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Conjugate Functions and Semiconformal mappings

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1. Introduction

Suppose that M is a Riemannian manifold and $f : M \to \mathbb{R}$ is a smooth function. We may ask whether there is a smooth function $g : M \to \mathbb{R}$ such that

$$(1) \qquad \|\nabla f\| = \|\nabla g\| \quad \text{and} \quad \langle \nabla f, \nabla g \rangle = 0.$$

In this case, we shall say that f and g are <u>conjugate</u> <u>functions</u>. Suppose M is 2-dimensional. If f has a conjugate, then f is harmonic. Locally, the converse is true: harmonic functions admit conjugates. We may ask what happens for manifolds of dimension $\geqslant 3$. Is there, for example, a partial differential equation that characterises those f that admit a conjugate? In this article we shall establish a second order partial differential inequality and, in case M is 3-dimensional, a third order partial differential equation that must be satisfied by a function f in order that it admit a conjugate. Our final aim (as yet not attained) is to find further differential equations characterising such functions. These (nonlinear) differential equations should provide a natural <u>conformally</u> <u>invariant</u> generalisation of harmonic function to dimensions greater than 2.

One motivation for the condition (1) comes from the theory of harmonic morphisms [**2**]. A mapping $F : M \to N$ between Riemannian manifolds is said to be a <u>harmonic</u> <u>morphism</u> if and only if harmonic functions locally defined on N pull back by F to harmonic functions locally defined on M. The Hopf fibration $S^3 \to S^2$ is a harmonic morphism. If the target manifold is $\mathbb{R}^2$ and we write $F = (f, g)$, it is clear that each of f and g should be harmonic (since the coördinate functions on $\mathbb{R}^2$ are harmonic). The other condition that F be a harmonic morphism in this case was derived by Jacobi in 1848. It is (1). It is especially natural to isolate this condition when one notices that it is <u>conformally</u> <u>invariant</u> (whereas being harmonic is only conformally invariant in dimension 2). The condition (1) on $F = (f, g)$ also has a good geometric interpretation. At points where F is a submersion, it says that its derivative dF is conformal on the subspace orthogonal to its null space: F is <u>semiconformal</u> (some authors use (<u>weakly</u>) <u>horizontally</u> <u>conformal</u>).

2. Examples

Here are some examples of pairs $F = (f, g)$ on $\mathbb{R}^3 \setminus x_1$-axis enjoying (1):–

(a): $f = {x_1}^2 - {x_2}^2 - {x_3}^2 \qquad g = 2x_1\sqrt{{x_2}^2 + {x_3}^2}$,

(b): $f = x_2 \dfrac{{x_1}^2 + {x_2}^2 + {x_3}^2}{{x_2}^2 + {x_3}^2} \qquad g = x_3 \dfrac{{x_1}^2 + {x_2}^2 + {x_3}^2}{{x_2}^2 + {x_3}^2}$,

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$$\textbf{(c):}\ f = \frac{(1-\|x\|^2/2)x_2 + \sqrt{2}x_1x_3}{{x_2}^2 + {x_3}^2} \qquad g = \frac{(1-\|x\|^2/2)x_3 + \sqrt{2}x_1x_2}{{x_2}^2 + {x_3}^2}.$$

This last one is the Hopf fibration conformally rearranged using stereographic coördinates: $\mathbb{R}^3 \hookrightarrow S^3 \to S^2 \leftrightarrow \mathbb{R}^2$. In none of these examples is f harmonic (whereas it is shown by Ababou, Baird, and Brossard [**1**] that if f and g are polynomial and conjugate on $\mathbb{R}^n$ for any $n \geqslant 3$, then they must be harmonic).

In this article we shall show, for example, that neither of

$$f = x_1x_2x_3 \qquad \text{nor} \qquad f = x_1^3 + x_2^3 + x_3^3 \tag{2}$$

on $\mathbb{R}^3$ admit a conjugate, even locally.

3. A differential inequality

The first function f of (2) may be dealt with by a differential inequality that must be satisfied in case f admit a conjugate. To express this inequality, let us write

$$f_i = \nabla_i f \qquad f_{ij} = \nabla_i\nabla_j f \qquad \text{and so on,}$$

where ∇_i is the metric connection (or just coördinate derivative on $\mathbb{R}^n$). Also, let us 'raise and lower' indices with the metric in the usual fashion and write a repeated index to denote a sum over that index. Thus, ${f^i}_i = \Delta f$ is the Laplacian and $f^i g_i = \langle \nabla f, \nabla g\rangle$. Given a smooth function f on a smooth n-manifold M, let

$$X \equiv (n-2)[2{f_i}^j f_j f^{ik} f_k - f^i f_i f^{jk} f_{jk}] + f^i f_i ({f^j}_j)^2. \tag{3}$$

Theorem 1 Suppose f is a smooth function defined on a Riemannian n-manifold. If f admits a conjugate, then it satisfies the partial differential inequality $X \leqslant 0$.

Proof Let us suppose that f admits a conjugate g and let us write $\omega_i \equiv \nabla_i g$, noting that $\omega_{ij} = \nabla_i\omega_j$ is symmetric in its indices. The equations (1) become

$$f^j\omega_j = 0 \quad \text{and} \quad \omega^j\omega_j = f^j f_j. \tag{4}$$

Differentiating again, we find

$$f^{ij}\omega_j + \omega^{ij} f_j = 0 \quad \text{and} \quad \omega^{ij}\omega_j = f^{ij} f_j. \tag{5}$$

Using that ω_{ij} is symmetric, we conclude that

$$f^{ij}\omega_i\omega_j + f^{ij} f_i f_j = 0. \tag{6}$$

We now regard (4) and (6) as algebraic equations on the 1-form ω_i and claim that $X \leqslant 0$ is a necessary constraint on f_i and f_{ij} in order to find a solution ω_i. Certainly, this will complete the proof.

Where f_i vanishes X also vanishes. We may suppose, therefore, we are at a point where $f_i \neq 0$ and we are required to show that $X \leqslant 0$ there. We may do this in local coördinates chosen conveniently at the point in question. Let us suppose that

$$f_1 = f_2 = \cdots = f_{n-1} = 0 \qquad f_n \neq 0 \qquad g_{ij} = \text{identity matrix}$$

at this point. If we let α be an index running over $1, 2, \cdots, n-1$, then we may rewrite X at our chosen point:–

$$\begin{aligned} X &= (f_n)^2\Big((n-2)[2f_{in}{f^i}_n - f^{jk}f_{jk}] + ({f^j}_j)^2\Big) \\ &= (f_n)^2\Big((n-2)[(f_{nn})^2 - f^{\alpha\beta}f_{\alpha\beta}] + (f_{nn} + {f^\alpha}_\alpha)^2\Big) \\ &= (f_n)^2\Big((n-1)(f_{nn})^2 - (n-2)f^{\alpha\beta}f_{\alpha\beta} + 2f_{nn}{f^\alpha}_\alpha + ({f^\alpha}_\alpha)^2\Big). \end{aligned}$$

Letting Q denote the $(n-1)\times(n-1)$ symmetric matrix $f_{nn}g^{\alpha\beta}+f^{\alpha\beta}$, we find that

$$(\operatorname{trace} Q)^2 = ((n-1)f_{nn} + f^{\alpha}{}_{\alpha})^2$$

and

$$\operatorname{trace} Q^2 = (n-1)(f_{nn})^2 + f^{\alpha\beta}f_{\alpha\beta} + 2f_{nn}f^{\alpha}{}_{\alpha},$$

whence

$$X = (f_n)^2\Big((\operatorname{trace} Q)^2 - (n-2)\operatorname{trace} Q^2\Big). \tag{7}$$

Now let us consider the three equations (4) and (6) in our special coördinates. The first one says that $\omega_n = 0$ and from the others, we find that

$$\begin{aligned}(f_{nn}g^{\alpha\beta}+f^{\alpha\beta})\omega_\alpha\omega_\beta &= f_{nn}\omega^i\omega_i + f^{ij}\omega_i\omega_j\\ &= f_{nn}\omega^i\omega_i - f^{ij}f_if_j \quad = \quad f_{nn}(\omega^i\omega_i-(f_n)^2) \quad = \quad 0.\end{aligned}$$

In particular, the symmetric matrix Q is indefinite. Thus, bearing (7) in mind, to finish the proof it suffices to show that for any $(n-1)\times(n-1)$ indefinite symmetric matrix Q,

$$(\operatorname{trace} Q)^2 \leqslant (n-2)\operatorname{trace} Q^2. \tag{8}$$

This is easily verified by diagonalising Q. □

Notice that when $n = 3$, the criterion (8) is not only necessary but also sufficient for the 2×2 symmetric matrix Q to be indefinite:–

$$(\operatorname{trace} Q)^2 - \operatorname{trace} Q^2 = 2\det Q.$$

In general, the constant $(n-2)$ in (8) is best possible.

When $n = 2$, the differential inequality $X \leqslant 0$ becomes

$$\|\nabla f\|^2(\Delta f)^2 \leqslant 0,$$

which can only happen if f is harmonic. Thus, we recover the well-known 2-dimensional criterion for f to admit a conjugate. In higher dimensions, it is sometimes the case that X is identically zero. In the examples (a) and (b) of §2 we find that $X = 0$. For (c), however,

$$f = \frac{(1-\|x\|^2/2)x_2 + \sqrt{2}x_1x_3}{{x_2}^2+{x_3}^2} \quad\Rightarrow\quad X = -\frac{(2+\|x\|^2)^2}{({x_2}^2+{x_3}^2)^4},$$

which is strictly negative.

4. Counterexamples

We may now dispose of the first of (2). We compute

$$f = x_1x_2x_3 \quad\Rightarrow\quad X = 6(x_1x_2x_3)^2.$$

There is no open set on which $X \leqslant 0$ so f does not admit a conjugate. This criterion, however, is not always sufficient:–

$$f = x_1^3 + x_2^3 + x_3^3 \quad\Rightarrow\quad X|_{(1,1,1)} = 7776 \quad\text{whilst}\quad X|_{(1,2,-2)} = -1944,$$

which does not rule out f having a conjugate near $(1,2,-2)$. Even worse,

$$f = \log\left(\frac{\sqrt{1+{x_2}^2+{x_3}^2}-1}{\sqrt{{x_2}^2+{x_3}^2}}\right) + 2\sqrt{1+{x_2}^2+{x_3}^2} \tag{9}$$

yields

$$X = -2\frac{(3+2{x_2}^2+2{x_3}^2)(1+2{x_2}^2+2{x_3}^2)^2}{({x_2}^2+{x_3}^2)^2(1+{x_2}^2+{x_3}^2)^4},$$

which is everywhere negative though, as we shall see, f does not admit a conjugate.

5. Some conformal invariants

Given the conformally invariant nature of the problem, it is not surprising that the quantity X is itself conformally invariant. This means that if the Riemannian metric g_{ij} is rescaled by an arbitrary positive smooth function, then X itself merely rescales. More specifically X is a conformally invariant density of weight -6 meaning that

$$g_{ij} \mapsto \hat{g}_{ij} = \Omega^2 g_{ij} \quad \Rightarrow \quad X \mapsto \hat{X} = \Omega^{-6}X.$$

We may check this directly as follows. The metric connection changes as

$$\hat{\nabla}_i\varphi_j = \nabla_i\varphi_j - \Upsilon_i\varphi_j - \Upsilon_j\varphi_i + \Upsilon^k\varphi_k g_{ij}$$

acting upon any 1-form φ_i, where $\Upsilon_i = \nabla_i \log\Omega$. Thus, while the exterior derivative $f_i = \nabla_i f$ of f is invariant ($\hat{f}_i = f_i$), the Hessian changes:–

$$\hat{f}_{ij} = f_{ij} - \Upsilon_i f_j - \Upsilon_j f_i + \Upsilon^k f_k g_{ij}.$$

Therefore,

$$\hat{f}^i\hat{f}_i = \Omega^{-2}f^i f_i \qquad \hat{f}_i{}^j\hat{f}_j = \Omega^{-2}[f_i{}^j f_j - \Upsilon_i f^j f_j]$$
$$\hat{f}_i{}^j\hat{f}_j\hat{f}^{ik}\hat{f}_k = \Omega^{-6}[f_i{}^j f_j f^{ik} f_k - 2f^k f_k\Upsilon^i f^j f_{ij} + \Upsilon^i\Upsilon_i(f^j f_j)^2]$$
$$\hat{f}^{ij}\hat{f}_{ij} = \Omega^{-4}[f^{ij}f_{ij} - 4\Upsilon^i f^j f_{ij} + 2\Upsilon^k f_k f^i{}_i + 2\Upsilon^i\Upsilon_i f^j f_j + (n-2)(\Upsilon^i f_i)^2]$$
$$\hat{f}^j{}_j = \Omega^{-2}[f^j{}_j + (n-2)\Upsilon^j f_j]$$

and all the terms involving Υ_i cancel in computing $\hat{X}$ from (3). From this direct calculation, the conformal invariance of X seems miraculous and, even in flat space $\mathbb{R}^n$, conformal invariants are somewhat thin on the ground. Nevertheless, it is possible, in principle, to list all conformal invariants of functions on $\mathbb{R}^n$ and this is done in [**3**]. Here are some further conformal invariants:–

$$J = f^i f_i \text{ (weight } -2) \quad \text{and} \quad Z = (n-2)f^{ij}f_i f_j + Jf^j{}_j \text{ (weight } -4).$$

They are of first and second order, respectively, and are invariant not only in flat space but also on an arbitrary Riemannian manifold. Any polynomial with consistent overall weight in known invariants is also invariant. Here are some third order invariants in flat space:–

$$R = Jf^i\nabla_i Z - 2Zf^i\nabla_i J \text{ (weight } -8)$$

and

$$S = Jf^i\nabla_i X - 3Xf^i\nabla_i J \text{ (weight } -10).$$

Again, they are invariant on any Riemannian manifold. Also, they are all even: invariant under change of orientation.

Here is an odd conformal invariant on flat $\mathbb{R}^3$:–

$$V = \epsilon^{jk\ell}(J\nabla_\ell(f^i H_{ij} f_k) - 3f^i H_{ij} f_k \nabla_\ell J)$$

where $\epsilon_{jk\ell}$ is the volume form and

$$H_{ij} = 2J\nabla_i\nabla_j J - 3(\nabla_i J)(\nabla_j J)$$

It has weight -11 and changes sign under change of orientation. As written, it seems that V is fourth order but the highest derivatives $\nabla_\ell\nabla_i\nabla_j J$ clash with the skew symmetry of $\epsilon^{jk\ell}$ and so V is actually only third order. It extends to an invariant on a general Riemannian 3-manifold with the addition of a 'curvature correction term' to H_{ij}:–

$$H_{ij} = 2J\nabla_i\nabla_j J - 3(\nabla_i J)(\nabla_j J) - 4R_{ij}J^2,$$

where R_{ij} is the Ricci curvature.

6. A partial differential equation

Theorem 2 Suppose f is a smooth function defined on an open subset of $\mathbb{R}^3$. If f admits a conjugate, then it satisfies, where $X < 0$, the following partial differential equation:–

$$2(ZS - 2XR + 2XZ^2 - 4JX^2)^2 + XV^2 = 0. \tag{10}$$

The proof is postponed until §8 below. Let us first give some applications.

7. Further counterexamples

It is a matter of calculation (most easily done with a computer) to see that Theorems 1 and 2 obstruct certain functions from admitting a conjugate. For $f = {x_1}^3 + {x_2}^3 + {x_3}^3$ of (2) we find that both X and the left hand side of (10) are non-zero polynomials

$$2(ZS - 2XR + 2XZ^2 - 4JX^2)^2 + XV^2 = 2^9 3^{30} 5^2 x_1^{28} x_2^4 + \cdots ,$$

whereas Theorems 1 and 2 would have their product vanishing identically in any open set on which f were to have a conjugate. Hence, nowhere does f admit a conjugate.

Similarly, though the function f defined on $\mathbb{R}^3 \setminus x_1$-axis by (9) everywhere passes the criterion of Theorem 1, it is immediately ruled out of having a conjugate by Theorem 2. The left hand side of (10) works out to be

$$128\frac{(1 + 2{x_2}^2 + 2{x_3}^2)^{16}}{({x_2}^2 + {x_3}^2)^{10}(1 + {x_2}^2 + {x_3}^2)^{14}}.$$

On the other hand,

$$f = \log\left(\frac{\sqrt{1 + {x_2}^2 + {x_3}^2} - 1}{\sqrt{{x_2}^2 + {x_3}^2}}\right) + \sqrt{1 + {x_2}^2 + {x_3}^2} \;\Rightarrow\; X = -\frac{2}{({x_2}^2 + {x_3}^2)^2},$$

which is everywhere negative and f also satisfies the differential equation of Theorem 2. In fact, f has a conjugate: $g = x_1 + \arctan(x_3/x_2)$.

8. Proof of Theorem 2

Recall our equations so far, namely (4) and (6):–

$$f^j\omega_j = 0 \qquad \omega^j\omega_j = f^j f_j \qquad f^{ij}\omega_i\omega_j + f^{ij}f_i f_j = 0. \tag{11}$$

Let us calculate the directional derivative $f^i\nabla_i$ of (6):–

$$f^{ijk}f_i f_j f_k + f^{ijk}f_i\omega_j\omega_k + 2f^{jk}f^i{}_j f_i f_k + 2f^{jk}\omega^i{}_j f_i\omega_k = 0.$$

Now use that ω_{ij} is symmetric and the second of equations (5) to replace $\omega^i{}_j f_i$ by $-f^i{}_j\omega_i$. We obtain:–

$$f^{ijk}f_if_jf_k + f^{ijk}f_i\omega_j\omega_k + 2f^{jk}f^i{}_jf_if_k - 2f^{jk}f^i{}_j\omega_i\omega_k = 0. \tag{12}$$

Notice that (11) and (12) are algebraic equations for ω_i, which at any point are constructed from the third jet of f at that point.

To proceed we need to suppose that the dimension is 3 and, for simplicity, we shall also suppose that the manifold is Euclidean space $\mathbb{R}^3$. We would like to solve the equations (11) for ω_j (since in dimension 3 there are 3 equations for 3 unknowns) and then eliminate it from (12) to give a vanishing expression in the third derivatives of f. At stated, this does not work because the sign of ω_j is not determined by (11). In fact, this is not a problem because changing the sign of ω_j also preserves (12). There is a more serious problem. As we shall soon see, there is not just a 2-fold ambiguity in ω_j from (11) but a 4-fold ambiguity, arising geometrically by viewing (11) as the intersection of two planar quadrics.

This more serious problem can be overcome as follows. Computer algebra is apparently insufficient to solve (11) in a useful way. Instead we shall choose special coördinates near a fixed basepoint, carry out the proposed solution and elimination in these special coördinates, and then guess the resulting partial differential equation in general coördinates, finally justifying our guesswork by computation in special coördinates.

We shall say that an orthonormal coördinate system is <u>special</u> at a given basepoint if and only if

$$f_1 = f_2 = f_{12} = 0 \quad \text{and} \quad f_3 \neq 0 \tag{13}$$

at that point. Evidently, this is possible at any point where $\|\nabla f\|$ is non-vanishing for we may take the x_3-axis to align with ∇f and then diagonalise the Hessian f_{ij} restricted to $(\nabla f)^\perp$ by rotation in the (x_1, x_2)-plane.

The first equation of (11) in special coördinates says that $\omega_3 = 0$ and the remaining two equations then read:–

$$(\omega_1)^2 + (\omega_2)^2 = (f_3)^2 \quad \text{and} \quad f_{11}(\omega_1)^2 + f_{22}(\omega_2)^2 + f_{33}(f_3)^2 = 0.$$

From these linear equations for $(\omega_1)^2$ and $(\omega_2)^2$, we find

$$(\omega_1)^2 = (f_3)^2\frac{f_{22}+f_{33}}{f_{22}-f_{11}} \qquad (\omega_2)^2 = (f_3)^2\frac{f_{11}+f_{33}}{f_{11}-f_{22}}, \tag{14}$$

provided that $f_{11} \neq f_{22}$. But if we compute X in special coördinates, we find

$$X = 2(f_3)^2(f_{11}+f_{33})(f_{22}+f_{33}),$$

whence $X < 0$ forces $f_{11} \neq f_{22}$. In fact, if we rewrite (14) as

$$(\omega_1)^2 = (f_3)^2\frac{2(f_3)^2(f_{22}+f_{33})^2}{2(f_3)^2(f_{22}+f_{33})^2 - X} \quad (\omega_2)^2 = (f_3)^2\frac{2(f_3)^2(f_{11}+f_{33})^2}{2(f_3)^2(f_{11}+f_{33})^2 - X},$$

then it is clear that when $X < 0$ the right hand sides of (14) are strictly positive and hence that there are precisely 4 solutions for the 1-form ω_j, as claimed above.

Now, let us also write out (12) in special coördinates:–

$$\begin{aligned}(f_3)^3 f_{333} + f_3[f_{113}(\omega_1)^2 + 2f_{123}\omega_1\omega_2 + f_{223}(\omega_2)^2] \\ + 2(f_3)^2[(f_{13})^2 + (f_{23})^2 + (f_{33})^2] \\ - 2[(f_{11})^2 + (f_{13})^2](\omega_1)^2 - 4f_{13}f_{23}\omega_1\omega_2 - 2[(f_{22})^2 + (f_{23})^2](\omega_2)^2 &= 0.\end{aligned}$$

If we rewrite this equation as

$$\begin{aligned}&(f_3)^3 f_{333} + 2(f_3)^2[(f_{13})^2 + (f_{23})^2 + (f_{33})^2] \\ &\quad + [f_3 f_{113} - 2(f_{11})^2 - 2(f_{13})^2](\omega_1)^2 \\ &\quad + [f_3 f_{223} - 2(f_{22})^2 - 2(f_{23})^2](\omega_2)^2 = 2[2f_{13}f_{23} - f_3 f_{123}]\omega_1\omega_2,\end{aligned} \tag{15}$$

then it is clear from (14) that the left hand of (15) is completely determined by the third jet of f at the basepoint, whilst the right hand side of (15) is determined up to sign. By squaring both sides and substituting for $(\omega_1)^2$ and $(\omega_2)^2$ according to (14), we certainly obtain a partial differential equation providing a necessary condition for f to have a conjugate where $X < 0$. The only problem with this partial differential equation is that it is written in special coördinates. It is, however, a matter of trial and error to express the left hand side of (15), once substituted from (14), as a rational expression in the conformal invariants J, Z, X, R, and S. It turns out to be

$$\frac{(ZS - 2XR + 2XZ^2 - 4JX^2)}{2(Z^2 - 2JX)}. \tag{16}$$

Concerning the square of the right hand side of (15),

$$(\omega_1)^2(\omega_2)^2 = -(f_3)^4\frac{(f_{22} + f_{33})(f_{11} + f_{33})}{(f_{22} - f_{11})^2} = -(f_3)^6\frac{X}{2(Z^2 - 2JX)}$$

and, in special coördinates, we find

$$V = -4(f_3)^5(f_{22} - f_{11})[f_3 f_{123} - 2f_{13}f_{23}]$$

whence

$$[f_3 f_{123} - 2f_{13}f_{23}]^2 = (f_3)^{-10}\frac{V^2}{16(f_{22} - f_{11})^2} = (f_3)^{-6}\frac{V^2}{16(Z^2 - 2JX)}.$$

Putting this together,

$$(2[f_3 f_{123} - 2f_{13}f_{23}]\omega_1\omega_2)^2 = -\frac{XV^2}{8(Z^2 - 2JX)^2}.$$

Together with (16) we conclude that

$$\frac{2(ZS - 2XR + 2XZ^2 - 4JX^2)^2}{8(Z^2 - 2JX)^2} = -\frac{XV^2}{8(Z^2 - 2JX)^2}$$

and (10) is obtained by equating numerators.

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