {[A]/A ε a}. Similarly when b&c ε {[A]/A ε a}, c ε {[A]/A ε a}. Hence, when b&c ε {[A]/A ε a}, b ε {[A]/A ε a} and c ε {[A]/A ε a}. The converse is given. RHS to LHS: Suppose b ≤ c and b ε {[A]/A ε a}. Then b&c = b and b&c ε {[A]/A ε a}, whence c ε {[A]/A ε a}. Hence, a is a T-CL-theory iff {[A]/A ε a} is a filter of <Σ, ≤, ~>.
(ii) a is a prime T-CL-theory iff (∀A,B)(A→B ε T & A ε a ⊃ B ε a) *and* (∀A,B)(A ε a & B ε a ⊃ A&B ε a) *and* (∀A,B)(AvB ε a ⊃ A ε a or B ε a}, i.e. iff, effectively applying (i), (∀b,c ε Σ)(b ε {[A]/A ε a} & c ε {[A]/A ε a} iff b&c ε {[A]/A ε a}) *and* (∀b,c ε Σ)(bvc ε {[A]/A ε a} iff b ε {[A]/A ε a} v c ε {[A]/A ε a}). Hence, a is a prime T-CL-theory iff {[A]/A ε a} is a prime filter of <Σ, ≤, ~>.

Corollary. T and T* are prime T-CL-theories and hence {T*} and {[A]/A ε T} are prime filters.
Proof: Given preceding lemmas it is enough to show that T and T* are closed under T-implication. But T is, by Ass. For T*, suppose A→B ε T and A ε T*. Then ~A ∉ T. Since $\vdash_{CL}$~B&(A→B) → ~A, when ~B ε T and A→B ε T, ~A ε T. Hence ~B ∉ T and B ε T*. Hence by the lemma {[A]/A ε T*} and {[A]/A ε T} are prime filters. But by Lemma 9.8.5, {[A]/A ε T*} = {T*}.

For any union a of equivalence classes of Σ, define a* wff thus: A ε a* iff ~A ∉ a, for every wff A. Then (T)* = T*, as previously defined. a* has the expected theoretical properties; it behaves like a prime T-CL-theory, as the next lemma shows.

Lemma 9.8.9.
Where a is a union of equivalence classes of Σ.
(i) a* is a union of equivalence classes of E;
(ii) a** = a;
(iii) if a is closed under T-implication then a* is closed under T-implication; and
(iv) if a is a prime T-CL-theory then a* is a prime T-CL-theory.
Proof: By an earlier lemma, a is closed under T-equivalence.
ad (i). It is enough to show that a* is closed under T-equivalence. Suppose A↔B ε T and A ε a*. Then ~A ∉ a. By Contraposition, ~B↔~A ε T. Then as a is closed under T-equivalence, if ~B ε a, ~A ε a. Hence ~B ∉ a, and B ε a*, as required.
ad (ii). For any wff A, A ε a** iff ~A ∉ a*, i.e. iff ~~A ε a, i.e. iff A ε a.
ad (iii). Let a be closed under T-implication. Suppose A→B ε T and A ε a*. Then ~A ∉ a. By Contraposition, ~B→~A ε T, whence ~B ∉ a, and B ε a*.
ad (iv). Let a be a prime T-CL-theory. By (iii), a* is closed under T-implication. As to Adjunction, suppose A ε a* and B ε a*. Then ~A ∉ a and ~B ∉ a. By primeness of a, ~Av~B ∉ a. Since ⊢~(A&B) → .~A v~B, ~(A&B) ∉ a, whence A&B ε a*. As to primeness, suppose AvB ε a*. Then ~(AvB) ∉ a. Since ⊢~A&~B → ~(AvB), ~A&~B ∉ a. Thus by Adjunction ~A ∉ a or ~B ∉ a, whence A ε a* or B ε a*.

The distinctive axioms schemes of CL can now be applied to severely restrict the structure of the De Morgan lattice on Σ.

Lemma 9.8.10.
(i) For all a ε Σ, a ≤ ~a or ~a ≤ a.
(ii) If, for t and a in Σ, $\overline{T}$ < t < a < ~t < T*, then a = ~a.
Proof: (i) Since $\vdash_{CL}$~A → .(~A→A) v (A→~A), when ~A ε T either ~A→A ε T or A→~A ε T, and hence when [A] ≠ T* either ~[A] ≤ [A] or [A] ≤ ~[A]. Then, for all equivalence classes for which a ≠ T*, ~a ≤ a or a ≤ ~a. But, by Lemma 9.8.6, ~T* ≤ T*, and hence, for all a ε Σ (i.e. whether a = T* or a ≠ T*), a ≤ ~a or ~a ≤ a.
For (ii), let $\overline{T}$ < t < a < ~t < T*. Suppose, for a first reductio, a < ~a. By Lemma 9.8.3, since t ≤ a, ~a ≤ ~t. If ~a = ~t, then by Lemma 9.8.3, a = t, which contradicts the assumption. Hence, unless a = ~a already, $\overline{T}$ < t < a < ~a < ~t < T*. Since $\vdash_{CL}$~A&B → .(~A →A) v (A→B), when ~A ε T and B ε T either ~A→A ε T or A→B ε T, and hence, when [A] ≠ T* and [B] ≠ $\overline{T}$, either ~[A] ≤ [A] or [A] ≤ [B]. Thus, for every a and b ε Σ, when a ≠ T* and b ≠ $\overline{T}$ either ~a ≤ a or a ≤ b. In particular, then, ~a ≤ a or a ≤ t; but neither

alternative is possible. Suppose next ~a < a. In a similar way T̄ < t < ~a < a < ~t < T*, whence a ≤ ~a or ~a ≤ t, neither of which is possible. Hence, by (i), a = ~a.

Corollary. If T̄ < t < a < ~t < T* and T̄ < t < b < ~t < T*, where a ≠ b, then a ≰ b and b ≰ a.

Proof. Let T̄ < t < a < ~t < T*, T̄ < t < b < ~t < T*, and a ≠ b. By the lemma, a = ~a and b = ~b. Hence, by corollary to Lemma 9.8.3, a ≰ b and b ≰ a.

Lemma 9.8.11. There are at most 2 elements x in Σ satisfying the ordering T̄< t < x < ~t < T*.

Proof: Suppose, per impossible, that a, b and c are 3 distinct elements satisfying the ordering. By the preceding corollary, the illustrated sub-lattice diagram obtains:

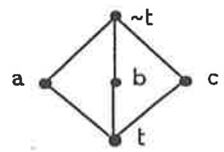

Figure 9.20 Non-distributive lattice

But, in contravention of Lemma 9.8.3, this lattice is not distributive, as the following example shows: a&(bvc) = a&~t = a, whilst (a&b)v(a&c) = tvt = t.

Lemma 9.8.12. There are no elements a such that T̄ < a < t < ~t < T* and there are no elements b such that T̄ < t < ~t < b < T*.

Proof: Suppose a were such that T̄ < a < t < ~t < T*. By Lemma 9.8.3, T̄ ≤ a ≤ t, ~t ≤ ~a ≤ T*. If in the latter ~t = ~a then t = a, which is impossible. If ~a = T* then a = T̄, which is again impossible. Hence, ~t < ~a < T*. Then by Lemma 9.8.10(ii), t = ~t, which is impossible. Suppose next that b were such that T̄ < t < ~t < b < T*. As in the previous part, T̄ < ~b < t < ~t < b < T*. So again by Lemma 9.8.10(ii), t = ~t, which is impossible.

Lemma 9.8.13. If T̄ < a < T* and T̄ < t < ~t < T* then t ≤ a ≤ ~t.

Proof: Suppose T̄ < a < T* and T̄ < t < ~t < T*. Then a ε {[A]/A ε T} and similarly t ε {[A]/A ε T}. For, to take the first, a ≠ T, by assumption, i.e. a is an equivalence class distinct from T̄. Hence a ⊆ T. If a were null the inclusion would be immediate; so it can be assumed that a is non-null. For let B be an element such that B ε a and B ∉ T̄; then B ε T. Now let C be any other element in a. Then since a is an equivalence class, B ↔ C ε T, so C ε T also.

Now a ε {[A]: A ε T} iff, for some B in a, B ε T, i.e. iff, since a $\subseteq$ T, some wff B is in a, which is true. By the corollary to Lemma 9.8.8, {[A]/A ε T} is a filter, and hence a&t ε {[A]/A ε T}. Hence, $\overline{T}$ < a&t ≤ t < ~t < T* and, by Lemma 9.8.12, a&t = t, whence t ≤ a. Since T* < ~a < ~$\overline{T}$, by Lemma 9.8.10, $\overline{T}$ < ~a < T*. Applying the above argument to ~a, t ≤ ~a, and hence a ≤ ~t, as required.

Lemma 9.8.14. There is at most one element t such that $\overline{T}$< t < ~t < T*.
Proof: Let $\overline{T}$ < t < ~t < T* and $\overline{T}$ < t' < ~t' < T*. By Lemma 9.8.13, t ≤ t' and t' ≤ t, and hence t = t'.

Lemma 9.8.15. If there are no elements x such that $\overline{T}$< x < ~x < T*, then the <Σ, ≤, ~> has at most 3 elements.
Proof: Assume there are no elements x such that $\overline{T}$ < x < ~x < T*. Similarly then, since ~~x = x, there are no x such that $\overline{T}$ < ~x < x < T*. Suppose a and b are elements such that $\overline{T}$ < a < T* and $\overline{T}$< b < T*. By Lemma 9.8.10(i) a ≤ ~a or ~a ≤ a. But if ~a < a or a < ~a initial assumptions are violated, so a = ~a. Similarly b = ~b. Further, as in Lemma 9.8.13, a ε {[A]/A ε T} and b ε {[A]/A ε T}. Thus since, by the corollary to Lemma 9.8.8, {[A]/A ε T} is a filter, then a&b ε {[A]/A ε T}. Hence, $\overline{T}$ < a&b ≤ a < T* and $\overline{T}$ < a&b ≤ b < T*. Also since a&b ≤ a, ~a ≤ ~(a&b) and hence $\overline{T}$ < a&b ≤ a ≤ ~(a&b) < T*. By initial assumption then, a&b = ~(a&b) = a. Similarly, $\overline{T}$ < a&b ≤ b ≤ ~(a&b) < T*, whence a&b = b, and hence a = b. That is, there is at most one element a such that $\overline{T}$ < a < T*, and the lattice <Σ, ≤, ~> has at most 3 elements.

There are 5 possible lattices of equivalence classes of Σ, as the following structure results prove. At this stage too we impose a (conventional) labelling on lattice elements.

Structure result 1 If the lattice <Σ, ≤, ~> has 2 elements then Σ = {T*, $\overline{T}$}, with $\overline{T}$ < T* and ~T* = $\overline{T}$
Proof: Both of T* and $\overline{T}$ are distinct elements of Σ and so are elements of every lattice. The ordering relation ≤ and negation function ~ are determined through Lemma 9.8.6.

Structure result 2 If the lattice <Σ, ≤, ~> has three elements Σ = {T*, $\overline{T}$, t}, with third element t and with $\overline{T}$ < t < T*, ~T* = $\overline{T}$ and ~t = t.
Proof: By Lemma 9.8.6, $\overline{T}$ < t < T* and ~T* = $\overline{T}$. Since, also applying previous lemmas, $\overline{T}$ < ~t < T*, the lattice would have more than 3 elements unless ~t = t.

Structure result 3 If the lattice <Σ, ≤, ~> has 4 elements then Σ = {T*, $\overline{T}$, t, ~t}, with $\overline{T}$ < t < ~t < T*, ~T = $\overline{T}$ and ~(t) = ~t.

Proof: If the lattice $<\Sigma, \leq, \sim>$ has 4 elements then, by Lemmas 9.8.14 and 15, there is exactly one element t such that $\overline{T} < t < \sim t < T^*$. The ordering relation and negation function are as required.

Structure result 4 If the lattice $<\Sigma, \leq, \sim>$ has 5 elements then $\Sigma = \{T^*, \overline{T}, t, \sim t, u\}$, with $\overline{T} < t < u < \sim t < T^*$, $\sim T^* = \overline{T}$, $\sim(t) = \sim t$, and $\sim u = u$.
Proof: If the lattice $<\Sigma, \leq, \sim>$ has 5 elements then, by Lemmas 9.8.14 and 15, there is exactly one element t such that $\overline{T} < t < \sim t < T^*$. By Lemma 9.8.13, the additional element u is such that $\overline{T} < t < u < \sim t < T^*$. The ordering relation and negation function are as required.

Structure result 5 If the lattice $<\Sigma, \leq, \sim>$ has 6 elements then $\Sigma = \{T^*, \overline{T}, t, \sim t, u, v\}$, with $\overline{T} < t < u < \sim t < T^*$, $t < v < \sim t$, $u \not\leq v$, $v \not\leq u$, $\sim T^* = \overline{T}$, $\sim t = \sim t$, $\sim u = u$ and $\sim v = v$.
Proof: If the lattice $<\Sigma, \leq, \sim>$ has 6 elements then, by Lemmas 9.8.14 and 15, there is exactly one element t such that $\overline{T} < t < \sim t < T^*$. By Lemma 9.8.13, the additional elements u and v are such that $t < u < \sim t$ and $t < v < \sim t$. By Lemma 9.8.10, $u = \sim u$ and $v = \sim v$. By the Corollary to Lemma 9.8.10, $u \not\leq v$ and $v \not\leq u$. Otherwise, the ordering relation and negation function are as required.

Structure result 6 There are no lattices $<\Sigma, \leq, \sim>$ with 7 or more elements.
Proof: Suppose the lattice $<\Sigma, \leq, \sim>$ has at least 7 elements. As in the proof of preceding structure results, there is exactly one element t such that $\overline{T} < t < \sim t < T^*$, and there are at least 3 elements x such that $t < x < \sim t$. But this contradicts Lemma 9.8.11.

The structure results determine all the lattices of equivalence classes of Σ. However for the purposes of proving completeness, what these finite lattices must supply are 4 prime T-CL-theories, theories which can represent or provide valuations for the 4 worlds, T, T*, a, a*, of the model structure <u>CL</u>, through equivalences of the form $I(A,a) = 1$ iff $A \varepsilon a$. The reason (connected with reduced modellings) that T-CL-theories are required rather than just CL-theories will become clear in the proof of the Interpretation Theorem to follow. Thus still required, in addition to the already defined T and T*, are two prime T-CL-theories, henceforth called a and a* in anticipation of the worlds they represent. Theories a and a* will be defined for each of the 5 finite lattices. In the smaller lattices, not all of T, T*, a and a* can be distinct. As with T and T*, a and a* are related by definition of * for unions of equivalence classes of Σ. Then, by Lemma 9.8.9, a is a prime T-CL-theory iff a* is a prime T-CL-theory. By Lemma 9.8.8, a and a* must be such that $\{[A]/A \varepsilon a\}$ and $\{[A]/A \varepsilon a^*\}$ are prime filters of $<\Sigma, \leq, \sim>$.

Since in the 4-world model structure <u>CL</u>, $T^* \leq a \leq T$ and $T^* \leq a \leq T$, we shall require of the corresponding theories that $T^* \subseteq a \subseteq T$ and $T^* \subseteq a^* \subseteq T$. Note that a and a* will be non-degenerate theories, i.e. neither null nor universal. Thus, the prime filters {[A]/A ε a} and {[A]/A ε a*} will be nontrivial, i.e. distinct from Ø and Σ.

The definitions of a and a* for the 5 lattices are as follows:

(I) 2-element lattice: a = a* = T = T*, there being only one non-trivial prime filter.

(II) 3-element lattice: a = T, a* = T*, there being only two non-trivial prime filters.

(III) 4-element lattice: a = a* = T* ∪ ~t, there being only three non-trivial prime filters.

(IV) 5-element lattice: a = T* ∪ ~t ∪ u, a* = T* ∪ ~t, there being four nontrivial prime filters.

(V) 6-element lattice: a = T* ∪ ~t ∪ u, a* = T* ∪ ~t ∪ v, there being four non-trivial prime filters.

In each case, a* = {a}*, according to the definition of * for unions of equivalence classes of Σ. Further, in each case, both a and a* are prime T-CL-theories.

Lemma 9.8.16. In all 5 lattices, (i) whenever ~A ε a and B ε a, A→B ε a, and (ii) whenever ~A ε a* and B ε a*, A→B ε a*.

Proof: There are first common preliminaries, and then each lattice is considered in turn. For (i), suppose ~A ε a and B ε a. Since $\vdash_{CL}$ ~A&B → . (~A→A) v (A→B), either ~A→A ε a or A→B ε a. Since $\vdash_{CL}$ (~A→A) → A, when A ∉ a, ~A→A ∉ a. Hence when A ∉ a, A→B ε a. Since $\vdash_{CL}$~A&B → . (B→~B) v (A→B), B→~B ε a or A→B ε a. But since $\vdash_{CL}$B→~B → ~B, when ~B ∉ a, B→~B ∉ a. Hence, when ~B ∉ a, A→B ε a.

For (ii), suppose ~A ε a* and B ε a*. Then, as for a above, ~A→A ε a* or A→B ε a*, A ∉ a* ⊃ A→B ε a*, B→~B ε a* or A→B ε a*, and ~B ∉ a* ⊃ A→B ε a*.

Lattices (I) and (III), where a = a*.

(i) Since ~A ε a ⊃ A ∉ a, A→B ε a, as required.

(ii) Since ~A ε a* ⊃ A ∉ a*, A→B ε a*.

Lattice (II)

(i) Since A ∉ a ⊃ A→B ε a, let A ε a. Since ~A ε a, A ε t. Since B ε a, B ε t or B ε T*. In either case, A → B ε T, since t ≤ t and t ≤ T*. Then A→B ε a, as required.

(ii) Since a* = T*, ~A ε T* and hence A ε T. Then A ∉ T*, A ∉ a*, and so A→B ε a*.

Lattice (IV)

(i) Since A ∉ a ⊃ A→B ε a, let A ε a. Since ~A ε a, A ε u. Since ~B ∉ a ⊃ A→B ε a, let ~B ε a. Since B ε a, B ε u. Hence, as A and B belong to the same equivalence class, A↔B ε T. Since ~A→A ∉ a ⊃ A→B ε a, let ~A→A ε a. Then ~A ↔A ε T and A↔B ε T. For A and B belong to the same equivalence class u; A and ~A do also, since ~A ε a and, since ~A→A ε a, A ε a. By

Lemma 9.8.2, (~A→A) ↔ (A→B) ε T, and hence A→A ε a. Again, by Lemma 9.8.2, (A→A) ↔ (A→B) ε T, and hence A→B ε a, as required.
(ii) Since a* = T*U~t, ~A ε T*U~t and hence A ε $\overline{T}$Ut. Then A ∉ a* and A→B ε a*, as required.

Lattice (V)

(i) Since A ∉ a ⊃ A→B ε a, let A ε a. Since ~A ε a, ~A ε u. Since ~B ∉ a ⊃ A→B ε a, let ~B ε a. Since B ε a, B ε u. Since ~A→A ∉ a ⊃ A→B ε a, let ~A→A ε a. By the same argument as for Lattice (IV) (i), A→B ε a.
(ii) Since A ∉ a* ⊃ A→B ε a*, let A ε a*. Since ~A ε a*, A ε v. Since ~B ∉ a* ⊃ A→B ε a*, let ~B ε a. Since B ε a*, B ε v. Since ~A→A ∉ a* ⊃ A→B ε a*. Again by the argument for Lattice (IV)(i), A→B ε a*.

Let the canonical valuation v in the 4-world m.s. CL be defined in essentially the standard way; i.e. for sentential parameter p, and each c in K = {T,T*,a,a*} and corresponding T-CL-theory, v(p,c) = 1 iff p ε c. More precisely v(p,c) = 1 iff p ε c' where c' is the T-CL-theory matching c. To satisfy the hereditariness requirement for v, it is enough that:

for every p, if v(p,a) = 1 then v(p,T) = 1; if v(p,a*) = 1 then v(p,T) = 1; and if v(p,T*) = 1 then v(p,a) = 1.

But this is guaranteed in all five lattices by the theory orderings T* ⊆ a ⊆ T and T* ⊆ a* ⊆ T. In so reducing the hereditariness requirements, and subsequently in the interpretation theorem the Brady simplification of the valuation and interpretation rules are taken advantage of (see §9.7).

Valuation v is extended as before to an interpretation I on all wff.

Theorem 9.8.2. (Interpretation theorem) For each c in K and corresponding T-CL-theory, I(A,c) = 1 iff A ε c, for every wff A.

Proof is by induction on wff from the given basis: The induction steps for ~ and & are standard; the main work goes into step for →. There are 4 subcases:
I) at T: I(A→B,T) = 1 iff (I(A,T) = 1 ⊃ I(B,T) = 1) & (I(A,T*) = 1 ⊃ I(B,T*) = 1 & (I(A,a) = 1 ⊃ I(B,a) = 1) & (I(A,a*) = 1 ⊃ I(B,a*) = 1); i.e. iff (A ε T ⊃ B ε T) & (A ε T* ⊃ B ε T*) & (A ε a ⊃ B ε a) & (A ε a* ⊃ B ε a*) i.e. provisionally, iff A→B ε T.

The provisional step is justified as follows:

RHS to LHS: Since T, T*, a and a* are all T-CL-theories, for each such theory, when A→B ε T, if A ε c then B ε c.

LHS to RHS: To show as required that when (A ∉ T or B ε T) and (A ∉ T* or B ε T*) and (A ∉ a or B ε a) and (A ∉ a* or B ε a*) then A→B ε T, it suffices to prove:

A ∉ T ⊃ A→B ε T ... (1)
B ε T* ⊃ A→B ε T ... (2)
B ε T & A ∉ T* & A ∉ a & A ∉ a* ⊃ A→B ε T ... (3)
B ε T & A ∉ T* & A ∉ a & B ε a* ⊃ A→B ε T ... (4)
B ε T & A ∉ T* & B ε a & A ∉ a* ⊃ A→B ε T ... (5)

B ε T & A ∉ T* & B ε a & B ε a* ⊃ A→B ε T ... (6)

For (1) and (2) apply, respectively, the CL theorems Av(A→B) and ~Bv (A→B).

ad (3). Since $\vdash_{CL}$~A&B → .(~A→A)v(A→B), when ~A ε T and B ε T then ~A→A ε T or A→B ε T. Hence, when B ε T and A ∉ T* then either ~A ∉ a or A ε a or else A→B ε T, since a is a T-CL-theory. Hence when B ε T and A ∉ T* and A ∉ a and A ∉ a*, A→B ε T.

ad (4). By Lemma 9.8.16, when ~A ε a* and B ε a*, A→B ε a*. Hence, when A ∉ a and B ε a*, A→B ε T, since a* ⊆ T.

ad (5). Similarly by Lemma 9.8.16, when ~A ε a and B ε a, A→B ε a. Hence, when A ∉ a* and B ε a, A→B ε T, since a ⊆ T.

ad (6). Since $\vdash_{CL}$~A&B → .(B→~B) v (A→B), when ~A ε T and B ε T, B→~B ε T or A→B ε T; and hence when B ε T and A ∉ T*, either B ∉ a or ~B ε a or else A→B ε T, a being a T-CL-theory. Hence when B ε T and A ∉ T* and B ε a and B ε a*, A→B ε T.

II) at T*: I(A→B,T*) = 1 iff I(A,T) = 1 ⊃ I(B,T*) = 1, i.e. iff A ε T ⊃ B ε T*, i.e. provisionally, iff A→B ε T*.

The provisional step is justified as follows:

RHS to LHS: Since $\vdash_{CL}$A→B → .~AvB, when A→B ε T*, ~A ε T* or B ε T*, and hence where A→B ε T*, when A ε T, B ε T*.

LHS to RHS: To show that when A ∉ T or B ε T* then A→B ε T*, it suffices to prove:

A ∉ T ⊃ A→B ε T* ... (1)

B ε T* ⊃ A→B ε T* ... (2)

ad (1). Since $\vdash_{CL}$A v (A→B), $\vdash_{CL}$A v (A → .~A→B) and so $\vdash_{CL}$A v (~A →. A→B). Then A ε T or ~A → .A→B ε T. Since T* is a T-CL-theory, A ε T or when ~A ε T* then A→B ε T*. Then A ∉ T, A→B ε T*.

ad (2). As in (1), $\vdash_{CL}$A v (~A → .A→B) whence $\vdash_{CL}$~B v (B → .A→B). Then ~B ε T or B → .A→B ε T and, since T* is a T-CL-theory, ~B ε T or when B ε T* then A→B ε T*. Hence, when B ε T*, A→B ε T*.

III) at a: I(A→B,a) = 1 iff (I(A,a*) = 1 ⊃ I(B,T*) = 1) & (I(A,T) = 1 ⊃ I(B,a) = 1), i.e. iff (A ε a* ⊃ B ε T*) & (A ε T ⊃ B ε a), i.e. provisionally, iff A→B ε a.

The provisional step is justified as follows:

RHS to LHS: (i) Since $\vdash_{CL}$~B → .A→B → ~A, when ~B ε T, A→B → ~A ε T; and so, since a is a T-CL-theory, where ~B ε T, then when A→B ε a, ~A ε a. Hence when A→B ε a and ~A ∉ a, ~B ∉ T, and so where A→B ε a, then, when A ε a*, B ε T*.

(ii) Since $\vdash_{CL}$A → .A→B → B) when A ε T, A→B → B ε T, and hence where A ε T when A→B ε a then B ε a. Thus where A→B ε a, then when A ε T, B ε a.

LHS to RHS: To show that when A ∉ a* or B ε T* and A ∉ T or B ε a, then A→B ε a, it suffices to prove

A ∉ a* & A ∉ T ⊃ A→B ε a ... (1)
A ∉ a* & B ε a ⊃ A→B ε a ... (2)
B ε T* & A ∉ T ⊃ A→B ε a ... (3)
B ε T* & B ε a ⊃ A→B ε a ... (4)

ad (1). Since $\vdash_{CL}$A v (~A → .A→B), A ε T or ~A → .A→B ε T, and hence A ε T or when ~A ε a then A→B ε a. Thus when A ∉ a* and A ∉ T, A→B ε a.

ad (2). By Lemma 9.8.16, when ~A ε a and B ε a, A→B ε a; and so when A ∉ a* and B ε a, A→B ε a.

ad (3). By Lemma 9.8.16, when ~A ε a and B ε a, A→B ε a; and, since T ⊆ a, when ~A ε T* and B ε T*, A→B ε a. Thus when B ε T* and A ∉ T, A→B ε a.

ad (4). Since $\vdash_{CL}$~B v (B → .A→B), ~B ε T or B → .A→B ε T; and hence either ~B ε T or when B ε a, A→B ε a. Thus, when B ε T* and B ε a, A→B ε a.

IV at a*: I(A→B,a*) = 1 iff (I(A,a) = 1 ⊃ I(B,T*) = 1) & (I(A,T) = 1 ⊃ I(B,a*) = 1), i.e. iff (A ε a ⊃ B ε T*) & (A ε T ⊃ B ε a*), i.e. provisionally iff A→B ε a*.

The final provisional step is justified as follows:

RHS to LHS: (i) Since $\vdash_{CL}$~B → .A→B → ~A, where ~B ε T when A→B ε a*, ~A ε a*. Thus when A→B ε a* and ~A ∉ a*, ~B ∉ T, and so where A→B ε a* when A ε a then B ε T*.

(ii) Since $\vdash_{CL}$A → .A→B → B, where A ε T then when A→B ε a*, B ε a*. Thus when, A→B ε a*, if A ε T, B ε a*.

LHS to RHS: To show that where (A ∉ a or B ε T*) and (A ∉ T or B ε a*), A→B ε a*, if suffices to prove

A ∉ a & A ∉ T ⊃ A→B ε a* ... (1)
A ∉ a & B ε a* ⊃ A→B ε a* ... (2)
B ε T* & A ∉ T ⊃ A→B ε a* ... (3)
B ε T* & B ε a* ⊃ A→B ε a* ... (4)

ad (1). Since $\vdash_{CL}$A v (~A → .A→B), either A ε T or when ~A ε a* A→B ε a*, and so when A ∉ a and A ∉ T, A→B ε T*.

ad (2). By Lemma 9.8.16, when ~A ε a* and B ε a*, A→B ε a*, and thus when A ∉ a and B ε a*, A→B ε a*.

ad (3). By Lemma 9.8.16, when ~A ε a* and B ε a*, A→B ε a*, and since T* ⊆ a*, when ~A ε T* and B ε T*, A→B ε a*. Thus when B ε T* and A ∉ T, A→B ε a*.

ad (4). Since $\vdash_{CL}$~B v (B → .A→B), either ~B ε T or when B ε a* then A→B ε a*; and so when B ε T* and B ε a*, A→B ε a*.

Theorem 9.8.3. (Completeness Theorem) If A is valid in CL then A is a theorem of CL.

Proof: Suppose A is a non-theorem of CL. By a lemma, there is a prime regular CL-theory T, such that A ∉ T. But T, together with T* and a and a*, as defined for each of the 5 lattices, determine a valuation v for the CL model structure. And this valuation extends to an interpretation I such that, for every wff B, I(B,T) = 1 iff B ε T. Hence, I(A,T) ≠ 1 for this model, and so A is invalid in the CL model structure.

Corollary. If A is valid in the CL matrices, CLM, then A is a theorem of CL. *Proof* applies Theorem 9.7.2.

A variant case: the axiomatization of Belnap's 8-valued matrix M_0 As with FD (of §3.1 of RLR1), so with M_0, an axiomatization based on material-implication, defined as customarily A ⊃ B =df ~A v B, which by no means indispensible, makes life easier. A hook-based system BM, for which matrix M_0 proves characteristic, has the same wff as CL and these postulates:

1. A→A.	2. A→B ⊃ .B→C → .A→C.
3. A→B ⊃ .C→A → .C→B.	4. A ⊃ .A→B → B.
5. A&(A→B) → B.	6. A&B → A.
7. A&B → B.	8. (A→B) & (A→C) ⊃ .A → B&C.
9. A&(BvC) → (A&B) v (A&B).	10. A→~A → ~A.
11. A→~B → .B→~A.	12. ~~A → A.
13. A&~A&B → .A→ B	14. ~A&(A→B) ⊃ .~A → Av(A→B).
15. A ⊃ (~A&B → A v (A→B)).	16. (~A→A)&(A→B) ⊃ .(B→A)v(C→B).
17. A&B ⊃ .(~A→A) v (A→B).	18. A ⊃ .(~A→A) v (A&~A → B).

A, A⊃B ⇒ B; A, A→B ⇒ B; A, B ⇒ A&B.

The axioms given suffice for the completeness proof to follow; but of course several axioms can be strengthened, e.g. schemes 2-4 to those of R, which BM extends, and axioms 14, 15, 17 and 18 to the following:

14'. ~A&(A→B) ⊃ .~A → .A→B.	15'. A → .~A&B → A v (A→B).
17'. A&B → .(~A→A) v (A→B).	18'. A → (~A→A) v (A&~A→B).

Plainly BM can be reaxiomatized with Material Detachment as sole rule. This affords a striking contract with CL, where γ fails (for consider, from CL2, ~A ⊃ .A→B).

Theorem 9.8.4. (Soundness) For every wff A of BM, if A is a theorem of BM then A is valid in M_0.

Proof is by the usual induction: As can be checked by hand or computer each of the axioms is valid in M_0, and the rules preserve validity.

Completeness proof preliminaries adapt those already given. The main variation from the CL argument lies in the partial displacement of → by ⊃ and in the appeal to theories closed under provable material implication (an innovation previously used in the investigation of first degree theories). Where proofs are simply ⊃-variants in arguments given for CL, or are repetitions or elementary variants of proofs laboured earlier, details will be omitted and at most the (⊃) theorems used cited.

A *BM*$^{\aleph}$*-theory* a is a set of wff closed under provable material implication and adjunction, i.e. whenever $\vdash_{BM}$ A⊃B and A ε a, B ε a, and whenever A ε a and B ε a, A&B ε a. Any BM$^{\aleph}$-theory is a BM-theory. A set b of wff is *BM*$^{\aleph}$*-derivable* from a set a of wff written a$\vdash_{BM\aleph}$b, iff for some A_1, ...,A_m ε a and some B_1, ...,B_n ε b, $\vdash_{BM}$ A_1& ... &A_m ... B_1v ... vB_n. A pair <a,b> is *BM*$^{\aleph}$*-maximal* iff a$\nvdash_{BM\aleph}$b and a and b are disjoint and exhaust the set of wff.

Lemma 9.8.17.
(1) If <a,b> is BM$^{\aleph}$-maximal then a is a prime BM$^{\aleph}$-theory.
(2) (Extension). If a$\nvdash_{BM\aleph}$b then there are sets a' and b' of formulae such that a ⊆ a', b ⊆ b' and <a',b'> is BM$^{\aleph}$-maximal.
(3) (Priming). Let T be the set of theorems of BM and let A be a non-theorem of BM. Then there is a BM$^{\aleph}$-theory T' such that T ⊆ T', A ∉ T' and T' is prime.
Proof: In each case replace → by ⊃ in proof of the corresponding results in Chapter 4 of RLR1.

Now let T be a prime BM$^{\aleph}$-theory as so determined by (3) and a given non-theorem A. Then Th ⊆ T and A ∉ T, where Th = {B: $\vdash_{BM}$B}. Where as usual T*=$_{df}$ {A: ~A ∉ T}, T = T*. Proof of this uses the BM theorems Bv~B and B&~B ⊃ A. In view of the theorems, A↔A, A&B ⊃ B&A and (A↔B) & (B↔C) ⊃ .A↔C, T-co-implication (A↔B ε T for wff A and B) is an equivalence relation on wff (cf. Lemma 9.8.1). As before, let [A] be the equivalence class determined by A, and Σ the set of equivalence classes.

Many of the lemmas proved earlier and their corollaries now go over intact, both in statement and in proof (except that, where noted, ⊃-analogues of →-theorems are sometimes used), namely Lemmas 9.8.2 (but replace → as main connective in higher degree system theorems by ⊃), 9.8.3, 9.8.7, 9.8.8, 9.8.9. Because T = T*, the De Morgan lattice, of Lemma 9.8.3(iv), has a truth filter, a factor of considerable impact in subsequent finitization.

Lemma 9.8.3(v). <Σ,≤,~,D> is an intensional lattice, with 'truth-filter' D = {[A]: A ε T}.
Proof involves showing that D is a consistent and exhaustive filter of Σ: Suppose a and b are in D; then there are wff A and B such that a = [A] and b = [B]. Firstly, a&b ε D iff [A&B] ε {[A]/A ε T}, i.e. iff A&B ε T, since T is a union of equivalence classes of Σ, i.e. iff A ε T and B ε T, i.e. iff [A] ε {[A]/A ε T} and [B] ε {[A]/A ε T}, i.e. iff a ε D and b ε D. Secondly, ~a ε D iff [~A] ε {[A]/A ε T}, i.e. ~A ε T, i.e. iff A ∉ T, since T = T*, i.e. iff [A] ∉ {[A]/A ε T}, i.e. iff a ∉ D.

Lemma 9.8.18.
(i) For every a ε Σ, ~a ≠ a;
(ii) For every a ε Σ, if ~a < a then a ε D.
Proof: (i) If ~a = a, then by Lemma 9.8.3(v), ~a ε D iff a ∉ D. Hence ~a ≠ a.
(ii) Since $\vdash_{BM}$~A→A → A, when ~A→A ε T, A ε T; and hence if ~[A] ≤ [A], A ε T, and so [A] ε D. So when ~a < a, a ε D.

Lemma 9.8.19. For all a ε D, if ~a ≮ a, then a&~a is the g.l.b. of Σ and av~a is the l.u.b. of Σ.
Proof: Since $\vdash_{BM}$A ⊃ .(~A→A) v (A&~A → B), when A ε T either ~A→A ε T or A&~A → B ε T. Hence, when [A] ε D either ~[A] ≤ [A] or [A]&~[A] ≤ [B]. Then, for a,b ε Σ, when a ε D either ~a < a or a&~a ≤ b. So for a ε D, if ~a ≮ a then a&~a ≤ b, for all b ε Σ, and hence a&~a is the g.l.b. of Σ. If a&~a ≤ b, for all b ε Σ, then a&~a ≤ ~b, for all b ε Σ, and hence ~~b ≤ ~(a&~a) and b ≤ av~a, for all b ε Σ. Thus, if a&~a is g.l.b. of Σ, av~a is the l.u.b. of Σ.

Lemma 9.8.20. For all a ε D, when ~a ≮ a then a is the g.l.b. of D and ~a is the l.u.b. of $\overline{D}$.
Proof: Since $\vdash_{BM}$ A&B ⊃ (~A→A) v (A→B), when A ε T and B ε T either ~A→A ε T or A→B ε T. Hence when [A] ε D and [B] ε D either ~[A] ≤ [A] or [A] ≤ [B]. Then, for a,b ε Σ, when a ε D and b ε D either ~a ≤ a or a ≤ b. So for a ε D, if ~a ≮ a then a ≤ b, for all b ε D, and hence a is the g.l.b. of D. For a ε D, if a ≤ b, for all b ε D, then, for any c ε $\overline{D}$, ~c ε D (by Lemma 9.8.3(v)), a ≤ ~c and c ≤ ~a. Also, by Lemma 9.8.3(v), a ε $\overline{D}$. Hence, if a is the g.l.b. of D then ~a is the l.u.b. of $\overline{D}$.

Corollary. There is at most one a ε D such that ~a ≮ a.

Lemma 9.8.21. If, for all a ε D, ~a < a, then D has at most 2 elements, one of which if the l.u.b. of Σ, and $\overline{D}$ has at most 2 elements, one of which is the g.l.b. of Σ, being the negation of the l.u.b. of Σ.
Proof: Let ~a < a, for all a ε D. Let a and b be 2 distinct elements of D. Since D is a filter, a&b ε D and either a&b < a or a&b < b. Let a&b < a. Since $\vdash_{BM}$(~A→A)&(A→B) ⊃ .(B→A) v (C→B), when ~A→A ε T and A→B ε T either B→A ε T or C→B ε T, and hence ~[A] ≤ [A] and [A] ≤ [B] ⇒ [B] ≤ [A] or [C] ≤ [B]. For a&b,a,c ε Σ, ~(a&b) ≤ a&b and a&b ≤ a ⇒ a ≤ a&b or c ≤ a. By above assumption, ~(a&b) ≤ a&b. Since a&b < a, a&b ≤ a and a ≰ a&b. Hence, c ≤ a, for all c ε Σ, and a is the l.u.b. of Σ. Similarly, if a&b < b then b is the l.u.b. of Σ. Hence, of any two distinct elements a of D, one is the l.u.b. of Σ, and there can be at most 2 distinct elements of D. Also, D has the l.u.b. of Σ as one of its elements, since D is non-null and if D has just

one element a then a is the l.u.b. of $\mathcal{Z}$ because $\sim a < a$ and $\sim a$ is the only other element of $\mathcal{Z}$, by Lemma 9.8.3(v).

By Lemma 9.8.3(v), $\bar{D} = \{a \in \mathcal{Z} / \sim a \in D\}$, and hence $\bar{D}$ has at most two distinct elements, one of which if the negation of the l.u.b. of $\mathcal{Z}$. However, if a is the l.u.b. of $\mathcal{Z}$ then, for all $b \in \mathcal{Z}$, $b \leq a$, and, for all $b \in \mathcal{Z}$, $\sim b \leq a$, and hence $\sim a \leq b$. Then $\sim a$ is the g.l.b. or $\mathcal{Z}$ and $\bar{D}$ has the g.l.b. of $\mathcal{Z}$ as one of its elements.

Structure result 1 If, for all $a \in D$, $\sim a < a$, then there are only two possible lattices:
(I) $\mathcal{Z} = \{l, \sim l\}$, with $\sim l < l$, $\sim(l) = \sim l$, and $D = \{l\}$.
(II) $\mathcal{Z} = \{l, a, \sim l, \sim a\}$, with $\sim l < \sim a < a < l$, $\sim(l) = \sim l$, $\sim(a) = \sim a$, and $D = \{l, a\}$.
Proof: Let $\sim a < a$, for all $a \in D$. Then, by Lemma 9.8.21, D has the l.u.b. of $\mathcal{Z}$, call it l, as one of its members, and $\bar{D}$ has the g.l.b. of $\mathcal{Z}$, $\sim l$, as one of its members.
(i) Let D and $\bar{D}$ have one element each. Then $D = \{l\}$, $\sim(l) = \sim l$, and $\sim l < l$. Note that $l = T$ and $\sim l = \bar{T}$.
(ii) Let D and $\bar{D}$ have two elements each. Call the other element of D, a. Then $\sim a \in \bar{D}$ Hence, $\sim l < \sim a < a < l$, $\sim(l) = l$ and $\sim(a) = a$.
By Lemmas 9.8.21 and 9.8.3(v), there are no possible lattices (I) and (II).

Since the structure result exhausts the possible lattices under the assumption $\sim a < a$, for $a \in D$, assume next that there is an element t of D such that $\sim t \nless t$. By the Corollary to Lemma 9.8.20, there is exactly one such element. By Lemma 9.8.20, t is the g.l.b. of D and $\sim t$ is the l.u.b. of $\bar{D}$. By Lemma 9.8.19, $t \& \sim t = \sim l$ and $t \vee \sim t = l$, where l is the l.u.b. of $\mathcal{Z}$. As shown in the proof of Lemma 9.8.21, $\sim l$ is the g.l.b. of $\mathcal{Z}$. Since D is a filter, $t < l$ and $t \in D$, $l \in D$ and hence $\sim l \in \bar{D}$. Also $t \nless \sim t$, since if $t < \sim t$ then $\sim t \in D$ which contradicts $t \in D$. Hence, the depicted *main* sub-lattice diagram results:

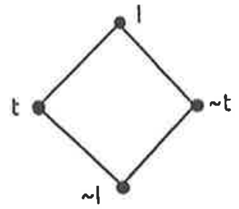

Figure 9.21 Sub-lattice diagram

The argument establishes:
Structure result 2 If, for some $t \in D$, $\sim t \nless t$, and the lattice $<\mathcal{Z}, \leq, l, D>$ has 4 elements then $\mathcal{Z} = \{l, t, \sim l, \sim t\}$, with $\sim l < t < l$, $\sim l < \sim t < l$, $t \nleq \sim t$, $\sim t \nleq t$, $\sim(l) =$

$\sim I$, $\sim(t) = \sim t$, and $D = \{I, t\}$.
In this case the lattice diagram is the sub-lattice diagram shown.

Lemma 9.8.22. There are no elements a such that $\sim I < a < t$ and no elements b such that $\sim t < b < I$.
Proof: Let a be such that $\sim I < a < t$. Since t is the g.l.b. of D, $a \in \bar{D}$ and, since $\sim t$ is the l.u.b. of $\bar{D}$, $a < \sim t$. Hence $a \leq t \& \sim t$ and $a \leq \sim I$, which contradicts $\sim I < a$. Next let b be such that $\sim t < b < I$. Since t is the l.u.b. of $\bar{D}$, $b \in D$ and, since t is the g.l.b. of D, $t < B$. Hence $t \vee \sim t \leq b$ and $I \leq b$, which contradicts $b < I$.

Lemma 9.8.23. If $t < a < I$, $t < b < I$ and $a \neq b$, then $a \& b = t$ and $a \vee b = I$.
Proof: Let $t < a < I$ and $t < b < I$.
(i) Suppose $a \& b \neq t$. Then $\sim(a \& b) < a \& b$. Since $\vdash_{BM} (\sim A \to A) \& (A \to B) \supset . (B \to A) \vee (C \to B)$, when $\sim A \to A \in T$ and $A \to B \in T$, $B \to A \in T$ or $C \to B \in T$, and hence when $\sim[A] \leq [A]$ and $[A] \leq [B]$, $[B] \leq [A]$ or $[C] \leq [B]$. Then, substituting appropriately, when $\sim(a \& b) \leq a \& b$ and $a \& b \leq a$, $a \leq a \& b$ or $I \leq a$. Hence, $a \leq a \& b$ and $a \leq b$. Also, when $\sim(a \& b) \leq a \& b$ and $a \& b \leq b$, $b \leq a \& b$ or $I \leq b$. Then, $b \leq a \& b$, $b \leq a$ and hence $a = b$. Transposing then yields $a \neq b \supset a \& b = t$.
(ii) Let $a \vee b \neq I$. Since $a \neq t$, $\sim a < a$, and since $b \neq t$, $\sim b < b$. Applying $\sim[A] \leq [A]$ and $[A] \leq [B] \supset [B] \leq [A]$ or $[C] \leq [B]$, $\sim a \leq a$ and $a \leq a \supset b \leq a \vee b \leq a$ or $I \leq a \vee b$, and hence $a \vee b \leq a$ and $b \leq a$. Also, $\sim b \leq b$ and $b \leq a \vee b \supset a \vee b \leq b$ or $I \leq a \vee b$, and hence $a \vee b \leq b$, $a \leq b$ and $a = b$. Transposing then yields $a \neq b \Longrightarrow a \vee b = I$.

Corollary. If $t < a < I$, $t < b < I$ and $a \neq b$ then $a \npreceq b$ and $b \npreceq a$.
Proof: Let $t < a < I$, $t < b < I$ and $a \neq b$. If $a \leq b$ then $a \& b = a$, which contradicts the lemma. Hence, $a \npreceq b$. If $b \leq a$ then $a \& b = b$, which contradicts the lemma. Hence, $b \npreceq a$.

Lemma 9.8.24. There are at most 2 elements a such that $t < a < I$.
Proof: Supposing otherwise, a, b, c are three distinct elements such that $t < a < I$, $t < b < I$, and $t < c < I$. By Lemma 9.8.23, the sub-lattice diagram below must be shown. But this sub-lattice is not distributive, since $a \& (b \vee c) = a \& I = a$, whereas $(a \& b) \vee (a \& c) = t \vee t = t$. Hence, there are at most 2 elements a such that $t < a < I$.

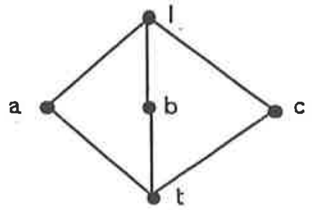

Figure 9.22 Non-distributive sub-lattice diagram

Structure result 3 If, for some t ε D, ~t ≮ t, and the lattice <Σ, ≤, ~, D> has 6 elements then Σ = {I, t, a, ~I, ~t, ~a}, with ~I < a < t < I, ~I < ~a < ~t < I, ~a < a, t ≰ ~a, ~a ≰ t, a ≰ ~t, ~t ≰ a, t ≰ ~t, ~t ≰ t, ~(I) = ~I, ~(t) = ~t, ~(a) = ~a, and D = {I,t,a}.
Proof: Again, the main sub-lattice diagram applies. D and $\overline{D}$ each have one additional element, by Lemma 9.8.3(v). Call the element in D, a. Then ~a ε D. Since a ε D, and ~a ε $\overline{D}$, t < a < I, and ~I < ~a < ~t. Also, ~t < I, and ~I < t. Since a ε D, and a ≠ t, ~a < a, by Lemma 9.8.20 Corollary. We have, from the diagram, ~t ≰ t and t ≰ ~t. Hence, t ≰ ~a, since t ≤ ~a, t ≤ ~t. Also, a ≰ ~t, since if a ≤ ~t then t ≤ ~a. As well, ~a ≰ t, since if ~a ≤ t then ~I < ~a < t, contradicting Lemma 9.8.22. Hence, ~t ≰ a, since if ~t ≤ a then ~a ≤ t.

Structure result 4 If, for some t ε D, ~t ≮ t, and the lattice <Σ, ≤, ~, D> has 8 elements then Σ = {I, t, a, b, ~I, ~t, ~a, ~b}, with ~I < t < a < I, ~I < ~a < ~t < I, t < b < I, ~I < ~b < ~t, ~a < a, ~b < b, t ≰ ~t, ~t ≰ t, t ≰ ~a, ~a ≰ t, t ≰ ~b, ~b ≰ t, ~t ≰ a, a ≰ ~t, ~t ≰ b, b ≰ ~t, a ≰ b, b ≰ a, ~a ≰ ~b, ~b ≰ ~a, a ≰ ~b, ~b ≰ a, ~a ≰ b, b ≰ ~a, ~(I) = ~I, ~(t) = ~t, ~(a) = ~(a) = ~(b) = ~b, and D = {I, t, a, b}.
Proof: Again, the main sub-lattice diagram applies, D and $\overline{D}$ each have two additional elements. Call the additional elements of D, a and b. Then ~a ε $\overline{D}$ and ~b ε $\overline{D}$. Since a,b ε D, and ~a,~b ε $\overline{D}$, t < a < I, t < b < I, ~I < ~a < ~t, ~I < ~b < ~t. Also, ~t < I, and ~I < t. Since a,b ε D, a ≠ t and b ≠ t, ~a < a and ~b < b, by Lemma 9.8.20 Corollary. By the diagram, t ≰ ~t and ~t ≰ t, t ≰ ~a, since if t ≤ ~a then t ≤ ~t. Similarly, t ≰ ~b. a ≰ ~t, since if a ≤ ~t then t ≤ ~a. Similarly, b ≰ ~t. And ~a ≰ t, since if ~a ≤ t then ~I < ~a < t, contradicting Lemma 9.8.22. Similarly, ~b ≰ t. ~t ≰ a, since if ~t ≤ a then ~a ≤ t. Similarly, ~t ≰ b, a ≰ b and b ≰ a, by Lemma 9.8.23 Corollary. Also, ~a ≰ ~b and ~b ≰ ~a, since if ~a ≤ ~b then b ≤ a, and if ~b ≤ ~a then a ≤ b. a ≰ ~b, since if a ≤ ~b then t ≤ ~b and ~b ε T_E, T_E being a filter. b ≰ ~a, since if b ≤ ~a then a ≤ ~b.
Let ~b ≤ a. Then a and ~t have no g.l.b. since if a&~t is the g.l.b. then

~a ≤ a&~t, ~b ≤ a&~t, ~av~b ≤ a&~t, ~t ≤ a&~t (by Lemma 9.8.23, ~(a&b) = ~t) and ~t ≤ a. Hence, ~b ≰ a and then ~a ≰ b.

Structure result 5 If, for some t ε d, ~t ≰ t, then the three lattices of structure results 2, 3 and 4 are the only possible ones.
Proof: By the argument for Structure result 2, there are at least 4 elements in the lattice, <Σ,≤,~,D>. By Lemma 9.8.3(v), further elements must be introduced in pairs, one element of the pair in D and the other element of the pair in D̅. Since, for each further element a of D, t < a < I, using Lemma 9.8.24, there are at most 2 further elements other than t and I, that can be added to D. Hence D has a maximum possible cardinality of 4 and Σ has a maximum possible cardinality of 8, leaving the possible lattices as those cited.

Structure results 1-5 determine the 5 lattices of equivalence classes of Σ. For the purposes of proving completeness, we need 3 prime T-BM-theories, which provide valuations for the 3 worlds of the model structure. Thus we still require, in addition to T, 2 prime T-BM-theories, which we will call a and a*, in anticipation of the worlds for which they will provide valuations. a and a* will be defined for each of the 5 lattices. a and a* will be related according to the definition of * for unions of equivalence classes of Σ. {[A]/A ε a} and {[A]/A ε a*} must both be prime filters of <Σ,≤,~,D>. The filters {[A]/A ε a} and {[A]/A ε a*} will both be non-trivial, i.e. ≠ ∅ and ≠ Σ, for reasons already outlined for CL. This will ensure that the T-BM-theories a and a* are non-degenerate.

The definition of a and a* for the 5 lattices are as follows:

(I) 2-element lattice: a = a* = T = Σ, there being only one non-trivial prime filter.

(II) 4-element lattice with ~a < a, for all a ε D (see result 1): a = I ∪ a ∪ ~a, a* = I, there being three non-trivial prime filters.

(III) 4-element lattice with ~a ≰ a, for some a ε D (see result 2): a = a* = I ∪ ~t, there being only two non-trivial prime filters.

(IV) 6-element lattice: a = I ∪ a ∪ ~t ∪ ~a, a* = I ∪ ~t, there being three non-trivial prime filters.

(V) 8-element lattice: a = I ∪ a ∪ ~t ∪ ~a, a* = I ∪ b ∪ ~t ∪ ~b, there being three non-trivial prime filters.

In each case a* = (a)*, according to the definition of * for unions of equivalence classes of Σ. Further, in each case a and a* are prime T-BM-theories.

For the model structure for BM, let the valuation ν be determined as follows: For all sentential variables p, ν(p,T) = 1 iff p ε T, ν(p,a) = 1 iff p ε a, and ν(p,a*) = 1 iff p ε a*.

Theorem 9.8.5. (Interpretation) The valuation v extends to an interpretation of I for all wff, such that, for every wff A, I(A,T) = 1 iff A ε T, I(A,a) = 1 iff A ε a and I(A,a*) = 1 iff A ε a*.

Proof: The proof is by induction from the given basis equating I with v for sentential variables. The induction steps for ~ and & are standard: that for ~ uses the definition of * and the identities a** = a and T* = T, and that for & the fact that T, a and a* are appropriate theories. The hard work concerns the steps for → for which there are three cases:

I) at T: I(A→B,T) = 1 iff (I(A,T) = 1 ⊃ I(B,T) = 1) & (I(A,a) = 1) ⊃ I(B,a) = 1) & (I(A,a*) = 1 ⊃ I(B,a*) = 1), i.e. iff (A ε T ⊃ B ε T) & (A ε a ⊃ B ε a) & (A ε a* ⊃ B ε a*), i.e. provisionally iff A→B ε T.

RHS to LHS: Since T, a and a* are all T-BM-theories, when A→B ε T, A ε x ⊃ B ε x for x = T, a and a*.

LHS to RHS: To show that when (A ∉ T or B ε T) and (A ∉ a or B ε a) and (A ∉ a* or B ε a*), A→B ε T, it suffices to prove

A ∉ T & A ∉ a & A ∉ a* ⊃ A→B ε T ... (1)
A ∉ T & A ∉ a & B ε a* ⊃ A→B ε T ... (2)
A ∉ T & B ε a & A ∉ a* ⊃ A→B ε T ... (3)
A ∉ T & B ε a & B ε a* ⊃ A→B ε T ... (4)
B ε T & A ∉ a & A ∉ a* ⊃ A→B ε T ... (5)
B ε T & A ∉ a & B ε a* ⊃ A→B ε T ... (6)
B ε T & B ε a & A ∉ a* ⊃ A→B ε T ... (7)
B ε T & B ε a & B ε a* ⊃ A→B ε T ... (8)

ad (1). Let A ∉ T, A ∉ a and A ∉ a*. For all 5 lattices A ε ~I, the g.l.b. of the lattice. Hence, for every wff B, [A] ≤ [B] and A→B ε T.

ad (2). Let A ∉ T, A ∉ a and B ε a*. Then, for lattices (I), (II), (III) and (IV), A ε ~I, and for lattice (V), A ε ~I ∪ ~b. For all 5 lattices, [A] ≤ [B] and A→B ε T. So it is similarly, under the conditions established, in the remaining cases (3)-(8).

ad (3). Let A ∉ T, B ε a and A ∉ a*. Then, for lattices (I) and (III), A ε ~I, and, for lattices (II), (IV) and (V), A ε ~I ∪ ~A.

ad (4). Let A ∉ T, B ε a and B ε a*. Then, for lattices (I) and (II), B ε I, and, for lattices (III), (IV) and (V), B ε I ∪ ~t.

ad (5). Let B ε T, A ∉ a and A ∉ a*. Then, for lattices (I) and (II), A ε ~I, and, for lattices (III), (IV) and (V), a ε ~I ∪ t.

ad (6). Let B ε T, A ∉ a and B ε a*. Then, for lattices (I), (II), (III) and (IV), B ε I, and, for lattice (V), B ε I ∪ b.

ad (7). Let B ε T, B ε a and A ∉ a*. Then, for lattices (I) and (III), b ε I, and for lattices (II), (IV) and (V), B ε I ∪ a.

ad (8). Let B ε T, B ε a and B ε a*. For all 5 lattices, B ε I.

II) at a: I(A→B,a) = 1 iff (I(A,a*) = 1 ⊃ (∀c ε K)I(B,c) = 1) & ((∃b ε K) I (A,b) = 1 ⊃ I(B,a) = 1); i.e. iff (A ε a* ⊃ B ε T & B ε a & B ε a*) & (A ε T v A ε a v A ε a* ⊃ B ε a), i.e. provisionally, iff A→B ε a.

RHS to LHS: It suffices to show the following:

$A\to B\ \varepsilon\ a\ \&\ A\ \varepsilon\ a^* \supset B\ \varepsilon\ T$ ... (1)

$A\to B\ \varepsilon\ a\ \&\ A\ \varepsilon\ a^* \supset B\ \varepsilon\ a$ ... (2)

$A\to B\ \varepsilon\ a\ \&\ A\ \varepsilon\ a^* \supset B\ \varepsilon\ a^*$ ... (3)

$A\to B\ \varepsilon\ a\ \&\ A\ \varepsilon\ T \supset B\ \varepsilon\ a$ ... (4)

$A\to B\ \varepsilon\ a\ \&\ A\ \varepsilon\ a \supset B\ \varepsilon\ a$ ... (5)

$A\to B\ \varepsilon\ a\ \&\ A\ \varepsilon\ a^* \supset B\ \varepsilon\ a$ ... (6)

<u>ad</u> (1). Since $\vdash_{BM}{\sim}B \ldots .A\to B \to {\sim}A$, ${\sim}B\ \varepsilon\ T \supset A\to B \to {\sim}A\ \varepsilon\ T$, and hence ${\sim}B\ \varepsilon\ T \supset (A\to B\ \varepsilon\ a \supset {\sim}A\ \varepsilon\ a)$. Then $A\to B\ \varepsilon\ a\ \&\ {\sim}A \not\varepsilon\ a \supset {\sim}B \not\varepsilon\ T$, and hence $A\to B\ \varepsilon\ a\ \&\ A\ \varepsilon\ a^* \supset B\ \varepsilon\ T$.

<u>ad</u> (2). Since $\vdash_{BM}A\to B \to .{\sim}A\vee B$, $A\to B\ \varepsilon\ a \supset {\sim}A\ \varepsilon\ a \vee B\ \varepsilon\ a$, and hence $A\to B\ \varepsilon\ a\ \&\ A\ \varepsilon\ a^* \supset B\ \varepsilon\ a$.

<u>ad</u> (3). Since $\vdash_{BM}(A\to B)\&{\sim}B \to {\sim}A$, $A\to B\ \varepsilon\ a\ \&\ {\sim}B\ \varepsilon\ a \supset {\sim}A\ \varepsilon\ a$, and hence $A\to B\ \varepsilon\ a\ \&\ A\ \varepsilon\ a^* \supset B\ \varepsilon\ a^*$.

<u>ad</u> (4). Since $\vdash_{BM}A \supset .A\to B \to B$, $A\ \varepsilon\ T \supset A\to B \to B\ \varepsilon\ T$, and hence $A\ \varepsilon\ T \supset (A\to B\ \varepsilon\ a \supset B\ \varepsilon\ a)$. Then, $A\to B\ \varepsilon\ a\ \&\ A\ \varepsilon\ T \supset B\ \varepsilon\ a$.

<u>ad</u> (5). Since $\vdash_{BM}A\&(A\to B) \to B$, $A\to B\ \varepsilon\ a\ \&\ A\ \varepsilon\ a \supset B\ \varepsilon\ a$.

<u>ad</u> (6). Since $\vdash_{BM}A\to B \to .{\sim}A\vee B$, $A\to B\ \varepsilon\ a \supset {\sim}A\ \varepsilon\ a \vee B\ \varepsilon\ a$, and hence $A\to B\ \varepsilon\ a\ \&\ A\ \varepsilon\ a^* \supset B\ \varepsilon\ a$.

LHS to RHS: To prove that when $(A \not\varepsilon\ a^*$ or $(B\ \varepsilon\ T$ and $B\ \varepsilon\ a$ and $B\ \varepsilon\ a^*))$ and $((A \not\varepsilon\ T$ and $A \not\varepsilon\ a$ and $A \not\varepsilon\ a^*)$ or $B\ \varepsilon\ a)$ then $A\to B\ \varepsilon\ a$, amounts to proving the following:

$A \not\varepsilon\ a^*\ \&\ A \not\varepsilon\ T\ \&\ A \not\varepsilon\ a \supset A\to B\ \varepsilon\ a$ ... (1)

$A \not\varepsilon\ a^*\ \&\ B\ \varepsilon\ a \supset A\to B\ \varepsilon\ a$ ... (2)

$B\ \varepsilon\ T\ \&\ B\ \varepsilon\ a\ \&\ B\ \varepsilon\ a^*\ \&\ A \not\varepsilon\ T\ \&\ A \not\varepsilon\ a\ \&\ A \not\varepsilon\ a^* \supset A\to B\ \varepsilon\ a$ ... (3)

$B\ \varepsilon\ T\ \&\ B\ \varepsilon\ a\ \&\ B\ \varepsilon\ a^* \supset A\to B\ \varepsilon\ a$ ... (4)

<u>ad</u> (1) and (3). Since $\vdash_{BM}{\sim}A\&(A\to B) \supset .{\sim}A \to A\vee(A\to B)$, ${\sim}A\ \varepsilon\ T\ \&\ A\to B\ \varepsilon\ T \supset .{\sim}A \to A\vee(A\to B)\ \varepsilon\ T$, and hence ${\sim}A\ \varepsilon\ T\ \&\ A\to B\ \varepsilon\ T \supset ({\sim}A\ \varepsilon\ a \supset .A\ \varepsilon\ a \vee A\to B\ \varepsilon\ a)$. Then $A \not\varepsilon\ T\ \&\ A \not\varepsilon\ a^*\ \&\ A \not\varepsilon\ a\ \&\ A\to B\ \varepsilon\ T \supset A\to B\ \varepsilon\ a$. But in establishing $I(A\to B,T) = 1$ iff $A\to B\ \varepsilon\ T$ above, we established (in case (1)) that $A \not\varepsilon\ T\ \&\ A \not\varepsilon\ a\ \&\ A \not\varepsilon\ a^* \supset A\to B\ \varepsilon\ T$. Hence, $A \not\varepsilon\ T\ \&\ A \not\varepsilon\ a\ \&\ A \not\varepsilon\ a^* \supset A\to B\ \varepsilon\ a$. (3) now follows.

<u>ad</u> (2). Since $\vdash_{BM}A\&{\sim}A\&B \to .A\to B$, $A\ \varepsilon\ a\ \&\ {\sim}A\ \varepsilon\ a\ \&\ B\ \varepsilon\ a \supset A\to B\ \varepsilon\ a$, and hence $A \not\varepsilon\ a^*\ \&\ B\ \varepsilon\ a\ \&\ A\ \varepsilon\ a \supset A\to B\ \varepsilon\ a$. Since $\vdash_{BM}A \supset .{\sim}A\ \&B \to .A\vee(A\to B)$, $A\ \varepsilon\ T \supset {\sim}A\&B \to .A\vee(A\to B)\ \varepsilon\ T$, and hence $A\ \varepsilon\ T \supset ({\sim}A\ \varepsilon\ a\ \&\ B\ \varepsilon\ a \supset A\ \varepsilon\ a \vee A\to B\ \varepsilon\ a)$. Then $A \not\varepsilon\ a^*\ \&\ B\ \varepsilon\ a\ \&\ A \not\varepsilon\ a\ \&\ A\ \varepsilon\ T \supset A\to B\ \varepsilon\ a$. By (1) above, $A \not\varepsilon\ a^*\ \&\ B\ \varepsilon\ a\ \&\ A \not\varepsilon\ a\ \&\ A \not\varepsilon\ T \supset .A\to B\ \varepsilon\ a$. Assembling these 3 results, $A \not\varepsilon\ a^*\ \&\ B\ \varepsilon\ a \supset A\to B\ \varepsilon\ a$.

<u>ad</u> (4). Since $\vdash_{BM}B\&(A\to B) \supset .B \to {\sim}B\vee(A\to B)$, $B\ \varepsilon\ T\ \&\ A\to B\ \varepsilon\ T \supset B \to {\sim}B\vee(A\to B)\ \varepsilon\ T$, and hence $B\ \varepsilon\ T\ \&\ A\to B\ \varepsilon\ T \supset (B\ \varepsilon\ a \supset {\sim}B\ \varepsilon\ a \vee A\to B\ \varepsilon\ a)$. Then $B\ \varepsilon\ T\ \&\ B\ \varepsilon\ a\ \&\ B\ \varepsilon\ a^*\ \&\ A\to B\ \varepsilon\ T \supset A\to B\ \varepsilon\ a$. But in establishing $I(A\to B,T) = 1$ iff $A\to B\ \varepsilon\ T$ above, we established (in case(8)) that $B\ \varepsilon\ T\ \&$

B ε a & B ε a* ⊃ A→B ε T. Hence, B ε T & B ε a & B ε a* ⊃ A→B ε a.
<u>III) at a*:</u> I(A→B,a*) = 1 iff (I(A,a) = 1 ⊃ (∀c ε K)I(B,c) = 1) & ((∃b ε K)I(A,b) = 1 ⊃ I(B,a*) = 1), i.e. iff (A ε a ⊃ B ε T & B ε a & B ε a*) & (A ε T v A ε a v A ε a* ⊃ B ε a*), i.e. provisionally, iff A→B ε a*.

RHS to LHS: It is enough to prove the following:

A→B ε a* & A ε a ⊃ B ε T ... (1)
A→B ε a* & A ε a ⊃ B ε a ... (2)
A→B ε a* & A ε a ⊃ B ε a* ... (3)
A→B ε a* & A ε T ⊃ B ε a* ... (4)
A→B ε a* & A ε a* ⊃ B ε a* ... (5)

<u>ad</u> (1). Since $\vdash_{BM}$~B ⊃ .A→B → ~A, ~B ε T ⊃ A→B → ~A ε T, and hence ~B ε T ⊃ .A→B ε a* ⊃ ~A ε a*. Then A→B ε a* & ~A ∉ a* ⊃ ~B ∉ T, and hence A→B ε a* & A ε a ⊃ B ε T.

<u>ad</u> (2). Since $\vdash_{BM}$(A→B)&~B → ~A, A→B ε a* & ~B ε a* ⊃ ~A ε a*, and hence A→B ε a* & A ε a ⊃ B ε a.

<u>ad</u> (3). By $\vdash_{BM}$A → B →. ~A v B.

<u>ad</u> (4). Since $\vdash_{BM}$A ⊃ .A→B → B, A ε T ⊃ A→B → B ε T, and hence A ε T ⊃ (A→B ε a* ⊃ B ε a*). Then, A→B ε a* & A ε T ⊃ B ε a*.

<u>ad</u> (5). By $\vdash_{BM}$(A→B)&A → B.

LHS to RHS: To prove that when (A ∉ a or (B ε T and B ε a and B ε a*)) and ((A ∉ T and A ∉ a and A ∉ a*) or B ε a*) then A→B ε a*, amounts to proving the following:

A ∉ a & A ∉ T & A ∉ a* ⊃ A → B ε a* ... (1)
A ∉ a & B ε a* ⊃ A→B ε a* ... (2)
B ε T & B ε a & B ε a* & A ∉ T & A ∉ a & A ∉ a* ⊃ A→B ε a* ... (3)
B ε T & B ε a & B ε a* ⊃ A→B ε a* ... (4)

<u>ad</u> (1) and (3). Since $\vdash_{BM}$~A&(A→B) ⊃ (~A → Av(A→B)), ~A ε T & A→B ε T ⊃ ~A → Av(A→B) ε T, and hence ~A ε T & A→B ε T ⊃ (~A ε a* ⊃ A ε a* v A→B ε a*). Then A ∉ T & A ∉ a & A ∉ a* & A→B ε T ⊃ A→B ε a*. In establishing I(A→B,T) = 1 iff A→B ε T above, we established (in case (1)) that A ∉ T & A ∉ a & A ∉ a* ⊃ A→B ε T. Hence, A ∉ T & A ∉ a & A ∉ a* ⊃ A→B ε a*. (3) now follows also.

<u>ad</u> (2). Since $\vdash_{BM}$A&~A&B → .A→B, A ε a* & ~A ε a* & B ε a* ⊃ A→B ε a, and hence A ∉ a & B ε a* & A ε a* ⊃ A→B ε a*. Since $\vdash_{BM}$A ⊃ .~A&B → .Av(A→B), A ε T ⊃ ~A&B → .Av(A→B) ε T, and hence A ε T ⊃ (~A ε a* & B ε a* ⊃ A ε a* v A→B ε a*). Then A ∉ a & B ε a* & A ∉ a* & A ε T ⊃ A→B ε a*. By (1) above, A ∉ a & B ε a* & A ∉ a* & A ∉ T ⊃ A→B ε a*. Assembling these 3 results, A ∉ a & B ε a* ⊃ A→B ε a*.

<u>ad</u> (4). Since $\vdash_{BM}$B&(A→B) ⊃ .B → .~Bv(A→B), B ε T & A→B ε T ⊃ B → ~Bv(A→B) ε T, and hence B ε T & A→B ε T ⊃ (B ε a* ⊃ .~B ε a* v A→B ε a*). Then B ε T & B ε a & B ε a* & A→B ε T ⊃ A→B ε a*. In

establishing I(A→B,T) = 1 iff A→B ε T, we established (in case (8)) that B ε T & B ε a & B ε a* ⊃ A→B ε T. Hence, B ε T & B ε a & B ε a* ⊃ A→B ε a*.

Theorem 9.8.6. (Completeness) For every wff A, if A is valid in the model structure for BM, $\vdash_{BM}$A.
Proof is as for Theorem 9.8.3: Let A be a non-theorem of BM. By an extension lemma, there is a prime BMℵ-theory T containing all the theorems of BM and such that A ∉ T. By Theorem 9.8.5, T (= T*), together with a and a*, as defined for each of the 5 lattices, determines a valuation *v* for the BM model structure, which extends to an interpretation I such that, for every wff B, I(B,T) = 1 iff B ε T. Hence, I(A,T) = 0, for this model; and so A is invalid in the BM model structure.

Corollary. Matrix M_0 is characteristic for BM.

Notes

1 §§1-6 incorporate Meyer and Routley [1973a], which in turn builds on, but does not depend on, Meyer and Routley [1972], where negation-free entailment logics are treated. In contrast to [1972], in [1973a] the adequacy of the semantics is not presupposed, but derived, and negation is encompassed and shown to involve no serious complications.

2 The connections made, which hold for a sweeping class of affixing systems, admit of course of further generalisation. The universal worlds semantics of Routley and Meyer [1976b] may be variously connected with general matrix semantics in the style of Lindenbaum.

3 The number of additional connectives required for free-wheeling algebraic analyses already indicates that such analyses are somewhat more limited than semantical analyses. If the addition of Boolean negation renders the analysis of negation correspondingly more straightforward, that will reveal that algebraic method (as presently conceived) are also more classically biased.

4 Much of this section follows, with minor adaption, C. Mortensen's work on finite algebras.

5 For one argument that the matrices are characteristic, see ENT1, p.470; for another, semantically grounded argument, see Brady [1982].

6 For a study of L3 see Brady [1982].

7 The simplification and the detour are based on the work of R. Brady.

8 Recall here and in what follows that a and a* are constants, signifying worlds in K.

9 The 4-semantics can be alternatively obtained from the prime De Morgan monoid, represented by the CL matrices, by two algebraic constructions and thus directly from the CL model structure. Firstly, a model structure is constructed from the De

Morgan monoid in the same fashion as in Theorem 9.4.2 above, except that the set K of set-ups is taken to be the set of all non-trivial prime filters instead of the set of all prime filters. The model structure, so constructed, is the model structure <u>CL'</u>. Secondly, a De Morgan monoid is constructed from the model structure by forming the algebra of ranges, as done in §9.4. The 4-semantics in fact amounts to this constructed De Morgan monoid, which is isomorphic to the original De Morgan monoid.

10 Another such chain comprises Sugihara matrices, a sequence of which RM3 is a member: See Mortensen [1980].

11 Certainly also, because they are finite-valued, the logics violate the rationality requirement of Routley and Wolf [1974]. The reason usually is that such logics, because of the axiom, that induce finiteness, violate the requirement of H-coherence, discussed in §3.7 in RLR1.

12 System BN4, and its analyses, are due to Brady and developed in his [1982], where fuller details may be found. Connective V may either be defined or taken as primitive.

13 Note however that Ass, $A\&(A\rightarrow B)\rightarrow B$, is *not* a theorem of BN4. (Take $A = n$ and $B = b$ or $= f$ in M4.) Thus BN4 is not an extension of C (p.289 of RLR1).

14 The detailed argument for this and for several other points that follow may be found in R.T. Brady, 'Truth-preservation in the Routley-Meyer semantics', on which this subsection is based.

As in §5.5 the matrix theory may be enlarged by classical negation '$\bar{\ }$' with matrix

	t	b	n	f .
$\bar{\ }$	f	n	b	t

Then, as before, $A>B$ has the same value as $\bar{A} \vee B$. Where constant f with invariant value f is also adjoined, $\bar{A}$ has the same value as $A > f$.

15 If for every Lm.s. $T = T^*$ or $T = u$ and for some Lm.s. $T \neq u$, then L is a conservative extension of classical logic S.

16 Rule RTP4 does not preserve the taking of either of the values, t or b, in TP_4, and hence one has to be wary of including it in such extensions of TP4 as the systems L of §5.5. If RTP4 is omitted, it can be replaced by taking all the theorems cited which involve negation (except the first) as extra axioms; in this resulting system all the rules preserve the taking of either of the values, t or b, in TP_4.

17 The problem of axiomatizing finite relevance-establishing matrices to obtain finite-valued relevant logics, and, in particular, of axiomatising the crystal lattice CL and M_0 was suggested by R. Routley. The solutions reported upon were obtained by R. Brady, upon whose work the two following subsections are almost entirely based. Routley now conjectures that more direct proofs which shortcut the algebraic detour can be obtained.

18 The second scheme has already been encountered, in the previous section, in the axiomatization of RM3, while the first scheme is a weakening of the other axiom scheme adopted for RM3.

19 Recall that for a lattice <L, ~>, F is a *filter* of <L, ~> iff, for all a,b ε L, a ε F and b ε F iff a&b ε F, where a&b is the meet of a and b in L. Also, F is a *prime filter* of <L, ~> iff, F is a filter of <L, ~> and, for all a,b ε L, avb ε F iff a ε F or b ε F, where avb is the join of a and b in L. The lemma established follows more directly using well-known equivalent characterizations of filter and prime-filter.

We now prove soundness for this semantics.

Theorem 12.3.3. If A is a theorem of $DJ^{to\otimes}$ then A is valid in the $DJ^{to\otimes}$-semantics.

Proof: The axioms of DJ are $DJ^{to\otimes}$-valid and the rules of DJ preserve $DJ^{to\otimes}$-validity, as in RLR1, for the unreduced semantics. Note that the Hereditary and Entailment Lemmas continue to hold, with the three additions, given their truth conditions and the semantic postulates p6 and p7. Further, the extra rules, R4, R5 and R6, for $DJ^{to\otimes}$ can be shown to preserve $DJ^{to\otimes}$-validity, using the truth conditions for 't', 'o' and '⊗', together with the above two Lemmas. The cases for 't' and 'o' can be found in RLR1, and the case for '⊗' is similar.

The *second step* is to remove from the $DJ^{to\otimes}$-semantics the truth conditions for 't', 'o' and '⊗', yielding what we will call the DJ-semantics. The following theorem is then clear.

Theorem 12.3.4. If A does not contain 't', 'o' nor '⊗', then if A is valid in the $DJ^{to\otimes}$-semantics then A is valid in the DJ-semantics.

The *third step* is to prove completeness for DJ with respect to the DJ-semantics. This is basically what is set out in §2.1 or in RLR1, pp.298-318. The only extra thing to check is the satisfaction of the postulates p6 and p7. Since the canonical O_c is the set of all prime DJ-theories containing all the theorems of DJ and $a \le b$ iff $a \subseteq b$, for prime DJ-theories a and b, then p6 holds. Since $R_c cda$ iff, for all formulae A,B, if $A \rightarrow B \in c$ and $A \in d$ then $B \in a$, for prime DJ-theories c, d and a, then p7 holds. Hence, the following theorem applies.

Theorem 12.3.5. If A is valid in the DJ-semantics then A is a theorem of DJ.

We are now in a position to put the previous theorems together.

Theorem 12.3.6. A is a theorem of DJ iff A is derivable in LDJ.

Proof: L→R. By Theorem 12.3.1. R→L. By Theorems 12.3.2-12.3.5, since A does not contain 't', 'o' nor '⊗'.

§12.4. Closed Set Logic

Chris Mortensen

(i) Introduction

Perhaps the most natural paraconsistent logic from the point of view of semantics is Closed Set Logic, CSL. CSL is paraconsistent but irrelevant, and thus looms as a serious rival to the whole class of relevant paraconsistent logics. In the following sections, we begin by describing its natural semantics, as a topological dual of intuitionist logic which has an equally natural semantics but which is neither relevant nor paraconsistent. We then review results on proof theory, due to Nicolas Goodman [1981]. The criticism that CSL lacks a reasonable implication is discussed and rejected. Then algebraic and category-theoretic representations due to Mortensen [1995], Lavers and James are sketched. (See Chapter 11, with Peter Lavers; Chapter 12, by William James, and Chapter 13 on duality and connections with the Routley star.)

(ii) Topology and Logic

If we consider the classical representation of dynamical systems in terms of phase spaces, we see that analysis is employed ubiquitously in the formulation and solution of differential equations of motion. Analysis is conducted on a topological space of points. Indeed, analysis is typically conducted on open sets of points: one proves for example things such as: if f and g are continuous functions on an open interval I then so are f+g, f-g etc. Thus analysis can seem to have an intuitionist character, since after all intuitionist logic is the logic of open sets (OSL).

Intuitionism begins with the assumption that *whatever is true, is true on open sets of points.* For example, if f is continuous on a trajectory which is an open set O of points, then 'f is continuous at x', is true at all t in O. Happily, this respects natural disjunction v and conjunction & operations, since open sets are closed w.r.t. unions and (finite) intersections. Open sets are not however closed w.r.t. set complement, so whatever negation is, it will not be Boolean in nature. Hence intuitionism takes instead *the union of all open sets included in the set complement.* This is guaranteed to exist by the fact that open sets are closed under arbitrary unions. From this, it is easy to see that the boundaries of open sets, at least those open sets which are not also closed, are places where both a proposition and its negation will fail to hold. This is the characteristic intuitionist property of negation. Intuitionist implication further follows from the definition: $A \rightarrow B =_{df} -A \cup B$. This yields an implication with reasonable properties including modus ponens. Furthermore, intuitionist negation is recovered by the special case that: $A \rightarrow \{\} = -A \cup \{\} = -A$. Equivalently, $A \rightarrow B$ is definable as $\bigcup\{C: C \cap A$ is a subset of $B\}$.

However, closed sets can also play a significant role. When we include the boundaries of sets, we are including the places where otherwise discontinuous change takes place, where the left hand limit fails to equal the right hand limit. The capacity for paraconsistency at these points is reflected in the nature of negation. The natural topological dual of the intuitionist idea is to begin with the assumption that *whatever is true, is true on closed sets of* *points*. This supposition similarly supports natural disjunction and conjunction operations, since (finite) unions and (arbitrary) intersections of closed sets are closed. However, negation needs to be tweaked, though in a different way from intuitionism. This is achieved by taking it to be ***the intersection of all closed sets including the set complement***. This is guaranteed to exist by the fact that arbitrary intersections of closed sets are closed. But now we can see how it is that closed set negation is paraconsistent. For supposing that a proposition holds all over a closed (and non-open) set of points including its boundary, its negation holds on the set complement and its boundary, which is the same boundary. That is, from the paraconsistent point of view, ***boundaries are places where propositions and their negations hold.*** If we say that a proposition B follows from another A in a model, just in case the set of worlds at which A holds is a subset of those at which B holds, then we can say that it is not the case that everything follows from a contradiction, because if A holds on a closed set C, then A&-A holds at the boundary of C. Hence no proposition holding on any closed set not containing C, follows from A&-A. This is the mark of a paraconsistent logic.

This is the ideal setting for classical discontinuous change. Classical dynamics certainly recognizes discontinuity, but deals with it piecewise, without any mathematics at the discontinuity itself. This is particularly true in the case of quantum measurement. The natural mark of a causal process is a continuous path in phase space, and quantum measurement is causal in some sense, yet it is not continuous if regarded as consistent. The simplest thing to do is to identify the classically distinct values of the function at the boundary or boundaries. This ensures that the left hand limit of the function equals its right hand limit, which is continuity. The space of values of the function at the point is thought of as 'rolled up' at the point of discontinuity so that the classically distinct values are identified on the surface of the cylinder, retaining much of its functionality as the additive group of reals modulo the circumference of the cylinder. The inconsistency arises by insisting that *as well* the value space must retain its character as the set of (classically distinct) reals. This means that the distinctness of the values will be retained as a mark of the classical discontinuity of the process.

The naturalness of this semantics for CSL and the fact that it is topologically dual to OSL, make it a real rival to the relevant logics, which struggle with the Routley star to provide their account of negation.

It should be recorded that open set implication does not topologically dualize to a kind of implication in the closed set environment. The natural CSL dual is given by:

$A \div B =_{df} A \cap -B$. Equivalently, one can define $A \div B$ as the intersection set of {C: A is a subset of C∪B}. This is known as *pseudo-difference* (see Curry [1963], McKinsey-Tarski [1946]). Modus ponens, which is arguably a core feature of any kind of implication, fails for ÷. Indeed, $A \div A = \{\}$, which thus holds nowhere. On the other hand, this has to be seen in the properly dual perspective. After all, intuitionist OSL fails to have a pseudo-difference operation. Hence for example OSL negation cannot be recovered in the natural way it can from pseudo-difference, by $-A =_{df} T \div A$, where T is the whole space. On the other hand, Boolean logic, the logic of general sets (or of clopen sets), has both.

(iii) Proof Theory

Proof theory for CSL has been worked out by Nicolas Goodman [1981]. The results reported in this section are due to him, except where otherwise indicated. In terms of theorems, we have:

Theorem 12.4.1. All tautologies of classical logic in the $\{\&, \vee, -, \div\}$ language hold on the whole space T; that is, all tautologies take T as their value in any semantical assignment of closed sets.

Thus, as Goodman points out, the differences between CSL and classical logic must come in the deducibility relation (or in the behaviour of implication). We already saw some such differences in the previous section. Goodman provides a Gentzen-style proof theory as follows:

(1) $\emptyset \Rightarrow \emptyset$
(2) From $\emptyset \Rightarrow \Gamma$ infer $\emptyset \Rightarrow \Gamma, \Sigma$
(3) From $\emptyset \Rightarrow \Gamma, \delta$ and $\delta \Rightarrow \Sigma$ infer $\emptyset \Rightarrow \Gamma, \Sigma$
(4) From $\emptyset \Rightarrow \Gamma$ infer $\emptyset \& \delta \Rightarrow \Gamma$ and infer $\delta \& \emptyset \Rightarrow \Gamma$
(5) From $\emptyset \Rightarrow \Gamma, \delta, \tau$ infer $\emptyset \Rightarrow \Gamma, \delta \vee \tau$
(6) From $\emptyset \Rightarrow \Gamma, \delta$ and $\emptyset \Rightarrow \Gamma, \tau$ infer $\emptyset \Rightarrow \Gamma, \delta \& \tau$
(7) From $\emptyset \Rightarrow \Gamma$ and $\delta \Rightarrow \Gamma$ infer $\emptyset \vee \delta \Rightarrow \Gamma$
(8) From $\emptyset \Rightarrow \Gamma, \delta$ infer $\emptyset \div \delta \Rightarrow \Gamma$ and conversely
(9) $\emptyset \Rightarrow T$

A soundness and completeness theorem in the following form can then be proved. Let Σ be any set of wffs, and write: $\Sigma \vdash \emptyset \Rightarrow \Theta_1, \ldots, \Theta_n$ to mean that the sequent $\emptyset \Rightarrow \Theta_1, \ldots, \Theta_n$ follows from the above rules together with the rules $T \Rightarrow \delta$ for each wff δ in Σ. Write $\Sigma \models \emptyset \Rightarrow \Theta_1, \ldots, \Theta_n$ to mean that in every assignment of closed sets to wffs in which the members of Σ are all assigned to the whole space S, $\emptyset$ is assigned to a subset of the union of the values of $\Theta_1, \ldots, \Theta_n$. Then:

Theorem 12.4.2. $\Sigma \vdash \emptyset \Rightarrow \Theta_1, \ldots, \Theta_n$ iff $\Sigma \models \emptyset \Rightarrow \Theta_1, \ldots, \Theta_n$

This shows that derivability respects the natural semantic ordering of set inclusion. Now, a desideratum for an object language implication is that it respect the deduction theorem, and thus also set inclusion in the intuitive semantics of closed sets. However, Goodman also shows:

Theorem 12.4.3. No connective → definable in terms of the connectives {&,v,-,÷} has the property of being true all over the whole space iff its antecedent is included in its consequent.

Goodman expresses the opinion that this means that CSL has no adequate notion of implication and so is too weak for much mathematics. But this is too pessimistic a conclusion. For one thing, it remains to be seen how much mathematics can be done independently of an object-language implication. Results in Mortensen [1995] indicate that the important mathematics goes on at the sub-logic level. Of course a deducibility relation is necessary, but that is guaranteed with the proof theory and the set ordering. Secondly, the fact that such a → connective would not be definable in terms of the other connectives surely cuts little ice. After all, in open set logic the connective & is not definable in terms of {v,-}, but we do not regard it as defective. Even in Boolean set logic, & is not definable from v alone.

(iv) Semantics

It is well known that algebraic semantics for open set logic are given by Heyting algebras. These are distributive lattices with maximal and minimal elements {T,F} and having a binary operation → satisfying:
$C \cap A \leq B$ iff $C \leq A \rightarrow B$. This enables -A to be defined in a natural way as A→F. Dually, we have Brouwerian algebras, which are distributive lattices with maximal and minimal elements, and having a binary operation ÷ satisfying: $A \leq C \cup B$ iff $A \div B \leq C$. This enables paraconsistent negation -A to be defined in an equally natural way as T÷A. It is obvious that CSL is sound and complete for the class of Brouwerian algebras iff OSL is sound and complete for the class of Heyting algebras. It is an interesting comment on the fixity of logician's ideas, that McKinsey-Tarski [1946] were aware of Brouwerian algebras, but used them as semantics for OSL by inverting the functor back again!

Recalling Goodman's criticism that CSL lacks a reasonable implication, it should be noted that there is a natural S5-style implication on any lattice given by: if $A \leq B$ then A→B = the whole space S, else A→B = { }.

Here too we can clearly see the source of the irrelevance of CSL: formulas like Av-A always take the top element T, and their denials always take F, so there are wffs which are deducible from everything, and wffs from which everything is deducible. Is this a difficulty for CSL? It is hard to feel so: inconsistency-control was always the more important part of the story in this writer's opinion, deductive relevance was along for the ride. This is reflected in the complexity of the relevant account of implication.

Using instead worlds-style semantics, it is known that Heyting algebras and open set topological spaces are obtained from S4-style worlds semantics by taking as propositions just the *hereditary sets*, that is those sets of worlds closed w.r.t. the transitive reflexive relation which determines the S4 semantics. The dual is straightforward: take instead the *antihereditary sets*, those sets of worlds closed backwards w.r.t. that relation. This aspect of the duality illustrates that the way to obtain Brouwerian algebras from Heyting algebras is simply to reverse the order. Order reversal swaps T for F, → for ÷ and ∪ for ∩. Naturally enough, the order reversal on any distributive lattice is a distributive lattice.

In turning to category-theoretic semantics, we notice a considerable Public Relations Exercise on behalf of intuitionist logic, deriving from the alleged fact that 'topos logic is intuitionist'. But if the duality between OSL and CSL is as deep as topological, then there should be as equally natural a representation of CSL as topos logic, or at least as the logic of structures equally natural with toposes. It turns out that either view can be taken. The remainder of this section follows Mortensen [1995].

An (elementary) *topos* is a category with initial and terminal objects, pullbacks, pushouts, exponentiation, and a *subobject classifier*, which is an object Ω of the category together with an arrow T: 1 --> Ω satisfying the condition that for every monic arrow f: a >--> b there is a unique *classifier* arrow chi_f : b --> Ω which makes the following diagram a pullback.

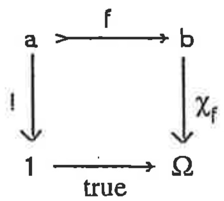

Figure 12.1 The subobject classifier

The subobjects of a given object are interpreted as monics (left-cancellable arrows, one-one functions) with the given object as their codomain. This is intuitively reasonable, since injections mark out the parts one-one. It turns out that there is a natural construction (see below) which makes the collection of arrows 1 --> Ω into a Heyting algebra, and thus provides a categorial semantics for OSL. The (monic) arrows 1 --> Ω represent the special case where a monic picks out an atomic part of its codomain, in this case the 'truth value object' Ω.

This ingenious construction cries out for topological dualization. To dualize, simply rename T with F, and relabel the classifier arrow chi_f as chi-bar_f.

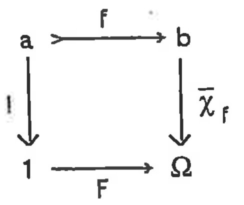

Figure 12.2 The complement classifier

That is, a *complement topos* classifies relative complements among the subobjects (monics) of a thing. It is not quite that simple, of course: the renaming carries with it some consequences. Reversing the order on the truth value object means also recovering the top element Tr in the way that intuitionistically the bottom element F is constructed. Negation swings around to having a paraconsistent character. It is important to note that the constructions for conjunction and disjunctions must be swapped around from their intuitionist duals, since unions and intersections reverse their roles. Intuitionist implication becomes once again pseudo-difference. It is immediate that this construction is a semantic representation of CSL. We give here the paraconsistent constructions, and omitting their well-known intuitionist duals.

True Tr :1 → Ω is the complement-character of the initial object 0.

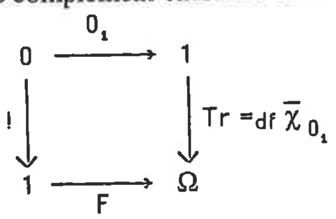

Figure 12.3 The True

This is plausible for a complement-classifier. It is the dual of the definition of ⊥ for toposes.

Negation ⌐ : Ω → Ω is the complement-character of *T*

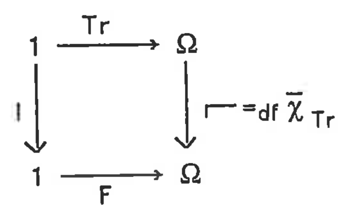

Figure 12.4 Negation

This dualizes $\neg$ for toposes.

Disjunction $\vee : \Omega \times \Omega \rightarrow \Omega$ is the complement-character of $\langle F,F\rangle: 1 \rightarrow \Omega \times \Omega$

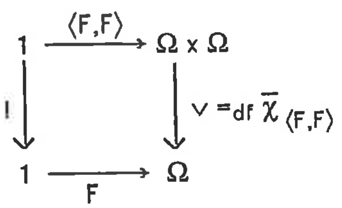

Figure 12.5 Disjunction

Compare with *Set*, where the complement of $\{(0,0)\}$ in 2×2 is $\{(1,1),(1,0),(0,1)\}$.

Conjunction $\wedge : \Omega \times \Omega \rightarrow \Omega$ is given by

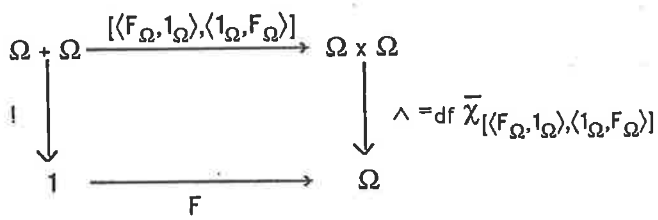

Figure 12.6 Conjunction

One can go on to develop presheaf and sheaf representations of Brouwerian algebras and CSL. This has been worked out by William James, see [1997]. The important consequence of his work is that any presheaf construction carries with it a natural paraconsistent logic via its complement-topos. Such constructions are very general, so this ensures many applications.

It goes without saying that a complement topos is really just a topos redescribed. Hence the existence of such structures is not in doubt. This justifies the claim made above that one can say that CSL is an equally natural paraconsistent logic of toposes, or just as easily say that CSL is the paraconsistent logic of structures equally natural with toposes. In either case, CSL has a natural categorial semantics.

Relevant logic should therefore take up the task of providing categorial semantics for its systems. Until this is achieved, relevant logic can only appear as more ad-hoc than its paraconsistent rival. The extent of the mathematical applications of CSL, and particularly the naturalness of its uses in phase spaces, argue that CSL throws out a serious challenge to the relevant logic project.

On the other hand, at bottom relevant (implicational) semantics and closed set semantics seem to be natural partners: the topological spaces supply the worlds and the ternary relations supply the implicational structure. Even the Routley functor (Routley star operator) plays a role in being a natural transformation between theories of open set and closed set logic ([1995] Chap 13). Further (unpublished) work by William James indicates that the task of conversion of ternary worlds semantics to categorial semantics is not as easy as it might seem. The obvious representation in categorial monoids, motivated by the algebraic Dunn monoids, has restricted properties (see [1997]). Further work is essential here.

Finally, one must not forget the connection with Brazilian logic. Da Costa invented the paraconsistent C-systems earlier than the Routleys realized the paraconsistent nature of relevant semantics. Da Costa was very far-sighted in proposing the C-systems as duals of intuitionism in which LNC and the other half of double negation fail. A good feature of these systems is that they enable an object-language representation of the local consistency of a formula. For a time, it seemed that there was a difficulty in that negation fails the replacement theorem, which prevents any attempt at a Lindenbaum algebra for an algebraic semantics. However, it can fairly be said that CSL is Brazilian-style in motivation. In any case, it is not clear that this is much of a difficulty for the C-systems. It was shown early by da Costa that there were set theories of C-systems in which the Russell Set exists, and more recent work by the Brazilian school including Beziau, Bueno, Carnelli, Coniglio, D'Ottaviano and others has indicated a rich range of mathematical applications.

In sum, closed set logic deserves careful consideration as a rival to relevant logic and other paraconsistent logics.

We now prove soundness for this semantics.

Theorem 12.3.3. If A is a theorem of $DJ^{to\otimes}$ then A is valid in the $DJ^{to\otimes}$-semantics.

Proof: The axioms of DJ are $DJ^{to\otimes}$-valid and the rules of DJ preserve $DJ^{to\otimes}$-validity, as in RLR1, for the unreduced semantics. Note that the Hereditary and Entailment Lemmas continue to hold, with the three additions, given their truth conditions and the semantic postulates p6 and p7. Further, the extra rules, R4, R5 and R6, for $DJ^{to\otimes}$ can be shown to preserve $DJ^{to\otimes}$ -validity, using the truth conditions for 't', 'o' and '⊗', together with the above two Lemmas. The cases for 't' and 'o' can be found in RLR1, and the case for '⊗' is similar.

The *second step* is to remove from the $DJ^{to\otimes}$ -semantics the truth conditions for 't', 'o' and '⊗', yielding what we will call the DJ-semantics. The following theorem is then clear.

Theorem 12.3.4. If A does not contain 't', 'o' nor '⊗', then if A is valid in the $DJ^{to\otimes}$ -semantics then A is valid in the DJ-semantics.

The *third step* is to prove completeness for DJ with respect to the DJ-semantics. This is basically what is set out in §2.1 or in RLR1, pp.298-318. The only extra thing to check is the satisfaction of the postulates p6 and p7. Since the canonical O_c is the set of all prime DJ-theories containing all the theorems of DJ and a ≤ b iff a ⊆ b, for prime DJ-theories a and b, then p6 holds. Since $R_c$cda iff, for all formulae A,B, if A→B ε c and A ε d then B ε a, for prime DJ-theories c, d and a, then p7 holds. Hence, the following theorem applies.

Theorem 12.3.5. If A is valid in the DJ-semantics then A is a theorem of DJ.

We are now in a position to put the previous theorems together.

Theorem 12.3.6. A is a theorem of DJ iff A is derivable in LDJ.

Proof: L→R. By Theorem 12.3.1. R→L. By Theorems 12.3.2-12.3.5, since A does not contain 't', 'o' nor '⊗'.

§12.4. Closed Set Logic

Chris Mortensen

(i) Introduction

Perhaps the most natural paraconsistent logic from the point of view of semantics is Closed Set Logic, CSL. CSL is paraconsistent but irrelevant, and thus looms as a serious rival to the whole class of relevant paraconsistent logics. In the following sections, we begin by describing its natural semantics, as a topological dual of intuitionist logic which has an equally natural semantics but which is neither relevant nor paraconsistent. We then review results on proof theory, due to Nicolas Goodman [1981]. The criticism that CSL lacks a reasonable implication is discussed and rejected. Then algebraic and category-theoretic representations due to Mortensen [1995], Lavers and James are sketched. (See Chapter 11, with Peter Lavers; Chapter 12, by William James, and Chapter 13 on duality and connections with the Routley star.)

(ii) Topology and Logic

If we consider the classical representation of dynamical systems in terms of phase spaces, we see that analysis is employed ubiquitously in the formulation and solution of differential equations of motion. Analysis is conducted on a topological space of points. Indeed, analysis is typically conducted on open sets of points: one proves for example things such as: if f and g are continuous functions on an open interval I then so are f+g, f-g etc. Thus analysis can seem to have an intuitionist character, since after all intuitionist logic is the logic of open sets (OSL).

Intuitionism begins with the assumption that *whatever is true, is true on open sets of points.* For example, if f is continuous on a trajectory which is an open set O of points, then 'f is continuous at x', is true at all t in O. Happily, this respects natural disjunction v and conjunction & operations, since open sets are closed w.r.t. unions and (finite) intersections. Open sets are not however closed w.r.t. set complement, so whatever negation is, it will not be Boolean in nature. Hence intuitionism takes instead *the union of all open sets included in the set complement.* This is guaranteed to exist by the fact that open sets are closed under arbitrary unions. From this, it is easy to see that the boundaries of open sets, at least those open sets which are not also closed, are places where both a proposition and its negation will fail to hold. This is the characteristic intuitionist property of negation. Intuitionist implication further follows from the definition: $A \rightarrow B =_{df} -A \cup B$. This yields an implication with reasonable properties including modus ponens. Furthermore, intuitionist negation is recovered by the special case that: $A \rightarrow \{\} = -A \cup \{\} = -A$. Equivalently, $A \rightarrow B$ is definable as $\bigcup\{C: C \cap A$ is a subset of $B\}$.

However, closed sets can also play a significant role. When we include the boundaries of sets, we are including the places where otherwise discontinuous change takes place, where the left hand limit fails to equal the right hand limit. The capacity for paraconsistency at these points is reflected in the nature of negation. The natural topological dual of the intuitionist idea is to begin with the assumption that *whatever is true, is true on closed sets of* *points.* This supposition similarly supports natural disjunction and conjunction operations, since (finite) unions and (arbitrary) intersections of closed sets are closed. However, negation needs to be tweaked, though in a different way from intuitionism. This is achieved by taking it to be *the intersection of all closed sets including the set complement.* This is guaranteed to exist by the fact that arbitrary intersections of closed sets are closed. But now we can see how it is that closed set negation is paraconsistent. For supposing that a proposition holds all over a closed (and non-open) set of points including its boundary, its negation holds on the set complement and its boundary, which is the same boundary. That is, from the paraconsistent point of view, *boundaries are places where propositions and their negations hold.* If we say that a proposition B follows from another A in a model, just in case the set of worlds at which A holds is a subset of those at which B holds, then we can say that it is not the case that everything follows from a contradiction, because if A holds on a closed set C, then A&-A holds at the boundary of C. Hence no proposition holding on any closed set not containing C, follows from A&-A. This is the mark of a paraconsistent logic.

This is the ideal setting for classical discontinuous change. Classical dynamics certainly recognizes discontinuity, but deals with it piecewise, without any mathematics at the discontinuity itself. This is particularly true in the case of quantum measurement. The natural mark of a causal process is a continuous path in phase space, and quantum measurement is causal in some sense, yet it is not continuous if regarded as consistent. The simplest thing to do is to identify the classically distinct values of the function at the boundary or boundaries. This ensures that the left hand limit of the function equals its right hand limit, which is continuity. The space of values of the function at the point is thought of as 'rolled up' at the point of discontinuity so that the classically distinct values are identified on the surface of the cylinder, retaining much of its functionality as the additive group of reals modulo the circumference of the cylinder. The inconsistency arises by insisting that *as well* the value space must retain its character as the set of (classically distinct) reals. This means that the distinctness of the values will be retained as a mark of the classical discontinuity of the process.

The naturalness of this semantics for CSL and the fact that it is topologically dual to OSL, make it a real rival to the relevant logics, which struggle with the Routley star to provide their account of negation.

It should be recorded that open set implication does not topologically dualize to a kind of implication in the closed set environment. The natural CSL dual is given by:

$A \div B =_{df} A \cap -B$. Equivalently, one can define $A \div B$ as the intersection set of {C: A is a subset of C∪B}. This is known as *pseudo-difference* (see Curry [1963], McKinsey-Tarski [1946]). Modus ponens, which is arguably a core feature of any kind of implication, fails for ÷. Indeed, $A \div A = \{\}$, which thus holds nowhere. On the other hand, this has to be seen in the properly dual perspective. After all, intuitionist OSL fails to have a pseudo-difference operation. Hence for example OSL negation cannot be recovered in the natural way it can from pseudo-difference, by $-A =_{df} T \div A$, where T is the whole space. On the other hand, Boolean logic, the logic of general sets (or of clopen sets), has both.

(iii) Proof Theory

Proof theory for CSL has been worked out by Nicolas Goodman [1981]. The results reported in this section are due to him, except where otherwise indicated. In terms of theorems, we have:

Theorem 12.4.1. All tautologies of classical logic in the $\{\&, \vee, -, \div\}$ language hold on the whole space T; that is, all tautologies take T as their value in any semantical assignment of closed sets.

Thus, as Goodman poin*s out, the differences between CSL and classical logic must come in the deducibility relation (or in the behaviour of implication). We already saw some such differences in the previous section. Goodman provides a Gentzen-style proof theory as follows:

(1) $\emptyset \Rightarrow \emptyset$
(2) From $\emptyset \Rightarrow \Gamma$ infer $\emptyset \Rightarrow \Gamma, \Sigma$
(3) From $\emptyset \Rightarrow \Gamma, \delta$ and $\delta \Rightarrow \Sigma$ infer $\emptyset \Rightarrow \Gamma, \Sigma$
(4) From $\emptyset \Rightarrow \Gamma$ infer $\emptyset \& \delta \Rightarrow \Gamma$ and infer $\delta \& \emptyset \Rightarrow \Gamma$
(5) From $\emptyset \Rightarrow \Gamma, \delta, \tau$ infer $\emptyset \Rightarrow \Gamma, \delta \vee \tau$
(6) From $\emptyset \Rightarrow \Gamma, \delta$ and $\emptyset \Rightarrow \Gamma, \tau$ infer $\emptyset \Rightarrow \Gamma, \delta \& \tau$
(7) From $\emptyset \Rightarrow \Gamma$ and $\delta \Rightarrow \Gamma$ infer $\emptyset \vee \delta \Rightarrow \Gamma$
(8) From $\emptyset \Rightarrow \Gamma, \delta$ infer $\emptyset \div \delta \Rightarrow \Gamma$ and conversely
(9) $\emptyset \Rightarrow T$

A soundness and completeness theorem in the following form can then be proved. Let Σ be any set of wffs, and write:
$\Sigma \vdash \emptyset \Rightarrow \Theta_1, \ldots, \Theta_n$ to mean that the sequent $\emptyset \Rightarrow \Theta_1, \ldots, \Theta_n$ follows from the above rules together with the rules $T \Rightarrow \delta$ for each wff δ in Σ. Write $\Sigma \models \emptyset \Rightarrow \Theta_1, \ldots, \Theta_n$ to mean that in every assignment of closed sets to wffs in which the members of Σ are all assigned to the whole space S, $\emptyset$ is assigned to a subset of the union of the values of $\Theta_1, \ldots, \Theta_n$. Then:

Theorem 12.4.2. $\Sigma \vdash \emptyset \Rightarrow \Theta_1, \ldots, \Theta_n$ iff $\Sigma \models \emptyset \Rightarrow \Theta_1, \ldots, \Theta_n$

This shows that derivability respects the natural semantic ordering of set inclusion. Now, a desideratum for an object language implication is that it respect the deduction theorem, and thus also set inclusion in the intuitive semantics of closed sets. However, Goodman also shows:

Theorem 12.4.3. No connective → definable in terms of the connectives {&,v,-,÷} has the property of being true all over the whole space iff its antecedent is included in its consequent.

Goodman expresses the opinion that this means that CSL has no adequate notion of implication and so is too weak for much mathematics. But this is too pessimistic a conclusion. For one thing, it remains to be seen how much mathematics can be done independently of an object-language implication. Results in Mortensen [1995] indicate that the important mathematics goes on at the sub-logic level. Of course a deducibility relation is necessary, but that is guaranteed with the proof theory and the set ordering. Secondly, the fact that such a → connective would not be definable in terms of the other connectives surely cuts little ice. After all, in open set logic the connective & is not definable in terms of {v,-}, but we do not regard it as defective. Even in Boolean set logic, & is not definable from v alone.

(iv) Semantics

It is well known that algebraic semantics for open set logic are given by Heyting algebras. These are distributive lattices with maximal and minimal elements {T,F} and having a binary operation → satisfying: $C \cap A \leq B$ iff $C \leq A \rightarrow B$. This enables -A to be defined in a natural way as A→F. Dually, we have Brouwerian algebras, which are distributive lattices with maximal and minimal elements, and having a binary operation ÷ satisfying: $A \leq C \cup B$ iff $A \div B \leq C$. This enables paraconsistent negation -A to be defined in an equally natural way as T÷A. It is obvious that CSL is sound and complete for the class of Brouwerian algebras iff OSL is sound and complete for the class of Heyting algebras. It is an interesting comment on the fixity of logician's ideas, that McKinsey-Tarski [1946] were aware of Brouwerian algebras, but used them as semantics for OSL by inverting the functor back again!

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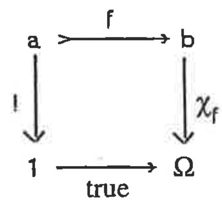

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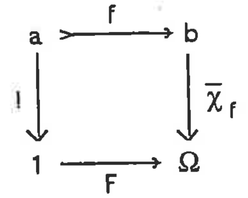

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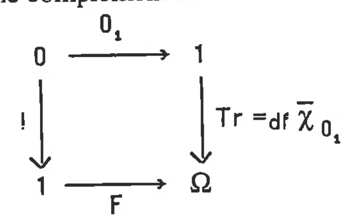

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This is plausible for a complement-classifier. It is the dual of the definition of ⊥ for toposes.

Negation ⌐— : Ω → Ω is the complement-character of *T*

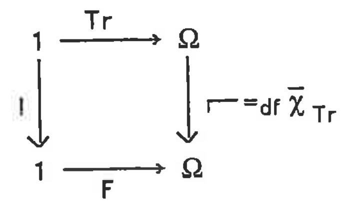

Figure 12.4 Negation

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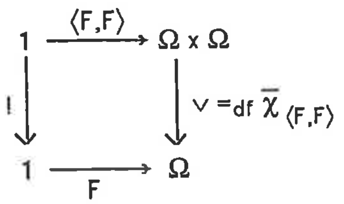

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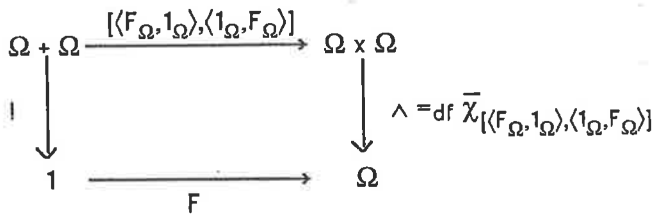

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Cubic logic, Ulam games, and paraconsistency

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ABSTRACT. *In this paper we call for attention to be paid to the link between logic and geometry. To apply this theme, we survey the connection between n-cubes, Lukasiewicz logics and Ulam games. We then extend what is known to the case where the number of permitted lies in a Ulam game exceeds 1. We conclude by identifying the precise sense in which these logics are paraconsistent.*

KEYWORDS: *geometry, paraconsistent logic, Ulam games, Lukasiewicz.*

1. Geometry and logics

Logicians have often steered clear of linking their subject with geometry. One can speculate as to reasons. Lattices appear widely in the semantics of logics, yet lattices have only rudimentary geometric representations (Hasse diagrams). Other geometric structures do not obviously lend themselves to interpretations for logics. Furthermore, inconsistent mathematical theories have tended to focus on algebraic number theory to the exclusion of geometry. In the end, perhaps it is simply that logicians, despite the heading over the doorway of Plato's Academy, have not been particularly well trained in the difficult subject of geometry.

This paper is a part of the project of geometrising logic: that is, to find geometrical models of logical theories, and conversely to develop logics and logical theories out of classes of geometrical objects. Looming in the background is a topic we will not tackle here, though it has been commenced elsewhere [MOR 02], namely to write inconsistent mathematical theories for inconsistent geometrical objects such as the impossible triangle and the like.

An obvious link between geometry and logic is the concept of a *group*. On the one hand the application of group theory to describe geometrical structures, as called for in Felix Klein's *Erlangen* program, is ubiquitous; while on the other hand there is a significant body of literature on group logics.

Indeed, the basic intuitive understanding of the group operations calls out for logical application. First, the idea that the elements of a group can be *transformations*, that is *actions* of a certain sort, lends itself to the thought that a group is a space of propositions, namely the propositions describing the carrying out of the actions. Second, the basic group operation for combining transformations is readily understood as "and then", which is undoubtedly a kind of conjunction. More than that, it is a kind of conjunction which is *associative but not necessarily commutative*. Indeed, it is commutative just in case the group is Abelian, and many interesting examples of non-Abelian groups are known. Third, the group inverse is an obvious candidate for a type of negation, one which obeys the law of Double Negation. These three ingredients suggest that there ought to be various rich classes of logical structures forthcoming.

There seem to be three approaches to logics arising from groups which can be found in the literature. First we have the construction of MV-algebras from groups by Chang and many others (usefully surveyed in [CIG 00]). (It should be noted that the construction proceeds by truncating groups to produce what are called greaps (*i.e.* group-heaps, see [MOR 02]). As is well-known, this construction produces the class of Lukasiewicz logics. Second we have the Abelian Logic of Meyer-Slaney [MEY 89]. Third there is the approach of Lewin-Sagastume [LEW 02], which is closer to MV-algebras than it is to Meyer-Slaney, though there are some interesting differences (see also [GAL 04]). We will not survey these three approaches here, though it can be said that they all derive from structures which are not simply groups but *lattice-ordered* groups. The group supplies the intensional structure (fusion, fission, strong implication and negation), while the lattice order supplies the extensional structure (conjunction, disjunction and the quantifiers). This then holds out the prospect that there is a natural generalisation to topological groups, (groups with an additional topological structure), where the group supplies the intensional structure and the topological space supplies conjunction, disjunction and a further negation which is of an open set (intuitionist) character, or alternatively of closed set (paraconsistent) character.

It must be stressed, of course, that one should not conflate the narrower topic of group logics with the broader program of geometrical interpretations of logical theories. The former is an obvious way to implement the latter, but there are likely to be many other fruitful directions thrown up by consideration of geometrical applications.

In this paper we propose to illustrate the connection between geometry and Lukasiewicz logics by extending results about Ulam games. We will see that different classes of geometrical structures are reflected in different sizes of Lukasiewicz logics. Ulam games are in a sense *prima facie* paraconsistent. In the last section, we will take up the question of just how paraconsistency manifests itself in these games. To begin, we must describe Ulam games.

2. Ulam games

Given a set S of numbers (called the search space), and an unknown number $x \in S$, the task is to find x by a series of questions. A maximum number of lies L in the answers are permitted (it is permitted to have fewer than L lies). The size of the search space S is denoted by n, and for simplicity we can assume that $S = \{1, ..., n\}$. The case $L = 0$ corresponds to the familiar game of Twenty Questions. For example if $L = 0, n = 10^6$, then x may be determined by a series of questions which successively halve the search space. The first question would be: does $x \in \{1, ...500,000\}$? After 20 questions x is found. We consider Ulam games with L lies. The following facts are known: (see *e.g.* [CIG 00])

(1) solving such games requires Lukasiewicz logics with $L + 2$ values;

(2) The case $L = 1$, corresponds to 3-valued Lukasiewicz logic arising naturally on the vertices, edges and faces of n-dimensional cubes (n-cubes), where n is the size of the search space.

In this paper we describe these results, then extend what is known by presenting geometrical modellings for cases $L > 1$. It will be seen that this makes a difference to the kinds of geometrical modelling to be given, and so is reflected in the size of the Lukasiewicz logic which characterises the problem.

3. Lukasiewicz logics

L should be L+1

The technique for solving such games when $L > 0$ utilizes $(L + 2)$-valued Lukasiewicz logic. The values of $(L + 2)$-valued logic can be usefully represented as $\{0, 1/L, 2/L, ..., L/(L + 1), 1\}$ with just 1 as the designated value. It is well-known that logical operators on these structures are given by:

$\neg x = 1 - x$

$x \vee y = \max(x, y)$ (disjunction)

$x \wedge y = \min(x, y)$ (conjunction)

$x \otimes y = (x + y - 1) \vee 0$ (fusion)

$x \oplus y = (x + y) \wedge 1$ (fission)

$x \Rightarrow y = \neg x \oplus y$ (implication)

4. Three example games

- Game I: Lies $L = 1$, space $S = \{1, 2\}$, that is $n = 2$
- Game II: $L = 1, n = 3$
- Game III: $L = 2, n = 3$

Game I: $L = 1$, $S = \{1, 2\}$.

Let x denote the unknown number, where the answerer, who may tell at most one lie, knows that $x = 1$. The questioner does not know what x is, but does know what L and S are. The Beginning Knowledge State of the questioner at time $t = 0$ (called K0) is a vector of ones corresponding to the numbers in S. That is, $K0 = (1, 1)$.

Question 1: $x \in \{1\}$?

Answer: Yes (the truth)

Then the next Knowledge State, $K1 = (1, 1/2)$

Rule for proceeding: a direct answer of "yes" is treated as we show here, whereas a direct answer of "no" is treated as an answer "yes" to the question about the complement-set of numbers. The values for any number to which the answer was yes stay the same, while the value for any number to which the answer is no drops to the next Łukasiewicz value (*i.e.* it reduces by $1/(L + 1)$). This is equivalent to the logical operation of fusion of the previous knowledge state point-by-point with 1 for those numbers which attracted the answer yes, and $L/(L + 1)$ for those numbers which attracted the answer no. We can obviously form the latter sequence as a vector, and thus write $Kt + 1 = Kt\otimes$ vector of answers to question $Qt + 1$. Continuing with Game I:

Question 2: $x \in \{2\}$?

Answer: Yes (a lie)

Next Knowledge State, $K2 = (1/2, 1/2)$

Question 3: $x \in \{2\}$?

Answer: No

Next Knowledge State, $K3 = (1/2, 0)$

The game concludes when the knowledge state is a vector of values only one of which is non-zero. Such knowledge states can be called solution states and $K3$ is obviously one such. The place of the non-zero value in a solution state corresponds to the hidden number. Thus here the game is finished and the answer $x = 1$ is read off from $K3$.

Game II: $L = 1$, $n = 3$.

(Let the hidden number be $x = 2$.) Initial Knowledge State $K0 = (1, 1, 1)$.

Question 1: $x \in \{1, 2\}$?

Answer: No (a lie)

Next Knowledge State $K1 = (1/2, 1/2, 1)$.

Question 2: $x \in \{1,3\}$?

Answer: No

Next Knowledge State $K2 = (0, 1/2, 1/2)$.

Question 3: $x \in \{3\}$?

Answer: No

Next Knowledge State $K3 = (0, 1/2, 0)$ which is a solution state. The solution is $x = 2$.

Game III: $L = 2$, $n = 3$.

(Let the hidden number be $x = 2$.) Initial Knowledge State $K0 = (1, 1, 1)$.

Question 1: $x \in \{1,2\}$?

Answer: No (first lie)

Next Knowledge State $K1 = (2/3, 2/3, 1)$.

Question 2: $x \in \{1,3\}$?

Answer: No

Next Knowledge State $K2 = (1/3, 2/3, 2/3)$.

Question 3: $x \in \{3\}$?

Answer: Yes (a second lie)

Next Knowledge State $K3 = (0, 1/3, 2/3)$.

Question 4: $x \in \{2\}$?

Answer: Yes

Next Knowledge State $K4 = (0, 1/3, 1/3)$.

Question 5: $x \in \{3\}$?

Answer: No

Next Knowledge State $K5 = (0, 1/3, 0)$ which is a solution state. The solution $x = 2$ is read off. The non-zero value even tells us that 2 lies were told. Had it been 1 then there would have been no lies; if 2/3 then 1 lie.

5. Cubic logic

It is known that for one lie, $L = 1$, this procedure can be modeled isomorphically by the vertices, edges and faces of an n-cube. The knowledge state vectors supply coordinates for vertices, edges and faces in a natural way.

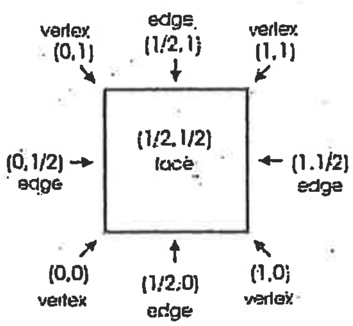

Figure 1. *Game I:* $L = 1$, $n = 2$

Knowledge states are isomorphic with the vertices, edges and faces of a square (2-cube). There are 4 vertices, 4 edges and 1 face. The vertices are correspond to knowledge states $(1,1)$, $(1,0)$, $(0,1)$ and $(0,0)$. The latter value is never achieved since the game ends with one non-zero value, but serves as a base point. The edges are $(1,1/2)$, $(1/2,1)$, $(1/2,0)$ and $(0,1/2)$, and the face (whole 2-cube) is $(1/2,1/2)$. The game begins at the point $(1,1)$, and moves down the edges, perhaps occupying the whole face, until one of the solution states is reached. The four solution states are the two one-dimensional edges connected to $(0,0)$, and their two nonzero endpoints.

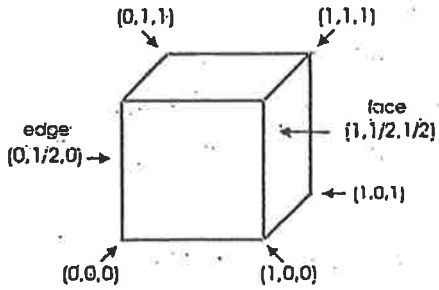

Figure 2. *Game II:* $L = 1$, $n = 3$

Knowledge states are isomorphic to the vertices, edges and faces of a 3-cube (including the whole cube itself). There are 8 vertices corresponding to the 8 knowledge states $(1,1,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(0,0,0)$. The 6 faces correspond to the states $(1,1,1/2)$, $(1,1/2,1)$, $(1/2,1,1)$, $(1,1/2,1/2)$, $(1/2,1,1/2)$ and $(1/2,1/2,1)$. There are 12 edges, corresponding to states where just one place is 1/2 and the remaining places are 1 or 0 The whole cube is $(1/2,1/2,1/2)$, which is a possible knowledge state. The 6 solution states are the 3 edges $(0,0,1/2)$, $(0,1/2,0)$ and $(1/2,0,0)$, and their 3 nonzero endpoints $(0,0,1)$, $(0,1,0)$ and $(1,0,0)$

6. The general case L > 1

When we move to general L, we take note that for $L = 1$ and general n, it is already known that the corresponding figure is an n-cube. Now for general L the solution becomes obvious from the correspondence with the geometry. Begin with the n-cube. It suffices to produce a figure which has the right number of Lukasiewicz-values, $L + 2$, mapped onto it. This can readily be done by taking the one-dimensional edges and partitioning each of them into $L + 1$ equal portions. Thus the edge $(0, 0, ...1/2, ...0, 0)$ is broken by the points $(0, 0, ...1/(L + 1), ...0, 0)$, $(0, 0, ...2/(L + 1), ...0, 0)...(0, 0, ...L/(L + 1), ...0, 0)$. That is, what was the middle Lukasiewicz value of 1/2 for the case $L = 1$ is split into L intermediate Lukasiewicz values $1/L + 1, ...L/L + 1$ for the general case.

Each of the vectors with only one non-zero entry is a solution vector, *i.e.* corresponds to a solution knowledge state. Evidently, there are $L + 1$ different solution state vectors having a single non-zero entry and zeros in all the same places, one for each of the non-zero Lukasiewicz values $1/(L + 1), ...(L + 1)/(L + 1)$. Moreover, there are n different sets of these, one for each place of the n-vector. Hence there are $n.(L + 1)$ different solution vectors. These correspond to the L partitions along each of the n one-dimensional edges, together with those having a single non-zero entry which is 1, which are the final endpoints. As an independent check, it is clear from the description of Ulam games given above that they have $n.(L + 1)$ different solution states.

This process can be likened to drawing lines across the 2-faces of the n-cube to join up the partitioning points (see illustrations below). However, there is one thing to note. In the case $L = 1$, every vertex, edge, face and the whole n-cube corresponds to a knowledge state. But for general L, the drawn lines do not correspond to knowledge states. Knowledge states only correspond to partitions of the "external" features, that is those which are named in the case $L = 1$.

We illustrate this with two diagrams, one for the case $L = 2$, $n = 2$, and the other for Game III, *viz* $L = 2$, $n = 3$.

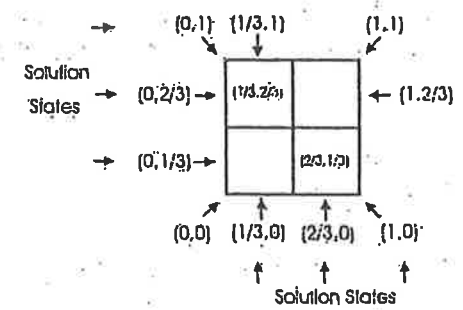

Figure 3. *The case* $L = 2$, $n = 2$

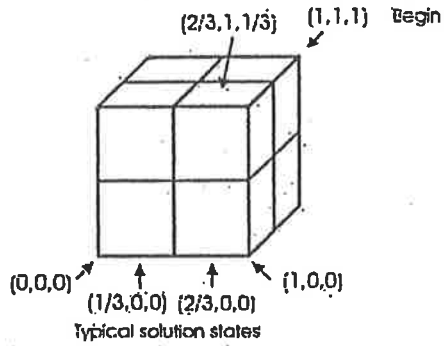

Figure 4. *Game III:* $L = 2$, $n = 3$

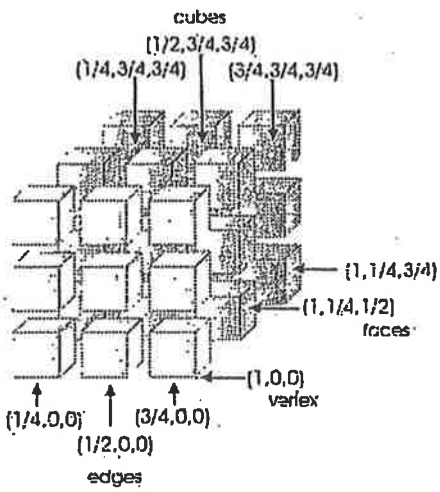

Figure 5. *The case* $L = 3$, $n = 3$

One can break up the n-cubes into arrays of L^n sub-n-cubes by separating along the lines drawn on the 2-faces, as in Figure 5. However, to avoid being misleading, it should be noted that not every edge and face produced is a possible knowledge state. This is for the same reason that in the previous examples the lines drawn on the 2-faces, as well as internal faces and cubes, do not correspond to possible knowledge states since they represent intersections between different Lukasiewicz values. In passing, this suggests a paraconsistent extension of Ulam games where more that one Lukasiewicz value is assigned to the same vector place in a knowledge state, which can be achieved by allowing multiple inconsistent answers to questions, say by allowing more than one person to answer questions simultaneously. We do not develop this extension to the theory here.

7. Paraconsistency

There is an obvious intuitive sense in which Ulam games have the flavour of paraconsistency about them. But what is that? After all, Lukasiewicz logics are not paraconsistent, or at least they are not paraconsistent with the usual designated value of $\{1\}$ alone. Of course, Lukasiewicz logics, like most others, can be made paraconsistent by designating enough values, say all those from 1/2 up, but that trivialises the problem. Daniele Mundici [MUN 02, p. 407] points out that Ulam games are fault-tolerant (paraconsistent) in the sense that the procedure allows successive opposite answers to the same question to be reflected in a knowledge state which is not impossible. If we define the *Pinocchio negation* $\sim$ of a knowledge state as that knowledge state brought forth by an opposite answer to the same question, then we can see that $x \otimes \sim x$ need not be an impossible knowledge state.

This is true, but it is just a little misleading, as we now explain. First, let us distinguish atomic knowledge states, as vectors of values which are either 1 or $L/(L+1)$. These are obviously the knowledge states produced by answers to a single question, before fusing with any pre-existing knowledge state. Now there is a natural Pinocchio negation definable on atomic knowledge states: the vector which interchanges the 1s with the $L/(L+1)$s. This corresponds to the opposite answer to the question. We note too that Pinocchio negation should not be confused with Lukasiewicz negation $\neg$, which replaces 1 by 0, and $L/(L+1)$ by $1/(L+1)$.

Unfortunately we have not been able to extend the Pinocchio negation operator to general knowledge states. Indeed, we doubt that it can be done in such a way as to preserve reasonable properties for negation (such as double negation). That is, we conjecture that Pinocchio negation is a reasonable negation operator only on atomic knowledge states. If that is so, then the observation that $x \otimes \sim x$ need not be an impossible knowledge state only makes sense when x is confined to atomic states.

But when we note this, it is now apparent why this should be so. Atomic knowledge states correspond 1-1 to the characteristic functions of subsets of the search space S. For example, let $S = \{1...5\}$ and ask: is x in the subset $\{1, 2, 3\}$? The answer yes is associated with the atomic knowledge state $(1, 1, 1, L/(L+1), L/(L+1))$, while the answer no is associated with the atomic knowledge state $(L/(L+1), L/(L+1), L/(L+1), 1, 1)$. Thus, *Pinocchio negation is but classical negation on the set of atomic knowledge states*, and these themselves form a Boolean algebra under set unions and intersections.

This does nothing to diminish the fault-tolerant nature of the Ulam game operators. But interestingly it re-locates it, in the *interaction* between an essentially classical negation $\sim$ and the Lukasiewicz fusion $\otimes$. It makes the paraconsistency to be more of a matter of the properties of conjunction, an approach which has certainly recommended itself to various significant scholars. It is an interesting further direction of inquiry how widespread such a phenomenon might be.

In conclusion, we note that while we have presented the geometry as it arises out of Ulam games, the situation can be viewed conversely, as presenting Lukasiewicz logics arising naturally out of various geometrical figures. Then the Ulam games serve to provide the linking functor for these connections. The association of Lukasiewicz logics with geometrical figures is thus in line with the program enunciated at the beginning of this essay, to demonstrate logics as arising naturally from geometry.

8. References

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