

# Use the information dimension, not the Hausdorff

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## Abstract

Multi-fractal patterns occur widely in nature. In developing new algorithms to determine multi-fractal spectra of experimental data I am lead to the conclusion that generalised dimensions  $D_q$  of order  $q \leq 0$ , including the Hausdorff dimension, are effectively *irrelevant*. The reason is that these dimensions are extraordinarily sensitive to regions of low density in the multi-fractal data. Instead, one should concentrate attention on generalised dimensions  $D_q$  for  $q \geq 1$ , and of these the information dimension  $D_1$  seems the most robustly estimated from a finite amount of data.

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# 1 Introduction

The characterisation of spatial distributions in terms of fractal concepts [12, 3] is becoming increasingly important. In particular, many distributions in nature are found to have the characteristics of a multi-fractal [6, 5, 14]: among many examples are galaxy clustering [1, 13], strange attractors [15], fluid turbulence [17], percolation [7], the shapes of neurons [9, 8], and plant distributions [2] and shapes [10].

In application, methods for estimating fractal dimensions are often unreliable. One source of error lies in largely unknown biases introduced by the finite size of data sets, addressed by Grassberger [4], and in the associated finite range of length-scales inherent in gathered data. In situations where thousands or tens of thousands of data points are known such biases may be minor; however, in some interesting problems, for example in the spatial clustering of underwater plants [2], only of the order of 100 data points are known and confidence in the fractal characterisation may be misplaced. We need to know more about factors that cause errors in dimension estimates.

Section 2 discusses the sensitivity of the multiplicative multi-fractal process to regions of very low probability (measure). Since such regions only rarely contribute a data point, an experimental sample cannot discern them but such regions do affect the generalised dimensions. Hence I argue that the determination from experimental data of generalised dimensions,  $D_q$ , for non-positive  $q$  is meaningless; for  $0 < q < 1$  computations are very sensitive to the sample; and thus the most robust fractal dimension is the information dimension  $D_1$ . The argument is supported in Section 3 by a maximum likelihood method [16] of estimating the multi-fractal properties of a data set. The method shows the enormous sensitivity of  $D_q$  for negative  $q$ . In contrast the information dimension is reliably estimated.

## 2 Poor conditioning of generalised dimensions of negative order

For example, consider the Hausdorff dimension,  $D_0$ , of multifractals generated by two different ternary multiplicative process.

- Consider first the process shown in Figure 1(a) where an interval is divided into three thirds and the “mass” of the original interval is assigned as follows: a fraction  $f_1 > 0$  to the left third; a fraction

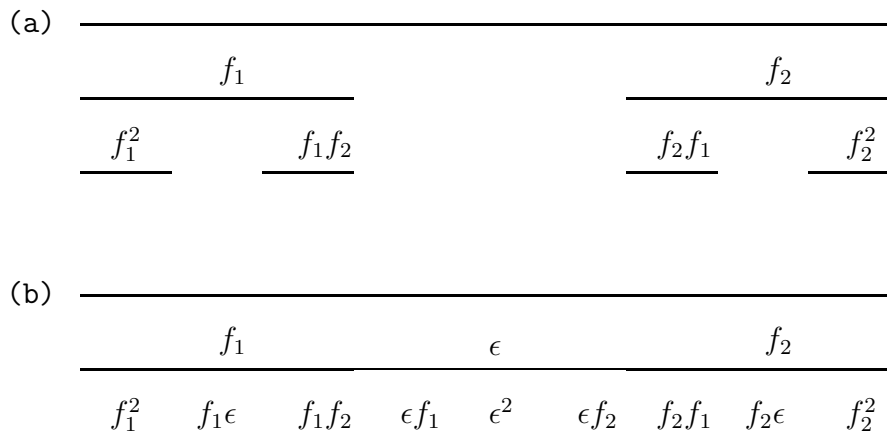


Figure 1: schematic diagram of the first few stages in the multiplicative multi-fractal process to illustrate the sensitivity of the Hausdorff dimension  $D_0$  with respect to low density regions, (b), as a perturbation of the same process with zero density regions, (a).

$f_2 = 1 - f_1 > 0$  to the right third; and none to the middle third. Repeat this subdivision recursively. This generates a multiplicative multifractal whose Hausdorff dimension of  $D_0 = \log_3 2 = 0.6309$  is precisely the same as the Cantor set because there is no “mass” in the middle thirds.

- Conversely, and perversely, consider the process shown in Figure 1(b) where for some small  $\epsilon$  the “mass” is assigned as follows: a fraction  $f_1 > 0$  is assigned to the left third; a fraction  $f_2 > 0$  is assigned to the rightmost third; and a small fraction  $\epsilon > 0$  is assigned to the middle third (such that  $f_1 + f_2 + \epsilon = 1$ ). Repeat recursively. This generates a multiplicative multi-fractal whose Hausdorff dimension is  $D_0 = 1$  because there is “mass” everywhere along the whole interval! Although the vast bulk of the “mass” can be covered by  $2^n$  intervals of length  $3^{-n}$ , we definitely do need  $3^n$  intervals in order to ensure coverage of the thinly spread “mass” that fills most of the original interval.

The importance of this for the analysis of an experimental data set of  $N$  sampled points is that one cannot tell the difference from the data between these two multi-fractal generating processes for an  $\epsilon = o(1/N)$ . Thus one cannot estimate the Hausdorff dimension  $D_0$  with any accuracy since either answer, 0.6309 or 1 could be correct.

Similar reasoning applies to generalised dimensions with negative  $q$ . Elementary arguments give that the generalised dimensions [6] of the multi-fractal generated by the second process above are

$$D_q = \begin{cases} \frac{-1}{q-1} \log_3 [f_1^q + f_2^q + \epsilon^q] & \text{if } q \neq 1, \\ -[f_1 \log_3 f_1 + f_2 \log_3 f_2 + \epsilon \log_3 \epsilon] & \text{if } q = 1. \end{cases} \quad (1)$$

It is readily appreciated that for negative order  $q$  and small  $\epsilon$ , the term  $\epsilon^q$  inside the logarithm dominates the evaluation of the generalised dimension  $D_q$ . Hence, all generalised dimensions for negative  $q$  are also extremely sensitive to small  $\epsilon$ . In a data set obtained from experiments, one cannot expect to distinguish between zero  $\epsilon$  and small non-zero  $\epsilon = o(1/N)$ , and yet the generalised exponents and multi-fractal spectrum are markedly different. See Figure 2 which plots the generalised dimensions for  $f_1 \approx 1/4$ ,  $f_2 \approx 3/4$  and various small  $\epsilon$ .

We can be more precise about the sensitivity to low density regions by computing the derivative of  $D_q$  with respect to  $\epsilon$ . For definiteness, suppose  $f_1 = \phi_1(1 - \epsilon)$  and  $f_2 = \phi_2(1 - \epsilon)$ . Then

$$\frac{\partial D_q}{\partial \epsilon} = \frac{-q}{q-1} \frac{\epsilon^{q-1} - (\phi_1^q + \phi_2^q)(1 - \epsilon)^{q-1}}{\log 3 [\epsilon^q + (\phi_1^q + \phi_2^q)(1 - \epsilon)^q]}. \quad (2)$$

For small, but non-zero,  $\epsilon \rightarrow 0$  this asymptotes to

$$\frac{\partial D_q}{\partial \epsilon} \sim \frac{1}{\log 3} \begin{cases} \frac{q}{q-1} & \text{if } 1 < q, \\ \frac{q}{(1-q)(\phi_1^q + \phi_2^q)} \epsilon^{q-1} & \text{if } 0 < q < 1, \\ \frac{q}{1-q} \epsilon^{-1} & \text{if } q < 0. \end{cases} \quad (3)$$

This derivative is unbounded as  $\epsilon \rightarrow 0$  for  $q < 1$ , and so any computation of  $D_q$  is only robust if  $q \geq 1$ .

The reason for this aberrant behaviour is clear. With a finite number of data points, it is impossible to tell the difference between truly empty space and space which is visited so rarely that no data point happens to fall within it. That is, one cannot tell the difference between empty space and space that should be filled in with very low probability. These differences dramatically affect the generalised dimensions  $D_q$  for  $q < 1$ . Thus for any experimental data set:

- estimating  $D_q$  for  $q \leq 0$  is nonsense (including the Hausdorff dimension);
- estimates of  $D_q$  for small positive  $q$  are sensitive; and

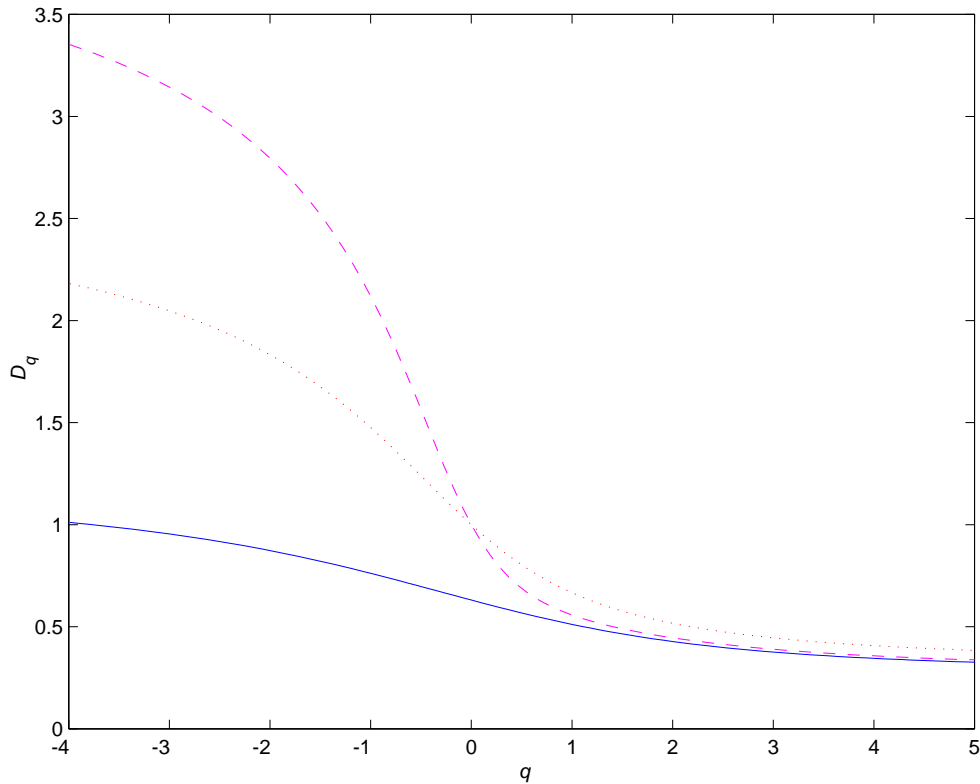


Figure 2: multi-fractal generalised dimensions  $D_q$  for the ternary multi-fractal process with  $f_1 = (1 - \epsilon)/4$ ,  $f_2 = (1 - \epsilon)3/4$  and  $\epsilon = 0$  (solid), 0.01 (dashed) and 0.05 (dotted). This figure shows that  $D_q$  for negative order  $q$  is extraordinarily sensitive to small influences: the curve of smaller  $\epsilon$  is the most changed.

- I only recommend the reporting of dimensions  $D_q$  for  $q \geq 1$  as being robust.

Out of all the generalised dimensions for order  $q \geq 1$ ,  $D_1$  is most representative of the fractal as a whole. For large order  $q$ , the computation of  $D_q$  is determined only by the very “densest” regions of the multi-fractal and so is not representative of the whole fractal. In the above multiplicative process,

$$D_q \sim -\log_3 \max(f_1, f_2, \epsilon) \quad \text{as } q \rightarrow \infty,$$

showing that the large  $q$  behaviour is dictated by the one parameter of the process that determines the character of the very densest clusters in the fractal. The very dense clusters occur rarely in the fractal; they have low fractal dimension as seen in the low  $f$  value typically associated with low values of  $\alpha$

in the multi-fractal spectrum. Because of this rareness, the computation from experimental data of  $D_q$  for large positive order  $q$  is unreliable. Then, conversely, the information dimension weights the data most uniformly, and so “knows” most about the fractal, without being overly sensitive to the possible occurrence of regions of very low probability. The information dimension seems most informative.

### 3 Fractal dimensions unbiased by finite size of data sets

Cronin & Roberts [16] proposed a novel method to eliminate biases, caused by finite sized data sets, in determining the multi-fractal properties of a given data set. Jelenik et al. [9, 8] used this method to explore the shape of neuron cells. The method compares characteristics of the inter-point distances in the data set with those of artificially generated multi-fractals. By maximising the likelihood that the characteristics are the same we model the multi-fractal nature of the data by the parameters of the artificial multi-fractal. By searching among artificial multi-fractals with precisely the same number of sample points as in the data, we anticipate that biases due to the finite sample size will be statistically the same in the data and in the artificial multi-fractals; hence predictions based upon the fitted multi-fractal parameters should be unbiased by the finite sample size.

The method also appears to give a reliable indication of the error in the estimates—a very desirable feature as also noted by Judd & Mees [11]. Most importantly for this paper, I generate finite size data sets with specific parameters for the following specific multiplicative multi-fractal process. Given parameters  $\rho \in [0, 0.5]$  and  $\phi \in [0, 0.5]$  a binary multiplicative multi-fractal is generated by the recursive procedure of dividing each interval into two halves, then assigning a fraction  $\phi$  of the points in the interval to a random sub-interval of length  $\rho$  in the left half, and the complementary fraction  $\phi' = 1 - \phi$  to a random sub-interval of length  $\rho$  in the right half. Such a process has generalised dimension

$$D_q = \frac{\log(\phi^q + \phi'^q)}{(q-1)\log\rho}, \quad (4)$$

and a multi-fractal spectrum  $f(\alpha)$  [5, §4] given parameterically in terms of  $0 < \xi < 1$  and  $\xi' = 1 - \xi$  as

$$f = \frac{\xi \log \xi + \xi' \log \xi'}{\log \rho}, \quad \alpha = \frac{\xi \log \phi + \xi' \log \phi'}{\log \rho}. \quad (5)$$

Here I chose  $\rho = 1/3$  and  $\phi = 1/4$  and sample the process with  $N = 100$ ; such a multi-fractal forms a finite data set whose parameters we need to estimate from the sample.

As explained in [16], we analyse such a sample by probing it with *exactly* the same multiplicative multi-fractal process, and seek the best fit parameters. Here the resulting estimate of the original parameters is then in error *only* due to the finite size of the sample of the original multi-fractal process. Because we fit the data with a process which we know includes the one that generated the data (a luxury rare in practise), there is no other error. Thus the spread in errors that we see is characteristic of only the errors induced by a finite sized sample, nothing else. In particular, observe that the deductions of the preceding section are indeed appropriate.

I repeat the sampling of the multi-fractal followed by a maximum likelihood estimate of the parameters 16 times. Figure 3 plots the estimates of the parameters. Observe that the whole sampling and estimation process appears unbiased in that the mean of the predictions is reasonably close to the correct values of the parameters.

Ultimately, experimenters want to examine multi-fractal properties of the data. Here these will be determined from the parameters  $(\rho, \phi)$  of the best fit multi-fractal substituted into analytic expressions such as (4) and (5). For each of the 16 realisations and their best-fit estimates, I plot the corresponding predicted generalised dimensions  $D_q$  in Figure 4. (The corresponding graphs of the multi-fractal spectra  $f(\alpha)$  are plotted in Figure 7 of [16] along with the true  $f(\alpha)$  curve.) Observe that the predicted dimensions for positive  $q$  (low  $\alpha$ ) are quite good for all realisations, especially near the information dimension,  $D_1$ . However, predicted dimensions for negative  $q$  (high  $\alpha$ ) are very poor; this is also the case for the Hausdorff dimension  $D_0$  (the maximum of the  $f(\alpha)$  curve). The negative  $q$  predictions are poor despite the fitting process “knowing” that there are no very low probability regions in this artificial process. In general applications one cannot know this and I expect the negative  $q$  (large  $\alpha$ ) predictions to be significantly worse. These numerical results convincingly support the arguments of the preceding section that we should use the information dimension, not the Hausdorff.

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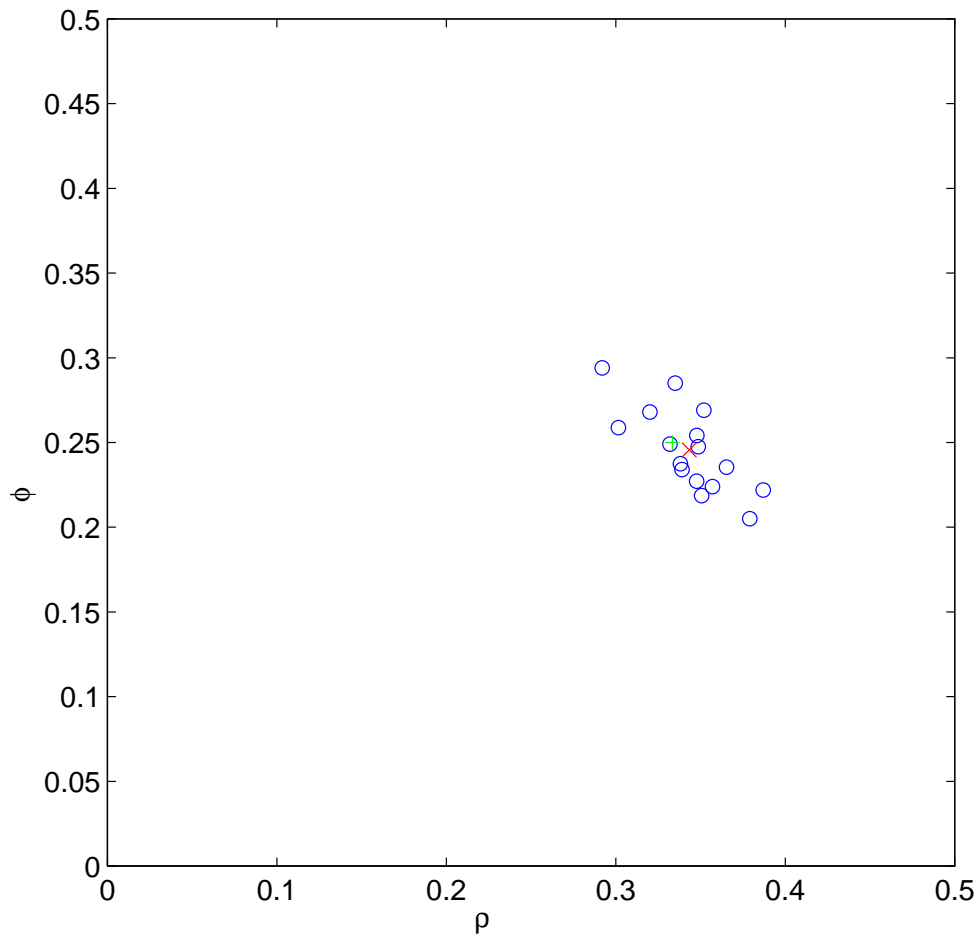


Figure 3: predicted multi-fractal parameters  $(\rho, \phi)$ , indicated by  $\circ$ 's, from the maximum likelihood match to an ensemble of 16 different realisations, each of  $N = 100$  data points, of a binary multiplicative multi-fractal with parameters  $\rho = 1/3$  and  $\phi = 1/4$ , indicated by  $+$ . The mean location of the predictions is indicated by a  $\times$ .

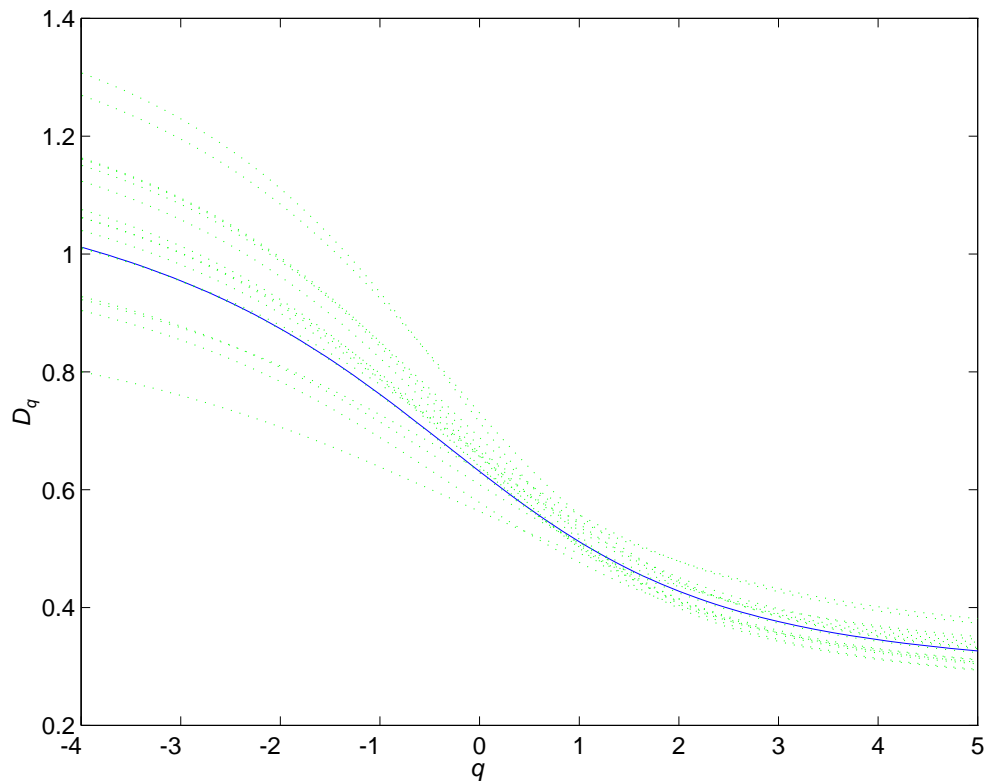


Figure 4: ensemble of multi-fractal generalised dimensions  $D_q$ , dotted, for each of the predictions plotted in Figure 3 made from samples of  $N = 100$  data points. For comparison the generalised dimensions for the actual fractal is plotted as the solid line. Observe the good estimation near the information dimension, but the large errors for negative order  $q$ .