The inertial dynamics of thin film flow of non-Newtonian fluids

A. J. Roberts*

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Abstract

Consider the flow of a thin layer of non-Newtonian fluid over a solid surface. I model the case of a viscosity that depends nonlinearly on the shear-rate; power law fluids are an important example, but the analysis here is for general nonlinear dependence. The modelling allows for large changes in film thickness provided the changes occur over a large enough lateral length scale. Modifying the surface boundary condition for tangential stress forms an accessible base for the analysis where flow with constant shear is a neutral critical mode, in addition to a mode representing conservation of fluid. Perturbatively removing the modification then constructs a model for the coupled dynamics of the fluid depth and the lateral momentum. For example, the results model the dynamics of gravity currents of non-Newtonian fluids even when the flow is not very slow.

Keywords: thin fluid flow; non-Newtonian fluid; inertia; power law rheology

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1 Introduction

Consider the two dimensional flow of a thin layer of fluid over a flat substrate. The fluid of thickness $\eta(x,t)$ spreads with mean lateral velocity $\bar{u}(x,t)$. Suppose the fluid has the non-Newtonian, power law, stress-strain relation that the stress $\propto$ (strain-rate)^s for some fixed exponent s: the exponent $s=1$ for a Newtonian fluid; $s<1$ is shear thinning; and $s>1$ is shear thickening. Such a power law is sometimes called Ostwald's or Norton's constitutive relation [5]. Then the systematic analysis developed in this article supports the nondimensional model

$$\frac{\partial\eta}{\partial t}+\frac{\partial}{\partial x}[\eta\bar{u}]=0\,, \tag{1}$$

$$\begin{aligned}\mathrm{Re}\left[\frac{\partial\bar{u}}{\partial t}+\frac{167-25/s}{96}\bar{u}\frac{\partial\bar{u}}{\partial x}+\frac{25-13/s}{96\eta}\bar{u}^2\frac{\partial\eta}{\partial x}\right]\\ \approx-\frac{5(25-1/s)c_s}{48\sqrt{2}\eta}\left(\frac{\sqrt{2}\bar{u}}{\eta}\right)^s\\ +\frac{19+1/s}{24}\left(g_1-g_2\frac{\partial\eta}{\partial x}\right)\,,\end{aligned} \tag{2}$$

where Re is the nondimensional Reynolds number, c_s is the coefficient of proportionality in the nonlinear stress-strain relation, and where g_1 and g_2 are the nondimensional components of gravity along and normal to the flat substrate. Fluid is conserved through (1). The momentum equation (2) incorporates effects of inertia, self-advection, bed drag and gravitational forcing; the dependence of the coefficients upon s models the subtle effects of the power law rheology.

This model not only applies to the flow of simple liquids, it applies to: gravity currents of suspensions with medium to high volume fractions as these are non-Newtonian [17]; power law rheologies are used to model ice flow [9, 18, e.g.] and at even a few metres per year the Reynolds number is significant for a thick glacier; and a modified model would apply to turbulent flow as the Smagorinsky large eddy closure of turbulence corresponds to the shear thickening case of exponent $s=2$ [8, Eqn. (6), e.g.]. This article puts models such as (1)–(2) within the sound support of modern dynamical systems theory, Section 3, to empower us to systematically control error, assess domains of validity, and to systematically account for further physical effects.

The analysis here encompasses not only power law fluids but a general nonlinear dependence of the stress upon the strain-rate as codified in Section 2. In contrast, almost all previous thin fluid film modelling use only a power law dependence. Some industrial plastics have a complicated non-monotonic dependence [3] that cannot be represented by a simple power law. Similarly, dense suspensions often have non-monotonic dependence [17]. The resultant model derived in Section 4 also applies to such complicated industrial plastics and dense suspensions.

The lubrication approximation of very slow flow, low Reynolds number, underpins previous theoretical models for non-Newtonian thin fluid films: Perazzo & Gratton [10] and Betelu & Fontelos [1] examined flow with surface tension; this followed experiments compared with travelling waves and similarity solutions by Gratton, Minotti & Mahajan [5]. Gratton et al. comment "the differences between Newtonian and non-Newtonian currents are significant and can clearly be observed in experiments". But the lubrication approximation, that creates models expressed only in terms of the fluid thickness $\eta(x,t)$, does not model inertia and so cannot resolve any wave-like dynamics. To model faster flows, potentially with wave effects, we must resolve the dynamics of both the fluid thickness and a measure of horizontal momentum [12, 16], we used η and $\bar{u}$ in (1)–(2). For example, Harris et al. [6] modelled particle driven gravity currents using shallow water equations that resolve the dynamics of both the fluid thickness and the mean lateral velocity. How-

*Computational Engineering and Science Research Centre, Department of Mathematics & Computing, University of Southern Queensland, Toowoomba, Queensland 4352, Australia. mailto:aroberts@usq.edu.au

ever, such modelling of essentially dissipative flows, albeit dissipative via turbulence, by the laminar inviscid foundation of shallow water equations appears a contradiction that demands resolution. This article shows how such models of non-Newtonian fluid flow may be put on a sound mathematical basis to empower accurate physical forecasts.

2 Differential equations to model non-Newtonian flow

Let the incompressible fluid have thickness $\eta(x,t)$, constant density ρ, a nonlinear rheology, and let the fluid flow with some varying velocity field $\mathbf{u}=(u,v)=(u_1,u_2)$ and pressure field p.

Nonlinear constitutive relation Define the strain-rate tensor [5, 17][1]

$$\dot\varepsilon_{ij}=\tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{3}$$

where $x_1=x$ and $x_2=y$ are distances along and normal to the solid substrate, respectively. Then the stress tensor for the fluid is $\sigma_{ij}=-p\delta_{ij}+2\rho\nu\dot\varepsilon_{ij}$: for a Newtonian fluid the kinematic viscosity ν is constant; but when the kinematic viscosity varies with strain-rate then we model shear thickening or shear thinning non-Newtonian fluids.

The important class of non-Newtonian fluids that we address has viscosity which depends only upon the magnitude $\dot\varepsilon$ of the second invariant of the strain-rate tensor [1]:

$$\dot\varepsilon^2=\sum_{i,j}\dot\varepsilon_{ij}^2\,. \tag{4}$$

For example, Bird et al. [2, see [1]] report that a solution of 0.5% Hydroxyethylcellulose is shear thinning: at 20 °C the solution has viscosity $\mu=m\dot\varepsilon^{s-1}$ for exponent $s=1/1.96$ and coefficient $m=0.84\,\mathrm{N\,s^s/m^2}$.

Partial differential equations Make equations nondimensional with respect to some velocity scale, a typical fluid thickness, and the fluid density. The nondimensional PDEs for the incompressible, two dimensional, fluid flow are firstly the continuity equation

$$\nabla\cdot\mathbf{u}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \tag{5}$$

and secondly the momentum equation

$$\mathrm{Re}\left(\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\nabla p+\nabla\cdot\boldsymbol{\tau}+\mathbf{g}\,, \tag{6}$$

where Re is the appropriate Reynolds number, $\boldsymbol{\tau}$ is the nondimensional deviatoric stress tensor, and $\mathbf{g}=(g_1,g_2)$ is the nondimensional forcing of gravity. For a fluid with a nonlinear stress-strain relation, the nondimensional deviatoric stress tensor

$$\tau_{ij}=2\nu(\dot\varepsilon)\dot\varepsilon_{ij}=\nu(\dot\varepsilon)\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right). \tag{7}$$

[1] Some, such as Betelu & Fontelos [1], use double this tensor.

Boundary conditions Solve these PDEs with nondimensional boundary conditions:

- on the bed of no-slip,[2]

$$\mathbf{u}=0\quad\text{on}\quad y=0\,; \tag{8}$$

- the kinematic condition on the free-surface of

$$\frac{\partial\eta}{\partial t}+u\frac{\partial\eta}{\partial x}=v\quad\text{on}\quad y=\eta\,; \tag{9}$$

- the stress normal to the free surface comes from constant environmental pressure and surface tension, that is,

$$\begin{aligned}-p+\frac{1}{1+\eta_x^2}&\left(\tau_{22}-2\eta_x\tau_{12}+\eta_x^2\tau_{11}\right)\\&=\frac{\mathrm{We}\,\eta_{xx}}{(1+\eta_x^2)^{3/2}}\quad\text{on}\quad y=\eta\,,\end{aligned} \tag{10}$$

  where We is a nondimensional Weber number characterising the importance of surface tension;

- and there must be no tangential stress at the free surface,

$$(1-\eta_x^2)\tau_{12}+\eta_x(\tau_{22}-\tau_{11})=0\quad\text{on}\quad y=\eta\,. \tag{11}$$

  This boundary condition of zero tangential stress implicitly is effectively one of zero shear at the surface; this is not appropriate for material with a finite yield stress. Here we assume the fluid yields for arbitrarily small stress.

3 Centre manifold theory supports the modelling

This section shows how to place models such as (1)–(2) on a sound theoretical base. Artificially modify the zero tangential stress free surface condition (11) to have an artificial forcing proportional to the local velocity, a forcing which we later remove by evaluating at parameter $\gamma=1$:

$$\begin{aligned}(1-\tfrac{1}{6}\gamma)&\left[(1-\eta_x^2)\tau_{12}+\eta_x(\tau_{22}-\tau_{11})\right]\\&=(1-\gamma)\frac{\nu(\mathrm{E})}{\eta}u\quad\text{on}\quad y=\eta\,.\end{aligned} \tag{12}$$

Evaluated at $\gamma=1$ this artificial right-hand side becomes zero so the boundary condition (12) reduces to the physical boundary condition (11). However, when the parameter $\gamma=0$ and the lateral gravity and lateral derivatives negligible, $g_1=\partial_x=0$, a neutral mode of the dynamics is the lateral shear $u=\sqrt{2}\mathrm{E}y$ where I define E to be proportional to the mean lateral strain-rate:

$$\mathrm{E}=\frac{1}{\sqrt{2}\eta}\int_0^\eta\frac{\partial u}{\partial y}\,dy=\frac{1}{\sqrt{2}\eta}u|_{y=\eta}\,.$$

[2] If modelling turbulent flows by a large eddy closure, we may justifiably replace this no-slip bed condition by a mixed boundary condition on the lateral velocity: $u\propto\frac{\partial u}{\partial y}$.

This neutral lateral shear mode arises because in pure shear flow $\tau_{12} = \nu u_y$ and hence the artificial free surface condition (12) reduces to $\nu u_y = \nu u/\eta$ on $y = \eta$. Conservation of fluid provides a second neutral mode in the dynamics. That is, when $\gamma = g_1 = \partial_x = 0$ then a two parameter family of equilibria exists corresponding to some uniform lateral shear flow, $u = \mathrm{E} y$, on a fluid of any constant thickness η. For large enough lateral length scales, these equilibria occur independently at each location x [11, 13, e.g.] and hence the space of equilibria are in effect parametrised by $\mathrm{E}(x)$ and $\eta(x)$. Provided we can treat lateral derivatives ∂_x as a modifying influence, that is provided solutions vary slowly enough in x, centre manifold theory [4, 7, 14, e.g.] assures us three things: this space of equilibria is perturbed to a slow manifold, on which the evolution is slow, that *exists for a finite range* of γ and g_1, and which may be parametrised by the mean lateral shear $\mathrm{E}(x,t)$ and the local thickness of the fluid $\eta(x,t)$; the slow manifold is *attractive for all nearby initial conditions*; and that a formal power series in the parameters γ, g_1 and derivatives ∂_x *approximates* the slow manifold. That is, the theory supports the existence, relevance and construction of slow manifold models such as (1)–(2).

4 Low order models of the dynamics

Computer algebra readily constructs slow manifold models as asymptotic solutions of the governing differential equations and boundary conditions.

4.1 Power law fluids

For simplicity, suppose the rheology is a nondimensional power law for the kinematic viscosity, $\nu = c_s \dot{\varepsilon}^{s-1}$.

Computer algebra [15, §3] derives that for such a power law fluid, the evolution of the fluid thickness η and the stress parameter E is

$$\begin{aligned} \frac{\partial\eta}{\partial t} &= -\frac{\partial}{\partial x}\left[\left(1+\tfrac{5\sqrt{2}}{48s}\gamma\right)\tfrac{1}{2}\eta^2\mathrm{E}\right] \\ &\quad + \mathcal{O}\big(\partial_x^2 + g_1^2 + \gamma^3\big)\,, \qquad (13)\\ \mathrm{Re}\,\frac{\partial \mathrm{E}}{\partial t} &= -\tfrac{5}{2}\left(\gamma + \tfrac{1-1/s}{4}\gamma^2\right)c_s\frac{\mathrm{E}^s}{\eta^2} \\ &\quad - \mathrm{Re}\sqrt{2}\left[\left(\tfrac{3}{8}+\tfrac{1-8/s}{96}\gamma\right)\eta\mathrm{E}\frac{\partial\mathrm{E}}{\partial x} - \tfrac{1}{6s}\gamma\mathrm{E}^2\frac{\partial\eta}{\partial x}\right] \\ &\quad + \sqrt{2}\left[\tfrac{3}{4} - \tfrac{1+1/s}{16}\gamma\right]\eta^{-1}\left(g_1 - g_2\frac{\partial\eta}{\partial x}\right) \\ &\quad + \mathcal{O}\big(\partial_x^2 + g_1^2 + \gamma^3\big)\,. \qquad (14) \end{aligned}$$

The nonlinear rheology primarily appears as a nonlinear drag on the bed. However, changes to the vertical profiles of velocity and pressure due to different power laws affect the coefficients of this model through their dependence upon exponent s.

In modelling the flow of thin fluid layers, researchers generally prefer to use the mean lateral velocity or the lateral fluid flux instead of the shear parameter E. Using the velocity fields computed simultaneously with (13)–(14) the computer algebra [15, §3] also derives the mean lateral velocity

$$\begin{aligned} \bar{u} &= \frac{1}{\eta}\int_0^\eta u\,dy = \frac{1}{\sqrt{2}}\left(1+\tfrac{5}{24s}\gamma + \tfrac{5(4-1/s)}{288s}\gamma^2\right)\eta\mathrm{E} \\ &\quad + \frac{\eta^2\mathrm{E}^{1-s}}{sc_s}\left[\tfrac{\mathrm{Re}}{160}\mathrm{E}\frac{\partial\mathrm{E}}{\partial x} + \tfrac{1}{48}\left(g_1 - g_2\frac{\partial\eta}{\partial x}\right)\right] + \cdots. \end{aligned}$$

Reverting this series to express E in terms of $\bar{u}$, and substituting into the model (13)–(14) leads to a model for the coupled evolution of $\eta(x,t)$ and $\bar{u}(x,t)$. Evaluating at the physically relevant $\gamma = 1$, to remove the artifice in the surface boundary condition (12), then gives the model (1)–(2) discussed in the Introduction of this article.

Computer algebra experiments [15, §1] suggest that the convergence of the asymptotic series in γ is markedly improved by the factor $(1 - \frac{1}{6}\gamma)$ on the left-hand side of the tangential stress boundary condition (12). This factor is equivalent to an Euler transformation of the asymptotic series. As shown in other similar applications [12, e.g.], evaluation at $\gamma = 1$ is valid provided the lateral derivatives are small enough.

Computer algebra [15, §3] may construct terms in the formal power series solutions to higher order in the parameters γ, g_1 and ∂_x to generate many valid approximations of varying orders of accuracy. For example, to resolve any effects of surface tension we need to compute terms in ∂_x^2 that are neglected in (2) and (14). With the support of centre manifold theory, researchers may choose an approximate model that suits the parameter regime of their application.

4.2 More general non-Newtonian fluids

We now return to the more general case where the viscosity ν of the fluid depends quite generally upon the magnitude of the shear-rate $\dot{\varepsilon}$, instead of being a simple power law. In this more general case the expressions for the modelling are much more complicated. The reason is the general nonlinear dependence of viscosity on strain-rate: for conciseness define

$$\bar{\mathrm{E}} = \frac{\sqrt{2}\bar{u}}{\eta}\,, \quad \bar{\nu} = \nu(\bar{\mathrm{E}}) \quad \text{and} \quad R_{\bar{\nu}} = \frac{1}{\bar{\nu} + \bar{\mathrm{E}}\bar{\nu}'}\,, \qquad (15)$$

where primes on $\bar{\nu}$ denote the derivatives $d/d\mathrm{E}$ of the viscosity $\nu(\mathrm{E})$ and evaluated at $\mathrm{E} = \sqrt{2}\bar{u}/\eta$.

The procedure is as for the power law case: computer algebra [15, §3] constructs the slow manifold and evolution thereon to some order of error; then revert the asymptotic series to find stress parameter E as a function of mean velocity $\bar{u}$; and substitute to express the model in terms of η and $\bar{u}$. Conservation of fluid again derives (1) (to any order of error). The dynamics of momentum then leads to

$$\begin{aligned} \mathrm{Re}\,\frac{\partial\bar{u}}{\partial t} &= -\left[\tfrac{5\gamma}{2} + \tfrac{5\gamma^2}{48}\bar{\mathrm{E}}\bar{\nu}R_{\bar{\nu}}^2\big(2\bar{\nu}' + \bar{\mathrm{E}}\bar{\nu}''\big)\right]\frac{\bar{\nu}\bar{u}}{\eta^2} \\ &\quad - \mathrm{Re}\left[\tfrac{7}{4} - \tfrac{13\gamma}{48} + \tfrac{\gamma}{96}\bar{\mathrm{E}}R_{\bar{\nu}}^2\big(38\bar{\nu}\bar{\nu}' + 12\bar{\mathrm{E}}\bar{\nu}'^2 + 13\bar{\mathrm{E}}\bar{\nu}\bar{\nu}''\big)\right]\bar{u}\frac{\partial\bar{u}}{\partial x} \\ &\quad - \mathrm{Re}\sqrt{2}\left[\tfrac{1}{8} - \tfrac{\gamma}{16} + \tfrac{13\gamma}{192}\bar{\mathrm{E}}^2R_{\bar{\nu}}^2\big(2\bar{\nu}'^2 - \bar{\nu}\bar{\nu}''\big)\right]\bar{\mathrm{E}}\bar{u}\frac{\partial\eta}{\partial x} \end{aligned}$$

$$+ \left[\tfrac{3}{4} + \tfrac{\gamma}{12} - \tfrac{\gamma}{24}\bar{E}\bar{\nu}R_{\bar{\nu}}^2\big(2\bar{\nu}' + \bar{E}\bar{\nu}''\big)\right]\left(g_1 - g_2\frac{\partial h}{\partial x}\right)$$
$$+ \mathcal{O}\big(\partial_x^2 + g_1^2 + \gamma^3\big)\,. \tag{16}$$

Evaluate this equation at $\gamma = 1$ to recover a physically relevant model of the dynamics of lateral momentum.

The power law model (2) is just one specific subclass of the general model (16): obtain (2) by the specific choice of a power law viscosity, $\nu(\dot\varepsilon) = c_s\dot\varepsilon^{s-1}$.

5 Conclusion

Following similar modelling for Newtonian thin films [12], this approach places the modelling of an important class of non-Newtonian fluids upon the sound basis of centre manifold theory [4, 7, 14, e.g.]. This modern dynamical system foundation empowers us to systematically derive the novel and accurate models (2), (14) and (16) for the lateral momentum of fluids with nonlinear rheology.

These models of thin fluid flow can be directly applied to flows as diverse as those of industrial plastics [3, e.g.], ice [9, 18, e.g.], and medium to dense suspensions [17, e.g.]. When you desire more accuracy than that presented here, computer algebra readily computes higher order approximations [15, §3]. Modifying the no-slip boundary condition on the bed, (8), will empower the modelling of turbulent layers of flow over a substrate via large eddy closures. There are enormous applications for this approach to modelling the dynamics of laterally extensive layers of fluids.

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