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A dynamical approximation for stochastic partial differential equations

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Random invariant manifolds provide geometric structures for understanding stochastic dynamics. In this paper, a dynamical approximation estimate is derived for a class of stochastic partial differential equations, by showing that the random invariant manifold is almost surely asymptotically complete. The asymptotic dynamical behavior is thus described by a stochastic ordinary differential system on the random invariant manifold, under suitable conditions. As an application, stationary states (invariant measures) are considered for a class of stochastic hyperbolic partial differential equations. © *2007 American Institute of Physics.*
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I. INTRODUCTION

Stochastic partial differential equations (SPDEs) arise as macroscopic mathematical models of complex systems under random influences. There have been rapid progresses in this area.[12,22,9,26,2,14] More recently, SPDEs have been investigated in the context of random dynamical systems (RDSs);[1] see, for example Refs. 3, 5, 4, 8, 7, 23, 10, and 11, among others.

Invariant manifolds are special invariant sets represented by graphs in state spaces (function spaces) where solution processes of SPDEs live. A random invariant manifold provides a geometric structure to reduce stochastic dynamics. Stochastic bifurcation, in a sense, is about the changes in invariant structures for RDSs. This includes qualitative changes of invariant manifolds, random attractors, and invariant measures or stationary states.

Duan *et al.*[10,11] have recently proved results on the existence and smoothness of random invariant manifolds for a class of SPDEs. In this paper, we further derive a dynamical approximation estimate between the solutions of SPDEs *and* the orbits on the random invariant manifolds. This is achieved by showing that the random invariant manifold is almost surely asymptotically complete (see Definition 4.1). The asymptotic dynamical behavior thus can be described by a stochastic ordinary differential system on the random invariant manifold under suitable conditions. In this approach one key assumption is that the global Lipschitz constant of nonlinear term is small enough.

If the invariant manifold is almost surely asymptotically complete, we can approximate the infinite dimensional system by a system restricted on the random invariant manifold which is in fact finite dimensional. That is, the infinite dimensional system is reduced to a finite dimensional system, which is useful for understanding the asymptotic behavior of the original stochastic system.[19]

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In this paper we first consider invariant manifolds for a class of infinite dimensional RDS defined by SPDEs, then reduce the random dynamics to the invariant manifolds. When the invariant manifolds are shown to be almost surely asymptotically complete (see Definition 4.1), we obtain dynamical approximations of the solutions of SPDEs by orbits on the invariant manifolds. Almost sure cone invariance concept (see Definition 4.2) is used to prove almost sure asymptotic completeness property of the random invariant manifolds.

Invariant manifolds are often used as a tool to study the structure of attractors,[6,25] which contain stationary solutions. As an application of our dynamical approximations result, we investigate the existence of stationary solutions of a hyperbolic SPDE in Sec. V. Specifically, we will consider the following stochastic hyperbolic PDE, with large diffusivity and highly damped term, on the space-time domain $[0,2\pi]\times(0,+\infty)$:

$$u_{tt}(t,x) + \alpha u_t(t,x) = \nu\Delta u(t,x) + f(u(t,x),x) + u(t,x)\circ \dot{W}(t), \tag{1.1}$$

with

$$u(0,x)=u_0, \quad u_t(0,x)=u_1, \quad u(t,0)=u(t,2\pi)=0, \quad t>0,$$

where ν and α are both positive, and $f\in C^2(\mathbb{R},\mathbb{R})$ is a bounded globally Lipschitz nonlinearity. Note that the stochastic Sine-Gordon equation ($f=\sin u$) is an example. When the damping is large enough, the existence of the stationary solutions for (1.1) is obtained by considering the stochastic system on the random invariant manifold.

This paper is organized as follows. We state the main result on dynamical approximation for a class of SPDEs in Sec. II. Then we recall background materials in RDSs and the existence result of invariant manifolds[10] in Sec. III. The main result is proved in Sec. IV, and an application in detecting stationary states is discussed in the final section, Sec. V.

II. MAIN RESULT

We consider the stochastic evolutionary system

$$du(t) = (Au(t) + F(u(t)))dt + u(t)\circ dW(t), \tag{2.1}$$

where A is the generator of a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ on real valued separable Hilbert space $(H,|\cdot|)$ with inner product $\langle\cdot,\cdot\rangle$, $F:\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous nonlinear function with $F(0)=0$ and Lipschitz constant L_F is assumed to be small, and $W(t)$ is a standard real valued Wiener process. Moreover, $\circ$ denotes the stochastic differential in the sense of Stratonovich. Suppose that $\sigma(A)$, the spectrum of operator A, splits as

$$\sigma(A) = \{\lambda_k,\ k\in\mathbb{N}\} = \sigma_c\cup\sigma_s, \quad \sigma_c,\ \sigma_s \neq \emptyset\ , \tag{2.2}$$

with

$$\sigma_c\subset\{z\in\mathbb{C}:\mathrm{Re}\, z\geq 0\}, \quad \sigma_s\subset\{z\in\mathbb{C}:\mathrm{Re}\, z<0\},$$

where $\mathbb{C}$ denotes the complex number set. σ_c is assumed to be a finite set. Denote the corresponding eigenvectors to $\{\lambda_k, k\in\mathbb{N}\}$ by $\{e_1,\ldots,e_n,e_{n+1},\ldots\}$. By the above assumptions, there is an A-invariant decomposition $H=H_c\oplus H_s$ such that for the restrictions $A_c=A|_{H_c}$, $A_s=A|_{H_s}$, one has $\sigma_c=\{z:z\in\sigma(A_c)\}$ and $\sigma_s=\{z:z\in\sigma(A_s)\}$. Moreover $\{e^{tA_c}\}$ is a group on H_c and there exist projections Π_c and Π_s such that $\Pi_c+\Pi_s=\mathrm{I}d_H$, $A_c=A|_{\mathrm{Im}\,\Pi_c}$, and $A_s=A|_{\mathrm{Im}\,\Pi_s}$. We also suppose that there are postive constants α and β with property $0\leq\alpha<\beta$, such that

$$|e^{tA_c}x|\leq e^{\alpha t}|x|, \quad t\leq 0, \tag{2.3}$$

$$|e^{tA_s}x| \leqslant e^{-\beta t}|x|, \quad t \geqslant 0. \tag{2.4}$$

For instance, $-A$ may be a strongly elliptic and symmetric second order differential operator on a smooth domain with zero Dirichlet boundary condition.

Since σ_c is a finite set, H_c is a finite dimensional space of dimension, say, $\dim H_c = n$. For $u \in H$ we have $u = u_c + u_s$ with $u_c = \Pi_c u \in H_c$ and $u_s = \Pi_s u \in H_s$. Furthermore, we assume that the projections Π_c and Π_s commute with A. Define the nonlinear map on H_c as

$$F_c : H_c \to H_c,$$

$$u_c \mapsto F_c(u_c) = \sum_{i=1}^{n} \langle F(u_c + 0), e_i \rangle e_i,$$

where 0 is the zero element in the vector space H_s. The concept of random invariant manifolds will be introduced in the next section. Let $\{\theta_t\}_{t\in\mathbb{R}}$ be the metric dynamical system generated by Wiener process $W(t)$, see (3.3). We will obtain the following main result.

Theorem 2.1: *(Dynamical approximations) Consider the following stochastic evolutionary system:*

$$du(t) = [Au(t) + F(u(t))]dt + u(t) \circ dW(t), \tag{2.5}$$

where the linear operator A and the nonlinearity F satisfy the conditions listed above. If the Lipschitz constant of the nonlinearity F is small enough, then this stochastic system has an n dimensional invariant manifold $\mathcal{M}(\omega)$ and there exists a positive random variable $D(\omega)$ and a positive constant k such that for any solution $u(t,\theta_{-t}\omega)$ of (2.5), there is an orbit $U(t,\theta_{-t}\omega)$ on the invariant manifold $\mathcal{M}(\omega)$, with the following approximation property:

$$|u(t,\theta_{-t}\omega) - U(t,\theta_{-t}\omega)| \leqslant D(\omega)|u(0) - U(0)|e^{-kt}, \quad t > 0, \quad \text{almost surely}. \tag{2.6}$$

Remark 2.2: The above result can be seen as a dynamical approximation for system (2.5). Any solution u of (2.5) can be approximated by an orbit U on the manifold $\mathcal{M}(\omega)$. In fact the function U can be represented as $u_c + \bar{h}^s(u_c)$, where u_c satisfies the following stochastic equation:

$$du_c(t) = [A_c u_c(t) + F_c(u_c(t) + \bar{h}^s(u_c(t),\theta_t\omega))]dt + u_c(t) \circ dW(t), \tag{2.7}$$

and, moreover,

$$\bar{h}^s : H_c \to H_s$$

is a random Lipschitz map; see Sec. IV. Here we also remark that $\bar{h}^s$ depends on ω, so (2.7) in fact is a nonautonomous stochastic differential equation on H_c.

III. RANDOM INVARIANT MANIFOLDS

Now we recall the basic concepts in RDSs and the basic result on the existence of random invariant manifolds for SPDEs from Duan *et al.*[10,11]

For our purpose we work on the canonical probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P})$, where the sample space Ω_0 consists of the sample paths of $W(t)$, that is,

$$\Omega_0 = \{w \in C([0,\infty),\mathbb{R}) : w(0) = 0\},$$

see Ref. 1. Consider the following stochastic evolutionary equations:

$$\begin{aligned} u_t &= Au + F(u) + u \circ \dot{W}(t), \\ u(0) &= u_0 \in H. \end{aligned} \tag{3.1}$$

This equation can be written in the following mild integral form:

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}F(u(s))ds + \int_0^t e^{A(t-s)}u(s) \circ dW(s). \tag{3.2}$$

By the assumption of A and F, we know that Eq. (3.2) has a unique solution $u(t,\omega;u_0) \in L^2(\Omega_0, C(0,T;H))$ for any $T>0$ in the sense of probability. For more about the solution of SPDEs, we refer to Ref. 9.

We now present some basics of RDSs. First we start with a driven dynamical system which models white noise: $\theta_t:(\Omega_0,\mathcal{F}_0,\mathbb{P}) \to (\Omega_0,\mathcal{F}_0,\mathbb{P})$, $t\in\mathbb{R}$, that satisfies the usual definition for a (deterministic) dynamical system:

- $\theta_0 = id$,
- $\theta_t\theta_s = \theta_{t+s}$ for all s, $t\in\mathbb{R}$, and
- the map $(t,\omega)\mapsto\theta_t\omega$ is measurable and $\theta_t\mathbb{P}=\mathbb{P}$ for all $t\in\mathbb{R}$.

A RDS on a metric space $(\mathcal{X};d)$ with Borel-algebra $\mathcal{B}$ over θ_t on $(\Omega_0,\mathcal{F}_0,\mathbb{P})$ is a measurable map:

$$\varphi:\mathbb{R}^+ \times \Omega_0 \times \mathcal{X} \to \mathcal{X},$$

$$(t,\omega,x) \mapsto \varphi(t,\omega)x,$$

such that

(i) $\varphi(0,\omega)=id$ (on $\mathcal{X}$) and
(ii) $\varphi(t+s,\omega)=\varphi(t,\theta_s\omega)\varphi(s,\omega)\,\forall t$, $s\in\mathbb{R}^+$, for almost all $\omega\in\Omega_0$ (cocycle property).

A RDS φ is continuous or differentiable if $\varphi(t,\omega)$: $\mathcal{X}\to\mathcal{X}$ is continuous or differentiable (see Ref. 1 for more details on RDS). As we consider the canonical probability space, the driven dynamical system θ_t can be defined as

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}, \tag{3.3}$$

where $\omega(\cdot)\in\Omega_0$ is a sample path of the Wiener process or Brownian motion $W(t)$.

For a continuous RDS $\varphi(t,\omega)$: $\mathcal{X}\to\mathcal{X}$ over $(\Omega_0,\mathcal{F}_0,\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$, we need the following notions to describe its dynamical behavior.

Definition 3.1: A collection $M=M(\omega)_{\omega\in\Omega}$ of nonempty closed sets $M(\omega)$, $\omega\in\Omega$, contained in $\mathcal{X}$ is called a random set if

$$\omega \mapsto \inf_{y\in M(\omega)} d(x,y)$$

is a real valued random variable for any $x\in\mathcal{X}$.

Definition 3.2: A random set $B(\omega)$ is called a tempered absorbing set for a RDS φ if for any bounded set $K\subset\mathcal{X}$ there exists $t_K(\omega)$ such that $\forall t\geq t_K(\omega)$,

$$\varphi(t,\theta_{-t}\omega,K) \subset B(\omega),$$

and for all $\varepsilon>0$,

$$\lim_{t\to\infty} e^{-\varepsilon t} d(B(\theta_{-t}\omega)) = 0, \quad a.e.\ \omega\in\Omega,$$

where $d(B)=\sup_{x\in B} d(x,0)$, with $0\in\mathcal{X}$, is the diameter of B.

For more about random set we refer to Ref. 8.

Definition 3.3: A random set $M(\omega)$ is called a positive invariant set for a RDS $\varphi(t,\omega,x)$ if

$$\varphi(t,\omega,M(\omega)) \subset M(\theta_t\omega) \quad \text{for } t\geq 0.$$

If $M(\omega)$ can be written as a graph of a Lipschitz mapping,

$$\psi(\cdot,\omega):\mathcal{X}_1 \to \mathcal{X}_2,$$

where $\mathcal{X}=\mathcal{X}_1 \oplus \mathcal{X}_2$, such that

$$M(\omega) = \{x_1 + \psi(x_1,\omega) | x_1 \in \mathcal{X}_1\},$$

then $M(\omega)$ is called a Lipschitz invariant manifold of φ.

For more about the random invariant manifold theory, see Ref. 10.

In order to apply the RDS framework, we transform the SPDE (3.1) into a random partial differential equation (random PDE). To this end, we introduce the following stationary process $z(\omega)$ which solves

$$dz + zdt = dW. \tag{3.4}$$

Solving Eq. (3.4) with initial value

$$z(\omega) = -\int_{-\infty}^{0} e^{\tau} w(\tau) d\tau,$$

we have a unique stationary process for (3.4),

$$z(\theta_t\omega) = -\int_{-\infty}^{0} e^{\tau}\theta_\tau\omega(\tau)d\tau = -\int_{-\infty}^{0} e^{\tau}\omega(t+\tau)d\tau + \omega(t).$$

The mapping $t \mapsto z(\theta_t\omega)$ is continuous. Moreover

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0 \text{ and } \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_\tau\omega)d\tau = 0 \quad \text{for a.e. } \omega \in \Omega_0.$$

We should point out that the above properties hold in a θ_t invariant set $\Omega \subset \Omega_0$ of full probability. For the proof see Ref. 10. In the following part to the end we consider the probability space $(\Omega,\mathcal{F},\mathbb{P})$ where

$$\mathcal{F} = \{\Omega \cap U : U \in \mathcal{F}_0\}.$$

The following random PDE on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ is a transformed version of the original SPDE (3.1):

$$v_t = Av + G(\theta_t\omega, v) + z(\theta_t\omega)v, \quad v(0) = x \in H, \tag{3.5}$$

where $G(\omega,v) := e^{-z(\omega)}F(e^{z(\omega)}v)$. It is easy to verify that G has the same Lipschitz constant as F and $G(0)=0$. By the classical evolutionary equation theory, Eq. (3.5) has a unique solution $v(t,\omega;x)$ which is continuous in x for every $\omega \in \Omega$. Then

$$(t,\omega,x) \mapsto \varphi(t,\omega)x := v(t,\omega;x)$$

defines a continuous RDS. We now introduce the transform

$$T(\omega,x) = xe^{-z(\omega)} \tag{3.6}$$

and its inverse transform

$$T^{-1}(\omega,x) = xe^{z(\omega)} \tag{3.7}$$

for $x \in H$. Then for the RDS $v(t,\omega;x)$ generated by (3.5),

$$(t,\omega,x) \to T^{-1}(\theta_t\omega, v(t,\omega;T(\omega,x)) := u(t,\omega;x)$$

is the RDS generated by (3.1). For more about the relation between (3.1) and (3.5) we refer to Ref. 10.

We now prove the existence of a random invariant manifold for (3.5) as in Refs. 10 and 11.

Let d_H be the metric induced by the norm $|\cdot|$. Then (H,d_H) is a complete separable metric space.

Now we briefly give an approach to obtain the random invariant manifold for system (3.5) by the Lyapunov-Perron method; for details see Ref. 11.

Projecting system (3.5) onto H_c and H_s, respectively, we have

$$\dot{v}_c = A_c v_c + z(\theta_t\omega)v_c + g_c(\theta_t\omega, v_c + v_s), \tag{3.8}$$

$$\dot{v}_s = A_s v_s + z(\theta_t\omega)v_s + g_s(\theta_t\omega, v_c + v_s), \tag{3.9}$$

where

$$g_c(\theta_t\omega, v_c + v_s) = \Pi_c G(\theta_t\omega, v_c + v_s)$$

and

$$g_s(\theta_t\omega, v_c + v_s) = \Pi_s G(\theta_t\omega, v_c + v_s).$$

Define the following Banach space for η, $-\beta < \eta < -\alpha$,

$$C_\eta^- = \{v:(-\infty,0] \to H: v \text{ is continuous and } \sup_{t\in(-\infty,0]} e^{-\eta t - \int_0^t z(\theta_s\omega)ds}|v(t)| < \infty\},$$

with norm

$$|v|_{C_\eta^-} = \sup_{t\in(-\infty,0]} e^{-\eta t - \int_0^t z(\theta_s\omega)ds}|v(t)|.$$

Define the nonlinear operator $\mathcal{N}$ on C_η^- as

$$\mathcal{N}(v,\xi)(t,\omega) = e^{A_c t + \int_0^t z(\theta_\tau\omega)d\tau}\xi + \int_0^t e^{A_c(t-\tau)+\int_\tau^t z(\theta_\varsigma\omega)d\varsigma} g_c(\theta_\tau\omega, v_c + v_s)d\tau$$
$$+ \int_{-\infty}^t e^{A_s(t-\tau)+\int_\tau^t z(\theta_\varsigma\omega)d\varsigma} g_s(\theta_\tau\omega, v_c + v_s)d\tau, \tag{3.10}$$

where $\xi \in H_c$. Then for any given $\xi \in H_c$ and each v, $\bar{v} \in C_\eta^-$, we have

$$|\mathcal{N}(v,\xi) - \mathcal{N}(\bar{v},\xi)|_{C_\eta^-} \leq \sup_{t\leq 0}\left\{ L_F\left(\int_0^t e^{(-\alpha-\eta)(t-s)}ds + \int_\infty^t e^{(-\beta-\eta)(t-s)}ds \right)\right\}|v-\bar{v}|_{C_\eta^-}$$
$$\leq L_F\left(\frac{1}{\eta+\beta} - \frac{1}{\alpha+\eta}\right)|v-\bar{v}|_{C_\eta^-}. \tag{3.11}$$

If

$$L_F\left(\frac{1}{\eta+\beta} - \frac{1}{\alpha+\eta}\right) < 1, \tag{3.12}$$

then by the fixed point argument,

$$v = \mathcal{N}(v) \tag{3.13}$$

has a unique solution $v^*(t,\omega;\xi) \in C_\eta^-$. Let $h^s(\xi,\omega) = \Pi_s v^*(0,\omega;\xi)$. Then

$$h^s(\xi,\omega) = \int_{-\infty}^{0} e^{-A_s\tau+\int_\tau^0 z(\theta_\varsigma\omega)d\varsigma} g_s(\theta_\tau\omega, v^*(\tau,\omega;\xi))d\tau. \tag{3.14}$$

$h^s(0,\omega)=0$ and h^s is Lipschitz continuous with Lipschitz constant CL_F; $C>0$ is a positive constant. Then we have the following result about the existence of random invariant manifold for the RDS $\varphi(t,\omega)$ generated by (3.5). For the detailed proof see Ref. 11.

Lemma 3.4: Suppose the assumptions on A and F in Sec. II and condition (3.12) hold. Then there exists a Lipschitz continuous random invariant manifold $\tilde{\mathcal{M}}(\omega)$ for $\varphi(t,\omega)$ which is given by $\tilde{\mathcal{M}}(\omega)=\{\xi+h^s(\xi,\omega):\xi\in H_c\}$.

Then by the transform T, as defined in (3.6), we have the following conclusion.

Lemma 3.5: (Invariant manifold for SPDE) $\mathcal{M}(\omega)=T^{-1}(\omega,\tilde{M}(\omega))$ is a Lipschitz continuous random invariant manifold for the SPDE (3.1).

Remark 3.6: Note that $\mathcal{M}(\omega)$ is independent of the choice of η.

IV. DYNAMICAL APPROXIMATIONS

In this section we prove Theorem 2.1 by showing that the invariant manifold obtained in the last section has the *almost sure asymptotic completeness* property. Then the dynamical behavior of (3.5) is determined by the system restricted on the invariant manifold.

The following concept is important in the study of the dynamical approximations of SPDEs.

Definition 4.1: (Almost sure asymptotic completeness) Let $\mathcal{M}(\omega)$ be an invariant manifold for a RDS $\varphi(t,\omega)$. The invariant manifold $\mathcal{M}$ is called almost surely asymptotically complete if for every $x\in H$, there exists $y\in\mathcal{M}(\omega)$ such that

$$|\varphi(t,\omega)x-\varphi(t,\omega)y| \leq D(\omega)|x-y|e^{-kt}, \quad t\geq 0,$$

for almost all $\omega\in\Omega$, where k is some positive constant and D is a positive random variable.

Now we introduce the *almost sure cone invariance* concept. For a positive random variable δ, define the following random set:

$$\mathcal{C}_\delta := \{(v,\omega)\in H\times\Omega : |\Pi_s v| \leq \delta(\omega)|\Pi_c v|\},$$

and the fiber $\mathcal{C}_{\delta(\omega)}(\omega)=\{v:(v,\omega)\in\mathcal{C}_\delta\}$ is called *random cone*. For a given RDS $\varphi(t,\omega)$, we give the following definition.

Definition 4.2: (Almost sure cone invariance) For a random cone $\mathcal{C}_{\delta(\omega)}(\omega)$, there is a random variable $\bar{\delta}\leq\delta$ almost surely such that for all x, $y\in H$,

$$x-y\in\mathcal{C}_{\delta(\omega)}(\omega)$$

implies

$$\varphi(t,\omega)x-\varphi(t,\omega)y\in\mathcal{C}_{\bar{\delta}(\theta_t\omega)}(\theta_t\omega) \quad \textit{for almost all } \omega\in\Omega.$$

Then the RDS $\varphi(t,\omega)$ has the cone invariance property for the cone $\mathcal{C}_{\delta(\omega)}(\omega)$.

Remark 4.3: Both asymptotic completeness and cone invariance are important tools to study the inertial manifold of deterministic infinite dimensional systems.[20,21,16] Here we modified both concepts for random systems.

Remark 4.4: Almost sure asymptotic completeness describes the attracting property of $\mathcal{M}(\omega)$ for RDS $\varphi(t,\omega)$. When this property holds, the infinite dimensional system $\varphi(t,\omega)$ can be reduced to a finite dimensional system on $\mathcal{M}(\omega)$, and the asymptotic behavior of $\varphi(t,\omega)$ can be determined by that of the reduced system on $\mathcal{M}(\omega)$.

For the RDS $\varphi(t,\omega)$ generated by the random PDE (3.5), we have the following result.

Lemma 4.5: For small Lipschitz constant L_F, RDS $\varphi(t,\omega)$ possesses the cone invariance property for a cone with a deterministic positive constant δ. Moreover if there exists $t_0>0$ such that x, $y\in H$ and

$$\varphi(t_0,\omega)x - \varphi(t_0,\omega)y \notin \mathcal{C}_\delta(\theta_{t_0}\omega),$$

then

$$|\varphi(t,\omega)x - \varphi(t,\omega)y| \leq D(\omega)e^{-kt}|x-y|, \quad 0 \leq t \leq t_0,$$

where $D(\omega)$ *is a positive tempered random variable and* $k=\beta-L_F-\delta^{-1}L_F>0$.

Note that the smallness condition on the Lipschitz constant L_F is specifically defined in (4.5) below.

Proof: Let v, $\bar{v}$ be two solutions of (3.5) and $p=v_c-\bar{v}_c$, $q=v_s-\bar{v}_s$; then

$$\dot{p} = A_c p + z(\theta_t\omega)p + g_c(\theta_t\omega, v_c+v_s) - g_c(\theta_t\omega,\bar{v}_c+\bar{v}_s), \tag{4.1}$$

$$\dot{q} = A_s q + z(\theta_t\omega)q + g_s(\theta_t\omega, v_c+v_s) - g_s(\theta_t\omega,\bar{v}_c+\bar{v}_s). \tag{4.2}$$

From (4.1) and (4.2) and by the property of A and F, we have

$$\frac{1}{2}\frac{d}{dt}|p|^2 \geq -\alpha|p|^2 + z(\theta_t\omega)|p|^2 - L_F|p|^2 - L_F|p|\cdot|q| \tag{4.3}$$

and

$$\frac{1}{2}\frac{d}{dt}|q|^2 \leq -\beta|q|^2 + z(\theta_t\omega)|q|^2 + L_F|q|^2 + L_F|p|\cdot|q|. \tag{4.4}$$

Then (4.4) $-\delta^2\times$ (4.3), we have

$$\frac{1}{2}\frac{d}{dt}(|q|^2-\delta^2|p|^2) \leq -\beta|q|^2 + z(\theta_t\omega)|q|^2 + L_F|q|^2 + L_F|p|\cdot|q| + \alpha\delta^2|p|^2 - z(\theta_t\omega)\delta^2|p|^2 + \delta^2 L_F|p|^2 + \delta^2 L_F|p|\cdot|q|.$$

Note that if $(p,q)\in\partial\mathcal{C}_\delta(\omega)$ [the boundary of the cone $\mathcal{C}_\delta(\omega)$], then $|q|=\delta|p|$ and

$$\frac{1}{2}\frac{d}{dt}(|q|^2-\delta^2|p|^2) \leq (\alpha-\beta+2L_F+\delta L_F+\delta^{-1}L_F)|q|^2.$$

If L_F is small enough such that

$$\alpha-\beta+2L_F+\delta L_F+\delta^{-1}L_F<0, \tag{4.5}$$

then $|q|^2-\delta^2|p|^2$ is decreasing on $\partial\mathcal{C}_\delta(\omega)$. Thus it is obvious that whenever $x-y\in\mathcal{C}_\delta(\omega)$, $\varphi(t,\omega)x-\varphi(t,\omega)y$ cannot leave $\mathcal{C}_\delta(\theta_t\omega)$.

We now prove the second claim. If there is $t_0>0$ such that $\varphi(t_0,\omega)x-\varphi(t_0,\omega)y\notin\mathcal{C}_\delta(\theta_{t_0}\omega)$, the cone invariance yields

$$\varphi(t,\omega)x-\varphi(t,\omega)y\notin\mathcal{C}_\delta(\theta_t\omega), \quad 0\leq t\leq t_0,$$

that is,

$$|q(t)|>\delta|p(t)|, \quad 0\leq t\leq t_0.$$

Then by (4.4) we have

$$\frac{1}{2}\frac{d}{dt}|q|^2 \leq -(\beta-L_F-\delta^{-1}L_F-z(\theta_t\omega))|q|^2, \quad 0\leq t\leq t_0.$$

Hence

$$|p(t)|^2 < \frac{1}{\delta^2}|q(t)|^2 \leq \frac{1}{\delta^2}e^{-2kt+\int_0^t z(\theta_s\omega)ds}, \quad 0 \leq t \leq t_0.$$

Then by the property of $z(\theta_t\omega)$, there is a tempered random variable $D(\omega)$ such that

$$|\varphi(t,\omega)x - \varphi(t,\omega)y| \leq D(\omega)e^{-kt}|x-y|, \quad 0 \leq t \leq t_0.$$

This completes the proof of the lemma. □

Before we prove Theorem 2.1, we need the following lemma, which implies the backward solvability of system (3.5) restricted on the invariant manifold $\tilde{\mathcal{M}}(\omega)$.

For any given final time $T_f > 0$, consider the following system for $t \in [0, T_f]$:

$$\dot{v}_c = A_c v_c + z(\theta_t\omega)v_c + g_c(\theta_t\omega, v_c + v_s), \quad v_c(T_f) = \xi \in H_c, \tag{4.6}$$

$$\dot{v}_s = A_s v_s + z(\theta_t\omega)v_s + g_s(\theta_t\omega, v_c + v_s), \quad v_s(0) = h^s(v_c(0)), \tag{4.7}$$

where h^s is defined as (3.14). Rewrite the above problem in the following equivalent integral form:

$$v_c(t) = e^{A_c(t-T_f)+\int_{T_f}^t z(\theta_\tau\omega)d\tau}\xi + \int_{T_f}^t e^{A_c(t-\tau)+\int_\tau^t z(\theta_\varsigma\omega)ds}g_c(\theta_\tau\omega, v(\tau))d\tau, \tag{4.8}$$

$$v_s(t) = e^{A_s t+\int_0^t z(\theta_\tau\omega)d\tau}h^s(v_c(0)) + \int_0^t e^{A_s(t-\tau)+\int_\tau^t z(\theta_\varsigma\omega)ds}g_s(\theta_\tau\omega, v(\tau))d\tau, \tag{4.9}$$

$t \in [0, T_f]$.

Lemma 4.6: Let (3.12) hold. Then for any $T_f > 0$, (4.8) and (4.9) has a unique solution $(v_c(\cdot), v_s(\cdot)) \in C(0, T_f; H_c \times H_s)$. Moreover for any $t \geq 0$, $(v_c(t, \theta_{-t}\omega), v_s(t, \theta_{-t}\omega)) \in \tilde{\mathcal{M}}(\omega)$ for almost all $\omega \in \Omega$.

Proof: The existence and uniqueness on small time interval can be obtained by a contraction argument, as in Lemma 3.3 of Ref. 10. Then the solution can be extended to any time interval; see Theorem 3.8 of Ref. 10. □

Now we complete the proof of the main result of this paper.

Proof of Theorem 2.1: It remains only to prove the almost sure asymptotic completeness of $\tilde{\mathcal{M}}(\omega)$. We fix a $\omega \in \Omega$. Consider a solution

$$v(t, \theta_{-t}\omega) = (v_c(t, \theta_{-t}\omega), v_s(t, \theta_{-t}\omega))$$

of (3.5). For any $\tau > 0$ by Lemma 4.6 we can find a solution of (3.5) $\bar{v}(t, \theta_{-t}\omega)$, lying on $\tilde{\mathcal{M}}(\omega)$ such that

$$\bar{v}_c(\tau, \theta_{-\tau}\omega) = v_c(\tau, \theta_{-\tau}\omega).$$

Then $\bar{v}(t, \theta_{-t}\omega)$ depends on $\tau > 0$. Write

$$\bar{v}_c(0; \tau, \omega) := \bar{v}_c(0, \omega)$$

and

$$\bar{v}_s(0; \tau, \omega) := \bar{v}_s(0, \omega).$$

By the construction of $\tilde{\mathcal{M}}(\omega)$,

$$|\bar{v}_s(0;\tau,\omega)| \leq \int_0^\infty e^{-\beta r-\int_0^{-r} z(\theta_\varsigma\omega)d\varsigma}|g_s(\theta_r\omega,v^*(-r))|dr \leq L_F\int_0^\infty e^{-(\beta+\eta)r}e^{\eta r}e^{-\int_0^{-r} z(\theta_\varsigma\omega)d\varsigma}|v^*(-r)|dr$$

$$:= N_{L_F}(\omega) \leq L_F|v^*|_{C_\eta^-}\int_0^\infty e^{-(\beta+\eta)r}dr \quad (\beta+\eta>0 \text{ by the choice of } \eta).$$

It is easy to see that $N_{L_F}(\omega)$ is a finite tempered random variable and $N_{L_F}(\omega)\sim O(L_F)$ almost surely, and since $\bar{v}_c(\tau,\theta_{-\tau}\omega)=v_c(\tau,\theta_{-\tau}\omega)$, by the cone invariance,

$$v(t,\theta_{-t}\omega)-\bar{v}(t,\theta_{-t}\omega) \notin \mathcal{C}_\delta(\omega), \quad 0\leq t\leq \tau.$$

Let $S(\omega)=\{\bar{v}_c(0;\tau,\omega):\tau>0\}$. Notice that

$$|\bar{v}_c(0;\tau,\omega)-v_c(0,\omega)| < \frac{1}{\delta}|\bar{v}_s(0;\tau,\omega)-v_s(0,\omega)| \leq \frac{1}{\delta}(N_{L_F}(\omega)+|v_s(0,\omega)|).$$

Then S is a random bounded set in finite dimensional space, that is, for almost all $\omega\in\Omega$, $S(\omega)$ is a bounded set in $\mathbb{R}^n$ and the bound may not be uniform to $\omega\in\Omega$. However, for almost all $\omega\in\Omega$, we can pick out a sequence $\tau_m\to\infty$ such that

$$\lim_{m\to\infty}\bar{v}_c(0;\tau_m,\omega)=V_c(\omega).$$

Moreover $V(\omega)$ is measurable with respect to ω. Let $V(t,\theta_{-t}\omega)=(V_c(t,\theta_{-t}\omega),V_s(t,\theta_{-t}\omega))$ be a solution of (3.5) with $V(0,\omega)=(V_c(\omega),h^s(V_c(\omega),\omega))$. Then $V(t,\theta_{-t}\omega)\in\tilde{\mathcal{M}}(\omega)$ and it is easy to check by a contradiction argument that

$$v(t,\theta_{-t}\omega)-V(t,\theta_{-t}\omega)\notin\mathcal{C}_\delta(\omega), \quad 0\leq t<\infty,$$

which means the almost sure asymptotic completeness of $\tilde{\mathcal{M}}(\omega)$.

This finishes the proof of Theorem 2.1.

Remark 4.7: This theorem implies that the random system (3.5) is asymptotically reduced to the following random ordinary differential equation on H_c:

$$\dot{v}_c = A_cv_c + z(\theta_t\omega)v_c + g_c(\theta_t\omega,v_c+h^s(\omega,v_c)), \tag{4.10}$$

where h^s is given by Eq. (3.14). Thus the SPDE (3.1) is asymptotically reduced to the following finite dimensional nonautonomous stochastic differential system on the invariant manifold H_c:

$$du_c=[A_cu_c+F_c(u_c+\bar{h}^s(u_c))]dt+u_c\circ dW(t), \tag{4.11}$$

where $\bar{h}^s(u_c)=e^{z(\theta_t\omega)}h^s(\theta_t\omega,v_c)$.

Remark 4.8: From the proof of Theorem 2.1, we see that the dynamical approximation estimate holds for the random PDE (3.5). In other words, we may apply Theorem 2.1 to random PDEs like (3.5), as long as appropriate conditions are satisfied.

V. AN APPLICATION: DETECTING STATIONARY STATES

In this final section, as an application of Theorem 2.1 and its consequences, we consider stationary solutions for a stochastic hyperbolic equation.

We intend to detect the stationary states of the following hyperbolic equation driven by multiplicative white noise:

$$u_{tt}+au_t=\nu\Delta u+bu+f(u)+u\circ\dot{W} \text{ in interval I}, \tag{5.1}$$

with

$$u(0) = u_0, \quad u_t(0) = u_1, \quad u(0) = u(2\pi) = 0,$$

where I is taken as the interval $(0,2\pi)$ for simplicity, both ν and $a>0$ are positive constants, and $W(t)$ is a scalar Wiener process. Moreover, $f \in C^{1,1}(\mathbb{R},\mathbb{R})$ is bounded with global Lipschitz constant L_f. For example, $f(x)=\sin x$, which yields the Sine-Gordon equation.

We study the existence of the stationary solutions of (5.1) by reducing the system to a finite dimensional system on an invariant manifold, which is almost surely asymptotically complete. In fact by Theorem 2.1, the dynamical behavior of (5.1) restricted on the invariant manifold is determined by a stochastic ordinary differential system. The existence and stability of the stationary solutions of the stochastic hyperbolic equation can be obtained from that of the stochastic ordinary differential system. A similar result for a parabolic system with large diffusivity holds.[13]

Let $\mathcal{H}=H_0^1(\mathrm{I})\times L^2(\mathrm{I})$. Rewrite system (5.1) as the following one order stochastic evolutionary equation in $\mathcal{H}$:

$$du = v\,dt, \tag{5.2}$$

$$dv = [-\nu Au + bu - av + f(u)]dt + u \circ dW(t), \tag{5.3}$$

where $A=-\Delta$ with Dirichlet boundary condition on I. Note that Theorem 2.1 cannot be applied directly to system (5.2) and (5.3). However, by Remark 4.8, we can still have the same result as Theorem 2.1 for the stochastic hyperbolic system (5.1). This will be clear after we transform (5.2) and (5.3) into the form of (3.5).

First we prove that system (5.2) and (5.3) generates a continuous RDS in $\mathcal{H}$. Let $\phi_1(t)=u(t)$, $\phi_2(t)=u_t(t)-u(t)z(\theta_t\omega)$ where $z(\omega)$ is the stationary solution of (3.4). Then we have the following random evolutionary equation:

$$d\phi_1 = [\phi_2 + \phi_1 z(\theta_t\omega)]dt, \tag{5.4}$$

$$d\phi_2 = [\nu\delta\phi_1 + b\phi_1 - a\phi_2 + f(\phi_1)]dt + [(z(\theta_t\omega) - az(\theta_t\omega) - z^2(\theta_t\omega))\phi_1 - z(\theta_t\omega)\phi_2]dt. \tag{5.5}$$

By a standard Galerkin approximation procedure as in Ref. 25, system (5.4) and (5.5) is well posed. In fact we give prior estimates. We multiply (5.4) with $\nu\phi_1$ in $H_0^1(\mathrm{I})$ and (5.5) with ϕ_2 in $L^2(\mathrm{I})$. Since f is bounded by a simple calculation, we have

$$\frac{d}{dt}[\nu|\phi_1|^2_{H_0^1(\mathrm{I})} + |\phi_2|^2_{L^2(\mathcal{D})}] \leqslant C(1 + |z(\theta_t\omega)|^2 + |z(\theta_t\omega)|)[\nu|\phi_1|^2_{H_0^1(\mathrm{I})} + |\phi_2|^2_{L^2(\mathrm{I})}]$$

for appropriate constant C. Then for any $T>0$, (ϕ_1,ϕ_2) is bounded in $L^\infty(0,T;\mathcal{H})$ which ensures the weak-star convergence by the Lipschitz property of f, for the details see Ref. 15. Let $\Phi(t,\omega)=(\phi_1(t,\omega),\phi_2(t,\omega))$; then $\varphi(t,\omega,\Phi(0))=\Phi(t,\omega)$ defines a continuous RDS in $\mathcal{H}$. Notice that the stochastic system (5.2) and (5.3) is conjugated to the random system (5.4) and (5.4) by the homeomorphism

$$T(\omega,(u,v)) = (u, v + uz(\omega)), \quad (u,v) \in \mathcal{H},$$

with inverse

$$T^{-1}(\omega,(u,v)) = (u, v - uz(\omega)), \quad (u,v) \in \mathcal{H}.$$

Then $\hat{\varphi}(t,\omega,(u_0,v_0))=T(\theta_t\omega,\varphi(t,\omega,T^{-1}(\omega,(u_0,v_0))))$ is the RDS generated by (5.2) and (5.3). For more relation about the two systems see Ref. 15.

Define

$$\mathcal{A}_\nu = \begin{pmatrix} 0 & -id_{L^2(\mathrm{I})} \\ \nu A - b & a \end{pmatrix}, \quad F(\Phi) = \begin{pmatrix} 0 \\ f(\phi_1) \end{pmatrix},$$

$$Z(\theta_t\omega) = \begin{pmatrix} z(\theta_t\omega) & 0 \\ z(\theta_t\omega) - az(\theta_t\omega) - z^2(\theta_t\omega) & -z(\theta_t\omega) \end{pmatrix},$$

where $-id_{L^2(\mathrm{I})}$ is the identity operator on the Hilbert space $L^2(\mathrm{I})$.

Then (5.4) and (5.5) can be written as

$$\frac{d\Phi}{dt} = -\mathcal{A}_\nu \Phi + Z(\theta_t\omega)\Phi + F(\Phi), \tag{5.6}$$

which is in the form of (3.5). Thus by Remark 4.8, we can still apply Theorem 2.1 here.

The eigenvalues of the operator A are $\lambda_k = k^2$ with corresponding eigenvectors $\sin kx$, $k = 1, 2, \ldots$. Then the operator $\mathcal{A}_\nu$ has the eigenvalues

$$\delta_k^{\pm} = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b - \nu k^2}, \quad k = 1, 2, \ldots$$

and the corresponding eigenvectors are $(1, \delta_k^{\pm})\sin kx$. Define subspace of $\mathcal{H}$,

$$\mathcal{H}_c = \mathrm{span}\{(1,0)^T \sin 2x\}, \quad \mathcal{H}_u = \mathrm{span}\{(1, \delta_1^-)^T \sin x\}$$

and $\mathcal{H}_{cu} = \mathcal{H}_c \oplus \mathcal{H}_u$. Write the projections from $\mathcal{H}$ to $\mathcal{H}_c$, $\mathcal{H}_u$, and $\mathcal{H}_{cu}$ as P_c, P_u, and P_{cu}, respectively. We also use the subspaces

$$\mathcal{H}_c^- = \mathrm{span}\{(1, \delta_2^{\pm})^T \sin 2x\}, \quad \mathcal{H}_u^- = \mathrm{span}\{(1, \delta_1^{\pm})^T \sin 2x\},$$

with the projections P_c^- and P_u^- from $\mathcal{H}$ to $\mathcal{H}_c^-$ and $\mathcal{H}_u^-$, respectively. Let $\mathcal{H}_{cu}^- = \mathcal{H}_c^- \oplus \mathcal{H}_u^-$ and $\mathcal{H}_s^- = \mathrm{span}\{(1, \delta_k^{\pm})^T \sin kx, k \in \mathbb{Z}^+ \backslash \{1,2\}\}$.

Here we consider a special case that $b = 4\nu$ and $\nu = a^2/4$. Then the operator $\mathcal{A}_\nu$ has one zero eigenvalue $\delta_2^- = 0$, one negative eigenvalue $\delta_1^- = a/2$, and the others are all complex numbers with positive real part. Since $\{(1, \delta_k^{\pm})\sin kx\}$ are not orthogonal, we introduce a new inner product (see Refs. 17, 18, and 24), which defines an equivalent norm on $\mathcal{H}$. For (u_1, v_1), $(u_2, v_2) \in \mathcal{H}_{cu}^-$, define

$$\langle (u_1,v_1),(u_2,v_2)\rangle_{cu} = 2\left(\langle -Au_1, u_2\rangle_0 + \frac{a^2}{4}\langle u_1, u_2\rangle_0 + \left\langle \frac{a}{2}u_1 + v_1, \frac{a}{2}u_2 + v_2 \right\rangle_0 \right)$$

and for (u_1, v_1), $(u_2, v_2) \in \mathcal{H}_s^-$ define

$$\langle (u_1,v_1),(u_2,v_2)\rangle_{s} = 2\left(\langle Au_1, u_2\rangle_0 - \frac{a^2}{4}\langle u_1, u_2\rangle_0 + \left\langle \frac{a}{2}u_1 + v_1, \frac{a}{2}u_2 + v_2 \right\rangle_0 \right),$$

where $\langle \cdot, \cdot \rangle_0$ is a usual inner product in $L^2(\mathrm{I})$. Then we introduce the following new inner product in $\mathcal{H}$ defined by

$$\langle U, \tilde{U}\rangle_{\mathcal{H}} = \langle U_{cu}, \tilde{U}_{cu}\rangle_{cu} + \langle U_s, \tilde{U}_s\rangle_s$$

for $U = U_{cu} + U_s$, $\tilde{U} = \tilde{U}_{cu} + \tilde{U}_s \in \mathcal{H}$. The new norm

$$\|U\|_{\mathcal{H}}^2 = \langle U, U\rangle_{\mathcal{H}}$$

is equivalent to the usual norm $\|\cdot\|_{H_0^1 \times L^2(D)}$. Moreover $\mathcal{H}_{cu}$ and $\mathcal{H}_s = \mathcal{H} \ominus \mathcal{H}_{cu}$ are orthogonal. The new norm has the following properties:[18]

(1) $\|U\|_{\mathcal{H}} = \sqrt{2}\|v\|_{L^2(D)}$ for $U = (0, v) \in \mathcal{H}$.

(2) $\|U\|_{\mathcal{H}} \geqslant \sqrt{2}a/2\|u\|_{L^2(D)}$ for $U=(u,v)\in\mathcal{H}$.
(3) In terms of the new norm the Lipschitz constant of f is $2L_f/a$.

A calculation yields that, in terms of the new norm, the semigroup $\mathcal{S}(t)$ generated by $\mathcal{A}_\nu$ satisfies (2.3) and (2.4) with $\alpha=0$, $\beta=a/2$, and $\dim H_c=\dim\mathcal{H}_{cu}=2$. Taking $\eta=-\beta-\alpha/2$, condition (3.12) for (5.1) becomes

$$\frac{8L_f}{a^2} < 1, \tag{5.7}$$

which holds if a is large enough.

Define the space C_η^- with $z(\theta_t\omega)$ replaced by $Z(\theta_t\omega)$. Then by the random invariant manifold theory in Sec. III above, we know system (5.6) has a two dimensional random invariant manifold $\tilde{\mathcal{M}}(\omega)$ in C_η^- provided a is large enough. $\tilde{\mathcal{M}}(\omega)$ can be denoted as the graph of a random Lipschitz map $h^s:\mathcal{H}_{cu}\to\mathcal{H}_s$ with Lipschitz constant CL_f/a. Then by Theorem 2.1, the dynamical behavior of (5.6), that is, (5.1), is determined by the following reduced system on $\mathcal{M}(\omega)$ which is a finite dimensional random system:

$$\frac{d\Phi_{cu}}{dt} = -P_{cu}\mathcal{A}_\nu\Phi_{cu} + Z(\theta_t\omega)\Phi_{cu} + P_{cu}F(\Phi_{cu}+h^s(\Phi_{cu})), \tag{5.8}$$

provided a is large enough. To see the reduced system more clearly, projecting system (5.1) to span$\{\sin x\}$ and span$\{\sin 2x\}$ by the projections Π_1 and Π_2, respectively, we have the following system:

$$\dot{u}_1 = v_1, \tag{5.9}$$

$$\dot{v}_1 = \frac{3a^2}{4}u_1 - av_1 + \Pi_1 f(u) + u_1\circ\dot{W}(t), \tag{5.10}$$

$$\dot{u}_2 = v_2, \tag{5.11}$$

$$\dot{v}_2 = -av_2 + \Pi_2 f(u) + u_2\circ\dot{W}(t). \tag{5.12}$$

By the new inner product in $\mathcal{H}$, define the following new inner product in $\mathbb{R}^2$ as

$$((x,y),(\tilde{x},\tilde{y})) = 2\left[\frac{a^2}{4}x\tilde{y} + \left(\frac{a}{2}x+y\right)\left(\frac{a}{2}\tilde{x}+\tilde{y}\right)\right].$$

Then projecting the above system (5.9)–(5.12) to $\mathcal{H}_{cu}$ by the new inner product in $\mathbb{R}^2$, we have the reduced system

$$\dot{u}_1 = \frac{a}{2}u_1 + \frac{2\sqrt{10}}{5}\Pi_1 f(u_1+u_2+\bar{h}^s(u_1,u_2)) + u_1\circ\dot{W}(t), \tag{5.13}$$

$$\dot{u}_2 = \Pi_2 f(u_1+u_2+\bar{h}^s(u_1,u_2)) + u_2\circ\dot{W}(t), \tag{5.14}$$

where $\bar{h}^s=e^{z(\omega)}h^s$.

Then the dynamical behavior of system (5.1) is determined by that of system (5.13) and (5.14) if a is large. So if system (5.13) and (5.14) has a stationary solution, so does system (5.1), and the stability property of the stationary solution is also determined by that of (5.13) and (5.14).

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