

THE BEHAVIOUR OF STOCHASTIC RUMOURS

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Abstract

This thesis presents results concerning the limiting behaviour of stochastic rumour processes.

The first result involves our published analysis of the evolution for the general initial conditions of the (common) deterministic limiting version of the classical Daley-Kendall and Maki-Thompson stochastic rumour models, [14].

The second result being also part of the general analysis in [14] involves a new approach to stiflers in the rumour process. This approach aims at distinguishing two main types of stiflers. The analytical and stochastic numerical results of two types of stiflers in [14] are presented in this thesis.

The third result is that the formulae to find the total number of transitions of a stochastic rumour process with a general case of the Daley-Kendall and Maki-Thompson classical models are developed and presented here, as already presented in [16].

The fourth result is that the problem is taken into account as an optimal control

problem and an impulsive control element is introduced to minimize the number of final ignorants in the stochastic rumour process by repeating the process. Our published results are presented in this thesis as appeared in [15] and [86].

Numerical results produced by our algorithm developed for the extended [MT] model and [DK] model are demonstrated by tables in all details of numerical values in the appendices.

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being made available in all forms of media, now or hereafter known.

Signed..... Date.....

”Sevgi sabırlıdır, sevgi şevkatlidir.

Sevgi kıskanmaz, övünmez, böbürlenmez.

Sevgi kaba davranmaz, kendi çıkarımı

aramaz. Sevgi haksızlığa sevinmez,

gerçek olanla sevinir. Sevgi

her şeye katlanır, her şeye inanır,

her şeyi umut eder, her şeye dayanır.”

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Chapter 1

Preliminaries

1.1 Structure of the thesis

In this Chapter, a sociological introduction to rumours is presented. Then, a historical background and motivation of the mathematical theory of epidemics and rumours and comparisons of distinction of stiflers in epidemics and rumours are presented. Some useful terms such as meaningful interaction, first type of stiflers and second type of stiflers are introduced. A general comparison of the deterministic modelling and stochastic modelling is given. A specific example as a real life application is introduced for the distinction of the stiflers. A general overview to the literature of mathematical theory of rumours is given.

Chapter 2 deals with analytical solutions for two types of stiflers and some convexity

properties of certain functions of these two types of stiflers. Two extensions are given in the treatment of the classical Daley-Kendall and Maki-Thompson Stochastic Rumour Models. First, arbitrary initial numbers of ignorants and spreaders are allowed. Secondly, stiflers are distinguished according to their provenance. In the analysis, analytical and numerical solutions are given for the general extended model introduced and analytical solution is motivated by the *Lambert W* function. Some convexity properties of first type of stiflers and second type of stiflers are also derived. An algorithm is proposed for a stochastic numerical solutions of the model. It is observed that the algorithm given converges in all cases considered.

Chapter 3 deals with the formulization of the number of transitions in the rumour process. The number of transitions of the stochastic rumour process is formulated for our extended models of both classical Daley-Kendall and Maki-Thompson Rumour Models. The extension includes general initial conditions and distinguished stiflers that are described in detail and worked on in Chapter 2.

In Chapter 4, an optimization formulation is given to describe the behaviour of stochastic rumours with two types of spreaders and two and more than two, multiple broadcasts initiated in the same process at different times.

Chapter 5 presents general conclusions of this thesis.

This thesis also includes six appendices containing some useful definitions and theo-

rems and mainly computational details and tables illustrating our stochastic numerical results. In particular, we give lower and upper bounds for the [DK] model by using two numerical methods, namely a *direct iteration method* and *Newton's method* in Appendix E. These two methods both converge to a lower bound at 0.0556483 and an upper bound, 0.203188. As expected, *Newton's method* converges more rapidly than *direct iteration method*.

The bibliography gives more than one hundred references, some of which are not cited in the body of the thesis, but which provide valuable discussion of certain theoretical results underlying the techniques used in the thesis.

1.2 Rumours in social life and beyond

Rumours are social facts that can play roles in many aspects of social life from family life to politics. The spread of rumours, which can be regarded as a specific spread of information or disinformation, is analyzed by using mathematical tools in this thesis. Before the mathematical background and analysis are given, we consider the social side of rumours in this introductory section.

The role of rumours may be illustrated by the following examples, the majority of which are drawn from Difonzo and Bordia [35]. They may be deliberately employed in the cases of conflicts for manipulation purposes – for example, they were used by

German agents in World War II to demoralize the French (Knapp [70]). Rumours can also play a significant role in the stock market [34]: rumours related to securities have affected stock price changes (for example, see Koenig [71]; Ross [100]). Rumour and legend are discussed (Cornwell and Hobbs [31]) with the main theme being irregular interactions between social psychology and folklore. The role of rumour can also be seen in ancient life. See [74] for the connections between rumour and communication in Roman politics. Another example concerns Haiti: rumours in politics have been a fundamental part of social life in Haiti for the last three decades. This country has been struggling in the grip of violence, fear and despotic repression where stories of violence and magic can paralyze people with fear and confusion. See Perice [87] for further details. How should an organization deal with ugly and potentially damaging rumours? This question, as well as the reining in of rumours in general, has been addressed by Difonzo, Bordia and Rosnow [34].

We conclude this section by citing the following sociological definition of rumour from [35]. “Rumour has been defined as information that: (a) is not verified, (b) is of local or current interest or importance, and (c) is intended primarily for belief. The first element of this definition pertains to the poor quality of the authenticating data for the information. Rumours are coloured by various shades of doubt because they are not accompanied by the “*secure standard of evidence*” (Allport and Postman [1]) that

would either confirm the rumour as truth or discredit it as falsehood. The difference between news and rumour is helpful here; *news* is always confirmed, but rumour is always unconfirmed (Rosnow [96] [97] [98] [99], Shibutani [102], DiFonzo [34]).”

1.3 The beginning of the study of epidemics

The study of rumours was initially motivated and influenced by the study of epidemics and it was long regarded, from the 1940s to the 1960s, as part of the general scheme of epidemics. We will outline a historical background of epidemics as given in [7] below.

Studies of epidemics date back to ancient Greek times. Hippocrates (459-377 B.C.) [63] is an example. However the main progress in the area starts in the 19th century. The most spectacular developments were made by Pasteur(1822-1895) [84] and Koch(1843-1910) [3] in bacteriological science. Nevertheless, some progress had already been made in the statistical analysis of records showing the incidence and locality of known cases of diseases. In the 17th century, John Graunt [59] and William Petty [88] had paid considerable attention to the London Bills of Mortality .

The above-mentioned work of Graunt and Petty may be taken to mark the beginning of vital and medical statistics and the understanding of large-scale phenomena connected with disease and mortality—but their approach was far from a connected theory of epidemics. Indeed, although this was a time of great progress in the field

of physics, particularly mechanics and astronomy, nearly 200 years passed before any real progress was achieved in the biological sphere. The next major advance came in 1855 when John Snow [105] showed, by studying the temporal and spatial pattern of cholera cases, that disease was being spread by the contamination of water supplies. Later, in 1873, William Budd [21] established a similar manner of spread for typhoid. Meanwhile, statistical returns had been made by William Farr(1807-1883) [42] [43], who studied empirical laws underlying the waxing and waning of epidemic outbreaks.

1.4 The mathematical theory of rumours and epidemics

The mathematical analysis, particularly stochastic analysis, of a rumour started with Daley and Kendall [32]. We shall call their model the [DK] model in this thesis. In fact, earlier literature on a deterministic rumour model dependent on epidemics was started by Rapoport who developed several models for the diffusion of information during the period 1948–54: see for example, Rapoport [92]; Rapoport and Rebhun [91].

The mathematical theory of epidemics began with Ross [101] whose mathematical model is essentially a deterministic model. Later, Kermack and McKendrick [68] and Soper [106] studied deterministic epidemic models. Bailey [6], in his analysis of

their (Ross, Kermack and McKendrick, Soper) work, reports that they assumed that for given numbers of susceptibles and infectious individuals, and given attack and removal rates, a certain definite number of fresh cases would occur in any specified time. However, it is widely realized that an appreciable element of chance enters into the conditions under which new infections or removals take place. Increasing use of stochastic modelling was made in the study of epidemics during the middle of last century (eg see Bailey [6]).

In these models we have, for any given instant of time, probability distributions for the total numbers of susceptible and infected individuals replacing the single point-values of the deterministic treatments. Stochastic models have a special importance in this context due to the fact that for epidemic processes stochastic means are not the same as the corresponding deterministic values.

1.5 The differences and similarities between the spread of rumours and epidemics

The spread of rumours and epidemics may be described in a variety of ways. Most work involves empirical and theoretical work to describe and formulate the spread of rumours or epidemics. Although rumours and epidemics are often thought of together,

the literature for rumours is much less developed than that for epidemics. This is partly because of the relative complexity of the rumour process. The complexity mainly arises from differences in the removal mechanism. This makes the analysis different from, and more difficult than, that of epidemics. In epidemics, an individual becomes an immune through death or isolation. On the other hand, in rumours, *spreaders* may become *stiflers* in two different ways. One is the case when two spreaders encounter each other, and the other is when a *spreader* encounters a *stifler*. The superficial similarities and differences between rumours and epidemics were discussed for the first time by Daley and Kendall [32]. Since then, the spread of rumours has been studied as a separate phenomenon from that of epidemics.

Examples of extension studies on rumour modelling include, in Belen, Kaya and Pearce [15], the behaviour of stochastic rumour as an optimal control problem. The general case of the rumour problem is analysed and an optimization formulation is given where the control input is impulsive. See also Chapter 4.

In Rhodes and Anderson [95], a rumour model is utilized to understand the spread of forest fires.

A remarkable work for the general stochastic epidemic is due to Siskind [103] who, more than three decades ago, gave a solution of the general stochastic epidemic¹.

¹without using Laplace transforms

1.6 The role of deterministic models and stochastic models for some real life problems

The models used to explain diffusion of information or the spread of rumours serve as a tool to understand a social phenomenon better. It is reasonable to model the spread of rumours as growth (or decay) processes. However, the predictions provided by the two types of model, deterministic and stochastic, are naturally different.

Stochastic as a word derives from the Greek word “*στοχαστης*” meaning “random” or “chance”. A *stochastic model* is a mathematical model which describes a natural phenomenon and predicts the possible outcomes of experiments or chance events occurring randomly in time that are weighted by their probabilities. Thus, stochastic models are applicable to any system involving random variability as time passes. There are various application areas from *geophysics* to *social science*. Some examples are as follows: stochastic models have been used for the prediction of the size and whereabouts of earthquakes [25]; stochastic models are used to describe and solve environmental and investment problems [77]; they have been applied in the study of learning [23]; life contingencies [64]; buying behaviour [79]; manufacturing systems [24]; operations research [62]; and the production of material handling systems in the construction industry [50]. Of more immediate relevance to the work presented in this

thesis, stochastic modelling has been widely applied in the study of epidemics and the study of diffusion of information or spread of rumours in the population of individuals (Bailey [6] [7] [8] [9], Daley and Gani [33], Frauenthal [46], Foster [45], Gani [52] [53], Kermack and McKendrick [68], Siskind [103]).

In comparing stochastic and deterministic models with an example, the deterministic model for a rate of growth of population provides a function giving the size of the population for any specified time while the stochastic model gives a probability distribution of population size for each time.

It is often found that the deterministic version of a model is required to make progress towards having a numerical solution. However, the deterministic model is regarded as an approximation.

1.7 Notation and terminology

Traditionally rumour modelling supposes a homogeneously mixed population without immigration, emigration, deaths, or births and we will adhere to this assumption. This closed population is classified into three classes called *spreaders*, *ignorants*, and *stiflers* who have ceased to spread the rumour, that correspond to *infectives*, *susceptibles* and *removed cases* respectively in epidemic theory. In this thesis, the notations i , s , and r are used for the number of ignorants, spreaders and stiflers respectively.

We shall call an interaction *meaningful* when it is between two spreaders *or* a spreader and an ignorant *or* a spreader and a stifler.

We shall differentiate stiflers into two types depending on the sort of interaction by which they became stiflers. Thus, a population of stiflers is further classified into two stifier sub-populations. Meaningful interaction between two spreaders results in either both becoming stiflers or one of them becoming a stifier while the other one remains a spreader. We shall refer to these as *stiflers of the first type* and denote their number by r_1 . If a spreader meets a stifier, then the spreader becomes a stifier of *second type* and we denote the number of this type of stifier by r_2 . Let r_0 be the total number of initial stiflers. Then $r_0 = r_{1,0} + r_{2,0} = 0$ in the standard formulation but we shall allow $r_0 > 0$ in this thesis.

We shall denote the size of population by either n or N . The former is more natural for our development while the latter, used in earlier literature, makes for simpler comparisons with that literature. Each of the notations of population size is re-introduced when it is used in the context in order to avoid confusion.

The result of a *meaningful interaction* is a *transition* from one *state* to another. With a closed population, the rumour process is finite in the sense that all of the spreaders become stiflers after a finite number of transitions. We denote this number of transitions by T and the number of spreaders at the end of the process is given by

$$s(T) = 0.$$

1.8 The classical methods in rumour models

A general method to solve a stochastic rumour problem is given as the *Principle of the Diffusion of Arbitrary Constants* by Daley and Kendall [32]. The differential equations represent the deterministic analogue of a stochastic process and if they can be explicitly solved, then this principle can be applied to other problems.

This method to estimate the rumour process which was constructed and developed by Daley and Kendall [32] is that of a *diffusion approximation*. They use the solution obtained from their deterministic model's ordinary differential equations to provide a constant of motion $\lambda(s, i)$ and replace this² by a random variable $\lambda(s(t), i(t))$. The distribution of the terminal value of λ is calculated³ as $\lambda_\infty = 2n + 1 + \Delta\lambda$. They refer to this as "the general method" for determining the value of this λ where the representative point on the path of a stochastic rumour takes small jumps from one deterministic path to another and each of these paths has its own λ -value". *Monte Carlo experiments* are also used to give numerical results which illustrate the effectiveness of the "principle" in estimating the evolution of the rumour process.

An important result [32] obtained by the use of the principle was that the proportion

²Here, i represents the number of ignorants and s represents the number of spreaders

³Here, $n + 1$ is the population size

of the population who never heard the rumour is 0.203188 asymptotically under the initial conditions where the initial number of spreaders is 1 and initial stiflers 0. See further [32].

Barbour [11] also discusses the principle of the diffusion of arbitrary constants. He derives a technique for obtaining analytical approximations to the first few cumulants of the continuous-time Markov lattice process when the population size becomes large, in cases where each transition rate is a multinomial expression in the lattice coordinates.

The second classical model for the spread of rumours, introduced by Maki and Thompson [78], is formulated as a discrete time Markov chain. We shall call their model the [MT] model. There is a typographical error in their publication, the proportion of ignorants to the total population is 0.238. When this is corrected, their paper gives that when the rumour process having the same initial condition as the [DK] model stops, the proportion of ignorants will be 0.203188, the same proportion as the [DK] model.

The ‘principle’ is not our main topic in this thesis— however, the system of differential equations which represents the deterministic analogue to our stochastic process in this thesis has the same characteristics as the [DK] model’s differential equations, and is solved explicitly. The numerical results obtained by our algorithm for the general stochastic rumour process also illustrate strong agreement with Daley and Kendall’s

results [32] and Maki and Thompson's results [78]. We reproduce those classical results in this thesis. See Table 3.4, Table 3.5 and other tables which appear throughout this thesis, including Appendices from Appendix C to Appendix F.

In the following paragraphs, we summarise the other main contributions to the mathematical theory of rumour modelling.

Bartholomew [12] discusses general epidemic models for the diffusion of news, rumours, and ideas. He gives both a deterministic approximation and a stochastic analysis for these models.

Dunstan [37] considers the general epidemic model developed (e.g. by Bailey [9]) for the diffusion of information and uses it as a model for the spread of rumours. Recursive expressions are found for the mean of the final size of each generation of hearers⁴. Simple expressions are found for the generation size and the asymptotic form of its final size in his deterministic model.

Sudbury [108] uses a martingale method for the [MT] model and shows that the proportion of the population never hearing the rumour converges in probability to 0.203 and the proportion of the transitions to the total population converges in probability to 1.594 as the population size becomes very large.

⁴*Hearer* : an individual who receives the information at any time. *Generation*: someone who receives the news direct from the source is called *first generation*, someone who hears from the first generation but not directly is called *second generation* and so on.

Later, Watson [113] examined the size of a rumour, defined as the number of individuals in the population eventually hearing the rumour.

Lefevre and Picard [75] examined a distribution of the final extent of a continuous-time rumour process on the basis of the [MT] model by using Gontcharoff polynomials. Pearce [85] gave a characterization for the time-dependent evolution of rumour processes. The method used is a probability generating function method for rumours and is more general than the [MT] and [DK] models.

Daley and Gani [33] reproduce the results in [32]. They suggest a k -fold stifling model and describe the deterministic versions of the k -fold stifling and (α, p) probability variants of the [DK] model, in which they suppose that a spreader attempts to spread the rumour with probability p and the spreader becomes a stifler with probability α . The basic model has $p = \alpha = 1$. It is assumed that a spreader does not decide to cease spreading the rumour until being involved in k stifling experiences and it is for this reason that the model is called k -fold stifling.

We give a solution for the k -fold stifling variant of [MT] model on page **125** in Appendix D.

More relevant works, Belen and Pearce [13] and Belen and Pearce [14], take care of analytical and numerical solutions for some refinements of the classical [MT] model with *general initial conditions*. Chapter 2 gives further details. Also in Belen, Kaya

and Pearce [15] an optimization formulation is given to describe the behaviour of a stochastic rumour by introducing two types of spreaders and two broadcasts initiated in the same process at different times and introducing impulsive control element in the process. Chapter 4 gives further details.

1.9 More general rumour models

Daley and Kendall were unable to find exact solutions to their models ([DK] model) and much of the literature has followed a variant of the [MT] model. In this thesis, we consider a further variant of both of these classical rumour models.

The spread of a rumour as a social process has many interesting aspects but these have not been considered in depth in mathematical treatments of rumours. One such aspect is the psychological effect of a rumour on the individual.

If an individual stops spreading a rumour then s/he probably has a psychological reason behind that decision. In some cases, an individual may stop spreading a rumour because of economic reasons. (For example, a spreader may be paid to spread the rumour once but when the payment stops later, the spreader may stop spreading the rumour.) It is a cynical truism that a spreader may be paid to disseminate a rumour and will desist when payment discontinues. In our work we consider psychological reasons behind the decisions for being stiflers and distinguish stiflers into two main

types.

We shall assume that ignorants, spreaders and stiflers are subject to homogeneous mixing and that meaningful interactions between individuals occur at a uniform rate proportional to their rate of encounter. Interaction between an ignorant and a spreader results in the ignorant becoming a spreader; that between a spreader and a stifter in the spreader becoming a stifter; that between two spreaders in one of them becoming a stifter. Other encounters are taken as being without effect. If there is initially at least one spreader and at least one non-spreader such a process of interactions results eventually in a population in which no spreaders remain. The proportion of the total population never to have heard the rumour at this stage (or subsequently) is given by a probability distribution on the interval $(0,1)$. A remarkable result relating to this has been outlined in section 1.8.

Surprisingly this result does not appear to have been explored further. If the effect of a single initial spreader is that nearly 80 per cent of a large population eventually hear a rumour, what would be the effect of a larger number of initial spreaders? Such a question is pertinent in the current age of mass media, where a once-off statement can proceed immediately to a large segment of a population, whence it is further disseminated by person to person interaction. Depending on the nature of the rumour, the notion that there are initially no stiflers may also be inappropriate.

The previous paragraph relates merely to variation of the initial conditions of the process. There is also a structural variation that suggests itself, namely, the possibility that spreaders who become stiflers by virtue of an interaction with other spreaders may in some way be different from those who become stiflers because of interaction with stiflers. For example, while both groups are stiflers, the former may still believe the rumour but see no point in further spreading it, while the latter may have actually ceased to believe the rumour. Here we pursue these possibilities. We allow general initial conditions and distinguish two types of new stifier according to the mechanism whereby they became stiflers. These two types occur either after an encounter between two spreaders or after an encounter of a spreader with a stifier.

The two mechanisms may represent different subpopulations which can be of interest in applications. For example, let us consider a homogeneously mixed population, whose $n + 1$ members are assumed to be the voters for certain political parties in a country. This example is motivated by an actual rumour about the former leader of the Christian Democrat Party in Germany, Helmut Kohl, that was related to donations given to the party [20] [117]. After this rumour was circulated his party lost many voters in the next local elections.

Let the parties be classified at time t_0 into two groups, namely *party A* of size $n_1(t_0)$ and *other parties* of size $n + 1 - n_1(t_0)$. Suppose that a rumour about the leader of

party A is introduced by one or more people in this population.

Other parties hope this will influence voters for party A to change their votes and so wish to spread it to as many people as possible. Clearly a spreader doesn't intend to vote for Party A.

- When two spreaders meet, each mentions the rumour to the other. In rumour theory it is usually presumed that as a result of this encounter one spreader will desist from further rumour-mongering ([MT] model) or that both will desist ([DK] model). Here, a first type of stifier occurs. This assumption is less restrictive than might appear to be the case at first sight, since the encounter rate can be nominated so as to reflect only that proportion of encounters associated with stifling.
- When a spreader encounters a stifier, the spreader thereafter desists from spreading the rumour. Again, use of an appropriate encounter rate allows for the scenario that only a proportion of such encounters stifle spreaders. The spreader is assumed to come to the belief that the society does not care about the allegation, is not affected by it, or that s/he can not change outcomes by spreading the allegation. Thus the spreader is discouraged and gives up supporting any party and *becomes a neutral*, i.e. a second type of stifier.

These motivations give rise to people who are either voters for the other parties or

neutral voters. This is a rationale for our assumption of two different types of stiflers in this specific example.

Suppose the initial population is large. It follows from the results of Chapter 2 and Chapter 4 that, according to the model described above, at the end of the rumour process other parties will take $0.324(= r_1(\infty))$ votes of the initial ignorant voters of party A. The proportion of the initial voters of party A who become neutral ($r_2(\infty)$) will be $0.473n_1(t_0)$ in our example.

The model also indicates that party A will retain $0.368n_1(t_0)$ votes of its initial voters. According to our analysis, the model gives the proportion of the first type of stiflers to total population as $r_1(\infty) = 0.5$, and the proportion of the second type of stiflers to total population as $r_2(\infty) = 0.5$, with respect to general initial conditions in which the initial proportions of stiflers and spreaders are greater than zero and α goes to 0.

Another example as an application for different type of stiflers is given in Chapter 2.

Chapter 2

Rumours with a general number of initial spreaders

2.1 Introduction

A rigorous treatment of the limiting behaviour of stochastic rumour processes proved unexpectedly tricky and the literature has mainly addressed technical questions of stochastic convergence, mostly via diffusion-type approximations and martingale arguments (see, for example, Barbour [11], Sudbury [108], Pittel [89] and Watson [113]). It is probably true to say that broader questions for those models are still largely unexplored. For example, the standard assumption that a rumour is initiated by a single spreader, while doubtless true in many concrete examples, is certainly inappropriate

for others in this current age of mass communication, where a rumour may be initiated by television or radio. In this chapter we adopt general initial conditions and consider the evolution of the models. We examine how the initial conditions bear on what proportion of ignorants and of the total population never get to hear the rumour.

We discover *inter alia* the perhaps surprising result that, even when the initial proportion of spreaders in the population tends to unity, the fraction of the subpopulation of initial ignorants that never hear the rumour does not approach zero.

Our second innovation derives from a closer perspective on stiflers. In stochastic rumours, stiflers are produced by two distinct mechanisms, the interaction of a spreader with a stifier and the interaction of two spreaders. However no attention is given to this differential genesis once a stifier has been produced. The two mechanisms may represent different subpopulations which can be of interest in applications. A homely example is supplied by women's fashion: a person may adopt a new fashion because it is new and distinctive, another may resist it for that reason until it has obtained widespread currency. The first person is likely to lose interest in the fashion and become a stifier when it is widespread (spreader–spreader interactions), the second when many others have already dropped the fashion (spreader–stifier interaction).

This example suggests it may sometimes be worthwhile to distinguish subpopulations amongst the ignorants and the spreaders as well. We eschew this initiation and

pursue only the distinction of two types of stiflers within the framework of existing models.

In order to uncover some new results without becoming enmeshed in technicalities we shall adopt a broad brush stroke and, after introducing stochastic rumours in Section (2.2), proceed using a continuous deterministic approximation via differential equations as in the seminal article of Daley and Kendall [32].

In Section (2.3), we treat the evolution of the deterministic model with time. In Section (2.4), we find the proportion of the initial ignorants who never hear the rumour and in Section (2.5), the proportion of the whole population who never hear the rumour. In Section (2.6) we calculate the final proportions of the population belonging to the two stifier subpopulations. Again we find a result not obvious by intuition: one of these proportions is a monotone function of the initial proportion of ignorants while the other is not!

2.2 The model

In this chapter, a stochastic rumour pertains to a fixed population of n individuals consisting of subpopulations of ignorants, spreaders and stiflers as we noted at page 10.

Homogeneous mixing of individuals occurs, with a given proportion of ignorant-

spreader interactions leading to the ignorant becoming a spreader and the same proportion of spreader–stifler interactions resulting in the spreader becoming a stifler. A similar phenomenon occurs with spreader–spreader interactions. In the [DK] model, the outcome is two stiflers. The [MT] model distinguishes between an initiating and a receptor spreader in such an interaction and only one spreader converts to being a stifler as a result of an encounter.

With each of these models a sequence of state transitions occurs. There are three types of transitions. Let i, s, r be the respective numbers of ignorants, spreaders and stiflers at a given moment. The ignorant–spreader and spreader–stifler interactions respectively may be expressed as state transitions

$$\begin{aligned} (i, s, r) &\longrightarrow (i - 1, s + 1, r), \\ (i, s, r) &\longrightarrow (i, s - 1, r + 1). \end{aligned} \tag{2.2.1}$$

The spreader–spreader interaction is

$$(i, s, r) \longrightarrow (i, s - 2, r + 2) \tag{2.2.2}$$

for the Daley-Kendall version of the process and

$$(i, s, r) \longrightarrow (i, s - 1, r + 1) \tag{2.2.3}$$

for the Maki-Thompson version. We remark that (2.2.1) and (2.2.3) are formally the same, though the first (the spreader–stifler interaction) occurs at rate sr and the latter

(the spreader–spreader interaction) at rate $s(s - 1)$. Such sequences lead inexorably (after a finite number of transitions) to states in which there are no spreaders left.

Note that we get $r = r_1 + r_2$ at this stage.

We now turn our attention to the limiting forms of these models as the total population size tends to infinity. We adopt a continuum formulation. Let $i(t), s(t), r(t)$ denote the proportions of the total population at time t that are respectively ignorants, spreaders and stiflers. With an appropriate choice of time scale, the common coefficient for interactions leading to a change of subpopulation of an individual can be taken as unity. The [DK] stochastic model and [MT] stochastic model lead to the same set of coupled deterministic subpopulation equations

$$\frac{di}{dt} = -is, \tag{2.2.4}$$

$$\frac{ds}{dt} = is - s^2 - sr = s(2i - 1), \tag{2.2.5}$$

$$\begin{aligned} \frac{dr}{dt} &= s^2 + sr \\ &= s(1 - i), \end{aligned} \tag{2.2.6}$$

which apply in the limit of a total population size tending to infinity. We adopt the initial conditions

$$i(0) = \alpha > 0, \quad s(0) = \beta > 0, \quad r(0) = \gamma, \quad \text{with} \quad \alpha + \beta + \gamma = 1. \tag{2.2.7}$$

We remark that (2.2.6) may be considered redundant, since

$$r = 1 - i - s. \quad (2.2.8)$$

It is convenient to introduce the parameter $\theta = \theta(\tau) := i/\alpha$, the ratio of the proportion of ignorants at time τ to the initial proportion. In the next section we address the dynamics and asymptotics of the continuum rumour process.

2.3 Evolution of the System

Theorem 1. *The evolution of the rumour process prescribed by (2.2.4)–(2.2.6) is given parametrically in terms of i by*

$$s = \beta - 2(i - \alpha) + \ln(i/\alpha) \quad (2.3.1)$$

$$= \beta - 2\alpha(\theta - 1) + \ln \theta \quad (2.3.2)$$

and (2.2.8).

The process evolves towards an asymptotic state $(i_\infty, 0, r_\infty)$, with

$$i \downarrow i_\infty = i_\infty(\alpha, \beta) \text{ as } \tau \rightarrow \infty$$

and

$$0 < i_\infty < 1/2. \quad (2.3.3)$$

The parameter $\theta_\infty := i_\infty/\alpha$ satisfies the transcendental equation

$$\theta_\infty e^{2\alpha(1-\theta_\infty)} = e^{-\beta}. \quad (2.3.4)$$

Further,

$$s \rightarrow 0 \quad \text{and} \quad r(\tau) \uparrow r_\infty = 1 - i_\infty \quad \text{as} \quad \tau \rightarrow \infty.$$

Proof. Equation (2.2.4) implies that i is a strictly decreasing function of τ and may therefore be used as an independent parameter. Combining (2.2.4) and (2.2.5) provides the relation

$$\frac{ds}{di} = \frac{1-2i}{i}, \quad (2.3.5)$$

which integrates to give (2.3.1). The value of r is then determined by (2.2.8).

Being strictly decreasing and bounded below by zero and above by unity, i must therefore tend to some limit $i_\infty < 1$ as $\tau \rightarrow \infty$. By (2.2.6), r is strictly increasing with time. Since it is bounded above by unity, it must tend to a limit $r_\infty > 0$ as $\tau \rightarrow \infty$. Also, since $i_\infty < 1$, we have from (2.2.6) that $s \rightarrow 0$ as $\tau \rightarrow \infty$, or equivalently $s \rightarrow 0$ as $i \rightarrow i_\infty$.

If $\alpha \leq 1/2$, then $i(\tau) < 1/2$ for all $\tau > 0$ since i is strictly decreasing, and hence $i_\infty < 1/2$. If $\alpha > 1/2$, then by (2.3.5) $ds/di < 0$ and

$$\frac{ds}{d\tau} = \frac{ds}{di} \cdot \frac{di}{d\tau} > 0$$

initially. Since $s \rightarrow 0$ as $\tau \rightarrow \infty$, $s(\tau)$ must first increase to a local (and global) maximum (at which time $i = 1/2$) and by (2.3.5) decrease thereafter. Because i is strictly decreasing, we thus have $i_\infty < 1/2$. Since $r_\infty > 0$, we have also that $i_\infty = 1 - r_\infty > 0$.

Finally, letting $\tau \rightarrow \infty$ in (2.3.2) yields

$$0 = \beta - 2\alpha(\theta_\infty - 1) + \ln \theta_\infty, \quad (2.3.6)$$

which is just (2.3.4).

Equation (2.3.6) may be expressed as

$$we^w = -2\alpha e^{-2\alpha-\beta}, \quad (2.3.7)$$

where $w := -2\alpha\theta_\infty$.

The equation

$$xe^x = y \quad (2.3.8)$$

has two real solutions when $-1/e < y < 0$ (see Figure 2.1). We have

$$-2\alpha e^{-2\alpha} < -2\alpha e^{-2\alpha-\beta}$$

for any $\alpha, \beta > 0$, so that $xe^x|_{x=-2\alpha} < xe^x|_{x=-2\alpha\theta}$ for $0 < \alpha < 1$. Hence one of the real solutions of (2.3.7) is less than -2α and the other greater than -2α . As we must have $0 < \theta < 1$, the physical solution to (2.3.7) is the one greater than -2α , that is, the

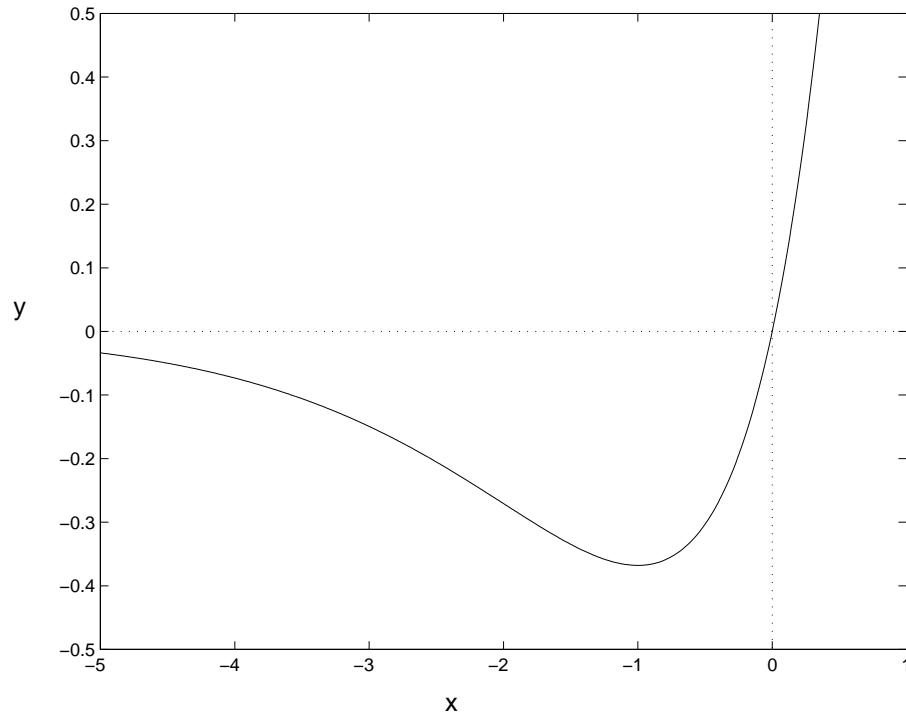


Figure 2.1: The graph of the equation $x = ye^y$

numerically smaller real solution of (2.3.8). The function $w = w(y)$ giving the unique real solution to (2.3.8) for $y \geq 0$ and the numerically smaller real solution for $y < 0$ has been in the literature for over 200 years and is known as the *Lambert w* function (see [29]). Lagrange's expansion provides an explicit series evaluation

$$w = \sum_{k=1}^{\infty} \frac{(-y)^k}{k!} k^{k-1}.$$

Thus for $\alpha, \beta > 0$

$$\theta = -\frac{1}{2\alpha} w(-2\alpha e^{-2\alpha-\beta}) = \sum_{k=1}^{\infty} \frac{(-2\alpha k)^{k-1}}{k!} \exp(-k(2\alpha + \beta)).$$

2.4 Proportion of Ignorants Never Hearing the Rumour

Theorem 2.

(a) For $\alpha + \gamma = 1 - \beta$ fixed, θ_∞ is strictly decreasing in α .

(b) For $\beta + \gamma = 1 - \alpha$ fixed, θ_∞ is strictly decreasing in β .

(c) For $\alpha + \beta = 1 - \gamma$ fixed, θ_∞ is strictly decreasing in α .

Proof. Consider situation (a). Implicit differentiation of (2.3.6) with respect to α provides

$$1 - (-1) - 2\alpha \frac{\partial \theta_\infty}{\partial \alpha} - 2\theta_\infty + \frac{1}{\theta_\infty} \frac{\partial \theta_\infty}{\partial \alpha} = 0$$

since

$$\gamma = 1 - \beta - \alpha$$

where $1 - \beta$ fixed.

$$\frac{\partial \theta_\infty}{\partial \alpha} \left(-2\alpha + \frac{1}{\theta_\infty}\right) + 2 - 2\theta_\infty = 0$$

$$\frac{\partial \theta_\infty}{\partial \alpha} = -2\theta_\infty \frac{1 - \theta_\infty}{1 - 2\alpha\theta_\infty}$$

which is negative since $\alpha\theta_\infty = i_\infty$, $0 < \alpha < 1$ and $0 < i_\infty < 1/2$. This establishes the

result in (a).

Similarly in the context of (b), we have

$$\beta - 2(\theta_\infty - 1 - \beta\theta_\infty + \beta - \gamma\theta_\infty + \gamma) + \ln\theta_\infty = 0$$

$$-\beta - 2\theta_\infty + 2 + 2\beta\theta_\infty + 2\gamma\theta_\infty - 2\gamma + \ln\theta_\infty = 0$$

$$-1 - 2\frac{\partial\theta_\infty}{\partial\beta} + 2\beta\frac{\partial\theta_\infty}{\partial\beta} + 2\theta_\infty + 2\gamma\frac{\partial\theta_\infty}{\partial\beta} - 2\theta_\infty + 2 + \frac{1}{\theta_\infty}\frac{\partial\theta_\infty}{\partial\beta} = 0$$

$$\frac{\partial\theta_\infty}{\partial\beta}(-2 + 2\beta + 2\gamma + \frac{1}{\theta_\infty}) = -1$$

$$\frac{\partial\theta_\infty}{\partial\beta} = -\frac{\theta_\infty}{1 + 2\alpha\theta_\infty} < 0$$

giving the requisite result.

Finally, for (c), suppose $\alpha + \beta = c$, fixed, so that $\theta_\infty = \theta_\infty(\alpha)$ and

$$c - \alpha - 2\alpha(\theta_\infty - 1) + \ln\theta_\infty = 0.$$

Implicit differentiation yields

$$0 - 1 - 2((\theta_\infty - 1) + \alpha\frac{\partial(\theta_\infty - 1)}{\partial\alpha}) + \frac{1}{\theta_\infty}\frac{\partial\theta_\infty}{\partial\alpha} = 0$$

$$-1 - 2\theta_\infty + 2 - 2\alpha \frac{\partial \theta_\infty}{\partial \alpha} + \frac{1}{\theta_\infty} \frac{\partial \theta_\infty}{\partial \alpha} = 0$$

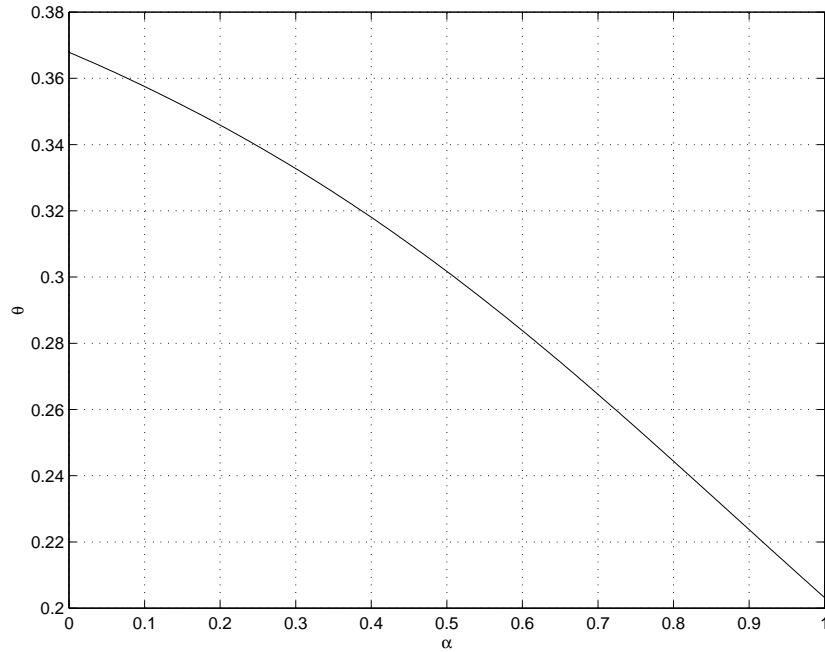
$$\frac{d\theta_\infty}{d\alpha} = -\frac{\theta_\infty(1 - 2\theta_\infty)}{1 - 2\alpha\theta_\infty} < 0. \quad (2.4.1)$$

Hence, the proof is complete.

Figure 2.2 depicts the situation for case (c) with the standard $\gamma = 0$, so that $\alpha + \beta = 1$. For simplicity, we commit an abuse of notation and set $\theta(\alpha) = \theta_\infty(\alpha)$. We have seen that θ is a strictly monotone decreasing function of α on $(0, 1)$. Its infimum satisfies the Daley–Kendall equation $2(1 - \theta) + \ln \theta = 0$ and is $\theta(1) \approx 0.2031878$. The other real solution $\theta = 1$ to this equation is aphysical, as noted in [32]. The supremum of θ is $\theta(0) = 1/e \approx 0.36787944$. That is, we have the somewhat surprising result that when nearly all the population are initially spreaders, it is still the case that a proportion $1/e$ of the initial ignorants never hear the rumour.

The infimum value $\theta(1)$ arises in the limit of total population tending to infinity for a fixed finite initial number of spreaders. The supremum value $\theta(0)$ arises similarly with a fixed finite number of ignorants.

Despite the suggestion from Figure 2.2, θ is not a concave function of α throughout $(0, 1)$. We may see this as follows. Implicit differentiation of (2.3.6) twice with respect

Figure 2.2: The behaviour of the function θ

to α yields

$$\theta(1 - 2\alpha\theta) \frac{d^2\theta}{d\alpha^2} = \frac{d\theta}{d\alpha} \left[4\theta^2 + \frac{d\theta}{d\alpha} \right]$$

or, using (2.4.1),

$$(1 - 2\alpha\theta) \frac{d^2\theta}{d\alpha^2} = \frac{d\theta}{d\alpha} \left[4\theta - \frac{1 - 2\theta}{1 - 2\theta\alpha} \right].$$

For $\alpha \approx 0$, the expression in brackets on the right is $\approx 4\theta(0) - [1 - 2\theta(0)] > 0$, so $d^2\theta/d\alpha^2$ is negative and θ is a strictly concave function of α . On the other hand, for $\alpha \approx 1$, the expression in brackets is $\approx 4\theta(1) - 1 < 0$, so $d^2\theta/d\alpha^2$ is positive and θ is a strictly convex function of α .

In the concluding section we examine the variation of $\zeta := i_\infty = \alpha\theta_\infty$ with the initial conditions.

2.5 Proportion of Total Population Never Hearing the Rumour

The dependence on initial conditions of the proportion ζ of the total population who never hear the rumour is also of interest.

Theorem 3.

- (a) For $\alpha + \gamma = 1 - \beta$ fixed, ζ is strictly increasing in α for $\alpha < 1/2$ and strictly decreasing in α for $\alpha > 1/2$.
- (b) For $\beta + \gamma = 1 - \alpha$ fixed, ζ is strictly decreasing in β .
- (c) For $\alpha + \beta = 1 - \gamma$ fixed, ζ is strictly increasing in α .

Proof. We may rewrite (2.3.6) as

$$\beta - 2(\zeta - \alpha) + \ln \zeta - \ln \alpha = 0. \quad (2.5.1)$$

The argument now follows that of Theorem 2. In (a), (b), (c) we have respectively from implicit differentiation of (4.8.2) that

$$\frac{\partial \zeta}{\partial \alpha} = \frac{\zeta}{\alpha} \cdot \frac{1 - 2\alpha}{1 - 2\zeta},$$

$$\frac{\partial \zeta}{\partial \beta} = \frac{-\zeta}{1 - 2\zeta},$$

$$\frac{d\zeta}{d\alpha} = \frac{\zeta}{\alpha} \cdot \frac{1 - \alpha}{1 - 2\zeta},$$

from which the conclusions follow directly, since $\zeta < 1/2$.

Corollary 1. *We have $\zeta^* := \sup \zeta = 1/2$. This occurs in the limiting case $\alpha = 1/2 = \gamma$, with $\beta = 0$.*

Proof. From (c) of Theorem 2, we have for fixed γ that ζ has supremum approached in the limit $\alpha = 1 - \gamma$ with $\beta = 0$. But from (a), we have in the limit $\beta = 0$ that ζ has supremum arising from $\alpha = 1/2$. This gives the second part of the enunciation.

From (4.8.2), ζ^* satisfies

$$0 = -2\zeta^* + 1 + \ln(2\zeta^*). \quad (2.5.2)$$

For $x > 0$, set

$$g(x) := \ln x - x + 1.$$

Then g is strictly increasing on $(0,1)$ and strictly decreasing on $(1,\infty)$. It follows that $x = 1$ is the only solution to $g(x) = 0$. We deduce from (2.5.2) that $\zeta^* = 1/2$.

Remark: The study of Daley-Kendall and Maki-Thompson places a lower bound for the proportion of ignorants at the end of the rumour process and our study an upper bound.

2.6 Two types of stiflers

We now proceed to distinguish between a spreader who became a stifler as a result of an interaction with another spreader and one who made the change following an

interaction with a stifter. Denote the proportions at time t of these two subpopulations in the total population by $r_1(t)$, $r_2(t)$ respectively, so that $r_1 + r_2 = r$. The asymptotic behaviour of r_1 and r_2 as $t \rightarrow \infty$ is then characterised by

$$\frac{dr_1}{dt} = s^2, \quad \frac{dr_2}{dt} = sr.$$

Coupling the former equation with (2.2.4) provides $dr_1/di = -s/i$, so that by

$$s = 1 + \alpha - 2i + \ln(i/\alpha), \tag{2.6.1}$$

$$\frac{dr_1}{di} = -\frac{1 + \alpha - 2i + \ln(i/\alpha)}{i}.$$

With the initial conditions $i(0) = \alpha$, $r_1(0) = 0$, this may be integrated to yield

$$r_1 = -(1 + \alpha) \ln(i/\alpha) + 2(i - \alpha) - (\ln(i/\alpha))^2/2.$$

For reference, we note that with the more general initial conditions $\gamma_1(0) = \gamma_1$ we derive that

$$r_1(\infty) - \gamma_1 = -(1 + \alpha) \ln \theta - 2\alpha(1 - \theta) - 1/2(\ln \theta)^2.$$

Since $i/\alpha \rightarrow \theta$ as $t \rightarrow \infty$, we derive

$$r_1(\infty) = -(1 + \alpha) \ln \theta - 2\alpha(1 - \theta) - \frac{1}{2}(\ln \theta)^2.$$

Define $\phi_1(\alpha) := r_1(\infty)$. Then in the basic $\gamma_1(0) = 0$ case,

$$\begin{aligned}
\frac{d\phi_1}{d\alpha} &= -\ln \theta - 2(1 - \theta) + \frac{1}{\theta} \frac{d\theta}{d\alpha} (2\alpha\theta - 1 - \alpha - \ln \theta), \\
&= -\ln \theta - 2(1 - \theta) \\
&= -(1 - \alpha)(1 - 2\theta),
\end{aligned}$$

where $i_\infty = \alpha\theta$ (the parameter θ represents the proportion of the ignorant subpopulation who never hear the rumour) was set and since $s_\infty = 0$,

$$1 + \alpha(1 - 2\theta) + \ln \theta = 0, \quad (2.6.2)$$

using (2.6.1).

Thus $d\phi_1/d\alpha < 0$. Hence ϕ_1 is a strictly decreasing function of α for $\alpha \in (0, 1)$ (see Figure 2.3). It has limit 0.32380.. as $\alpha \rightarrow 1$ and limit 0.5 as $\alpha \rightarrow 0$. To see the latter, substitute $\alpha = 0$ in (2.6.2). Then

$$\theta = 1/e, \text{ so } \phi_1(0) = -\ln \theta - \frac{1}{2}(\ln \theta)^2|_{\theta=1/e} = 1/2.$$

Further,

$$\frac{d^2\phi_1}{d\alpha^2} = \left(2 - \frac{1}{\theta}\right) \frac{d\theta}{d\alpha} > 0, \quad (2.6.3)$$

so that ϕ_1 is a strictly convex function of α .

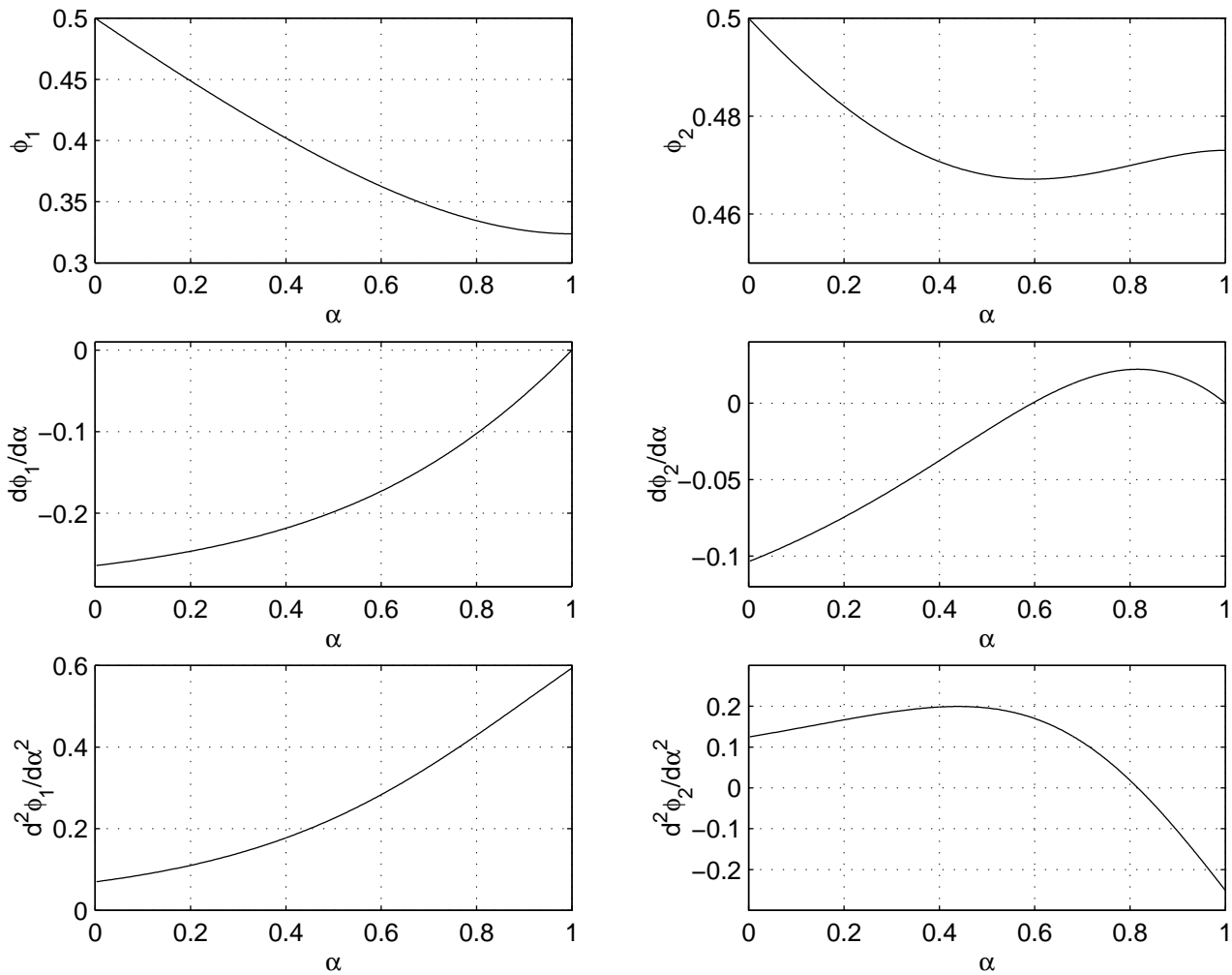


Figure 2.3: The graphs of ϕ_1 , ϕ_2 , and their derivatives as functions of α .

We now consider

$$\phi_2(\alpha) := r_2(\infty) \tag{2.6.4}$$

$$= 1 - i(\infty) - r_1(\infty)$$

$$= 1 - \alpha\theta + (1 + \alpha)\ln\theta + 2\alpha(1 - \theta) + \frac{1}{2}(\ln\theta)^2.$$

We have

$$\begin{aligned}\frac{d\phi_2}{d\alpha} &= \frac{1}{\theta} \frac{d\theta}{d\alpha} [-3\alpha\theta + (1 + \alpha) + \ln \theta] - 3\theta + \ln \theta + 2, \\ &= -\alpha \frac{d\theta}{d\alpha} - 3\theta + \ln \theta + 2,\end{aligned}$$

by (2.6.2). Hence, noting that $\frac{d\beta}{d\alpha} = \theta + \alpha \frac{d\theta}{d\alpha}$, we obtain

$$\begin{aligned}\frac{d\phi_2}{d\alpha} &= -\frac{d\beta}{d\alpha} - 2\theta + \ln \theta + 2 \\ &= \frac{1 - \alpha}{1 - 2\theta} \left[(1 - 2\theta)^2 + \frac{d\theta}{d\alpha} \right],\end{aligned}$$

where the derivation of the last equality is detailed in Appendix B. From (2.6.2) it then

follows that $d\phi_2/d\alpha$ can vanish only when

$$(1 - 2\theta)^2 = \frac{\theta(1 - 2\theta)}{1 - 2\theta\alpha},$$

that is, $(1 - 2\theta)(1 - 2\alpha\theta) = \theta$ or $2\alpha\theta = (1 - 3\theta)/(1 - 2\theta)$.

We have also from (2.6.2) that $\alpha = -(1 - \log \theta)/(1 - 2\theta)$ and so

$$2\alpha\theta = -(1 + \ln \theta)2\theta/(1 - 2\theta).$$

Equating the two expressions for $2\alpha\theta$, we see that $d\phi_2/d\alpha$ can vanish only when $(1 - \theta)/2\theta = -\ln \theta$.

Defining $h := 1/\theta$, this relation becomes

$$\frac{h - 1}{2} = \ln h, \tag{2.6.5}$$

which is easily seen to have a unique solution h^* on $(1, \infty)$. Relation (2.6.5) can be expressed as

$$-\frac{1}{2}e^{-1/2} = -\frac{1}{2}he^{-h/2}$$

so that $-h^*/2 = w(-\frac{1}{2}e^{-1/2})$, from which we obtain $h^* \approx 3.512$. It follows that

$\theta = \theta^* = -[2w(-e^{-1/2})]^{-1} \approx 0.2847$, and finally

$$\alpha^* = \frac{1}{2\theta^*} \frac{1 - 3\theta^*}{1 - 2\theta^*} \approx 0.5952.$$

We can readily verify that $d\phi_2/d\alpha$ is negative for $\alpha = 0^+$ and positive for $\alpha = 1^-$.

Hence ϕ_2 is strictly decreasing for $\alpha \in (0, \theta^*)$ and strictly increasing on $(\theta^*, 1)$ (see Figure 2.3).

The second derivative $d^2\phi_2/d\alpha^2$ can be shown to be

$$\frac{d^2\phi_2}{d\alpha^2} = \frac{\theta(1-2\theta)}{(1-2\alpha\theta)^3} \left(4 + 3\alpha - 10\alpha\theta + 8\alpha^2\theta^2 - 4\alpha^2\theta - \frac{1}{\theta} \right).$$

The solution to $d^2\phi_2/d\alpha^2 = 0$ in the interval $(0, 1)$ can numerically be shown to be $\alpha^{**} \approx 0.8163$. The graph of $d^2\phi_2/d\alpha^2$ is also shown in Figure 2.3. It follows that, as a function of α , ϕ_2 is quasiconvex on $(0, 1)$.

The graphs of ϕ_1 and ϕ_2 are depicted together in Figure 2.4. It is interesting to observe that ϕ_1 and ϕ_2 approach $1/2$ when $\alpha \rightarrow 0$. So we find it worthwhile to examine the dynamics of r_1 and r_2 relative to each other during the process for this particular

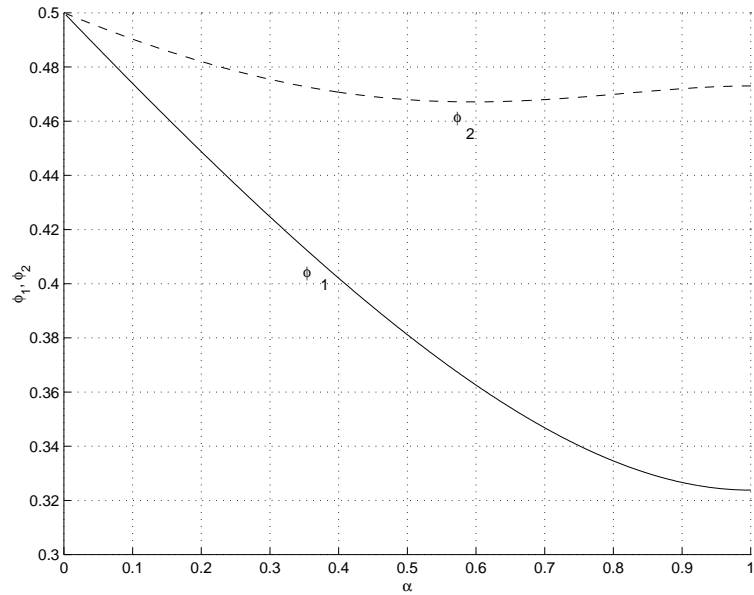


Figure 2.4: The graphs of ϕ_1 and ϕ_2 depicted together.

case. When $\alpha \rightarrow 0$, the equations describing the rumour process can be re-written as

$$\frac{ds}{dt} = -s \quad (2.6.6)$$

$$\frac{dr_1}{dt} = s^2 \quad (2.6.7)$$

and $r_2 = 1 - r_1 - s$. The solution to Equation (2.6.6) is obtained also using $s(0) = 1$ as

$$s(t) = e^{-t}$$

and then the solution to (2.6.7) can be shown to be

$$r_1(t) = \frac{1}{2} (1 - e^{-2t}) . \quad (2.6.8)$$

Furthermore

$$r_2(t) = 1 - e^{-t} - \frac{1}{2} (1 - e^{-2t}) . \quad (2.6.9)$$

Now (2.6.8) and (2.6.9) are combined to give the dynamics of r_1 and r_2 relative to each other:

$$r_2 = 1 - r_1 - \sqrt{1 - 2r_1}. \quad (2.6.10)$$

The dynamical relationship between r_1 and r_2 in (2.6.10) has been illustrated in Figure 2.5. The evolutions of r_1 and r_2 with respect to s are also plotted in Figure 2.6. Note that from (2.6.6) and (2.6.7), $dr_1/ds = -s$, and from $r_2 = 1 - r_1 - s$, $dr_2/ds = s - 1$; in other words, when s decreases, the sum of the rates of changes of r_1 and r_2 with respect to s is -1. While $d^2r_1/ds^2 = -1$, $d^2r_2/ds^2 = 1$, or $d^2r_1/ds^2 = -d^2r_2/ds^2$, which is visualised in Figure 2.6.

Hence, from the application point of view, spreaders encounter stiflers more than encountering each other while the process goes further, as may be expected. Therefore the number of second type of stiflers is greater than the number of first type of stiflers throughout most of the process. When the number of ignorants in the beginning of the process is very small the number of first type of stiflers is almost equal to the number of second type of stiflers, whereas half of the population is first type and other half is second type of stiflers at the end of the stochastic rumour process.

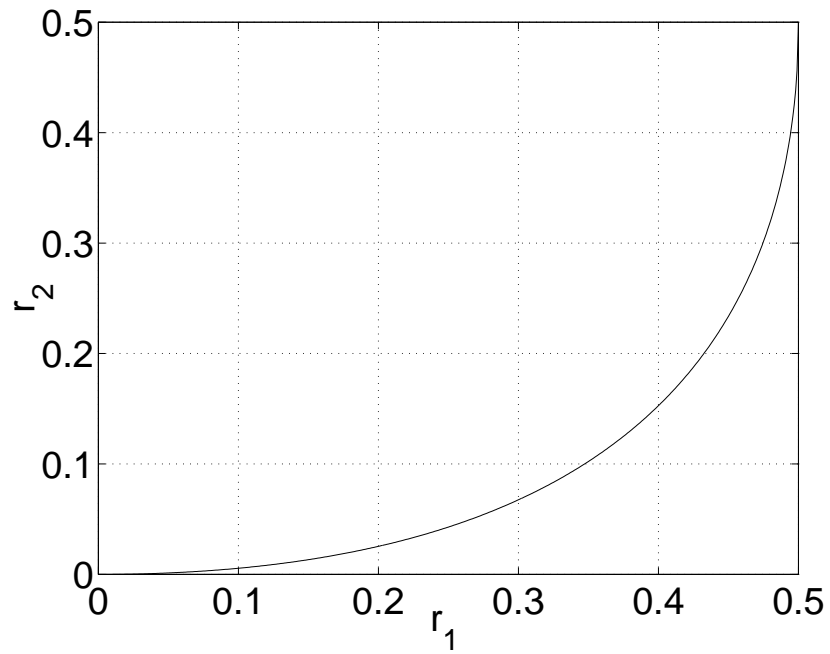


Figure 2.5: The dynamical relationship between r_1 and r_2 for the case when $\alpha \rightarrow 0$.

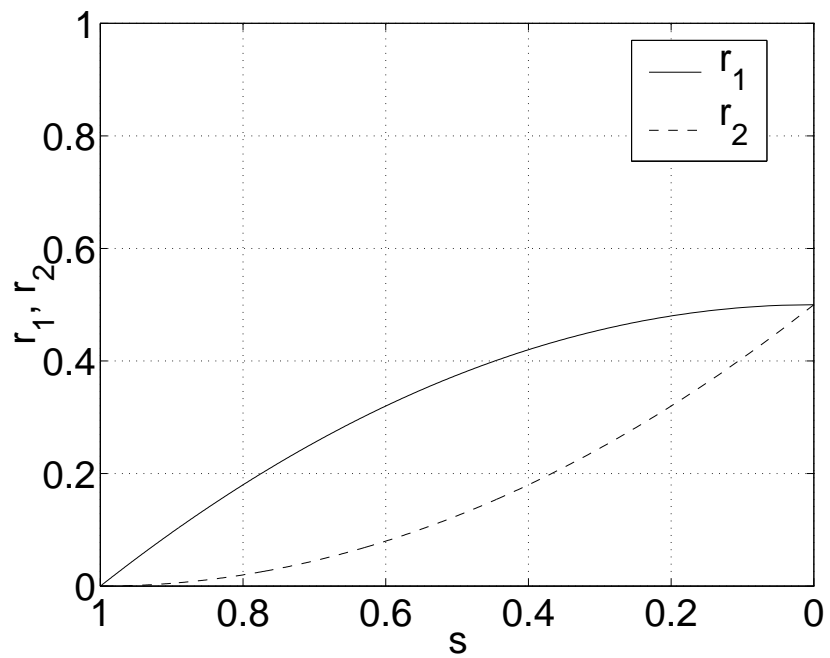


Figure 2.6: The evolutions of r_1 and r_2 with respect to s .

Chapter 3

Stochastic rumour process and transitions

3.1 Transitions and transition probabilities

3.1.1 Introduction

In general, every transition in a stochastic system has its own *transition probability*.

Transition probabilities have been a special and important concept in the study of Markov processes. Most of the definitions given for transition probabilities are based on a Markov chain, which may be either discrete-time or continuous-time (we use both a discrete-time Markovian [MT] model and a continuous-time [MT] model in this thesis), and they involve construction of the conditional probabilities. For example, consider

a homogeneous Markov chain [7], for which $P(X_k = m | X_{k-1} = n) = p_{nm}$ where the order of the subscripts in p_{nm} corresponds to the direction of the transition, namely one has $n \rightarrow m$. Here, p_{nm} denotes the conditional probability for a system, which is at state n in the $(k-1)$ th observation and at state m in the k th observation, $n, m \in \Sigma$. These probabilities are known as the transition probabilities, which are assumed to be independent of k . In other words, p_{nm} as a transition probability at stage k depends only on X_{k-1} but not on the previous random variables.

Table 3.1 details the interactions between individuals of the population and the elements of the sample space for the classical rumour process. With a slight abuse of notation we shall write $s \rightleftharpoons i$ to denote an interaction between a spreader and an ignorant, with similar notation for other meaningful interactions. Note that in Table 3.1 T_k^{is} is the number of transitions at a k th state where a k th transition occurs because of the interaction between an ignorant and a spreader— other possibilities follow in similar notation.

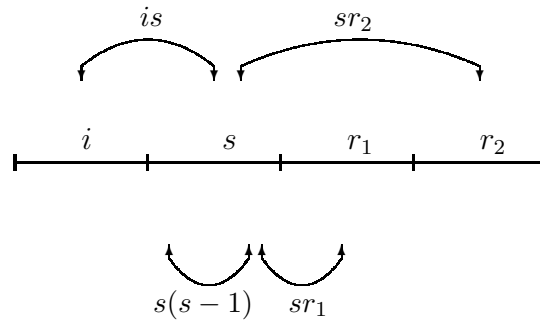


Figure 3.1: Interactions between classes and the number of outcomes

In [16], Belen and Pearce give analytical formulae for the number of transitions for both an extended [DK] and an extended [MT] model. The next section includes this work.

3.2 The number of transitions in [DK] and [MT] models

In reality, the variables i , s , r_1 and r_2 governing the rumour process are integers. There would also be a finite number of meaningful interactions altogether before the process stops when $s = 0$, as opposed to infinite-time in the continuous time model. In certain applications it may be of interest to know the number of meaningful interactions, referred to as transitions in this thesis. In applications, it may be an answer to the question: how long does it take to spread a rumour or how many meaningful interactions are required to make the rumour known as much as possible in the population?

It may be important or crucial for some cases; for example, if a rumour is an advertisement introducing a new product then the company may like to know a size of rumour process where its product is known when the process stops after a certain number of transitions. They can know, before they introduce their product to the market, the length of time required for their product to become known by assuming certain time ratios (which are taken as equal to each other and 1 in our analysis) between states while each transition occurs, or how many times the advertisement of their product will be circulated with respect to arbitrary initial conditions in the population size that varies from small size up to large size going to ∞ in a certain time period. So that, for example, they may produce a reasonable number of brochures (being equal to the number of transitions) to circulate in the population.

Thus, it is reasonable to consider the models in a discrete-time framework.

Theorem 3.2.1 (Extended DK model). *Let T denote the number of transitions for the [DK] model extended to general initial conditions. Then*

$$T = 2(r_{2,T} - r_{2,0}) + \frac{3}{2}(r_{1,T} - r_{1,0}) - s_0 - 1$$

in terms of (s, r_1, r_2)

$$T = -(i_T - i_0) + (r_{2,T} - r_{2,0}) + \frac{1}{2}(r_{1,T} - r_{1,0}) - 1$$

in terms of (i, r_1, r_2)

where $i_T, s_T, r_{1,T}, r_{2,T}$ are the numbers of ignorants, spreaders, first type of stiflers

and second type of stiflers when the rumour process stops. We evaluate r_1 and r_2 - see Chapter 2.

Proof of Theorem 3.2.1 :

Let $a, b, c, d \in \mathbf{R}$ and $n > 0$ be the size of population. Also suppose $i_0 = \alpha n$, $s_0 = \beta n$, $r_{1,0} = \gamma_1 n$, $r_{2,0} = \gamma_2 n$ are the initial values of ignorants, spreaders, first type of stiflers and second type of stiflers at time $t = 0$ respectively where α, β, γ_1 and γ_2 are non-negative real numbers. Then we construct an equation such that

$$a\Delta s_k + b\Delta i_k + c\Delta r_{1,k} + d\Delta r_{2,k} = 1 \quad (3.2.1)$$

where $\Delta s_k = s_{k+1} - s_k$, $\Delta i_k = i_{k+1} - i_k$, $\Delta r_{1,k} = r_{1,k+1} - r_{1,k}$, $\Delta r_{2,k} = r_{2,k+1} - r_{2,k}$ subject to the initial values $s_0, i_0, r_{1,0}, r_{2,0}$ at time $t = 0$.

The ratio constants a, b, c, d are found by using the following equations

$$a(+1) + b(-1) + 0 + 0 = 1$$

$$0 + a(-2) + c(+2) + 0 = 1$$

$$0 + a(-1) + 0 + d(+1) = 1$$

where $\Delta s_k, \Delta i_k, \Delta r_{1,k}$ and $\Delta r_{2,k}$ are given according to the possible transitions which may occur from a state to another state. Hence, $a = b+1$, $-a+c = 1/2$, and $-a+d = 1$.

We may choose $b = 0$ so $a = 1$, $c = 3/2$. $d = 2$.

The equation 3.2.1 may be expressed as

$$as_{k+1} + bi_{k+1} + cr_{1,k+1} + dr_{2,k+1} = as_k + bi_k + cr_{1,k} + dr_{2,k} + 1. \quad (3.2.2)$$

Substituting a, b, c, d into (3.2.2) and then iterating gives

$$s_T + 2r_{2,T} + \frac{3}{2}r_{1,T} = s_0 + 2r_{2,0} + \frac{3}{2}r_{1,0} + T + 1.$$

Using initial conditions and the fact that $s_T = 0$;

$$T = 2r_{2,T} + \frac{3}{2}r_{1,T} - \beta n - 2\gamma_2 n - \frac{3}{2}\gamma_1 n - 1$$

A similar method is used for verifying the other expression for T , the number of transitions in terms of (s, r_1, r_2) . The result follows. \square

See also stochastic numerical solutions obtained by our algorithm, for T and for size of each subpopulation in different sizes of population, tabulated in Table 3.2, Table 3.4 and Table 3.5. These numerical solutions illustrate agreement with Theorem 3.2.1. This agreement can be shown by substituting numerical values of $r_{2,T}, r_{1,T}, r_{2,0}, r_{1,0}, i_T, i_0$ and s_0 tabulated in the tables, into the formula derived for T .

Corollary 3.2.1. *Let $\beta = \frac{s_0}{n}$ and $\lambda = \lim_{n \rightarrow \infty} \frac{T}{n}$ and $\gamma_1 = \gamma_2 = 0$.*

If $\beta = \frac{1}{n}$ then $\lambda = 1.431$, and

If s_0 is sufficiently large and $s_0 \neq n$ that is, in the limit $\alpha \rightarrow 0$ then we get $\lambda \approx 0.748$.

Theorem 3.2.2 (Extended MT model).

$$T = 2(r_T - r_0) - s_0 - 1$$

where r_T is the total number of stiflers when the rumour process stops.

Proof of Theorem 3.2.2

Let $a, b, c, d \in \mathbf{R}$ and $n > 0$ be the size of the population. We construct an equation such that

$$a\Delta s_k + b\Delta i_k + c\Delta r_{1,k} + d\Delta r_{2,k} = 1 \quad (3.2.3)$$

where $\Delta s_k = s_{k+1} - s_k$, $\Delta i_k = i_{k+1} - i_k$, $\Delta r_{1,k} = r_{1,k+1} - r_{1,k}$, $\Delta r_{2,k} = r_{2,k+1} - r_{2,k}$.

The following three equations are used to find the ratio constants a, b, c, d :

$$b(-1) + a(+1) = 1$$

$$a(-1) + c(+1) = 1$$

$$a(-1) + d(+1) = 1.$$

Hence, $a = b+1, c = 1+a, d = 1+a$. Substituting a, b, c, d into (3.2.3) and iterating gives

$$s_m + 2r_m = s_0 + 2r_0 + m + 1, \quad 1 < m \leq T$$

where m is an integer. Note that $c = d$. The process stops at $m = T$ with $s_T = 0$, so that

$$T = 2(r_T - r_0) - s_0 - 1,$$

where $r_T = r_{1,T} + r_{2,T}$ and $r_0 = r_{1,0} + r_{2,0}$. The proof is completed by the substitution of initial values. □

Corollary 3.2.2. *Let $\beta = \frac{s_0}{n}$ and $\lambda = \lim_{n \rightarrow \infty} \frac{T}{n}$.*

If $\beta = \frac{1}{n}$ then $\lambda = 1.593$, and

In the limit $\alpha \rightarrow 0$, we get $\lambda \approx 0.995$

See also Table 3.3, Table 3.4 and Table 3.5 for stochastic numerical results obtained by using our algorithm, illustrating T as well as the number of each subpopulation. These numerical simulations verify the analytical expression for T given by Theorem 3.2.2. The verifications can be done by substituting numerical values of r_T , r_0 , and s_0 tabulated in the tables, into the formula derived for T . Other numerical results for our extended (to the general initial conditions) models, reproduced in a different form or for a different size of population, are given also in Chapter 2 and Appendices E - F.

Text entries in Tables 3.4–3.5 should be read as follow: ‘ $i(\text{MT})$ ’ the number of final ignorants in the [MT] model, ‘ $i(\text{DK})$ ’ the number of final ignorants in the [DK] model, ‘ $r(\text{MT})$ ’ the number of final stiflers in the [MT] model, ‘ $r_1(\text{DK})$ ’ the number of final first type of stiflers in the [DK] model, ‘ $r_2(\text{DK})$ ’ the number of final second type of stiflers in the [DK] model, ‘MT# Tran’ the number of transitions in the [MT] model, ‘DK# Tran’ the number of transitions in the [DK] model, ‘MT CPU[sec], DK CPU[sec]’ computation time for each model.

Note that these numerical simulations are given for different population sizes from very small (e.g. $n = 10$) to large (e.g. 10 billion).

By referring to our stochastic numerical results tabulated in Table 3.6 - Table 3.7, we observe that there is always a proportion of the population who never knows the rumour even when the initial proportion of the spreaders is very large and is close to the whole population (but never equals this size, to be able to initiate the rumour process). According to these tabulated results, our stochastic numerical results also suggest that there is always an ignorant proportion of population that will never be 0 regardless of the initial values of spreaders (except for the extreme cases where $s_0 = 0$ or $s_0 = n$), at the end of the rumour process. More detailed analytical and numerical solutions are considered in Chapter 2.

The range of the proportion of transitions is between 0.748 and 1.000 when the rumour process stops. This is significant when time constraints are considered to be important in an application. Note that the number of transitions is quite relevant to the total time in which the rumour process takes place.

Interaction 1	Interaction 2	Interaction 3
$i \rightleftharpoons s$	$s \rightleftharpoons s$	$s \rightleftharpoons r (= r_1 \cup r_2)$
$i_{k+1} = i_k - 1$	$i_{k+1} = i_k$	$i_{k+1} = i_k$
$s_{k+1} = s_k + 1$	$s_{k+1} = s_k - 1$ [MT]	$s_{k+1} = s_k - 1$
$r_{1,k+1} = r_{1,k}$	$r_{1,k+1} = r_{1,k} + 1$ [MT]	$r_{1,k+1} = r_{1,k}$
$s_{k+1} = s_k + 1$	$s_{k+1} = s_k - 2$ [DK]	$s_{k+1} = s_k - 1$
$r_{1,k+1} = r_{1,k}$	$r_{1,k+1} = r_{1,k} + 2$ [DK]	$r_{1,k+1} = r_{1,k}$
$r_{2,k+1} = r_{2,k}$	$r_{2,k+1} = r_{2,k}$	$r_{2,k+1} = r_{2,k} + 1$
$T_{k+1}^{is} = T_k^{is} + 1$	$T_{k+1}^{is} = T_k^{is}$	$T_{k+1}^{is} = T_k^{is}$
$T_{k+1}^{ii} = T_k^{ii}$	$T_{k+1}^{ss} = T_k^{ss} + 1$	$T_{k+1}^{ss} = T_k^{ss}$
$T_{k+1}^{sr} = T_k^{sr}$	$T_{k+1}^{sr} = T_k^{sr}$	$T_{k+1}^{sr} = T_k^{sr} + 1$
$\frac{\rho i}{\rho i + \rho \binom{s}{2} + \rho r}$	$\frac{\rho \binom{s}{2}}{\rho i + \rho \binom{s}{2} + \rho r}$	$\frac{\rho r}{\rho i + \rho \binom{s}{2} + \rho r}$
$\frac{\rho i}{\rho i + \rho(s-1) + \rho r}$	$\frac{\rho(s-1)}{\rho i + \rho(s-1) + \rho r}$	$\frac{\rho r}{\rho i + \rho(s-1) + \rho r}$

Table 3.1: Possible meaningful interactions, transitions and the relative transition probabilities for different transitions at rate ρ and $r = r_1 + r_2$.

n	i_0	s_0	$r_{1,0}$	$r_{2,0}$	T	i_T	$r_{1,T}$	$r_{2,T}$
10^2	30	20	40	10	41	18	44	38
350	105	70	140	35	133	69	156	125
360	108	72	144	36	137	71	160	129
10^3	300	200	400	100	380	200	438	362
10^4	3000	2000	4000	1000	3699	2046	4416	3538
10^6	300 000	200 000	400 000	100 000	373540	202774	441822	355404

Table 3.2: Numerical results for extended [DK] model with respect to different sizes of population and arbitrary initial conditions.

n	i_0	s_0	r_0	T	i_T	r_T
10^2	30	20	50	43	18	82
350	105	70	175	139	70	280
360	108	72	180	141	73	287
10^3	300	200	500	393	203	797
10^4	3000	2000	5000	3927	2036	7964
10^6	300 000	200 000	500 000	394687	202656	797344

Table 3.3: Numerical results for extended [MT] model with respect to different size of population and different initial conditions.

n	i (MT)	i (DK)	r (MT)	r_2 (DK)	r_1 (DK)
10^1	1	4	9	4	2
10^2	16	28	84	42	30
10^3	215	200	785	458	342
10^4	2038	2065	7962	4723	3212
10^5	20329	20242	79671	47276	32482
10^6	203477	202616	796523	473736	323648
10^7	2035639	2032267	7964361	4729347	3238386
10^8	20322212	20321461	79677788	47298973	32379566
10^9	203203310	203198549	796796690	472988473	323812978
10^{10}	2031882549	2031882608	7968117451	4730060624	3238056768

Table 3.4: Comparison actual values of [MT] and [DK] ignorants and stiflers.

n	MT # Trans.	DK # Trans.	MT CPU [sec]	DK CPU [sec]
10^1	17	10	0.03	0.01
10^2	167	128	0.01	0.03
10^3	1569	1428	0.04	0.05
10^4	15923	14263	0.11	0.11
10^5	159341	143274	0.83	0.80
10^6	1593045	143294 3	8.67	7.72
10^7	15928721	14316272	84.20	76.19
10^8	159355575	143167294	846.51	763.07
10^9	1593593379	1431696412	8496.55	7793.22
10^{10}	15936234901	14317206399	82770.67	126485.20

Table 3.5: Comparison of actual values of [MT] and [DK] transitions.

β	i_T/i_0	T	r_2	r_1	i_T
0.00000001	0.203203	143168591	47298016	32381706	20320278
0.1	0.223786	133386623	47195466	32663794	20140737
0.2	0.244404	124162273	46981587	33466066	19552348
0.3	0.264720	115597426	46785959	34683672	18530369
0.4	0.283909	107803419	46710545	36254886	17034569
0.5	0.301819	100762294	46797471	38111568	15090961
0.6	0.318171	94448624	47077723	40195452	12726825
0.7	0.332792	88800655	47552581	42463662	9983757
0.8	0.346016	83720908	48202776	44876904	6920320
0.9	0.357546	79154742	49035858	47388684	3575458
0.99	0.366287	75401329	49901519	49732194	366287

Table 3.6: Computational results for the extended [DK] model with respect to the population size $n := 10^8$ and $r_{1,0} = 0$ and $r_{2,0} = 0$.

β	T	i_T/i_0	$r_{1,T}/n$	$r_{2,T}/n$
0.000000001	1593579410	0.2032102942	0.3237645626	0.4730251431
0.1	1497258565	0.2237452418	0.3266044557	0.4720248282
0.5	1198266799	0.3017331958	0.3811696768	0.4679637551
0.9	1028493619	0.3575319052	0.4739302993	0.4903165102
0.99	1002679785	0.3660106957	0.4973381758	0.4990017414
0.999	1000283769	0.3581149876	0.4997091591	0.4999327362

Table 3.7: Computational results for the extended [MT] model with respect to the

population size $n := 10^9$ and $r_{1,0} = 0$ and $r_{2,0} = 0$.

Chapter 4

Impulsive control of rumours

4.1 Introduction

In Chapter 2, we analysed the evolution from general initial conditions of the deterministic limiting version of the classical stochastic rumour models, Daley-Kendall and Maki-Thompson. Rumour models can be used to describe a variety of phenomena, such as the dissemination of information, disinformation or memes and changes in political persuasion and the stock market, for some of which the standard assumption of a single initial spreader and no initial stiflers is inappropriate. Results for a rumour process with general initial conditions are also relevant for the present study, which envisages a second rumour process developing in a situation created by a first.

In an age of mass communication, it is natural to consider the initiation of a rumour

by means of television, radio or the internet (Frost [49]). We use the generic term *broadcast* to refer to such an initiation. This chapter envisages a control ingredient being added to a rumour model by the introduction of one or more subsequent broadcasts, with the intention of reducing the final proportion of the population never hearing the rumour. The rumour process is started by a broadcast to subscribers, who constitute the initial spreaders. Those ignorants who become spreaders after an encounter with a spreader we term nonsubscriber spreaders. We wish to determine, for given initial proportions of ignorants, spreaders and stiflers in the population, when to effect a second broadcast so as to minimise the final proportion of ignorants.

Two basic scenarios are considered. In the first, the recipients of the second broadcast are again the subscribers: a subscriber who had become a stifer is reactivated as a subscriber spreader. In Scenario 2, the recipients of the second broadcast are all individuals who were once spreaders.

As before, we describe the process in the continuum limit corresponding to a total population size tending to infinity. The resultant differential equations with each scenario can be expressed in state-space form, with the upward jump in subscriber spreaders modelled by an impulsive control input. Since we are dealing with an optimal control problem, a natural approach would be to employ a Pontryagin-like maximum principle furnishing necessary conditions for an extremum of an impulsive control sys-

tem (see, for example, Blaquièrre [18] and Rempala and Zapczyk [94]). However, because of the tractability of the dynamical system equations, we are able to solve the given impulsive control problem without resorting to this theory.

The distinction between subscriber and nonsubscriber spreaders is of some independent interest. Before the second broadcast, these groups may be identified with two types of spreaders in an ordinary rumour, those who were spreaders initially and those who began as ignorants but became spreaders from an encounter with a spreader.

In Section 4.2.1 we refine the standard rumour model, studying the evolution of the proportions of these two groups in the population. In Section 4.3 we formulate Scenario 1 and derive associated results. We prove that the optimal time for the second broadcast at the end of the first process, that is, when the proportion of spreaders in the population drops to zero. Section 4.4 parallels Section 4.3 for Scenario 2.

The development utilises the refinement of the basic rumour presented in the following section. An alternative approach with multiple broadcasts is given in Section 4.6. This is published in [86].

4.2 Impulsive control of rumours for two broad-casts

4.2.1 Refinement of the rumour model

In this section we consider the evolution of the proportions of subscriber and nonsubscriber spreaders in a standard rumour model.

The Daley–Kendall model considers a population of n individuals with subpopulations of ignorants, spreaders and stiflers. Denote the respective sizes of these subpopulations by i , s and r . We define in addition the respective numbers of subscriber and nonsubscriber stiflers by s_1 and s_2 , so that $s = s_1 + s_2$. There is homogeneous mixing of individuals. The interactions which result in changes of subpopulation in time $d\tau$, along with their associated probabilities, are as follows.

Interaction	Transition	Probability
$i \rightleftharpoons s$	$(i, s_1, s_2, r) \mapsto (i - 1, s_1, s_2 + 1, r)$	$is d\tau + o(d\tau)$
$s_1 \rightleftharpoons s_1$	$(i, s_1, s_2, r) \mapsto (i, s_1 - 2, s_2, r + 2)$	$s_1(s_1 - 1) d\tau + o(d\tau)$
$s_2 \rightleftharpoons s_2$	$(i, s_1, s_2, r) \mapsto (i, s_1, s_2 - 2, r + 2)$	$s_2(s_2 - 1) d\tau + o(d\tau)$
$s_1 \rightleftharpoons s_2$	$(i, s_1, s_2, r) \mapsto (i, s_1 - 1, s_2 - 1, r + 2)$	$s_1 s_2 d\tau + o(d\tau)$
$s_1 \rightleftharpoons r$	$(i, s_1, s_2, r) \mapsto (i, s_1 - 1, s_2, r + 1)$	$s_1 r d\tau + o(d\tau)$
$s_2 \rightleftharpoons r$	$(i, s_1, s_2, r) \mapsto (i, s_1, s_2 - 1, r + 1)$	$s_2 r d\tau + o(d\tau)$

We now adopt a continuum formulation appropriate for $n \rightarrow \infty$. Let $i(\tau)$, $s(\tau)$ ($= s_1(\tau) + s_2(\tau)$), $r(\tau)$ denote respectively the proportions of ignorants, spreaders and stiflers in the population at time $\tau \geq 0$. The evolution of the limiting form of the model is prescribed by the deterministic dynamic equations

$$\frac{di}{d\tau} = -i s, \quad (4.2.1)$$

$$\frac{ds_1}{d\tau} = -s_1(1 - i), \quad (4.2.2)$$

$$\frac{ds_2}{d\tau} = -s_2(1 - 2i) + s_1 i, \quad (4.2.3)$$

$$\frac{dr}{d\tau} = s(1 - i) \quad (4.2.4)$$

with initial conditions

$$i(0) = \alpha > 0, \quad s_1(0) = \beta > 0, \quad s_2(0) = 0 \quad \text{and} \quad r(0) = \gamma \geq 0 \quad (4.2.5)$$

satisfying

$$\alpha + \beta + \gamma = 1. \quad (4.2.6)$$

We remark that (4.2.2) and (4.2.3) may be combined to provide

$$\frac{ds}{d\tau} = -s(1 - 2i). \quad (4.2.7)$$

It is convenient to introduce the parameter $\theta = \theta(\tau) := i/\alpha$, the ratio of the proportion of ignorants at time τ to the initial proportion. As earlier in Chapter 2 we have the following dynamics and asymptotics for the basic continuum rumour process.

Theorem 4.2.3. *The evolution of the rumour process prescribed by (4.2.1), (4.2.7) and (4.2.4) subject to (4.2.5) and (4.2.6) is given parametrically in terms of i by*

$$s = \beta - 2(i - \alpha) + \ln(i/\alpha) \quad (4.2.8)$$

$$= \beta - 2\alpha(\theta - 1) + \ln \theta \quad (4.2.9)$$

and $r = 1 - i - s$.

The process evolves towards an asymptotic state $(i_\infty, 0, r_\infty)$, with

$$i \downarrow i_\infty = i_\infty(\alpha, \beta) \text{ as } \tau \rightarrow \infty$$

and

$$0 < i_\infty < 1/2. \quad (4.2.10)$$

The parameter $\theta_\infty := i_\infty/\alpha$ satisfies the transcendental equation

$$\theta_\infty e^{2\alpha(1-\theta_\infty)} = e^{-\beta}. \quad (4.2.11)$$

Further,

$$s \rightarrow 0 \quad \text{and} \quad r(\tau) \uparrow r_\infty = 1 - i_\infty \quad \text{as } \tau \rightarrow \infty.$$

Where we wish to indicate that s is regarded as a function of i (defined by (4.2.8)),

we put

$$s = S(i) := \beta - 2(i - \alpha) + \ln(i/\alpha).$$

The dynamics of the process are specified by augmenting Theorem 4.2.3 with the following result.

Theorem 4.2.4. *In the rumour model prescribed by (4.2.1)–(4.2.4) with initial conditions (4.2.5), (4.2.6) and*

$$s_1(0) = \beta, \quad s_2(0) = 0 \quad (4.2.12)$$

we have

$$s_1 = s \exp\left(-\int_i^\alpha \frac{du}{S(u)}\right) \quad (4.2.13)$$

and

$$s_2 = s \left[1 - \exp\left(-\int_i^\alpha \frac{du}{S(u)}\right)\right]. \quad (4.2.14)$$

We have that s_1 is strictly decreasing, with the asymptotics of s_1 and s_2 given by

$$(i) \quad \lim_{i \rightarrow i_\infty} \frac{s_1}{s} = 0,$$

$$(ii) \quad \lim_{i \rightarrow i_\infty} \frac{s_2}{s} = 1.$$

Proof. Equations (4.2.1) and (4.2.2) combine to yield

$$\frac{1}{s_1} \frac{ds_1}{di} = \frac{1}{is} (1 - i)$$

or

$$d(\ln s_1) = \frac{1}{is} (1 - 2i) di + \frac{1}{s} di.$$

Also we have on combining (4.2.1) and (4.2.7) that

$$\frac{ds}{di} = \frac{1-2i}{i}, \quad (4.2.15)$$

from which we obtain

$$d(\ln s_1) = \frac{1}{s} ds + \frac{1}{s} di .$$

Equation (4.2.13) now follows on integration and use of the initial condition (4.2.12).

Equation (4.2.14) now follows from $s_2 = s - s_1$.

That s_1 is strictly decreasing follows from (4.2.2). Because $s \rightarrow 0$ as $\tau \rightarrow \infty$, we have $s_1 \rightarrow 0$ and $s_2 \rightarrow 0$. By l'Hôpital's rule,

$$\begin{aligned} \lim_{i \rightarrow i_\infty} \frac{s_1}{s} &= \lim_{i \rightarrow i_\infty} \frac{ds_1/di}{ds/di} \\ &= \lim_{i \rightarrow i_\infty} \frac{s_1(1-i)/(is)}{(1-2i)/i} \\ &= \left(\lim_{i \rightarrow i_\infty} \frac{1-i}{1-2i} \right) \lim_{i \rightarrow i_\infty} \frac{s_1}{s}. \end{aligned}$$

The last step follows from (4.2.10), which also gives (i). Part (ii) follows from $s = s_1 + s_2$.

□

The asymptotics for s_1 and s_2 show that the spreader population changes in the course of time from consisting entirely of subscribers to consisting entirely of nonsubscribers.

Suppose that the second broadcast is made when $i = i_b$. Denote by s_{1b} and s_b respectively the proportions of subscriber spreaders and all spreaders just before that

broadcast. Also denote by s_{1b}^+ and s_b^+ respectively the corresponding proportions immediately after the broadcast. Then by (4.2.13)

$$s_{1b} = s_b \exp \left[- \int_{i_b}^{\alpha} \frac{du}{S(u)} \right] \quad (4.2.16)$$

and by (4.2.8)

$$s_b = \beta - 2(i_b - \alpha) + \ln(i_b/\alpha) . \quad (4.2.17)$$

At any time after the second broadcast

$$s = s_b^+ - 2(i - i_b) + \ln(i/i_b) . \quad (4.2.18)$$

4.3 Results for Scenario 1

We now proceed to address Scenario 1 in which only the original subscribers become spreaders again. Under this Scenario

$$s_{1b}^+ = \beta ,$$

and so

$$s_b^+ = s_{1b}^+ + s_b - s_{1b} = \beta + s_b \left[1 - \exp \left(- \int_{i_b}^{\alpha} \frac{du}{S(u)} \right) \right] . \quad (4.3.1)$$

The rumour process stops when $s = 0$. Let i_f, i_∞ denote respectively the final proportion of ignorants in the population subsequent to the second broadcast and the proportion of ignorants at the end of a single rumour process. The problem of finding the optimum broadcast time, or equivalently the value $i = i_b$ when the second

broadcast is made, can be posed as

$$P_1 : \begin{cases} \min_{i_b} & i_f \\ \text{subject to} & s_b^+ - 2(i_f - i_b) + \ln(i_f/i_b) = 0, \\ \text{where} & s_b^+ = \beta + s_b \left[1 - \exp\left(-\int_{i_b}^{\alpha} \frac{du}{S(u)}\right) \right] \\ \text{and} & s_b = \beta - 2(i_b - \alpha) + \ln(i_b/\alpha). \end{cases}$$

Theorem 4.3.1. *For $0 < \alpha \leq 1 - \beta < 1$, the minimum in Problem P_1 is given uniquely*

by $i_b = i_\infty = i_\infty(b)$.

Proof. Implicit differentiation with respect to i_b of the constraint equation in Prob-

lem P_1 provides

$$\frac{\partial s_b^+}{\partial i_b} - 2 \left(\frac{\partial i_f}{\partial i_b} - 1 \right) + \frac{1}{i_f} \frac{\partial i_f}{\partial i_b} - \frac{1}{i_b} = 0. \quad (4.3.2)$$

From (4.3.1) we have

$$\frac{\partial s_b^+}{\partial i_b} = \frac{\partial s_b}{\partial i_b} - \left(\frac{\partial s_b}{\partial i_b} + 1 \right) \exp\left(-\int_{i_b}^{\alpha} \frac{du}{S(u)}\right),$$

so that by (4.2.16)

$$\frac{\partial s_b^+}{\partial i_b} = \frac{\partial s_b}{\partial i_b} - \left(\frac{\partial s_b}{\partial i_b} + 1 \right) \frac{s_{1b}}{s_b}. \quad (4.3.3)$$

Also from (4.2.17) we have

$$\frac{\partial s_b}{\partial i_b} = \frac{1 - 2i_b}{i_b}. \quad (4.3.4)$$

Elimination of $\partial s_b/\partial i_b$ and $\partial s_b^+/\partial i_b$ between (4.3.2)–(4.3.4) provides

$$\frac{\partial i_f}{\partial i_b} = \frac{i_f}{1 - 2i_f} \frac{1 - i_b}{i_b} \frac{s_{1b}}{s_b}. \quad (4.3.5)$$

By Theorem 4.2.3 with initial state the state entered immediately after the second broadcast, we have $0 < i_f < 1/2$ and so $i_f/(1 - 2i_f) > 0$ and $i_\infty < i_b < \alpha$. The terms $(1 - i_b)/i_b$ and s_{1b}/s_b are also positive, so $\partial i_f/\partial i_b > 0$ for $i_\infty < i_b < \alpha$. All three quotients on the right in (4.3.5) are bounded above. By Theorem 4.2.4 (i), $\partial i_f/\partial i_b \rightarrow 0$ as $i_b \rightarrow i_\infty$. Hence i_f is minimised uniquely by the choice $i_b = i_\infty$. This completes the proof. \square

Corollary 4.3.1. *Put*

$$i_\omega := \inf \{i_f : i_b \in [i_\infty, \alpha]\}.$$

If the second broadcast time is chosen to coincide with the first, the situation reduces to one of a single broadcast and i_f becomes i_∞ . Since i_f is strictly increasing as a function of i_b , we have

$$i_\omega \leq i_f \leq i_\infty \quad \text{for } i_b \in [i_\infty, \alpha].$$

Generally, (4.2.9) gives

$$\frac{i_\infty}{\alpha} \exp[2(\alpha - i_\infty)] = \exp(-\beta)$$

for the first broadcast and

$$\frac{i_\omega}{i_\infty} \exp[2(i_\infty - i_\omega)] = \exp(-\beta)$$

for an optimal second broadcast. Multiplying these relations together provides

$$\frac{i_\omega}{\alpha} \exp[2(\alpha - i_\omega)] = \exp(-2\beta). \tag{4.3.6}$$

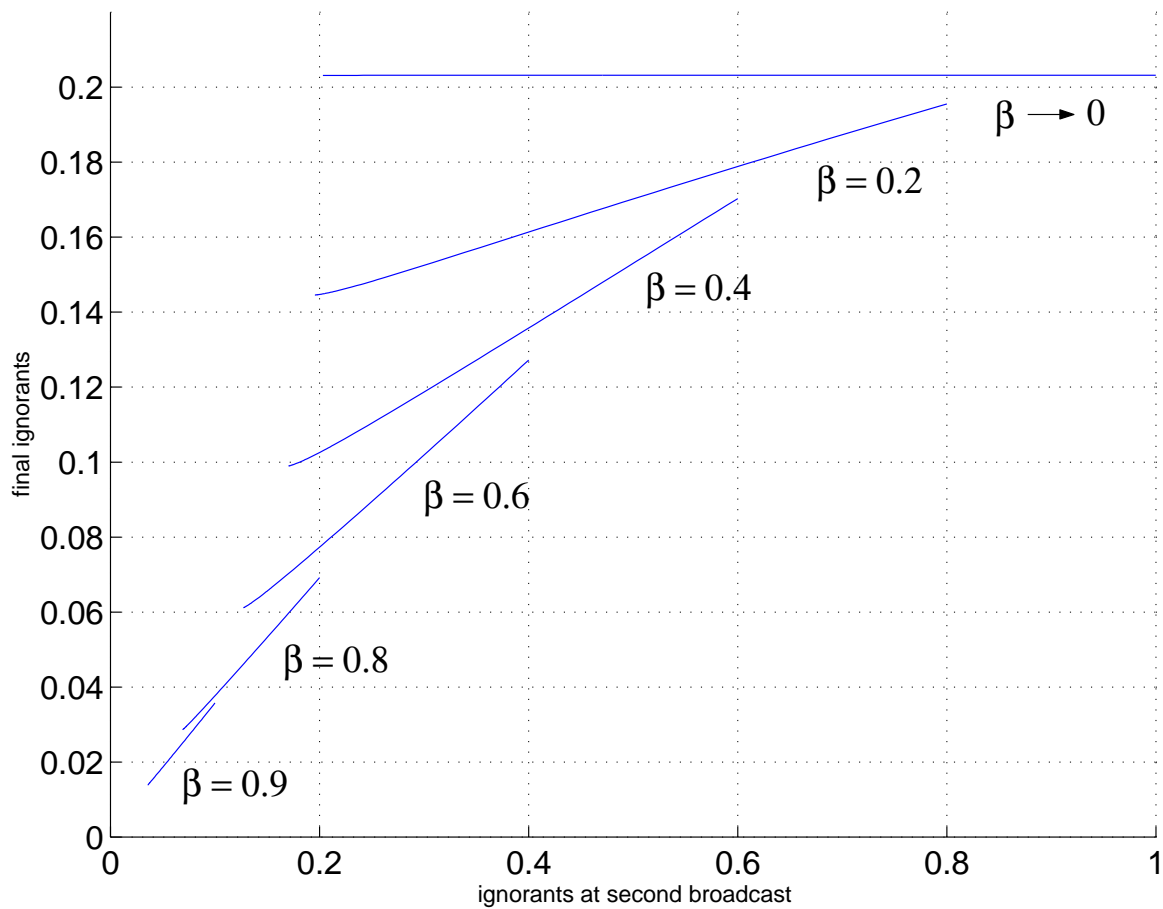


Figure 4.3.1: i_f vs i_b for various values of β under Scenario 1.

If $\alpha + 2\beta \leq 1$, this relation has a physical interpretation: the final proportion of ignorants after two broadcasts, the second being optimally timed, is the same as that resulting from a single broadcast with initial conditions

$$i(0) = \alpha \quad \text{and} \quad s(0) = 2\beta.$$

Figure 4.3.1 presents (for $\alpha + \beta = 1$ with six different values of β) graphs of the final proportion i_f of ignorants after the second broadcast as a function of the proportion i_b of ignorants at the time of the second broadcast.

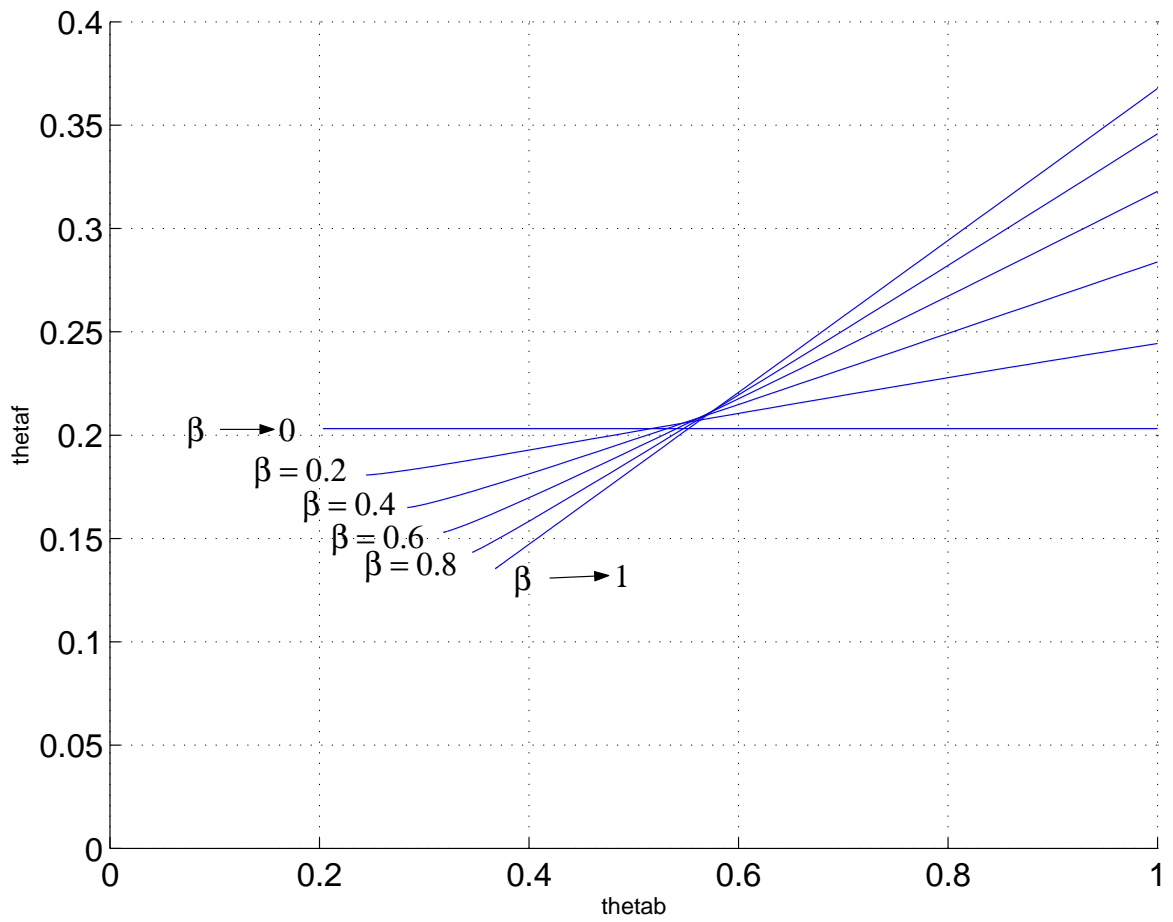


Figure 4.3.2: θ_f vs θ_b for various values of β under Scenario 1.

Corollary 4.3.2. *Suppose $\alpha \rightarrow \alpha_0 \neq 0$ and $\beta \rightarrow 0$. By (4.3.6), the limiting value of i_ω satisfies*

$$\frac{i_\omega}{\alpha_0} \exp[2(\alpha_0 - i_\omega)] = 1$$

and so $i_\omega = i_\infty$. By Corollary 1, $i_f = i_\infty$ for each choice of $i_b \in [i_\infty, \alpha]$, that is, the second broadcast is ineffective at any time.

Intuitively this is not surprising. In the limiting case $\beta \rightarrow 0$, the reactivation of subscriber stiflers does not change the state, so i_f has the same value for every $i_b \in [i_\infty, \alpha_0]$. In the case $\alpha \rightarrow 1$, the limit $i_\infty \approx 0.203$. We have accordingly the following result.

Theorem 4.3.2. *In the limiting case $\beta \rightarrow 0$, $\alpha \rightarrow \alpha_0 > 0$, any $i_b \in [i_\infty, \alpha_0]$ is a solution of Problem P_1 .*

In the limit as $\alpha \rightarrow 0$ and $\beta \rightarrow \beta_0 \neq 0$, we have from (4.2.11) that

$$\theta_\infty = \exp(-\beta_0),$$

so that for any feasible time for the second broadcast

$$\exp(-\beta_0) < \theta_b \leq 1. \tag{4.3.7}$$

Lemma 4.3.1. *Suppose $\alpha \rightarrow 0$ and $0 < \beta_0 \leq 1$. Then*

$$\lim_{\beta \rightarrow \beta_0} s_{1b}/s_b = 1.$$

Proof. Since $i = \alpha \theta$, (4.2.16) can be rewritten as

$$\frac{s_{1b}}{s_b} = \exp \left[-\alpha \int_{\theta_b}^1 (1/s) d\theta \right],$$

so it suffices to show that

$$h := \lim_{\alpha \rightarrow 0} \int_{\theta_b}^1 (1/s) d\theta < \infty.$$

From (4.2.9),

$$h = \int_{\theta_b}^1 \frac{d\theta}{\beta_0 + \ln \theta} = \exp(-\beta_0) \int_{\exp(\beta_0)\theta_b}^{\exp(\beta_0)} \frac{du}{\ln u}. \quad (4.3.8)$$

By Abramowitz and Stegun [2], Section 5.1, we may evaluate

$$\int_{\exp(\beta_0)\theta_b}^{\exp(\beta_0)} \frac{du}{\ln u} = E_i(\ln(\exp(\beta_0))) - E_i(\ln(\exp(\beta_0)\theta_b)),$$

where

$$E_i(x) := \gamma + \ln(\ln x) + \sum_{k=1}^{\infty} \frac{\ln^k x}{k \cdot k!}$$

and γ is Euler's constant.

Hence (4.3.8) becomes

$$h = \exp(-\beta_0) \left[\ln \left(\frac{\beta_0}{\beta_0 + \ln(\theta_b)} \right) + \sum_{k=1}^{\infty} \frac{\beta_0^k}{k \cdot k!} - \sum_{k=1}^{\infty} \frac{\ln^k(\exp(\beta_0)\theta_b)}{k \cdot k!} \right]. \quad (4.3.9)$$

When $\theta_b = 1$, $h = 0$. On the other hand when (4.3.7) holds with strict inequality, the leading term in brackets is finite. Both series in (4.3.9) converge absolutely. Hence h is finite as required. \square

The result given by Lemma 4.3.1 provides an interesting contrast to Theorem 4.2.4(i).

Theorem 4.3.3. *When $\alpha \rightarrow 0$ and $\beta \rightarrow \beta_0 \neq 0$, Problem P_1 possesses the unique solution $\theta_b = \theta_\infty = e^{-\beta_0}$.*

Proof. With $i_f = \alpha\theta_f$ and $i_b = \alpha\theta_b$, (4.3.5) becomes

$$\frac{\partial\theta_f}{\partial\theta_b} = \frac{\theta_f}{1 - 2\alpha\theta_f} \frac{1 - \alpha\theta_b}{\theta_b} \frac{s_{1b}}{s_b}.$$

Hence by Lemma 4.3.1, we have that in the limit

$$\frac{\partial\theta_f}{\partial\theta_b} = \frac{\theta_f}{\theta_b}, \quad (4.3.10)$$

which is always positive. This completes the proof. \square

Remark 1. *We may deduce from (4.3.10) that $\partial^2\theta_f/\partial\theta_b^2 = 0$ in the limit $\beta \rightarrow \beta_0$, so that θ_f is linear in θ_b . For $\alpha \rightarrow 0$ and $\beta \rightarrow \beta_0$, we have by Lemma 4.3.1 that the first constraint equation in Problem P_1 becomes*

$$\beta_0 + \ln \frac{\theta_f}{\theta_b} = 0.$$

This yields $\theta_f = e^{-\beta_0} \theta_b$. As may be seen in Figure 4.3.2 for the special case $\alpha + \beta = 1$, the slope of the graph of θ_f vs θ_b has the constant value $1/e$ for $\beta \rightarrow 1$. The slope is not constant for $0 < \beta < 1$.

4.4 Results for Scenario 2

Under Scenario 2 $s_b^+ = \alpha + \beta - i_b$ and so (4.2.18) becomes

$$s = \alpha + \beta + i_b - 2i + \ln(i/i_b)$$

for any time after the second broadcast.

The optimisation problem now has a much simpler form than in Problem P₁, namely

$$P_2 : \begin{cases} \min_{i_b} i_f \\ \text{subject to } \alpha + \beta + i_b - 2i_f + \ln(i_f/i_b) = 0. \end{cases}$$

When there are no stiffers initially in the population, that is, when $\alpha + \beta = 1$, Problem P₂ is independent of β , in contrast to Problem P₁. Parallels to Theorems 4.3.1, 4.3.2 and 4.3.3 for this scenario can be combined into a single result, which we give below.

Theorem 4.4.1. *When $\alpha \rightarrow \alpha_0$ with $0 < \alpha_0 \leq 1$, $i_b = i_\infty$ is the unique solution to Problem P₂. When $\alpha \rightarrow 0$ and $\beta \rightarrow \beta_0$, with $0 < \beta_0 \leq 1$, $\theta_b = \theta_\infty = e^{-\beta_0}$ is the only solution.*

Proof. First suppose $\alpha \rightarrow \alpha_0$ with $0 < \alpha_0 \leq 1$. Implicit differentiation of the constraint equation in Problem P₂ gives

$$\left(\frac{1}{i_f} - 2\right) \frac{\partial i_f}{\partial i_b} = \frac{1}{i_b} - 1$$

or

$$\frac{\partial i_f}{\partial i_b} = \frac{i_f}{1 - 2i_f} \frac{1 - i_b}{i_b}. \quad (4.4.1)$$

Since $0 < i_f < 1/2$ and $0 < i_b < 1$, we have $\partial i_f / \partial i_b > 0$, which implies, given $i_\infty < i_b < \alpha_0$, that $i_b = i_\infty$ is the unique solution.

Next consider $\alpha \rightarrow 0$ and $\beta \rightarrow \beta_0$, with $0 < \beta_0 \leq 1$, and let $i_f = \alpha \theta_f$ and $i_b = \alpha \theta_b$.

Substitution into (4.4.1) gives

$$\frac{\partial \theta_f}{\partial \theta_b} = \frac{\theta_f}{\theta_b} > 0$$

which implies, given $e^{-\beta_0} < \theta_b < 1$, that $\theta_b = \theta_\infty = e^{-\beta_0}$ is the only solution. \square

Figure 4.4.2 presents (for $\alpha + \beta = 1$ with three different values of β) graphs of i_f after the second broadcast as a function of i_b .

Remark 2. *As with Scenario 1, we have $\partial^2 \theta_f / \partial \theta_b^2 = 0$ and through the constraint equation in Problem P_2 we obtain $\theta_f = e^{-\beta_0} \theta_b$. In the limit as $\beta \rightarrow 1$, the slope of the graph of θ_f vs θ_b has the constant value $1/e$, as illustrated in Figure 4.4.1.*

Remark 3. *Consider the case when there are no stiflers initially, that is, when $\alpha + \beta =$*

1. *In Scenario 2, there are no stiflers left immediately after the second broadcast, all having become spreaders again. The larger the proportion of ignorants to have encountered the rumour before the second broadcast, the larger the proportion of stiflers present. So choosing a broadcast time with the highest proportion of stiflers (as in Theorem 4.4.1), that is, at the end of the process, is intuitively reasonable in order to achieve the lowest possible i_f .*

The results for Scenario 1 are less obvious. Intuitively one might want the ratio s/i to be as large as possible at the start of a process to increase the dispersal of the rumour.

However a second broadcast at the end of the first process does not necessarily make s/i larger than at earlier stages of the first process.

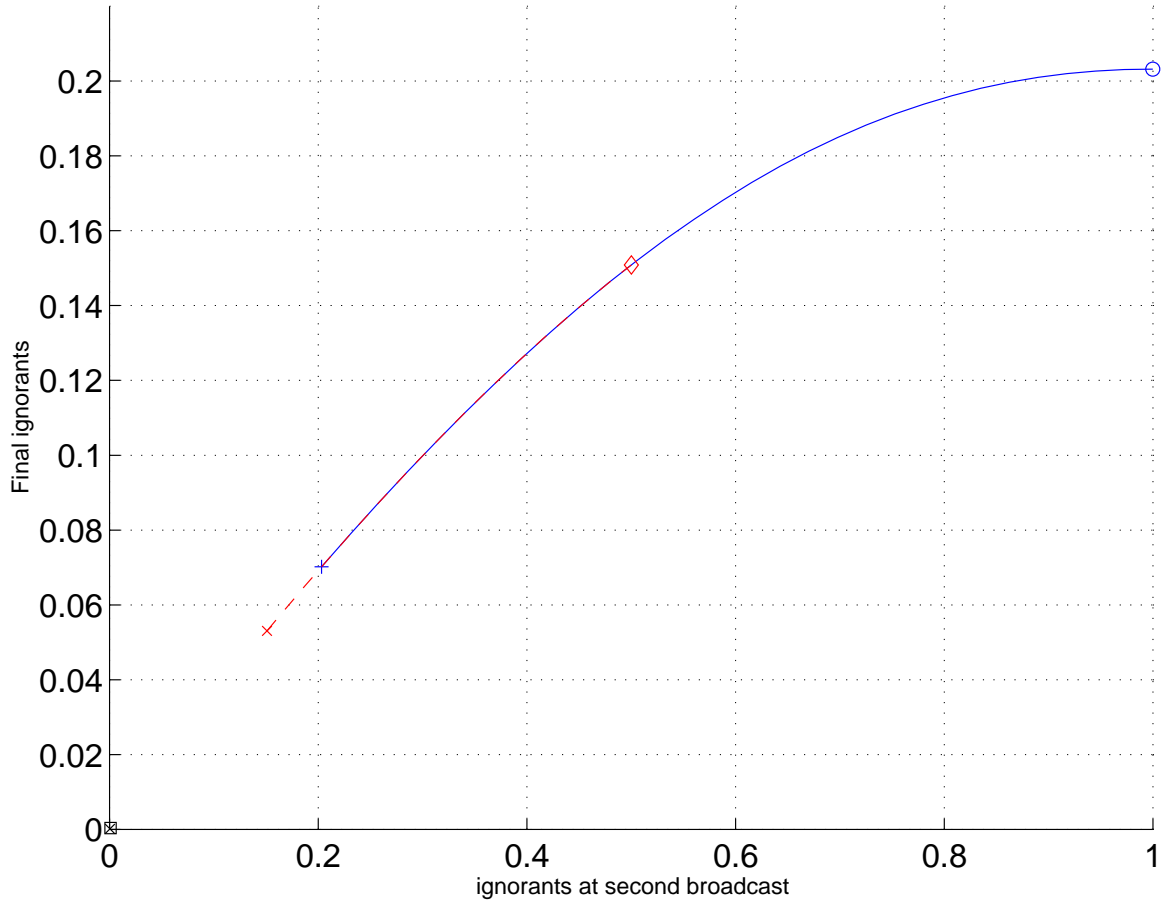


Figure 4.4.1: i_f vs i_b for various values of β under Scenario 2. The curve segment starting with \circ and ending with $+$ corresponds to the case $\beta \rightarrow 0$; the curve segment starting with \diamond and ending with \times corresponds to $\beta = 0.5$. The case $\beta \rightarrow 1$ is given by a point at the origin.

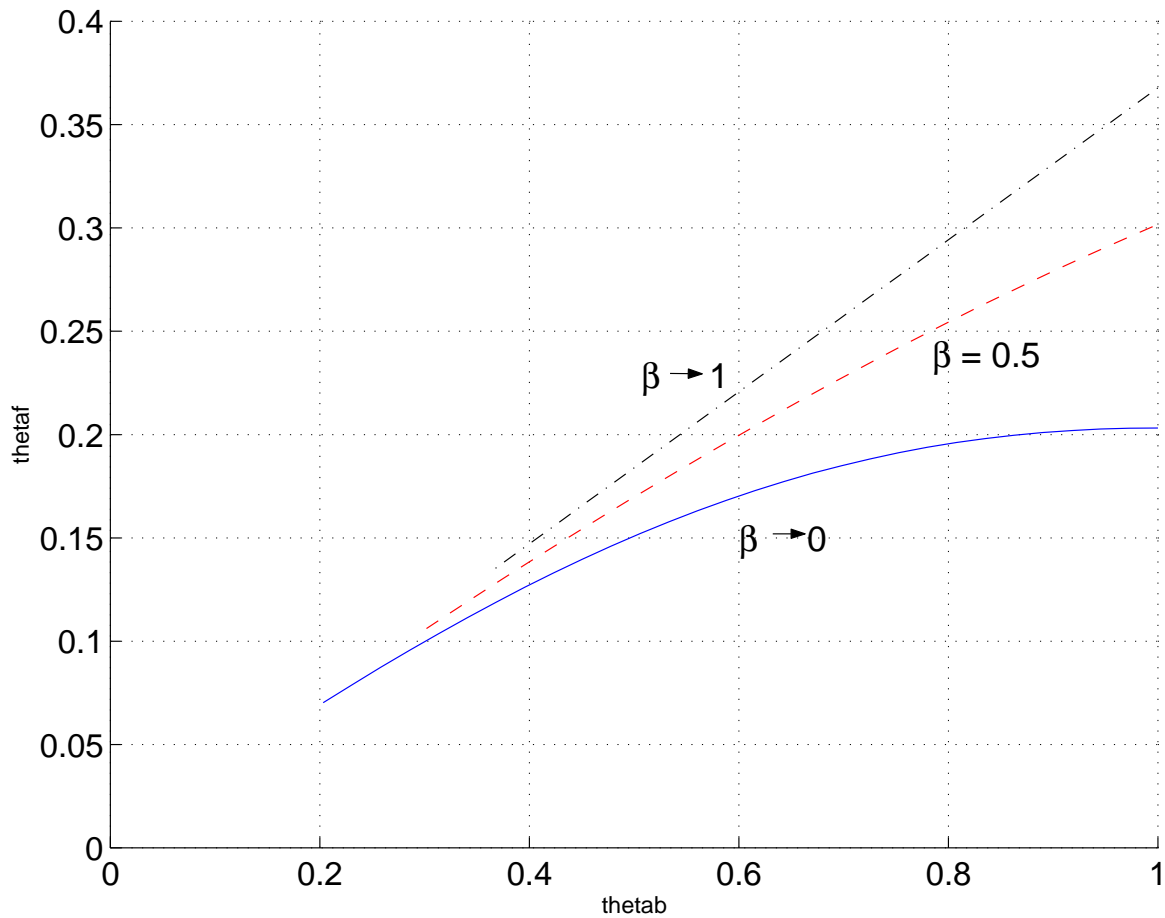


Figure 4.4.2: θ_f vs θ_b for various values of β under Scenario 2.

4.5 Conclusions of two broadcasts

We have introduced an impulsive control model of a rumour process and considered two consecutive broadcasts, the first one starting the rumour process. In both the cases when spreaders are reactivated from the subscriber stiflers (Scenario 1) as well as from those stiflers who once were spreaders (Scenario 2), we have shown that optimal time

for the second broadcast to minimize the final proportion of ignorants is always at the end of the process started by the first broadcast. In other words, a second rumour process commences once the first process terminates.

Some of the auxiliary results we obtained are worth mentioning here, because of their practical significance. One result shows that the spreader population changes from consisting entirely of subscribers to consisting entirely of nonsubscribers at the termination of a process. This can perhaps be interpreted as to why the end of the first process is the best time for a second broadcast. Another result implies that the final proportion of ignorants after two broadcasts, the second being optimally timed, is the same as that resulting from a single broadcast with twice the initial proportion of spreaders, provided this is allowed by the physical constraints. If one considers the initial proportion of spreaders as the "resource" available to start the process, then it may be best to allocate as much of this resource as possible at the beginning of the process.

4.6 Multiple broadcasts

A model with two broadcasts was envisaged in the previous Sections in this Chapter.

This has been published as [15]. A control ingredient is incorporated, the timing of the second broadcast.

This section presents a generalisation of this model to a general number $n > 1$ of broadcasts, with the intention of reducing the final proportion of the population never hearing the rumour. The rumour process is started by a broadcast to a subpopulation, the *subscribers*, who commence spreading the rumour. We wish to determine when to effect subsequent broadcasts $2, 3, \dots, n$ so as to minimise the final proportion of ignorants in the population.

Two basic scenarios are considered as it was for the two broadcasts also.

To obtain some results without becoming too enmeshed in probabilistic technicalities, we follow Daley and Kendall and, after an initial discrete description of the population, describe the process in the continuum limit corresponding to a total population tending to infinity. Exactly the same formulation occurs in the continuum limit if one starts with the Maki-Thompson formulation.

The analysis in this section is considerably simpler than that of sections 4.3 and 4.4 where we discussed , a number of other issues neglected in the present more streamlined account.

We may summarize the key results as follows.

Theorem 4.6.1. *In the rumour process prescribed by (2.2.4)–(2.2.7), (a) i is strictly decreasing with time with limiting value ζ satisfying*

$$0 < \zeta < 1/2; \tag{4.6.1}$$

(b) ζ is the smallest positive solution to the transcendental equation

$$\frac{\zeta}{\alpha} e^{2(\alpha-\zeta)} = e^{-\beta}; \quad (4.6.2)$$

(c) s is ultimately strictly decreasing to limit 0.

The limiting case $\alpha \rightarrow 1$, $\beta \rightarrow 0$, $\gamma \rightarrow 0$ is the classical situation treated by Daley and Kendall. In this case (4.6.2) becomes

$$\frac{\xi}{\alpha} e^{2(1-\xi)} = 1.$$

This is the equation used by Daley and Kendall to determine that in their classical case $\xi \approx 0.2031878$.

In the next Section we introduce two useful preliminary results. In Sections 4.8 and 4.9 we treat Scenarios 1 and Scenarios 2 respectively. Finally, in Section 4.10, we compare the two scenarios.

4.7 Technical Preliminaries

We shall make repeated use the following theorem.

Theorem 4.7.1. *Suppose $\alpha, \beta > 0$ with $\gamma \geq 0$ in a single-rumour process. Then we have the following.*

(a) For $\alpha + \gamma = 1 - \beta$ fixed, ζ is strictly increasing in α for $\alpha \leq 1/2$.

(b) For $\alpha + \gamma = 1 - \beta$ fixed, ζ is strictly decreasing in α for $\alpha \geq 1/2$.

(c) For $\alpha + \beta = 1 - \gamma$ fixed, ζ is strictly increasing in α .

This is part of [Chapter 2, Theorem 3 at page **34**] , except that the statements there corresponding to (a) and (b) are for $\alpha < 1/2$ and $\alpha > 1/2$ respectively. The extensions to include $\alpha = 1/2$ follow trivially from the continuity of ζ as a function of α .

It is also convenient to articulate the following lemma, the proof of which is immediate.

Lemma 4.7.1. *For $x \in [0, 1/2]$, the map $x \mapsto xe^{-2x}$ is strictly increasing.*

4.8 Scenario 1

We now address a compound rumour process in which $n > 1$ broadcasts are made under Scenario 1. We shall show that the final proportion of the population never hearing a rumour is minimised when and only when the second and subsequent broadcasts are made at the successive epochs at which $s = 0$ occurs. We refer to this procedure as control policy \mathcal{S} . It is convenient to consider separately the cases $0 < \alpha \leq 1/2$ and $\alpha > 1/2$. Throughout this and the following two sections, ξ denotes the final proportion of the population hearing none of the sequence of rumours.

Theorem 4.8.1. *Let $0 < \alpha \leq 1/2$ with $\beta > 0$ and $\gamma \geq 0$. Suppose Scenario 1 applies and $n > 1$ broadcasts are made. Then*

(a) ξ is minimised if and only if the control process \mathcal{S} is adopted;

(b) under control policy \mathcal{S} , ξ is a strictly increasing function of α .

Proof. Let \mathcal{T} be an optimal control policy, with successive broadcasts occurring at times $\tau_1 \leq \tau_2 \leq \dots \leq \tau_n$. We denote the proportion of ignorants in the population at τ_k by i_k ($k = 1, \dots, n$), so that $i_1 = \alpha$. Since i is strictly decreasing during the course of each rumour and is continuous at a broadcast epoch, we have from applying Theorem 4.6.1 to each broadcast in turn that

$$i_1 \geq i_2 \geq \dots \geq i_n > \xi > 0, \quad (4.8.1)$$

all the inequalities being strict unless two consecutive broadcasts are simultaneous.

Suppose if possible that $s > 0$ at time $\tau_n - 0$. Imagine the broadcast about to be made at this epoch were postponed and s allowed to decrease to zero before that broadcast is made. Denote by ξ' the corresponding final proportion of ignorants in the population. Since i decreases strictly with time, the final broadcast would then occur when the proportion of ignorants had a value

$$i'_n < i_n. \quad (4.8.2)$$

Now in both the original and modified systems we have that $s = \beta$ at $\tau_n + 0$. By Theorem 4.7.1(a), (4.8.2) implies $\xi' < \xi$, contradicting the optimality of policy \mathcal{T} .

Hence we must have $s = 0$ at $\tau_n - 0$ and so by Theorem 4.6.1 that

$$\frac{1}{2} > i_n > \xi .$$

Applying Theorem 4.7.1(a) again, to the last two broadcasts, gives that i_n is a strictly increasing function of i_{n-1} and that ξ is strictly increasing in i_n . Hence ξ is strictly increasing in i_{n-1} .

If $n = 2$, we have nothing left to prove, so suppose $n > 2$. We shall derive the desired results by backward induction on the broadcast labels. We suppose that for some k with $2 < k \leq n$ we have

- (i) $s = 0$ at time $\tau_j - 0$ for $j = k, k + 1, \dots, n$;
- (ii) ξ is a strictly increasing function of i_{k-1} .

To establish the inductive step, we need to show that $s = 0$ at $\tau_{k-1} - 0$ and that ξ is a strictly increasing function of i_{k-2} . The previous paragraph provides a basis $k = n$ for the backward induction.

If $s > 0$ at $\tau_{k-1} - 0$, then we may envisage again modifying the system, allowing s to reduce to zero before making broadcast $k - 1$. This entails that, if there is a proportion i'_{k-1} of ignorants in the population at the epoch of that broadcast, then

$$0 < i'_{k-1} < i_{k-1} .$$

By (ii) this gives $\xi' < \xi$ and hence contradicts the optimality of \mathcal{T} , so we must have

$s = 0$ at $\tau_{k-1} - 0$. Theorem 4.7.1(a) now yields that i_{k-1} is a strictly increasing function of i_{k-2} , so that by (ii) ξ is a strictly increasing function of i_{k-2} . Thus we have the inductive step and the theorem is proved. \square

For the counterpart result for $\alpha > 1/2$, it will be convenient to extend the notation of Theorem 4.7.1 and use $\zeta(i)$ to denote the final proportion of ignorants when a single rumour beginning with state $(i, \beta, 1 - i - \beta)$ has run its course.

Theorem 4.8.2. *Let $\alpha > 1/2$ with $\beta > 0$ and $\gamma \geq 0$. Suppose Scenario 1 applies and $n > 1$ broadcasts are made. Then*

- (a) ξ is minimised if and only if the control process \mathcal{S} is adopted;
- (b) under control policy \mathcal{S} , ξ is a strictly decreasing function of α .

Proof. First suppose that $i_n \geq 1/2$. By Theorem 4.6.1 and (4.8.1), this necessitates that $s > 0$ at time $\tau_2 - 0$. If we withheld broadcast 2 until $s = 0$ occurred, the proportion i'_2 of ignorants at that epoch would then satisfy

$$i'_2 = \zeta(i_1) \leq \zeta(i_n) = \xi < 1/2.$$

The relations between consecutive pairs of terms in this continued inequality are given by the definition of ζ , Theorem 4.7.1(b), the definition of ζ again, and Theorem 4.6.1 applied to broadcast n .

Hence policy \mathcal{S} would give rise to ξ' satisfying

$$\xi' < i'_n \leq i'_2 \leq \xi ,$$

contradicting the optimality of \mathcal{T} . Thus we must have $i_n < 1/2$ and so

$$i_1 \geq i_2 \geq \cdots \geq i_k \geq 1/2 > i_{k+1} \geq \cdots \geq i_n > \xi$$

for some k with $1 \leq k < n$.

Suppose if possible $k > 1$. Then arguing as above gives

$$i'_2 = \zeta(i_1) \leq \zeta(i_k) \leq i_{k+1} < 1/2 .$$

The second inequality will be strict unless $s = 0$ at time $\tau_{k+1} - 0$. This leads to

$$i'_3 = \zeta(i'_2) \leq \zeta(i_{k+1}) \leq i_{k+2} < 1/2,$$

and proceeding recursively we obtain

$$i'_{n-k+1} \leq i_n < 1/2$$

and so

$$i'_{n-k+2} \leq \xi .$$

Thus we have $\xi' < \xi$, again contradicting the optimality of \mathcal{T} . Hence we must have

$k = 1$, and so

$$i_1 > 1/2 \geq i_2 \geq i_3 \geq \cdots \geq i_n > \xi .$$

Consider an optimally controlled rumour starting from state $(i_2, \beta, 1 - i_2 - \beta)$. By Theorem 4.8.1(b), ξ is a strictly increasing function of i_2 . For \mathcal{T} to be optimal, we thus require that i_2 be determined by letting the initial rumour run its full course, that is, that $s = 0$ at $\tau_2 - 0$. This yields Part (a). Since $\alpha > 1/2$, Theorem 4.7.1(b) gives that, with control policy \mathcal{S} , i_2 is a strictly decreasing function of α . Part (b) now follows from the fact that ξ is a strictly increasing function of i_2 . \square

Remark 4. For an optimal sequence of n broadcasts under Scenario 1, Theorems 4.6.1, 4.8.1 and 4.8.2 provide

$$\frac{i_k}{i_{k-1}} e^{2(i_{k-1} - i_k)} = e^{-\beta} \quad \text{for } 1 < k \leq n \quad (4.8.3)$$

and

$$\frac{\xi}{i_n} e^{2(i_n - \xi)} = e^{-\beta}. \quad (4.8.4)$$

Multiplying these relations together provides

$$\frac{\xi}{\alpha} e^{2(\alpha - \xi)} = e^{-n\beta},$$

which may be rewritten as

$$\xi e^{-2\xi} = \alpha e^{-(2\alpha + n\beta)}. \quad (4.8.5)$$

By Lemma 4.7.1, the left-hand side is a strictly increasing function of ξ for $\xi \in [0, 1/2]$.

Hence (4.8.5) determines ξ uniquely. \square

Remark 5. Equations (4.8.3), (4.8.4) may be recast as

$$i_k e^{-2i_k} = i_{k-1} e^{-(\beta+2i_{k-1})} \quad \text{for } 2 \leq k \leq n \quad (4.8.6)$$

and

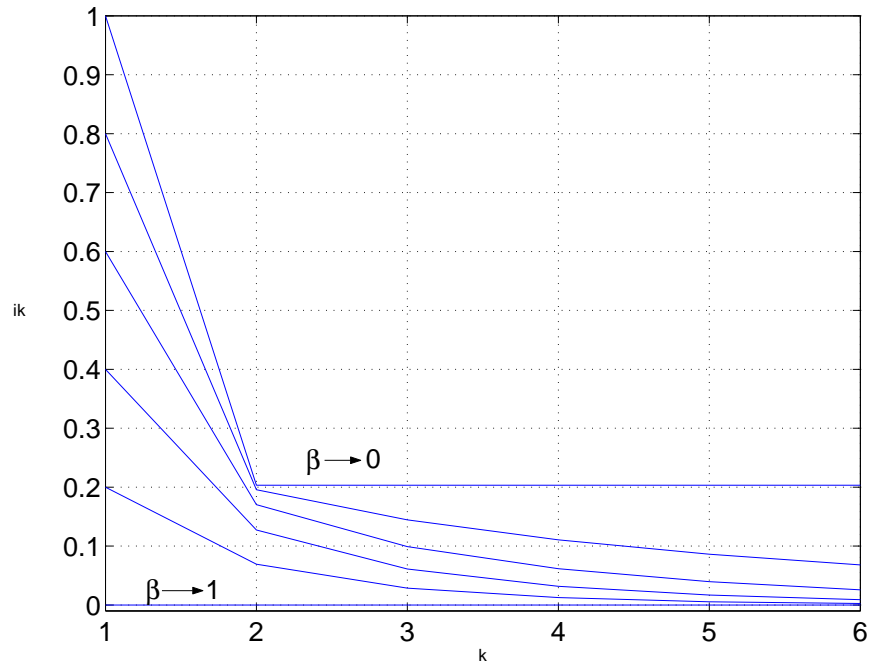
$$\xi e^{-2\xi} = i_k e^{-(\beta+2i_k)} . \quad (4.8.7)$$

Consider the limiting case $\beta \rightarrow 0$ and $\gamma \rightarrow 0$, which gives the classical Daley–Kendall limit of a rumour started by a single individual. Since $i_k \leq 1/2$ for $2 \leq k \leq n$ and $\xi \leq 1/2$, we have by Lemma 4.7.1 that in fact

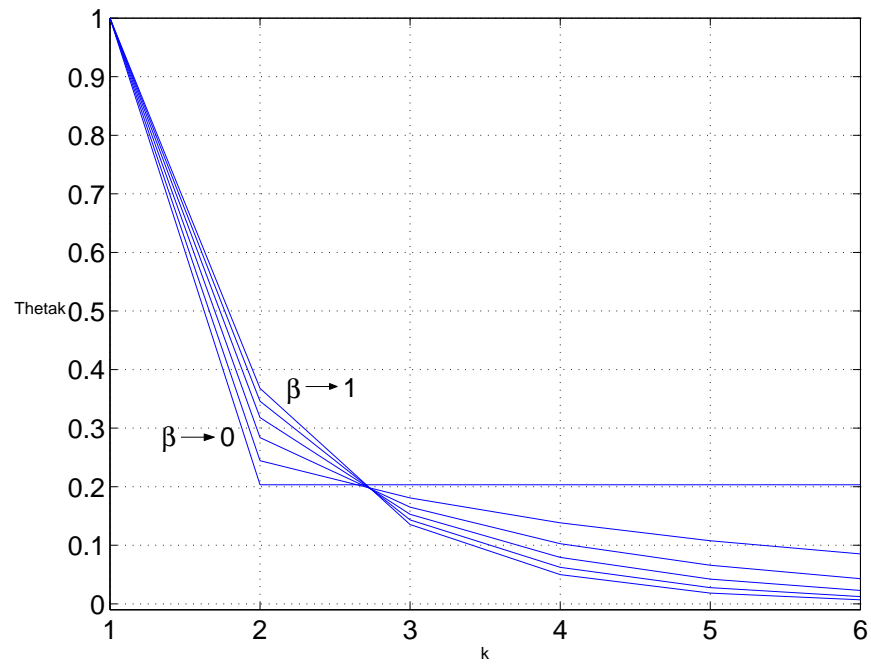
$$i_k = \xi \quad \text{for } 2 \leq k \leq n.$$

If $\alpha \leq 1/2$, then the above equality actually holds for $1 \leq k \leq n$. This is also clear intuitively: in the limit $\beta \rightarrow 0$ the reactivation taking place at the second and subsequent broadcast epochs does not change the system physically. This cannot occur for $\beta > 0$, which shows that when the initial broadcast is to a perceptible proportion of the population, as with the mass media, the effects are qualitatively different from those in the situation of a single initial spreader. \square

For the standard case of $\gamma = 0$, i.e. for $\alpha + \beta = 1$, the behaviour of i_k , with $n = 5$ broadcasts, is depicted in Figure 4.8.1(a). In generating the graphs Equation (4.8.6) has been solved with initial conditions $\beta = 0, 0.2, 0.4, 0.6, 0.8, 1$. Comments given in Remark 5 can also be verified from the figure.



(a)



(b)

Figure 4.8.1: An illustration of Scenario 1 with $\alpha + \beta = 1$ and 5 broadcasts. In each simulation β is incremented by 0.2.

Next we will examine the dependence of ξ on the initial conditions. Equation (4.8.5)

can be rewritten as

$$n\beta + 2(\alpha - \xi) + \ln \xi - \ln \alpha = 0 \quad (4.8.8)$$

The following theorem and its corollary are given in [Chapter 2, Theorem 3 and Corollary 1] for a single broadcast ($n = 1$). We re-state them here for multiple broadcasts ($n \geq 2$) and provide a suitably modified proof. The proof of the corollary coincides with that of the single broadcast case.

Theorem 4.8.3. *For any $n \geq 1$, ξ has the following properties.*

(a) *For $\alpha + \gamma = 1 - \beta$ fixed, ξ is strictly increasing in α for $\alpha < 1/2$ and strictly*

decreasing in α for $\alpha > 1/2$.

(b) *For $\beta + \gamma = 1 - \alpha$ fixed, ξ is strictly decreasing in β .*

(c) *For $\alpha + \beta = 1 - \gamma$ fixed, ξ is strictly increasing in α .*

Proof. In each part we will use the fact that $\xi < 1/2$. In part (a) implicit differentiation of (4.8.8) yields

$$\frac{\partial \xi}{\partial \alpha} = \frac{\xi}{\alpha} \frac{1 - 2\alpha}{1 - 2\xi},$$

which is positive for $\alpha < 1/2$ and negative for $\alpha > 1/2$, furnishing the required statement. Similarly in the context of (b) we have

$$\frac{\partial \xi}{\partial \beta} = -\frac{n\xi}{1 - 2\xi} < 0,$$

giving the required result. For (c) one gets

$$\frac{\partial \xi}{\partial \alpha} = \frac{\xi}{\alpha} \frac{1 + (n-2)\alpha}{1 - 2\xi} > 0 ,$$

for any $n \geq 1$, which completes the proof. \square

Corollary 4.8.1. *For any $n \geq 1$, we have $\bar{\xi} := \sup \xi = 1/2$. This occurs in the limiting case $\alpha = 1/2 = \gamma$, with $\beta = 0$.*

Figure 4.8.1(a) provides a graphical illustration of Theorem 4.8.3(c) for $\gamma = 0$, i.e. for $\alpha + \beta = 1$.

For $1 \leq k \leq n$, let $\theta_{k+1} := i_{k+1}/i_k$ denote the quotient of the proportion of ignorants at the end of the k th process by that at the beginning of that process. Also set $\theta_1 := i_1/\alpha = 1$. The product

$$\Theta_{k+1} = \theta_1 \cdots \theta_k \theta_{k+1}$$

is the ratio i_{k+1}/α of the proportion of ignorants at the end of the k th process to that at the beginning of the first, for $1 \leq k \leq n$. Note that $\Theta_1 = \theta_1 = 1$. Given n processes, we define the limiting value $\eta := \Theta_{n+1}$. The relevant equations can be re-written with these parameters as follows. For the k th process,

$$\Theta_{k+1} e^{-2\alpha \Theta_{k+1}} = e^{-(2\alpha + k\beta)} , \quad 1 \leq k \leq (n-1) , \quad (4.8.9)$$

and, for the n th process,

$$\eta e^{-2\alpha \eta} = e^{-(2\alpha + n\beta)} . \quad (4.8.10)$$

Put $w = -2\alpha\eta$. Then

$$w e^w = -2\alpha e^{-(2\alpha+n\beta)},$$

the solution of which is given by the so-called Lambert w function (Corless *et al.* [29], Chapter [2]). A direct application of the series expression given in Chapter [2] and Appendix C provides

$$w = \sum_{j=1}^{\infty} \frac{(2\alpha e^{-(2\alpha+n\beta)})^j}{j!} j^{j-1}.$$

Remark 6. In the case of almost no initial ignorants, i.e. when $\alpha \rightarrow 0$,

$$\eta = e^{-n\beta}. \quad (4.8.11)$$

The proportion of ignorants to those at the beginning decays exponentially at a rate equal to the product of the number n of broadcasts with the proportion β of initial spreaders. Two further cases, namely (i) $\beta \rightarrow 0$ and (ii) $\beta \rightarrow 1$, are interesting to consider. These are elaborated below.

- (i) In the case of $\beta \rightarrow 0$ there is only one spreader in the infinitely large population, and so almost all of the initial population consists of stiflers, i.e. $\gamma \rightarrow 1$. One gets

$$\eta = 1.$$

The number of introduced broadcasts during the rumour process may vary but, the proportion of ignorants at the beginning remains unchanged.

(ii) In the case of $\beta \rightarrow 1$ almost all of the initial population consists of spreaders,

and we obtain

$$\eta = e^{-n} .$$

Consider Equation (4.8.11) again. For $0 < \beta < 1$, as well as for $\beta \rightarrow 1$, we note that

$\eta \rightarrow 0$ as $n \rightarrow \infty$. □

For the standard case of $\gamma = 0$, the behaviour of Θ_k is illustrated in Figure 4.8.1(b) by solving (4.8.9) with different initial conditions and for 5 broadcasts. In particular Remark 6(ii) stated above can be observed from the figure.

Remark 7. Given initial proportions α of ignorants and β of subscribers, with $0 < \beta < 1$ or with $\beta \rightarrow 1$, the required number n of broadcasts to achieve a target proportion η or less of ignorants can be obtained through (4.8.10) as

$$k = \left\lceil -\frac{1}{\beta} [\ln(\eta) + 2\alpha(1 - \eta)] \right\rceil .$$

□

Equation (4.8.10) can be rewritten as

$$n\beta + 2\alpha(1 - \eta) + \ln \eta = 0 . \tag{4.8.12}$$

Theorem 4.8.4. *The limiting value η has the following properties.*

(a) *For $\alpha + \gamma = 1 - \beta$ fixed and any $n \geq 1$, η is strictly decreasing in α .*

(b) For $\beta + \gamma = 1 - \alpha$ fixed and any $n \geq 1$, η is strictly decreasing in β .

(c) For $\alpha + \beta = 1 - \gamma$ fixed, η is strictly decreasing in α for $n = 1$ and strictly increasing in α for $n \geq 2$.

Proof. We will use the fact that $\eta < 1/2$ (see Chapter 2) and that $\alpha\eta < 1/2$. In part

(a), implicit differentiation of (4.8.13) gives

$$\frac{\partial\eta}{\partial\alpha} = -\frac{\eta(2-n-2\eta)}{1-2\alpha\eta} \quad (*)$$

for fixed $\alpha + \beta = 1 - \gamma$. We observe that for $n = 1$ $(2 - n - 2\eta) > 0$, and so $\partial\eta/\partial\alpha < 0$.

On the other hand, $(2 - n - 2\eta) < 0$ for $n \geq 2$, and we get $\partial\eta/\partial\alpha > 0$.

$$\frac{\partial\eta}{\partial\alpha} = -\frac{2\eta(1-\eta)}{1-2\alpha\eta},$$

which is negative, furnishing the required result. In part (b) similarly through

$$\frac{\partial\eta}{\partial\beta} = -\frac{n\eta}{1-2\alpha\eta} < 0,$$

we get the required result. With the condition of fixed $\alpha + \beta = 1 - \gamma$ in (c) we obtain

$$\frac{\partial\eta}{\partial\alpha} = -\frac{\eta(2-n-2\eta)}{1-2\alpha\eta}.$$

We observe that for $n = 1$ $(2 - n - 2\eta) > 0$, and so $\partial\eta/\partial\alpha < 0$. On the other hand,

$(2 - n - 2\eta) < 0$ for $n \geq 2$, and we get $\partial\eta/\partial\alpha > 0$. This completes the proof. \square

Now we touch the convexity properties of Theorem 4.8.4. Implicit differentiation of

$$n\beta + 2\alpha(1-\eta) + \ln\eta = 0 \tag{4.8.13}$$

with respect to α yields

$$\frac{\partial \eta}{\partial \alpha} = -\frac{\eta(2-n-2\eta)}{1-2\alpha\eta} \quad (*)$$

for fixed $\alpha + \beta = 1 - \gamma$. We observe that for $n = 1$ $(2 - n - 2\eta) > 0$, and so $\partial\eta/\partial\alpha < 0$.

On the other hand, $(2 - n - 2\eta) < 0$ for $n \geq 2$, and we get $\partial\eta/\partial\alpha > 0$. Implicit

differentiation of 4.8.13 twice with respect to α yields

$$\frac{\partial^2 \eta}{\partial \alpha^2} = \frac{\partial \eta}{\partial \alpha} \left(-\frac{2-n-2\eta}{(1-2\alpha\eta)^2} + 2\frac{\eta}{1-2\alpha\eta} \right) - 2\frac{\eta}{1-2\alpha\eta}(2-n-2\eta).$$

Letting

$$A = \frac{-2+n+2\eta+2\eta-4\alpha\eta^2}{(1-2\alpha\eta)^2},$$

for $\alpha \approx 0$ and $n \geq 2$, $4\eta + n - 2 > 0$ so, $A > 0$ for $n \geq 2$. For $\alpha \approx 0$ and

$n = 1$, $4\eta - 1 = 4e^{-\beta} - 1$ since $\eta = e^{-n\beta}$. So, $4\eta - 1 > 0$ as $\beta \rightarrow 0$ or 1. Thus $A > 0$

for $n = 1$. Letting

$$B = -2\frac{\eta}{1-2\alpha\eta}(2-n-2\eta) = \frac{4\eta^2 + 2n\eta - 4\eta}{1-2\alpha\eta},$$

for $\alpha \approx 0$ and $n \geq 2$, $B > 0$. α , for $\alpha \approx 0$, $n \geq 2$.

$$B = -2\eta(2-n-2\eta) = -2\eta(1-2\eta).$$

As $\beta \rightarrow 1$, $B < 0$.

On the other hand, as $\beta \rightarrow 0$ $B > 0$. However when $\alpha \approx 0$, $\beta \approx 0$ is not the most expected case and this case may be ignored.

As a conclusion, for $n \geq 2$ and $\alpha \approx 0$,

$$\frac{\partial^2 \eta}{\partial \alpha^2} = \frac{\partial \eta}{\partial \alpha} A + B$$

is positive since $\frac{\partial \eta}{\partial \alpha} > 0$, $A > 0$ and $B > 0$. Hence η is a strictly convex function of α ,

for $\alpha \approx 0$, $n \geq 2$. On the other hand, for $n = 1$ and $\alpha \approx 0$,

$$\frac{\partial^2 \eta}{\partial \alpha^2} < 0$$

since $\frac{\partial \eta}{\partial \alpha} < 0$, $A > 0$ and $B < 0$. Hence η is a strictly concave function of α for

$\alpha \approx 0$ and $n = 1$.

Now we look at the cases for

$$\alpha = 1, \{n = 1 \text{ or } n \geq 2\}.$$

For $\alpha = 1$ and $n = 1$,

$$\frac{\partial^2 \eta}{\partial \alpha^2} = -\frac{\partial \eta}{\partial \alpha} - 2\eta < 0$$

since $\frac{\partial \eta}{\partial \alpha} = -\frac{\eta(1-2\eta)}{1-2\eta} = -\eta$ and $\eta > 0$. It is negative for both $\beta \rightarrow 0$ or $\beta \rightarrow 1$

analytically, but $\beta \rightarrow 1$ is not the case in the process while $\alpha \rightarrow 1$. Hence, η is a

strictly concave function of α .

For $\alpha = 1$ and $n \geq 2$,

$$(1 - 2\eta) \frac{\partial^2 \eta}{\partial \alpha^2} = \frac{\partial \eta}{\partial \alpha} X + Y,$$

where

$$X = \frac{4\eta(1 - \eta) + n - 2}{1 - 2\eta} > 0$$

(since $\eta < 1/2$) and

$$Y = 4\eta^2 + 2\eta n - 4\eta \geq 0.$$

Also $\frac{\partial \eta}{\partial \alpha} > 0$ so, $\frac{\partial^2 \eta}{\partial \alpha^2} > 0$.

Hence, η is a strictly convex function of α . It is noted that for any α and $n = 1$, η is strictly concave function of α and for any α , $n \geq 2$, η is a strictly convex function of α .

A graphical illustration of Theorem 4.8.4(c) for $\gamma = 0$ can be seen in Figure 4.8.1(b).

4.9 Scenario 2

Theorem 4.9.1. *Let $\alpha, \beta > 0, \gamma \geq 0$ and suppose Scenario 2 applies and $n > 1$ broadcasts are made. Then*

(a) ξ is minimised if and only if control policy \mathcal{S} is adopted;

(b) under control policy \mathcal{S} , ξ is a strictly increasing function of α .

Proof. The argument closely parallels that of Theorem 4.8.1. The proof follows verbatim down to (4.8.2). We continue by noting that in either the original or modified system $r = \gamma$ at time $\tau_n + 0$. By Theorem 4.7.1(c), (4.8.2) implies $\xi' < \xi$, contradicting the optimality of control policy \mathcal{T} . Hence we must have $s = 0$ at time $\tau_n - 0$. The rest of the proof follows the corresponding argument in Theorem 4.8.1 but with Theorem 4.7.1(c) invoked in place of Theorem 4.7.1(a). \square

Remark 8. The determination of ξ under Scenario 2 with control policy \mathcal{S} is more involved than that under Scenario 1. For $1 \leq k \leq n$, set $\beta_k = s(\tau_k + 0)$.

Then $i_k + \beta_k = 1 - \gamma = \alpha + \beta$, so that Theorem 4.6.1 yields

$$\frac{i_k}{i_{k-1}} e^{2(i_{k-1} - i_k)} = e^{-(\alpha + \beta - i_{k-1})} \quad \text{for } 1 < k \leq n$$

and

$$\frac{\xi}{i_n} e^{2(i_n - \xi)} = e^{-(\alpha + \beta - i_n)} .$$

We may recast these relations as

$$i_k e^{-2i_k} = i_{k-1} e^{-(\alpha + \beta + i_{k-1})} \quad \text{for } 1 < k \leq n \quad (4.9.1)$$

and

$$\xi e^{-2\xi} = i_n e^{-(\alpha + \beta + i_n)} . \quad (4.9.2)$$

Since $i_k, \xi \in (0, 1/2)$ for $1 < k \leq n$, Lemma 4.7.1 yields that (4.9.1), (4.9.2)

determine $i_2, i_3, \dots, i_n, \xi$ uniquely and sequentially from $i_1 = \alpha$. \square

For the standard case of $\gamma = 0$, i.e. $\alpha + \beta = 1$, Figure 4.9.1(a) depicts the behaviour of i_k , with $n = 5$ broadcasts, by solving (4.9.1). In generating the graphs the initial values $\beta = 0, 0.2, 0.4, 0.6, 0.8, 1$ have been used.

As in the case of Scenario 1, we will examine the dependence of ξ on the initial conditions. Equations (4.9.1)-(4.9.2) can be rewritten as

$$\beta + \alpha + i_n - 2\xi + \ln \xi - \ln i_n = 0 ; \quad (4.9.3)$$

$$\beta + \alpha + i_{k-1} - 2i_k + \ln i_k - \ln i_{k-1} = 0, \quad 1 < k \leq n . \quad (4.9.4)$$

We give the following theorem as a companion of Theorem 4.8.3.

Theorem 4.9.2. *For any $n > 1$, ξ has the following properties.*

(a) *For $\alpha + \gamma = 1 - \beta$ fixed, ξ is strictly increasing in α for $\alpha < 1/3$ and strictly decreasing in α for $\alpha > 1/2$.*

(b) *For $\beta + \gamma = 1 - \alpha$ fixed, ξ is strictly decreasing in β .*

(c) *For $\alpha + \beta = 1 - \gamma$ fixed, ξ is strictly increasing in α .*

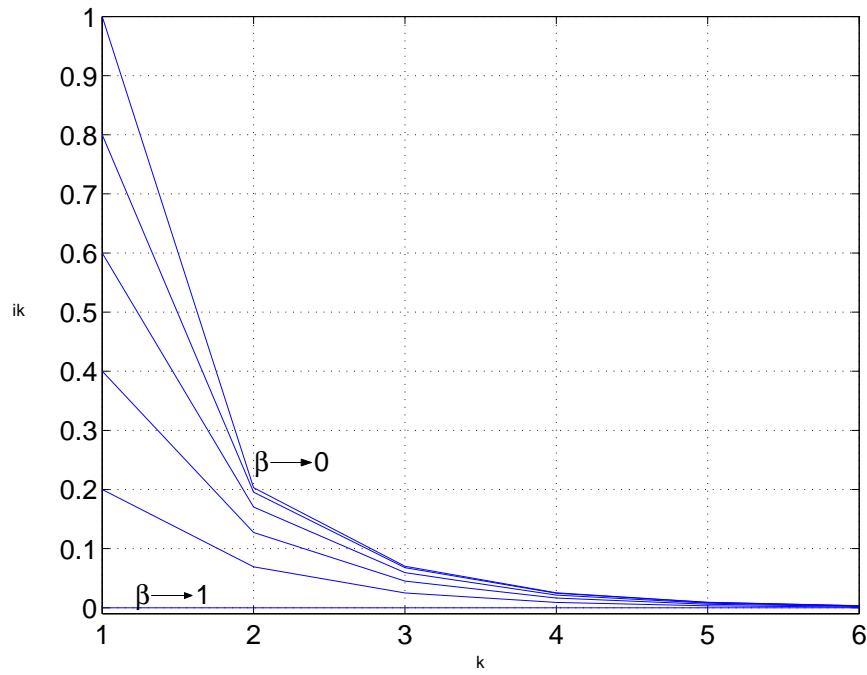
Proof. In each part we use the facts that $\xi < 1/2$ and $i_k < 1/2$, $1 < k \leq n$.

(a) Implicit differentiation of (4.9.3) yields

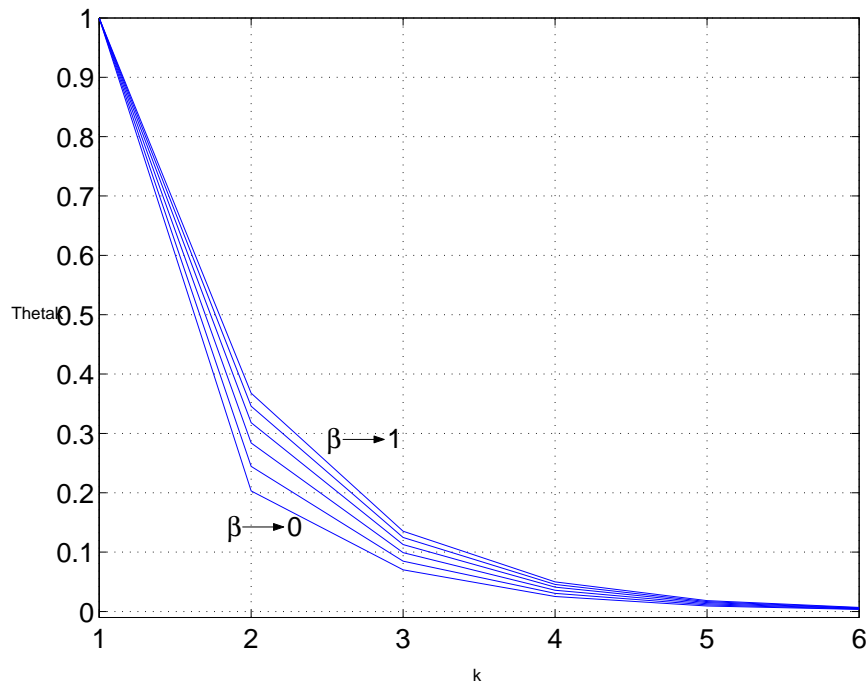
$$\frac{\partial \xi}{\partial \alpha} = -\frac{\xi}{i_n(1-2\xi)} \left[i_n - (1-i_n) \frac{\partial i_n}{\partial \alpha} \right] . \quad (4.9.5)$$

Observe that

$$\frac{\partial i_n}{\partial \alpha} = \frac{\partial i_n}{\partial i_{n-1}} \frac{\partial i_{n-1}}{\partial i_{n-2}} \dots \frac{\partial i_3}{\partial i_2} \frac{\partial i_2}{\partial \alpha} . \quad (4.9.6)$$



(a)



(b)

Figure 4.9.1: An illustration of Scenario 2 with $\alpha + \beta = 1$ and 5 broadcasts. In each simulation β is incremented by 0.2.

Implicit differentiation of (4.9.4) gives

$$\frac{\partial i_k}{\partial i_{k-1}} = \frac{i_k}{i_{k-1}} \frac{1 - i_{k-1}}{1 - 2i_k}, \quad 2 < k \leq n,$$

and

$$\frac{\partial i_2}{\partial \alpha} = \frac{i_2}{\alpha} \frac{1 - 2\alpha}{1 - 2i_2}.$$

Substituting these into (4.9.6) and simplifying we get

$$\frac{\partial i_n}{\partial \alpha} = i_n \frac{(1 - i_{n-1}) \cdots (1 - i_2)(1 - 2\alpha)}{(1 - 2i_n) \cdots (1 - 2i_2) \alpha}.$$

Now the term in square brackets in (4.9.5) can be rewritten as

$$i_n - (1 - i_n) \frac{\partial i_n}{\partial \alpha} = i_n \left[1 - \frac{(1 - i_n)(1 - i_{n-1}) \cdots (1 - i_2)(1 - 2\alpha)}{(1 - 2i_n)(1 - 2i_{n-1}) \cdots (1 - 2i_2) \alpha} \right].$$

We note that

$$\frac{1 - i_k}{1 - 2i_k} > 1, \quad 2 < k \leq n.$$

Furthermore

$$\frac{1 - 2\alpha}{\alpha} > 1 \quad \text{for } \alpha < 1/3,$$

and

$$\frac{1 - 2\alpha}{\alpha} < 0 \quad \text{for } \alpha > 1/2.$$

Therefore $[i_n - (1 - i_n) \partial i_n / \partial \alpha]$ is negative for $\alpha < 1/3$ and so $\partial \xi / \partial \alpha > 0$. Furthermore

$[i_n - (1 - i_n) \partial i_n / \partial \alpha]$ is positive for $\alpha > 1/2$ resulting in $\partial \xi / \partial \alpha < 0$. These furnish the

required statement.

(b) Implicit differentiation of (4.9.3) with respect to β gives

$$\frac{\partial \xi}{\partial \beta} = -\frac{\xi}{i_n(1-2\xi)} \left[i_n - (1-i_n) \frac{\partial i_n}{\partial \beta} \right].$$

Following a procedure similar to that used in the proof of part (a) it can be shown that

$$\frac{\partial i_n}{\partial \beta} = -i_n \frac{(1-i_{n-1}) \cdots (1-i_2)}{(1-2i_n) \cdots (1-2i_2)},$$

which is negative. Therefore $[i_n - (1-i_n) \partial i_n / \partial \beta] > 0$ and so $\partial \xi / \partial \beta > 0$, providing the required result.

(c) With $\alpha + \beta = 1 - \gamma$ fixed, implicit differentiation of (4.9.3) with respect to α yields

$$\frac{\partial \xi}{\partial \alpha} = \frac{\xi}{i_n} \frac{1-i_n}{1-2\xi} \frac{\partial i_n}{\partial \alpha}.$$

Since

$$\frac{\partial i_k}{\partial i_{k-1}} = \frac{i_k}{i_{k-1}} \frac{1-i_{k-1}}{1-2i_k} > 0, \quad 1 < k \leq n,$$

where $i_2 = \alpha$, we have

$$\frac{\partial i_n}{\partial \alpha} = \frac{\partial i_n}{\partial i_{n-1}} \frac{\partial i_{n-1}}{\partial i_{n-2}} \cdots \frac{\partial i_3}{\partial i_2} \frac{\partial i_2}{\partial \alpha} > 0 \tag{4.9.7}$$

and so $\partial \xi / \partial \alpha > 0$, completing the proof. \square

Theorem 4.9.2(c) has been illustrated in Figure 4.9.1(a) for $\gamma = 0$, i.e. for $\alpha + \beta = 1$.

Now convexity of ξ in Theorem 4.9.2 may be addressed as follows. Implicit differentiation of

$$\frac{\partial \xi}{\partial \alpha} = \frac{\xi}{i_n} \frac{1-i_n}{1-2\xi} \frac{\partial i_n}{\partial \alpha}$$

twice with respect to α yields

$$\frac{\partial^2 i_n}{\partial \alpha^2} \left(\frac{\xi(1-i_n)}{(1-2\xi)i_n} \right) + \frac{\partial i_n}{\partial \alpha} \frac{\partial}{\partial \alpha} \left(\frac{\xi(1-i_n)}{(1-2\xi)i_n} \right) = \frac{\partial^2 \xi}{\partial \alpha^2}.$$

$$\frac{\partial^2 \xi}{\partial \alpha^2} = \frac{\partial^2 i_n}{\partial \alpha^2} \left(\frac{\xi(1-i_n)}{(1-2\xi)i_n} \right) + \frac{\partial i_n}{\partial \alpha} \left(\frac{1-i_n}{i_n} \frac{\partial \xi}{\partial \alpha} \frac{1}{(1-2\xi)^2} - \frac{\xi}{1-2\xi} \frac{\partial i_n}{\partial \alpha} \frac{1}{i_n^2} \right)$$

$$\frac{\partial^2 \xi}{\partial \alpha^2} = \frac{\partial^2 i_n}{\partial \alpha^2} C + \frac{\partial i_n}{\partial \alpha} D$$

where

$$C = \frac{\xi(1-i_n)}{(1-2\xi)i_n} < 0$$

and

$$D = \frac{1-i_n}{i_n} \frac{\partial \xi}{\partial \alpha} \frac{1}{(1-2\xi)^2} - \frac{\xi}{1-2\xi} \frac{\partial i_n}{\partial \alpha} \frac{1}{i_n^2} < 0.$$

Also $\frac{\partial^2 i_n}{\partial \alpha^2} > 0$ for $\alpha \approx 1$ and $1/2 < i_n < 1$ by the Equation (4.9.7) and $\frac{\partial i_n}{\partial \alpha} > 0$. So, $\frac{\partial^2 \xi}{\partial \alpha^2} < 0$ for $\alpha \approx 1$.

Hence ξ is a strictly concave function of α . However,

if $0 < i_n < 1/2$ or $\alpha \approx 0$, then ξ can be either concave or convex.

We have not been able to resolve the concavity question for ξ generally if $1/2 < i_n < 1$ or $\alpha \approx 0$.

Using the notation that was introduced for Scenario 1, the recursive equations (4.9.1)-(4.9.2) can be rewritten as

$$\eta e^{-2\alpha\eta} = \Theta_n e^{-(\alpha+\beta+\Theta_n)}, \quad (4.9.8)$$

$$\Theta_k e^{-2\alpha\Theta_k} = \Theta_{k-1} e^{-(\alpha+\beta+\Theta_{k-1})}, \quad 1 < k \leq n, \quad (4.9.9)$$

where $\Theta_1 = 1$.

Remark 9. In the case of almost no initial ignorants in the population, i.e. when $\alpha \rightarrow 0$, Equations (4.9.8)-(4.9.9) reduce to

$$\eta = \Theta_n e^{-\beta}, \quad \Theta_k = \Theta_{k-1} e^{-\beta},$$

which in turn gives

$$\eta = e^{-n\beta}$$

This equation is the same as that obtained in Remark 6 made for Scenario 1. The rest of the discussion given in Remark 6 also holds for Scenario 2. \square

Figure 4.9.1(b) illustrates the above remark for $\alpha + \beta = 1$.

Remark 10. Given initial proportions α of ignorants and β of subscribers, the required number n of broadcasts necessary to achieve a target proportion ϵ or less of ignorants may be evaluated by solving (4.9.8)-(4.9.9) recursively to obtain the smallest positive integer n for which

$$\eta \leq \epsilon.$$

\square

4.10 Comparison of Scenarios

We now compare the eventual proportions ξ and ξ^* respectively of the population never hearing a rumour when n broadcasts are made under control policy \mathcal{S} with Scenarios 1 and 2. For clarity we use the superscript $*$ to distinguish quantities pertaining to Scenario 2 from the corresponding quantities for Scenario 1.

Theorem 4.10.1. *Suppose $\alpha, \beta > 0$ and $\gamma \geq 0$ are given and that a sequence of n broadcasts is made under control policy \mathcal{S} . Then*

(a) *if $n > 2$, we have*

$$i_k^* < i_k \quad \text{for } 2 < k \leq n;$$

(b) *if $n \geq 2$, we have*

$$\xi^* < \xi.$$

Proof. From (4.8.6), (4.8.7) (under Scenario 1) and (4.9.1), (4.9.2) (under Scenario 2), we see that ξ may be regarded as i_{n+1} and ξ^* as i_{n+1}^* , so it suffices to establish Part (a). This we do by forward induction on k .

Suppose that for some $k > 2$ we have

$$i_{k-1}^* \leq i_{k-1}. \tag{4.10.1}$$

A basis is provided by the trivial relation $i_2^* = i_2$. We have the defining relations

$$i_k^* e^{-2i_k^*} = i_{k-1}^* e^{-(\alpha+\beta+i_{k-1}^*)} \tag{4.10.2}$$

and

$$i_k e^{-2i_k} = i_{k-1} e^{-(\beta+2i_{k-1})} . \quad (4.10.3)$$

The inequality

$$i_{k-1}^* < \alpha$$

may be rewritten as

$$\beta + 2i_{k-1}^* < \alpha + \beta + i_{k-1}^* ,$$

so that

$$e^{-(\alpha+\beta+i_{k-1}^*)} < e^{-(\beta+2i_{k-1}^*)} .$$

Hence we have using (4.10.2) that

$$i_k^* e^{-2i_k^*} < i_{k-1}^* e^{-(\beta+2i_{k-1}^*)} .$$

Lemma 4.7.1 and (4.10.1) thus provide

$$i_k^* e^{-2i_k^*} < i_{k-1} e^{-(\beta+2i_{k-1})} .$$

By (4.10.3) and a second application of Lemma 4.7.1 we deduce that $i_k^* < i_k$, the

desired inductive step. This completes the proof. \square

Theorem 4.10.1 can be verified for the case of $\gamma = 0$ by comparing the graphs in Figures 4.8.1(a) and 4.9.1(a).

Chapter 5

Conclusions

We have mainly studied elaboration of the two important classical models developed for rumours in the literature, Daley-Kendall ([DK]) and Maki-Thompson([MT]) models.

The main contribution and results of this thesis may be summarized briefly as follows.

- Partitioning stiflers into two classes according to whether they are formed by spreader - spreader or spreader - stifier interactions.
- Adopting general initial conditions for classical stochastic rumour processes
- Allowing repeated rumours
- Introducing impulsive control of repeated rumours

All of the above give rise to a wealth of new structural theory and dynamics in a field in which there have been hitherto very few general results.

The structural innovations make rumour theory more amenable to nontrivial modelling. We believe that the new structural results will make it more feasible to answer questions arising out of such modelling.

The algorithms devised have been shown to be efficient through numerical results simulating analytic solutions.

The ideas in this thesis lead naturally to consideration of a number of variant models of rumour processes including

- People meeting more than two at a time
- Competing rumours and counter rumours
- Changing the veracity of the rumour
- Non-homogeneous populations.

Appendix A

A *stochastic process* is a probabilistic experiment that involves time. In other words, each sample point (i.e. possible outcome) of the experiment is a function of time, called a sample function. The *sample space* is the set of all possible sample functions, and the *events* are subsets of the sample space.

A stochastic *rumour process* is a type of system which evolves in time as a result of chance interactions between the individuals in a closed population. The behaviour of such a system may be described by a vector-valued family of random variables, u_k , such that

$$u_k = (i_k, s_k, r_{1,k}, r_{2,k})^\top$$

at time t_k for all $k = 0, 1, 2, \dots$, where $(\cdot)^\top$ stands for the transpose of the given vector, the component i_k measures the number of people in the population who have never heard the rumour, s_k the number who spread it, $r_{1,k}$ the number of *first type of stiflers* and $r_{2,k}$ the number of *second type of stiflers*.

The family u_k can be envisaged as a path of the rumour spreading *randomly* in space

Σ . That is, the rumour process as a stochastic process is a family u_k of random variables indexed by $t_k \in \mathbf{R}^+ \cup \{0\}$. The random variables, u_k at time t_k for all $k = 0, 1, 2, \dots$, map the *sample space* Ω into some set Σ . Here Σ is the *state space*. For each $t \in [0, \infty)$, the value of u_k is a point in Σ . Note that continuous time processes and discrete time processes are not distinguished at this stage in this section.

In the discrete case, let $i_k, s_k, r_{1,k}$ and $r_{2,k}$ be the number of ignorants, spreaders, first-type stiflers and second-type stiflers at the k th transition. The k th transition is denoted by the mapping $\phi : \{i_k, s_k, r_{1,k}, r_{2,k}\} \mapsto \{i_{k+1}, s_{k+1}, r_{1,k+1}, r_{2,k+1}\}$, for $k = 0, 1, 2, \dots$. That is, u_k is mapped to u_{k+1} by ϕ . Here $u_k : \Omega \longrightarrow \Sigma$ and each quadruplet u_k of numbers represents a new *state*.

The final outcome of the rumour process is defined by the mapping $\phi^T : \{i_0, s_0, r_{1,0}, r_{2,0}\} \longrightarrow \{i_T, s_T, r_{1,T}, r_{2,T}\}$. The mapping ϕ^T is the T -fold composition, $\phi^T = \phi \circ \phi \circ \dots \circ \phi$.

The analysis of mathematical models established for certain rumour processes under prescribed assumptions has shown that asymptotically for a very large population a fixed proportion of the population remains ignorant about the rumour. We have already mentioned two such classical cases in the main body of the thesis which have been introduced by Daley and Kendall [32] and Maki and Thompson [78] (the [DK] model and the [MT] model respectively).

Differential equations, both ordinary and partial, occur frequently in stochastic

models. There are various solution methods for these stochastic equations, for example the method of *Langevin* and that of *Chapman-Kolmogorov*. These methods and others are discussed in [54]. Langevin's equation is mentioned very briefly since it has historical importance for the study of stochastic equations and their applications in Physics, although it and its other related results are not used in this thesis. By referring to [54], Langevin's equation was the first example of a stochastic differential equation, a differential equation with a random term. Langevin solved the equation of motion for the position of the particle given by Newton's law $m \frac{\partial^2 x}{\partial t^2} = -6\pi\eta a \frac{\partial x}{\partial t} + X$ where $\frac{\partial x}{\partial t}$ is the velocity of the particle, a is the diameter of the particle, X a fluctuating force, η is the viscosity and m is the mass of the particle. His solution method was quite different from that of Einstein [38] and his solution is a random function.

Another example of stochastic differential equations is provided by *birth-death equations*. We give a brief introduction to these equations here. In [54], a probability distribution, $P(x, y, t)$, for the number of individuals at a given time is assumed such that

$$\text{Prob}(x \rightarrow x + 1; y \rightarrow y) = k_1 a x \Delta t,$$

$$\text{Prob}(x \rightarrow x - 1; y \rightarrow y + 1) = k_2 x y \Delta t,$$

$$\text{Prob}(x \rightarrow x; y \rightarrow y - 1) = k_3 y \Delta t,$$

$$\text{Prob}(x \rightarrow x; y \rightarrow y) = 1 - (k_1 a x + k_2 x y + k_3 y) \Delta t$$

where Δt is the time increment in which change occurs, x is the number of prey, y is the number of predators, a is the amount of food of the prey and k_1, k_2, k_3 are rate constants. Thus the probability laws are replaced by simple rate laws. These simple rate laws are then employed in the Chapman-Kolmogorov equations. In the Chapman-Kolmogorov equations, the probability at $t + \Delta t$ is expressed as a sum of terms, each of which represents the probability of a previous state multiplied by the probability of a *transition* from that *state* to *state* (x, y) . Thus

$$\begin{aligned} \frac{P(x, y, t + \Delta t) - P(x, y, t)}{\Delta t} &= k_1 a(x - 1)P(x - 1, y, t) \\ &\quad + k_2(x + 1)(y - 1)P(x + 1, y - 1, t) \\ &\quad + k_3(y + 1)P(x, y + 1, t) \\ &\quad - (k_1 a x + k_2 x y + k_3 y)P(x, y, t) \\ &\rightarrow \frac{\partial P(x, y, t)}{\partial t} \text{ as } \Delta t \rightarrow 0 . \end{aligned}$$

This type of model has a wide application—in fact to any system to which a population of individuals may be attributed.

Appendix B

In Chapter 2 we required the derivative of the function ϕ describing the second type of stiflers. The details of the calculation of this derivative are given below. Recall:

$$\phi_2 = 1 - \alpha\theta + (1 + \alpha)\ln\theta + 2\alpha(1 - \theta) + \frac{1}{2}\ln^2\theta.$$

Hence

$$\frac{d\phi_2}{d\alpha} = -\left(\theta + \alpha\frac{d\theta}{d\alpha}\right) + \left(\ln\theta + (1 + \alpha)\frac{1}{\theta}\frac{d\theta}{d\alpha}\right) + (2(1 - \theta) - 2\alpha)\frac{d\theta}{d\alpha} + \ln\theta\frac{1}{\theta}\frac{d\theta}{d\alpha}$$

$$= -\theta - \alpha\frac{d\theta}{d\alpha} + \ln\theta + (1 + \alpha)\frac{1}{\theta}\frac{d\theta}{d\alpha} + 2(1 - \theta) - 2\alpha\frac{d\theta}{d\alpha} + \ln\theta\frac{1}{\theta}\frac{d\theta}{d\alpha}$$

$$= \frac{1}{\theta}\frac{d\theta}{d\alpha}(-\alpha\theta - 2\alpha\theta + (1 + \alpha) + \ln\theta) - \theta + 2 - 2\theta + \ln\theta$$

$$= \frac{1}{\theta}\frac{d\theta}{d\alpha}(1 + \alpha - 3\alpha\theta + \ln\theta) + 2 - 3\theta + \ln\theta$$

where $\alpha = -\frac{1+\ln\theta}{1-2\theta}$ and $\frac{d\theta}{d\alpha} = -\frac{\theta(2\theta-1)^2}{1+2\theta\ln\theta}$. Consequently,

$$\begin{aligned}
\frac{d\phi_2}{d\alpha} &= -\frac{(2\theta-1)^2}{1+2\theta\ln\theta} \left(1 - \frac{1+\ln\theta}{1-2\theta} + 3\frac{1+\ln\theta}{1-2\theta}\theta + \ln\theta \right) \\
&\quad + 2 - 3\theta + \ln\theta \\
&= -\frac{2\theta-1}{1+2\theta\ln\theta} \left(-(1-2\theta) + 1 + \ln\theta - 3(1+\ln\theta)\theta - (1-2\theta)\ln\theta \right) \\
&\quad + 2 - 3\theta + \ln\theta \\
&= \frac{1-2\theta}{1+2\theta\ln\theta} (2\theta - 3\theta - 3\theta\ln\theta + 2\theta\ln\theta) \\
&\quad + 2 - 3\theta + \ln\theta \\
&= \frac{1-2\theta}{1+2\theta\ln\theta} (-\theta - \theta\ln\theta) + 2 - 3\theta + \ln\theta \\
&= \frac{(1-2\theta)(-\theta - \theta\ln\theta) + (1+2\theta\ln\theta)(2 - 3\theta + \ln\theta)}{1+2\theta\ln\theta} \\
&= \frac{-\theta - \theta\ln\theta + 2\theta^2 + 2\theta^2\ln\theta + 2 - 3\theta + \ln\theta}{1+2\theta\ln\theta} \\
&\quad + \frac{4\theta\ln\theta - 6\theta^2\ln\theta + 2\theta\ln^2\theta}{1+2\theta\ln\theta} \\
&= \frac{-4\theta + 3\theta\ln\theta + 2\theta^2 - 4\theta^2\ln\theta + 2 + \ln\theta + 2\theta\ln^2\theta}{1+2\theta\ln\theta}.
\end{aligned}$$

Appendix C

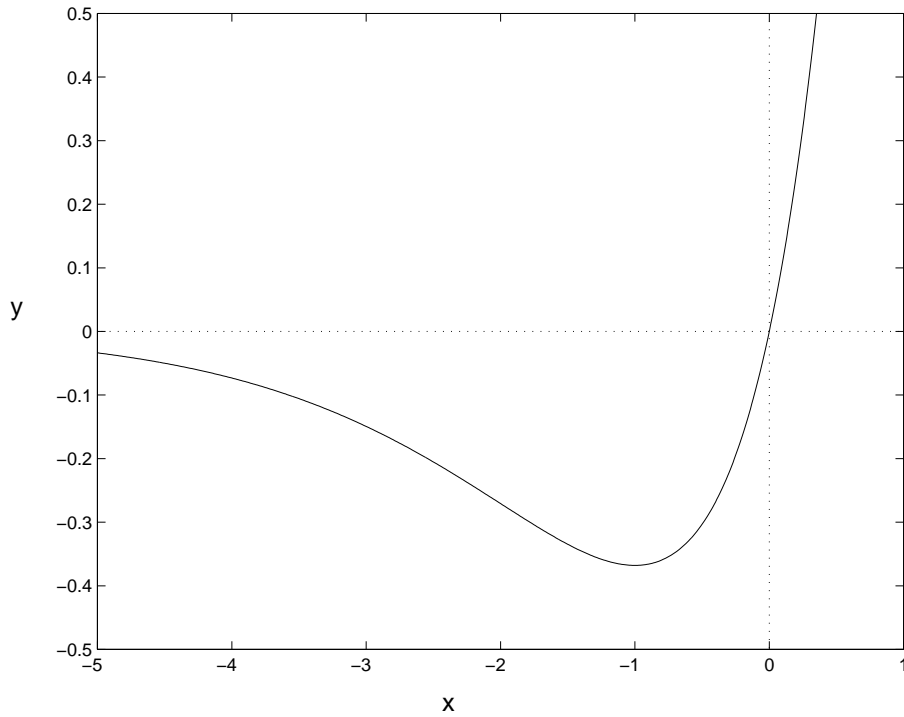
In Chapter 2, we gave the solution to the spread of a rumour for our extended model in a closed population. This required the evaluation of the Lambert- w function. Here we present the details of this calculation. In doing so we require the following result.

Theorem C.1 (LAGRANGE'S THEOREM). [109]

Let $f(z)$ and $\phi(z)$ be regular on and inside a closed contour C surrounding a point a , and let w be such that the inequality $|w\phi(z)| < |z - a|$ is satisfied at all points z on C . Then the equation $\zeta = a + w\phi(\zeta)$, regarded as an equation in ζ , has one root in the region enclosed by C . Further, any function of ζ regular on and inside C can be expanded as a power series in w by the formula

$$f(\zeta) = f(a) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \frac{d^{n-1}}{da^{n-1}} \{f'(a)[\phi(a)]^n\}. \quad (\text{C.1})$$

The Lambert- w function is illustrated in Figure C.1. To evaluate the Lambert- w function, we solve $ye^y = x$ by using a Lagrange expansion for the Belen-Pearce (2000)

Figure C.1: The Lambert- w function.

model as follows. Let

$$x = a + t\psi(x) = a + \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{d^{n-1}}{da^{n-1}} (\psi(a))^n.$$

Now, putting $x \rightarrow w$, $a \rightarrow 0$, $t \rightarrow x$, we have $\psi(x) = e^{-x}$ and

$$\begin{aligned} w &= \sum_{n=1}^{\infty} \frac{x^n}{n!} \frac{d^{n-1}}{da^{n-1}} (e^{-na}) \Big|_{a=0} \\ &= \sum_{n=1}^{\infty} \frac{x^n}{n!} (-n)^{n-1} e^{-na} \Big|_{a=0} \\ &= \sum_{n=1}^{\infty} \frac{(-x)^n}{n!} n^{n-1}. \end{aligned}$$

Hence, $\lambda = 2 - \alpha + \sum_{n=1}^{\infty} \frac{(2(1-\alpha)e^{-2+\alpha})^n}{nn!}$. Now let us check the conditions of Lagrange's

theorem: there is a contour C around a and t such that $|t\psi(z)| < |z - a|$ on C . That

is, $|xe^{-z}| < |z|$ on a contour C enclosing w with centre at the origin. The proof is

complete.

Special Case: Solution with zero initial proportion of stiflers and with arbitrary initial proportions of ignorants and spreaders:

We suppose initial conditions $i(0) = \alpha > 0$, $s(0) = 1 - \alpha$, $r(0) = 0$. From (2.2.4) and (2.2.5), we deduce that

$$\frac{ds}{di} = \frac{1 - 2i}{i}. \quad (\text{C.2})$$

Since $i > 0$ throughout the process, the right-hand side of (C.2) is well-defined. Integration with use of the initial conditions leads to

$$s = 1 + \alpha - 2i + \ln(i/\alpha). \quad (\text{C.3})$$

Set $i_\infty = \alpha\theta$. The parameter θ represents the proportion of the ignorant subpopulation who never hear the rumour. Since $s_\infty = 0$, we have

$$1 + \alpha(1 - 2\theta) + \ln \theta = 0 \quad (\text{C.4})$$

or

$$we^w = -2\alpha e^{-1-\alpha}, \quad (\text{C.5})$$

where $w := -2\alpha\theta$.

The equation

$$we^w = x \quad (\text{C.6})$$

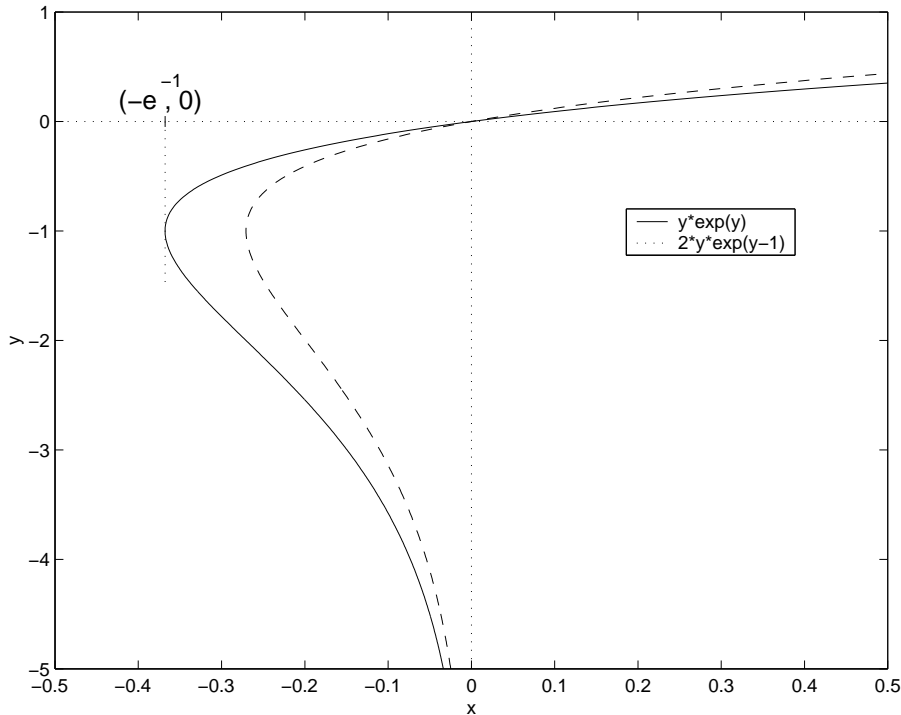


Figure C.2: The graph of the equation $ye^y = x$

has two real solutions when $-1/e < x < 0$ (see Figure C.2). For $0 < \alpha < 1$ we have

$$-2\alpha e^{-2\alpha} < -2\alpha e^{-1-\alpha},$$

so that $ye^y|_{y=-2\alpha} < ye^y|_{y=-2\alpha\theta}$ for $0 < \alpha < 1$. Hence one of the real solutions of (C.5) is less than -2α and the other greater than -2α . As we must have $0 < \theta < 1$, the physical solution to (C.5) is the one greater than -2α , that is, the real solution of (C.6) which is smaller in magnitude. Our solutions are depicted in Figure C.2.

The function $w = w(x)$ giving the unique real solution to (C.6) for $x \geq 0$ and the real solution of smaller magnitude for $x < 0$ has been in the literature for over 200 years and is known as the *Lambert w* function (see [29]).

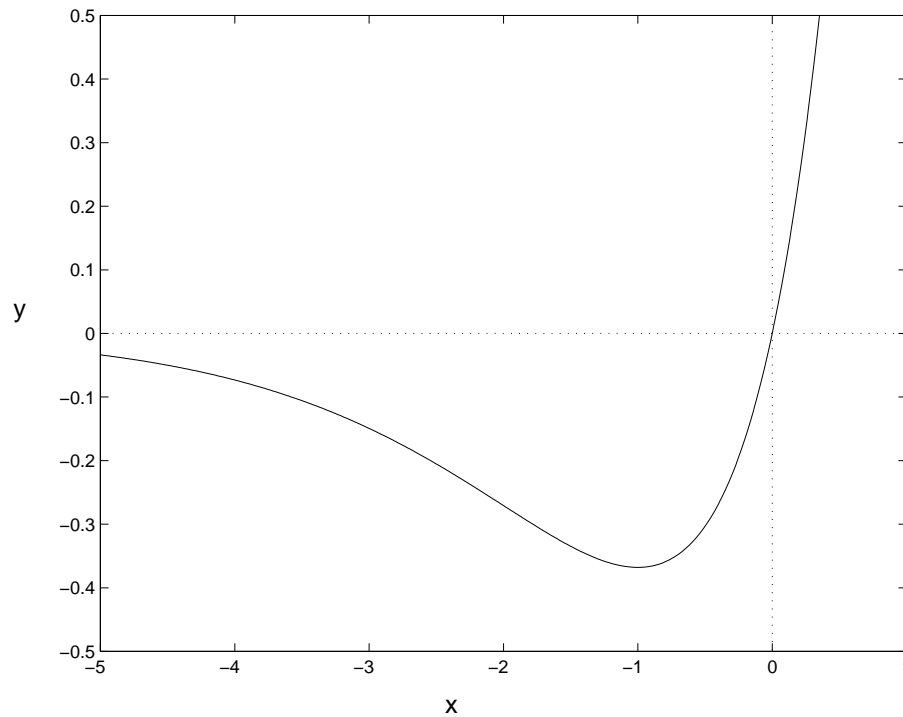


Figure C.3: The graph of the equation $ye^y = x$

Lagrange's expansion provides an explicit series evaluation for w :

$$w = \sum_{k=1}^{\infty} \frac{(-x)^k}{k!} k^{k-1}.$$

Thus for $0 < \alpha < 1$

$$\theta = -\frac{1}{2\alpha} w(-2\alpha e^{-1-\alpha}) = \sum_{k=1}^{\infty} \frac{(-2\alpha k)^{k-1}}{k!} e^{-k-k\alpha}.$$

We have that for $0 < \alpha < 1$, (C.4) may be used to define a single-valued function

$\theta = \theta(\alpha)$ with $0 < \alpha < 1$. In fact the range of θ is more circumscribed. First suppose,

if possible, that $\theta > 1/2$. Then (C.4) gives $1 + \ln \theta = \alpha(2\theta - 1) < 2\theta - 1$, since $\alpha < 1$.

However we have by elementary calculus that $1 + \ln \theta \geq 2\theta - 1$ for $1/2 \leq \theta \leq 1$, which

is a contradiction. Therefore

$$\theta \leq 1/2. \tag{C.7}$$

Indeed we must have $\theta < 1/2$, since (C.4) yields $1 + \ln \theta = 0$ for $\theta = 1/2$, which is also a contradiction. Differentiation of (C.4) provides

$$\frac{d\theta}{d\alpha} = -\frac{\theta(1-2\theta)}{1-2\theta\alpha}, \tag{C.8}$$

which by our foregoing discussion must be negative. Hence θ is a strictly monotone decreasing function of α on $(0, 1)$ (see Figure C.4). Its infimum satisfies the Daley-Kendall equation $2(1-\theta) + \ln \theta = 0$ and is $\theta(1) \approx 0.2031878$. The other real solution $\theta = 1$ to this equation is not feasible in our context. Despite the suggestion from Figure C.4, it can be shown that θ is not a concave function of α . The supremum of α is

$$\theta(0) = 1/e \approx 0.36787944. \tag{C.9}$$

That is, we have the somewhat surprising result that when nearly all the population are initially spreaders, it is still the case that a proportion $1/e$ of the initial ignorants never hear the rumour. See also Figure C.4.

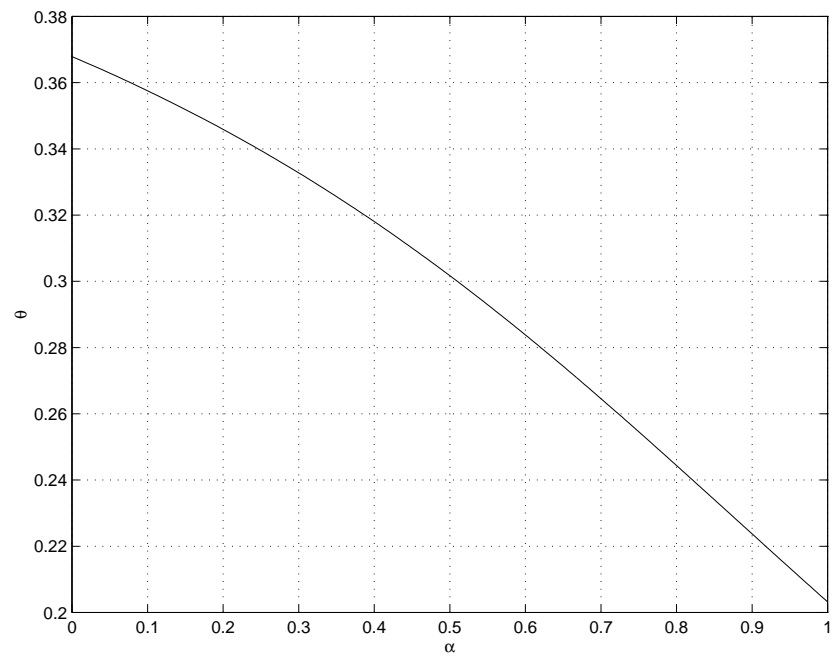


Figure C.4: The behaviour of the function θ as a function of α

Appendix D

In this appendix, the variant of the [MT] model called k -fold stifling by Daley and Gani [33] which was originally solved for 2-fold stifling by Carnal [26], is briefly presented. The term k -fold stifling means that it is assumed that a spreader does not decide to cease spreading the rumour until being involved in k stifling interactions. The proportion of final ignorants to the population is 0.05952021 ([26]) if the spreader hears the rumour twice before becoming a stifier. If the spreader hears the rumour k times before becoming a stifier during the process then the proportion of final ignorants can be formulated for all $k = 1, 2 \dots$ as

$$-\frac{1}{(k+1)}w\left(-(k+1)e^{-(k+1)}\right). \quad (\text{D.1})$$

Note that (D.1) is one of two solutions of the equation $ye^{(k+1)(1-y)} = 1$. The other solution is 1. A graphical depiction of the solutions is given by Figure D.1.

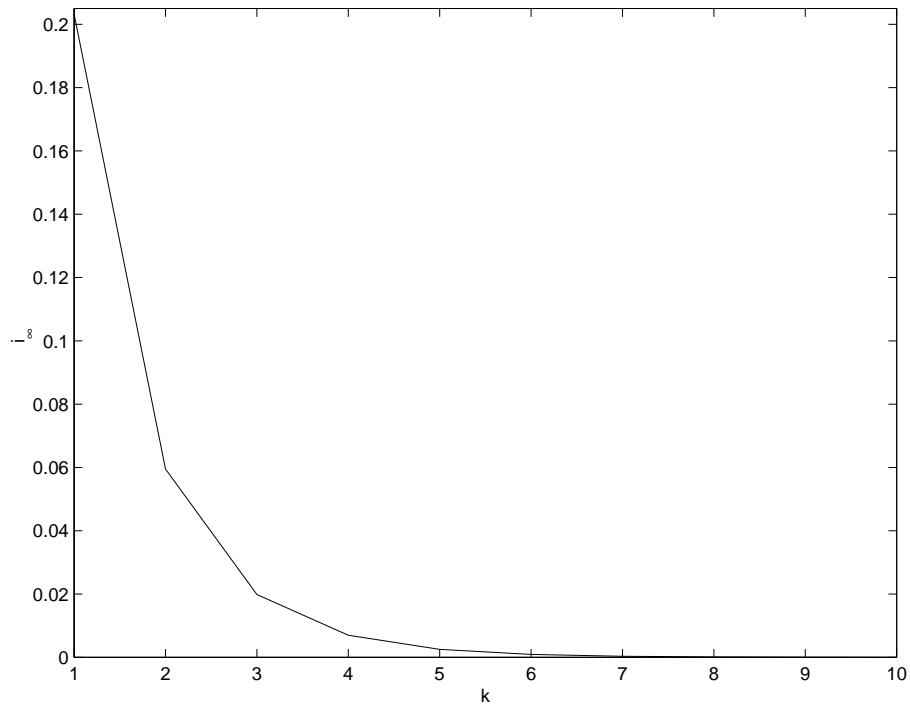


Figure D.1: k -fold variant of the model

Appendix E

In this appendix we give upper and lower bounds for the [DK] model by using two numerical methods, namely a *direct iteration method (DIM)* and *Newton's method (NM)*. Recall from equation (2.6.2) that the final proportion of ignorants in the [DK] model is

$$\varphi(\theta) = 2\theta - \ln \theta. \tag{E.1}$$

We use direct iteration to find the fixed point of the equation $\phi(\theta) = \theta$, yielding the final proportion of ignorants. Since we have $0 < \theta < \frac{1}{2}$ from equation (C.7), choose $\theta_0 = 0.3$. Table E.1 gives the first few iterations of DIM. Clearly $\varphi(\theta) \rightarrow 0.0556483$.

Table E.1: Convergence for the lower bound of [DK] by DIM-1

θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8
0.09	0.060	0.05610	0.0556986	0.0556539	0.0556489	0.0556483	0.0556483

Table E.2: Convergence for the lower bound of [DK] by DIM-2

θ_1	θ_2	θ_3	θ_4	θ_5
0.165	0.06929	0.05718	0.05581	0.0556663
θ_6	θ_7	θ_8	θ_9	
0.0556503	0.0556485	0.0556483	0.0556483	

Table E.3: Convergence for upper bound of [DK] by DIM-1

θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8
0.246	0.2216	0.211	0.206	0.204	0.20365	0.20337	0.20326
θ_9	θ_{10}	θ_{11}	θ_{12}	θ_{13}	θ_{14}	θ_{15}	
0.20322	0.2032005	0.2031930	0.2031899	0.2031887	0.2031882	0.203188	

For purposes of comparison, suppose now that

$$\theta > \frac{1}{2} \text{ and } \theta_0 = 0.6.$$

Table E.2 gives iterations of the DIM for $\theta_0 = 0.6$. Again we observe that $\varphi(\theta) \rightarrow 0.0556483$.

From the classical result we know that $\theta \leq 0.203188$. This is also illustrated in Tables E.3 and E.4 by the convergence to this value of the DIM given initial values $\theta_0 = 0.3$ and $\theta_0 = 0.2$ respectively. The convergence of θ to 0.0556483 illustrated

Table E.4: Convergence for upper bound of [DK] by DIM-2

θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7
0.122	0.104	0.1012	0.1005	0.1004	0.1003	0.1003

Table E.5: Convergence for upper bound by Newton's method

θ_1	θ_2	θ_3	θ_4
0.1934065	0.2027975	0.2031872	0.2031878

in Tables E.1 and E.2 therefore suggests that this value is a lower bound for θ , i.e.

$$0.0556483 \leq \theta \leq 0.203188.$$

For purposes of comparison, in Tables E.5 and E.6 we report our calculations using NM instead DIM for the upper and lower bounds using initial values $\theta_0 = 0.3$ and $\theta_0 = 0.2$ respectively. We have

$$\varphi(\theta) = 2\theta - \ln \theta$$

Table E.6: Convergence for lower bound by Newton's method

θ_1	θ_2	θ_3	θ_4	θ_5
0.06	0.0554598	0.0556479	0.0556483	0.0556483

from equation (E.1). Thus

$$\varphi'(\theta) = 2 - \frac{1}{\theta} = \begin{cases} > 0, & \theta > \frac{1}{2} \\ = 0, & \theta = \frac{1}{2} \\ < 0, & 0 < \theta < \frac{1}{2} \end{cases} \quad (\text{E.2})$$

We now apply NM to solve the equation $\varphi(\theta) = 2\theta - \ln(\theta) = c$, where the value of c is determined by the initial values of the rumour process. As we have seen, θ certainly assumes values in the range $0.05 \leq \theta \leq 1$, yielding values of c in the range $1.6 < c < 3$.

For a given value of c , the recurrence relation for the NM is then

$$\theta_{n+1} = \theta_n - \frac{\varphi(\theta_n)}{\varphi'(\theta_n)},$$

where

$$\varphi'(\theta_n) = 2 - \frac{1}{\theta_n}, \text{ that is, } \theta_{n+1} = \theta_n - \frac{2\theta_n - \ln \theta_n - c}{2 - \frac{1}{\theta_n}}.$$

As an example, with $c = 3$ we

obtain

$$\theta_{n+1} = \theta_n - \frac{2\theta_n - \ln \theta_n - 3}{2 - \frac{1}{\theta_n}},$$

yielding the numerical results presented in Tables E.5 – E.6. Note that, as expected,

Newton's converges more rapidly than DIM.

Appendix F

In this appendix we give computational results for both the [MT] model and the [DK] model.

In the tables in this appendix, β , T , i_T , i_0 , r_1 and r_2 are respectively the proportion of the spreaders to total population, the number of transitions, the number of ignorants when the rumour process stops, the initial number of ignorants, the number of first type of stiflers and the number of second type of stiflers. The classical results for the [MT] model with a single initial spreader are illustrated in Table F.1.

The data in this table like all data in other tables in this thesis were obtained by the algorithm developed and used to obtain the stochastic simulation results. Table F.2 gives simulation results for the extended [MT] model for the proportion of final ignorants (i_T) to initial ignorants (i_0). The population sizes are 10^5 , 10^6 , 10^7 and 10^8 , and β represents the proportion of initial spreaders to total population and varies from 0.00001 to 0.999. In chapter 2, as seen in Table F.2, the value of i_T/i_0 is approximately 0.358, and is effectively obtained for the population sizes 10^6 , 10^7 , 10^8 .

n	i	r	T	CPU Time [sec]
10^1	1	9	17	0.030
10^2	16	84	167	0.010
10^3	215	785	1569	0.040
10^4	2038	7962	15923	0.110
10^5	20329	79671	159341	0.83
10^6	203477	796523	1593045	8.670
10^7	2035639	7964361	15928721	84.200
10^8	20322212	79677788	159355575	846.000
10^9	203203310	796796690	1593593379	8496.55

Table F.1: The [MT] model for different population sizes n , with one initial spreader

β	10^5	10^6	10^7	10^8
0.00001	0.2054520547	0.2028982043	0.2030227184	0.2031623274
0.1	0.2274777740	0.2238677740	0.2235050052	0.2237780839
0.2	0.2469999939	0.2448125035	0.2441484928	0.2443652004
0.3	0.2660999894	0.2656899989	0.2643445730	0.2645373940
0.4	0.2869499922	0.2852266729	0.2836728394	0.2837518752
0.5	0.3035399914	0.3035820127	0.3014937937	0.3016579449
0.6	0.3210499883	0.3190949857	0.3176445067	0.3180365860
0.7	0.3357666731	0.3345266581	0.3322723210	0.3327519000
0.8	0.3466500044	0.3475799859	0.3458814919	0.3458819985
0.9	0.3574000001	0.3594399989	0.3565210104	0.3574211001
0.99	0.3610000014	0.3632000089	0.3652099967	0.3655700088
0.999	0.3499999940	0.3580000103	0.3573000133	0.3583199978

Table F.2: i_T/i_0 results for the extended [MT] model

β	T	i_T/i_0	r_1/n	r_2/n
0.000000001	1593579410	0.2032102942	0.3237645626	0.4730251431
0.1	1497258565	0.2237452418	0.3266044557	0.4720248282
0.5	1198266799	0.3017331958	0.3811696768	0.4679637551
0.9	1028493619	0.3575319052	0.4739302993	0.4903165102
0.99	1002679785	0.3660106957	0.4973381758	0.4990017414
0.999	1000283769	0.3581149876	0.4997091591	0.4999327362

Table F.3: The extended [MT] model with the population $n:=10^9$.

By using the algorithm for our extended [MT] model, we illustrate in Table F.3 the total number of transitions T when the rumour process stops, the proportions of final stiflers of first type (r_1) and stiflers of second type (r_2) to total population, and also the proportion of final ignorants to initial ignorants (i_T/i_0). In this table, β represents the same type of proportion of the population as in Table F.2. The approximations presented in these results are best in the case of very large population size (10^9), a very large number of initial spreaders (the proportion is 0.999) and nil initial stiflers. Here the proportion of final transitions to total population is approximately 1, the proportions of final first type of stiflers and final second type of stiflers are both approximately 0.5 and i_T/i_0 is approximately 0.358.

The computational results for the extended [MT] model are summarized in Tables F.4, F.5 and F.6 for the population sizes 10^5 , 10^7 and 10^8 respectively. It is observed that the proportion of transitions to total population is approximately 1 for any size of population when the initial proportion of spreaders is very large (from 0.99 to 0.9999). In other words, when the number of initial spreaders is very close to the total population size, the ignorants constitute a small proportion of the population, the number of initial stiflers is zero, and the number of transitions is almost equal to the total population size. The final proportions of first type of stiflers and of second type of stiflers both tend towards 0.50000 in the case of a large initial number of spreaders. It is also seen that the best approximation for i_T/i_0 (that is, the value closest to 0.368) occurs when $\beta=0.99$. In the case of population size 10^7 this value is 0.3652, and for population size 10^8 it is 0.3656. If β is taken as 0.99999 or 0.999999 as extreme cases the algorithm reports no ignorants left. We note, however, that these are extreme cases with an unrealistic assumption on the initial proportion of spreaders (which is almost unity).

Under the classical assumption that the initial number of spreaders is just 1, we expect (and Tables F.4, F.5 and F.6 confirm) that $i_T/i_0=i_T/n \approx 0.203$. We also observe that in this case the final proportion of first type of stiflers and the final proportion of second type of stiflers are approximately 0.324 and 0.473 respectively.

n	β	T	i_T/i_0	i_T/n	r_1/n	r_2/n
10^5	0.00001	158908	0.2054520547	0.2054499984	0.3214600086	0.4730899930
10^5	0.1	149053	0.2274777740	0.2047300041	0.3249199986	0.4703499973
10^5	0.2	140479	0.2469999939	0.1976000071	0.3337900043	0.4686099887
10^5	0.3	132745	0.2660999894	0.1862699986	0.3455300033	0.4681999981
10^5	0.4	125565	0.2869499922	0.1721699983	0.3620299995	0.4657999873
10^5	0.5	119645	0.3035399914	0.1517699957	0.3799299896	0.4683000147
10^5	0.6	114315	0.3210499883	0.1284199953	0.4013000131	0.4702799916
10^5	0.7	109853	0.3357666731	0.1007300019	0.4234699905	0.4758000076
10^5	0.8	106133	0.3466500044	0.0693299994	0.4467999935	0.4838699996
10^5	0.9	102851	0.3574000001	0.0357399993	0.4709900022	0.4932700098
10^5	0.99	100277	0.3610000014	0.0036100000	0.4947200119	0.5016700029
10^5	0.999	100029	0.3499999940	0.0003500000	0.4969300032	0.5027199984
10^5	0.9999	100003	0.3000000119	0.0000300000	0.4971199930	0.5028499961
10^5	0.99999	100000	0.0000000000	0.0000000000	0.4971500039	0.5028499961

Table F.4: Final results for the extended [MT] model with $0 < \beta < 1$ and populationsize $n = 10^5$

n	β	T	i_T/i_0	i_T/n	r_1/n	r_2/n
10^7	0.0000001	15939544	0.2030227184	0.2030227035	0.3239650130	0.4730122983
10^7	0.1	14976909	0.2235050052	0.2011545002	0.3268232942	0.4720222056
10^7	0.2	14093623	0.2441484928	0.1953188032	0.3346703947	0.4700107872
10^7	0.3	13299175	0.2643445730	0.1850412041	0.3468995094	0.4680593014
10^7	0.4	12595925	0.2836728394	0.1702037007	0.3626425862	0.4671536982
10^7	0.5	11985061	0.3014937937	0.1507468969	0.3813633919	0.4678896964
10^7	0.6	11458843	0.3176445067	0.1270578057	0.4021945000	0.4707477093
10^7	0.7	11006365	0.3322723210	0.0996816978	0.4247615933	0.4755567014
10^7	0.8	10616473	0.3458814919	0.0691763014	0.4488945901	0.4819290936
10^7	0.9	10286957	0.3565210104	0.0356521010	0.4740580022	0.4902898967
10^7	0.99	10026957	0.3652099967	0.0036521000	0.4974032938	0.4989446104
10^7	0.999	10002853	0.3573000133	0.0003573000	0.4998098016	0.4998328984
10^7	0.9999	10000421	0.2890000045	0.0000289000	0.5000535250	0.4999175966
10^7	0.99999	10000099	0.0000000000	0.0000000000	0.5000826120	0.4999173880
10^7	0.999999	10000009	0.0000000000	0.0000000000	0.5000844002	0.4999155998

Table F.5: Final results for the extended [MT] model with $0 < \beta < 1$ and populationsize 10^7 (n).

n	β	T	i_T/i_0	i_T/n	r_1/n	r_2/n
10^8	0.00000001	159367532	0.2031623274	0.2031623274	0.3237699270	0.4730677307
10^8	0.1	149719941	0.2237780839	0.2014002800	0.3266167045	0.4719829857
10^8	0.2	140901565	0.2443652004	0.1954921633	0.3345487118	0.4699591100
10^8	0.3	132964763	0.2645373940	0.1851761788	0.3468223512	0.4680014253
10^8	0.4	125949775	0.2837518752	0.1702511162	0.3625861704	0.4671627283
10^8	0.5	119834205	0.3016579449	0.1508289725	0.3812034130	0.4679676294
10^8	0.6	114557073	0.3180365860	0.1272146255	0.4020895064	0.4706958532
10^8	0.7	110034885	0.3327519000	0.0998255685	0.4247498512	0.4754245877
10^8	0.8	106164719	0.3458819985	0.0691763982	0.4488214850	0.4820021093
10^8	0.9	102851577	0.3574211001	0.0357421115	0.4739440680	0.4903137982
10^8	0.99	100268859	0.3655700088	0.0036557000	0.4973450303	0.4989992678
10^8	0.999	100028335	0.3583199978	0.0003583200	0.4997186661	0.4999229908
10^8	0.9999	100004437	0.2781000137	0.0000278100	0.4999540746	0.5000180602
10^8	0.99999	100000999	0.0000000000	0.0000000000	0.4999830425	0.5000169873
10^8	0.999999	100000099	0.0000000000	0.0000000000	0.4999845922	0.5000153780

Table F.6: Final results for the extended [MT] model with $0 < \beta < 1$ and $n = 10^8$

n	i	r_1	r_2	No. of Transitions	CPU Time [sec]
10^1	1	3	6	13	0.010
10^2	19	47	34	143	0.030
10^3	208	478	314	1425	0.010
10^4	2099	4759	3142	14229	0.070
10^5	20538	47134	32328	142758	0.440
10^6	203502	472284	324214	1430887	5.410
10^7	2028546	4732210	3239244	14323284	49.950
10^8	20321534	47301372	32377094	143168383	496.660
10^9	203204023	473042539	323753438	1431715233	5385.750

Table F.7: The classical [DK] model with first type and second type of stiflers and one initial spreaders

Table F.7 gives the results produced by our algorithm for the case of the classical [DK] model with a single initial spreader. As expected, these results give the final proportion of ignorants as approximately 0.203 and the final proportion of transitions to the total population size as approximately 1.4317. The simulations are done for different population sizes from 10 up to 10^9 .

Tables F.8 and F.9 give computational results for i_T/i_0 , for different β values from 0.1 to 0.99999 and for population sizes 10^6 , 10^7 , 10^8 , and 10^9 . For $0.1 \leq \beta \leq 0.9$ we

β	10^6	10^7	10^8
0.1	0.224050	0.224034	0.223786
0.2	0.244833	0.244594	0.244404
0.3	0.264567	0.264900	0.264720
0.4	0.283243	0.284402	0.283909
0.5	0.302304	0.302087	0.301819
0.6	0.318300	0.318492	0.318171
0.7	0.332823	0.332724	0.332792
0.8	0.345940	0.345874	0.346016
0.9	0.356889	0.357849	0.357546
0.99	0.364700	0.367780	0.366287

Table F.8: i_T/i_0 for the extended [DK] model with $0 < \beta < 0.99$

β	10^6	10^7	10^8	10^9
0.999	0.376000	0.365500	0.365500	0.367169
0.9999	0.489918	0.355940	0.365539	0.369200
0.99999	0.699050	0.379484	0.352521	

Table F.9: i_T/i_0 for the extended [DK] model with $0.999 < \beta < 0.99999$

increment β by 0.1. For larger values we use an increment of 9×10^{-m} for $m = 2, \dots, 5$.

It is a surprising result that, in contrast to the results obtained using the [MT] model, i_T/i_0 is not approximately 0 when $\beta = 0.99999$ for any population size between 10^6 and 10^9 . Instead, it appears to approximate the analytic solution (0.368), particularly when $\beta = 0.99$.

Tables F.10, F.11, F.12, and F.13 summarize the computational results for the extended [DK] model. We observe that the proportion of final transitions to total population is approximately 0.75 and the proportions of first type stiflers and second type of stiflers both tend to approximately 0.499. The proportion of final ignorants to total population does not go to 0. For large β , the best approximations for final ignorants are obtained for the larger population sizes. In particular, for $\beta = 0.999999$ and $n = 10^7$ we obtain $i_T = 1$. For $\beta = 0.999999$ and $n = 10^8$ we have $i_T = 36$, and for $\beta = 0.9999$ and $n = 10^9$ we have $i_T = 36920$.

We again observe that the proportions of final first type stiflers and of final second type of stiflers are approximately 0.324 and 0.473, respectively, when the process is initiated by a single spreader.

We conclude that both the extended [MT] and [DK] models, under the classical initial conditions of a single spreader and no stiflers, have the same final proportions of first type stiflers (0.324) and second type stiflers (0.473), even though the interactions

β	i_T/i_0	T	r_1	r_2	i
0.000001	0.203182	1432198	473944	322874	203182
0.1	0.224050	1333808	472554	325800	201645
0.2	0.244833	1241213	470027	334106	195867
0.3	0.264567	1156431	468456	346346	185197
0.4	0.283243	1079115	468068	361986	169946
0.5	0.302304	1007107	467670	381178	151152
0.6	0.318300	944317	470597	402082	127320
0.7	0.332823	888077	475696	424456	99847
0.8	0.345940	837392	482351	448460	69188
0.9	0.356890	791724	490516	473794	35689
0.99	0.364700	754033	499010	497342	3647
0.999	0.376000	750312	499755	499868	376
0.9999	0.490000	749947	499842	500108	49
0.99999	0.700000	749927	499856	500136	7
0.999999	0.000000	749934	499867	500132	0

Table F.10: The extended [DK] model with $n = 10^6$

β	i_T/i_0	T	r_1	r_2	i
0.0000001	0.203424	14313756	4730240	3235518	2034241
0.1	0.224034	13335884	4720692	3263000	2016307
0.2	0.244594	12414751	4699758	3343490	1956752
0.3	0.264900	11559352	4681607	3464092	1854300
0.4	0.284402	10775354	4669947	3623640	1706413
0.5	0.302087	10074313	4679928	3809638	1510434
0.6	0.318492	9442580	4707061	4018972	1273966
0.7	0.332724	8881608	4757730	4244098	998171
0.8	0.345874	8373504	4822255	4485996	691748
0.9	0.357849	7915118	4903784	4738366	357849
0.99	0.367780	7540732	4991801	4971420	36778
0.999	0.365500	7504853	5000674	4995670	3655
0.9999	0.356000	7501260	5001589	4998054	356
0.99999	0.380000	7500852	5001619	4998342	38
0.999999	0.100000	7500823	5001630	4998368	1

Table F.11: The extended [DK] model with $n = 10^7$

β	i_T/i_0	T	r_1	r_2	i
0.00000001	0.203203	143168591	47298016	32381706	20320278
0.1	0.223786	133386623	47195466	32663794	20140737
0.2	0.244404	124162273	46981587	33466066	19552348
0.3	0.264720	115597426	46785959	34683672	18530369
0.4	0.283909	107803419	46710545	36254886	17034569
0.5	0.301819	100762294	46797471	38111568	15090961
0.6	0.318171	94448624	47077723	40195452	12726825
0.7	0.332792	88800655	47552581	42463662	9983757
0.8	0.346016	83720908	48202776	44876904	6920320
0.9	0.357546	79154742	49035858	47388684	3575458
0.99	0.366287	75401329	49901519	49732194	366287
0.999	0.365500	75041799	49993248	49970202	36550
0.9999	0.365600	75005863	50002693	49993650	3656
0.99999	0.353000	75002218	50003494	49996152	353
0.999999	0.360000	75001836	50003579	49996384	36

Table F.12: The extended [DK] model with $n = 10^8$

β	i_T/i_0	T	r_1	r_2	i
0.999	0.367169	750405421	499912349	499720482	367169
0.9999	0.369200	750046240	500003209	499959870	36920

Table F.13: The extended [DK] model with $n = 10^9$

between two spreaders are different in the two models. The proportions of first type stiflers and second type stiflers under the assumption of a large number of initial spreaders ($\beta \rightarrow 1$) are both approximately 0.5. The proportions of total transitions to population size are 1.593 and 1.432 for the classical [MT] and [DK] models respectively, while the corresponding proportions for the extended models in the case that $\beta \rightarrow 1$ are approximately 1 and 0.750. For small β , the proportion of final ignorants to initial ignorants in both models in both versions (classical and extended) does not tend to 0; rather, it tends to 0.203. For large β ($\beta \rightarrow 1$) we have $i_T/i_0 \approx 0.368$.

Glossary

BROADCAST The generic term broadcast refers to an initiation of a rumour by means of television, radio or the internet.

CLOSED POPULATION A homogeneously mixed population without immigration, emigration, deaths, or births.

FIRST TYPE OF STIFLERS Meaningful interaction between two spreaders results in either both becoming stiflers or one of them becoming a stifter while the other one remains a spreader. This type of stiflers being sub population of stiflers are referred as first type of stiflers.

IGNORANTS People in the population who haven't heard the rumour

***k*-FOLD STIFLING** *k*-fold stifling refers to an interaction in which a spreader does not decide to cease spreading the rumour until being involved in *k* stifling interactions.

MEANINGFUL INTERACTION An interaction is called meaningful when it is

between two spreaders or a spreader and an ignorant or a spreader and a stifler.

NONSUBSCRIBER The ignorants who become spreaders after an encounter with a spreader we term this type of spreaders as nonsubscriber spreaders.

SECOND TYPE OF STIFLERS Meaningful interaction between a stifler and spreader results in spreader becoming a stifler while the stifler remains a stifler. This type of stiflers being sub population of stiflers are referred as second type of stiflers.

SPREADERS People in the population who have heard and spread the rumour.

STIFLERS People in the population who are the former spreaders have ceased to spread the rumour.

STOCHASTIC RUMOUR PROCESS A stochastic rumour process is a type of system which evolves in time as a result of chance interactions between the individuals in a closed population.

SUBSCRIBER People who are the initial spreaders when the rumour process is started by a broadcast to the spreaders are already portion of the whole population as spreaders initially.

TRANSITION The result of a meaningful interaction is a transition from one state to another.

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