



Gauge fixing and BRST formalism in non-Abelian gauge theories

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*To the morning star of my life,
my beloved sister Samantha*

Abstract

In this Thesis we present a comprehensive study of perturbative and non-perturbative non-Abelian gauge theories in the light of gauge-fixing procedures, focusing our attention on the BRST formalism in Yang-Mills theory. We propose first a model to re-write the Faddeev-Popov quantisation method in terms of group-theoretical techniques and then we give a possible way to solve the no-go theorem of Neuberger for lattice Yang-Mills theory with double BRST symmetry. In the final part we present a study of the Batalin-Vilkovisky quantisation method for non-linear gauges in non-Abelian gauge theories.

Statement of Originality

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

A Marco Ghiotti

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Introduction

Since the seminal work by Yang and Mills in 1954 [YM54], non-Abelian gauge theories have been indisputably of enormous importance in particle, and most generally, in theoretical physics. It is well known that all the four forces discovered so far (electromagnetic, weak, strong and gravitation force) are mediated by bosons or simply gauge fields (respectively photons, W^\pm and Z , gluons, gravitons): apart from the photon, all the other vector particles belong to a specific representation of non-Abelian gauge theory. The fundamental property of these theories, regardless of their commutativity, lies in the fact that the Lagrangian of the model is invariant under a local redefinition of the gauge field, known as a gauge transformation for gauge (or general coordinate transformation for gravity), as Dirac noticed in the light of quantum electrodynamics (QED) many years ago.

This local gauge invariance is responsible for one of the most challenging problems to solve in current theoretical physics: the understanding of the non-perturbative regime of non-Abelian theories. As pointed out by Gribov in the late 70-s [Gri78], quantum chromodynamics (QCD), the most accurate model for the strong interaction, suffers from a topological obstruction whenever one deals with low energies. This is due to the fact that the QCD Lagrangian, described by a Yang-Mills interaction, being left unchanged by a local gauge transformation, contains an infinite overcounting of physically equivalent gauge configurations, grouped together in different gauge orbits (equivalence classes). To take into account only gauge inequivalent configurations, ideally one would need to find an unambiguous procedure to consider only one representative per gauge orbit: such a method is called gauge-fixing. Unfortunately, as Gribov discovered and explained first in the physics language, and then Singer [Sin78] in a more mathematical manner, there is no analytic gauge-fixing procedure which guarantees the non-perturbative regime of QCD to be free of such an ambiguity, called the Gribov ambiguity. The topological obstruction can be easily understood if one considers the gauge-fixing term as a hypersurface which intersects the functional space of gauge orbits. If the fixing procedure were correct, then the surface would intersect only once per orbit. This is the case only for high-energy or perturbative QCD. Beyond this regime, when we are entitled to move far from configurations around the trivial gauge field $A_\mu = 0$

(and thus infinitesimal gauge transformations), even finite gauge transformations play a decisive role: consequently, many intersections are found along the orbit.

Surprisingly, a very interesting consequence of gauge-fixing had been already observed by four physicists three years before the paper of Gribov: Becchi, Rouet and Stora [BRS76], and independently Tyutin [Tyu], using the Faddeev-Popov quantisation method [DN67] to integrate out the infinite gauge redundancy in the path integral representation of QCD, discovered a new global structure, now called BRST formalism. To exponentiate the Faddeev-Popov operator appearing in the integrand of the path integral due to a functional change of variables, they introduced a pair of anticommuting fields, the ghost fields. These fields were the ones Kac and Feynman were looking for to guarantee the unitarity of the \mathcal{S} -matrix in non-Abelian scattering processes. Alongside the ghost fields, a Nakanishi-Lautrup field was used to rewrite the gauge-fixing term, such that the QCD path integral was constituted by a quartet of fields. However, this formalism was only fully understood some years later. In fact, mainly due to the extensive works of Atiyah [AJ78], Witten [Wit82] and Schwarz [Sch78], it was noticed that the theory of invariant polynomials, namely Donaldson theory and knot theory, could be reformulated in more physical terms: this was the birth of topological quantum field theory (TQFT). Within this theory, the BRST formalism was regarded as the most trivial example of supersymmetry, and the BRST charge, generator of the new global symmetry, regarded as a nilpotent supersymmetric operator.

All of a sudden, QCD, and generally speaking non-Abelian gauge theories, started being studied and analyzed from many different angles: Kugo and Ojima [KO79] first interpreted the BRST transformations as the quantum version of the classical gauge transformation. Then, they discovered the generalisation to non-Abelian gauge theory of the Gupta-Bleuler formalism in QED. They were thus able to give a prescription for confinement of quarks (known as the Kugo-Ojima criterion): this non-perturbative mechanism is responsible for avoiding quarks as free particles at low energies and therefore confining them into the hadrons such as nucleons and pions observed in nature, via the action of QCD self-interacting gauge fields, the gluons.

Moreover, mathematicians realised that the well-known theory of principal bundles could be re-expressed in terms of supersymmetric structures. The classical geometry of gauge theory started being enriched with ghost fields, extended coordinates and Grassmann manifolds. In [BT81], [QdUH⁺81] and [BTM82] it was pointed out how the classical theory of gauge geometry had as a logical implementation that of superfields and superconnections.

Schwarz and Witten constructed two different theories involving BRST formalism to study non-Abelian theories as supersymmetric ones by means of

topological path integrals: mathematical/geometrical theorems such the Hodge decomposition of connections, the Poincaré-Hopf and Gauss-Bonnet theorems became soon familiar to many physicists. The connection between the Gribov ambiguity and topological field theory then naturally emerged through this massive quantity of work: at the non-perturbative level, all of the equivalent configurations (Gribov copies) sum up to give a vanishing topological path integral [Fuj79, BBRT91]. This result is regarded as the vanishing Euler character of the gauge-group manifold on which QCD is evaluated [Sch99].

Although much important work has been carried out, in the understanding of non-perturbative gauge theories, some fundamental questions still remain to be answered: in particular, the mechanism responsible for quark confinement seems to be the hardest and most challenging of all. Nowadays the study of non-perturbative QCD is performed through different approaches: Dyson-Schwinger equations [AvS01] and lattice gauge theory [Wil74] perhaps are the more successful ones for practical purposes. In the former, functional and operator identities are pursued by the observation that the path integral of a total functional derivative with respect to one of the fields involved vanishes. In this scenario, the BRST formalism is largely adopted to facilitate highly complicated computations. However, the Gribov ambiguity is not fully avoided or solved: Dyson-Schwinger calculations are performed in the region where the Faddeev-Popov operator is positive definite, and thus the complication of going beyond this region is thus by-passed. On the other side, in lattice QCD, the discretisation of the space-time manifold determines a natural regularisation of the path integral, and therefore no gauge-fixing procedure is required. It is the stochastic nature of the numerical computation of the path integral (e.g. by means of Monte-Carlo algorithms) which self-consistently guarantees to have an insignificant probability to generate two gauge configurations on the same orbit. It is for instance not fully understood yet how the BRST formalism can be implemented on the lattice: this is due to a no-go theorem discovered in mid 80-s by Neuberger [Neu87]. In this paper, he noticed that because of the underlying BRST invariance, the gauge/fixing Y-M partition function can be shown to be independent of any gauge parameter. The consequence of this property is that the lattice path integral of the gauge-fixing action ratios turns out to be exactly zero. It seems then that the BRST formalism cannot be straightforwardly adopted at low energies. Yet, if we wish to construct a complete and organic theory of the strong force, it is then required to comprehend why the BRST formalism apparently only works in perturbative QCD. The motivation behind this Thesis is then to comprehend much more clearly the topological nature of the BRST algebra, both perturbatively and non-perturbatively in non-Abelian gauge theories.

This Thesis is structured as follows: Chapter I is dedicated to the introduction of constrained systems, first starting from the classical point of view of Lagrangian and Hamiltonian systems. We then adopt the covariant formalism to provide the quantum version of Maxwell theory and Y-M theory in the light of functional constraints. Chapter II is entirely devoted to the path integral representation of non-Abelian gauge theories. We also analyze the structure of the functional configuration space of the Y-M path integral both from the analytic and topological point of view. Special emphasis is given to the Gribov problem. In Chapter III starting from the Faddeev-Popov quantisation method, we describe how the BRST formalism emerges in non-Abelian gauge theories, focusing our attention on the supersymmetric structure of it and on the Kugo-Ojima criterion as the generalisation of the Gupta-Bleuler formalism. We give a detailed explanation of the BRST algebra with respect to linear and non-linear gauges. Chapter IV is dedicated to our first work conducted on the interpretation of the Faddeev-Popov quantisation method in terms of group-theoretical techniques. We proposed a supersymmetric manner to re-write the Faddeev-Popov operator, entering the Y-M path integral through a functional determinant, using the Nicolai map. In Chapter V, we move to lattice gauge theory, on which we present a model to circumvent and possibly solve the Neuberger problem, which so far prevented us from using the BRST formalism in lattice gauge theory. To conclude this Thesis, in Chapter VI we present the Batalin-Vilkovisky formalism both in continuum and lattice Yang-Mills theory in the light of non-linear gauges. We will then re-propose the same methodology of Chapter V in this framework.

Symmetries and Constraints in Euclidian Gauge Theories

In this chapter, we will present the theory of constrained systems, starting with a Euclidian classical theory: in this framework, we will first deal with the Hamiltonian formalism and then we will move to the one commonly adopted in this Thesis, the Lagrangian formalism. Gauge theories will be analyzed, considering first the Abelian gauge theory of electromagnetism and then its non-Abelian generalisation, Yang-Mills theory.

2.1 Classical constrained systems: Hamiltonian and Lagrangian formalism

Generally speaking, all the possible physical information we require and need to extract from a theory is encoded into the action S . This can be expressed either through the Lagrangian L or, by appropriate Legendre transformations, through the Hamiltonian H , leading to two different formalisms: the former is of the covariant formalism, whereas the latter the canonical one. Regardless of the particular type of formalism one wishes to adopt, one of the fundamental questions to answer in complex physical systems is how we proceed in the case of one or more constraints affecting the action and consequently, either the Lagrangian or the Hamiltonian. This problem can be immediately addressed at the classical level: we shall show how such constraints affect the dimensionality of the configuration space, how they will enter the equations of motion and furthermore, how they influence Noether's theorem. Consider for this purpose a local Euclidian Lagrangian [IZ,NO90] over a finite-dimensional space of generalised commuting (bosonic) variables q , whose base space is $t \in \mathbb{R}$: we call the *free* Lagrangian, L_0 , the part of L which is being described only by the generalised coordinates $q(t)$, and their first derivatives $\dot{q}(t)$ at most quadratically (this for the canonical formalism to be applicable). The remaining part of L contains higher-order terms in $q(t)$ and it is called the *interaction* Lagrangian, L_I . The classical action of the system is then defined as the integral over the base space

of $L(t)$

$$S \equiv \int dt L(t) = \int dt L(q_n(t), \dot{q}_n(t)). \quad (2.1)$$

Consider now a local variation of both $q(t)$ and $\dot{q}(t)$ in the action

$$\begin{aligned} \delta S &= \int dt \left[\frac{\partial L}{\partial q_n} \delta q_n + \frac{\partial L}{\partial \dot{q}_n} \delta \dot{q}_n \right] \\ &= \int dt \left[\frac{\partial L}{\partial q_n} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} \right] \delta q_n. \end{aligned} \quad (2.2)$$

where the sum over n is understood and we performed an integration by parts to obtain the last line, once appropriate boundary conditions on $q_n(t)$ are imposed. The action principle, or Hamilton principle of least action, states that the path satisfying the classical equations of motion is the one extremising S , such that $\delta S = 0$. Because of the variations δq_n are independent, we obtain the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_n} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} = 0, \quad (2.3)$$

which are a set of n de-coupled second-order differential equations. In the case when we deal with fields, they will be also called *field equations*. To reduce (2.3) to first-order differential equations, we introduce the canonical momentum conjugate

$$p_n \equiv \frac{\partial L}{\partial \dot{q}_n}, \quad (2.4)$$

and we suppose we can invert this relation, i.e. expressing velocities in terms of positions and momenta. The Hamiltonian is obtained through a Legendre transformation as

$$H(p_n, q_n) \equiv p_n \dot{q}_n(p, q) - L(q_n, \dot{q}_n(p, q)). \quad (2.5)$$

The Euler-Lagrange equations are being translated into the symplectic space of the Hamiltonian formalism: they now become the Hamilton equations. To see this, insert (2.5) into the action and vary both momenta and positions to get

$$\delta S = \int dt \left(\dot{q}_n \delta p_n - \dot{p}_n \delta q_n - \frac{\partial H}{\partial p_n} \delta p_n - \frac{\partial H}{\partial q_n} \delta q_n \right), \quad (2.6)$$

with

$$\frac{\partial H}{\partial p_n} = \dot{q}_n, \quad \frac{\partial H}{\partial q_n} = -\dot{p}_n. \quad (2.7)$$

2.1 Classical constrained systems: Hamiltonian and Lagrangian formalism 7

These are the Hamilton equations, which determine a symplectic structure¹ over the configuration space: the dimensionality of this space becomes then twice the original one because of the presence of momenta p . In this phase-space, it is convenient to introduce a way to associate elements at different points: this is achieved by the Poisson Brackets (PB) [VH05], defined as

$$\{F, G\} = \frac{\partial F}{\partial q_n} \frac{\partial G}{\partial p_n} - \frac{\partial F}{\partial p_n} \frac{\partial G}{\partial q_n}, \quad (2.8)$$

where repeated indices are understood to be summed. Any transformation of the canonical variables (p, q) which leaves these brackets unchanged is called canonical. The importance of such an analytic operator lies in the fact that it determines immediately if there is a symmetry. In fact, any function $F(p, q)$, whose total derivative vanishes, is a constant of motion if it has vanishing Brackets with H ,

$$\begin{aligned} \frac{dF}{dt} &= \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q} \dot{q} + \frac{\partial F}{\partial p} \dot{p} \\ &= \frac{\partial F}{\partial t} + \{F, H\} \quad \Rightarrow \quad \{H, F\} = 0 = \frac{\partial F}{\partial t}. \end{aligned} \quad (2.9)$$

At the quantum level, (2.9) describes the Heisenberg equation of motion, whereas we shall see that the Poisson Brackets will become, according to the formalism adopted, either the Lie Brackets in the language of gauge theories or the BV Brackets in the case of supersymmetry. With respect to the two canonical variables, q_n and p_n , the PB with H read

$$\{q_n, H\} = \dot{q}_n \quad \{p_n, H\} = \dot{p}_n. \quad (2.10)$$

To introduce the fundamental concept of a constraint, we start with a pedagogical summary in the Hamiltonian formalism, and then we will present it in the light of the covariant Lagrangian formalism.

Therefore, expand the time derivative in the Euler-Lagrange equations as

$$\frac{\partial L}{\partial q_n} - \frac{\partial^2 L}{\partial \dot{q}_n \partial q^m} \dot{q}^m - \frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}^m} \ddot{q}^m = 0. \quad (2.11)$$

This is a set of n coupled second-order differential equations in q^m with functional coefficients, whose Lipschitz condition of solvability depends on the invertibility of the matrix coefficient of \ddot{q}^m

$$\det \left(\frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}^m} \right) = \det \left(\frac{\partial p_n}{\partial \dot{q}^m} \right) \neq 0. \quad (2.12)$$

¹A symplectic vector space is a vector space V equipped with a nondegenerate, skew-symmetric, bilinear form, called the symplectic form. Its dimension must necessarily be even since every skew-symmetric matrix of odd size has determinant zero.

Whenever such relation does not hold, it means that there are one or more momenta, say m , which can be expressed in terms of the remaining momenta as

$$p_a = p_a(p_i, q_n) \quad 1 \leq a \leq m \leq i \leq n, \quad (2.13)$$

and that for each momentum p_a , there is a velocity \dot{q}^a which cannot be solved in terms of p_i or q_n . We will indicate the set of all these constraints as $\phi_a(p_n, q_n) = 0$. The corresponding constrained Hamiltonian \bar{H} is obtained in the same formal way as for the unconstrained one: such Hamiltonian has the peculiarity that it does not depend on the constrained velocities

$$\frac{\partial \bar{H}}{\partial \dot{q}^a} = 0, \quad (2.14)$$

implying that we can always add to \bar{H} any function of such velocities without affecting the Hamilton equations. The unconstrained Hamiltonian will be then expressed in terms of \bar{H} as

$$H = \bar{H} - \lambda^a(t)\phi_a, \quad (2.15)$$

with λ^a being a Lagrange multiplier, generally depending on t , whose role is to enforce the constraint $\phi_a = 0$. For the dynamics of such a system, in order to be consistent with the symplectic structure thus determined, not only have the canonical variables to respect the constraints, but also the constraints themselves in a self-consistent way such that

$$\frac{d\phi_a}{dt} = \{\phi_a, H\} \simeq 0, \quad (2.16)$$

where by $\simeq 0$ we mean “weakly zero”, provided the constraints are enforced. In the case in which such a condition does not hold, then we keep going by enforcing a new constraint, say ρ_a , defined as $\rho_a \equiv \{\phi_a, H\}$, until we find PB consistent with the condition (2.16). All these new constraints should be added *a posteriori* into the Hamiltonian, associated with the appropriate Lagrange multiplier as $H = \bar{H} - \sum_a^m \lambda_a \phi_a$, for N constraints. The difference between the Hamiltonian formalism and the Lagrangian is that the Lagrange multipliers plugged into H immediately manifest the condition of a constraint, to which we must associate arbitrary trajectories to the coordinates whose dynamics is undetermined by the equations of motion. This assigning goes under the name of gauge-fixing. Though in the Lagrangian formalism such a procedure is unavoidable in the presence of constraints, in the case of the Hamiltonian formalism, what is actually needed is H and to specify the constraints. Even in this classical theory, we

can introduce the concept of a gauge transformation: by this we mean a map of phase space coordinates as

$$\begin{aligned} p_n &\rightarrow p_n + \delta p_n(p_m, q^m, t) = p_n - \epsilon^a(t) \frac{\partial \phi_a}{\partial q_n} \equiv \bar{p}_n \\ q_n &\rightarrow q_n + \delta q_n(p_m, q^m, t) = q_n + \epsilon^a(t) \frac{\partial \phi_a}{\partial p_n} \equiv \bar{q}_n, \end{aligned} \quad (2.17)$$

such that the action S (and the equations of motion) should be left invariant under these transformations. In general, the action is not invariant under these gauges, and so we have to enforce the symmetry through an appropriate transformation rule for all the Lagrange multipliers and the Hamiltonian, respectively

$$\delta \lambda^a = -\frac{d}{dt} \epsilon^a \quad \delta H = \epsilon^a \{H, \phi_a\}. \quad (2.18)$$

Together with (2.17), (2.18) leave the entire system invariant.

2.1.1 Noether's theorem and charge algebra

The Noether theorem is a crucial theorem to understand how both external and internal symmetries play a decisive role in field theories. It states that to any continuous one-parameter set of invariances of the Lagrangian is associated a local conserved current. Integrating the fourth component of this current over three-space generates a conserved charge. From this charge, one can then construct a Lie charge algebra. In the case we have also internal symmetries, as it will be the case in gauge theories, this internal charge algebra will play a fundamental role in the quantisation procedure: it will give rise to the Lie charge algebra of BRST, a special case of a supersymmetric algebra. In this section we present the appearance of a charge algebra in the canonical formalism: consider a functional G over the finite-dimensional phase-space (p, q) : suppose $\{G, H\} = 0$ is satisfied, then we know from previous arguments that we are dealing with a symmetry over the time, whose infinitesimal generator is the Hamiltonian. The Hamiltonian itself is left invariant ($\delta H = 0$) under the following transformations

$$\delta q^i = \{q^i, G\} = \frac{\partial G}{\partial p_i} \quad \delta p_i = \{p_i, G\} = -\frac{\partial G}{\partial q^i}. \quad (2.19)$$

The difference from the Lagrangian formalism is that here constants of motion generate symmetries, and not viceversa, and moreover we have explicit expression for the variations ($\delta q, \delta p$). This is the inverse Noether theorem. The algebra spanned by these symmetries is obtained by using the Jacobi identity

$$\{\{G_\alpha, G_\beta\}, H\} + \{\{H, G_\alpha\}, G_\beta\} + \{\{G_\beta, H\}, G_\alpha\} = 0, \quad (2.20)$$

provided $\{G_\alpha\}$ is a complete set of generators. In fact, we have the identity

$$\{G_\alpha, G_\beta\} = P_{\alpha\beta} = -P_{\beta\alpha}, \quad (2.21)$$

with $P_{\alpha\beta}$ an antisymmetric polynomial in the constants of motion G_α

$$P_{\alpha\beta} = c_{\alpha\beta} + f_{\alpha\beta}{}^\gamma G_\gamma + \frac{1}{2}g_{\alpha\beta}{}^{\gamma\delta} G_\gamma G_\delta + \dots \quad (2.22)$$

All the coefficients of the Taylor expansion are constants, and therefore we have vanishing PB's with H at all orders of the expansion. $c_{\alpha\beta}$ is called the *central charge* of the expansion. Calling the generic infinitesimal variations (2.19) δ_α , for any functional F over the phase-space

$$\delta_\alpha F = \{F, G_\alpha\}, \quad (2.23)$$

the commutation relations yield

$$[\delta_\alpha, \delta_\beta]F = \{\{F, G_\beta\}, G_\alpha\} - \{\{F, G_\alpha\}, G_\beta\}. \quad (2.24)$$

Due to the Jacobi identity, we can write

$$\begin{aligned} [\delta_\alpha, \delta_\beta]F &= \{F, \{G_\alpha, G_\beta\}\} = C_{\alpha\beta}{}^\gamma \delta_\gamma F \\ &= \frac{\partial P_{\alpha\beta}}{\partial G_\gamma} = f_{\alpha\beta}{}^\gamma + g_{\alpha\beta}{}^{\gamma\delta} G_\delta + \dots \end{aligned} \quad (2.25)$$

where we made use of the antisymmetry of the coefficients of the Taylor expansion. It is clear that in order to have (2.25) fulfilled in the case of local symmetries, the central charge has to vanish identically, $c_{\alpha\beta} = 0, \forall \alpha, \beta$, generating a *first class* constraint [HT]. We shall only deal with closed algebras, so only the linear term in the Taylor expansion will be considered, whereas for more general algebras, such open ones, one can consider an arbitrary number of powers in G .

Algebraically speaking, whenever we have a vector space, with an antisymmetric product and Jacobi identity fulfilled, we are in the presence of a Lie algebra (For a more formal definition see Appendix A). So, whenever we have a local symmetry $\dot{G}_\alpha = \{G_\alpha, H\} = 0$ with time-dependent parameters, the generators $\{G_\alpha\}$ turn into a constraint

$$G_\alpha(q, p) = 0. \quad (2.26)$$

Defining on the hypersurface, spun by the generators of the symmetry, a set of equivalence classes, the constraints commute with the Hamiltonian through the corresponding PB over these classes. This is a necessary condition only *on-shell* (on the physical hypersurface), whereas *off-shell* this is no longer true. When

we deal with BRST algebra, we will introduce the concept of cohomology and the means to select the correct physical subspace, thanks to the Kugo-Ojima criterion [NO90]. To anticipate such a criterion, in the light of the canonical formalism, we can assert that the condition for selecting only the physical subspace is achieved by the two simultaneous conditions

$$\begin{cases} \{G_\alpha, G_\beta\} = P_{\alpha\beta}(G) = 0 \\ \{G_\alpha, H\} = Z_\alpha(G) = 0 \end{cases} \Leftrightarrow G = 0. \quad (2.27)$$

2.2 Covariant formalism in Abelian gauge-theories: Maxwell theory

As a further example of constrained theories, we now focus on gauge theories. First we will start with an Abelian case and then we will treat non-Abelian theories.

The electromagnetic theory invented by Maxwell in 1864 can be regarded as the prototype of Abelian gauge theories. It contains two symmetries: Lorentz invariance and gauge symmetry. The first was recognised only after the discovery of special relativity by Einstein in 1905, whereas the second one needed to wait for quantum mechanics and general relativity to be fully appreciated. The work of Yang and Mills in mid 50's shed light to more insights of this symmetry. The quantisation of electromagnetism led Dirac first, and then Feynman, through the path integral representation, to set up the bases of quantum electrodynamics (QED). It is therefore not surprising to use this theory of commuting c -numbers as a starting framework to study and appreciate all the subtleties of gauge theories.

The covariant formulation of electromagnetism starts by considering the electric field E and the magnetic field B not as isolated fields, but put together into a four-dimensional antisymmetric tensor $F_{\mu\nu} = -F_{\nu\mu}$, the *electromagnetic tensor* or *field-strength tensor*, such that $E_i = F_{i0}$ and $B_i = \epsilon_{ijk}F_{jk}$. The Lagrangian density of the theory is $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$, and the Maxwell equations of motion take the compact form

$$\begin{aligned} \partial_\mu(\tilde{F}^{\mu\nu}) &= 0 \\ \partial_\mu F^{\mu\nu} &= -j^\nu, \end{aligned} \quad (2.28)$$

with $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ the dual of $F^{\mu\nu}$ and $\epsilon^{\mu\nu\rho\sigma}$ is the four dimensional total antisymmetric tensor. Current conservation appears as a natural compatibility condition $\partial_\mu j^\mu = 0$. To make Lorentz invariance appear more naturally, let us

transform the first-order equations of motion into equivalent second-order ones: this can be achieved by introducing a four-potential A_μ , the photon field, such that $\vec{E} = -\vec{\nabla}A^0 - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \text{curl } \vec{A}$. Being Maxwell theory an Abelian theory, the photon field A_μ , called also the *gauge* or *vector boson potential*, does not self-couple. Therefore, $F_{\mu\nu}$ can be covariantly expressed as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (2.29)$$

and the equations of motion become written in terms of A_μ only as

$$\square A_\mu - \partial_\mu \partial^\nu A_\nu = -j_\mu, \quad (2.30)$$

with \square being the D'Alembertian operator $\square = \partial^\mu g_{\mu\nu} \partial^\nu$ and $g_{\mu\nu}$ being the metric. In Euclidian space, the metric is trivial, such that its signature simply reads $g_{\mu\nu} = (1, 1, 1, 1)$. In Maxwell theory there are two first-class constraints: first, in the Lagrangian there is no time-derivative of the gauge potential A_0 . Therefore, its conjugate momentum is vanishing

$$\pi^0 = \frac{\partial L}{\partial \dot{A}_0} = 0. \quad (2.31)$$

The momentum field π^i conjugate to A_i is the electric field and it has the equal time PB with A_i as follows

$$\{A_i(x), E_j(y)\} = \delta_{ij} \delta(x - y), \quad x^0 = y^0. \quad (2.32)$$

The fields E_j are not all independent, but are also subject to the Gauss law constraint

$$\partial_i E_i = 0. \quad (2.33)$$

These two constraints have also mutual vanishing PB.

The most important feature of this Abelian theory is the invariance of the Lagrangian under a local redefinition of A_μ , called *gauge transformation*, defined as

$$\begin{aligned} A_\mu(x) &\rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) \\ &\equiv \Lambda A_\mu, \end{aligned} \quad (2.34)$$

with $\Lambda(x)$ any smooth function defined on \mathbb{R}^{42} . This local invariance was fully discovered in the case of QED, when the electromagnetic field is coupled to

²The global gauge-invariance of \mathcal{L} is straightforward, and in a more accurate language is called gauge invariance of the *first kind*, whereas the local invariance is called of the *second kind*.

electrons, through the Dirac Lagrangian. Though, in the course of this Thesis we will only consider pure gauge theories, where the only degrees of freedom are the gauge potentials, and therefore we will not bother about the presence of physical fermions³.

Having defined our classical covariant electromagnetic theory, we would wish now to quantise it, keeping the covariance manifest. This quantisation procedure presents three fundamental problems to overcome [NO90]

1. the incompatibility between classical equations and quantum principles;
2. appearance of an indefinite metric;
3. subsidiary conditions to select the physical subspace.

1.: this incompatibility derives from the fact that if one wishes to quantise Maxwell theory, the equations of motion have to be modified if A_μ is a nontrivial field.

2.: since A_μ is supposed to be a massless field, it is impossible to avoid negative norm without violating manifest covariance. This has to do with the compactness of the little group of the Poincaré group. One can show [NO90] the impossibility of covariantly quantizing A_μ in the positive-metric Hilbert space \mathcal{H} . To solve this problem, Gupta [Gup50] and Bleuler [Ble50] proposed a formalism in the case of indefinite norm. This method will be then translated in the language of cohomological BRST, when we will deal with BRST quantisation of non-Abelian gauge theory in the course of the next chapter.

3.: if an indefinite-metric is necessary to quantise Maxwell theory, and a positive-definite metric is indispensable for the probabilistic interpretation of quantum states, some subsidiary conditions have to be imposed on the subspace, called “physical”, of the entire configuration space with indefinite metric. It is possible to select such a physical subspace, independently of the time and with positive norms.

The fundamental postulate for these three conditions to be valid is the separability of the Hilbert physical subspace, isomorphic to the space of square-integrable functions, called L^2 -space, i.e. functions rapidly decrease at infinity⁴. As ex-

³We will nonetheless encounter other types of fermions, which will turn out to be unphysical, called *ghost* fields, generated by the quantisation procedure *a la* Faddeev-Popov in the case of Y-M theory. These unphysical degrees of freedom will play a decisive role in Topological Field Theory (TFT), Supersymmetry (SUSY) and BRST.

⁴We will moreover see that, in the case of non-Abelian gauge transformations, we will have to further restrict the functional space to be Sobolev [STSF82, Ada, Maz], where even the first derivative has to be taken square-integrable.

plained in [NO90], without this postulate, quantum field theory could not be properly formulated.

2.2.1 Dirac quantisation method

We wish here to briefly present a very useful quantisation procedure in the presence of first and second-class constraints, developed by Dirac [NO90]. As we saw in (2.4) and (2.12), these equations are solvable but only partially, leaving some variables undefined, our constraints. They can be either of first-class, if the PB between the constraint ϕ_a and any quantity A can be expressed entirely in terms of linear combinations of ϕ_a 's, otherwise they are second-class. In quantum field theory, it is better if we can avoid dealing with first-class constraints: being always possible to find a quantity χ , such that $(\phi'_a, \chi) \neq 0$, this can be interpreted as an annihilation condition for certain state vectors (i.e. $\phi'|f\rangle = 0$) and consequently inconsistent with the existence of the quantum vacuum. As we previously saw in the language of the Hamiltonian formalism, first-class constraints are the generators of gauge transformations, and therefore they become second-class by adding gauge-fixing constraints. In the Lagrangian formalism, if we gauge-fix, losing the local gauge-invariance, then no first-class constraint remains left, and we will only deal with second-class constraints. To manage these second-class constraints into the quantisation procedure, Dirac proposed a way to deal with them: he introduced a generalisation of the Poisson Brackets, named after him as Dirac Brackets, defined as follows

$$(A, B)_D \equiv (A, B)_P - (A, \phi_a)_P (\mathcal{A}^{-1})_{ab} (\phi_b, B)_P, \quad (2.35)$$

where \mathcal{A} is the matrix of Poisson Brackets among the constraints ϕ_i , A and B are two general function. These new brackets have the same analytic and algebraic properties of PB, such as antisymmetry, Leibniz rule and Jacobi identity, but with the additional property that $(\phi_a, C)_D = 0$, for any C . Quantisation is carried out by replacing the Dirac Brackets by $-i$ times a commutator

$$(A, B)_D \rightarrow -i[A, B]. \quad (2.36)$$

The Dirac method therefore does not bother whether or not there is constraint in the theory, and for this reason it is a very useful method in quantizing theories with constraints.

2.2.2 Covariant Quantum Theory of Maxwell Theory

Following the works of Gupta [Gup50] and then Bleuler [Ble50], we can now define the covariant operator formalism of the electromagnetic field: by intro-

ducing an additional field b , called the Nakanishi-Lautrup field, we can quantise the theory by means of a covariant gauge-fixing. This term, though spoiling the local gauge invariance, allows to invert the differential operator $g_{\mu\nu}\square - \partial_\mu\partial_\nu$, which governs the two-point function of A_μ , otherwise non invertible because it is a projector operator. The gauge-fixed Lagrangian density becomes

$$\mathcal{L}_{\text{gf}} = \mathcal{L}_0 + b\partial_\mu A^\mu + \frac{1}{2}\alpha b^2, \quad (2.37)$$

where α is a positive real number, called *gauge parameter*. According to its particular value we can define different covariant gauges, for instance $\alpha = 1$ is the Feynman (or Fermi) gauge, $\alpha = 0$ is the Landau gauge, and for generic positive α we have the Lorentz gauge. The field equations then read

$$\begin{aligned} \partial_\mu F^{\mu\nu} - \partial^\nu b &= -j^\nu \\ \partial^\mu A_\mu + \alpha b &= 0, \end{aligned} \quad (2.38)$$

also called the quantum Maxwell equations. What is remarkable in this framework is that, by taking a total divergence of the first equation in (2.38), due to the anti-symmetry of the field-strength tensor and to the conservation of the Noether current, we obtain an additional condition

$$\square b = 0, \quad (2.39)$$

implying b is massless, despite the fact that A_μ is an interacting field. Eq. (2.39) is a central feature of Abelian gauge-theory in covariant gauges, and it is exactly the generalisation of this condition which will establish the appearance of the Ojima criterion in non-Abelian gauge theory for selecting the appropriate Hilbert physical subspace.

The quantisation procedure in canonical formalism elevates A_μ to a canonical variable: its canonical momentum conjugate is

$$\begin{aligned} \pi^\mu &= \frac{\partial \mathcal{L}_{\text{gf}}}{\partial \dot{A}_\mu} = F^{0\mu} + g^{0\mu}b \\ &= g^{0\mu}b, \end{aligned} \quad (2.40)$$

where the last line is due to the antisymmetry of $F^{\mu\nu}$. The canonical commutation relation at zero-time will be $[\pi^\mu(x), A_\nu(y)]_0 = i\delta_\nu^\mu\delta(\vec{x} - \vec{y})$ and otherwise vanishing. As a consequence, \dot{A}_k ($k = 1, 2, 3$) and b are directly expressible in terms of π^μ , whereas A_0 and \dot{b} are not. As mentioned before, the appearance of the b -field into covariant Maxwell theory sets up the basis for the Gupta-Bleuler condition: it can be derived by the observation that b , satisfying the

free-field equation (2.39), can be represented through a conserved local current, from which the following integral representation of b follows

$$b(y) = \int d\vec{z} [\partial_0^z D(z-y) \cdot b(z) - D(y-z) \partial_0 b(z)]. \quad (2.41)$$

By taking the various commutation relations at equal time (ETCR) with respect to the other fields of the theory and making use of (2.41), one observes that

$$[\Phi(x), b(y)] = -i\mathcal{L}(\Phi)^x D(x-y), \quad (2.42)$$

where Φ is any local quantity and \mathcal{L} is a differential operator. Thus, b can be regarded as a generator of local gauge transformations. Splitting the contribution of negative/positive frequency part in b , the Gupta-Bleuler condition yields

$$b^{(+)}(x)|f\rangle = 0 \quad b^{(-)} = (b^{(+)})^\dagger, \quad (2.43)$$

implying that the physical subspace $\mathcal{V}_{\text{phys}}$ (time-independent and Poincaré invariant) of the total Hilbert space is constituted by the totality of states $|f\rangle$ satisfying (2.43).

2.3 Non-Abelian gauge theories: a survey into Yang-Mills theory

Once the canonical formalism for the electromagnetic force is being translated in the quantum language, elevating all the canonical variables to quantum operators by means of a suitable quantisation procedure, the theory of constrained systems with gauge symmetry that interests us the most is Yang-Mills (Y-M) theory. Originally proposed by Yang and Mills in 1954 [YM54] in the context of isospin structures for $SU(2)$ theories, after some initial reluctance in the physics community, it soon became the fundamental model to establish a connection between electromagnetism and the weak force as a unified theory. It then circumvented parton models, and today it is believed the best candidate for the description of the strong force, especially after the brilliant and successful work in the 70's and 80's devoted to the proof of its renormalisation at all orders [tHV72], asymptotic freedom [GW73, Pol74] and to the understanding of non-perturbative phenomena such quark confinement [Wil74]. The reason why gauge theories are so important in particle physics is due to the fact that the 4 fundamental forces existing in nature are believed to be mediated by exchanging particles, called vector bosons. These integer-spin particles are subjected to dynamics described by gauge theories, being either Abelian, as in the case of

photons, or non-Abelian, as in the case of W^\pm or Z bosons, as well gluons and gravitons.

In this Thesis we will focus only on Y-M theory as the preferred model for describing quantum chromo-dynamics (QCD), which can be regarded as the generalisation of QED: while in the latter the underlying Lie (gauge) group is the Abelian Lie group $U(1)$, in the former the Lie group is a non-Abelian gauge group, the special unitary $SU(N)$. This is a crucial difference, as we will see, because now the gluons, matrix-valued vector bosons of the theory, are self-interacting, differently from the case of photons. This will be clear once we will show the Y-M field-strength tensor. To begin with, let us introduce some concepts of Lie group and algebra theory (for more details we remind the reader to see Appendix A): consider the semi-simple⁵, compact Lie group $SU(N)$: given a complete set of anti-Hermitian generators X_a for the algebra of $SU(N)$, $su(N)$, a group element of $SU(N)$ can be written through the local exponential (analytic) map as

$$g = \exp \left(\sum_{a=1}^N \theta^a(x) X_a \right), \quad (2.44)$$

where the various X_a satisfy the following commutation and anti-commutation relations

$$[X_a, X_b] = f_{ab}^c X_c \quad \{X_a, X_b\} = -\frac{1}{N_c} \delta_{ab} - i d_{ab}^c X_c, \quad (2.45)$$

with $f_{ab}^c = -f_{ba}^c$ and $d_{ab}^c = d_{ba}^c$. The local functions $\theta^a(x)$ are taken to be smooth over the manifold under consideration. The normalisation condition depends on the specific representation⁶ we use for the Lie group: for a generic representation ρ we obtain

$$\text{Tr}(X_a X_b) = -\rho \delta_{ab}. \quad (2.46)$$

In the fundamental representation $\rho = 1/2$ and in the adjoint $\rho = N_c$. The dimension of this algebra is relevant only if gauge fields are coupled with fermions. Each gluon field is then a matrix-valued Lorentz vector⁷, defined in terms of

⁵Semi-simplicity is equivalent to that for the Killing form [NO90], entering the Lagrangian density as $\mathcal{L}_{YM} = -\frac{1}{4} K_{ab} F_{\mu\nu}^a F^{\mu\nu b}$, and defined as $K_{ab} \equiv -\text{Tr}(\text{ad}(T_a) \text{ad}(T_b)) = -f_{ad}^c f_{bc}^d$ to be non-degenerate. The Killing form can be diagonalised as $K_{ab} = \delta_{ab}$, and w.r.t. this diagonalizing basis, upper and lower indexes in the structure constants do not make any difference any more, as long as we are concerned with compact Lie groups.

⁶We remind that a representation ρ of a Lie group G on a vector space V is a homeomorphism of Lie groups $\rho : G \rightarrow \text{Aut}V$.

⁷According to [CR, STSF82, Ada, Maz], the gauge fields A_μ have to be matrix-valued functions belonging to the space L^2 , the space of square-integrable functions, but the gauge transformations rather to the more restrictive W_1^2 , the space of Sobolev norm. Being $f \in W_1^2$, then $|f|^2 \equiv \int \frac{d^d x}{x^2} |f|^2 + \int d^d x |\partial f|^2 < \infty$.

the algebra generators through the adjoint map as

$$A_\mu(x) = A_\mu^a(x)X_a. \quad (2.47)$$

In Appendix B we shall briefly explain the geometric interpretation of the gauge field as the component of a Lie algebra-valued differential 1-form ω , called the connection form over a principal bundle, which determines the profound relation between Y-M theory and the theory of principal bundles [Nab, CR]. This algebraic and geometric structure appears in the four-dimensional free Euclidian action of YM theory, also called in a more geometric language the Y-M functional, as

$$\begin{aligned} S_{\text{YM}} &= \int_M d^4x \mathcal{L}(x) = \frac{1}{2} \text{Tr} \int_M d^4x F_{\mu\nu} F^{\mu\nu} \\ &= -\frac{1}{4} \int_M d^4x F_{\mu\nu}^a F_a^{\mu\nu}, \end{aligned} \quad (2.48)$$

where the trace over the gauge group ensures S to be a scalar quantity. The manifold M , is generally supposed to be oriented, compact, without boundary and endowed with a metric (in our case we have the flat, trivial Euclidian metric $\delta_{\mu\nu}$). The field-strength tensor $F_{\mu\nu}$, component of a differential 2-form Ω , is formally generated from ω , (and therefore from the potential A_μ) by the Maurer-Cartan equation

$$\begin{aligned} \Omega &= d\omega + \frac{1}{2}[\omega, \omega] \\ &= \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu. \end{aligned} \quad (2.49)$$

It can be geometrically interpreted as the Riemannian curvature in the principal bundle (also denoted as F_A), where the parallel transport along a curve is not commutative. In Maxwell theory, being the Lie group Abelian, we have $[A, A] = 0$ and this is the reason why the curvature is simply $F_A = dA$. According to the particular representation of $SU(N)$ we choose, $F_{\mu\nu}$ can assume different expressions: among the various irreducible representations, usually the fundamental and the adjoint are preferred. In the course of this Thesis we will adopt generally the adjoint representation, unless otherwise specified ⁸, In this

⁸The difference between these two commonly used representations lies in the fact that in the fundamental representation we use standard $N \times N$ matrices, where N is the dimensionality of the Lie group and these matrices form a complete set of generators of the algebra. In the case of the adjoint representation, the matrices representing the basis elements are formed from the structure constants f^{abc} , defined through the commutation relations among the generators (2.45). Therefore, each matrix has now a dimensionality $N^2 - 1 \times N^2 - 1$, and this representation provides an overcomplete set of generators.

representation, the covariant derivative is written as $\mathcal{D}_\mu = \partial_\mu + g[A_\mu, \cdot]$, with g the coupling constant of the theory; consequently, the field-strength tensor becomes

$$\begin{aligned} F_{\mu\nu} &= \partial_{[\mu} A_{\nu]} + g[A_\mu, A_\nu] \\ &= \frac{1}{g}[\mathcal{D}_\mu, \mathcal{D}_\nu] \\ &= \text{ad}(F_{\mu\nu}) = F_{\mu\nu}^a X_a. \end{aligned} \quad (2.50)$$

In electromagnetism, the field-strength satisfies the relation $\mathcal{P}(\partial_\rho F^{\mu\nu}) = 0$, where \mathcal{P} stands for cyclic permutation of Lorentz indices. In Y-M, however, this identity generalises to

$$\begin{aligned} 0 &= [\mathcal{D}_\mu, F_{\nu\rho}] + [\mathcal{D}_\rho, F_{\mu\nu}] + [\mathcal{D}_\nu, F_{\rho\mu}] \\ &= \frac{1}{g}\mathcal{P}[\mathcal{D}_\mu, [\mathcal{D}_\nu, \mathcal{D}_\rho]], \end{aligned} \quad (2.51)$$

as a consequence of the Jacobi's identity and (2.50). This is the analogue of the homogeneous Maxwell equations. It must be stressed that if $F_{\mu\nu}$ and A_ρ satisfy (2.51), $F_{\mu\nu}$ is not necessarily the strength tensor associated with A_ρ . This implies that, differently from the Abelian case, here $F_{\mu\nu}$ does not determine uniquely all gauge-invariant quantities.

2.3.1 Local gauge invariance

Requiring the action (2.48) to be invariant under a local gauge transformation (being the pull-back of the connection form ω through a local section σ over the principal bundle), we have a prescription for the transformation law of the gauge field as

$${}^g A_\mu = g^\dagger A_\mu g + \frac{1}{g} g^\dagger \partial_\mu g, \quad (2.52)$$

where $g = g(x)$ is a nonsingular local group element ⁹ of $SU(N)$ (cf. (2.44)). Given an infinitesimal group element $g(x) \underset{\theta \rightarrow 0}{\simeq} \mathbf{1} + \theta^a(x) X_a + o(\theta^2)$, the infinitesimal version of (2.52), is reminiscent of the canonical formalism of symmetries $\delta_\alpha q^i = \{q^i, G_\alpha\}_{\text{Poisson}}$, where G_α are the generators of the symmetry (cf. 2.27), is

$$\delta_g A_\mu^a = \partial_\mu \delta^{ab} \theta^b - g f^{abc} \theta^b A_\mu^c \equiv \mathcal{D}_\mu^{ab} \theta^b. \quad (2.53)$$

This invariance requirement is probably the most important property of Y-M theory and of any other local gauge theory. Comparing with electromagnetism,

⁹In a non rigorous language, we indicate $g(x)$ as the gauge transformation, but actually it is only a group element. The proper gauge transformation we refer to is (2.52).

we remember that the two fundamental physical fields of the theory, the magnetic field B^a and the electric field E^a can be arbitrarily defined through the introduction of an unphysical vector-gauge potential A_μ . If we demand the system not to change under a rotation in gauge space, then B and E are left unchanged by a suitable redefinition of the vector potential. The Lagrangian, which encodes all the physical information of the system, must not change as well: to be precise, under the aforementioned redefinition of A_μ by (2.52), \mathcal{L} changes only by total derivative modulo boundary terms which vanish if the fields vanish sufficiently fast at spatial infinity. The field-strength tensor transforms covariantly under the action of g

$${}^g F_{\mu\nu} = g^\dagger F_{\mu\nu} g. \quad (2.54)$$

An other interesting property of the field-strength tensor is that if it is vanishing in a neighbourhood of a point (flat connections), then the gauge field is a pure gauge

$$F_{\mu\nu} = 0 \Leftrightarrow \exists g(x) : A_\mu(x) = g^\dagger(x) \partial_\mu g(x). \quad (2.55)$$

It follows that if A_μ is a pure gauge, then we have vanishing curvature. In topological field theory, flat connections are studied in great details in BF theories [BBRT91]. The field equations become

$$L_a^\mu = \partial_\nu F_a^{\mu\nu} - g f_{ab}^c A_\nu^b F_c^{\nu\mu} = 0, \quad (2.56)$$

and are covariant, in the sense that if A_μ is a solution, so is ${}^g A_\mu$. Because of the invariance of the action under (2.52), the E-L equations are not independent¹⁰ but rather fulfill non-Abelian Bianchi identities

$$\begin{cases} d_A F = 0 \\ *d_A *F = 0, \end{cases} \quad (2.57)$$

with $*$ the Hodge operator¹¹ and d_A the covariant derivative [Nab] (see Appendix B). As usual, we introduce the momenta conjugate

$$\Pi_a^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu^a} = F_a^{0\mu}, \quad (2.58)$$

¹⁰It must be stressed here that the canonical energy momentum tensor $\Theta^{\mu\nu} = -2\text{Tr}(F^{\mu\rho}\partial^\nu A_\rho - \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma})$ is not gauge invariant. By subtracting a term $\Delta\Theta = -2\partial_\rho\text{Tr}(F^{\mu\rho}A^\nu)$, we can restore the gauge invariance of $\tilde{\Theta} \equiv \Theta - \Delta\Theta$. This gauge-invariant energy momentum tensor is equal to $\tilde{\Theta} = F_a^{\mu\rho}F_\rho^{a\nu} - \frac{1}{4}g^{\mu\nu}F^{a\mu\sigma}F^{a\sigma\rho}$ called the Beltrami tensor [IZ].

¹¹The action of $*$ on $F_{\mu\nu}$ determines its dual $*F_{\mu\nu} = \tilde{F}_{\mu\nu}$, fundamental to study instantons and solitons. For instance, a more geometric expression for the Y-M action is $S = \|F_{\text{YM}}\|^2 = -\frac{1}{4}\int_M F \wedge *F$ [Nab]. The existence of a norm $\|\cdot\|$ is guaranteed by the fact that we have a metric on the space of gauge configurations \mathcal{A} , denoted by \mathcal{A} . In the next chapter we will study the analytic and algebraic properties of this functional space.

from which we immediately see, as in the case of the Abelian Maxwell theory, that Π_a^0 is a primary constraint, and A_0^a is its canonical conjugated variable. Though, more insight can be obtained from the equations of motion directly: consider for instance the action (2.48) re-expressed in terms of the Y-M electric and magnetic fields E and B as [IZ]

$$\begin{aligned} S &= \frac{1}{2} \text{Tr} \int d^4x \mathcal{L}(x) \\ &= \frac{1}{2} \text{Tr} \int d^4x \left[\partial^0 \mathbf{A} \cdot \mathbf{E} + \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) - A^0 (\nabla \cdot \mathbf{E} + [\mathbf{A}, \mathbf{E}]) \right], \end{aligned} \quad (2.59)$$

where $\frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2)$ is the free energy. The canonical variables p and q become \mathbf{A} and \mathbf{E} , whereas A^0 plays the role of a Lagrange multiplier for the constraint identified with $\nabla \cdot \mathbf{E} + [\mathbf{A}, \mathbf{E}] = \mathcal{D}[\mathbf{A}]\mathbf{E}$. This constraint comes from the equations of motion (2.57) by setting the Lorentz index $\nu = 0$. In the Hamiltonian language we would say that, given a constraining manifold, whose constraints having vanishing PB with H or among themselves, they determine equivalent pairs of canonical variables if

$$\frac{d\mathbf{A}}{du} = \{\Gamma, \mathbf{A}\} \quad \frac{d\mathbf{E}}{du} = \{\Gamma, \mathbf{E}\} \quad (2.60)$$

$$\Gamma = \nabla \cdot \mathbf{E} + [\mathbf{A}, \mathbf{E}] = 0.$$

Even though this has been shown in a non-covariant way, it is interesting to see how the Y-M electric field and the gauge field play a decisive part in the canonical formalism of constraints. The next step is then to gauge-fix the action: in the Hamiltonian formalism, this is achieved by selecting a gauge and then taking the PB with respect to (2.60) one constructs the appropriate path integral. In the Lagrangian formalism, we consider the method proposed by Faddeev and Popov [DN67]: we will dedicate the entire next chapter in analyzing this procedure, the appearance of the ghost fields into the Y-M path integral and of course we will give extensive details on the Gribov problem [Gri78].

Euclidian solutions to the classical equations of motion:

Instantons

In a four dimensional manifold M we can define the quantity

$$n = -\frac{1}{64\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a = -\frac{1}{32\pi^2} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu} \quad (2.61)$$

the *instanton winding number*. This quantity is called ‘‘topological’’ because, differently from the Y-M functional (2.48), it does not depend on the metric.

Writing (2.61) as $n = -\text{Tr} \int d^4x F \wedge F$ and comparing with (2.48), we notice that there is no Hodge operator $*$: it is this operator which endows the Y-M functional a metric and not n . An other important property of n is that it does not depend on A , but only on the algebraic structure of the manifold and of the principal bundle. The importance of working on a four dimensional manifold lies in the fact that applying the Hodge operator to F , which is a differential 2-form, gives an other differential 2-form. In particular, we have these two particular case

$$F_A = \pm * F_A, \quad (2.62)$$

respectively called *self-dual* and *anti self-dual* curvatures. Moreover, F_A can be decomposed into its dual and anti self-dual part as $F_A = F_A^{\text{sd}} + F_A^{\text{asd}}$. It is possible to show that

$$\text{Tr} \int d^4x F \wedge *F \geq \left| \text{Tr} \int d^4x F \wedge F \right| \quad (2.63)$$

and that (2.63) is minimised when

$$\begin{aligned} \text{Tr} \int d^4x F_A \wedge *F_A = \text{Tr} \int d^4x F_A \wedge F_A & \quad F_A^{\text{asd}} = 0 \\ \text{Tr} \int d^4x F_A \wedge *F_A = -\text{Tr} \int d^4x F_A \wedge F_A & \quad F_A^{\text{sd}} = 0. \end{aligned} \quad (2.64)$$

The first case implies that $F_A = F_A^{\text{sd}}$, called *instanton*, whereas the second case $F_A = F_A^{\text{asd}}$, called *anti-instanton*. These two minimizing solutions of the action are also solutions of the Y-M equations of motion (2.57). The solution proposed by Belavin, Polyakov, Schwartz and Tyupkin [BPST75] for $n = \pm 1$ reads

$$A_\mu = \frac{x^2}{x^2 + \lambda^2} [\partial_\mu g(x)] g^\dagger(x), \quad (2.65)$$

with $g(x) = \frac{x_0 \pm i\sigma \cdot \mathbf{x}}{(x^2)^{1/2}}$.

Path integrals in Y-M theory

We present here a very powerful tool in theoretical physics, known as the path integral formalism (in Euclidean space), which we will largely adopt in the next chapters. For a more detailed description, consult for example [IZ, Riv]. The conceptualisation of path integrals in physics is mainly due to three scientists: Dirac, Feynman and Kac. In the 30's Dirac proposed the idea and Kac and Feynman established the mathematical basis to provide us with a unified view of quantum mechanics, field theory and statistical models. The basic idea behind the path integral approach to QFT is rather simple: at the quantum mechanical level, instead of pretending to solve the Schrödinger equation at general times t , one may first attempt to solve the easier problem at infinitesimal time δt . The time-evolution operator, decomposed into its potential and kinetic part, is divided in N discrete time intervals $\delta t = t/N$. The exponential of the time-operator can then be factorised into N parts, such that the eigenstates of each component are known independently. One then considers the amplitude of the time-evolution operator, split into N time intervals, between initial and final state: inserting the completeness relation of the eigenstates of the position and momentum operator, the potential and kinetic operators thus act (to the left and to the right respectively) on the corresponding eigenstates. In this way, the matrix element of the time-evolution operator has been expressed as $2N - 1$ dimensional integrals over eigenvalues. At each time step $t_n = n\delta t$, $n = 1, \dots, N$ we are integrating over coordinates parametrising the classical phase space (q_n, p_n) . Therefore, the integral kernel (the propagator) of the time evolution operator could be expressed as a sum over all possible paths connecting two points, q' and q'' with a weight factor provided by the exponential of the action. Mathematicians, such as Wiener, Kac himself, Cameron and Martin, dealing with stochastic processes already knew this approach, as far as the analytic continuation is concerned. Though, mathematicians were more reluctant to accept straightforwardly such path representation, because of its intrinsic and pathological difficulties, mainly due to the highly non-trivial definition of an infinite functional measure (and also of the infinite sum of phases). Historically, a semi-classical approach in solving this infinity was provided by the WKB method, in which the solution of the Feynman kernel is based on the fact that the harmonic oscillator (the quadratic Lagrangian) is exactly solvable and its solution is only determined by

the classical path and not by the summation over all the paths. Nonetheless, a correct mathematical interpretation and definition of the functional measure is still lacking: there have been many attempts [Unz86, Fuj79, Or196, Or104, Bae94], and references therein, to give a definite and rigorous definition of such a quantity, trying to find a relation with Lebesgue theory, measure theory and complex analysis in Hilbert spaces [Ada, Maz]. In [Unz86] it has been pointed out that a possible correct definition for a functional flat measure of bosonic fields could have the form of $[\mathcal{D}\Phi] = \prod_x \left[\det \left(\frac{\delta^2 \mathcal{L}}{\delta(\partial_0 \Phi) \delta(\partial_0 \Phi)} \right) \right]^{\frac{1}{2}} [d\Phi]$, whereas for fermions $[\mathcal{D}\Psi] = \prod_x \left[\det \left(\frac{\delta^2 \mathcal{L}}{\delta(\partial_0 \Psi) \delta(\partial_0 \Psi)} \right) \right]^{-\frac{1}{2}} [d\Psi]$ ¹. In the case of curved spaces, we may replace the former functional measures by $[d\Phi] = \prod_x (g^{00})^{1/2} (g^{\mu\nu})^{1/4} d\Phi$, with $g^{\mu\nu}$ the Riemannian metric tensor. Even these two objects can produce some problems, especially concerning their regularity, because of the infinity arising from the number of space-times in M . The most common technique to deal with such a regularisation problem is by virtue of the zeta-function [EVZ98], also adopted in the regularisation of functional determinants. The crucial problem is that, in any functional space, finding a converging Cauchy series through which the underlying metric is defined, and consequently the concept of distance between two elements of the space, requires a huge effort of mathematics techniques, not always successful. Therefore, what physicists do, and sometimes even mathematicians, is to postulate the existence of a converging distance, a well-defined measure, and the only property openly required is the translational invariance of it, up to boundary terms [Riv]. Though plagued by all these intrinsic and structural problems, path integrals are very elegant and suggestive and, they are ideally suited to

- implement the symmetries of the theory directly
- calculate correlations functions,
- incorporate constraints simply,
- analyze and explore field topology,
- isolate relevant dynamical variables,
- describe the non-zero temperature regime.

To clarify the notation we will adopt, we define as *functional* $F[\phi]$ of a real classical field $\phi(x)$ a rule that associates a number (generally complex) to each

¹The symbol of “det” stands for a functional determinant: a standard matrix determinant over the entire base-space M .

real configuration $\phi(x)$. Functionals, naively speaking, include as particular examples integrals of functions as $F[\phi] = \int dx f(\phi(x))$. By *functional differentiation* we denote $\delta F[\phi]/\delta\phi(y)$ as the formal limit (assuming the ratio exists) $\delta F[\phi]/\delta\phi(y) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon}(F[\phi'] - F[\phi])$, with $\phi' = \phi(x) + \epsilon\delta(x - y)$. The fundamental quantity in path integrals is the generating functional of the theory, formally defined as the integral of the action over all of the possible constituent fields, taking values over all the possible space-time points

$$\mathcal{Z}[J] \equiv \int_{\mathcal{M}} [d\Phi] e^{-S[\Phi] + \int_M J(x) \cdot \Phi(x)}, \quad (3.1)$$

where \mathcal{M} is the functional configuration space of Φ and M the space-time base space on which Φ is evaluated. As seen in the previous chapter, in the case of the Hamilton principle of least action, the path configurations extremising the path integrals are the solutions of the classical equations of motion. In path integrals, though, we consider all the possible paths represented by the functional measure: these quantum fluctuations have to be imagined as wrapped around the classical flux tube connecting the two boundary points in the configuration space, whose contribution is weighted by the exponential of the action. From $\mathcal{Z}[J]$ one can extract all the possible Green's functions of the theory by taking the appropriate functional derivative (according to the spin-statistics of each field) with respect to the external sources $J(x)$. Unfortunately, these Green's functions are cumbersome quantities to use, and in general the diagrams that constitute a Green's function are disconnected (diagrams of two or more sub-diagrams that are not linked by propagators). Moreover, being interested in *one-particle irreducible* (1PI) diagrams (diagrams that cannot be separated by cutting single propagators), the isolation of connected 1PI diagrams is easily obtained by functional derivatives with respect to the external source $J(x)$ by means of a Legendre transformation of the effective action. To practically evaluate $\mathcal{Z}[J]$ from path integrals we may retain in the exponent the quadratic part of the action, expand the rest in a Schwinger series and apply Wick's theorem: this is valid as long as the coupling constant of the theory is small. If that is not the case, since we are only able to exactly calculate path integrals whose action is Gaussian, we may use different techniques, such as the steepest descent or stationary space methods.

We are now ready to translate the quantisation procedure by means of the covariant operators we investigated in the last chapter into the language of path integrals. This formalism is fundamental in analyzing and studying the topological nature of Y-M theory, which constitutes the main subject of this Thesis. We will first introduce the Faddeev and Popov method to quantise non-Abelian gauge theories in Euclidian space; then we will present in details the

Gribov ambiguity which plagues these theories whenever one attempts to attack their non-perturbative nature. At last, we will concentrate on the functional configuration space \mathcal{A} , paying particular attention to its stratification through the various Gribov regions C_i .

3.1 Faddeev-Popov quantisation of non-Abelian gauge theories

In late 60's, in the attempt to quantise non-Abelian theories, Faddeev and Popov [DN67] proposed an original method based on covariant path integrals. As Feynman noted in [Fey63], in the gravitational field and Yang-Mills theory, diagrams with closed loops depend non-trivially on the longitudinal parts of Green's functions and scattering amplitudes are neither unitary nor transverse. Alongside Feynman, also De Witt [DeW64] proposed a remedy to circumvent this problem. Though, they were not able to give a prescription for arbitrary diagrams. The Faddeev and Popov method was developed exactly to generalise Feynman and DeWitt's arguments. We will follow their work in the light of Euclidian path integrals ², emphasising the role of the functional measure and of the configuration space: these two subjects, alongside the underlying local gauge-invariance of the Y-M action (and of the functional measure) will turn out to be of extreme importance to understand the appearance of the famous Gribov ambiguity [Gri78].

The FP quantisation procedure essentially deals with non-Abelian theories subject to constraints. In fact, as we noted in the previous chapter, the time component of the momentum conjugate $\Pi_\mu^a(x)$ is subjected to a vanishing constraint due to the antisymmetry of the field-strength tensor. This condition is not consistent with the assumed commutation relations. Thus the simple-minded application of the canonical quantisation of non-Abelian gauge theory fails. This difficulty arises as long as we rely on a gauge invariant Lagrangian, such that \mathcal{L}_{YM} remains invariant under an infinitesimal gauge transformation, changing A_μ^a into $A_\mu^a + \mathcal{D}_\mu^{ab}\theta^b$. We previously observed how in the canonical operator formalism, the introduction of a gauge-fixing term corresponds to a constraint that

²We choose the Euclidian metric because we will shortly analyze non-perturbative problems of Y-M, and for this purpose we will adopt lattice gauge theories *a la* Wilson, in which the Euclidian metric, by means of a Wick rotation from Minkowski space, is required to perform proper numerical simulations. This rotation alludes to the fact that a multiplication with the imaginary unit can be interpreted as a $\pi/2$ -rotation in the complex plane. Therefore imaginary time representations of Lagrangian actions are denoted as Euclidean actions, whereas standard (real time) as Minkowski actions.

could eliminate this unnecessary gauge freedom. We would then wish to incorporate in the path integral only those gauge connections A that are unrelated by gauge transformations. This is a more difficult task in a non-Abelian one. To start with the functional quantisation of Y-M theory, we introduce in a flat (Euclidean) metric, the gauge-unfixed Y-M generating functional

$$\mathcal{Z}_{\text{YM}}[J] \equiv \int_{\mathcal{A}} [dA] e^{\frac{1}{2} \text{Tr} \int_M \mathcal{L}_{\text{YM}} + \text{Tr} \int_M J^\mu(x) \cdot A_\mu(x)}. \quad (3.2)$$

Here, the path integral is considered over the configuration space \mathcal{A} spanned by the gauge fields A_μ , defined on a Riemannian manifold M for now left as general as possible. In order to remove the gauge freedom of the Lagrangian density³ and to preserve the manifest covariance, we may then choose the Lorentz condition

$$\partial^\mu A_\mu^a = 0, \quad (3.3)$$

such that the gauge freedom is eliminated because (3.3) is no longer gauge invariant⁴. There are some variety of gauges other than the Lorentz gauge. Among them, the following noncovariant gauges are frequently used: Coulomb (radiation) gauge $\partial^i A_i^a = 0$ ($i = 1, 2, 3$), axial gauge $A_3^a = 0$ and temporal gauge $A_0^a = 0$. Being interested in manifestly covariant gauges, we will adopt the Lorentz gauge henceforth. To incorporate the gauge constraint (3.3) into the generating functional (3.2), we use the standard Lagrange multiplier method well known in analytic dynamics as follows

$$\mathcal{L}_{\text{gf}} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi} (\partial^\mu A_\mu^a)^2. \quad (3.4)$$

Though the gauge-fixing Lagrangian density so constructed manifestly breaks gauge invariance, all the physical quantities extracted from it should of course be gauge-independent and therefore one has the freedom to fix the value of the gauge parameter ξ arbitrarily (for instance $\xi = 1$ corresponds to Feynman gauge, whereas $\xi \rightarrow 0$ to Landau gauge). According to (3.4), the equations of motion will change, and consequently the expression of the momentum conjugate as

$$\Pi_\mu^a = -F_{0\mu}^a - \frac{1}{\xi} \delta_{0\mu} (\partial^\nu A_\nu^a), \quad (3.5)$$

which circumvents the vanishing condition for the time component of Π_μ^a . In such a way, the commutation relations are satisfied and the path integral quantisation is apparently well-defined. Yet, in the case of non-Abelian gauge theories,

³It is worthwhile noting that the source term $J^\mu(x) \cdot A_\mu(x)$ is not gauge invariant.

⁴Of course in (3.2) the gauge invariance is already broken by the presence of the source term. Nonetheless, as we will focus on the sourceless generating functional, it is important to stress this lost gauge invariance by means of the gauge-fixing term.

due to the self-interaction of the gauge field A , we also have, differently from the case of QED, a three-body and a four-body interaction in the Lagrangian (3.4). At one-loop level, the gauge-field contribution to the self-energy part $\Pi_{\mu\nu}^{ab}(q)$ for gauge fields A_μ^a does not satisfy the requirement for gauge invariance $q^\mu \Pi_{\mu\nu}^{ab}(q) = 0$. The reason why such requirement fails is due to the non correct method of extrapolating the physical polarisation for the gauge field even with the gauge-fixing Lagrangian (3.4). Feynman first [Fey63] and then De Witt [DeW64] pointed out this difficulty in the early stages of the development of quantisation of non-Abelian theories. To solve this problem, Faddeev and Popov tried to incorporate in (3.2) an appropriate gauge-fixing condition with the double purpose to eliminate the infinite redundancy of gauge transformed fields affecting the path integral and to guarantee the elimination of the unphysical polarisation states of the gauge field.

As we know well, the Y-M Lagrangian density is gauge invariant by construction, whereas the gauge-fixing and source terms are not. The measure requires special attention: performing an infinitesimal gauge transformation we find

$$\begin{aligned} [d^g A] &= [dA] \det \left(\frac{\delta^g A_\mu^a}{\delta A_\nu^b} \right) \\ &= [dA] \det(\delta^{ab} - f^{abc} \theta^c) \\ &= [dA] (1 + \text{Tr} L + \dots + \det L) \\ &= [dA] (1 + O(\theta^2)), \end{aligned} \tag{3.6}$$

with $L = -f^{abc} \theta^c$. To analyze the problem connected with the infinite gauge measure, it is possible to disregard from (3.2) the source term and dealing just with $\mathcal{Z}_{\text{YM}}[0]$. Furthermore, in the light of covariant gauges, the condition (3.3) can be generalised to the case of a differential operator G^μ

$$G^\mu A_\mu^a(x) = \chi^a(x), \tag{3.7}$$

with $\chi^a(x)$ a matrix-valued local function not depending on gauge transformations. The essential requirement for the gauge condition (3.7) is to be single-valued: this would guarantee the bijectivity of the map between the space of gauge configurations A and the space of gauge connections modulo gauge-transformations, $A/[^g A]$, satisfying (3.7). Basically, it is demanded that the gauge-fixing hypersurface generated by (3.7) should intersect each gauge orbit once and only once. A single-valued gauge-fixing condition is called in the literature *ideal*. Yet, it is not difficult to show that for any field A satisfying (3.7), there are many others, obtained by a gauge transformation of A , satisfying the same condition

$$G^\mu {}^g A_\mu^a(x) = \chi^a(x) \quad \Leftrightarrow \quad G^\mu A_\mu^a(x) = \chi^a(x). \tag{3.8}$$

It seems then the requirement of an ideal gauge-fixing condition immediately fails: in perturbation theory, where the FP method is discussed, this inconvenience is circumvented by considering only infinitesimal fluctuations around the trivial gauge configuration $A_\mu = 0$. In fact, any configuration ${}^g 0 = g^\dagger \partial_\mu g$ satisfying (3.8) lies sufficiently far from the intersection between the hypersurface and the trivial orbit, and therefore the gauge-fixing condition can still be regarded single-valued. However, beyond perturbation theory, when finite gauge transformations are important, single-valued gauge conditions are considered impossible to be found (thus the adjective “ideal”) as long as Y-M theory is evaluated on S^4 , the standard manifold for physical processes [Sin78]. This pathological problem affecting the non-perturbative regime of non-Abelian theories is called the *Gribov ambiguity* and will be detailed in the next section. For the moment, we will only deal with perturbation theory and consequently we are allowed to neglect such obstruction.

Faddeev and Popov proposed a way to take into account the condition (3.7) in the sourceless generating functional $\mathcal{Z}_{\text{YM}}[0]$, generalising the well known formula in standard calculus for an appropriate change of variables,

$$\left| \det \left(\frac{\partial f_i}{\partial x_j} \right) \right|_{\vec{f}=\vec{0}}^{-1} = \int dx_1 \dots dx_n \delta^{(n)}(\vec{f}(\vec{x})) \quad (3.9)$$

where the map is supposed to be bijective, i.e. single-valued and $\det \left(\frac{\partial f_i}{\partial x_j} \right)$ is the Jacobian of the transformation. Being the Jacobian independent of local coordinates, we then write

$$1 = \int dx_1 \dots dx_n \left| \det \left(\frac{\partial f_i}{\partial x_j} \right) \right|_{\vec{f}=\vec{0}} \delta^{(n)}(\vec{f}(\vec{x})). \quad (3.10)$$

The generalisation of (3.10) in the language of non-Abelian gauge theory can be cast in the form

$$1 = \int [dg] \left| \det \left(\frac{\delta F[{}^g A]}{\delta g} \right) \right|_{F=0} \delta(F[{}^g A]) \quad (3.11)$$

with $F[{}^g A] = 0$ the local gauge condition (3.7). Some remarks are necessary here. The functional integration $\int [dg]$ we perform in (3.11) is over the group space and it is called the Haar measure [Nab, NO90, TS04, Nak, Smi02]. This measure is invariant under a gauge transformation: in fact, for any functional of the gauge group g , we can distinguish between left or right invariant measure, according to the kind of group action w.r.t. g_0 we perform on the group element g . The left invariant measure satisfies for instance the following condition

$$\int dg f(g) = \int dg f(g_0^{-1} g) = \int (dg_0 g) f(g) = \int dg f(g), \quad (3.12)$$

and similarly for the right action. In general, left and right invariant measures are not necessarily equal, but it is possible to prove [Mut98] (and references therein) that for compact groups, simple and semi-simple groups, and also finite groups this is the case. According to the parametrisation for $SU(N)$ one adopts, we can give a more practical expression for the Haar measure. For instance, instead of using group elements g , we can perform the integration over the local gauge functions $\theta^a(x)$ appearing through the exponential map connecting the group $SU(N)$ to its algebra $su(N)$. In this case, the Haar measure is proportional to $\sqrt{\det g_{ab}} \prod_{a,x} d\theta^a(x)$ [Nak, Smi02], with g_{ab} being the metric in $SU(N)$ ⁵. The same result can be also achieved by parametrising the group through Euler angles, showing that the $SU(N)$ volume can be written as $2^{N-1/2} \pi^{\frac{(N-1)(N+2)}{2}} \sqrt{N} \prod_{k=1}^{N-1} \frac{1}{k!}$ [TS04].

The second comment is on the absolute value of the Jacobian. Faddeev and Popov did not consider it because they were interested in quantising Y-M theory in the perturbative regime. The reason why in perturbation theory we can remove the absolute value lies in the positive definiteness of the Jacobian in (3.11), also known as the determinant of the Faddeev-Popov (FP) operator $\mathcal{M}[A]$. Under an infinitesimal gauge transformation, in local coordinates, this operator is

$$\begin{aligned} \mathcal{M}_{ab}(x, y) &= \frac{\delta}{\delta \theta^a(y)} F_b[\theta A(x)] \\ &= \frac{\delta}{\delta \theta^a(y)} \left[-\frac{\delta F_b}{\delta A_c^\mu(x)} \mathcal{D}_{cd}^\mu \theta_d(x) \right] \\ &= -\frac{\delta F_b}{\delta A_c^\mu(x)} \mathcal{D}_{ac}^\mu \delta^4(x - y). \end{aligned} \quad (3.13)$$

If we adopt the covariant condition (3.7), we obtain

$$\begin{aligned} \mathcal{M}_{ab}(x, y) &= -\partial_\mu \mathcal{D}_{ab}^\mu \delta^4(x - y) \\ &= -[\square \delta_{ab} - g f_{abc} \partial^\mu A_\mu^c(x)] \delta^4(x - y). \end{aligned} \quad (3.14)$$

It is now clear why, in the case of the trivial orbit and with appropriate boundary conditions, the FP operator has definite sign, because it is just the Laplacian operator multiplied tensorially with the Lie group. It is therefore redundant to keep the absolute value in perturbation theory. We will see, however, that beyond this regime not only is the absolute value necessary, but it is its very presence

⁵Chosen an arbitrary parametrisation for the group element, the metric in the group space can be written as $g_{kl} = \frac{1}{\rho} \text{Tr} \left(\frac{\partial g}{\partial \theta^k} \frac{\partial g^\dagger}{\partial \theta^l} \right)$, with ρ the normalisation in the given representation. Under coordinate transformations $\theta^k = f^k(\theta')$, the metric is covariant, $g_{kl} = g_{mn} \frac{\partial \theta'^m}{\partial \theta^k} \frac{\partial \theta'^n}{\partial \theta^l}$.

which determines the Gribov ambiguity [GKW05]. According to the gauge condition (3.3), the FP operator is not independent of A . The determinant of $\mathcal{M}[A]$ has also an interesting geometric interpretation: in [BV81a, DZ91, DV80], it was shown how it is related to the volume of the gauge orbit. It is worthwhile noting that the generalisation of the linear covariant gauge condition (3.3) to (3.7) does not affect the form of the operator \mathcal{M} . In the case of non-covariant gauges, temporal gauge has a constant FP operator but Gauss' law is lost, whereas Coulomb gauge has not. It is an easy task to show that in QED, the FP operator is simply the Laplacian operator and therefore $\det \mathcal{M}$ results in an overall constant factor for the path integral. This is the reason why in QED, in the presence of linear gauges, there is no Gribov ambiguity.

Inserting then (3.11) into the gauge-fixing generating functional, we obtain

$$\mathcal{Z}_{\text{gf}}[0] = \int [dg][dA] \det \left(\frac{\delta F[gA]}{\delta g} \right)_{F=0} \delta(F[gA]) e^{-\int_M \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}}. \quad (3.15)$$

Making use of the local gauge invariance of the Y-M action and of the functional Haar measure, which allows us to show that also the functional Jacobian does not depend on the gauge transformation g ⁶, we can re-write (3.15) as

$$\mathcal{Z}_{\text{gf}}[0] = \int [dg] \int [d^g A] \det \left(\frac{\delta F[gA]}{\delta g} \right)_{F=0} \delta(F[gA]) e^{-\int_M \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}}, \quad (3.16)$$

where we have left the dependence of g also in the measure. In this way we have factored out the infinite measure over the gauge group⁷. The importance of this passage is clearly manifest when one deals with expectation values of gauge-invariant operators (thus physical observables), where we need to calculate

$$\langle \mathcal{O} \rangle = \frac{\int_{\mathcal{A}} [dA] \mathcal{O}[A] e^{-S_{\text{YM}}[A]}}{\mathcal{Z}_{\text{YM}}}. \quad (3.17)$$

Making use of the Faddeev-Popov method, we re-write the ratio as

$$\langle \mathcal{O} \rangle = \frac{\int [dA] \det \left(\frac{\delta F[gA]}{\delta g} \right) \delta(F[gA]) \mathcal{O}[A] e^{-\int_M \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}}}{\int [dA] \det \left(\frac{\delta F[gA]}{\delta g} \right) \delta(F[gA]) e^{-\int_M \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}}}, \quad (3.18)$$

⁶According to the left-invariant measure, one can show that any integral over the gauge group is gauge invariant. In fact, as mentioned above, it is easy to show that $\int dg \Phi(g) = \int (dg_0 g) \Phi(g)$, regardless whether $\Phi(g)$ is a gauge-invariant function or not.

⁷The Haar measure of the continuous group \mathcal{G} of gauge transformations is infinite because, though the Lie group $SU(N)$ is compact, a gauge transformation belongs to the functional space $\Omega^0(M, \text{ad}P)$. Therefore $[dg]$ also includes an infinite product over all the $x \in M$, $[dg] = \lim_{x \rightarrow \infty} \prod_x dg(x)$. In lattice Y-M theory, the discretisation of the space-time allows to make sense of such an infinity, and consequently the Haar measure becomes regulated naturally.

where $\int [dg]$ has been cancelled both from numerator and denominator, due to the gauge-invariance of $\mathcal{O}[A]$. Therefore, since gauge-invariant quantities should not be sensitive to changes of auxiliary conditions, it is possible to average over the local functions $\chi^a(x)$ of (3.7) with a Gaussian weight, substituting the delta function in the integrand of (3.16) as

$$\int [d\chi] \delta(G^\mu A_\mu^a - \chi^a) e^{-\frac{1}{2\xi} \int_M (\chi^a)^2} = e^{-\int_M \frac{1}{2\xi} (G^\mu A_\mu^a)^2}. \quad (3.19)$$

The complete gauge-fixing generating functional without sources is consequently

$$\mathcal{Z}_{\text{gf}}[0] = \int [d^g A] \det \left(\frac{\delta F[gA]}{\delta g} \right) e^{-\int_M \left\{ \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi} (G^\mu A_\mu^a)^2 \right\}}. \quad (3.20)$$

The perturbative expansion of $\det \mathcal{M}$ ⁸ leads to non local interactions between gauge fields. To express these interactions as local ones, we perform a manipulation which takes into account Grassmann fields, unphysical and fictitious, playing only an algebraic role. A more detailed explanation of Grassmann fields and algebra will be given in the next chapter, when we will introduce the BRST formalism. For the sake of comprehension, we just wish to remind the reader that it is possible to write a functional determinant of any $N \times N$ matrix operator Q ⁹ over a complete set of dimension $2N$ of anti-commuting generators, called Grassmann, such that

$$\det Q = \int \prod_{i=1}^k d\bar{\eta}_k d\eta_k e^{-\bar{\eta} Q \eta} \quad (3.21)$$

In the language of Feynman diagrams, these fields have been called by Feynman FP ghosts: though anti-commuting, they are Lorentz scalar, and therefore do not satisfy the spin-statistics theorem. As far Feynman diagrams are concerned, they are allowed to run around loops but not in external lines. They do not add to the spectrum of observable particles in the theory. The indisputable importance of these unphysical fields lies in their role played to guarantee the unitarity of the \mathcal{S} -matrix. As seen in the previous chapter, the decomposition into positive and negative frequency states of the B -field led us to a subsidiary condition $B^+(x)|\text{phys}\rangle = 0$, whose role was to specify and select the physical states. This condition could be associated to the Gupta-Bleuler condition, which guarantees

⁸We remind the reader that, given a functional determinant $\det A$, this can be exponentiated as $\det A = \exp(\text{Tr} \log A)$, which can be diagrammatically represented as infinite non-local loops.

⁹In the case of Grassmann fields, the operator Q is not required to have special properties (apart from singularities). On the contrary, in the case of Gaussian bosonic integration, when the field is real, Q has to be positive definite.

the unitarity of QED. In the attempt to generalise QED to the case of non-Abelian gauge theories, such as QCD for instance, the non-linear self-interaction of the gauge fields causes the subsidiary condition on the B -field to fail. Feynman pointed out that this could affect the breakdown of the unitarity of the \mathcal{S} -matrix. In fact, due to this self-interaction, it is not guaranteed that the contribution of unphysical degrees of freedom of A_μ , the longitudinal and temporal modes, to intermediate states could cancel out. Feynman himself and De Witt found in the context of perturbation theory that this problem concerning unitarity could be explained by the absence of massless scalar fermions to closed loops in Feynman diagrams. It is then thanks to Faddeev and Popov that these missing unphysical particles showed up through their quantisation method. Furthermore, in the context of Y-M theory renormalisation, it is due to the work of Veltman and 't Hooft [tHV72] that we can prove now that ghosts allow exact cancellation at all orders of the longitudinally and temporally polarised modes in the intermediate states, where the intermediate states include transverse vector particles. In this way unitarity is preserved. To insure global invariance, these ghost fields belong to the adjoint representation of the Lie group under consideration.

Under this manipulation, (3.20) assumes the original form presented by Faddeev and Popov

$$\mathcal{Z}_{\text{gf}}[0] = \int [dA][d\bar{\eta}][d\eta] e^{-\int_M \left\{ \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \bar{\eta}^a \mathcal{M}_{ab} \eta^b + \frac{1}{2\xi} (G^\mu A_\mu^a)^2 \right\}}, \quad (3.22)$$

and in the presence of sources

$$\mathcal{Z}_{\text{gf}}[J, \zeta, \bar{\zeta}] = \int [dA][d\bar{\eta}][d\eta] e^{-\int_M \left\{ \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \bar{\eta}^a \mathcal{M}_{ab} \eta^b + \frac{1}{2\xi} (G^\mu A_\mu^a)^2 - J^a A_\mu^a - \bar{\zeta}^a \eta^a + \bar{\eta}^a \zeta^a \right\}}, \quad (3.23)$$

Our quantisation procedure *a la* Faddeev and Popov is then completed, as well as providing a generating functional, through the exponentiation of the FP operator, able to generate appropriate Feynman diagrams in the context of perturbation theory. Though, as pointed out by Gribov [Gri78] and then explained in the language of principal bundles by Singer [Sin78], (3.23) makes only sense in the high-energy regime, but it fails to be applicable beyond it, when non-perturbative effects have to be taken into account. This will be explained in the next section.

3.2 The Gribov Ambiguity

As we saw in the last chapter, the constraints we have to impose when quantising Maxwell's electrodynamics do not change the energy spectrum. This is

so because we can reduce the number of degrees of freedom to be quantised by taking advantage of the gauge-invariance of the classical theory. This procedure is called *gauge fixing*. The choice of a gauge fixing term is arbitrary, but it leads to different problematic. For example, Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$ is not compatible with the Poisson Brackets $\{A_i(\vec{x}), \Pi_j(\vec{y})\} = \delta_{ij} \delta(\vec{x} - \vec{y})$, because the spatial divergence of the δ -function does not vanish. This implies that the quantisation of the theory is achieved only at the price of modifying the commutation relations. Moreover, differently from manifest covariant gauges, such as Lorentz or Landau gauges, Coulomb gauge spoils the Lorentz invariance. Another gauge which breaks the Lorentz invariance is temporal or Weyl gauge: in this gauge one imposes the condition $A_0 = 0$, which causes the Gauss Law to be lost. We have also seen how the Faddeev-Popov method of quantising non-Abelian theories provides a way to avoid the infinite gauge measure. Though, such a procedure is plagued by a topological obstruction, which prevents us from going beyond perturbation theory. As we will see, this problem is a common problem in any non-Abelian theory which is evaluated on a configuration space over equivalence classes of gauge-transformations.

In the late seventies, in fact, it was first pointed out by V. Gribov in his seminal work [Gri78] that once Y-M theory is gauge-fixed by means of Coulomb gauge, one has to face a degeneracy of such gauge, i.e. the gauge orbits can intersect the Coulomb gauge hypersurface at more than one point. Following Gribov, we consider two gauge-equivalent fields \vec{A} and \vec{A}'

$$\vec{A}' = g^\dagger \vec{A} g + g^\dagger \vec{\nabla} g. \quad (3.24)$$

Because of the non-linearity of (3.24), a transverse field potential satisfying the Coulomb-gauge condition may actually happen to be a pure gauge, which should not be separately counted as an additional physical degree of freedom. He explicitly constructed such a transverse field for $SU(2)$, and showed the uncertainty in the gauge-fixing procedure when the QCD coupling constant g becomes of order of unity, i.e. in the non-perturbative regime. In terms of the FP method, this uncertainty arises when the FP operator acquires zero-mode solutions, which occurs in the infrared region, where the vacuum enhancement of the dressed Coulomb gluon propagator becomes catastrophically large [Zwa04]. To see the appearance of this uncertainty, we start first with a covariant gauge, and then we limit ourselves to the three-dimensional case. The divergence of A'_μ is given by

$$\partial_\mu A'^\mu = (\partial_\mu g^\dagger) A^\mu g + g^\dagger (\partial_\mu A^\mu) g + g^\dagger A^\mu \partial_\mu g + (\partial_\mu g^\dagger) \partial^\mu g + g^\dagger \square g. \quad (3.25)$$

The requirement for the gauge transformation g to not change the divergence of

both A_μ and A'_μ leads to the condition

$$\partial_\mu(g^\dagger(\mathcal{D}^\mu[A]g)) = 0. \quad (3.26)$$

This second order partial differential equation, for large values of A_μ , i.e. for distant configurations from perturbation theory (which is evaluated around $A_\mu = 0$), will produce several non-trivial (different from constant gauge transformations) solutions. Among these solutions, we can distinguish three types: 1) solutions belonging to gauge orbits intersecting the gauge-fixing hypersurface only once, which correspond to ideal gauge conditions. 2) solutions belonging to gauge orbits never intersecting the gauge-fixing hypersurface and 3) solutions belonging to gauge orbits intersecting the gauge-fixing hypersurface more than once. The third case is what Gribov discovered in Coulomb gauge, which would cause the ambiguity in the gauge-fixing procedure. These redundant solutions have been called *Gribov copies*, i.e. for each configuration satisfying the gauge condition, there are gauge-equivalent configurations that satisfy the same condition. Therefore, the gauge-fixing procedure fails in removing all the unnecessary gauge degrees of freedom. It is pedagogical to show that, within perturbation theory, for an infinitesimal gauge transformation ($g \simeq 1 + X^a \theta_a(x) + O(\theta^2)$) the divergence (3.25) becomes

$$\square\theta - (\partial^\mu\theta)A_\mu + A_\mu(\partial^\mu\theta) = 0 \quad \text{or} \quad \partial^\mu(\partial_\mu\theta + [A_\mu, \theta]) = 0. \quad (3.27)$$

We see that the condition for the appearance of Gribov copies is equivalent to the requirement for the operator $-\partial^\mu(\partial_\mu\theta + [A_\mu, \theta])$, the Faddeev-Popov operator, to have zero eigenvalues (zero modes). It is also interesting to notice [SS05] that the eigenvalue equation for the FP operator ¹⁰ resembles a Schrödinger equation as

$$-\partial^\mu(\partial_\mu\alpha + [A_\mu, \alpha]) = \epsilon[A]\alpha, \quad (3.28)$$

with the gauge potential A playing the role of a potential. Being the eigenvalue ϵ a functional of A (as well as the eigenfunction α), we expect, for sufficient large values of A , the zero-energy solution ($\epsilon[A] = 0$) to exist.

3.2.1 Gribov pendulum

In his work, Gribov considered the three-dimensional case with Lie group $SU(2)$ to explore explicit solutions to the Coulomb gauge. Moreover, to simplify the

¹⁰Performing a gauge transformation on A^μ , the FP operator w.r.t. the gauge transformed field becomes $\mathcal{M}_{FP}[gA] = \square - \partial_\mu[gA^\mu, \cdot] = \square - \partial_\mu[A^\mu - g^\dagger[\mathcal{D}^\mu, g], \cdot] = \mathcal{M}_{FP} + \partial_\mu[g^\dagger[\mathcal{D}^\mu, g], \cdot]$ Applying n-times a gauge transformation on A gives $g^n A_\mu = (g_n)^\dagger A_\mu g_n - (g_n)^\dagger \partial_\mu g_n$ where $g_n = \prod_{i=1}^n g_i$. This implies that the FP operator for the n-th gauge transformation is $\mathcal{M}_{FP}[g^n A] = \square - \partial_\mu[(g_n)^\dagger A_\mu g_n - (g_n)^\dagger \partial_\mu g_n, \cdot] = \mathcal{M}_{FP}[A] + \partial_\mu[(g_n)^\dagger[\mathcal{D}^\mu, g_n], \cdot]$.

parametrisation, he chose a time-independent and spherically symmetric gauge field A_i , $i = 1, 2, 3$, such that it only depends on the unit vector $n_i = x_i/r$, with $r = \sqrt{x^i x_i}$. Under such assumptions, he considered for the following parametrisation

$$A_i = f_1(r) \frac{\partial \hat{n}}{\partial x_i} + f_2(r) \hat{n} \frac{\partial \hat{n}}{\partial x_i} + f_3(r) \hat{n} n_i, \quad (3.29)$$

with $\hat{n} = i n_i \sigma_i$ and the functions $f_i(r)$ supposed to be smooth on the domain of r . Since

$$\frac{\partial \hat{n}}{\partial x_i} = \frac{i}{r} (\sigma_i - (\vec{n} \cdot \vec{\sigma}) n_i), \quad (3.30)$$

it follows that the gauge field can be written as

$$A_i = \frac{i}{r} f_1(r) \sigma_i - i \frac{i}{r} f_1(r) (\vec{n} \cdot \vec{\sigma}) n_i - \frac{i}{r} f_2(r) (\vec{n} \cdot \vec{\sigma}) \sigma_i + \frac{i}{r} f_2(r) n_i + i f_3(r) (\vec{n} \cdot \vec{\sigma}) n_i. \quad (3.31)$$

In the case in which $f_1 = f_3 = 0$, (3.31) simplifies to

$$A_i = \frac{i}{r^2} \epsilon_{ijk} x_j \sigma_k f_2(r), \quad (3.32)$$

which is purely transverse $\vec{\nabla} \cdot \vec{A} = 0$, due to the antisymmetry of the tensor ϵ_{ijk} .

The condition for the existence of copies

$$\begin{aligned} \vec{A}' &= g^\dagger \vec{A} g + g^\dagger \vec{\nabla} g \\ \vec{\nabla} \vec{A}' &= \vec{\nabla} \vec{A}, \end{aligned} \quad (3.33)$$

together with a suitable parametrisation for the $SU(2)$ gauge transformation g ($g = e^{\frac{i}{2} \alpha(r) \vec{n} \cdot \vec{\sigma}}$) determine an explicit form for the gauge transformed field A'_i as follows

$$\begin{aligned} A'_i &= \left(f_1 \cos \alpha + \left(f_2 + \frac{1}{2} \right) \sin \alpha \right) \frac{\partial \hat{n}}{\partial x_i} + \left(\left(f_2 + \frac{1}{2} \cos \alpha \right) - f_1 \sin \alpha - \frac{1}{2} \right) \hat{n} \frac{\partial \hat{n}}{\partial x_i} \\ &+ \left(f_3 + \frac{1}{2} \frac{d\alpha}{dr} \right) \hat{n} n_i. \end{aligned} \quad (3.34)$$

The condition on the divergence of the gauge fields determines instead a second-order differential equation in α

$$\alpha''(r) + \frac{2}{r} \alpha'(r) - \frac{4}{r^2} \left(\left(f_2 + \frac{1}{2} \right) \sin \alpha + (f_1 \cos \alpha - 1) \right) = 0. \quad (3.35)$$

Performing a logarithmic change of variables $\tau = \log r$, (3.35) assumes the form of the equation for a pendulum in the presence of a damping term, proportional to the velocity

$$\alpha''(\tau) + \alpha'(\tau) - 4 \left(\left(f_2 + \frac{1}{2} \right) \sin \alpha + (f_1 \cos \alpha - 1) \right) = 0. \quad (3.36)$$

This equation is called the *Gribov pendulum*. The presence of sinusoidal functions makes the pendulum equation to be highly non-linear: no analytic solution in closed form is known [CU01, GS01], but only numerical ones. The only possibility to analyze such pendulum is to simplify the problem by imposing particular boundary conditions and approximation. Regardless these analytic difficulties, we can illustrate the situation qualitatively: at each point in the pendulum trajectory, three forces are applied: $4f_1$ and $4f_2 + 2$ respectively in the longitudinal and transverse direction, whereas f_1 onto determines a perturbation as sketched in Figure 3.2.1

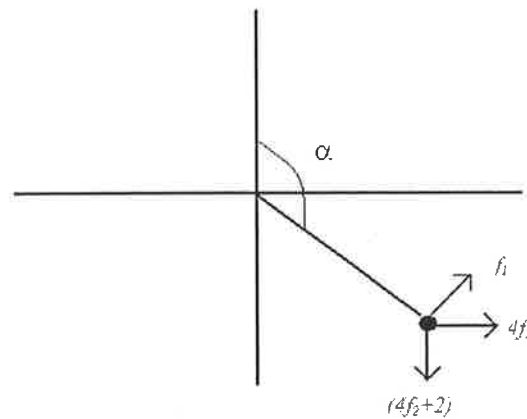


Figure 3.1: The Gribov pendulum

To simplify further the calculation, we could adopt the pure transverse configuration (3.32), such that the Gribov pendulum equation becomes

$$\alpha''(\tau) + \alpha'(\tau) - 4 \left(f + \frac{1}{2} \right) \sin \alpha = 0, \quad (3.37)$$

in which only the force proportional to $4f + 2$ is applied on the pendulum. The smooth function $f(e^\tau)$ is necessary to preserve the regularity of solutions [Sci77, Hen79]: A_i is required to be regular at $r = 0$, implying $\alpha(e^\tau) \rightarrow_{r \rightarrow 0} n\pi + O(r^2)$ and to go to zero at infinity faster than $1/r$, implying $\alpha(+\infty) - \alpha(-\infty)$ being either 0 or $\pm\pi/2$ [Sci77]. To be more precise, the condition at infinity can be of two types, according to the boundary conditions one needs to impose: we distinguish between weak (WBC) or strong (SBC) boundary conditions. Both conditions require A_i to be regular at the origin, such that $f(r) \rightarrow_{r \rightarrow 0} O(r)$, but while WBC impose $f(r) \rightarrow_{r \rightarrow \infty} \text{constant}$, SBC impose $f(r) \rightarrow_{r \rightarrow \infty} 0$. For

more details about the various types of Gribov copies one can obtain according to WBC or SBC, we refer the reader for [SS05].

To conclude this section, it is worthwhile addressing the solution found by Henyey in [Hen79]. As noted in the previous chapter, a fundamental role in Euclidean Y-M theory is played by instantons: these are classical solutions to the equations of motion of pure Euclidean Y-M theories which have finite action. To fully understand their importance, we have to introduce some basic concepts concerning the topology of Euclidean Y-M theory. The boundary of the four-dimensional Euclidean space-time at infinity ($r \rightarrow \infty$) is given by the three-sphere S_∞^3 . The gauge field A_μ , when $r \rightarrow \infty$, becomes a pure gauge, i.e. $A_\mu \rightarrow_{r \rightarrow \infty} g^\dagger \partial_\mu g + O(1/r^2)$, and the corresponding field-strength tensor vanishes, $F_{\mu\nu}(g^\dagger \partial_\mu g) = 0$. With such boundary condition for the gauge configuration, it is possible to show [GM86,Nab] the existence of a map between S_∞^3 and $SU(2)$: being the topology of the three-sphere the same of $SU(2)$ (topological equivalent) this map can be characterised by the *winding number* ν (also called the Pontryagin number) corresponding to the discrete *homotopy*¹¹ $\pi_3(S^3) = \mathbb{Z}$. One of the major achievements in the Yang-Mills theory was the discovery of the relation between instanton solutions and their classification by the winding number ν [BPST75,tH74,Uhl82b]. The relevance of Henyey's work relies on the fact that he was able to explicitly obtain Gribov's copies with vanishing winding number and which fall off faster than $1/r$ for $r \rightarrow \infty$. Starting with a gauge field $A_i = ia_i(r)\sigma_3$, satisfying the Coulomb condition, and adopting a suitable parametrisation for the $SU(2)$ gauge transformation, he obtained a differential equation of the second order similar to (3.35). Adopting polar coordinates for the various parameters, he showed that the function a , specifying the gauge field can be put in the form

$$a(r, \theta) = \frac{1}{2r \sin \theta} - \frac{1}{\sin(2rb \sin \theta)} \left(b + r^2 \sin^2 \theta \left(b'' + \frac{4}{r} b' \right) \right), \quad (3.38)$$

where the function b only depends on the radius and is defined as $b(r) = \frac{k}{(r^2 + r_0^2)^{3/2}}$ and $k < \frac{3^{3/2}\pi}{4} r_0^2$. As long as such a function b exists [Sci79], then $a(r, \theta)$ fulfills the boundary and regularity conditions required in Euclidean Y-M theory, being regular at the origin $r = 0$ and decaying at infinity faster than $1/r^2$.

¹¹Given two continuous maps from the hypersphere S^n to M , ϕ and φ , they are said to be homotopic if there exists a map $F(x, t)$, with $0 \leq t \leq 1$, which interpolates continuously between them, namely $F(x, 0) = \phi$ and $F(x, 1) = \varphi$. The homotopy between ϕ and φ is denoted by the symbol $\phi \sim \varphi$. The set of homotopy classes is denoted by $\pi_n(M)$. When $M = S^n$, the equivalence homotopy classes are labelled by the winding number ν : two maps $\phi, \varphi : S^n \rightarrow S^n$ can be continuously deformed into one another iff both maps cover S^n the same number of times as x covers it once.

3.3 The functional spaces \mathcal{A} and \mathcal{A}/\mathcal{G}

The functional analysis of the Gribov ambiguity leads to examine the configuration space \mathcal{A} , the functional space of all gauge connections A_μ ¹², in more detail. This is an infinite-dimensional affine space, on which it is possible to fix a point and coordinate axis such that every point in the space can be represented as an n -tuple of its coordinates. Not only is \mathcal{A} affine, but also a Hilbert space: in fact, if we denote by $\Omega_k^p(M, \text{ad}P)$ the k -Sobolev completion of $\Omega^p(M, \text{ad}P)$ [Uhl82b, Ada, Maz, Nab] the space of smooth sections of degree p , then \mathcal{A} assumes the structure of a Hilbert space, as an affine space modelled over $\Omega^1(M, \text{ad}P)$.

Since all physically relevant quantities are gauge invariant, the objects of interest are the families of gauge related connections rather than the connections themselves. For this purpose we denote by \mathcal{G} the group of all gauge transformations, whose elements $g \in \Omega^0(M, \text{ad}P)$ in local coordinates determine the group (adjoint) action on A_μ as

$$\begin{aligned} {}^g A_\mu(x) &= g^\dagger(x) A_\mu(x) g(x) + \frac{1}{g} g^\dagger(x) \partial_\mu g(x) \\ &= A_\mu(x) + \frac{1}{g} g^\dagger(x) \mathcal{D}_\mu g(x). \end{aligned} \quad (3.39)$$

The fundamental question is how we define the connections and the functions belonging to the group of gauge transformations [GS01]: Uhlenbeck [Uhl82a], considering Sobolev gauge connections $A_\mu \in W^{1,p}(\mathbb{B}^n, \mathcal{G})$, $n \geq p/2$ and gauge transformations with one more weak derivative $g \in W^{2,p}(\mathbb{B}^n, G)$ on the n -dimensional unit ball \mathbb{B}^n , was able to prove that with such a setting, providing appropriate curvatures F_A , ${}^g A \in W^{1,p}(\mathbb{B}^n, \mathcal{G})$ belongs to the Coulomb gauge. The restriction to Sobolev norms was also suggested in a very interesting work [STSF82], in which it was shown under which conditions it was possible to find the absolute minima of the Y-M functional. This issue will be of great importance when, in the next section, we will discuss about the various Gribov regions and the so-called fundamental modular region (FMR). On the other side, Dell'Antonio and Zwanziger [DZ91] considered less restrictive conditions on the gauge connections, provided with a standard $L^2(\mathbb{R}^n)$ norm, and they proved the existence of gauge copies of A_μ in the Coulomb gauge on the non-compact n -dimensional Euclidean space \mathbb{R}^n .

Regardless of these subtle distinctions, the set of all physically inequivalent connections is determined by the orbit space (a manifold) $\mathfrak{A} = \mathcal{A}/\mathcal{G}$, i.e.

¹²The gauge connections, as explained in the previous chapter, are the components of a smooth matrix-valued differential one-form $\Omega^1(M, \text{ad}P)$.

the set of equivalence classes where A and A' are equivalent if there exists a $g \in \mathcal{G}$ such that $A' = {}^gA$. The high non-triviality of this Riemannian space¹³ is the reason for which we encounter the Gribov ambiguity in non-Abelian gauge theory. Following Singer [Sin78], we try now to highlight the fundamental topological obstruction that Gribov discovered in the light of Coulomb gauge. In the Feynman approach to quantisation of Y-M theory, one would want to make sense of $\int \mathcal{D}A \{ \cdot \} e^{-\|F_A\|^2} / \int \mathcal{D}A e^{-\|F_A\|^2}$, where $\|F_A\|^2$ is the Yang-Mills functional (2.48), and the integrand of the numerator may be constant on orbits of \mathcal{G} . These orbits are expected to have an infinite measure though and this introduces a difficulty in evaluating the ratio. One then would like to perform the integral over \mathfrak{A} , but this turns out to be intractable and this is the reason why we choose an arbitrary gauge-fixing condition. This procedure would be consistent if one would be able to choose in a continuous manner one gauge connection on each orbit. When changing variables from \mathcal{A} to \mathfrak{A} we introduce in the functional integral a Jacobian, which is interpreted as the integral of a probability measure along the fibers. Gribov observed that by choosing a Coulomb gauge with appropriate boundary conditions at ∞ , there exist gauge transformed connections belonging to a trivial principal bundle P over \mathbb{R}^4 , with Lie group $SU(2)$, that intersect the Coulomb hypersurface not only in the vicinity of trivial configurations, as $A_\mu = 0$, but also at a large distance from 0. What Singer showed is that, this scenario is not only valid and applicable to the case of Coulomb gauge, but more generally, if the conditions at ∞ amount to studying Riemannian surfaces as $S^4 = \mathbb{R}^4 \cup \infty$ (the unit sphere in \mathbb{R}^5), then topological considerations imply that no gauge exists. Thus, the Gribov ambiguity for the Coulomb gauge will occur in all the other gauges, and no continuous gauge fixing is possible. In practice, the topological obstruction occurs when one tries to invert the projective map $\sigma : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{G}$: due to (3.39) σ is mapping an affine space to a non-affine one, such that any $A \in \mathcal{A}$ is being mapped onto the orbit \mathcal{G} of A . Yet, σ^{-1} maps back to one $A \in \mathcal{A}$ each representative of the same orbit without distinction, and therefore such a function is not bijective. This topological obstruction therefore prevents one from introducing affine coordinates in a global way¹⁴.

¹³A very detailed description of the Riemannian structure of the gauge configuration space in Y-M theory can be found in [BV81a, DV80].

¹⁴This problem occurs also in General relativity when one considers diffeomorphisms of the metric tensor [Nab, Nak].

3.4 The Gribov regions C_i and the fundamental modular region Λ

Following Gribov [Gri78], it is possible to define on \mathcal{A}/\mathcal{G} different regions C_i , according to the number of negative eigenvalues of the the Faddeev-Popov operator. To see this, consider the eigenvalue equation for the Faddeev-Popov operator (3.28): for small values of A_μ it is solvable for small and positive $\epsilon[A]$ only. More precisely, one can show that for small A_μ , $\epsilon_i[A] > 0$. As the gauge field increases its magnitude, one of the eigenvalues turns out to vanish, and then becoming negative as the field increases further. Therefore the magnitude of A_μ insures the existence of negative energy states, i.e. bound states. Supposing to keep going with increasing A_μ , some other eigenvalues will start vanishing and subsequently changing sign. If we divide the orbit space into regions C_i ($i = 0, 1, \dots, N$), the *Gribov regions* with i denoting the number of negative eigenvalues $\epsilon[A]$ for the F-P operator $-\partial^\mu(\partial_\mu + [A_\mu, \cdot])$, we may obtain the following schematic picture. The various lines denoted by l_i correspond to the so-called *Gribov horizons*: the

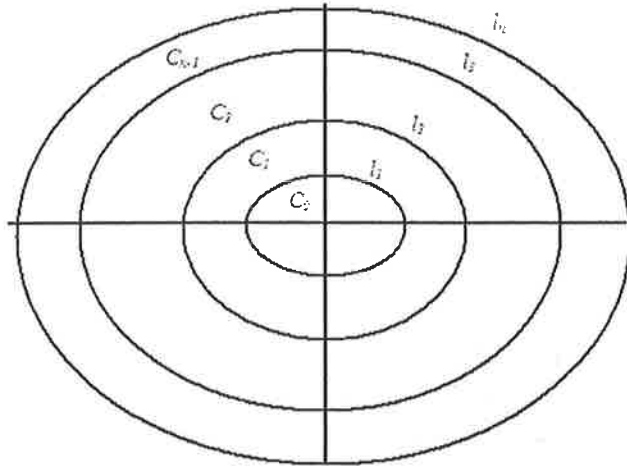


Figure 3.2: The Gribov regions C_i and the fundamental modular region Λ

label indicates the number of vanishing eigenvalues of the corresponding F-P operator \mathcal{M} . Therefore, when passing from C_i to C_{i+1} , one crosses one horizon and the overall sign of $\det \mathcal{M}[A]$ changes. Moreover, it is possible to show that for any configuration lying within the region C_{i+1} close to the boundary l_{i+1} there is an equivalent configuration within the region C_i close to the same boundary l_{i+1} . It is important to notice that in the first region C_0 , there is no negative

eigenvalue, or put in another form, the lowest eigenvalue is positive, guaranteeing the condition for the positiveness of the Faddeev-Popov determinant such that for a given gauge-condition $F[A]$, the zeroth Gribov region is defined as $C_0 = \{A_\mu \in \mathfrak{A}, F[A] = 0, \epsilon_i[A] > 0 \mid -\partial^\mu(\partial_\mu + [A_\mu, \cdot]) > 0\}$. As originally suggested by Gribov himself and rephrased in the language of path integrals, the Y-M functional integral over gauge-inequivalent configurations should be restricted to an appropriate region, where the gauge-fixing condition would be guaranteed unambiguously. Gribov suggested to restrict the integration over C_0 : yet, as proved in [STSF82], this region wouldn't necessarily guarantee to find unique solutions to the gauge condition for each orbit. Therefore, it is necessary to find a better way to evaluate the Y-M path integral in such a way that the integration region only selects one single representative for each gauge orbit. It was shown in [STSF82] first and then developed in [DZ91], that there is a functional method to determine such a region. Suppose we define a covariant L^2 -vector Morse potential¹⁵ along the gauge orbit

$$V[gA] \equiv \|gA\|^2 = - \int_M \text{tr} \left((g^\dagger A_\mu g + g^\dagger \partial_\mu g)^2 \right). \quad (3.40)$$

Expanding around the minimum of eq.(3.40), writing $g(x) = \exp(X(x))$, one easily finds:

$$\begin{aligned} \|gA\|^2 &= \|A\|^2 + 2 \int_M \text{tr}(X \partial_\mu A_\mu) + \int_M \text{tr}(X^\dagger \mathcal{M}[A] X) \\ &\quad + \frac{1}{3} \int_M \text{tr}(X [[A_\mu, X], \partial_\mu X]) + \frac{1}{12} \int_M \text{tr}([D_\mu X, X][\partial_\mu X, X]) + O(X^5). \end{aligned} \quad (3.41)$$

At any local minimum the vector potential is therefore transverse, $\partial_\mu A_\mu = 0$, and $\mathcal{M}[A]$ is a positive operator. The set of all these vector potentials is by definition the Gribov region C_0 . Using the fact that $\mathcal{M}[A]$ is linear in A , C_0 is seen to be a convex subspace¹⁶ of the set of transverse connections Γ . Its

¹⁵A smooth function $u : M \rightarrow \mathbb{R}$ is called Morse if all its critical points are non-degenerate. Morse functions exist on any smooth manifold, and in fact form an open dense subset of smooth functions on M .

¹⁶If A_1 and A_2 are two gauge fields inside Ω , they by definition satisfy the Faddeev-Popov eigenvalue equation, $-\mathcal{M}(A_{1,2})\phi_n = \lambda_n(A_{1,2})\phi_n$, such that $\lambda_n > 0, \forall n$ ($\mathcal{M} \equiv \partial^\mu(\partial_\mu - [A_\mu, \cdot])$). To prove convexity it suffices to show that, given a real parameter $s \in [0, 1]$, through which we relate A_1 to A_2 as $sA_1 + (1-s)A_2 = \bar{A}$, the field \bar{A} always belongs to Ω , regardless s and the particular choice of the two starting fields. This is easy to see because, by definition, A_1 and A_2 belong to C_0 , and so they have positive eigenvalues. This implies their combination by s is always positive, whichever value for the parameter we pick up and therefore $\bar{A} \in \Omega$. In addition to this convexity, there's another theorem which claims there's always an equivalent

boundary ∂C_0 is called the Gribov horizon. At the Gribov horizon, the *lowest* eigenvalue of the Faddeev-Popov operator vanishes, and points on ∂C_0 are hence associated with coordinate singularities. Any point on ∂C_0 can be seen to have a finite distance to the origin of field space and in some cases even uniform bounds can be derived [Zwa94, DZ91]. The Gribov region is then defined as the set of *local* minima of the norm functional (3.40) and needs to be further restricted to the *absolute* minima to form a fundamental domain, which will be denoted by Λ . The fundamental domain is clearly contained within the Gribov region and therefore Λ is proven to be convex too. We can define Λ in terms of the absolute minima over $g \in \mathcal{G}$ of $\|gA\|^2 - \|A\|^2 = \langle g, \mathcal{M}[A]g \rangle$ as

$$\Lambda = \{A \in \mathcal{A} \mid \min_{g \in \mathcal{G}} \langle g, \mathcal{M}[A]g \rangle = 0\}. \quad (3.42)$$

A different approach in restricting the integration region to Λ may come from stochastic quantisation as explained in [Zwa04]. A detailed overview of the analytic properties of the Gribov and fundamental modular region can be found in [vB92, Zwa94, Zwa04] and references therein. For the purpose of this thesis we only focus on some elementary properties of these regions. As Λ is contained in C_0 , this means Λ is also bounded in each direction and has a boundary $\partial\Lambda$. Convexity of Λ allows us to consider rays extending from the origin of Λ , set to $A_\mu = 0$ out to C_0 , crossing the common boundary, such that at some point along the ray, this absolute minimum has to pass the local minimum. At the point they are exactly degenerate, there are two gauge equivalent vector potentials with the same norm, both at the absolute minimum. As in the interior the norm functional has a unique minimum, again by continuity, these two degenerate configurations have to both lie on the boundary of Λ . This is the generic situation. If the degeneracy at the boundary is continuous along non-trivial directions one necessarily has at least one non-trivial zero eigenvalue for $\mathcal{M}[A]$ and the Gribov horizon will touch the boundary of the fundamental domain at these so-called singular boundary points. It is interesting to note in the case of stochastic quantisation in [Zwa04], it was suggested that in the thermodynamic limit, as the number of configurations tends to increase, the Y-M functional integral would be dominated by configurations lying on the common boundary of Λ and C_0 .

The final comment we would like to point out here concerns the practical realisation of such a fundamental modular region. As the Gribov region is associated with the local minima, and since the space of gauge transformations

field in the second Gribov region, C_1 , for a field A_μ inside C_0 and close to the Gribov horizon l_1 . These two geometric properties of C_0 ensure us that the common cartoon which displays this region is correct.

resembles that of a spin model, the analogy with spin glasses makes it unreasonable to expect that the Gribov region is free of further gauge copies [vB92]. Unfortunately restrictions to a subset of the transverse gauge fields is a rather non-local procedure. This cannot be avoided since it reflects the non-trivial topology of field space [Sin78]. Early after the discovery of Gribov of the degeneracy in quantising non-Abelian gauge theories, within the context of U gauge, in [BMRS78] an unambiguous way was proposed to select single representatives for each orbit. Though, this gauge fails in being covariant and hard to put in a close analytic form. Further proposals of eliminating the Gribov ambiguity are very frequent in literature, for instance see [MSV04, GS01, Zwa04]. Nonetheless, the most rigorous scenario in which Gribov copies can be consistently and practically avoided is Lattice Gauge Theory (LET), according to Wilson's procedure [Wil74]. In the course of the next chapters we will often deal with such formalism: for now, it suffices to say that in LGT, it is well known that due to the discretisation procedure adopted, no gauge-fixing is required. The Y-M path integral is calculated over an ensemble of links $U_{x,\mu}$ randomly generated by appropriate Monte Carlo algorithms. It is then possible to show that the probability to generate two link configurations lying on the same gauge orbit is statistically negligible. The fundamental difference between continuous and lattice gauge theories lies therefore in the fact that in the former formalism it is possible to simulate numerically the dependence of the gluon and ghost propagators on Gribov copies [AdFF01, AdFF02, LSWP98, BBLW00, BHW02]. Appearance of Gribov copies are studied in Landau gauge in [MPR91, Sha84] whereas, for instance, a proposal to eliminate the ambiguity of gauge-equivalent configurations can be found in [Tes98] in the light of a simple toy model using BRST arguments. It is however a difficult task to show how the continuous limit of LGT can be obtained, maintaining the theory free of Gribov copies.

BRST formalism in Yang-Mills Theory

Soon after the work of Faddeev and Popov [DN67], the attention of the physics community was focused greatly on the appearance of these fictitious and unphysical particles, called FP ghosts. As Feynman suggested early on, these particles were meant to be necessary to guarantee the unitarity of the \mathcal{S} -matrix in non-Abelian gauge theories. De Witt had also suggested that this breakdown of unitarity was due to missing contributions of a pair of massless scalar (or vector in the case of the gravitational field) fermions to closed loops in Feynman diagrams. It was further realised that the Ward-Takahashi identities for Abelian theories, as well as Slavnov-Taylor identities for the non-Abelian case, both indispensable to prove renormalisability of the respective theories, should necessarily involve these unphysical ghosts. Though ghost particles were thus the missing particles physicists were after, the geometric structure of gauge theory seemed to be plagued by unphysical modes which do not follow the spin-statistics for fermions. This chapter will be then entirely dedicated to the BRST formalism, introduced independently in [BRS76] and [Tyu] in the mid-1970s: this quantisation method will be analyzed in the light of covariant Y-M theory, firstly with linear gauges, such as Landau gauge and successively with a more general class of non-linear gauges, such as the Curci-Ferrari gauge. We will also present the Kugo-Ojima criterion for selecting the appropriate physical states.

4.1 Faddeev-Popov ghosts and the birth of a new symmetry

In the last chapter we saw how FP ghosts appeared in the path integral representation of Y-M theory, through the introduction of two Grassmann fields in order to exponentiate the determinant of the FP operator. As we know, the Lagrangian appearing in (3.23) has lost its local gauge invariance by the introduction of a gauge-fixing term: it would be nonetheless desirable to maintain the infinitesimal gauge invariance of the theory. The extension of this sym-

metry to the case of finite transformations can be understood heuristically by performing the same transformation many times as $\Delta A = \lim_{n \rightarrow \infty} \delta_n A$ with $\delta A = \mathcal{D}[A]\theta$. This infinite repetition of infinitesimal variations can be avoided by introducing a Grassmann parameter in the definition of δA in the following way: in the previous chapter we saw that by the exponential map we can define a local relation between the group and its algebra as $g(x) = e^{X^a \theta_a(x)}$. Suppose now we introduce in the exponent a parameter ϵ as $g(x) = e^{\epsilon X^a \theta_a(x)}$ and we expand the exponential in a Taylor series, $g(x) = e^{\epsilon X^a \theta_a(x)} = \mathbb{I} + \epsilon X^a \theta_a(x) + \frac{1}{2}(\epsilon X^a \theta_a(x))^2 + \dots + \frac{1}{n!}(\epsilon X^a \theta_a(x))^n$. If we are allowed to take $(\epsilon X^a)^2 = 0$ not as an approximate relation, but as an exact one, then we notice that the infinitesimal form becomes exact by itself being identical with its finite one. This constraint mimics the infinitesimal form of the original local gauge invariance, whereas it does not reproduce its finite form which has been broken by the gauge-fixing procedure. It is well known that in differential geometry, an object which is endowed with such nilpotency condition is a differential form [Nab, Nak]: these forms constitute a finite-dimensional Grassmann algebra equipped with exterior product (see Appendix). This anti-commuting nature underlying classical gauge transformations led in mid 70's Becchi, Rouet and Stora [BRS76] and independently Tyutin [Tyu] to construct a coherent formalism in covariant non-Abelian theories to solve in more algebraic way Slavnov-Taylor identities and to prove the renormalisability of the theory. Moreover, the canonical quantisation of Yang-Mills theory and its correct application to the Fock space of instantaneous field configurations were elucidated by Kugo and Ojima. Later works by many authors, notably Thomas Suchcker and Edward Witten, have clarified the geometric significance of the BRST operator and related fields and emphasised its importance to topological quantum field theory and string theory.

The BRST formalism is based on the use of the Faddeev-Popov ghosts to construct a nilpotent operator δ and its associated Noether charge Q_B , the generator of quantum gauge transformations. Furthermore, the Grassmann nature of δ identifies it as a supersymmetric operator and consequently the BRST formalism is considered an example of a supersymmetric theory (SUSY). Another important property of this formalism is its understanding in terms of differential geometry and fiber bundles. In [BT81] it was pointed out how the gauge-fixing procedure by means of FP ghosts would enlarge the Riemannian structure of the principal bundle inherited by Y-M theory into a supersymmetric space, extended to include Grassmann degrees of freedom [DJT82, DJ82]. In [QdUH⁺81, HQR-MdU82, BTM82], these ideas were confirmed and expanded in the context of superfield formalism and covariant quantisation for Y-M theory. Topologically speaking, the central idea of the BRST construction is to identify the solutions of the gauge constraints with the cohomology classes of a certain nilpotent op-

erator, the BRST operator δ [KvH91], generated by a pair of anticommuting Lorentz scalar fields, the FP ghosts.

Following the original works of Becchi, Rouet and Stora [BRS76] and Tyutin [Tyu], we want to show how the Grassmann structure shows up naturally in gauge transformations: consider for this purpose the linearly covariant gauge-fixed Lagrangian ¹

$$\mathcal{L} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{FP}} = \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi} (\partial_\mu A_\mu^a)^2 + i\bar{c}^a \mathcal{M}_{ab} c^b, \quad (4.1)$$

and a local infinitesimal gauge transformation

$$\delta A_\mu^a(x) = \partial_\mu \theta^a(x) - g f^{abc} A_\mu^c(x) \theta^b(x) \equiv \mathcal{D}_\mu^{ab}[A] \theta^b(x). \quad (4.2)$$

Substituting $\theta^a(x) = c^a(x)$ in (4.2), with c^a being a local Grassmann field, we obtain

$$\delta A_\mu^a(x) = \mathcal{D}_\mu^{ab}[A] c^b(x). \quad (4.3)$$

It is worthy noting the role played by the ghost field: it replaces the classical θ^a gauge function to provide the quantum version of (4.2). Under such a transformation, it is rather trivial to show the invariance of \mathcal{L}_{YM} under (4.3), because the ghost field does not affect the original gauge invariance. Conversely, the variation of the Faddeev-Popov Lagrangian yields

$$\delta(\mathcal{L}_{\text{FP}}) = i\partial^\mu \bar{c}^a (\mathcal{D}_\mu^{ab} c^b). \quad (4.4)$$

So (4.1) is not invariant under the local infinitesimal gauge transformation (4.3) with an arbitrary gauge function θ^a . In [BRS76, Tyu] it was proposed to “gauge” transform also the two ghosts. For this purpose consider the following analogy with differential geometry in the case of an infinite-dimensional Lie group: in Y-M theory we deal with an infinite-dimensional Lie group \mathcal{G} of gauge transformations, together with its Lie algebra \mathfrak{g} . On \mathfrak{g} , we can define a *Maurer-Cartan* differential form ω , which is a left-invariant 1-form, whose functional nature is due to the fact that $\mathfrak{g} = \mathcal{C}^\infty(M, su(N))$, or a section of the fiber bundle $\Gamma(M \times_{SU(N)} su(N))$. On this algebra, we can define a coboundary operator δ (dual of a derivative operator), which acts on elements of \mathfrak{g} according to the following anti-derivation rule

$$\delta(\varphi_1 \wedge \varphi_2) = \delta(\varphi_1) \wedge \varphi_2 + (-1)^{\text{deg}\varphi_1} \varphi_1 \wedge \delta(\varphi_2), \quad (4.5)$$

¹We start the BRST formalism without the Nakanishi-Lautrup field b , also called the *on-shell* BRST, i.e. when we consider the equations of motion for the b -field. Then we will show how the auxiliary field plays the role of insuring the BRST invariance *off-shell*.

and satisfies the nilpotency condition

$$\delta^2 = 0. \quad (4.6)$$

From the last chapter, we know that such a Maurer-Cartan form ω satisfies $\delta\omega = -\frac{1}{2}\omega \times \omega = -\frac{1}{2}[\omega, \omega]$. By rewriting $\omega(x) = g c(x)$, we chose the following on-shell transformations

$$\begin{aligned} \delta c^a(x) &= -\frac{g}{2}(c(x) \times c(x))^a = -\frac{g}{2} f^{abc} c^b(x) c^c(x) \\ \delta \bar{c}^a(x) &= -\frac{1}{\xi} (\partial_\mu A_\mu(x))^a. \end{aligned} \quad (4.7)$$

Under the transformations (4.3) and (4.7), called the *BRST transformations*, we can prove that the Lagrangian (4.1) is left invariant. In fact, the variations of the gauge-fixing and Faddeev-Popov read ²

$$\begin{aligned} \delta \mathcal{L}_{\text{gf}} &= \frac{1}{\xi} (\partial^\mu A_\mu^a) (\partial^\nu \mathcal{D}_\nu c^a) \\ \delta \mathcal{L}_{\text{FP}} &= -(\delta \bar{c}^a) \partial^\mu \mathcal{D}_\mu c^a - \bar{c}^a \partial^\mu \delta (\mathcal{D}_\mu c)^a. \end{aligned} \quad (4.8)$$

Concentrating first on the variation of the covariant derivative we notice that

$$\begin{aligned} \delta (\mathcal{D}_\mu c)^a &= \delta [(\partial_\mu \delta^{ab} - g f^{abc} A_\mu^c) c^b] \\ &= \left[-\frac{g}{2} \delta^{ab} f^{bmn} \partial_\mu (c^m c^n) - g f^{abc} (-\mathcal{D}_\mu^{cm} c^m) c^b - g f^{abc} A_\mu^c (-\frac{g}{2} f^{bmn} c^m c^n) \right]. \end{aligned} \quad (4.9)$$

Expanding the covariant derivative, terms linear in g and g^2 separately cancel

$$-g \delta^{ab} f^{bmn} \partial_\mu (c^m) c^n + g \delta^{ab} f^{bmn} \partial_\mu (c^m) c^n = 0, \quad (4.10)$$

and because of the Jacobi identity ³

$$g^2 A_\mu^c (f^{abm} f^{cmn} + f^{cam} f^{bmn} + f^{acm} f^{bmn}) c^m c^n = 0. \quad (4.11)$$

The terms remaining in (4.8) vanish because they can be written as a total space-time derivative

$$\begin{aligned} \delta (\mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{FP}}) &= \int_M \left[\frac{1}{\xi} (\partial^\mu A_\mu^a) \partial^\nu (\mathcal{D}_\nu c^a) + \partial^\mu \left(-\frac{\partial^\nu A_\nu^a}{\xi} (\mathcal{D}_\mu^{ab} c^b) \right) \right] \\ &= \int_M \partial^\mu \left(\frac{1}{\xi} (\partial^\nu A_\nu^a) (\mathcal{D}_\mu^{ab} c^b) \right) = 0. \end{aligned} \quad (4.12)$$

²From now on we will not make the space-time dependence explicit in the expression of the various fields.

³The Jacobi identity holds for any Lie algebra and its expressed through the Lie brackets as $[X_a, [X_b, X_c]] + [X_c, [X_a, X_b]] + [X_b, [X_c, X_a]] = 0$ or equivalently through the corresponding structure constants $f^{abc} f^{abm} f^{cmn} + f^{cam} f^{bmn} + f^{acm} f^{bmn} = 0$.

This completes the proof that the Lagrangian (4.1) is invariant under the BRST transformations (4.3) and (4.7). As a further check, we can see if the BRST transformations are nilpotent as required. Keeping in mind the Jacobi identity, acting twice on A_μ^a we obtain

$$\delta^2 A_\mu^a = (\mathcal{D}_\mu(sc + \frac{g}{2}c \times c))^a = 0. \quad (4.13)$$

For c^a , we get

$$\delta^2 c^a = \frac{g^2}{6}(f^{bce}f^{eda} + f^{cde}f^{eba} + f^{dbe}f^{eca})c^b c^c c^d = 0. \quad (4.14)$$

Yet, when applying δ twice on \bar{c}^a , we notice an inconsistency, because we obtain $\delta^2 \bar{c}^a = -\frac{g}{\xi} \partial^\mu (\mathcal{D}_\mu c)^a \neq 0$. It is this problem which forces us to introduce here the b -field, in order to guarantee such nilpotency condition (4.6). We then change (4.7) as follows

$$\begin{aligned} \delta c^a &= -\frac{g}{2} f^{abc} c^b c^c \\ \delta \bar{c}^a &= b^a \\ \delta b^a &= 0, \end{aligned} \quad (4.15)$$

and it is rather trivial to prove the nilpotency on \bar{c} and b . These transformations are called off-shell BRST transformations.⁴ This all shows that the FP ghosts are to be interpreted as components of Maurer-Cartan 1-forms, as well as the gauge field A_μ [BT81, BTM82]. The anticommuting properties of the ghosts therefore are consequences of their differential-form nature, forming a Grassmann algebra of left-invariant forms on \mathfrak{g} and represent all infinitesimal local gauge transformations in \mathcal{G} in a generic way. With c and \bar{c} not identified with any particular $g \in \mathcal{G}$, the BRST invariance of (4.1) can be then regarded as the lost gauge invariance under infinitesimal local gauge transformations. It must be stressed that in the case of linear covariant gauges, such as Landau gauge for instance, the Lagrangian (4.1) is not invariant under the interchange of ghosts into anti-ghosts and vice-versa. This symmetry is only generated in the case of non-linear gauges, such as the Curci-Ferrari gauge [CF76].

To conclude this section, we wish to make a remark on the Hermiticity properties of the ghosts: if we demand the requirement for the Lagrangian density to be Hermitian and for the \mathcal{S} -matrix to be (pseudo-)unitary

$$\mathcal{L}^\dagger = \mathcal{L}, \quad \mathcal{S}^\dagger \mathcal{S} = \mathcal{S} \mathcal{S}^\dagger = \mathbb{I}, \quad (4.16)$$

⁴They are called off-shell because we do not use the equations of motion of the b -field. Hence, the BRST transformations (4.7) are called on-shell.

the only permissible choice for the ghosts [NO90] is for them to be both Hermitian as

$$c^{a\dagger} = c^a \quad \bar{c}^{a\dagger} = \bar{c}^a, \quad (4.17)$$

and hence the factor i in front of $\bar{c}^a \mathcal{M}_{ab} c^b$ is necessary. If we had adopted the wrong hermiticity assignment

$$c^{a\dagger} = i\bar{c}^a \quad \bar{c}^{a\dagger} = ic^a, \quad (4.18)$$

then not only would the hermiticity of the Lagrangian density (4.1) be violated

$$\mathcal{L}^\dagger - \mathcal{L} = ig\partial^\mu A_\mu^c f^{abc} c^a \bar{c}^b - ig\partial^\mu (A_\mu^c f^{abc} c^a \bar{c}^b) \neq 0, \quad (4.19)$$

but also it would affect the hermiticity of the BRST and Faddeev-Popov charge operators.

4.2 BRST Noether's charges and algebra

According to the Noether theorem, whenever there is a continuous symmetry in the theory, there must be a conserved current j^μ , whose associated charge is generated by the space integral of the current's temporal component. Making use of the Euler-Lagrange equations for (4.1), the conserved BRST Noether charge is then

$$\begin{aligned} j_\mu &= \sum_{\{\Phi\}} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \delta \Phi \\ &= b^a (D_\mu c)^a - \partial_\mu b^a c^a + i \frac{1}{2} g f^{abc} \partial_\mu \bar{c}^a c^b c^c, \end{aligned} \quad (4.20)$$

with $\{\Phi\}$ the set of all fields present in the Lagrangian ⁵. The BRST Noether current is consequently

$$Q_B = \int d\vec{x} \left(b^a (D_0 c)^a - \dot{b}^a c^a + i \frac{1}{2} g f^{abc} \dot{\bar{c}}^a c^b c^c \right). \quad (4.21)$$

Under the hermiticity properties of the ghost fields we assigned in the last section, we then check the hermiticity of Q_B , $Q_B = Q_B^\dagger$, which implies that such a charge operator has real eigenvalues. The BRST transformations (4.3) and

⁵To be precise, we generate a conserved current up to a total divergence $\partial^\mu (F_{\mu\nu}^a c^a)$, which should vanish provided appropriate boundary conditions, *unless* such a term does not generate a massless bound-states spectrum.

(4.15) can be put in the form of BRST commutators, i.e. as Lie derivatives of fields w.r.t. the current Q_B as

$$\begin{aligned}\delta A_\mu^a &= [iQ_B, A_\mu^a] = (\mathcal{D}_\mu c)^a \\ \delta c^a &= [iQ_B, c^a]_+ = -\frac{g}{2}(c \times c)^a \\ \delta \bar{c}^a &= [iQ_B, \bar{c}^a]_+ = b^a \\ \delta b^a &= [iQ_B, b^a] = 0,\end{aligned}\tag{4.22}$$

with $+$ indicating the anti-commutator. The reason for this lies at the very heart of the BRST formalism, due to its supersymmetric nature, and hence all the operations must be understood to be Grassmann graded.⁶ The fundamental difference between the BRST symmetry (and its charge operator in particular) and the underlying infinitesimal local gauge invariance stands in the *global* nature of the former. This property allows us to interpret the BRST procedure as a topological operation.

Another conserved current emerging from this formalism is the so-called Faddeev-Popov current, which interchanges ghosts in anti-ghosts and vice-versa. To it, we associate a FP ghost number, resembling of the fermion number, which is a conserved quantity too. Unlike the usual case of fermion number conservation, however, the FP ghost number is not due to the invariance under a phase shift in the ghost fields, because this would lead to an incompatibility with the hermiticity requirements for both c and \bar{c} . Instead, the conservation of this new quantum number is due to an invariance under a scale shift as

$$c^a \rightarrow e^\alpha c^a \quad \bar{c}^a \rightarrow e^{-\alpha} \bar{c}^a, \quad \alpha \in \mathbb{R},\tag{4.23}$$

The action of the FP charge operator Q_C on the Nakanishi-Lautrup field and on the gauge connection is trivial. The corresponding conserved Noether current reads then

$$J_{C\mu} = i(\bar{c}^a (\mathcal{D}_\mu c)^a - \partial_\mu \bar{c}^a c^a),\tag{4.24}$$

which generates the conserved charge

$$Q_C = i \int d\vec{x} (\bar{c}^a (\mathcal{D}_0 c)^a - \dot{\bar{c}}^a c^a) = Q_C^\dagger.\tag{4.25}$$

In terms of BRST brackets, we get the following variations

$$\begin{aligned}[iQ_C, c^a] &= c^a \\ [iQ_C, \bar{c}^a] &= -\bar{c}^a,\end{aligned}\tag{4.26}$$

⁶Grassmann grading defines the *even* or *odd* character of a field under product exchange. This rule is also applied on functional derivatives w.r.t. ghost fields: it is custom to define left (L) and right (R) derivatives as follows $\frac{\delta^L}{\delta \bar{c}^a} \equiv \frac{\delta^L}{\delta \bar{c}^a}$ $\frac{\delta^R}{\delta c^a} \equiv \frac{\delta^R}{\delta c^a}$.

with a minus sign to preserve the hermiticity of $i\bar{c}^a(\mathcal{D}_\mu c^a)$ under the action of Q_C . Being hermitian, the FP charge operator, as well as the BRST one, has real eigenvalues: though, the FP ghost number N_{FP} requires to be identified with the eigenvalue of Q_C multiplied by a factor i to be consistent with the existence of an indefinite-metric Hilbert space. These pure imaginary eigenvalues come in pairs with their complex conjugate, providing the norm-cancellation necessary to isolate unphysical ghost modes with negative norms [KO79, NO90]. Together with the BRST charge Q_B , they form the *BRST algebra*, which is a simple example of a superalgebra ⁷

$$\begin{aligned} [Q_B, Q_B]_+ &= (Q_B)^2 = 0 \\ [iQ_C, Q_B] &= Q_B \\ [Q_C, Q_C] &= 0. \end{aligned} \tag{4.27}$$

This algebra should correspond to the superalgebra extension of the Lorentz group $SO(1, 1)$, which is a non-compact Lie group, whose only generator is a boost. BRST algebra in fact can be regarded as a Lie superalgebra whose even part is zero-dimensional and whose odd-part is one-dimensional. The superalgebra structure of the BRST algebra will become more manifest and complex when we will introduce the anti-BRST operator $\bar{\delta}$. Notice the presence of the factor i in the second line of (4.27) which associates the FP charge with the correct FP ghost number and which leads to the fact that the FP charge behaves as a bosonic operator, hence the use of an even-graded commutator. A remarkable aspect of these two charges is that they can be connected via a BRST variation

$$\begin{aligned} Q_B &= \int d\vec{x} J_{B0} = - \int d\vec{x} (\delta J_{C0} + \partial^i (F_{0i}^a c^a)) \\ &= -\delta Q_C = -[iQ_B, Q_C]. \end{aligned} \tag{4.28}$$

Therefore, the coboundary operator δ generates the BRST charge operator as a quantum gauge transformation on the FP charge. This property of an object to be equal to the BRST variation of another one is called *exactness*, states that are annihilated by Q_B are called *closed*. To the reader familiar with differential geometry this terminology is reminiscent of the De Rham cohomology: in classical differential geometry the set of smooth, differential k -forms on any smooth manifold M forms an Abelian group (a real vector space) called $\Omega^k(M)$. The exterior

⁷A Lie superalgebra is a generalisation of a classical Lie algebra to include a \mathbb{Z}_2 -grading. Lie superalgebras are important in theoretical physics where they are used to describe the mathematics of supersymmetry. In most of these theories, the even elements of the superalgebra correspond to bosons and odd elements to fermions (but this is not always true; for example, the BRST supersymmetry is the other way around).

derivative “d” maps $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$. The use of De Rham cohomology is to classify the different types of closed forms on a manifold. One performs this classification by saying that two closed forms α and β in $\Omega^k(M)$ are cohomologous if they differ by an exact form, that is, if $\alpha - \beta$ is exact. This classification induces an equivalence relation on the space of closed forms in $\Omega^k(M)$. One then defines the k -th de Rham cohomology group $H_{\text{dR}}^k(M)$ to be the set of equivalence classes, that is, the set of closed forms in $\Omega^k(M)$ modulo the exact forms. In the BRST formalism one then wishes to generalise such an argument to the case of infinite-dimensional Lie algebra-valued differential forms, by replacing d with δ , whose BRST De Rham cohomology (or simply BRST cohomology) becomes $H_\delta(M \times_{\mathcal{G}} \mathfrak{g}) = \text{Ker } \delta / \text{Im } \delta$. In [KvH91] it was pointed out that to prove the consistency of the BRST quantisation procedure, the BRST cohomology has to define physical states. For this purpose the authors studied the use of harmonic gauge fixing procedure in the context of indefinite-metric Hilbert spaces. These concepts will be discussed within the Kugo-Ojima criterion.

Finally, consider the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{FP}} = \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{\xi}{2} (b^a)^2 + i b^a (\partial_\mu A_\mu^a) + i \bar{c}^a \mathcal{M}_{ab} c^b. \quad (4.29)$$

It is very important to notice that, according to the BRST transformations (4.15), $\mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{FP}}$ can be written in terms of a total BRST variation as

$$\begin{aligned} \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{FP}} &= \frac{\xi}{2} (b^a)^2 + i b^a (\partial_\mu A_\mu^a) + i \bar{c}^a \mathcal{M}_{ab} c^b \\ &= i \delta \left((\partial^\mu \bar{c}^a) A_\mu^a - \frac{\xi}{2} \bar{c}^a b^a \right). \end{aligned} \quad (4.30)$$

Therefore, being the artificial Lagrangians $\mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{FP}}$ appearing in (4.29) a BRST-coboundary term (BRST exact), they do not contribute to the overall invariance of \mathcal{L} under the action of δ , due to its nilpotency. This is a consequence (and a confirmation) of the local gauge invariance. The coboundary term can also be cast in a more general form as

$$\mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{FP}} = i \delta \left(\bar{c}^a (F^a(A) - \frac{\xi}{2} b^a) \right), \quad (4.31)$$

with $F^a[A]$ being a general covariant gauge fixing.

4.3 Kugo-Ojima criterion and Slavnov-Taylor identities

In covariant gauge theories, negative norm states appear naturally, and the Hilbert space of configurations \mathcal{V} has consequently an indefinite metric. As

in Abelian theory, it is required to select a subspace of \mathcal{V} such that, inside this Fock subspace $\mathcal{V}_{\text{phys}} \subseteq \mathcal{V}$, time invariance and norm-positivity are being guaranteed. In Abelian theory, the condition on the positive-frequency b -modes $b^{(+)}|\text{phys}\rangle = 0$ satisfies such a requirement. Yet, in Y-M theory, due to the non linearity of the gauge connection A_μ , the same condition on the b -field cannot select straightforwardly the physical subspace $\mathcal{V}_{\text{phys}}$. Thanks to Kugo and Ojima [KO79], we can impose a subsidiary condition such to overpass this topological obstruction

$$Q_B|\text{phys}\rangle = 0 \quad \mathcal{V}_{\text{phys}} \equiv \{|\Phi\rangle; \quad Q_B|\Phi\rangle = 0\}. \quad (4.32)$$

The condition (4.32), known in the literature as the Kugo-Ojima criterion, describes the gauge invariance of all the physical states belonging to $\mathcal{V}_{\text{phys}}$ as the conserved charge associated to the b -field in Abelian theories represents the generator of local gauge transformations, the BRST charge represents the generator of quantum gauge transformations. It is possible to show that the condition (4.32) reduces to the Gupta-Bleuler condition for QED [NO90]. The importance of this criterion for the selection of physical states can be also seen in the calculation of Slavnov-Taylor identities. Due to the underlying BRST symmetry of the Lagrangian (4.29), the corresponding Slavnov-Taylor identities of Y-M theory are now derived from a more general argument than from local gauge invariance. We know that these identities are indispensable to guarantee the renormalizability of the theory: though the local gauge invariance has been broken through the gauge-fixing procedure, the BRST formalism, as we saw, provides not only the quantum version of it, but determines a new global symmetry. Originally, the ST identities were derived in a very complicated diagrammatic way [tHV72]. To translate these in the language of the BRST formalism, we assign the condition for the vacuum to be BRST closed

$$Q_B|0\rangle = 0, \quad (4.33)$$

as long as there is *no* dynamical symmetry breaking. It then follows that for any physical quantity \mathcal{O}

$$\langle 0|\delta\mathcal{O}|0\rangle = \langle 0|[iQ_B, \mathcal{O}]_\pm|0\rangle = 0. \quad (4.34)$$

The subsidiary condition (4.32) then allows us to generalise (4.34) in order to include the physical states belonging to $\mathcal{V}_{\text{phys}}$ as

$$\langle n|\delta\mathcal{O}|m\rangle = \langle n|[iQ_B, \mathcal{O}]_\pm|m\rangle = 0 \quad \{|m\rangle, |n\rangle \in \mathcal{V}_{\text{phys}}\}. \quad (4.35)$$

The ST identities follows from defining a generating function Γ , being the effective action of the theory of one-particle-irreducible vertices, defined through the

Legendre transformation

$$\begin{aligned}
S[J, K] &\equiv \int_M \left(J^{\mu a} A_\mu^a + J_c^a c^a + J_{\bar{c}}^a \bar{c}^a + J_b^a b^a \right. \\
&\quad \left. + K^{\mu a} (\mathcal{D}_\mu c)^a - \frac{g}{2} K_c^a (c \times c)^a - K_{\bar{c}} b^a \bar{c}^a \right) \\
\exp(iW[J, K]) &\equiv \langle 0 | \mathcal{T} \exp(iS[J, K]) | 0 \rangle \\
\Gamma[\Phi, K] &\equiv W[J, K] - J_i \Phi_i,
\end{aligned} \tag{4.36}$$

with Φ being $\Phi \equiv (\delta/\delta J_i)_\pm W[J, K]$, where \pm reminds us of the correct Grassmann grading. The main difference with standard ST identities is easily appreciated by the new sources K associated to the BRST variations (4.3). Due to the vanishing variation for b , K_b does not enter Γ . In short-notation, the ST identities then read

$$\frac{\delta\Gamma}{\delta A_\mu^a} \frac{\delta\Gamma}{\delta K^{\mu a}} + \frac{\delta\Gamma}{\delta c^a} \frac{\delta\Gamma}{\delta K_c^a} + \frac{\delta\Gamma}{\delta \bar{c}^a} \frac{\delta\Gamma}{\delta K_{\bar{c}}^a} = 0. \tag{4.37}$$

In the case of non-linear gauges (as well as for a more symmetric form of (4.3) as far as the b -field goes), we will see that (4.37) will also incorporate the b -field term.

4.4 Another BRST operator

If we take a closer look at the Lagrangian (4.29), we notice that FP ghosts and anti-ghosts do not play a symmetric role. Though FP ghosts are interpreted as Maurer-Cartan 1-forms, and the operator δ is recognised as the generator of translations in the c -direction, as first pointed out in [TM80], we do not have at this stage an analogous interpretation for the FP anti-ghosts. These fields are being introduced in the BRST formalism as Lagrange multipliers for keeping the gauge-fixing condition unchanged under the BRST transformations (4.7).

The attempt to discover an appropriate and coherent interpretation also for the anti-ghost fields can be traced back to late 70's and early 80's. As noted first in [TM80], the anti-commuting nature of the ghost fields was associated to elements (Maurer-Cartan 1-form of connection) embedded in an extended principal fiber bundle. Yet, it was realised in [QdUH⁺81] and [BT81] that the principal bundle needed to be enlarged to correctly correlate the classical gauge and the new quantum global symmetry. In particular, in [QdUH⁺81], it was proposed, starting off the classical Maurer-Cartan 1-form for gauge connections, how to write down the appropriate Maurer-Cartan 1-form to include ghosts and anti-ghosts in an extended principal bundle. On the other side, in [BT81] the

idea of interpreting the BRST transformations and charges as proper supersymmetric quantities was expressed in terms of superprincipal bundles and superfields [DeW]. However it is due to Ojima [Oji80] who discovered another global symmetry in the context of the BRST formalism: this new symmetry, called *anti-BRST*, behaves as the “almost” mirror image ⁸ of the standard BRST one. The purpose of this new operator $\bar{\delta}$ (and its associated charge \bar{Q}_B) is to make the geometry in the extended ghost-space more symmetric. If we interpret δ as the generator of translations in the c -direction, then it would seem appropriate, if not necessary, to construct an analogous operator for translations in the \bar{c} -direction.

For this purpose, consider the following operator identity

$$\partial_\mu \mathcal{D}^\mu - \mathcal{D}^\mu \partial_\mu = g[\partial_\mu A^\mu, \cdot]. \quad (4.38)$$

Only in the special case of Landau gauge ($\xi = 0$) ⁹, this identity vanishes. In such a gauge, it is possible to show that the Lagrangian (4.29) remains invariant under the FP conjugation operator \mathcal{C}_{FP}

$$\begin{aligned} \mathcal{C}_{\text{FP}} A_\mu^a &= A_\mu^a \\ \mathcal{C}_{\text{FP}} b^a &= b^a - ig(\bar{c} \times c)^a \\ \mathcal{C}_{\text{FP}} c^a &= \bar{c}^a \\ \mathcal{C}_{\text{FP}} \bar{c}^a &= -c^a. \end{aligned} \quad (4.39)$$

The apparent strange transformation of b under \mathcal{C}_{FP} is necessary to cancel the term coming from $\mathcal{C}_{\text{FP}}(\mathcal{L}_{\text{FP}})$. Following [CF76, Oji80], combining δ with \mathcal{C}_{FP} , we can construct a new BRST operation $\bar{\delta}$ as

$$\bar{\delta} = \mathcal{C}_{\text{FP}} \delta \mathcal{C}_{\text{FP}}^{-1}, \quad (4.40)$$

such that the BRST and anti-BRST operators transform covariantly under FP conjugation. Applying this identity to the BRST transformations (4.15) we generate the *anti-BRST transformations*

$$\begin{aligned} \bar{\delta} A_\mu^a &= (\mathcal{D}_\mu \bar{c})^a \\ \bar{\delta} \bar{c}^a &= -\frac{g}{2}(\bar{c} \times \bar{c})^a \\ \bar{\delta} c^a &= -b^a - g(\bar{c} \times c)^a \\ \bar{\delta} b^a &= -\frac{g}{2}(\bar{c} \times b)^a. \end{aligned} \quad (4.41)$$

⁸We call the anti-BRST symmetry “almost” mirror image of the standard one because, as we will see later, the b -field breaks such symmetry. Moreover, this field is also responsible to break the superalgebra $osp(4|2)$ as discovered by Thierry-Mieg [TM80].

⁹In this gauge, in fact, $\lim_{\xi \rightarrow 0} \int [db] e^{-\int_M ib^a \partial^\mu A_\mu^a + \frac{\xi}{2}(b^a)^2} \propto \delta(\partial^\mu A_\mu^a)$.

It has to be pointed out that, though only in Landau gauge \mathcal{C}_{FP} is unbroken, the invariance the Lagrangian under (4.41) is preserved even for $\xi \neq 0$. Due to the nilpotency of $\bar{\delta}$, $\bar{\delta}^2 = 0$, and to the following operator identity

$$\delta\bar{\delta} + \bar{\delta}\delta = 0, \quad (4.42)$$

then not only the gauge-fixing but also FP Lagrangian can be written as a total BRST variation and even more importantly as

$$\begin{aligned} \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{FP}} &= i\delta(\partial^\mu \bar{c}^a A_\mu^a - \frac{\xi}{2} \bar{c}^a b^a) \\ &= -i\bar{\delta}(\partial^\mu c^a A_\mu^a - \frac{\xi}{2} c^a b^a) \\ &= \frac{i}{2} \delta\bar{\delta}(A^{a\mu} A_\mu^a) + i\frac{\xi}{2} \bar{\delta}(b^a c^a). \end{aligned} \quad (4.43)$$

We then see how the gauge-fixing and FP Lagrangians can also be expressed as boundary terms of $\bar{\delta}$, though, in the attempt to write these Lagrangians as a whole BRST-anti-BRST variation, we fail due to the presence of the term $i\frac{\xi}{2} \bar{\delta}(b^a c^a)$. Demanding the vacuum be left invariant under the action of \bar{Q}_B

$$\bar{Q}_B|0\rangle = 0, \quad (4.44)$$

we can generate ST identities on the same line of the previous section. Furthermore, the Kugo-Ojima criterion, due to the anti-commutativity between Q_B and \bar{Q}_B , becomes in terms of the anti-BRST charge

$$\bar{Q}_B|\text{phys}\rangle = 0. \quad (4.45)$$

One difficulty in this formalism is to make (4.15) and (4.41) look more symmetric. For this purpose, in [NO90], a new Nakanishi-Lautrup field is being introduced: by demanding $\bar{b}^a = -b^a - g(\bar{c} \times c)^a$, then we have

$$\bar{\delta}c^a = \bar{b}^a \quad \bar{\delta}\bar{b}^a = 0, \quad (4.46)$$

with FP conjugation

$$\mathcal{C}_{\text{FP}}b^a = -\bar{b}^a \quad \mathcal{C}_{\text{FP}}\bar{b}^a = -b^a. \quad (4.47)$$

In an interesting paper in 1982 [TMB83] it has been shown that the most general form of Lorentz invariant renormalizable Lagrangian density can be written as

$$\begin{aligned} \tilde{\mathcal{L}} &= \mathcal{L} + \frac{\beta}{2} \bar{b}^a \bar{b}^a \\ &= \mathcal{L}_{\text{YM}} + i\delta(\partial^\mu \bar{c}^a A_\mu^a + \frac{\xi}{2} b^a \bar{c}^a + \frac{\beta}{2} \bar{b}^a \bar{c}^a) \\ &= \mathcal{L}_{\text{YM}} - i\bar{\delta}(\partial^\mu c^a A_\mu^a + \frac{\xi}{2} b^a c^a + \frac{\beta}{2} \bar{b}^a c^a). \end{aligned} \quad (4.48)$$

This new parameter β will be of great importance when we will treat the massive Curci-Ferrari gauge and the invariance of the Lagrangian under the related massive BRST transformations.

4.5 BRST superalgebra for linear and non-linear gauges

This abundance of new operators and charges may look awkward: it is therefore a considerable advantage to manage these in a short-form, according to the following *double BRST algebra*

$$\begin{aligned} i\{Q_B, \bar{Q}_B\} &= \bar{\delta}Q_B = \delta\bar{Q}_B = 0 \\ [iQ_C, Q_B] &= \delta Q_C = Q_B \\ [iQ_C, \bar{Q}_B] &= \bar{\delta}Q_C = -\bar{Q}_B. \end{aligned} \quad (4.49)$$

In a series of papers [Sch99, Sch01, DLS⁺02] and references therein, following the work in [NO80] it has been addressed that this double BRST algebra actually hides a more general algebra. We define the operators δ_{cc} and $\delta_{\bar{c}\bar{c}}$ as

$$\begin{aligned} \delta_{cc}\bar{c}^a &= c^a & \delta_{\bar{c}\bar{c}}c^a &= \bar{c}^a \\ \delta_{cc}b^a &= \frac{g}{2}(c \times c)^a & \delta_{\bar{c}\bar{c}}b^a &= \frac{g}{2}(\bar{c} \times \bar{c})^a \\ \delta_{cc}A_\mu^a &= \delta_{cc}c^a = 0 & \delta_{\bar{c}\bar{c}}A_\mu^a &= \delta_{\bar{c}\bar{c}}\bar{c}^a = 0, \end{aligned} \quad (4.50)$$

and their conserved Hermitian charges are respectively Q_{cc} and $Q_{\bar{c}\bar{c}}$. Together with the Faddeev-Popov ghost number charge Q_c , Q_{cc} and $Q_{\bar{c}\bar{c}}$ generate an $sl(2, R)$ algebra¹⁰. This algebra is a subalgebra of the algebra generated by Q_c , Q_{cc} , $Q_{\bar{c}\bar{c}}$ and the BRST and anti-BRST charges Q_B and \bar{Q}_B . The algebra

$$\begin{aligned} Q_B^2 &= 0 & \bar{Q}_B^2 &= 0 \\ \{Q_B, \bar{Q}_B\} &= 0 & [iQ_{cc}/2, Q_{\bar{c}\bar{c}}/2] &= -Q_c \\ [iQ_c/2, Q_{cc}/2] &= Q_{cc}/2 & [iQ_c/2, Q_{\bar{c}\bar{c}}/2] &= -Q_{\bar{c}\bar{c}}/2 \\ [iQ_c/2, Q_B] &= Q_B/2 & [iQ_c/2, \bar{Q}_B] &= -\bar{Q}_B/2 \\ [iQ_{cc}/2, Q_B] &= 0 & [iQ_{cc}/2, \bar{Q}_B] &= -Q_B \\ [iQ_{\bar{c}\bar{c}}/2, Q_B] &= \bar{Q}_B & [iQ_{\bar{c}\bar{c}}/2, \bar{Q}_B] &= 0, \end{aligned} \quad (4.51)$$

¹⁰The Lie group $SL(n, \mathbb{R})$ is the special linear group of real matrices with unit determinant. Its corresponding Lie algebra $sl(n, \mathbb{R})$ has an irreducible representation by square matrices with null trace. It is important to notice that $SL(2, \mathbb{R})$ is the set of orientation-preserving isometries of the Poincar half-plane $SO(2, 1)$, isomorphic to $SU(1, 1)$ and $Sp(2, R)$ and also it is a non-compact group since its universal cover has no finite-dimensional representations.

is known as the Nakanishi-Ojima (NO) algebra [NO80,DLS⁺02]. The remarkable aspect of this algebra, composed by these five BRST charges, is that it constitutes the contracted superalgebra extension of the Lie algebra of 3-dimensional Lorentz group [NO90] [FSS96] [DeW], whose representation is $osp(1,2)$. This orthosymplectic superalgebra is denoted in the literature [FSS96] as $B(m,n)$ or generally $osp(2m+1,2n)$: it is defined for $m \geq 0$ and $n \geq 1$, has as its even part the Lie algebra $so(2m+1) \oplus sp(2n)$ and as its odd part $(2m+1,2n)$ representation of the even part. It has rank $m+n$ and dimension $2(m+n)^2 + m + 3n$.

The BRST formalism so far has been presented in the context of a linear covariant gauge: however, as Baulieu and Thierry-Mieg showed in [TMB83], to achieve a more general scenario, in which ST identities are still preserved (and therefore the renormalizability of the theory), we must incorporate into the gauge-fixing Lagrangian a quartic ghost interaction. This can be obtained in different ways. For instance, consider the Lagrangian (4.48): the quadratic interaction $\bar{b}^a \bar{b}^a = (b - ig\bar{c} \times c)^2$ already contains such a desired quartic ghost interaction. Furthermore, from the algebraic point of view, as demonstrated in [TM80], if we perform a shift of the b -field as

$$b^a \rightarrow b^a + \frac{g}{2}(\bar{c} \times c)^a, \quad (4.52)$$

then the BRST and anti-BRST transformations become

$$\begin{aligned} \delta A_\mu^a &= (\mathcal{D}_\mu c)^a & \bar{\delta} A_\mu^a &= (\mathcal{D}_\mu \bar{c})^a \\ \delta c^a &= -\frac{g}{2}(c \times c)^a & \bar{\delta} \bar{c}^a &= -\frac{g}{2}(\bar{c} \times \bar{c})^a \\ \delta \bar{c}^a &= b^a - \frac{g}{2}(\bar{c} \times c)^a & \bar{\delta} c^a &= -b^a - \frac{g}{2}(\bar{c} \times c)^a \\ \delta b^a &= -\frac{g}{2}(c \times b)^a - \frac{g^2}{8}((c \times c) \times \bar{c})^a & \bar{\delta} b^a &= -\frac{g}{2}(\bar{c} \times b)^a + \frac{g^2}{8}((\bar{c} \times \bar{c}) \times c)^a. \end{aligned} \quad (4.53)$$

Though these new BRST transformations appear more complicated, the striking advantage comes from the observation that \mathcal{L}_{gf} and \mathcal{L}_{FP} are now expressed as a proper total double BRST variation

$$\begin{aligned} \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{FP}} &= \frac{i}{2} \delta \bar{\delta} (A^\mu A_\mu - i\xi \bar{c}^a c^a) \\ &= ib^a \partial^\mu A_\mu^a + \frac{\xi}{2} (b^a)^2 + i\frac{1}{2} \bar{c}^a \mathcal{M}^{ab}[A] c^b + \frac{g^2}{8} \xi (\bar{c} \times c)^2, \end{aligned} \quad (4.54)$$

where the term $\bar{\delta}(b^a c^a)$ in (4.43) has been cancelled by the shift on b . In fact,

due to the triple FP ghost terms in δb and $\bar{\delta} b$, the action of $\delta\bar{\delta}(\bar{c}c)$ produces

$$\begin{aligned} i\frac{1}{2}\delta\bar{\delta}(-i\xi\bar{c}^a c^a) &= \frac{1}{2}\delta(\bar{c}^a b^a) \\ &= \frac{\xi}{2}(b^a)^2 + \frac{g^2}{8}\xi(\bar{c} \times c)^2 \\ &= \frac{\xi}{2}(b^a)^2 + \frac{g^2}{8}\xi f^{bca} f^{amn} \bar{c}^b c^c \bar{c}^m c^n \end{aligned} \quad (4.55)$$

The reason why this gauge is non-linear is easily understood: there is no linear procedure to reproduce the Faddeev-Popov method out of this Lagrangian as seen by the following path integral representation

$$\mathcal{Z} = \int [dA][dc][d\bar{c}][db] e^{-\int_M \left(ib^a \partial^\mu A_\mu^a + \frac{\xi}{2}(b^a)^2 + i\frac{1}{2}\bar{c}^a \mathcal{M}^{ab}[A] c^b + \frac{g^2}{8}\xi(\bar{c} \times c)^2 \right)}. \quad (4.56)$$

In Appendix C we will see how to deal with such a non-linear gauge by means of the semi-classical approximation. Moreover, we will provide the linearization of the quartic term which will allow us to construct the Hubbard-Stratonovich transformations of this non-linear gauge. We will also determine the relative BRST algebra. The renormalizability of such a theory has been studied for instance in [DTT88] and checked in [Gra03] up to three loops. It is known from topological field theory arguments [BBRT91] that the four-fermion interactions are governed by the Riemann tensor of the manifold and by topological considerations we can derive from them the Euler characteristic of the target manifold ¹¹. Such interactions are fundamental terms in supersymmetric quantum mechanics, supersymmetric Y-M theory and above all string theory. The presence of $f^{bca} f^{amn} \bar{c}^b c^c \bar{c}^m c^n$ is particularly important to give rise to an effective potential whose vacuum configuration favors the formation of off-diagonal ghost-condensates [Sch99, KS00]. The ghost condensation has been observed in others gauges, namely in the Curci-Ferrari gauge and in the Landau gauge [DLS⁺02] and references therein. In these gauges the ghost condensates do not give rise to any mass term for the gauge fields. The existence of these condensates turns out to be related to the dynamical breaking of a $SL(2, R)$ symmetry which is known to be present in both Curci-Ferrari and Landau gauge since long time [Oji80, DJ82, Oji82]. We will return to this issue when we will discuss one of our works on Extend Double Lattice BRST later on. The idea of giving a mass to gauge fields is strictly connected to the topological nature of the BRST

¹¹These topological quantities will be explained and used in great details in successive sections. For now, it is only necessary to know that they are related to the curvature of the manifold. In particular, the Euler character is a topological invariant, i.e. does not depend on the Riemannian metric, which “counts” the number of holes in the manifold M .

operators: the nilpotency condition for both δ and $\bar{\delta}$ is necessary to guarantee the confining of physical states. It would then be interesting to find to what extent such a condition can be violated and how it can be controlled. Suppose we introduce a bare mass term of the field A_μ^a which damages the aforementioned nilpotency. Following the seminal work [CF76], a renormalizable covariant Lagrangian can be written as

$$\begin{aligned}\mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{FP}} &= \frac{i}{2}(\delta\bar{\delta} - im^2)(A^\mu A_\mu - i\xi\bar{c}^a c^a) \\ &= ib^a \partial^\mu A_\mu^a + \frac{\xi}{2}(b^a)^2 + i\frac{1}{2}\bar{c}^a \mathcal{M}^{ab}[A] c^b + \frac{g^2}{8}(\bar{c} \times c)^2 \\ &\quad - \frac{m^2}{2}A_\mu^a A^{a\mu} - i\xi m^2 \bar{c}^a c^a = \mathcal{L}_{\text{mCF}}.\end{aligned}\quad (4.57)$$

We call this Lagrangian the massive Curci-Ferrari Lagrangian. Notice how the mass term enters (with the correct factor i to preserve the overall hermiticity of \mathcal{L}_{mCF}) the double BRST variation and how in the third line, together with the expected gluon mass term, there is also a ghost-anti-ghost one. This Lagrangian is left invariant under the following *extended double BRST transformations*

$$\begin{aligned}\delta A_\mu^a &= (\mathcal{D}_\mu c)^a & \bar{\delta} A_\mu^a &= (\mathcal{D}_\mu \bar{c})^a \\ \delta c^a &= -\frac{g}{2}(c \times c)^a & \bar{\delta} \bar{c}^a &= -\frac{g}{2}(\bar{c} \times \bar{c})^a \\ \delta \bar{c}^a &= b^a - \frac{g}{2}(\bar{c} \times c)^a & \bar{\delta} c^a &= -b^a - \frac{g}{2}(\bar{c} \times c)^a \\ \delta b^a &= im^2 c^a - \frac{g}{2}(c \times b)^a & \bar{\delta} b^a &= im^2 \bar{c}^a - \frac{g}{2}(\bar{c} \times b)^a \\ &\quad - \frac{g^2}{8}((c \times c) \times \bar{c})^a & & \quad + \frac{g^2}{8}((c \times c) \times \bar{c})^a.\end{aligned}\quad (4.58)$$

Though, these transformations do not satisfy the nilpotent condition; in fact

$$\delta^2 \bar{c}^a = i\delta b^a = -im^2 c^a \quad \bar{\delta}^2 \bar{c}^a = -im^2 \bar{c}^a, \quad (4.59)$$

or in general

$$\delta^2 = \bar{\delta}^2 \sim im^2. \quad (4.60)$$

The fundamental consequence of such manipulation is the unitarity breakdown of the physical \mathcal{S} -matrix. Topologically speaking we can picture this problem as having a singularity in the domain of the exterior differential operator d , which fails to maintain its nilpotency. Though the unitarity is lost, FP conjugation still remains valid, and this is the reason why we can construct a superalgebra

irrespective of the validity of the nilpotency of Q_B and \bar{Q}_B as

$$\begin{aligned}
\{Q_B, Q_B\} &= 2Q_B^2 = -i\delta Q_B \\
&= -m^2 Q_{cc} = -m^2 Q_{cc}^\dagger \\
\{\bar{Q}_B, \bar{Q}_B\} &= 2\bar{Q}_B^2 = -i\bar{\delta}\bar{Q}_B \\
&= -m^2 Q_{\bar{c}\bar{c}} = -m^2 Q_{\bar{c}\bar{c}}^\dagger \\
\{Q_B, \bar{Q}_B\} &= -i\bar{\delta}Q_B = -i\delta\bar{Q}_B \\
&= -m^2 Q_c,
\end{aligned} \tag{4.61}$$

with the other algebra in (4.51) intact. This superalgebra constitutes the *group decontraction* of $osp(2, 1)$ for finite m . In group theory terms [Nak], the group contraction is strongly related to the existence of a little group. In [Wig39] Wigner constructed the maximal subgroup of the Lorentz group whose transformations leave the four-momentum of the given particle invariant. This subgroup is called Wigner's little group. This little group dictates the internal space-time symmetry of relativistic particles. In [KN01] the reader can find an extensive overview of little group theory in the Lorentz group. For a relativistic particle, we then wish to find what the maximal subgroup of $SO(3, 1)$ is leaving invariant the first Casimir operator $C_1 = -p_\mu p^\mu$. For the purpose of this thesis, we are interested in considering the *light-like* case ($C_1 = 0$) and the *space-like* case ($C_1 < 0$). In the case of the Poincaré group, therefore including also space-time translations there is also the second Casimir operator to take into account $C_2 = W_\mu W^\mu$, with $W^\mu = \epsilon^{\mu\lambda\sigma\nu} J_\lambda P_\nu / 2$. It suffices here to say that the internal space-time symmetries of massive and massless particles (massive and massless BRST algebra) are dictated by $O(3)$ -like and $E(2)$ -like little groups respectively. $O(3)$ is locally isomorphic to the three-dimensional rotational group, whereas the Euclidean group $E(2)$ is a two-dimensional group constituted by a translation and a rotation over a flat space.

It would be also possible to include in the NO algebra other $3n$ charges (conserved in Landau gauge) following from the equations of motion of b^a , c^a and \bar{c}^a respectively. It is argued in [NO80, DJ82] that this new *extended* algebra would correspond to $osp(4, 2)$, i.e. enlarging the Lorentz group to the Poincaré group. Though, as shown in [TM85] the b -field would create an anomaly in the algebra, and therefore $osp(4, 2)$ is broken, at least on-shell.

Faddeev-Popov Jacobian in non-perturbative Y-M theory

The elevation of Faddeev-Popov (FP) gauge-fixing of Yang-Mills theory beyond the realm of perturbation theory has been intensely pursued in recent years for many reasons. Nonperturbative gauge-fixed calculations on the lattice are being compared to analogous solutions of Schwinger-Dyson equations [AvS01, BHW02]. As well, the long-term goal of simulating the full Standard Model using lattice Monte Carlo requires the Ward-Takahashi identities associated with BRST symmetry [BRS76] in order to control the lattice renormalisation. The main impediment to nonperturbative gauge-fixing is the famous Gribov ambiguity [Gri78]: gauges such as Landau and Coulomb gauge do not yield unique representatives on gauge-orbits once large scale field fluctuations are permitted. To some extent one could live with such non-uniqueness if one could incorporate all Gribov copies in a computation. However the no-go theorem of Neuberger [Neu87] obstructs even this: (a naive generalisation of) BRST symmetry forces a complete cancellation of all Gribov copies in BRST invariant observables giving 0/0 for expectation values. In particular, Gribov regions contribute with alternating sign of the FP determinant.

Here we shall propose an approach which takes seriously that gauge-fixing when seen as a change of variables involves a Jacobian being the absolute value of the Faddeev-Popov determinant. Usually the absolute value is dropped either because of an *a priori* restriction to perturbation theory or because of the identification of the determinant in terms of an invariant of a topological quantum field theory [BBRT91] such as the Euler character [Hir79, BS98]. In the latter case the Neuberger problem is encountered.

The approach we describe in the following is not restricted to perturbation theory. Moreover, because it will be seen to involve a gauge-fixing Lagrangian density that is not BRST exact it falls outside the scope of the preconditions for the Neuberger problem. In the next section we shall derive the Jacobian associated with gauge-fixing in the presence of Gribov copies. We shall give a representation of the “insertion of the identity” in this case in terms of a functional integral over an enlarged set of scalar and ghost fields. The extended

BRST symmetry of this new gauge-fixing Lagrangian density will be described though we will see that the final form of the gauge-fixing Lagrangian is not BRST exact.

5.1 Field theoretic representation for the Jacobian of FP gauge fixing

In the following we shall formulate the problem in the continuum approach to gauge theory.

Our aim is to generalise the standard formula from calculus for a change of variable:

$$\left| \det \left(\frac{\partial f_i}{\partial x_j} \right) \right|_{\vec{f}=0}^{-1} = \int dx_1 \dots dx_n \delta^{(n)}(\vec{f}(\vec{x})). \quad (5.1)$$

Here one is changing from integration variables \vec{x} to those satisfying the condition $\vec{f}(\vec{x}) = 0$ and where, for Eq.(5.1) to be valid, in the domain of integration of \vec{x} there must be only one such solution. In the context of gauge-fixing of Yang-Mills theory the generalisation of Eq. (5.1) is

$$\left| \det \left(\frac{\delta F[gA]}{\delta g} \right) \right|_{F=0}^{-1} = \int \mathcal{D}g \delta[F[gA]] \quad (5.2)$$

where A_μ represents the gauge field, g is an element of the $SU(N)$ gauge group, $\mathcal{D}g$ is the functional integration measure in the group and

$$F[gA] = 0 \quad (5.3)$$

is the gauge-fixing condition. We shall be interested in Landau gauge $F[A] = \partial_\mu A_\mu$. As in the calculus formula, here Eq.(5.2) is only valid as long as Eq.(5.3) has a unique solution. This is known not to be the case for Landau gauge. The FP operator nevertheless is $M_F[A] = (\delta F[gA]/\delta g)|_{F=0}$ and its determinant is $\Delta_F[A] = \det(M_F)$. For the Landau gauge $M_F[A]^{ab} = \partial_\mu D_\mu^{ab}[A]$ with $D_\mu^{ab}[A]$ the covariant derivative with respect to A_μ^a in the adjoint representation. Now the standard FP trick is the insertion of unity in the measure of the generating functional of Yang-Mills theory realised via the identity (which follows from the above definitions):

$$1 = \int \mathcal{D}g \Delta_F[gA] \delta[F[gA]]. \quad (5.4)$$

By analogy with standard calculus, in the presence of multiple solutions to the gauge-fixing condition Eq.(5.4) must be replaced by

$$N_F[A] = \int \mathcal{D}g \delta(F[gA]) \left| \det M_F[gA] \right|, \quad (5.5)$$

where $N_F[A]$ is the number of different solutions for the gauge-fixing condition $F[gA] = 0$ on the orbit characterised by A , where A is any configuration on the gauge orbit in question for which $\det M_F \neq 0$. It is known that Landau gauge has a fundamental modular region (FMR), namely a set of unique representatives of every gauge orbit which is moreover convex and bounded in every direction [DZ91, STSF82]. The following discussion can be found in more detail in [vB95]. Denoted Λ , the FMR is defined as the set of absolute minima of the functional $V_A[g] = \int d^4x (gA)^2$ with respect to gauge transformations g . The stationary points of $V_A[g]$ are those A_μ satisfying the Landau gauge condition. The boundary of the FMR, $\partial\Lambda$, is the set of degenerate absolute minima of $V_A[g]$. Λ lies within the Gribov region C_0 where the FP operator is positive definite. The Gribov region is comprised of all of the local minima of $V_A[g]$. The boundary of C_0 , the Gribov horizon ∂C_0 , is where the FP operator M_F (which corresponds to the second order variation of $V_A[g]$ with respect to infinitesimal g) acquires zero modes. When the degenerate absolute minima of $\partial\Lambda$ coalesce, flat directions develop and M_F develops zero modes. Such orbits cross the intersection of $\partial\Lambda$ and ∂C_0 . The interior of the fundamental modular region is a smooth differentiable and everywhere convex manifold. Orbits crossing the boundary of the FMR on the other hand will cross that boundary again at least once corresponding to the degenerate absolute minima. Though, at present, there is no practical computational algorithm for constructing the FMR, it exists and we will make use of it for labelling orbits, i.e., A_u are defined to be configurations in the FMR, $A_u \in \Lambda$. Since every orbit crosses the fundamental modular region once we are guaranteed to have $N_F \geq 1$. In turn the gA_u fulfilling the constraint of Eq. (5.3) would be every other gauge copy of A_u along its orbit. Eq.(5.5) is equal to the number of Gribov copies on a given orbit, $N_{GC} = N_F - 1$, except that copies lying on any of the Gribov horizons ($\Delta_F = 0$) do not contribute to N_F . The finiteness of N_F in the presence of a regularisation leading to a finite number of degrees of freedom (such as a lattice formulation) can be argued as follows. Consider two neighboring Gribov copies corresponding to a single orbit. If they contribute to N_F they cannot lie on the Gribov horizon. Therefore they do not lie infinitesimally close to each other along a flat direction, namely they have a finite separation. This is true then for all copies on an orbit contributing to N_F : all copies contributing to N_F have a finite separation. But the g which create the copies of A_u belong to $SU(N)$ which has a finite group volume. Thus for each space-time point there is a finite number of such g . We conclude then for a regularised formulation that N_F is finite. Consider then the computation of the expectation value of a gauge-invariant operator $O[A]$ over an ensemble of gauge-field configurations A_u which is this set of unique representatives of gauge orbits discussed above. Note that for a gauge-invariant observable, it makes no

difference whether $A_u \in \Lambda$ or if the A_u 's are any other unique representatives of the orbits. The expectation value on these configurations

$$\langle O[A] \rangle = \frac{\int \mathcal{D}A_u O[A_u] e^{-S_{YM}}}{\int \mathcal{D}A_u e^{-S_{YM}}} \quad (5.6)$$

is well-defined. Since in any regularised formulation N_F is a finite positive integer, we can legitimately use Eq.(5.5) to resolve the identity analogous to the FP trick and insert into the measure of integration for an operator expectation value. We thus have

$$\langle O[A] \rangle = \frac{\int \mathcal{D}A_u \frac{1}{N_F[A_u]} \int \mathcal{D}g \delta(F[gA]) \left| \det M_F[gA] \right| O[A] e^{-S_{YM}[A]}}{\int \mathcal{D}A_u \frac{1}{N_F[A_u]} \int \mathcal{D}g \delta(F[gA]) \left| \det M_F[gA] \right| e^{-S_{YM}[A]}}. \quad (5.7)$$

We can now pass $N_F[A_u]$ under the group integration $\mathcal{D}g$ and combine the latter with $\mathcal{D}A_u$ to obtain the full measure of all gauge fields $\mathcal{D}(gA_u)$ which we can write now as $\mathcal{D}A$. N_F is certainly gauge-invariant: it is a property of the orbit itself. So $N_F[A_u] = N_F[gA_u] = N_F[A]$. Thus we can write

$$\langle O[A] \rangle = \frac{\int \mathcal{D}A \frac{1}{N_F[A]} \delta(F[A]) \left| \det M_F[A] \right| O[A] e^{-S_{YM}[A]}}{\int \mathcal{D}A \frac{1}{N_F[A]} \delta(F[A]) \left| \det M_F[A] \right| e^{-S_{YM}[A]}}. \quad (5.8)$$

Perturbation theory can be recovered from this of course by observing that only A fields near the trivial orbit, containing $A = 0$ and for which $S_{YM}[A] = 0$, contribute significantly in the perturbative regime: the curvature of the orbits in this region is small so that the different orbits in the vicinity of $A = 0$ intersect the gauge-fixing hypersurface $F = 0$ the same number of times. Then the number of Gribov copies is the same for each orbit, N_F is independent of A_u and we can cancel N_F out of the expectation value. In that case

$$\langle O[A] \rangle = \frac{\int \mathcal{D}A \delta(F[A]) \left| \det M_F[A] \right| O[A] e^{-S_{YM}[A]}}{\int \mathcal{D}A \delta(F[A]) \left| \det M_F[A] \right| e^{-S_{YM}[A]}}. \quad (5.9)$$

In turn, observing that fluctuations near the trivial orbit cannot change the sign of the determinant, the modulus can also be dropped and one recovers the usual starting point for a standard BRST invariant formulation of Landau gauge perturbation theory. Note that perturbation theory is built on the gauge-fixing surface in the neighbourhood of $A = 0$, which for a gauge-invariant quantity will be equivalent to averaging over the Gribov copies of $A = 0$ as in Eq. (5.9). For the non-perturbative regime, the orbit curvature increases significantly and

in general there is no reason to expect that N_F would be the same for each orbit. Moreover the determinant can change sign. Let us focus on the partition function appearing in Eq. (5.8)

$$Z_{\text{gauge-fixed}} = \int \mathcal{D}A N_F^{-1}[A] \left| \det(M_F[A]) \right| \delta(F[A]) e^{-S_{YM}} \quad (5.10)$$

The objective is to generalise the BRST formulation of Eq.(5.10) such that it is valid beyond perturbation theory taking into account the modulus of the determinant. We thus start with the following representation:

$$\left| \det(M_F[A]) \right| = \text{sgn}(\det(M_F[A])) \det(M_F[A]). \quad (5.11)$$

This representation goes under the name of the *Nicolai map* [Nic80, BBRT91].

5.1.1 The Nicolai map and Topological Field Theory

Soon after the seminal work of Gribov [Gri78], the attention on gauge-fixing procedures in non-Abelian gauge theories [Sin78] led physicists to examine more in depth the strong relation between these theories and Topological Field Theory (TFT). It was immediately realised that in 4-dimensional gauge theory certain topological aspects play an important role: in late 70's and early 80's an enormous amount of work, mainly due to Donaldson, Schwarz and Witten, allowed the physics community to discover how Y-M theory could be explained in terms of topologically invariant quantities, such as polynomials and knots [Sch78, Wit82, Don83, Don90, Wit89]. The discovery of solutions to the Y-M classical equations of motion, called instantons, [BPST75] and the analysis of monopole structures in gauge theory [tH74, tH76] spread light into a world dense of interesting topological properties [Uhl82b, Pol77, AJ78], as well as a better understanding of the geometrical/mathematical background of low dimensional manifolds of Yang-Mills theory in terms of the well known theory of principal bundles [BV81a, DV80, AB82]. The study of these relations among mathematics, topology and physics has become known as Topological Quantum Field Theory (TQFT). In the following sections we will largely adopt [BBRT91] as a leading guide: as TQFT is a considerable subject to cover, we will try to focus on those parts which directly concerns supersymmetric aspects of Yang-Mills theory, the coherent and appropriate scenario on which the BRST formalism resides.

In topological quantum field theories we are only interested in those observables that only depend on global features of the space on which these theories are defined. Consequently, the observables are independent of any Riemannian metric characterising the underlying manifolds. The study of quantities which are

topological invariant was first started by Euler, in 1736 when he published a paper on the solution of the Knigsberg bridge problem entitled *Solutio problematis ad geometriam situs pertinentis* which translates into English as *The solution of a problem relating to the geometry of position*¹. The quantum version of Euler's ideas deals largely with the path integral representation of topological invariants such as the Ray-Singer torsion [Sch78] or Morse theory and its relations with supersymmetric quantum mechanics [Wit82]. As we will see, in the construction of TQFT, one can adopt two main different frameworks, the Witten-type or Schwarz-type: for the Witten type theories, also called *cohomological*, one combines certain topological shift symmetry with any other local symmetry, whereas in Schwarz type models, called *quantum*, the attention is focused on the usual gauge symmetry. We will only consider Witten-type theory throughout this work. The necessary ingredients to construct a topological field theory are

- a collection of Grassmann-graded² fields $\{\Phi\}$ defined on a Riemannian manifold M with metric g ,
- a nilpotent operator Q , $Q^2 = 0$, odd w.r.t. Grassmann grading,
- the physical Hilbert space is defined by the condition $Q|phys\rangle = 0$, and its physical states are defined to be Q -cohomology classes³,
- an energy-momentum tensor which is Q -exact, i.e. corresponds to the variation of a functional $V_{\alpha,\beta}$ of fields w.r.t. Q ⁴

$$T_{\alpha\beta} = \{Q, V_{\alpha,\beta}(\Phi, g)\}. \quad (5.12)$$

The existence of a Nicolai map is admitted in a Witten-type theory, such that the path integral can be restricted to the moduli space of classical solutions (instantons). Nicolai has proven that for theories with a global supersymmetry there exists a non-linear and, in general, non-local mapping of the bosonic fields which trivialises the bosonic part of the action, and whose determinant cancels the Pfaffian⁵ (in the case of Majorana spinors) of the fermionic fields present.

¹The paper not only shows that the problem of crossing the seven bridges in a single journey is impossible, but generalises the problem to show that, in today's notation, *A graph has a path traversing each edge exactly once if exactly two vertices have odd degree.*

²Grassmann grading defines the *even* or *odd* character of a field under product exchange.

³A state annihilated by Q is said to be Q -closed, while a state of the form $Q|\chi\rangle$ is called Q -exact; this equivalence relation partitions the physical Hilbert space into Q -cohomology classes, states which are Q -closed modulo Q -exact states.

⁴The variation $\delta\mathcal{O} = \{Q, \mathcal{O}\}$ corresponds to a Grassmann-graded commutator of fields.

⁵The Pfaffian of an even $2n$ -dimensional antisymmetric matrix M_{ij} is defined as $Pf(M) = \epsilon_{i_1, \dots, i_{2n}} M_{i_2}^{i_1} \dots M_{i_{2n}}^{i_{2n-1}}$, with the property that the determinant of the matrix M is equal to the square of the Pfaffian.

Consider the following action

$$S = \int d\tau \left[i \left(\frac{d\phi^i}{d\tau} + s g^{ij}(\phi) \frac{\partial V}{\partial \phi^j} \right) B_i + \frac{1}{2} g^{ij}(\phi) B_i B_j - \frac{1}{4} R_{ijkl} \bar{\psi}^i \psi^k \bar{\psi}^j \psi^l - i \bar{\psi}_i \left(\delta_j^i \frac{\mathcal{D}}{\mathcal{D}\tau} + s g^{ik}(\phi) \frac{\mathcal{D}^2 V}{\mathcal{D}\phi^k \mathcal{D}\phi^j} \right) \psi^j \right]. \quad (5.13)$$

After integrating out the fermions, the partition function takes the form

$$Z = \int_{\phi} e^{-S(\phi)} Pf[\phi]. \quad (5.14)$$

The existence of a Nicolai map for such a theory tells us that there exists a map $\phi \rightarrow \xi(\phi)$ such that the Jacobian of the transformation compensates the Pfaffian (up to a sign). The partition function Z then assumes the topological form

$$Z = \int_{\xi} e^{-\int \frac{1}{2} \xi^2} \times (\text{winding number of the mapping}), \quad (5.15)$$

where the winding number is the number of times ξ runs over its range as ϕ is varied. In [Nic80] Nicolai was able to show this map only up to third order in the coupling constant for $N = 1$ super Yang-Mills theory in 4-dim. This approximation is due to the highly non-local character of the map, which can be found analytically only in low dimensional cases. In the above case, suppose we use the following change of variables, showing the instanton sector of the theory

$$\phi \rightarrow \xi = \frac{d\phi}{d\tau} + s \frac{\partial V}{\partial \phi}. \quad (5.16)$$

With this change of variables we get

$$Z = \int_{\xi} e^{-\frac{1}{2} \int d\tau \xi^2} \det \left(\frac{\delta \xi}{\delta \phi} \right) \left[\det \left(\frac{\delta \xi}{\delta \phi} \right) \right]^{-1} = \pm 1. \quad (5.17)$$

The ratio of functional determinants is then ± 1 , which can be regarded, when dealing with Y-M theory, as a topological manner to consider the Gribov problem. In the next chapter, we will analyze this ratio problem from the topological point of view of the Poincaré-Hodge and Gauss-Bonnet theorem. There, we will discuss the fundamental topological obstruction in non-perturbative Y-M theory which determines the *exact* cancellation of this ratio, also known as the Neuberger problem.

5.2 The Nicolai map in the Faddeev-Popov Jacobian

As mentioned, the factor $\det(M_F[A])$ in Eq.(5.11) is represented as a functional integral via the usual Lie algebra valued ghost and anti-ghost fields in the adjoint representation of $SU(N)$. Let us label these as c^a, \bar{c}^a . It is usual also (see for example [NO90]) to introduce a Nakanishi-Lautrup auxiliary field b^a . Thus the effective gauge-fixing Lagrangian density ⁶

$$\mathcal{L}_{\text{det}} = -b^a \partial_\mu A_\mu^a + \frac{\xi}{2} b^a b^a + \bar{c}^a M_F^{ab} c^b \quad (5.18)$$

yields [NO90]

$$\lim_{\xi \rightarrow 0} \int \mathcal{D}\bar{c}^a \mathcal{D}c^a \mathcal{D}b^a e^{-\int d^4x \mathcal{L}_{\text{det}}} = \delta(F[A]) \det(M_F[A]). \quad (5.19)$$

In order to write the factor $\text{sgn}(\det(M_F[A]))$ in terms of a functional integral weighted by a local action, we consider the following Lagrangian density

$$\mathcal{L}_{\text{sgn}} = iB^a M_F^{ab} \varphi^b - i\bar{d}^a M_F^{ab} d^b + \frac{1}{2} B^a B^a \quad (5.20)$$

with \bar{d}^a, d^a being new Lie algebra valued Grassmann fields and φ^a, B^a being new auxiliary commuting fields. Consider in Euclidean space the path integral

$$\mathcal{Z}_{\text{sgn}} = \int \mathcal{D}\bar{d}^a \mathcal{D}d^a \mathcal{D}\varphi^a \mathcal{D}B^a e^{-\int d^4x \mathcal{L}_{\text{sgn}}}. \quad (5.21)$$

Completing the square in the Lagrangian density of Eq.(5.20), the B field can be integrated out in the partition function leaving an effective Lagrangian density

$$\mathcal{L}'_{\text{sgn}} = \frac{1}{2} \varphi^a ((M_F)^T)^{ab} M_F^{bc} \varphi^c - i\bar{d}^a M_F^{ab} d^b, \quad (5.22)$$

where $(M_F)^T$ denotes the transpose of the FP operator. Integrating all remaining fields now it is straightforward to see that the partition function Eq.(5.21) amounts to just

$$\mathcal{Z}_{\text{sgn}} = \frac{\det(M_F)}{\sqrt{\det((M_F)^T M_F)}} = \text{sgn}(\det(M_F)). \quad (5.23)$$

⁶We will use throughout this Chapter a different convention from the one we will adopt in the next Chapter. In this Chapter, in fact we adopt Hermitian generators for the algebra, and Hermitian ghost fields.

Thus the representation Eq.(5.21) can be used for the first factor of Eq.(5.11). The Lagrangian density of Eq.(5.20) therefore combines with the standard BRST structures of Eq.(5.18) coming from the determinant itself in Eq.(5.11) so that an equivalent representation for the partition function based on Eq.(5.10) is

$$\mathcal{Z}_{\text{gauge-fixed}} = \int \mathcal{D}A_\mu^a \mathcal{D}\bar{c}^a \mathcal{D}c^a \mathcal{D}\bar{d}^a \mathcal{D}d^a \mathcal{D}b^a \mathcal{D}\varphi^a (N_F[A])^{-1} e^{-S_{\text{YM}} - S_{\text{det}} - S_{\text{sgn}}} \quad (5.24)$$

with S_{det} and S_{sgn} the actions corresponding to the above Lagrangian densities Eqs. (5.18,5.20).

5.3 A new extended BRST

The symmetries of the new Lagrangian density, \mathcal{L}_{sgn} , are essentially a boson-fermion supersymmetry and can be seen from Eq.(5.20). In analogy to the standard BRST transformations typically denoted by s , we shall denote them by the Grassmann graded operator t

$$\begin{aligned} t\varphi^a &= d^a \\ td^a &= 0 \\ t\bar{d}^a &= B^a \\ tB^a &= 0, \end{aligned} \quad (5.25)$$

such that

$$t\mathcal{L}_{\text{sgn}} = 0. \quad (5.26)$$

Eqs.(5.25) realise the infinitesimal form of shifts in the fields. The operation t is nilpotent: $t^2 = 0$. Using Eqs.(5.25) we can give the following form for the Lagrangian density \mathcal{L}_{sgn} ,

$$\mathcal{L}_{\text{sgn}} = t \left(\bar{d}^a (iM_F^{ab} \varphi^b + \frac{1}{2} B^a) \right). \quad (5.27)$$

The question now is how to combine this with the standard BRST transformations

$$\begin{aligned} sA_\mu^a &= D_\mu^{ab} c^b \\ sc^a &= -\frac{1}{2} g f^{abc} c^b c^c \\ s\bar{c}^a &= b^a \\ sb^a &= 0. \end{aligned} \quad (5.28)$$

The transformations due to t and s are completely decoupled except that the latter also act on the gauge field on which the FP operator M_F depends. We propose the following unification of these symmetry operations. Consider an operation \mathcal{S} block-diagonal in s and t : $\mathcal{S} = \text{diag}(s, t)$. The operator acts on the following multiplet fields:

$$\mathcal{A}^a = \begin{pmatrix} A_\mu^a \\ \varphi^a \end{pmatrix}, \mathcal{C}^a = \begin{pmatrix} c^a \\ d^a \end{pmatrix}, \bar{\mathcal{C}}^a = \begin{pmatrix} \bar{c}^a \\ \bar{d}^a \end{pmatrix}, \mathcal{B}^a = \begin{pmatrix} b^a \\ B^a \end{pmatrix}. \quad (5.29)$$

We see that these fields transform under \mathcal{S} completely analogously to the standard BRST operations

$$\begin{aligned} \mathcal{S}\mathcal{A}^a &= \mathcal{D}^{ab}\mathcal{C}^b \\ \mathcal{S}\mathcal{C}_i^a &= \mathcal{F}_{ijk}^{abc}\mathcal{C}_j^b\mathcal{C}_k^c \\ \mathcal{S}\bar{\mathcal{C}}^a &= \mathcal{B}^a \\ \mathcal{S}\mathcal{B}^a &= 0, \end{aligned} \quad (5.30)$$

where $i, j, k = 1, 2$ label the elements of the multiplets, and

$$\begin{aligned} \mathcal{D}^{ab} &= \text{diag}(D_\mu^{ab}, \delta^{ab}) \\ \mathcal{F}_{111}^{abc} &= -\frac{1}{2}gf^{abc}, \quad \mathcal{F}_{ijk}^{abc} = 0 \quad \text{for } ijk \neq 111. \end{aligned} \quad (5.31)$$

Note that nilpotency is satisfied, $\mathcal{S}^2 = 0$. We shall refer to this type of operation as an *extended* BRST transformation which we distinguish from the BRST-anti-BRST or double BRST algebra of the Curci-Ferrari model [CF76, TM80]. We can thus formulate the gauge-fixing Lagrangian density for the Landau gauge as

$$\mathcal{L}_{\text{gf}} = \text{Tr} \mathcal{S} \begin{pmatrix} \bar{c}^a F^a & 0 \\ 0 & \bar{d}^a (iM_F^{ab}\varphi^b + \frac{1}{2}B^a) \end{pmatrix}. \quad (5.32)$$

This approach admits also an extended anti-BRST operation:

$$\begin{aligned} \bar{\mathcal{S}}\mathcal{A}^a &= \mathcal{D}^{ab}\bar{\mathcal{C}}^b \\ \bar{\mathcal{S}}\bar{\mathcal{C}}_i^a &= \mathcal{F}_{ijk}^{abc}\bar{\mathcal{C}}_j^b\bar{\mathcal{C}}_k^c \\ \bar{\mathcal{S}}\mathcal{C}^a &= -\mathcal{B}^a \\ \bar{\mathcal{S}}\mathcal{B}^a &= 0. \end{aligned} \quad (5.33)$$

Writing $\bar{\mathcal{S}} = \text{diag}(\bar{s}, \bar{t})$ we can extract the standard anti-BRST \bar{s} -operations, in Landau gauge, [TMB83, BTM82]

$$\begin{aligned} \bar{s}A_\mu^a &= D_\mu^{ab}\bar{c}^b \\ \bar{s}\bar{c}^a &= -\frac{1}{2}gf^{abc}\bar{c}^b\bar{c}^c \\ \bar{s}c^a &= -b^a \\ sb^a &= 0 \end{aligned} \quad (5.34)$$

and those corresponding to \bar{t} :

$$\begin{aligned}\bar{t}\varphi^a &= \bar{d}^a \\ \bar{t}\bar{d}^a &= 0 \\ \bar{t}d^a &= -B^a \\ \bar{t}B^a &= 0.\end{aligned}\tag{5.35}$$

Moreover, the ghosts and anti-ghosts in this extended structure also fulfill the criteria for being Maurer-Cartan one-forms,

$$S\bar{C} + \bar{S}C = 0.\tag{5.36}$$

However there is no extended BRST-anti-BRST (or double) symmetric form of the gauge-fixing Lagrangian density Eq. (5.32), unlike the two pieces of which it consists. Such a representation exists in the s -sector of Landau gauge:

$$\mathcal{L}_{\text{gf},s} = \frac{1}{2}s\bar{s}A_\mu^a A_\mu^a.\tag{5.37}$$

In the t -sector, the corresponding structure is

$$\mathcal{L}_{\text{gf},t} = \frac{1}{2}t\bar{t}[\varphi^a M_F^{ab}\varphi^b + \bar{d}^a d^a].\tag{5.38}$$

However the complete Landau gauge-fixing Lagrangian density can only be expressed via a trace, namely as

$$\mathcal{L}_{\text{gf}} = \frac{1}{2}\text{Tr}S\bar{S}\mathcal{W}\tag{5.39}$$

with

$$\mathcal{W} = \text{diag}(A_\mu^a A_\mu^a, \varphi^a M_F^{ab}\varphi^b + \bar{d}^a d^a).\tag{5.40}$$

Nevertheless this compact representation formulates the modulus of the determinant in Landau gauge fixing in terms of a local Lagrangian density and follows as closely as possible the standard BRST formulation without the modulus.

Decontracted Double Lattice BRST, the Curci-Ferrari Mass and the Neuberger Problem

In 1974, Wilson [Wil74] formulated Euclidian gauge theories on the lattice in order to shed light on the confinement mechanism in QCD and to study the non-perturbative regime of non-Abelian gauge theories. To construct the proper lattice gauge theory of QCD, we need first to discretize the space-time, then the transcription of the gauge and fermion fields¹ succesively the action and the re-definition of the functional measure. Finally the transcription of the operators to probe the physics. For a detailed analysis of lattice gauge theory we refer to [Gup97] and [Smi02]. Here we just wish to give the basic properties and definitions of this theory which will be used in the following sections. To start with, we need to stress that the lattice procedure provides a cutoff which naturally regularizes the ultraviolet divergences of quantum field theories. As with any regulator, it must be removed after renormalisation: the continuum version of any lattice theory is provided by taking the adopted lattice spacing to zero. In the wide range of possible lattice regularisation, the simplest one consists in taking the isotropic cubic grid, where there is no distinction between the space lattice spacing a_S and the time one a_T . Moreover, on the lattice, we sacrifice Lorentz invariance, but all the other internal symmetries are preserved, particularly local gauge invariance². Having said that, any four-dimensional integral can be written in terms of the lattice spacing a as

$$\int d^4x \rightarrow a^4 \sum_n, \quad (6.1)$$

where the space-time coordinate x_μ has been replaced by a set of integers n_μ , such that $x_\mu = an_\mu$ and \sum_n corresponds to a finite sum over the lattice sites n .

¹Dealing only with pure gauge theories we will not consider fermion fields in this introduction to lattice gauge theory.

²Requiring gauge invariance at all a is necessary otherwise one would have many more parameters to tune (such as the gluon mass for instance) and there would arise many more operators at any given order in a . The lattice action will be also invariant under charge conjugation \mathcal{C} , parity \mathcal{P} and time reversal \mathcal{T} .

The construction of gauge fields is somewhat tricky and requires some attention: observing that a particle moving on a contour picks up a phase factor, Wilson formulated gauge fields on a space-time lattice introducing the concept of link variables $U_\mu(x)$, connected to this phase factor. These links are the fundamental variables on the lattice, they live in the Lie group G of the theory, connecting x to $x + \hat{\mu}$, defined as

$$\begin{aligned} U_\mu(x) &= \text{P exp} \left\{ g \int_{x_\mu}^{x+\hat{\mu}} X^a A_\nu^a(x) dz^\nu \right\} \\ &= U_\mu(x, x + \hat{\mu}), \end{aligned} \quad (6.2)$$

with X^a the $N^2 - 1$ anti-hermitian generators of the Lie algebra \mathfrak{g} . P denotes the path ordering, such that

$$U_\mu(x, x - \hat{\mu}) \equiv U_{-\mu}(x) = U_\mu^\dagger(x - \hat{\mu}, x) \quad (6.3)$$

Under a gauge transformation $g(x)$, the link variable transforms as

$$\mathcal{U} U_\mu(x) \equiv g(x) U_\mu(x) g^\dagger(x + \hat{\mu}). \quad (6.4)$$

With these definitions, there are two types of gauge invariant objects (which can be of arbitrary size and shape and over any representation of the Lie group) that one can construct on the lattice 1) a string of path-ordered product of links capped by a fermion and an antifermion; 2) closed Wilson loops, whose simplest example is the *plaquette*, a 1×1 loop

$$W_{\mu\nu}^{1 \times 1} = \text{ReTr} \left(U_\mu(x) U_\nu(x + \hat{\mu}) U_\mu^\dagger(x + \hat{\nu}) U_\nu^\dagger(x) \right). \quad (6.5)$$

A gauge invariant action has to build up out of loops and strings, with the physical constraint that in the limit $\lim_{a \rightarrow 0}$, we recover the continuum theory (in the case of QCD, the Y-M action). Consider for this purpose the Wilson loop (6.5), where the average field A_μ is defined at the midpoint of the link

$$\begin{aligned} W_{\mu\nu}^{1 \times 1} &= \text{ReTr} \left(U_\mu(x) U_\nu(x + \hat{\mu}) U_\mu^\dagger(x + \hat{\nu}) U_\nu^\dagger(x) \right) \\ &= e^{ag A_\mu(x + \frac{\hat{\mu}}{2}) + A_\nu(x + \frac{\hat{\nu}}{2}) - A_\mu(x + \frac{\hat{\mu}}{2}) - A_\nu(x + \frac{\hat{\nu}}{2})}. \end{aligned} \quad (6.6)$$

Expanding about $x + \frac{\hat{\mu} + \hat{\nu}}{2}$ gives

$$\begin{aligned} W_{\mu\nu}^{1 \times 1} &= \exp \left\{ a^2 g (\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{a^4 g}{12} (\partial_\mu^3 A_\nu - \partial_\nu^3 A_\mu) + \dots \right\} \\ &= 1 + a^2 g F_{\mu\nu} - \frac{a^4 g^2}{2} F_{\mu\nu} F^{\mu\nu} + O(a^6) + \dots \end{aligned} \quad (6.7)$$

Summing over the Lorentz indices we can write

$$\begin{aligned} S[U] &= \frac{6}{g^2} \sum_x \sum_{\mu < \nu} \text{ReTr} \frac{1}{3} (1 - W_{\mu\nu}^{1 \times 1}) \\ &= \frac{a^4}{4} \sum_x \sum_{\mu < \nu} F_{\mu\nu} F^{\mu\nu} \rightarrow \frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu}. \end{aligned} \quad (6.8)$$

Historically, lattice calculations are generally presented in terms of the coupling $\beta = 6/g^2$ (for $SU(3)$).

6.1 Double BRST on the lattice

In the covariant continuum formulation of gauge theories, in terms of local field systems, one has to deal with the redundant degrees of freedom due to gauge invariance. Within the language of local quantum field theory, the machinery for that is based on the so-called Becchi-Rouet-Stora-Tyutin (BRST) symmetry which is a global symmetry and can be considered the quantum version of local gauge invariance [NO90, AFRvS03]. In short, one starts out from the representations of a BRST algebra on indefinite metric spaces with assuming the existence (and completeness) of a nilpotent BRST charge Q_B . The physical Hilbert space can then be defined as the equivalence classes of BRST closed (which are annihilated by Q_B) modulo exact states (which are BRST variations of others). In QED this machinery reduces to the usual Gupta-Bleuler construction. For the generalisation thereof, in non-Abelian gauge theories, all is well in perturbation theory also. Beyond perturbation theory, however, there is a problem with such a construction that has not been fully and comprehensively addressed as yet. It relates to the famous Gribov ambiguity [Gri78], the existence of so-called Gribov copies that satisfy the Lorenz condition [JO01] (or any other local gauge fixing condition) but are related by gauge transformations, and are thus physically equivalent. As a result of this ambiguity, the usual definitions of a BRST charge fail to be globally valid.

A rigorous non-perturbative framework is provided by lattice gauge theory. Its strength and beauty derives from the fact that gauge-fixing is not required. However, in order to arrive at a non-perturbative definition of non-Abelian gauge theories in the continuum, from a lattice formulation, we need to be able to perform the continuum limit in a formally watertight way. And there is the gap in our present understanding. The same problem as described above comes back to haunt us in another dress when attempting to fix a gauge via BRST formulations on the lattice. There it is known as the Neuberger problem which asserts that the expectation value of any gauge invariant (and thus physical)

observable in a lattice BRST formulation will always be of the indefinite form $0/0$ [Neu87].

The BRST algebra requires the introduction of further unphysical degrees of freedom. These are the Faddeev-Popov ghosts and anti-ghosts which violate the Spin-Statistics Theorem of local quantum field theory on positive definite metric (Hilbert) spaces. Contrary to what the name anti-ghost might suggest, however, in the usual linear covariant gauges the treatment of ghosts and anti-ghosts is completely asymmetric. On the other hand, it is also known for many years that it is possible to extend the BRST algebra to be entirely symmetric w.r.t. ghosts and anti-ghosts. This additional symmetry arises naturally in the Landau gauge but can also be extended to more general gauges the so-called Curci-Ferrari gauges at the expense of quartic ghost self-interactions. The most interesting feature of these gauges for our purpose, however, is that they allow the introduction of a mass term for ghosts [CF76]. While such a Curci-Ferrari mass m breaks the nilpotency of the BRST and anti BRST charges, which is known to result in a loss of unitarity and which therefore meant that this relatively old model received little attention for many years, it also serves to regulate the Neuberger zeroes in a lattice formulation. In [KvSW05] this was exemplified in a simple Abelian toy-model where the zeroes in the numerator and denominator of expectation values become proportional to m^2 and allow to compute a finite value for $m^2 \rightarrow 0$ via l'Hospital's rule.

For the $SU(N)$ gauge theory on a finite four-dimensional lattice things are naturally much more complicated than in the toy model. In this Chapter we developed a full lattice formulation of the time-honored model by Curci and Ferrari with its decontracted double BRST/anti-BRST and ghost-mass term, as announced in [GvSW06]. We first extend Neuberger's no-go-theorem to include the ghost/anti-ghost symmetric case of the non-linear covariant Curci-Ferrari gauges for $m^2 = 0$, a case originally excluded by Neuberger. At non-vanishing Curci-Ferrari mass the partition function of the model used as the gauge-fixing device can be shown to be polynomial in m^2 and thus non-vanishing. In this way regularising the Neuberger zeroes, the leading power of that polynomial can be extracted from a suitable number of derivatives (w.r.t. m^2) before the limit $m^2 \rightarrow 0$ is taken, in the spirit of l'Hospital's rule. This gives rise to a modified lattice BRST model without Neuberger problem.

For the topological lattice formulation of the double BRST symmetry of the ghost/anti-ghost symmetric covariant gauges we start out from the standard gauge-fixing functional $V_U[g]$ of covariant gauges which here assumes the role of

a Morse potential on a gauge orbit,

$$V_U[g] = -\frac{1}{2\rho} \sum_i \sum_{j \sim i} \text{tr} U_{ij}^g = -\frac{1}{\rho} \sum_{x,\mu} \text{Re tr} U_{x,\mu}^g. \quad (6.9)$$

Here, in the first form, $U_{ij} \in SU(N)$ is the directed link variable connecting nearest neighbour sites i and j . The sum $j \sim i$ denotes summation over all nearest neighbours j of site i . We assume periodic boundary conditions. The double sum thus runs twice over all links $\langle ij \rangle$, and with $U_{ij}^\dagger = U_{ji}$ it is therefore equivalent to the simple sum over links in the second form, where $U_{x,\mu}$ stands for the same link field U at position x in direction μ . The constant ρ is the normalisation of the $SU(N)$ generators X . We use anti-Hermitian $[X^a, X^b] = f^{abc} X^c$ with $\text{tr} X^a X^b = -\rho \delta^{ab}$. We explicitly only need the fundamental representation, where $\rho = \rho_{\text{fund}} = 1/2$.

As usual, under gauge transformations the link variables U transform

$$U_{ij} \rightarrow U_{ij}^g = g_i^\dagger U_{ij} g_j. \quad (6.10)$$

BRST transformations s and anti-BRST transformations \bar{s} in the topological setting do not act on the link variables U directly, but on the gauge transformations g_i like infinitesimal right translations in the gauge group with real ghost and anti-ghost Grassmann fields c_i^a, \bar{c}_i^a as parameters, respectively,

$$sg = g X^a c^a = gc, \quad \bar{s}g = g X^a \bar{c}^a = g\bar{c}, \quad (6.11)$$

where we introduced Lie-algebra valued, anti-Hermitian ghost fields $c_i \equiv X^a c_i^a$ with $c_i^\dagger = -c_i$, and analogous anti-ghost fields $\bar{c}_i \equiv X^a \bar{c}_i^a$. For consistency, we furthermore require

$$sg^\dagger = (sg)^\dagger = -cg^\dagger, \quad \bar{s}g^\dagger = (\bar{s}g)^\dagger = -\bar{c}g^\dagger. \quad (6.12)$$

For the gauge-transformed link variables this then implies

$$sU_{ij}^g = -c_i U_{ij}^g + U_{ij}^g c_j, \quad \bar{s}U_{ij}^g = -\bar{c}_i U_{ij}^g + U_{ij}^g \bar{c}_j. \quad (6.13)$$

The BRST transformations for (anti)ghosts and Nakanishi-Lautrup fields b are straightforward lattice analogues (per site) of their continuum counterparts,

$$sc^a = -\frac{1}{2}(c \times c)^a, \quad (6.14)$$

$$s\bar{c}^a = b^a - \frac{1}{2}(\bar{c} \times c)^a, \quad (6.15)$$

$$sb^a = -\frac{1}{2}(c \times b)^a - \frac{1}{8}((c \times c) \times \bar{c})^a. \quad (6.16)$$

The relatively obvious notation of using the “cross-product” herein refers to the structure constants for $SU(N)$, *e.g.* $(\bar{c} \times c)^a \equiv f^{abc} \bar{c}^b c^c$.

In the ghost/anti-ghost symmetric gauges as considered here, the anti-BRST variations are obtained by substituting $c \rightarrow \bar{c}$ and $\bar{c} \rightarrow -c$ according to Faddeev-Popov conjugation. Thus,

$$\bar{s}c^a = -b^a = \frac{1}{2}(\bar{c} \times c)^a, \quad (6.17)$$

$$\bar{s}\bar{c}^a = -\frac{1}{2}(\bar{c} \times \bar{c})^a, \quad (6.18)$$

$$\bar{s}b^a = -\frac{1}{2}(\bar{c} \times b)^a + \frac{1}{8}((\bar{c} \times \bar{c}) \times c)^a. \quad (6.19)$$

The action of the topological lattice model for gauge fixing a la Faddeev-Popov with double BRST invariance can then be written in compact form as

$$S_{\text{GF}} = i s \bar{s} \left(V_U[g] + i \frac{\xi}{2\rho} \sum_i \text{tr} \bar{c}_i c_i \right). \quad (6.20)$$

This is the lattice counterpart of the continuum gauge-fixing Lagrangian

$$\mathcal{L}_{\text{GF}} = \frac{i}{2} s \bar{s} (A_\mu^a A_\mu^a - i \xi \bar{c}^a c^a) \quad \text{with} \quad S_{\text{GF}} = \int d^D x \mathcal{L}_{\text{GF}} \quad (6.21)$$

in D Euclidean dimensions.

Performing the anti-BRST variation first, we obtain

$$\begin{aligned} \bar{s} V_U[g] &= \frac{1}{2\rho} \sum_i \sum_{j \sim i} \text{tr} (\bar{c}_i U_{ij}^g - \bar{c}_j U_{ij}^g) \\ &= \frac{1}{2\rho} \sum_i \sum_{j \sim i} \bar{c}_i^a \text{tr} (X^a (U_{ij}^g - U_{ji}^g)) \\ &= - \sum_i \bar{c}_i^a F_i^a(U^g), \end{aligned} \quad (6.22)$$

where

$$F_i^a(U^g) = -\frac{1}{2\rho} \sum_{j \sim i} \text{tr} (X^a (U_{ij}^g - U_{ji}^g)) \quad (6.23)$$

is, of course, the standard gauge-fixing condition of covariant gauges which reduces in the continuum limit to

$$F_i^a(U^g) \xrightarrow{a \rightarrow 0} a^2 \partial_\mu A_\mu^{g^a} + \mathcal{O}(a^4). \quad (6.24)$$

As we know, the gauge-fixing condition is derived by considering the first derivative of the Morse potential $V_U[g]$ with respect to the group element g . However,

since this functional derivation takes into account matrix elements, its computation requires special attention to the matrix-order, and so it can turn out to be ambiguous. To avoid this complication we define the one-parameter subgroup of the Lie group as $g_t(x) = e^{t\omega_x}$, with $t \in \mathbb{R}$. Through this parameter t we then have to compute a simple 1-dimensional derivation, bypassing the non-commuting nature of the matrix elements. Adopting standard notation (x, μ) , we write

$$\begin{aligned} \frac{d}{dt} V_U[g] &= -\frac{1}{\rho} \sum_{x,\mu} \text{Re tr}[(\omega_x - \omega_{x+\mu})^{g_t} U_{x,\mu}] \\ &= -\frac{1}{2\rho} \sum_{x,\mu} \text{tr}[\omega_x ({}^{g_t}U_{x,\mu} - {}^{g_t}U_{x,\mu}^\dagger) - \omega_{x+\mu} ({}^{g_t}U_{x,\mu} - {}^{g_t}U_{x,\mu}^\dagger)] \\ &= -\frac{1}{\rho} \sum_{x,\mu} \text{Re tr} \left\{ \omega_x \left[\frac{1}{2} ({}^{g_t}U_{x,\mu} - {}^{g_t}U_{x,\mu}^\dagger) - \frac{1}{2} ({}^{g_t}U_{x-\mu,\mu} - {}^{g_t}U_{x-\mu,\mu}^\dagger) \right] \right\}. \end{aligned} \quad (6.25)$$

From now on we will drop g_t from the link notation. Defining $A_{x,\mu} = \frac{1}{2}(U_{x,\mu} - U_{x,\mu}^\dagger)_{\text{traceless}} = A_{x,\mu}^a X^a$, we then have

$$\begin{aligned} \frac{d}{dt} V_U[g] &= -\frac{1}{\rho} \sum_x \omega_x^a \sum_\mu \text{tr} \{ X^a (A_{x,\mu} - A_{x-\mu,\mu}) \} \\ &= \sum_x \omega_x^a \sum_\mu (A_{x,\mu}^a - A_{x-\mu,\mu}^a) \\ &= (\omega, \nabla \cdot A). \end{aligned} \quad (6.26)$$

Turning back to the BRST variations, with Eqs. (6.17), (6.18) we furthermore have

$$\bar{s}(\bar{c}^a c^a) = \bar{c}^a b^a, \quad (6.27)$$

and therefore, for the gauge-fixing action, we obtain the alternative form

$$S_{\text{GF}} = -i \sum_i s \left(\bar{c}_i^a (F_i^a(U^g) + \frac{i\xi}{2} b_i^a) \right). \quad (6.28)$$

As in the continuum formulation, in this form it looks exactly like the gauge-fixing action of standard Faddeev-Popov theory for the linear covariant gauge. The specific features of the ghost/anti-ghost symmetric framework show when working out the remaining BRST variation. From the first term we have (i),

$$\begin{aligned} \sum_i (s\bar{c}_i^a) F_i^a &= -\frac{1}{2\rho} \sum_i \sum_{j \sim i} \text{tr} (b_i (U_{ij}^g - U_{ji}^g)) \\ &\quad + \frac{1}{4\rho} \sum_i \sum_{j \sim i} \text{tr} (\{\bar{c}_i, c_i\} (U_{ij}^g - U_{ji}^g)). \end{aligned} \quad (6.29)$$

The first term implements the gauge-fixing condition as in standard Faddeev-Popov theory. The second term, containing the anticommutator $\{\bar{c}, c\}$, is characteristic of ghost/anti-ghost symmetry because it combines with the remaining quadratic ghost terms to produce a Hermitian Faddeev-Popov operator (for any gauge parameter ξ). To see this explicitly, consider (ii),

$$\begin{aligned} \sum_i \bar{c}_i^a (sF_i^a) &= \frac{1}{2\rho} \sum_i \sum_{j \sim i} \text{tr} (\bar{c}_i c_i U_{ij}^g \\ &\quad - \bar{c}_i U_{ij}^g c_j + c_j U_{ji}^g \bar{c}_i - c_i \bar{c}_i U_{ji}^g), \end{aligned} \quad (6.30)$$

so that the difference (i) - (ii) yields

$$\begin{aligned} \sum_i s(\bar{c}_i^a F_i^a) &= -\frac{1}{2\rho} \sum_i \sum_{j \sim i} \text{tr} (b_i (U_{ij}^g - U_{ji}^g)) \\ &\quad + \frac{1}{2\rho} \sum_i \sum_{j \sim i} \text{tr} (\bar{c}_i U_{ij}^g c_j - c_i U_{ij}^g \bar{c}_j - [\bar{c}_i, c_i] \frac{1}{2} (U_{ij}^g + U_{ji}^g) + c_i \bar{c}_i \frac{1}{2} (U_{ij}^g + U_{ji}^g)) \\ &\equiv \sum_i b_i^a F_i^a + \sum_{i,j} \bar{c}_i^a M_{\text{FP}ij}^{ab} c_j^b \end{aligned} \quad (6.31)$$

which defines the lattice Faddeev-Popov operator M_{FP} of the ghost/anti-ghost symmetric Curci-Ferrari gauges. Following the same method we used to derive the gauge-fixing condition, we want here to show how to derive the Faddeev-Popov operator M_{FP} , which is obtained by the second derivative with respect the real parameter t of the Morse potential $V_U[g]$ as follows

$$\begin{aligned} \frac{d^2}{dt^2} V_U[g] &= \sum_x \omega_x^a \sum_\mu \frac{d}{dt} (A_{x,\mu}^a - A_{x,\mu}^{a\dagger}) \\ &= \frac{1}{\rho} \sum_x \omega_x^a \sum_\mu \left\{ \frac{1}{4\rho} (\text{tr}(U_{x,\mu} + U_{x,\mu}^\dagger)(\omega_{x+\mu}^a - \omega_x^a) - \text{tr}(U_{x-\mu,\mu} + U_{x-\mu,\mu}^\dagger)(\omega_x^a - \omega_{x-\mu}^a)) \right. \\ &\quad - \frac{\rho}{2} f^{abc} (A_{x,\mu}^c (\omega_{x+\mu}^b + \omega_x^b) - A_{x-\mu,\mu}^c (\omega_x^b + \omega_{x-\mu}^b)) \\ &\quad \left. + \frac{i}{4} d^{abc} (\text{tr}(X^c(U_{x,\mu} + U_{x,\mu}^\dagger))(\omega_{x+\mu}^b - \omega_x^b) \text{tr}(X^c(U_{x-\mu,\mu} + U_{x-\mu,\mu}^\dagger))(\omega_x^b - \omega_{x-\mu}^b)) \right\}. \end{aligned} \quad (6.32)$$

Changing notation to $U_{ij} \equiv U_{x,\mu}$ and $A_{ij} \equiv A_{x,\mu}$ we can write

$$\begin{aligned} \frac{d^2}{dt^2} V_U[g] &= \frac{1}{\rho} \sum_i \omega_i^a \sum_{j \sim i} \frac{1}{4\rho} \left\{ \text{tr}(U_{ij} + U_{ji})(\omega_j^a - \omega_i^a) - \frac{\rho}{2} f^{abc} A_{ij}^c (\omega_j^b + \omega_i^b) \right. \\ &\quad \left. + \frac{i}{4} d^{abc} \text{tr}(X^c(U_{ij} + U_{ji}))(\omega_j^b - \omega_i^b) \right\}. \end{aligned} \quad (6.33)$$

Therefore, the Faddeev-Popov operator can be obtained as a double derivative with respect to the gauge function ω^a of $\frac{d^2}{dt^2}V_U[g]$

$$(M_{\text{FP}})_{ij}^{ab} = \frac{\partial}{\partial \omega_j^b} \frac{\partial}{\partial \omega_i^a} \frac{d^2}{dt^2} V_U[g]. \quad (6.34)$$

Note that the terms in (6.30) can be written in the form

$$\begin{aligned} \sum_i \bar{c}_i^a (sF_i^a) &= \frac{1}{4\rho} \sum_{i,j \sim i} \bar{c}_i^a \left\{ \text{tr}([X^a, X^b](U_{ij}^g - U_{ji}^g)) (c_i^b + c_j^b) \right. \\ &\quad \left. + \text{tr}(\{X^a, X^b\}(U_{ij}^g + U_{ji}^g)) (c_i^b - c_j^b) \right\} \end{aligned} \quad (6.35)$$

This, of course, corresponds to the widely used Faddeev-Popov operator of lattice Landau gauge, as first derived in [Zwa94]. It differs by the quadratic ghost terms in (6.29) from the ghost/anti-ghost symmetric one, M_{FP} in (6.31), which can be written in the alternative form,

$$\begin{aligned} \sum_{i,j} \bar{c}_i^a M_{\text{FP}ij}^{ab} c_j^b &= -\frac{1}{4\rho} \sum_{x,\mu} \left\{ \text{tr}(\{X^a, X^b\}(U_{x,\mu}^g + U_{x,\mu}^{g\dagger})) \times (\bar{c}_{x+\hat{\mu}}^a - \bar{c}_x^a)(c_{x+\hat{\mu}}^b - c_x^b) \right. \\ &\quad \left. + \text{tr}([X^a, X^b](U_{x,\mu}^g - U_{x,\mu}^{g\dagger})) \times (\bar{c}_x^a(c_{x+\hat{\mu}}^b - c_x^b) - (\bar{c}_{x+\hat{\mu}}^a - \bar{c}_x^a)c_x^b) \right\}, \end{aligned} \quad (6.36)$$

In the continuum limit this reduces to the ghost/anti-ghost symmetric Faddeev-Popov operator

$$M_{\text{FP}ij}^{ab} \xrightarrow{a \rightarrow 0} -a^2 \frac{1}{2} (\partial D^{ab} + D^{ab} \partial) \delta(x-y) + \mathcal{O}(a^4).$$

To complete the derivation of the gauge-fixing action in the ghost/anti-ghost symmetric framework, we furthermore need work out the BRST variation of $s\bar{s}(\bar{c}^a c^a) = s(\bar{c}^a b^a)$ from (6.14)-(6.16). This, however, is done in exactly the same way as in the continuum, the result is (iii),

$$s(\bar{c}^a b^a) = b^a b^a + \frac{1}{4} (\bar{c} \times c)^2. \quad (6.37)$$

Putting together all terms from (i) to (iii) we obtain the full gauge-fixing action with extended double BRST invariance on the lattice in the form,

$$S_{\text{GF}} = \sum_i \left\{ -ib_i^a F_i^a(U^g) - i\bar{c}_i^a M_{\text{FP}i}^a[c] + \frac{\xi}{2} b_i^a b_i^a + \frac{\xi}{8} (\bar{c}_i \times c_i)^2 \right\}, \quad (6.38)$$

where we introduced the short-hand notation that

$$M_{\text{FP}_i^a}[c] \equiv -\frac{1}{4\rho} \sum_{j \sim i} \left\{ \text{tr} \left([X^a, X^b] (U_{ij}^g - U_{ji}^g) \right) c_j^b \right. \\ \left. + \text{tr} \left(\{X^a, X^b\} (U_{ij}^g + U_{ji}^g) \right) (c_i^b - c_j^b) \right\}, \quad (6.39)$$

which corresponds to the ghost/anti-ghost symmetric Faddeev-Popov operator in (6.36), in particular, we have

$$\sum_i \bar{c}_i^a M_{\text{FP}_i^a}[c] = \sum_{i,j} \bar{c}_i^a M_{\text{FP}_{ij}^{ab}} c_j^b. \quad (6.40)$$

The full symmetry of the ghost/anti-ghost symmetric Curci-Ferrari gauges [CF76, TM80] is given by a semidirect product of a global $SL(2, \mathbb{R})$, which includes ghost number and Faddeev-Popov conjugation, with the BRST/anti-BRST symmetries as used above³. This is the global symmetry of the Landau gauge, and it is sometimes referred to as extended BRST symmetry.

Among the general class of all covariant gauges [TMB83], with a Lagrangian which is polynomial in the fields, Lorentz, globally gauge and BRST invariant, and renormalisable in $D = 4$, the ghost/anti-ghost symmetric case is special and interesting in that it allows to smoothly connect to the Landau gauge for $\xi \rightarrow 0$, without changing the global symmetry properties.

In particular, introducing with [TMB83] a second gauge parameter $\beta \in [0, 1]$, to interpolate between the various generalised covariant gauges, the linear covariant gauges of standard Faddeev Popov theory correspond to the line $\beta = 0$ in the two gauge-parameter plane (ξ, β) . Along this line, the global symmetry changes abruptly when reaching the Landau gauge limit; and for $\beta = 1$, one obtains a mirror image of standard Faddeev-Popov theory with the roles of ghosts and anti-ghosts interchanged. The ghost/anti-ghost symmetric gauges discussed here then correspond to the line $\beta = 1/2$. For $\xi = 0$ the distinction is an illusion. The whole interval for $\beta \in [0, 1]$ at $\xi = 0$ is equivalent and corresponds to the Landau gauge. The important difference is, however, that the $SL(2, \mathbb{R})$ symmetric line at $\beta = 1/2$ provides a unique class of covariant gauges which share the full extended BRST symmetry of the Landau gauge for any value of ξ . The limit $\xi \rightarrow 0$ is thus a smooth one, as far as the symmetries are concerned, only along this line. The price to pay are the quartic ghost self-interactions in (6.38) which again vanish only in the Landau gauge limit.

For a further discussion of the general ghost creating gauges, and their geometrical interpretation, see [TM80]. The one-loop renormalisation was first discussed in [TMB83], for explicit calculations of renormalisation constants and

³Also see Appendix A of Ref. [AFRvS03].

anomalous dimensions of the ghost/anti-ghost symmetric case up to including the three-loop level, see [dBSvNW96, Gra03]. The Dyson-Schwinger equations of these gauges were studied in [AFRvS03].

6.2 The Neuberger problem

Following Neuberger, we introduce an auxiliary parameter t in the Euclidean partition function to be used as the gauge-fixing device via the Faddeev-Popov procedure of inserting unity into the unfixed partition function of $SU(N)$ lattice gauge theory. The gauge-fixing action of the double BRST invariant model given by (6.20) consists of two terms both of which are separately BRST (and anti-BRST) exact. Multiplying the 1st term in (6.20) by the real parameter t amounts to a mere redefinition of the Morse potential which should have no further effect. We can therefore write the gauge-fixing partition function with double BRST,

$$Z_{\text{GF}}(t) = \int d[g, b, \bar{c}, c] \exp \left\{ -is\bar{s} \left(t V_U[g] + i \frac{\xi}{2\rho} \sum_i \text{tr} \bar{c}_i c_i \right) \right\}, \quad (6.41)$$

which is independent of the set of link variables $\{U\}$ and the gauge parameter ξ because of its topological nature. Moreover, the t independence is really not different from the ξ independence here, and it is thus rather obvious. Explicitly, the derivative with respect to t (or ξ) produces the expectation value of a BRST exact operator which vanishes, *i.e.*,

$$\begin{aligned} Z'_{\text{GF}}(t) &= \int d[g, b, \bar{c}, c] (-is\bar{s} V_U[g]) \exp \left\{ -is\bar{s} \left(t V_U[g] + i \frac{\xi}{2\rho} \sum_i \text{tr} \bar{c}_i c_i \right) \right\} \\ &= 0, \end{aligned} \quad (6.42)$$

provided the BRST operators are nilpotent (property that we will see lost in the case of the massive Curci-Ferrari gauge). At $t=0$ on the other hand, we obtain with (6.27) and (6.37),

$$Z_{\text{GF}}(0) = \mathcal{N} \int d[b, \bar{c}, c] \times \exp \left\{ - \sum_i \left(\frac{\xi}{2} b_i^a b_i^a + \frac{\xi}{8} (\bar{c}_i \times c_i)^2 \right) \right\}, \quad (6.43)$$

where the volume of the gauge group on the lattice, from the invariant integrations $\prod_i dg_i$ via the Haar measure over $g_i \in SU(N)$ per site i , is absorbed in the constant \mathcal{N} . The Gaussian integrations over the Nakanishi-Lautrup fields b are also well-defined and produce a factor $(2\pi/\xi)^{(N^2-1)/2}$ per site.

One might be tempted to conclude at this point that the quartic ghost self-interactions in (6.43) might remove the uncompensated Grassmann integrations of the linear covariant gauges where no such self-interactions occur. The ghost/anti-ghost integrations at $t = 0$ also factorise into independent integrations $d\bar{c}_i^a dc_i^a$ over $2(N^2 - 1)$ Grassmann variables per site. For $N = 3$, for example, the 4th order term of the exponential in (6.43) produces a monomial in \bar{c}_i^a, c_i^a which contains each of these 16 Grassmann variables exactly once, so that their integration might produce a non-vanishing result. This is not the case, however. Working out the prefactor of this monomial, as we will do explicitly in the more general case with including a non-vanishing Curci-Ferrari mass m below, one finds that the prefactor of this term in (6.43) vanishes in the massless case and thus,

$$Z_{\text{GF}}(0) = 0. \quad (6.44)$$

Because of the t -independence (6.42), this implies the vanishing of the gauge-fixing partition function (6.41) of the ghost/anti-ghost or $SL(2, \mathbb{R})$ symmetric formulation with double BRST invariance in the same way as that of standard Faddeev-Popov theory observed in [Neu87]. As for the latter, the sign-weighted sum over all Gribov copies, as originally proposed to generalise the Faddeev-Popov procedure in presence of Gribov copies [Hir79, Fuj79], vanishes.

This cancellation of Gribov copies is well-known [Sha84]. The fact that it also arises here, in the ghost/anti-ghost symmetric formulation with its quartic self-interactions, directly relates to the topological interpretation [BS98, Sch99] of the Neuberger zero: Z_{GF} can be viewed as the partition function of a Witten-type topological model to compute the Euler character χ of the gauge group. On the lattice the gauge group is a direct product of $SU(N)$'s per site, and because the Euler character factorises,

$$Z_{\text{GF}} = \chi(SU(N)^{\#\text{sites}}) = \chi(SU(N))^{\#\text{sites}} = 0^{\#\text{sites}}.$$

For $t = 0$ the action in (6.41) decouples from the link-field configuration and $Z_{\text{GF}}(0)$, albeit computing the same topological invariant, has of course no effect in terms of fixing a gauge. In the present formulation, with $Z_{\text{GF}}(0)$ in (6.43), the independent Grassmann integrations per site of the quartic-ghost term which contains the curvature of $SU(N)$ each compute its Euler character via the Gauss-Bonnet theorem [BBRT91]. This explicitly produces one factor of zero per site on the lattice. And it provides the topological explanation for the vanishing of the prefactor of the corresponding monomial of degree $2(N^2 - 1)$ in the Grassmann variables \bar{c}, c , which could otherwise exist in the expansion of the exponential in (6.43) for all odd N . For $N = 3$, for example, the zero in this prefactor arises, upon normalordering, from a cancellation of 368 non-vanishing individual terms

when expanding the square of the square of the quartic ghost self-interaction. This cancellation would be rather unnatural to arise accidentally, without such explanation.

The vanishing of the gauge-fixing partition function at $t = 0$ part in Neuberger's argument, in the ghost/anti-ghost symmetric gauges with $SL(2, \mathbb{R}) \rtimes$ double BRST symmetry, therefore most directly reflects the topological origin of the Neuberger zero. Eq. (6.43) precisely represents a product of one Gauss-Bonnet integral expression for $\chi(SU(N))$ per site of the lattice.

Note that the gauge parameter ξ can be removed completely from the expression for $Z_{\text{GF}}(0)$ in Eq. (6.43) by a rescaling $\sqrt{\xi} b \rightarrow b$ and $\sqrt[4]{\xi} \bar{c} \rightarrow \bar{c}$, $\sqrt[4]{\xi} c \rightarrow c$, which leaves the integration measure unchanged. The same rescaling for the full gauge-fixing partition function $Z_{\text{GF}}(t)$ in (6.41), which amounts to replacing the action in S_{GF} in (6.38) by

$$S_{\text{GF}}(t) = \sum_i \left\{ -itb_i^a F_i^a(U^g) - it\bar{c}_i^a M_{\text{FP}_i}^a[c] + \frac{\xi}{2} b_i^a b_i^a + \frac{\xi}{8} (\bar{c}_i \times c_i)^2 \right\}, \quad (6.45)$$

furthermore shows that t and ξ really represent a single parameter $t/\sqrt{\xi}$. Setting $t = 0$ in Neuberger's argument is therefore the same as the $\xi \rightarrow \infty$ limit which is usually what is considered as the Gauss-Bonnet limit in topological quantum field theory [BBRT91]. As mentioned above, there is no gauge-fixing in this limit, but it provides a simple way to compute the value (zero here) of the partition function which is independent of $t/\sqrt{\xi}$.

In the opposite limit, that of the Landau gauge $\xi \rightarrow 0$ or $t/\sqrt{\xi} \rightarrow \infty$, of course, $Z_{\text{GF}}(t)$ still reduces to the sign-weighted sum over all Gribov copies as usual [Hir79, Fuj79],

$$Z_{\text{GF}}(t) \rightarrow \sum_{\text{copies } \{g^{(i)}\}} \text{sign}(\det M_{\text{FP}}(U^{g^{(i)}})), \quad (6.46)$$

which because of the t (and ξ) independence (6.42) thus computes the same topological zero [Sha84, BS98, Sch99], in this case via the Poincaré-Hopf theorem [BBRT91].

6.3 The massive Curci-Ferrari model on the lattice

In the previous section we have seen that the quartic ghost self-interactions of the $SL(2, \mathbb{R}) \rtimes$ double BRST symmetric Curci-Ferrari gauges have no effect on

the disastrous conclusion of the 0/0 problem in lattice BRST. They rather serve to reveal most clearly the topological origin of this problem.

We will demonstrate explicitly below that this zero can be regularised, however, by introducing a Curci-Ferrari mass m , as proposed in [KvSW05, GvSW06]. The gauge-fixing action S_{GF} is thereby once more replaced by

$$S_{\text{mGF}}(t) = i(s\bar{s} - im^2) \left(t V_U[g] + i\xi \sum_i \text{tr } \bar{c}_i c_i \right) \quad (6.47)$$

(where we dropped in the 2nd term the factor $1/(2\rho) = 1$, in the fundamental representation). The BSRT and anti-BRST transformations of U^g , \bar{c} and c in Eqs. (6.13), (6.14), (6.15) and (6.17), (6.18) of Sect. 6.1 remain unchanged. Those for the Nakanishi-Lautrup b -fields, Eqs. (6.16) and (6.19), are replaced by [TM80],

$$sb^a = im^2 c^a - \frac{1}{2}(c \times b)^a - \frac{1}{8}((c \times c) \times \bar{c})^a, \quad (6.48)$$

$$\bar{s}b^a = im^2 \bar{c}^a - \frac{1}{2}(\bar{c} \times b)^a + \frac{1}{8}((\bar{c} \times \bar{c}) \times c)^a. \quad (6.49)$$

In the derivation of the explicit form for $S_{\text{mGF}}(t)$, using these modified anti-BRST transformations, the only modification in comparison to Sect. 6.1, arises from $s(\bar{c}^a b^a)$ in (6.37), which now becomes,

$$s(\bar{c}^a b^a) = -im^2 \bar{c}^a c^a + b^a b^a + \frac{1}{4}(\bar{c} \times c)^2. \quad (6.50)$$

The additional first term on the right contributes an additional term $-i\frac{\xi m^2}{2}\bar{c}_i^a c_i^a$ to the gauge-fixing Lagrangian, *c.f.*, Eq. (6.28). Together with the same contribution from the explicit mass term in (6.47) we therefore obtain *twice that* as the ghost mass-term of the massive Curci-Ferrari model (this subtlety will be worth remembering for later). The action of the massive Curci-Ferrari model therefore becomes, explicitly,

$$\begin{aligned} S_{\text{mGF}}(t) = m^2 t V_U[g] + \sum_i \left\{ -it b_i^a F_i^a(U^g) - it \bar{c}_i^a M_{\text{FP}_i^a}[c] \right. \\ \left. + \frac{\xi}{2} b_i^a b_i^a - im^2 \xi \bar{c}_i^a c_i^a + \frac{\xi}{8} (\bar{c}_i \times c_i)^2 \right\}. \end{aligned} \quad (6.51)$$

BRST and anti-BRST transformations are no-longer nilpotent at finite m^2 , but we have [NO90, CF76, TMB83]

$$\begin{aligned} s^2 = im^2 \sigma^+, \quad \bar{s}^2 = -im^2 \sigma^-, \\ s\bar{s} + \bar{s}s = -im^2 \sigma^0, \end{aligned} \quad (6.52)$$

where σ^\pm and σ^0 generate the global $SL(2, \mathbb{R})$ including ghost number and Faddeev-Popov conjugation. The Curci-Ferrari mass decontracts the $sl(2, \mathbb{R}) \times$ double BRST algebra of the massless case to the $osp(1|2)$ superalgebra extension of the Lie algebra of the 3-dimensional Lorentz group $SL(2, \mathbb{R})$. Conversely, the $m^2 \rightarrow 0$ limit is interpreted as a Wigner-Inonu contraction of the simple superalgebra $osp(1|2)$ [NO90, TM80]. The BRST and anti-BRST invariance of the massive Curci-Ferrari action in (6.47) itself follows readily from this algebra as given in (6.52), noting that only \bar{c} and c transform non-trivially under the $SL(2, \mathbb{R})$.

We emphasise that this algebra decontraction has from the very beginning been known to lead to a breakdown of unitarity when attempting a BRST cohomology construction of a physical Hilbert space in analogy to the massless case [CF76]. In fact, explicit examples exist for states of negative norm surviving in any such construction [dBSvNW96, Oji82]. They do not belong to BRST quartets and can therefore not be removed by the quartet mechanism [NO90]. Only through the algebra contraction by $m^2 \rightarrow 0$ do these states reduce to zero norm components which have no effect on the physical S -matrix elements.

Here we deliberately do not want to interpret the mass parameter by Curci and Ferrari as a physical mass. It rather serves to meaningfully define a limit $m^2 \rightarrow 0$ on the lattice, perhaps in parallel with the continuum limit, to recover nilpotent (anti-)BRST transformations.

To study the parameter dependence, we first define the partition function of the massive Curci-Ferrari model, explicitly listing all three parameters (even though these again really only represent 2 independent ones as we will show below),

$$Z_{\text{mGF}}(t, \xi, m^2) = \int d[g, b, \bar{c}, c] \exp \{ - S_{\text{mGF}}(t) \} , \quad (6.53)$$

with $S_{\text{mGF}}(t)$ from (6.47) or (6.51). We note in passing that the terms proportional to m^2 in the massive Curci-Ferrari action (6.51) are given by

$$\mathcal{O}(t, \xi) \equiv tV_U[g] - i\xi \sum_i \bar{c}_i^a c_i^a , \quad (6.54)$$

or, in the continuum,

$$\mathcal{O}(t, \xi) = \int d^D x \left(\frac{t}{2} A_\mu^a(x) A_\mu^a(x) - i\xi \bar{c}^a(x) c^a(x) \right) . \quad (6.55)$$

For $t = 1$ this coincides with the on-shell BRST invariant (at $m^2 = 0$) operator proposed by Kondo as a possible candidate for a dimension 2 condensate [KS00]. The doubling of the explicit ghost mass-term in (6.20), by the BRST variation of $\bar{c}b$ in (6.50) as mentioned above, is crucial here. Without this difference in

the relative factor of 2 between the two terms in $\mathcal{O}(t, \xi)$ and the gauge fixing functional

$$-iW_{\text{GF}} = tV_U[g] - i\frac{\xi}{2} \sum_i \bar{c}_i^a c_i^a, \quad (6.56)$$

one could not have both, the on-shell BRST invariance of \mathcal{O} and the gauge-fixing action in (6.20) from the double BRST variation $S_{\text{GF}} = s\bar{s}W_{\text{GF}}$, at the same time.

The observation that the mass terms in (6.51) are given by $m^2\mathcal{O}(t, \xi)$ could in principle be used to obtain the expectation value of Kondo's operator from the derivative

$$\langle \mathcal{O}(t, \xi) \rangle = -\frac{\partial}{\partial m^2} \ln Z_{\text{mGF}}(t, \xi, m^2) \Big|_{m^2=0}, \quad (6.57)$$

upon insertion into the unfixed partition function of lattice gauge theory, *i.e.*, with taking the additional expectation value in the gauge-field ensemble. As any other observable at $m^2 = 0$ this expectation value as it stands, unfortunately, of course also suffers from Neuberger's 0/0 problem of lattice BRST.

In order to demonstrate that the Curci-Ferrari mass regulates the Neuberger zero, for $t = 0$ we will verify by explicit calculation that

$$Z_{\text{mGF}}(0, \xi, m^2) \neq 0. \quad (6.58)$$

In fact, from (6.53), (6.51),

$$Z_{\text{mGF}}(0, \xi, m^2) = \mathcal{N} \int d[b, \bar{c}, c] \exp \left\{ - \sum_i \left(\frac{\xi}{2} b_i^a b_i^a - im^2 \xi \bar{c}_i^a c_i^a + \frac{\xi}{8} (\bar{c}_i \times c_i)^2 \right) \right\}, \quad (6.59)$$

which again factorises into independent Grassmann (and b -field) integrations per site on the lattice. Using the same rescaling $\sqrt{\xi} b \rightarrow b$ and $\sqrt[4]{\xi} \bar{c} \rightarrow \bar{c}$, $\sqrt[4]{\xi} c \rightarrow c$ as mentioned in the last section, we obtain,

$$Z_{\text{mGF}}(0, \xi, m^2) = \left(V_N (2\pi)^{(N^2-1)/2} I_N(m^2 \sqrt{\xi}) \right)^{\#\text{sites}}, \quad (6.60)$$

where V_N is the group volume of $SU(N)$, and

$$I_N(\hat{m}^2) = \int \prod_{a=1}^{N^2-1} d(i\bar{c}^a) dc^a \exp \left\{ i\hat{m}^2 \bar{c} \cdot c - \frac{1}{8} (\bar{c} \times c)^2 \right\}, \quad (6.61)$$

where we used the rather obvious abbreviations $\bar{c}c = \bar{c}^a c^a$, $(\bar{c} \times c)^a = f^{abc} \bar{c}^b c^c$, and $\hat{m}^2 = m^2 \sqrt{\xi}$. Note that we define the Grassmann integration measure to include the imaginary unit i with the real anti-ghosts \bar{c} so as to reproduce the result of

integrating over complex conjugate Grassmann variables $c^a \pm i\bar{c}^a$. Expanding the exponential and collecting the relevant powers in the ghost/anti-ghost variables, for $SU(2)$ we obtain

$$I_N(\widehat{m}^2) = \int \prod_{a=1}^{N^2-1} d(i\bar{c}^a) dc^a \left\{ \frac{\widehat{m}^6}{3!} (i\bar{c} \cdot c)^3 + \frac{1}{8} (\bar{c} \times c)^2 \widehat{m}^2 (i\bar{c} \cdot c) \right\}. \quad (6.62)$$

Due to the anti-symmetry of the ghost fields we notice that $(i\bar{c} \cdot c)^3 = 6 \prod_a (i\bar{c}^a \cdot c^a)$ and $\epsilon^{abc} \bar{c}^b c^c \epsilon^{ade} \bar{c}^d c^e = +(\bar{c} c)^2$. Therefore the term combining the quartic and the quadratic interaction simply becomes $(\bar{c} \times c)^2 (i\bar{c} \cdot c) = (i\bar{c} c)^3$. According to these considerations $I_N(\widehat{m}^2)$ becomes for $SU(2)$

$$I_2(\widehat{m}^2) = \frac{3}{4} \widehat{m}^2 \left(1 + \frac{4}{3} \widehat{m}^4 \right). \quad (6.63)$$

For $SU(3)$ the computation is a bit more tedious. First of all notice that the quartic term can be written $f^{abc} \bar{c}^b c^c f^{ade} \bar{c}^d c^e = \frac{2}{N_c} (\bar{c}^a c^c)^2 + d^{abc} (\bar{c}^b c^c) d^{ade} (\bar{c}^d c^e)$ or equivalently, adopting the fundamental representation, with Hermitian generators $f^{abc} \bar{c}^b c^c f^{ade} \bar{c}^d c^e = 2 \text{tr} ((T^a f^{abc} \bar{c}^b c^c)^2)$. This last term can also be cast into a more convenient form as

$$\begin{aligned} 2 \text{tr} ((T^a f^{abc} \bar{c}^b c^c)^2) &= -2 \text{tr} ([T^b, T^c] \bar{c}^b c^c)^2 \\ &= -2 \text{tr} (\bar{c} c + c \bar{c})^2 \\ &= -2 \text{tr} (\{\bar{c}, c\})^2. \end{aligned} \quad (6.64)$$

Also, using the Jacobi Identity

$$f^{abc} \bar{c}^b c^c f^{ade} \bar{c}^d c^e + f^{acd} c^c \bar{c}^d f^{abe} \bar{c}^b c^e + f^{adb} \bar{c}^d \bar{c}^b f^{ace} c^c c^e = 0 \quad (6.65)$$

we can write

$$(\bar{c} \times c)^2 = -\frac{1}{2} (\bar{c} \times \bar{c})(c \times c). \quad (6.66)$$

Therefore, the quartic interaction, in the fundamental representation reads

$$\begin{aligned} (\bar{c} \times c)^2 &= -\frac{1}{2} f^{abc} \bar{c}^b \bar{c}^c f^{ade} c^d c^e \\ &= -\text{tr} (T^a f^{abc} \bar{c}^b \bar{c}^c T^h f^{hde} c^d c^e) \\ &= \text{tr} (\{\bar{c}, \bar{c}\} \{c, c\}) \\ &= 4 \text{tr} (\bar{c}^2 c^2). \end{aligned} \quad (6.67)$$

Similarly, for the quadratic term, we have

$$\bar{c}^a c^a = 2 \operatorname{tr}(\bar{c}c). \quad (6.68)$$

The integral over the ghost fields then assumes the compact form

$$I_N(\widehat{m}^2) = \int \prod_{a=1}^{N^2-1} d(i\bar{c}^a) dc^a e^{2\widehat{m}^2 \operatorname{tr}(\bar{c}c) + \frac{\xi^2}{2} \operatorname{tr}(c^2 c^2)}. \quad (6.69)$$

The result for $SU(3)$, using Mathematica to compute all the possible combinations, is

$$I_3(\widehat{m}^2) = \frac{45}{64} \widehat{m}^4 \left(1 + 4\widehat{m}^4 + \frac{64}{15} \widehat{m}^8 + \frac{64}{45} \widehat{m}^{12} \right). \quad (6.70)$$

In both cases we factored the leading power for $\widehat{m}^2 \rightarrow 0$. $I_N(\widehat{m}^2)$ is polynomial in $\widehat{m}^2 = m^2 \sqrt{\xi}$ of degree $N^2 - 1$, for all N . The successively lower powers of \widehat{m}^2 decrease by 2 in each step in this polynomial, reflecting an increasing power of the quartic ghost self-interactions contributing to each term. Therefore, the polynomials $I_N(\widehat{m}^2)$ are odd/even in \widehat{m}^2 for N even/odd.

Because the polynomial is odd for all even N there can thus not be an order-zero term in the first place. The powers of the quartic interactions alone never match the number of independent Grassmann variables, and the Neuberger zero at $\widehat{m}^2 = 0$ arises rather trivially for even N (for the same reason that the Euler character of an odd-dimensional manifold necessarily vanishes).

For N odd, $I_N(\widehat{m}^2)$ is an even polynomial which could in principle have an order zero, constant term. The fact that this term is absent, *e.g.*, as explicitly verified for $SU(3)$ in (6.70), reflects the vanishing of the Euler character of $SU(N)$ also for odd N , as mentioned above (the even dimension $N^2 - 1$ of the algebra is deceiving in this case as, for example, the parameter space of $SU(3)$ can roughly be thought of consisting of odd-dimensional S^3 and S^5).

In any case, the polynomials $I_N(\widehat{m}^2)$ do not have a constant term and therefore vanish with $\widehat{m}^2 \rightarrow 0$, *i.e.*, $I_N(0) = 0$, as expected. Moreover, the scaling argument used here and in the last section shows that the partition function (6.53) of massive Curci-Ferrari model can only depend on two of the three parameters,

$$Z_{\text{mGF}}(t, \xi, m^2) = f(t/\sqrt{\xi}, \xi m^4). \quad (6.71)$$

An independent route of deriving this generic form, from the equations of motions, will be presented below. In this section we explicitly obtained $f(0, y)$ with $y = \widehat{m}^4$ to constrain this function $f(x, y)$ of two variables along the $x = t/\sqrt{\xi} = 0$ line, and verified that

$$Z_{\text{mGF}}(0, \xi, m^2) = f(0, \xi m^4) \propto \begin{cases} (\xi m^4)^{\#\text{sites}/2}, & N = 2 \\ (\xi m^4)^{\#\text{sites}}, & N = 3 \end{cases}$$

for $m^2 \rightarrow 0$. Because of the topological explanation of the zero obtained in this limit, *i.e.*, $f(0,0) = 0$, as discussed in the last section, this actually constrains f to vanish along the entire $y = 0$ line, $f(x,0) = 0$ for all $x = t/\sqrt{\xi}$.

For $x = 0$ we could furthermore define a non-vanishing, finite limit

$$\lim_{m^2 \rightarrow 0} (\xi m^4)^{-N_{\text{tot}}} Z_{\text{mGF}}(0, \xi, m^2) = \text{const.} \quad (6.72)$$

with an appropriate power $N_{\text{tot}} = \#$ of sites on a finite lattice for odd N , or half that for even N . This constant could in principle be inserted into the unfixed lattice gauge theory measure without harm, *i.e.*, avoiding the zero in (6.44). Because $x = 0$, however, this still has no effect in terms of gauge-fixing by the Faddeev-Popov procedure either. We need to get away from $x = 0$, at least by a small amount, to suppress those parts of the gauge orbits with large violations of the Lorenz condition. At finite Curci-Ferrari mass m^2 we no-longer have the t -independence (or x -independence) of (6.42). We can therefore not conclude at this point yet that the constant in (6.72) will essentially remain unchanged when going to some finite $x \neq 0$ as we must.

We are not quite there yet, and we will therefore have to have a closer look at the parameter dependence of the massive Curci-Ferrari model in the next section.

6.4 Parameter Dependences

From Eqs. (6.53) and (6.47) or (6.51) we immediately obtain the following (logarithmic) derivatives,

$$\begin{aligned} t \frac{\partial}{\partial t} \ln Z_{\text{mGF}}(t, \xi, m^2) &= -i \langle (s\bar{s} - im^2) t V_U[g] \rangle_{m^2}, \\ 2\xi \frac{\partial}{\partial \xi} \ln Z_{\text{mGF}}(t, \xi, m^2) &= -i \langle (s\bar{s} - im^2) (-i\xi \sum_i \bar{c}_i^a c_i^a) \rangle_{m^2}, \\ m^2 \frac{\partial}{\partial m^2} \ln Z_{\text{mGF}}(t, \xi, m^2) &= -\langle m^2 \mathcal{O}(t, \xi) \rangle_{m^2}, \end{aligned} \quad (6.73)$$

where the subscripts m^2 on the right denote expectation values within the Curci-Ferrari model at finite mass. In particular, the derivative w.r.t. m^2 in the last line differs from (6.57) only in that m^2 has not been set to zero here yet. All these expectation values can, in general, depend on the link-field configuration $\{U\}$ which acts as a background field to the model. Independence of $\{U\}$ is only guaranteed to hold in the topological limit $m^2 \rightarrow 0$.

From the definition of \mathcal{O} in (6.54), we thus find that

$$\left(t \frac{\partial}{\partial t} + 2\xi \frac{\partial}{\partial \xi} - m^2 \frac{\partial}{\partial m^2} \right) \ln Z_{\text{mGF}}(t, \xi, m^2) = -i \langle s\bar{s} \mathcal{O}(t, \xi) \rangle_{m^2}.$$

The standard argument that the expectation value of an (anti-)BRST exact operator vanishes does not hold at finite m^2 . Neither are BRST and anti-BRST variations nilpotent, nor is \mathcal{O} invariant under the BRST or anti-BRST transformations. However, the equations of motion for (anti-)ghost and Nakanishi-Lautrup fields on the lattice, *i.e.*, their lattice Dyson-Schwinger equations, can be used to show that, indeed,

$$\langle s\bar{s}\mathcal{O}(t,\xi)\rangle_{m^2} = 0, \quad (6.74)$$

even at finite m^2 . In fact, consider the variation of the massive Curci-Ferrari action w.r.t. the b -field

$$\frac{\delta S_{\text{mGF}}}{\delta b_i^a} = itF_i^a(U^g) + \xi b_i^a = \xi(b')_i^a, \quad (6.75)$$

such that

$$\begin{aligned} \left\langle \frac{\delta S_{\text{mGF}}}{\delta b_i^a} b_j^b \right\rangle_{m^2} &= \xi \left\langle (b')_i^a \left((b')_i^a - \frac{it}{\xi} F_j^b(U^g) \right) \right\rangle_{m^2} \\ &= -i(N_c^2 - 1)N_{\# \text{sites}} \delta^{ab} \delta_{ij}, \end{aligned} \quad (6.76)$$

where we used the fact that $\langle (b')_i^a (b')_i^a \rangle_{m^2}$ corresponds to a Gaussian integral, whose result is simply $\frac{1}{\xi} \delta^{ab} \delta_{ij}$. For the anti-ghost field we have, where all the functional derivatives are understood left graded, we write

$$\begin{aligned} \frac{\delta S_{\text{mGF}}}{\delta \bar{c}_i^a} &= itM_{\text{FP}i}^a[c] - im^2 c_i^a + \xi \frac{g^2}{4} f^{abc} c_i^b f^{cde} \bar{c}_i^d c_i^e \\ &= itM_{\text{FP}i}^a[c] - im^2 c_i^a - \xi \frac{g^2}{4} ((\bar{c} \times c) \times c)_i^a. \end{aligned} \quad (6.77)$$

Consequently we have the lattice DS equations as follows

$$\begin{aligned} \left\langle \frac{\delta S_{\text{mGF}}}{\delta \bar{c}_i^a} \bar{c}_j^b \right\rangle_{m^2} &= it \langle M_{\text{FP}i}^a[c] \bar{c}_j^b \rangle_{m^2} - im^2 \langle c_i^a \bar{c}_j^b \rangle_{m^2} - \left\langle \xi \frac{g^2}{4} ((\bar{c} \times c) \times c)_i^a \bar{c}_j^b \right\rangle_{m^2} \\ &= i(N_c^2 - 1)N_{\# \text{sites}} \delta^{ab} \delta_{ij}. \end{aligned} \quad (6.78)$$

Putting together Eq. (6.76) and (6.78), we exactly obtain Eq. (6.74). Therefore,

$$\left(t \frac{\partial}{\partial t} + 2\xi \frac{\partial}{\partial \xi} - m^2 \frac{\partial}{\partial m^2} \right) Z_{\text{mGF}}(t, \xi, m^2) = 0. \quad (6.79)$$

This differential equation entails that we can write the partition function of the model in the generic form (6.71).

As we already did in the previous sections, we therefore continue to use the new parameters $x = t/\sqrt{\xi}$ and $\hat{m}^2 = m^2\sqrt{\xi}$ from now on, writing

$$Z_{\text{mGF}} \equiv Z_{\text{mGF}}(x, \hat{m}^2). \quad (6.80)$$

Again using rescaled fields $\sqrt{\xi} b \rightarrow b$, $\sqrt[4]{\xi} \bar{c} \rightarrow \bar{c}$, $\sqrt[4]{\xi} c \rightarrow c$ and $\sqrt[4]{\xi} \bar{s} \rightarrow \bar{s}$, $\sqrt[4]{\xi} s \rightarrow s$, the (anti-)BRST transformations of Eq. (6.14) – (6.19) remain formally unchanged, and m^2 is replaced by \hat{m}^2 in those of the massive model in Eqs. (6.48), (6.49). Correspondingly, all other relations above are then converted by the formal replacements $\xi \rightarrow 1$, $t \rightarrow x$ and $m^2 \rightarrow \hat{m}^2$. In particular,

$$\begin{aligned} S_{\text{mGF}}(x) &= i(s\bar{s} - i\hat{m}^2) \left(x V_U[g] - \frac{i}{2} \sum_i \bar{c}_i^a c_i^a \right) \\ &= \sum_i \left\{ -ix b_i^a F_i^a(U^g) - ix \bar{c}_i^a M_{\text{FP}_i^a}[c] \right. \\ &\quad \left. + \frac{1}{2} b_i^a b_i^a + \frac{1}{8} (\bar{c}_i \times c_i)^2 \right\} + \hat{m}^2 \mathcal{O}(x), \end{aligned} \quad (6.81)$$

with

$$\mathcal{O}(x) = x V_U[g] - i \sum_i \bar{c}_i^a c_i^a \quad (6.82)$$

The two independent derivatives left, are readily read off in an analogous way to give

$$\begin{aligned} \frac{\partial}{\partial x} \ln Z_{\text{mGF}}(x, \hat{m}^2) &= -i \langle (s\bar{s} - i\hat{m}^2) V_U[g] \rangle_{\hat{m}^2}, \\ \frac{\partial}{\partial \hat{m}^2} \ln Z_{\text{mGF}}(x, \hat{m}^2) &= -\langle \mathcal{O}(x) \rangle_{\hat{m}^2}. \end{aligned} \quad (6.83)$$

In absence of a topological argument for the gauge parameter independence at finite Curci-Ferrari mass, the best we can do to achieve independence of $x = t/\sqrt{\xi}$ is to allow an x dependent mass parameter $\hat{m}^2 \equiv \hat{m}^2(x)$. In particular, the $x = 0$ results of the previous section are then to be interpreted as being expressed in terms of $\hat{m}^2(0)$. These results will remain unchanged for $x \neq 0$, if we adjust the mass function $\hat{m}^2(x)$ with x in the partition function Z_{mGF} , accordingly. That is, if

$$\begin{aligned} 0 &= \frac{d}{dx} Z_{\text{mGF}}(x, \hat{m}^2(x)) \\ &= \left(\frac{\partial}{\partial x} + \frac{d\hat{m}^2}{dx} \frac{\partial}{\partial \hat{m}^2} \right) Z_{\text{mGF}}(x, \hat{m}^2(x)). \end{aligned} \quad (6.84)$$

From Eqs. (6.83) we see that this requires that

$$\frac{d\hat{m}^2}{dx} = -i \frac{\langle (s\bar{s} - i\hat{m}^2) V_U[g] \rangle_{\hat{m}^2}}{\langle \mathcal{O}(x) \rangle_{\hat{m}^2}}. \quad (6.85)$$

This is not a very profound insight. The crucial question at this point is, whether the tuning of the Curci-Ferrari mass parameter with x is possible independent of

the link configuration $\{U\}$ which is far from obvious here. Otherwise we would have to choose a different trajectory in the parameter space (x, \widehat{m}^2) for different gauge orbits which would be of little use then, as far as the Faddeev-Popov gauge-fixing procedure is concerned. If it is possible, on the other hand, we can then use the value of the mass $\widehat{m}_0^2 = \widehat{m}^2(0)$ at $x = 0$ to regulate the Neuberger zero and use the x and $\{U\}$ independent, non-vanishing and finite constant

$$\lim_{\widehat{m}_0^2 \rightarrow 0} (\widehat{m}_0^4)^{-N_{\text{tot}}} Z_{\text{mGF}}(x, \widehat{m}^2(x)) = \text{const.} \quad (6.86)$$

as the starting definition of Faddeev-Popov gauge fixing on the lattice. Then, of course, we would also expect that there should be a topological meaning to this constant which is so far, however, unfortunately unknown to us.

All we can offer at the moment is to verify that all is well at $x = 0$, where we can do the explicit calculations. It is relatively straight-forward to show in this way that

$$\left. \frac{d\widehat{m}^2}{dx} \right|_{x=0} = \text{const } x + O(x^2) \quad (6.87)$$

with the constant independent of $\{U\}$. While this is merely necessary, but not sufficient, it demonstrates that we can get away from $x = 0$, at least infinitesimally. This is of qualitative importance as a non-zero value of $x = t/\sqrt{\xi}$, no matter how small, corresponds to a large but finite ξ at $t = 1$ and thus eliminates the gauge freedom. The study and result for the second derivative will be presented in the next publication.

Batalin-Vilkovisky Formalism In Y-M Theory

We have shown so far how the quantisation of Y-M theory can be pursued by means of several formalisms: canonical, covariant operator, path integral and BRST formalism. Yet, the most general algebraic way to quantise a gauge theory is achieved by the Batalin-Vilkovisky (BV) formalism [BV81b, BV83]. This formalism provides a Grassmann-graded canonical formalism, by means of a new canonical structure, called *anti-bracket*, which generalises and elevates the standard Hamiltonian formalism to a more algebraic scenario. Furthermore, the Batalin-Vilkovisky method encompasses the Faddeev-Popov quantisation and can be entirely formulated in the light of BRST and anti-BRST symmetry. The new fields introduced in this formalism, the *anti-fields*, necessary to build up the global canonical structure, are identified with functional derivatives of an anti-fermion gauge-fixing term. We will first introduce the main ingredients of this formalism, and after that we will present the BV construction of Euclidian 4-dimensional Y-M theory in the framework of non-linear gauges. At last, we will provide the lattice version of our model.

7.1 The Appearance Of Anti-Fields

To introduce a canonical formalism, graded with respect to the Poisson brackets first we have to deal with the concept of Grassmann parity, which defines the *even* or *odd* character of a field under product exchange. Given a set of fields $\Phi^A(x)$ of Grassmann parity $\epsilon(\Phi^A) = \epsilon_A$, then we associate to them the corresponding anti-fields $\Phi_A^*(x)$ of opposite parity, as $\epsilon(\Phi_A^*) = \epsilon_A + 1$. Fields and anti-fields play the role of conjugate variables in the Hamiltonian framework, and therefore it is natural to define a canonical conjugation as

$$(\Phi^A, \Phi_B^*) = \delta_B^A \quad (\Phi^A, \Phi^B) = (\Phi_A^*, \Phi_B^*) = 0, \quad (7.1)$$

where the conjugation is defined through the BV brackets

$$(F, G) = \frac{\delta^r F}{\delta \Phi^A} \frac{\delta^l G}{\delta \Phi_A^*} - \frac{\delta^r F}{\delta \Phi_A^*} \frac{\delta^l G}{\delta \Phi^A}, \quad (7.2)$$

where r and l denote respectively right and left derivative. These brackets determine therefore a canonical structure onto the BV formalism, providing it a Jacobi identity

$$\sum_{\mathcal{P}(F,G,H)} (-1)^{(\epsilon_F+1)(\epsilon_H+1)} (F, (G, H)) = 0, \quad (7.3)$$

and a Leibnitz rule

$$\begin{aligned} (F.GH) &= (F, G)H + (-1)^{\epsilon_G(\epsilon_F+1)} G(F, H) \\ (FH.G) &= F(H, G)H + (-1)^{\epsilon_H(\epsilon_G+1)} (F, G)H. \end{aligned} \quad (7.4)$$

To the antifields we can also associate a ghost number as

$$\mathcal{G}h(\Phi_A^*) = -\mathcal{G}h(\Phi^A) - 1 \quad (7.5)$$

and it is constrained such that the quantum action carries total ghost number zero. The BV quantisation prescription amounts to solve the *quantum Master Equation*

$$\frac{1}{2}(W, W) = i\hbar\Delta W, \quad (7.6)$$

where Δ a second-order odd differential (nilpotent) operator defined as

$$\Delta = (-1)^{\epsilon_A+1} \frac{\delta^r}{\delta\Phi^A} \frac{\delta^r}{\delta\Phi_A^*}. \quad (7.7)$$

and $W = W[\Phi, \Phi^*]$ a generic quantum action, that is supposed to be expandable in powers of \hbar

$$W = S_{\text{cl}} + \sum_{n=0}^{\infty} \hbar^n S_{\text{qu}}^{(n)} \quad (7.8)$$

Here, S_{cl} is the classical action obtained by setting all the anti-fields to zero, and S_{qu} its quantum fluctuation. To the lowest order in \hbar , the ‘‘classical’’ Master Equation reduces to

$$(S, S) = 0. \quad (7.9)$$

The path integral representation of the BV formalism starts from the consideration that the classical action, as previously observed, is generated while setting $\Phi_A^* = 0$. Therefore, by defining the correct gauge-fixing anti-fermion, we can write

$$\mathcal{Z} = \int [d\Phi_A][d\Phi_A^*] \delta\left(\Phi_A^* - \frac{\delta^r \bar{\Psi}}{\delta\Phi_A}\right) \exp\left[-\frac{1}{\hbar}W\right]. \quad (7.10)$$

After integrating out the anti-field Φ_A^* using the delta-function, one can verify that the action is left invariant under usual BRST transformations.

7.2 Non-linear gauges in BV formalism

It is well known that the pure Yang-Mills action $S[A] = \frac{1}{2} \int_M F_{\mu\nu} F^{\mu\nu}$, in a certain irreducible representation ρ of $SU(N)$, is left invariant under a gauge transformation ${}^g A = g^\dagger A g + g^\dagger \partial g$. An interesting question to ask ourselves is what happens if the gauge field A is being shifted as $A \rightarrow A - \tilde{A}$: does the gauge symmetry still remain and moreover, how does this shift-symmetry affect the underlying BRST structure? It is this background gauge manipulation of the Y-M action which poses the bases of the Batalin-Vilkovisky method of quantisation. The appearance of the anti-ghost fields in $S[A - \tilde{A}]$ can be observed by gauge-fixing iteratively the gauge-symmetry and the shift-symmetry, as done for instance in [AD93]. Though pedagogically interesting, we prefer to give a more heuristic approach to it. Suppose to gauge fix with a non-linear gauge the Euclidian Y-M action: we then insert in the path integral representation the following covariant, non-linear gauge-fixing Euclidean Lagrangian with ghost/anti-ghost symmetry

$$\mathcal{L}_{\text{gf}}[A, b, c, \bar{c}] = ib^a \partial A^a + \frac{\xi}{2} b^2 + \frac{i}{2} \bar{c}^a \{ \partial, \mathcal{D} \}^{ab} c^b + \frac{\xi}{8} (\bar{c} \times c)^2. \quad (7.11)$$

where all the fields are in the adjoint representation of $SU(N)$. This Lagrangian is left invariant under the BRST and anti-BRST transformations (4.53) and can be written either as a BRST coboundary term

$$\mathcal{L}_{\text{gf}}[A, b, c, \bar{c}] = s \bar{\Psi}, \quad \bar{\Psi} = i \bar{c}^a \left(\partial_\mu A^{a\mu} - i \frac{\xi}{2} b^a \right), \quad (7.12)$$

or a double BRST coboundary term

$$\mathcal{L}_{\text{gf}}[A, b, c, \bar{c}] = s \bar{s} W, \quad W = \frac{i}{2} (A_\mu^a A^{a\mu} - i \xi \bar{c}^a c^a). \quad (7.13)$$

Performing now a shift in all the fields ¹

$$A \rightarrow A - \tilde{A} \quad c \rightarrow c - \tilde{c} \quad \bar{c} \rightarrow \bar{c} - \tilde{\bar{c}} \quad b \rightarrow b - \tilde{b}, \quad (7.14)$$

we notice that the quantum action $S[A, c, \bar{c}, b]$ becomes also invariant under the shift symmetry

$$\begin{aligned} s\Phi(x) &= \alpha(x) & \bar{s}\Phi(x) &= \gamma(x) \\ s\tilde{\Phi}(x) &= \alpha(x) - \beta(x) & \bar{s}\tilde{\Phi}(x) &= \gamma(x) - \chi(x), \end{aligned} \quad (7.15)$$

¹From now on we will leave component notation implicit, unless needed in a specific computation.

with $\Phi(x)$ the set of all fields and $\tilde{\Phi}(x)$ the set of shifted ones. Here $\beta(x)$ and $\chi(x)$ represent the original BRST and anti-BRST variations of $\Phi(x)$, whereas $\alpha(x)$ and $\gamma(x)$ correspond to some collective fields generating the field-shift. Let us focus first on the BRST construction of the Batalin-Vilkovisky Lagrangian. The BRST transformations (4.53), according to the shift (7.14), assume the form

$$\begin{aligned}
sA_\mu &= \psi_\mu & s\tilde{A}_\mu &= \psi_\mu - \mathcal{D}_\mu^{(A-\tilde{A})}(c - \tilde{c}) \\
sc &= \epsilon & s\tilde{c} &= \epsilon + \frac{1}{2}[c - \tilde{c}, c - \tilde{c}] \\
s\bar{c} &= \bar{\epsilon} & s\tilde{\bar{c}} &= \bar{\epsilon} - (b - \tilde{b}) + \frac{1}{2}[\bar{c} - \tilde{\bar{c}}, c - \tilde{c}] \\
sb &= \epsilon_b & s\tilde{b} &= \epsilon_b + \frac{1}{2}[c - \tilde{c}, b - \tilde{b}] + \frac{1}{8}[[c - \tilde{c}, c - \tilde{c}], \bar{c} - \tilde{\bar{c}}]. \quad (7.16)
\end{aligned}$$

A few remarks here are needed: first of all, the choice of transformations we have made is consistent with the hermiticity of the ghost fields, chosen to be real, to satisfy the hermiticity of the Lagrangian and therefore the unitarity of the \mathcal{S} -matrix. Moreover, to generate a ghost/anti-ghost gauge-fixing action, we have shifted the b -field as $b \rightarrow b - \frac{1}{2}[\bar{c}, c]$: such a symmetry is different from standard covariant linear gauges, as Landau gauge, where the action is not symmetric under the exchange of ghost into anti-ghost fields. This operation also affects the linearity of the gauge chosen, and as a consequence we obtain a nonlinear gauge, whose main feature is to generate a quartic ghost interaction in the action. This term is required from topological considerations to produce the most general renormalisable covariant action with an underlying BRST symmetry, as showed in [BTM82]. Finally, the choice we made to associate the covariant derivative only to the shifted field makes the original gauge symmetry of the original gauge field to be carried entirely by the collective field. The transformation of the original gauge field is then taken always just as a shift.

To enforce the overall invariance under s of the quantum action, we require more fields: among some new Nakanishi-Lautrup fields, it is important to notice the appearance of the anti-fields, denoted with an asterisk

$$\begin{aligned}
s\psi_\mu &= 0 & s\epsilon &= 0 \\
s\bar{\epsilon} &= 0 & s\epsilon_b &= 0 \\
sA_\mu^* &= B_\mu & sB_\mu &= 0 \\
sc^* &= B & sB &= 0 \\
s\bar{c}^* &= \bar{B} & s\bar{B} &= 0 \\
sb^* &= B_b & sB_b &= 0, \quad (7.17)
\end{aligned}$$

where the multiplet $(\psi_\mu, \epsilon, \bar{\epsilon}, \epsilon_b)$ is the ghost multiplet associated with the shift

symmetry for (A_μ, c, \bar{c}, b) , $(A_\mu^*, c^*, \bar{c}^*, b^*)$ are the anti-ghosts² and (B_μ, B, \bar{B}, B_b) the corresponding auxiliary fields. Having such an abundance of fields with different ghost number, it is worthwhile to provide the table

$$\begin{array}{llll}
\mathcal{G}h(A_\mu) = \mathcal{G}h(\bar{A}_\mu) = 0 & \mathcal{G}h(A_\mu^*) = -1 & \mathcal{G}h(\psi_\mu) = 1 & \mathcal{G}h(B_\mu) = 0 \\
\mathcal{G}h(c) = \mathcal{G}h(\bar{c}) = 1 & \mathcal{G}h(c^*) = 0 & \mathcal{G}h(\epsilon) = 2 & \mathcal{G}h(B) = 1 \\
\mathcal{G}h(\bar{c}) = \mathcal{G}h(\bar{\bar{c}}) = -1 & \mathcal{G}h(\bar{c}^*) = -2 & \mathcal{G}h(\bar{\epsilon}) = 0 & \mathcal{G}h(\bar{B}) = -1 \\
\mathcal{G}h(b) = \mathcal{G}h(\tilde{b}) = 0 & \mathcal{G}h(b^*) = -1 & \mathcal{G}h(\epsilon_b) = 1 & \mathcal{G}h(B_b) = 0.
\end{array}$$

It is interesting to notice that in the BV formalism, the Nakanishi-Lautrup fields associated to the FP ghosts are Grassmann, with Ghost number respectively ± 1 . We can therefore write in close-form the various Ghost number and Grassmann parity for the general field Φ as

$$\mathcal{G}h(\Phi^*) = \mathcal{G}h(\Phi) - 1 \quad \epsilon(\Phi^*) = \epsilon(\Phi) + 1. \quad (7.18)$$

We may also construct the following table which summarises the relations among the BV fields

Field	Collective Field	Anti-Ghost (anti-field)	Ghost	Nakanishi-Lautrup Field
A_μ	\bar{A}_μ	A_μ^*	ψ_μ	B_μ
c	\bar{c}	c^*	ϵ	B
\bar{c}	$\bar{\bar{c}}$	\bar{c}^*	$\bar{\epsilon}$	\bar{B}
b	\tilde{b}	b^*	ϵ_b	B_b

The physical requirement for the gauge-fixing Lagrangian is obtained by demanding all the fields associated to the shift symmetry to vanish. We thus recover the original theory, by choosing for instance the following shift-symmetry gauge-fixing Lagrangian

$$\begin{aligned}
\tilde{\mathcal{L}}_{\text{GF}} = i & \left[B_\mu \bar{A}_\mu - A_\mu^* \left(\psi^\mu - \mathcal{D}_\mu^{(A-\bar{A})}(c - \bar{c}) \right) + \bar{B} \bar{c} + \bar{c}^* \left(\epsilon + \frac{1}{2} [c - \bar{c}, c - \bar{c}] \right) \right. \\
& + B \bar{c} + c^* \left(\bar{\epsilon} - (b - \tilde{b}) + \frac{1}{2} [\bar{c} - \bar{\bar{c}}, c - \bar{c}] \right) \\
& \left. + B_b \tilde{b} - b^* \left(\epsilon_b + \frac{1}{2} [c - \bar{c}, b - \tilde{b}] + \frac{1}{8} [[c - \bar{c}, c - \bar{c}], \bar{c} - \bar{\bar{c}}] \right) \right]. \quad (7.19)
\end{aligned}$$

It is an easy task to check that the Lagrangian $\tilde{\mathcal{L}}_{\text{GF}}$ is left invariant under the BRST transformations (7.16) and (7.17). Moreover, the requirement for

²These antifields are the usual antighosts of the collective fields enforcing the Dyson-Schwinger equations through shift symmetries.

$\tilde{\mathcal{L}}_{\text{GF}}$ to be Hermitian is guaranteed by the factor i in front of the trace. By performing the integration over the B -fields, we generate Delta functions in the shift-symmetry fields, which will set them to zero.

All the field couplings in (7.19) amount to an overall vanishing Ghost number as required. Suppose the gauge-fixing anti-fermion $\bar{\Psi}$ to depend upon only the original fields as

$$\begin{aligned}\mathcal{L}_{\text{GF}} &= i s \bar{\Psi} = i \left(s A_\mu \frac{\delta \bar{\Psi}}{\delta A_\mu} + s c \frac{\delta \bar{\Psi}}{\delta c} + s \bar{c} \frac{\delta \bar{\Psi}}{\delta \bar{c}} + s b \frac{\delta \bar{\Psi}}{\delta b} \right) \\ &= i \left(\frac{\delta \bar{\Psi}}{\delta A_\mu} \psi^\mu + \frac{\delta \bar{\Psi}}{\delta c} \epsilon + \frac{\delta \bar{\Psi}}{\delta \bar{c}} \bar{\epsilon} + \frac{\delta \bar{\Psi}}{\delta b} \epsilon_b \right).\end{aligned}\quad (7.20)$$

Thus, integrating out the auxiliary fields we set all the fields associated to the shift symmetry to zero. The remaining Lagrangian is the *BV gauge-fixing Lagrangian*

$$\begin{aligned}\mathcal{L}_{\text{BV}} &= \tilde{\mathcal{L}}_{\text{GF}} + \mathcal{L}_{\text{GF}} \\ &= i \left\{ A_\mu^* \mathcal{D}^\mu c + \frac{1}{2} \bar{c}^* [c, c] - c^* \left(-b + \frac{1}{2} [\bar{c}, c] \right) - b^* \left(\frac{1}{2} [c, b] + \frac{1}{8} [[c, c], \bar{c}] \right) \right. \\ &\quad \left. - A_\mu^* \psi^\mu + \bar{c}^* \epsilon + c^* \bar{\epsilon} - b^* \epsilon_b \right\} + i \left(\frac{\delta \bar{\Psi}}{\delta A_\mu} \psi^\mu + \frac{\delta \bar{\Psi}}{\delta c} \epsilon + \frac{\delta \bar{\Psi}}{\delta \bar{c}} \bar{\epsilon} + \frac{\delta \bar{\Psi}}{\delta b} \epsilon_b \right) \\ &= i \left\{ A_\mu^* \mathcal{D}^\mu c + \frac{1}{2} \bar{c}^* [c, c] - c^* \left(-b + \frac{1}{2} [\bar{c}, c] \right) + b^* \left(\frac{1}{2} [c, b] + \frac{1}{8} [[c, c], \bar{c}] \right) \right. \\ &\quad \left. - \left(A_\mu^* - \frac{\delta \bar{\Psi}}{\delta A_\mu} \right) \psi^\mu + \left(\bar{c}^* + \frac{\delta \bar{\Psi}}{\delta c} \right) \epsilon + \left(c^* + \frac{\delta \bar{\Psi}}{\delta \bar{c}} \right) \bar{\epsilon} - \left(b^* - \frac{\delta \bar{\Psi}}{\delta b} \right) \epsilon_b \right\}.\end{aligned}\quad (7.21)$$

To obtain the conditions on the anti-fields, it is sufficient to integrate out the ghosts associated with the shift symmetry

$$\begin{aligned}A_\mu^* &= \frac{\delta \bar{\Psi}}{\delta A_\mu} & s A_\mu &= i \frac{\delta \mathcal{L}_{\text{BV}}}{\delta A_\mu^*} \\ \bar{c}^* &= -\frac{\delta \bar{\Psi}}{\delta c} & s c &= -i \frac{\delta \mathcal{L}_{\text{BV}}}{\delta c^*} \\ c^* &= -\frac{\delta \bar{\Psi}}{\delta \bar{c}} & s \bar{c} &= -i \frac{\delta \mathcal{L}_{\text{BV}}}{\delta \bar{c}^*} \\ b^* &= \frac{\delta \bar{\Psi}}{\delta b} & s b &= i \frac{\delta \mathcal{L}_{\text{BV}}}{\delta b^*}\end{aligned}\quad (7.22)$$

Once we define the explicit form of the anti-fermion gauge-fixing $\bar{\Psi}$, according to (7.22) we automatically determine the identification of the anti-fields with

BRST fields as

$$\begin{aligned}
A_\mu^* &= -i\partial_\mu \bar{c} \\
\bar{c}^* &= 0 \\
c^* &= i \left(\partial_\mu A^\mu - i \frac{\xi}{2} b \right) \\
b^* &= \frac{\xi}{2} \bar{c}.
\end{aligned} \tag{7.23}$$

Equations (7.23) clarify the geometric interpretation of the anti-fields on the line of Maurer-Cartan 1-forms. We are now able to write the total action as a BRST variation of a gauge-fixing anti-fermion function as a proper Witten-type theory

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + s\bar{\Psi} \\
&= \mathcal{L}_0 + is \left(A_\mu^* \tilde{A}^\mu + \bar{c}^* \tilde{c} + c^* \tilde{\bar{c}} + b^* \tilde{b} \right) \\
&= \mathcal{L}_0 + is \left(\Phi^* \tilde{\Phi} \right),
\end{aligned} \tag{7.24}$$

such that $\mathcal{G}h(A_\mu^* \tilde{A}^\mu + \bar{c}^* \tilde{c} - \tilde{\bar{c}} c^* + b^* \tilde{b}) = -1$ as expected. It is worth noting the difference with the ordinary BRST-exact gauge-fixing term

$$\bar{\Psi} = i\bar{c} \left(\partial_\mu A^\mu - i \frac{\xi}{2} b \right) = i\bar{c} c^*. \tag{7.25}$$

7.3 Including Double BRST Algebra

As noted by Nakanishi and Ojima [NO80, Oji80] the quantum action is also invariant under an additional symmetry, known as anti-BRST, whose relation with the BRST operator is given by Faddeev-Popov conjugation

$$\bar{s} = C_{\text{FP}} s C_{\text{FP}}^{-1}. \tag{7.26}$$

Following the structure of (7.16), we demand that $\bar{s}(\Phi - \tilde{\Phi})$ reproduces the anti-BRST variations of ordinary fields (A_μ, c, \bar{c}, b) : for instance we might have this algebra

$$\begin{aligned}
\bar{s}A_\mu &= A_\mu^* + \mathcal{D}_\mu^{(A-\tilde{A})}(\bar{c} - \tilde{c}) & \bar{s}\tilde{A}_\mu &= A_\mu^* \\
\bar{s}c &= c^* - (b - \tilde{b}) - \frac{1}{2}[\bar{c} - \tilde{c}, c - \tilde{c}] & \bar{s}\tilde{c} &= c^* \\
\bar{s}\bar{c} &= \bar{c}^* - \frac{1}{2}[\bar{c} - \tilde{c}, \bar{c} - \tilde{\bar{c}}] & \bar{s}\tilde{\bar{c}} &= \bar{c}^* \\
\bar{s}b &= b^* - \frac{1}{2}[\bar{c} - \tilde{c}, b - \tilde{b}] + \frac{1}{8}[[\bar{c} - \tilde{c}, \bar{c} - \tilde{\bar{c}}], c - \tilde{c}] & \bar{s}\tilde{b} &= b^*.
\end{aligned} \tag{7.27}$$

Imposing the condition of invariance on $\tilde{\mathcal{L}}_{\text{GF}}$, we generate the other variations: consider for example

$$\begin{aligned}\bar{s} \left[B_\mu \tilde{A}_\mu - A_\mu^* \left(\psi^\mu - \mathcal{D}_\mu^{(A-\bar{A})}(c - \tilde{c}) \right) \right] &= B_\mu \bar{s} \tilde{A}_\mu + A_\mu^* \bar{s} \left(\psi^\mu - \mathcal{D}_\mu^{(A-\bar{A})}(c - \tilde{c}) \right) \\ &= B_\mu A_\mu^* + A_\mu^* \bar{s} \psi_\mu - A_\mu^* \bar{s} \left(\mathcal{D}_\mu^{(A-\bar{A})}(c - \tilde{c}) \right).\end{aligned}\quad (7.28)$$

Here we have imposed vanishing variation on B_μ , as suggested in [AD93] and [BD95]. In order to have invariance under \bar{s} we need to impose this variation on the field ψ_μ

$$\begin{aligned}\bar{s} \psi_\mu &\equiv -B_\mu + \bar{s} \left(\mathcal{D}_\mu^{(A-\bar{A})}(c - \tilde{c}) \right) \\ &= -B_\mu - \mathcal{D}_\mu^{(A-\bar{A})}(b - \tilde{b}) - \frac{1}{2} \mathcal{D}_\mu^{(A-\bar{A})}[(\bar{c} - \tilde{c}), (c - \tilde{c})] + [\mathcal{D}_\mu^{(A-\bar{A})}(\bar{c} - \tilde{c}), (c - \tilde{c})].\end{aligned}\quad (7.29)$$

Since the ghost fields are anti-commuting, then the following identity holds $[\bar{c}, c] = [c, \bar{c}]$,³ implying that

$$\bar{s} \psi_\mu = -B_\mu - \mathcal{D}_\mu^{(A-\bar{A})}(b - \tilde{b}) + \frac{1}{2} [\mathcal{D}_\mu^{(A-\bar{A})}(\bar{c} - \tilde{c}), (c - \tilde{c})] - \frac{1}{2} [\mathcal{D}_\mu^{(A-\bar{A})}(c - \tilde{c}), (\bar{c} - \tilde{c})],\quad (7.30)$$

where we used the fact that $-\frac{1}{2} [(\bar{c} - \tilde{c}), \mathcal{D}_\mu^{(A-\bar{A})}(c - \tilde{c})] = -\frac{1}{2} [\mathcal{D}_\mu^{(A-\bar{A})}(c - \tilde{c}), (\bar{c} - \tilde{c})]$. According to [BD95], the anti-variation $\bar{s} \psi_\mu$ in a linear gauge is

$$\bar{s} \psi_\mu = -B_\mu - \mathcal{D}_\mu^{(A-\bar{A})}(b - \tilde{b}) - [\mathcal{D}_\mu^{(A-\bar{A})}(c - \tilde{c}), (\bar{c} - \tilde{c})].\quad (7.31)$$

Therefore, in the presence of non-linear gauges, this variation becomes more symmetric, as far as the action of the covariant derivative onto the FP ghosts. It is worth checking the nilpotency of this transformation:

$$\begin{aligned}\bar{s}^2 \psi_\mu &= \bar{s} \left(-B_\mu - \mathcal{D}_\mu^{(A-\bar{A})}(b - \tilde{b}) - \frac{1}{2} \mathcal{D}_\mu^{(A-\bar{A})}[(\bar{c} - \tilde{c}), (c - \tilde{c})] + [\mathcal{D}_\mu^{(A-\bar{A})}(\bar{c} - \tilde{c}), (c - \tilde{c})] \right) \\ &= \bar{s} \left[\mathcal{D}_\mu^{(A-\bar{A})} \left(-(b - \tilde{b}) - \frac{1}{2} [(\bar{c} - \tilde{c}), (c - \tilde{c})] \right) \right] + \bar{s} \left([\mathcal{D}_\mu^{(A-\bar{A})}(\bar{c} - \tilde{c}), (c - \tilde{c})] \right).\end{aligned}\quad (7.32)$$

³In fact, using component notation $[\bar{c}, c]^a = f^{abc} \bar{c}^b c^c$. Changing the index b into c , and using the antisymmetry of the structure constants we get $f^{abc} \bar{c}^b c^c = f^{acb} \bar{c}^c c^b = -f^{acb} c^b \bar{c}^c = f^{abc} c^b \bar{c}^c \equiv [c, \bar{c}]$.

Using the identity $\bar{s} \left(\mathcal{D}_\mu^{(A-\bar{A})}(\bar{c} - \tilde{c}) \right) = 0$ we obtain

$$\begin{aligned}
\bar{s}^2 \psi_\mu &= (\bar{s} \mathcal{D}_\mu^{(A-\bar{A})}) \left(-(b - \tilde{b}) - \frac{1}{2} [(\bar{c} - \tilde{c}), (c - \tilde{c})] \right) \\
&\quad - \left[\mathcal{D}_\mu^{(A-\bar{A})}(\bar{c} - \tilde{c}), -(b - \tilde{b}) - \frac{1}{2} [(\bar{c} - \tilde{c}), (c - \tilde{c})] \right] \\
&= \left[\mathcal{D}_\mu^{(A-\bar{A})}(\bar{c} - \tilde{c}), -(b - \tilde{b}) - \frac{1}{2} [(\bar{c} - \tilde{c}), (c - \tilde{c})] \right] \\
&\quad - \left[\mathcal{D}_\mu^{(A-\bar{A})}(\bar{c} - \tilde{c}), -(b - \tilde{b}) - \frac{1}{2} [(\bar{c} - \tilde{c}), (c - \tilde{c})] \right] \\
&= 0
\end{aligned} \tag{7.33}$$

For the FP ghosts, we apply the same procedure, supposing the variations with respect to the two auxiliary fields to vanish:

$$\begin{aligned}
&\bar{s} \left\{ \bar{B}\tilde{c} + \bar{c}^* \left(\epsilon + \frac{1}{2} [c - \tilde{c}, c - \tilde{c}] \right) + B\tilde{c} + c^* \left(\bar{\epsilon} - (b - \tilde{b}) + \frac{1}{2} [\bar{c} - \tilde{c}, c - \tilde{c}] \right) \right\} \\
&= -B\bar{c}^* + \bar{c}^* \bar{s}\epsilon + \bar{c}^* \frac{1}{2} \bar{s}[c - \tilde{c}, c - \tilde{c}] \\
&\quad - \bar{B}c^* + c^* \bar{s}\bar{\epsilon} + c^* \left(-\bar{s}(b - \tilde{b}) + \frac{1}{2} \bar{s}[\bar{c} - \tilde{c}, c - \tilde{c}] \right).
\end{aligned} \tag{7.34}$$

We separate the two contributions for ϵ and $\bar{\epsilon}$: for the second line we obtain ⁴

$$\begin{aligned}
\bar{c}^* \bar{s}\epsilon &= B\bar{c}^* - \bar{c}^* \frac{1}{2} \bar{s}[c - \tilde{c}, c - \tilde{c}] \\
&= \bar{c}^* B - \bar{c}^* \frac{1}{2} [\bar{s}(c - \tilde{c})](c - \tilde{c}) + \bar{c}^* \frac{1}{2} (c - \tilde{c}) \bar{s}(c - \tilde{c}) \\
&= \bar{c}^* B - \bar{c}^* [(c - \tilde{c}), (b - \tilde{b})] - \frac{1}{2} \bar{c}^* [(c - \tilde{c}), [(\bar{c} - \tilde{c}), (c - \tilde{c})]],
\end{aligned} \tag{7.35}$$

implying that

$$\begin{aligned}
\bar{s}\epsilon &= B - [(c - \tilde{c}), (b - \tilde{b})] - \frac{1}{2} [(c - \tilde{c}), [(\bar{c} - \tilde{c}), (c - \tilde{c})]] \\
&= B - [(c - \tilde{c}), (b - \tilde{b})] - \frac{1}{4} [[(c - \tilde{c}), (c - \tilde{c})], (\bar{c} - \tilde{c})].
\end{aligned} \tag{7.36}$$

Checking the nilpotency we find an inconsistency: in fact

$$\begin{aligned}
\bar{s}^2 \epsilon &= \bar{s} \left(B - [(c - \tilde{c}), (b - \tilde{b})] - \frac{1}{2} [(c - \tilde{c}), [(\bar{c} - \tilde{c}), (c - \tilde{c})]] \right) \\
&= \frac{1}{4} [[(\bar{c} - \tilde{c}), (c - \tilde{c})], [(\bar{c} - \tilde{c}), (c - \tilde{c})]] \neq 0.
\end{aligned} \tag{7.37}$$

⁴It is worth noting that $[c, [\bar{c}, c]] = f^{abc} f^{cmn} c^b \bar{c}^m c^n$. Using the Jacobi Identity $f^{abc} f^{cmn} = -f^{mac} f^{cbn} - f^{bmc} f^{can}$, we obtain $f^{abc} f^{cmn} c^b \bar{c}^m c^n = -f^{mac} f^{cbn} c^b \bar{c}^m c^n - f^{bmc} f^{can} c^b \bar{c}^m c^n$. Rearranging the indices in the third term we get $f^{abc} f^{cmn} c^b \bar{c}^m c^n = -f^{mac} f^{cbn} c^b \bar{c}^m c^n - f^{abc} f^{cmn} c^b \bar{c}^m c^n$ and thus $2f^{abc} f^{cmn} c^b \bar{c}^m c^n = -f^{mac} f^{cbn} c^b \bar{c}^m c^n$ or similarly $2[c, [\bar{c}, c]] = +[\bar{c}, [c, c]] = -[[c, c], \bar{c}]$.

This means that we have to replace $\bar{s}B = 0$ with

$$\bar{s}B = -\frac{1}{4}[[(\bar{c} - \tilde{c}), (c - \tilde{c})], [(\bar{c} - \tilde{c}), (c - \tilde{c})]], \quad (7.38)$$

which is nilpotent

$$\bar{s}^2 B = 0, \quad (7.39)$$

because of the following identities

$$\begin{aligned} [[[\bar{c}, \tilde{c}], c], [\bar{c}, c]] &= -[[\bar{c}, c], [[\bar{c}, \tilde{c}], c]] \\ [[\bar{c}, b], [\bar{c}, c]] &= -[[\bar{c}, c], [\bar{c}, b]]. \end{aligned} \quad (7.40)$$

Similarly for $\bar{\epsilon}$ we obtain

$$\begin{aligned} c^* \bar{s} \bar{\epsilon} &= \bar{B} c^* - c^* \left(-\bar{s}(b - \tilde{b}) + \frac{1}{2} \bar{s}[\bar{c} - \tilde{c}, c - \tilde{c}] \right) \\ &= c^* \bar{B} - c^* [(\bar{c} - \tilde{c}), (b - \tilde{b})] + \frac{1}{4} c^* [[\bar{c} - \tilde{c}, \bar{c} - \tilde{c}], c - \tilde{c}]. \end{aligned} \quad (7.41)$$

It is clear that this transformation is nilpotent because

$$\bar{s}^2 b = \bar{s} \left(-\frac{1}{2} [\bar{c}, b] + \frac{1}{8} [[\bar{c}, \tilde{c}], c] \right) = 0. \quad (7.42)$$

For the b and \tilde{b} fields we have, according to the identities $[c, [\bar{c}, b]] = [[\bar{c}, b], c]$ and $[c, [\tilde{c}, b]] = [[\tilde{c}, c], b] + [[c, b], \tilde{c}]$, derived from the anti-symmetric property of the structure constants and the Jacobi Identity,

$$\begin{aligned} b^* \bar{s} \epsilon_b &= -B_b b^* - b^* \bar{s} \left(\frac{1}{2} [(c - \tilde{c}), (b - \tilde{b})] \right) - b^* \bar{s} \left(\frac{1}{8} [[c - \tilde{c}, c - \tilde{c}], \bar{c} - \tilde{c}] \right) \\ b^* \bar{s} \epsilon_b &= -b^* B_b + \frac{1}{4} b^* [[\bar{c} - \tilde{c}, c - \tilde{c}], b - \tilde{b}] + \frac{1}{16} b^* [[\bar{c} - \tilde{c}, \bar{c} - \tilde{c}], [c - \tilde{c}, c - \tilde{c}]]. \end{aligned} \quad (7.43)$$

Therefore, we generate the following anti-transformations on the ghosts

$$\begin{aligned} \bar{s} \psi_\mu &= -B_\mu - \mathcal{D}_\mu^{(A-\bar{A})}(b - \tilde{b}) + \frac{1}{2} [\mathcal{D}_\mu^{(A-\bar{A})}(\bar{c} - \tilde{c}), (c - \tilde{c})] - \frac{1}{2} [\mathcal{D}_\mu^{(A-\bar{A})}(c - \tilde{c}), (\bar{c} - \tilde{c})] \\ \bar{s} \epsilon &= B - [(c - \tilde{c}), (b - \tilde{b})] + \frac{1}{4} [[(c - \tilde{c}), (c - \tilde{c})], (\bar{c} - \tilde{c})] \\ \bar{s} \bar{\epsilon} &= \bar{B} - [(\bar{c} - \tilde{c}), (b - \tilde{b})] + \frac{1}{4} [[(\bar{c} - \tilde{c}), (\bar{c} - \tilde{c})], (c - \tilde{c})] \\ \bar{s} \epsilon_b &= -B_b + \frac{1}{4} [[\bar{c} - \tilde{c}, c - \tilde{c}], b - \tilde{b}] + \frac{1}{16} [[\bar{c} - \tilde{c}, \bar{c} - \tilde{c}], [c - \tilde{c}, c - \tilde{c}]] \end{aligned} \quad (7.44)$$

and on the anti-fields

$$\begin{aligned}
\bar{s}A_\mu^* &= 0 & \bar{s}B_\mu &= 0 \\
\bar{s}c^* &= 0 & \bar{s}B &= 0 \\
\bar{s}\bar{c}^* &= 0 & \bar{s}\bar{B} &= 0 \\
\bar{s}b^* &= 0 & \bar{s}B_b &= 0.
\end{aligned} \tag{7.45}$$

We may notice an usual structure for the anti-variations of the fields associated to the shift-symmetry. These equations look similar to the usual form we saw for the anti-BRST transformations (4.41), $\bar{s}c = -b - \frac{1}{2}[\bar{c}, c]$. The vector ghost ψ_μ behaves as a gauge field due to the presence of the covariant derivative: yet, due to its Ghost number ($\mathcal{G}h(\psi_\mu) = 1$), the ghost field appearing in (4.3) ($sA_\mu = \mathcal{D}_\mu^{(A)}c$) has been replaced by the Nakanishi-Lautrup field in the second term, plus an additional coupling of the covariant derivative with the FP ghost in the adjoint representation. The anti-transformations for ϵ and $\bar{\epsilon}$ are obtained one from the other by ghost exchange operation, whereas for ϵ_b we notice an additional coupling with $\times (c - \bar{c})$ with respect to terms appearing in $\bar{s}b$.

In order to achieve a more symmetric for (7.27), (7.44) and (7.45), we can adopt the same procedure used to symmetrise standard BRST transformations in Chapter 3. Consider in fact the following shift in B_μ

$$\begin{aligned}
B_\mu \rightarrow & B_\mu - \frac{1}{2}\mathcal{D}_\mu^{(A-\bar{A})}(b - \tilde{b}) + \frac{1}{4}[\mathcal{D}_\mu^{(A-\bar{A})}(\bar{c} - \tilde{\bar{c}}), (c - \tilde{c})] \\
& - \frac{1}{4}[\mathcal{D}_\mu^{(A-\bar{A})}(c - \tilde{c}), (\bar{c} - \tilde{\bar{c}})]
\end{aligned} \tag{7.46}$$

Apply now this shift to $\bar{s}\psi_\mu$ and to sA_μ^* and what we get is

$$\begin{aligned}
\bar{s}\psi_\mu &= -B_\mu - \frac{1}{2}\mathcal{D}_\mu^{(A-\bar{A})}(b - \tilde{b}) + \frac{1}{4}[\mathcal{D}_\mu^{(A-\bar{A})}(\bar{c} - \tilde{\bar{c}}), (c - \tilde{c})] - \frac{1}{4}[\mathcal{D}_\mu^{(A-\bar{A})}(c - \tilde{c}), (\bar{c} - \tilde{\bar{c}})] \\
sA_\mu^* &= B_\mu - \frac{1}{2}\mathcal{D}_\mu^{(A-\bar{A})}(b - \tilde{b}) + \frac{1}{4}[\mathcal{D}_\mu^{(A-\bar{A})}(\bar{c} - \tilde{\bar{c}}), (c - \tilde{c})] - \frac{1}{4}[\mathcal{D}_\mu^{(A-\bar{A})}(c - \tilde{c}), (\bar{c} - \tilde{\bar{c}})]
\end{aligned} \tag{7.47}$$

which tell us how the anti-field of the gauge-field behaves as the anti-ghost for the ghost field associated to the shift symmetry, as well as in ordinary BRST, \bar{c} is the anti-ghost of c . The only difference in the geometric interpretation of the BRST operators as differential operators in the superspace lies in the different sign with respect to the auxiliary field b (Note we have generated the same difference in sign with respect to B_μ). The same approach can be adopted to all

the other remaining auxiliary fields associated with the shift symmetry

$$\begin{aligned}
\bar{s}\epsilon &= B - \frac{1}{2}[c - \tilde{c}, b - \tilde{b}] - \frac{1}{8}[[c - \tilde{c}, c - \tilde{c}], \bar{c} - \tilde{\bar{c}}] \\
\bar{s}\bar{\epsilon} &= \bar{B} - \frac{1}{2}[\bar{c} - \tilde{\bar{c}}, b - \tilde{b}] - \frac{1}{8}[[\bar{c} - \tilde{\bar{c}}, \bar{c} - \tilde{\bar{c}}], c - \tilde{c}], \\
\bar{s}\epsilon_b &= -B_b - \frac{1}{4}[[\bar{c} - \tilde{\bar{c}}, c - \tilde{c}], b - \tilde{b}] + \frac{1}{16}[[[\bar{c} - \tilde{\bar{c}}, \bar{c} - \tilde{\bar{c}}], c - \tilde{c}], c - \tilde{c}],
\end{aligned} \tag{7.48}$$

and the anti-fields

$$\begin{aligned}
sc^* &= B + \frac{1}{2}[c - \tilde{c}, b - \tilde{b}] - \frac{1}{8}[[c - \tilde{c}, c - \tilde{c}], \bar{c} - \tilde{\bar{c}}] \\
s\bar{c}^* &= \bar{B} + \frac{1}{2}[\bar{c} - \tilde{\bar{c}}, b - \tilde{b}] - \frac{1}{8}[[\bar{c} - \tilde{\bar{c}}, \bar{c} - \tilde{\bar{c}}], c - \tilde{c}] \\
sb^* &= B_b - \frac{1}{4}[[\bar{c} - \tilde{\bar{c}}, c - \tilde{c}], b - \tilde{b}] + \frac{1}{16}[[[\bar{c} - \tilde{\bar{c}}, \bar{c} - \tilde{\bar{c}}], c - \tilde{c}], c - \tilde{c}].
\end{aligned} \tag{7.49}$$

We notice a geometric feature in the transformations for the two fields associated respectively to the Y-M ghost and anti-ghost: in this case, the difference in sign is not with respect to the auxiliary fields, B and \bar{B} , but in the other of the transformations. Thus is due to the fact that the anti-fields associated with c and \bar{c} have an even Grassmann parity, reflected in the BRST and anti-BRST transformations. The anti-fields b^* behaves as usual. The transformations for the four Nakanishi-Lautrup fields are more complicated: let us see in details sB_μ

$$\begin{aligned}
sB_\mu &= s \left\{ -\frac{1}{2}\mathcal{D}_\mu^{(A-\bar{A})}(b - \tilde{b}) + \frac{1}{2}[\mathcal{D}_\mu^{(A-\bar{A})}(\bar{c} - \tilde{\bar{c}}), (c - \tilde{c})] \right\} \\
&= -\frac{1}{2} \left\{ \mathcal{D}_\mu^{(A-\bar{A})}s(b - \tilde{b}) - [\mathcal{D}_\mu^{(A-\bar{A})}(c - \tilde{c}), b - \tilde{b}] \right\} \\
&\quad + \frac{1}{2} \left\{ s \left(\mathcal{D}_\mu^{(A-\bar{A})}(\bar{c} - \tilde{\bar{c}}) \right), (c - \tilde{c}) \right\} - \frac{1}{2}[\mathcal{D}_\mu^{(A-\bar{A})}(\bar{c} - \tilde{\bar{c}}), s(c - \tilde{c})]
\end{aligned} \tag{7.50}$$

The term $s \left(\mathcal{D}_\mu^{(A-\bar{A})}(\bar{c} - \tilde{\bar{c}}) \right)$ vanishes and so we obtain

$$\begin{aligned}
sB_\mu &= -\frac{1}{2}\mathcal{D}_\mu^{(A-\bar{A})} \left\{ -\frac{1}{2}[c - \tilde{c}, b - \tilde{b}] - \frac{1}{8}[[c - \tilde{c}, c - \tilde{c}], \bar{c} - \tilde{\bar{c}}] \right\} \\
&\quad + \frac{1}{2}[\mathcal{D}_\mu^{(A-\bar{A})}(c - \tilde{c}), b - \tilde{b}] + \frac{1}{4}[\mathcal{D}_\mu^{(A-\bar{A})}(\bar{c} - \tilde{\bar{c}}), [c - \tilde{c}, c - \tilde{c}]].
\end{aligned} \tag{7.51}$$

Using the Leibnitz rule for the covariant derivative sB_μ assumes the more compact form

$$sB_\mu = \frac{1}{2} \tag{7.52}$$

$$sB = s \left\{ \frac{1}{2}[c - \tilde{c}, b - \tilde{b}] + \frac{1}{8}[[c - \tilde{c}, c - \tilde{c}], \bar{c} - \tilde{\bar{c}}] \right\}$$

$$= \tag{7.53}$$

$$\bar{s}B = \tag{7.54}$$

$$s\bar{B} = s \left\{ \frac{1}{2}[\bar{c} - \tilde{\bar{c}}, b - \tilde{b}] - \frac{1}{8}[[\bar{c} - \tilde{\bar{c}}, \bar{c} - \tilde{\bar{c}}], c - \tilde{c}] \right\}$$

$$= \tag{7.55}$$

$$\bar{s}\bar{B} = \tag{7.56}$$

$$sB_b = s \left\{ -\frac{1}{4}[[\bar{c} - \tilde{\bar{c}}, c - \tilde{c}], b - \tilde{b}] + \frac{1}{16}[[[\bar{c} - \tilde{\bar{c}}, \bar{c} - \tilde{\bar{c}}], c - \tilde{c}], c - \tilde{c}] \right\}$$

$$= \tag{7.57}$$

$$\bar{s}B_b = \tag{7.58}$$

We may be naively tempted to symmetrise also the ghosts associate with the shift symmetry, but that would cause an ambiguity in the definition of the covariant, one of the essential features of the BV formalism, as pointed out previously. As a final part of this section, we show how to write the full Lagrangian as double BRST-exact quantity. We remind the reader that in the last section we wrote the full Lagrangian as

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + s\bar{\Psi} \\ &= \mathcal{L}_0 + s \left\{ -A_\mu^* \tilde{A}^\mu + \bar{c}^* \tilde{c} + c^* \tilde{\bar{c}} - b^* \tilde{b} \right\}. \end{aligned} \tag{7.59}$$

By demanding the BRST symmetry to be unbroken, we can then generate the gauge-fixing anti-fermion as

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + s\bar{\Psi} = \mathcal{L}_0 - (-1)^{\epsilon(\Phi)} s(\Phi^* \tilde{\Phi}) \\ &= \mathcal{L}_0 + s\bar{s}\Sigma = \mathcal{L}_0 + \frac{1}{2} s\bar{s} \left(-\tilde{A}_\mu \tilde{A}^\mu + \tilde{c}\bar{c} + \tilde{\bar{c}}c - \tilde{b}b \right) \\ &= \mathcal{L}_0 - (-1)^{\epsilon(\Phi)} \frac{1}{2} s\bar{s}(\tilde{\Phi} \tilde{\Phi}) \end{aligned} \tag{7.60}$$

Conclusions

This Thesis has been devoted to two main subjects: the study of gauge fixing methods in non-Abelian gauge theory and the BRST formalism, both in perturbative and non-perturbative QCD.

We have thus found a representation for Landau gauge-fixing corresponding to the FP trick being an actual change of variables with appropriate determinant. The resulting gauge-fixing Lagrangian density enjoys a larger extended BRST and anti-BRST symmetry. However it cannot be represented rigorously as a BRST exact object, rather the sum of two such objects corresponding to different BRST operations. This means that some of the BRST machinery is not available to this formulation, such as the Kugo-Ojima criterion for selecting physical states. We discuss cursorily now the perturbative renormalisability of the present formulation of the theory. Note that the procedure leading to Eq. (5.32) does not introduce any new coupling constants; only the strong coupling constant g is present in $M_F[A]$ coupling the Yang-Mills field to both the new ghosts and scalars. The dimensions of the new fields are

$$[\varphi] = L^0, \quad [d] = [\bar{d}] = L^{-1}, \quad [B] = L^{-2}. \quad (8.1)$$

Most importantly in this context, the kinetic term for the new boson fields φ^a is *quartic* in derivatives:

$$\mathcal{L}_{\text{kin}} = \varphi^a (\partial^2)^2 \varphi^a, \quad (8.2)$$

which is renormalisable, by power counting, since φ^a are dimensionless. Such a contribution is seemingly harmless in the ultraviolet regime: for large momenta propagators will vanish like $1/p^4$. Moreover it should play an important role in guaranteeing the decoupling of such contributions in perturbative diagrams. That such a decoupling should occur is clear from Eq. (5.11): in the perturbative regime fluctuations about $A_\mu = 0$ will not feel the $\text{sgn}(\det M_F[A])$, so that the field theory constructed in this way must be equivalent to the perturbatively renormalisable Landau gauge fixed theory. For example in the computation of the running coupling constant we expect that this property will lead to a complete decoupling of the t -degrees of freedom so that the known Landau gauge result emerges from just the gluon and standard ghost sectors. Naturally, the new degrees of freedom will be relevant in the infra-red regime, which will be the object of future study

Regarding the BRST formalism in non-perturbative non-Abelian theories, we showed that the massive Curci-Ferrari model with its decontracted double BRST symmetry can be formulated on the lattice without the 0/0 problem. The parameter m^2 is not interpreted as a physical mass but rather serves to meaningfully define a limit $m^2 \rightarrow 0$ in the spirit of l'Hospital's rule. At finite m^2 the topological nature of the gauge-fixing partition function seems lost. It is possible, however, to tune the Curci-Ferrari mass with the gauge parameter ξ so that the limit $m^2 \rightarrow 0$ can be defined along a certain trajectory in parameter space independent of ξ . An interesting open question might then be the topological interpretation of the model within the decontracted double BRST $osp(1|2)$ superalgebra framework.

In the Batalin-Vilkovisky formalism for non-linear gauges, we showed how the BRST and anti-BRST transformations assume a more complicated form than with respect to standard linear gauges, such as Landau gauge. We have constructed an algebraic BRST structure which still preserves the required nilpotency, allowing us to write the complete B-V Lagrangian in the form of a coboundary term, both for BRST and anti-BRST transformations. The natural implementation of this work leads to derive the lattice algebraic structure of this theory, mimicking the Curci-Ferrari model we have already proposed, in which the background lattice gauge-fixing has to be translated in the language of the anti-field formalism.

Appendix A

Connection on a principal bundle

In this appendix we will enlist briefly the major topics of gauge theory from the geometric and topological point of view. This is necessary to understand the rich geometric structure of Y-M theory. Moreover, its generalisation to super-space is essential to the comprehension of super-symmetry, BRST and topological field theory (TFT). This will be covered in Appendix C. We assume the reader being familiar with basic concepts of topology, such as manifolds, tangent and cotangent spaces. For this we remind the interested reader to [Nab]. As previously said, Y-M theories can be regarded as the quantum theory of principal bundles, on which we construct connections, covariant derivatives and curvature forms. To start with we define a principal bundle: a differentiable principal fiber bundle over a manifold M with group structure G consists of a manifold P and an action of G on P satisfying the following conditions

- G acts freely on P without fixed points, i.e. $gx = x$ implies $g = I$ (only the identity element fixes any x), $P \times G \rightarrow P$ is denoted by $P \times G \ni (u, a) \rightarrow ua \in P$;
- M is the quotient space of P by the equivalence relation induced by G , $M = P/G$, and the canonical projection $\pi : P \rightarrow M$ is differentiable;
- P is locally trivial ($P \cong \mathbb{R}^n l$).

To any element A of the algebra \mathfrak{g} of G , we associate a vector $\Sigma(A)$ on P , the fundamental vector field corresponding to A . $\Sigma(A)$ is actually generated by the right action of G on P ¹: if $A \in \mathfrak{g}$, then $\exp(tA)$ is a one-parameter subgroup of G , acting on P as

$$\Sigma(A)_u \cdot f = \left. \frac{d}{dt} f(u_t) \right|_{t=0}, \quad (9.1)$$

where $u_t = R_{\exp(tA)}(u)$. $\Sigma(A)_u$ is a vector tangent to P at u (tangent to the fiber). Call G_u the subspace of $T_u(P)$ of vectors tangent to the fiber through u ,

¹Left and right actions of a group element are diffeomorphisms defined as $L_g(h) = hg$ and $R_g(h) = hg$.

at u .

$$\Sigma : \mathfrak{g} \rightarrow G_u \quad \text{is an isomorphism.} \quad (9.2)$$

A connection in P is a choice of a supplementary linear subspace Q_u in $T_u(P)$ to G_u

$$T_u(P) = G_u \oplus Q_u \quad (9.3)$$

where $Q_{ua} = (R_a)_*Q_u$ is a *push-forward*² and depends differentially on u . Q_u is called the horizontal space and G_u the vertical space. Choosing a Q_u amounts to choosing a basis in $T_u(P)$, though this distribution is not, in general integrable. Geometrically, this corresponds to the non triviality of parallel transport using the holonomy group of the principal bundle.

Connection form

A connection form is a Lie-algebra valued 1-form ω such that

- ω applied on any fundamental vector field $\Sigma(A)$ reproduces A , i.e. $\omega(\Sigma(A)) = A$;
- $(R_a^*\omega)(X) = \text{Ad}_{a^{-1}} \cdot \omega(X)$. The horizontal subspace Q_u is the kernel of ω , that is to say that X_u is horizontal iff $\omega(X_u) = 0$;

where $(R_a^*\omega)(X)$ is a *pull-back*³. It's possible to express ω , the connection form on P , by a family of local forms, each one being defined in an open subset of the base-space manifold M . Let $\{U_\alpha\}$ be a covering of M , we choose in P the preferred set of local sections σ_α and the corresponding transition functions $\psi_{\alpha\beta}$: for each α and β . we define a Lie-algebra-valued 1-form on U_α by

$$\omega_\alpha = \sigma_\alpha^* \omega \quad \text{pull-back of } \omega \text{ through } \sigma_\alpha \quad (9.4)$$

where

$$\omega_\beta = \text{Ad}_{\psi_{\alpha\beta}^{-1}} \cdot \omega_\alpha + \psi_{\alpha\beta}^{-1} d\psi_{\alpha\beta} \quad (9.5)$$

in $U_\alpha \cap U_\beta$. If ω is a connection form on $P = M \times G$, we can construct from a global section σ_1 of P the form on M

$$\omega_1 = \sigma_1^*(\omega). \quad (9.6)$$

²Let M and N be two smooth manifolds, with dimension m and n respectively. Let $f : M \rightarrow N$ be a smooth function. Then, the *differential* or *push-forward* f_* (or df) of f in the point $p \in M$ is the application $f_* : T_p M \rightarrow T_{f(p)} N$. The push-forward defines then a change of variables in tangent spaces.

³The transpose action of the push-forward is the pull-back f^* (or δf), defined as $f^* : T_{f(p)}^* N \rightarrow T_p^* M$. Contrary to the push-forward, we cannot pass from the cotangent space $T_p^* M$ to $T_{f(p)}^* N$, but only the other way round, linking a change of variables for cotangent spaces, dual of tangent spaces.

If we now use a G -valued function g on M to transform σ_2 into $\sigma_2(x) = \sigma_1(x) \cdot g(x)$, we can define a new 1-form on M

$$\omega_2 = \sigma_2^*(\omega) \quad (9.7)$$

we have

$$\omega_2 = \text{Ad}_{g^{-1}} \cdot \omega_1 + g^{-1} dg. \quad (9.8)$$

Geometrical interpretation of gauge potentials

On a 4-dim manifold, the connection form ω , defined on an open subset of M , U_α , can be expressed as

$$\omega_\alpha = A_\alpha^\mu(x) dx_\mu \quad (9.9)$$

whose Lie-valued components transform as

$$A'^\mu(x) = \text{Ad}_{g^{-1}} \cdot A^\mu + g^{-1} \partial^\mu g \quad (9.10)$$

which are the components of the transformed connection form

$$\omega'_\alpha = \sigma_\alpha'^* \omega = A'^\mu(x) dx_\mu. \quad (9.11)$$

A change of σ by the action of some G -valued function g on M can be viewed as a change of coordinates in the principal fiber bundle P , and it induces a transformation of the components A^μ similar to the usual gauge transformation of potentials. Then the gauge potential naturally becomes the component of a geometrical object of a definite type: a connection form on P .

Covariant derivative

The concept of covariant derivative is strongly related to the horizontal lift of the derivative ∂_μ . A vector field \tilde{X} is the lift of a vector field X on M , which is the horizontal field on P , which projects onto X , s.t.

$$\pi_*(\tilde{X}_u) = X_{\pi(u)} \quad \text{where} \quad \pi : P = M \times G \rightarrow M. \quad (9.12)$$

Suppose we choose a local chart U_α on M , with local coordinates $\{x^\mu\}$. Then, we construct vector fields, with generators as $\partial_\mu = \frac{\partial}{\partial x^\mu}$, whose lift $\tilde{\partial}_\mu$ lies on $\pi^{-1}(U_\alpha) = U_\alpha \times G$. If σ_α is section over U_α , then

$$\omega_\alpha(\partial_\mu) = \sigma_\alpha^* \omega(\partial_\mu) = \omega(\sigma_{\alpha*} \partial_\mu) = (A_\alpha^\nu(x) dx_\nu, \partial_\mu) = A_{\alpha\mu} = \omega(\Sigma(A_{\alpha\mu})) \quad (9.13)$$

Hence

$$\omega(\sigma_{\alpha*} \partial_\mu - \Sigma(A_{\alpha\mu})) = 0, \quad (9.14)$$

where $\sigma_{\alpha*}\partial_\mu - \Sigma(A_{\alpha\mu})$ is evidently horizontal. Then

$$\tilde{\partial}_\mu \Big|_u = \sigma_{\alpha*}\partial_\mu - \Sigma(A_{\alpha\mu}) \quad \text{with} \quad u = \sigma_\alpha(x). \quad (9.15)$$

We can identify $\sigma_{\alpha*}\partial_\mu$ with ∂_μ and $-\Sigma(A_{\alpha\mu})$ with the Lie-algebra-valued element A_μ , to recover the usual covariant derivative

$$\mathcal{D}_\mu = \partial_\mu - A_\mu. \quad (9.16)$$

So, any point on the local section σ_α , defined by $\pi^{-1}(U_\alpha) = U_\alpha \times G$, can be thought of as

$$u_0 = \sigma_\alpha(x_0) = (x_0, e) = \sigma_{\alpha*}\partial_\mu \oplus \Sigma(A_{\alpha\mu}). \quad (9.17)$$

This point u_0 is generated by the curve on the fiber $\pi^{-1}(U_\alpha) \Big|_{x_0}$

$$P \supseteq \pi^{-1}(x_0) \ni u_t = u_0 \exp(tA_\mu) = \left(\underbrace{x_0}_{\text{point in } M}, \underbrace{e^{tA_\mu}}_{\text{element of } G} \right). \quad (9.18)$$

If f is a function on $\pi^{-1}(U_\alpha)$, then the restriction of this function to $\pi^{-1}(x_0)$ is a function F defined on G , because it's e^{tA_μ} which localises $\pi^{-1}(U_\alpha)$ to $\pi^{-1}(x_0)$. The directional derivative along u_t is clearly the action of the Lie algebra element A_μ on F at e . Thus, the covariant derivative is section-dependent. There's also another way to interpret the covariant derivative, which follows from the adjoint action on any element of P

$$\partial_\mu \psi = \partial_\mu \psi - \lim_{t \rightarrow 0} \frac{1}{t} [e^{-tA_\mu} \psi(u_0) e^{tA_\mu} - \psi(u_0)] \quad (9.19)$$

for any function ψ on P , s.t. $\psi(ua) = \text{Ad}_{a^{-1}}\psi$. It is also important to notice that while the commutator of two fundamental vectors is still a fundamental vector, showing that this map preserves space and algebra structure, it is not true that the commutator of two horizontal vector fields is still horizontal. In particular

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = -(\partial_{[\mu} A_{\nu]}) + [A - \mu, A_\nu] = -F_{\mu\nu} \quad (9.20)$$

is a fundamental vector field on the bundle space written as a Lie algebra element, and moreover

$$F_{\beta\mu\nu} = \text{Ad}_{\psi_{\alpha\beta}^{-1}} \cdot F_{\alpha\mu\nu} \quad \rightarrow \quad F'_{\mu\nu}(x) = \text{Ad}_{g^{-1}} \cdot F_{\mu\nu}(x) \quad (9.21)$$

on $U_\alpha \cap U_\beta$.

Curvature form

From the commutator of two covariant derivatives, which is a fundamental vector on P , s.t. $\omega([\mathcal{D}_\mu, \mathcal{D}_\nu]) = [\mathcal{D}_\mu, \mathcal{D}_\nu]$, we can construct a Lie-algebra valued 2-form Ω . Locally, on $U_\alpha \cap U_\beta$

$$\Omega_\alpha = \frac{1}{2} F_{\alpha\mu\nu} dx^\mu \wedge dx^\nu \quad (9.22)$$

with

$$\Omega_\beta = \text{Ad}_{\psi_{\alpha\beta}^{-1}} \Omega_\alpha. \quad (9.23)$$

To connect the curvature form to the connection form, we need to introduce the covariant exterior derivative d_ω , as

$$\Omega = d_\omega \omega = d\omega + \frac{1}{2} [\omega, \omega]. \quad (9.24)$$

If X and Y are two tangent vector to the bundle, then

$$\Omega(X, Y) = d\omega(X, Y) + \frac{1}{2} [\omega(X), \omega(Y)]. \quad (9.25)$$

Let's decompose X and Y into their vertical and horizontal components

$$X = hX \oplus vX \quad Y = hY \oplus vY \quad (9.26)$$

then, what we get is

$$\begin{aligned} \Omega(X, Y) &= d\omega(hX, hY) + d\omega(vX, vY) + d\omega(hX, vX) + d\omega(vX, hY) \\ &\quad + \frac{1}{2} [\omega(vX), \omega(vY)] + \frac{1}{2} [\omega(hX), \omega(hY)] \\ &= d\omega(hX, hY). \end{aligned} \quad (9.27)$$

Though $d^2 = 0$, $\mathcal{D}^2 \neq 0$, whereas $\mathcal{D}\Omega = 0, \forall \omega$ (Bianchi identity), using the Jacobi identity.

Group of gauge transformations

Gauge transformations are equivariant automorphisms of some G -bundle P . The 1-forms of connections are the physical interesting objects, whose components are the gauge potentials. Choosing a particular G -bundle automatically defines the set of Chern classes. ($P_k, k \in \mathbb{Z}$). In this context, the gauge transformations assume the role of elements of an infinite-dimensional Lie group, called \mathcal{G} , whose group composition is smooth

$$\Phi : P \rightarrow P, \quad \Phi \in \mathcal{C}^\infty(\text{Ad}P). \quad (9.28)$$

This group composition can be expressed as follows

$$\begin{aligned} \forall g \in \mathcal{G} | g : P \rightarrow P \Rightarrow \exists \gamma \in \text{Map}(\text{Ad}P), \gamma : P \rightarrow G, \\ g(u) = u \cdot \gamma(u), \quad u \in P, \gamma(ua) = a^{-1}\gamma(u)a, \end{aligned} \quad (9.29)$$

$$\forall g, h \in \mathcal{G}, g \circ h(u) = u \cdot (\gamma_h \cdot \gamma_g)(u).$$

Locally, the mapping $\gamma : P \rightarrow G$ can be written as

$$\gamma_\beta(x) = \psi_{\alpha\beta}^{-1}(x) \gamma_\alpha(x) \psi_{\alpha\beta}(x), \forall x \in U_\alpha \cap U_\beta, \{U_i \in I\} \subseteq P. \quad (9.30)$$

This representation is in 1-1 correspondence with the sections of the bundle \mathcal{B} associated with P with standard fiber G , G acting on itself by the adjoint map ($a(g) = aga^{-1}$). The group \mathcal{G} of gauge transformations can be identified with the set $\Gamma(\mathcal{B})$ of sections of \mathcal{B} , which is not a principal bundle though, because the action of G is not free. \mathcal{B} will have global sections and unit element (x, e) . The Lie algebra of \mathcal{G} , $\text{Lie}\mathcal{G}$. As we know, elements of sections of tangent and cotangent bundles are respectively vector fields and forms. Consider the constant unit section s of \mathcal{B} : through any point of \mathcal{B} passes one fiber. Using the local triviality of $\text{mat}\mathcal{B}$ over patches U_α , we may identify the fiber with the group G . Tangent vectors to the fiber s follow immediately, as well as parallel transport and all the operations on vector fields. These fields are elements of the algebra of G , vectors to a fiber $\pi_{\text{mat}\mathcal{B}}^{-1}(x)$, with $x \in U_\alpha$. On the transition $U_\alpha \cap U_\beta$, the map is of course

$$A_\beta = \text{Ad}_{\psi_{\alpha\beta}^{-1}} \cdot A_\alpha. \quad (9.31)$$

The field we have just determined on \mathcal{B} can be identified as a section of an associated bundle E to P , where the fiber is \mathfrak{g} and the adjoint action of G on \mathfrak{g} . Then, $\Gamma(E)$ is the Lie algebra of $\mathcal{G} \equiv \gamma(\mathcal{B})$. $\Gamma(E)$ is an infinite-dimensional module. Any section of \mathcal{B} can be written as

$$\mathcal{C}^\infty(\text{Ad}P) = \mathcal{G} \equiv \Gamma(\mathcal{B}) \ni s = \exp(\sigma), \sigma \in \Gamma(E) \quad (9.32)$$

$$\sigma : U_\alpha \rightarrow \mathfrak{g}.$$

At last, there's a particular class of gauge transformations, those which have values in the center Z of G : for such a transformation we have on some local chart U_α

$${}^g A_\mu^\alpha = g_\alpha^{-1} A_\mu^\alpha g_\alpha + g_\alpha^{-1} \partial_\mu g_\alpha = A_\mu^\alpha. \quad (9.33)$$

This can be also written as

$$\partial_\mu g_\alpha + [A_\mu^\alpha, g_\alpha] = \mathcal{D}_\mu^\alpha g_\alpha = 0, \quad (9.34)$$

i.e. $\nabla g_\alpha = 0$. Then, g_α belongs to the center of the holonomy group of the connection under consideration. In fact

$$\mathcal{D}_{[\mu}\mathcal{D}_{\nu]}g = [F_{\mu\nu}, g] = 0. \quad (9.35)$$

Covariant derivative in background

As an exercise, consider the covariant derivative, whose connection is a pure gauge, acting on a generic function ω in the adjoint representation

$$\mathcal{D}_\mu[g^\dagger\partial_\mu g]\omega = \partial_\mu\omega + [g^\dagger\partial_\mu g, \omega]. \quad (9.36)$$

Explicitly, it becomes

$$\begin{aligned} \partial_\mu\omega + [g^\dagger\partial_\mu g, \omega] &= \partial_\mu\omega + g^\dagger\partial_\mu g\omega - \omega g^\dagger\partial_\mu g \\ &= g^\dagger g\partial_\mu\omega g^\dagger + g^\dagger\partial_\mu g\omega g^\dagger - g^\dagger g\omega g^\dagger\partial_\mu g g^\dagger \\ &= g^\dagger[g\partial_\mu\omega g^\dagger + \partial_\mu g\omega g^\dagger - g\omega g^\dagger\partial_\mu g g^\dagger] \\ &= g^\dagger[g\partial_\mu\omega g^\dagger + \partial_\mu g\omega g^\dagger + g\omega\partial_\mu g^\dagger] \\ &= g^\dagger[\partial_\mu(g\omega g^\dagger)]g, \end{aligned} \quad (9.37)$$

where we have used the identity

$$g^\dagger g = \mathbb{I} \quad \Rightarrow \quad \partial_\mu(g^\dagger g) = 0. \quad (9.38)$$

To evaluate the operator \mathcal{D}^2 , we use then this compact expression

$$\mathcal{D}_\mu[g^\dagger\partial_\mu g] = g^\dagger[\partial_\mu(g\omega g^\dagger)]g$$

↓

$$\mathcal{D}^2(\omega) = g^\dagger[\partial_\mu(g[g^\dagger[\partial_\mu(g\omega g^\dagger)]g]g^\dagger)]g = g^\dagger(\square[g\omega g^\dagger])g. \quad (9.39)$$

Appendix B

Hubbard-Stratonovich transformations to linearize the quartic-ghost interaction

The massive Curci-Ferrari gauge-fixing Lagrangian density presents, being a specific example of a broader class of non-linear gauges, the feature of a quartic-ghost interaction: this is necessary in a non-linear gauge to preserve renormalizability. This non-linearity also contributes to prevent from applying straightforwardly Grassmann integration, which would turn out into the more common form of a functional determinant of the Faddeev-Popov operator, as it happens for instance in Landau gauge. To avoid such a problem, we will perform a linearisation of the quartic term, in the framework of path integral linearisation technique, making use of the Hubbard-Stratonovich transformations. To begin with, we choose a $SU(N)$ Lagrangian density, ghost/anti-ghost symmetric, quantized in the massive Curci-Ferrari gauge

$$\mathcal{L}_{\text{mCF}} = \text{tr} \left\{ \frac{\xi}{2} b^2 + ibF[{}^gA] + \frac{m^2}{2} ({}^gA)^2 + \frac{i}{2} \bar{c} \{ \partial, \mathcal{D} \} c - im^2 \xi \bar{c} c + \frac{g^2}{8} \xi (\bar{c} \times c)^2 \right\}, \quad (10.1)$$

which is left invariant under the following BRST and anti-BRST matrix transformations

$$\begin{aligned} sA_\mu &= -\mathcal{D}_\mu c & \bar{s}A_\mu &= -\mathcal{D}_\mu \bar{c} \\ sc &= -\frac{g}{2} c \times c & \bar{s}\bar{c} &= -\frac{g}{2} \bar{c} \times \bar{c} \\ s\bar{c} &= b - \frac{g}{2} \bar{c} \times c & \bar{s}c &= -b - \frac{g}{2} \bar{c} \times c \\ sb &= im^2 c - \frac{g}{2} c \times b & \bar{s}b &= im^2 \bar{c} - \frac{g}{2} \bar{c} \times b \\ & - \frac{g^2}{8} (c \times c) \times \bar{c} & & + \frac{g^2}{8} (\bar{c} \times \bar{c}) \times c. \end{aligned} \quad (10.2)$$

The two transformations relative to the Nakanishi-Lautrup field b are responsible of the presence of the quartic ghost interaction in (10.1). In the ghost/anti-ghost symmetric case, the Faddeev-Popov is $1/2\{\partial, \mathcal{D}\}$ rather than just $\partial\mathcal{D}$ as in standard linear gauges. As pointed out in [NO90], the presence of m^2 in the

BRST transformations and consequently in Eq. (10.1) spoils the nil-potency of both the BRST operators, such that their mutual anti-commutativity is given by [TM80, CF76, NO90]

$$\begin{aligned} \{s, s\} &= -m^2\delta = -m^2\delta^\dagger & \{\bar{s}, \bar{s}\} &= -m^2\bar{\delta} = -m^2\bar{\delta}^\dagger \\ \{s, \bar{s}\} &= -m^2\delta_{\text{FP}}, \end{aligned} \quad (10.3)$$

with $\delta, \bar{\delta}$ and the Faddeev-Popov ghost number operator δ_{FP} generating a $SL(2, \mathbb{R})$ algebra [DLS⁺02, Sch99]

$$\begin{aligned} [\delta, \delta_{\text{FP}}] &= -2\delta & [\bar{\delta}, \delta_{\text{FP}}] &= -2\bar{\delta} \\ [\delta, \bar{\delta}] &= \delta_{\text{FP}}. \end{aligned} \quad (10.4)$$

In [DJ82] and [NO90] it was argued that the 5 charges, obtained from the 5 operators we just showed, constituted a super-symmetric Lie algebra $OSp(4|2)$: though in [TM80], it was actually found that the b field broke down such a symmetry, and therefore its super-symmetric algebra. To introduce the Hubbard-Stratonovich transformations, we simplify the above Lagrangian employing the case of $SU(2)$ (the generalisation to $SU(N)$ only invokes the introduction of f^{abc} as structure constants), such that the quartic interaction becomes

$$\text{tr} \frac{g^2}{8} \xi (\bar{c} \times c)^2 = \frac{g^2}{8} \epsilon^{abc} \bar{c}^b c^c \epsilon^{dmn} \bar{c}^m c^n \text{tr} X^a X^d. \quad (10.5)$$

Adopting anti-Hermitian algebra generators and a normalisation $\text{tr} X^a X^d = -\frac{1}{2} \delta^{ad}$, the Lagrangian density of Eq. (10.1) becomes

$$\begin{aligned} \mathcal{L}_{\text{mCF}} &= -\frac{1}{2} \left\{ \frac{\xi}{2} (b^a)^2 + i b^a F^a [gA] + \frac{m^2}{2} (gA^a)^2 + \frac{i}{2} \bar{c}^a \{\partial, \mathcal{D}\}^{ab} c^b \right. \\ &\quad \left. - i m^2 \xi \bar{c}^a c^a + \frac{g^2}{8} \epsilon^{abc} \bar{c}^a c^b \epsilon^{cmn} \bar{c}^m c^n \right\}. \end{aligned} \quad (10.6)$$

In [Sch99, DLS⁺02] the quartic interaction was linearized in the light of Maximal Abelian gauge, though the BRST formulation of the corresponding Lagrangian wasn't explicitly revealed. In particular, it was only showed how to find the relative BRST transformation for the ϕ field in order to preserve the invariance of the Lagrangian under BRST and in that gauge, the coupling involves ± 2 scalar fermions rather than scalar bosons. This is because of the maximal abelian decomposition of the various couplings and consequently of the structure constants. We will further on see how the complete $SU(N)$ structure constants will play an important role as far as their convolution is concerned. Moreover,

it wasn't showed again in [DLS⁺02] how to generate the Lagrangian density over the two BRST operators, which a crucial thing to achieve in order to prove the topological nature of a BRST-based theory. Therefore, our objective here is to demonstrate that in the case of non-linear gauges, such as the massive Curci-Ferrari, it is possible to re-write Eq. (10.6) as a total BRST-anti-BRST variation, and we will then present the extensive BRST transformations. To begin with, let's use Hubbard-Stratonovich transformations to linearize Eq. (10.5)

$$e^{-\int \frac{g^2}{8} \epsilon^{abc} \bar{c}^a c^b \epsilon^{cmn} \bar{c}^m c^n} = \mathcal{C} \int \mathcal{D}\phi e^{-\int \frac{\xi}{2} \phi^a \phi^a - i g \frac{\xi}{2} \phi^a \epsilon^{abc} \bar{c}^b c^c}, \quad (10.7)$$

with $\mathcal{C} = \left(\det \frac{2\pi}{\xi}\right)^{1/2}$. The scalar field ϕ^a has vanishing ghost number and is required to be hermitian to maintain the total Hermiticity of the Lagrangian density. It is then left invariant under FP charge operator δ_{FP} . Both ghosts and anti-ghosts functions are chosen to be hermitian, such that $(c^a)^\dagger = c^a$ and $(\bar{c}^a)^\dagger = \bar{c}^a$, see *e.g.* [AFRvS03, Oji80, KO79] and references therein. The local Lagrangian density becomes then

$$\begin{aligned} \mathcal{L}_{\text{mCF}} = & -\frac{1}{2} \left\{ \frac{\xi}{2} b^a b^a + i b^a F^a [gA] + \frac{m^2}{2} (gA^a)^2 + \frac{i}{2} \bar{c}^a \{\partial, \mathcal{D}\}^{ab} c^b - im^2 \xi \bar{c}^a c^a \right. \\ & \left. + \frac{\xi}{2} \phi^a \phi^a - i g \frac{\xi}{2} \phi^a \epsilon^{abc} \bar{c}^a c^b \right\}, \end{aligned} \quad (10.8)$$

which implies that we need 3 additional ϕ fields in $SU(2)$ and $N^2 - 1$ in $SU(N)$. The partition function in Euclidean space-time, which will be a functional depending on a certain background gauge field A_μ reads

$$\begin{aligned} Z_{\text{mCF}}[A] &= \mathcal{C} \int \mathcal{D}g \mathcal{D}\bar{c} \mathcal{D}c \mathcal{D}b \mathcal{D}\phi e^{2\text{tr}S[gA, b, \bar{c}, c, \phi]} \\ &= \mathcal{C} \int \mathcal{D}\mu e^{2\int \mathcal{L}_{\text{mCF}}}. \end{aligned} \quad (10.9)$$

Before performing the integration in the ghost fields, which is defined over real ghost fields as $\mathcal{D}\bar{c}\mathcal{D}c \equiv \prod_x \prod_a i \bar{c}^a(x) c^a(x)$, we wish to separate the contribution of the ghost zero and non-zero modes with respect to the eigenvalue equation of the Faddeev-Popov operator. In non-linear gauges, the presence of a quartic ghost term allows us to absorb two zero modes without causing any harm as far the Grassmann integration is concerned. In addition to that, the diagonal quadratic ghost term, $-im^2 \xi \bar{c}^a c^a$ can absorb an additional zero mode: this is the reason why in the massive Curci-Ferrari gauge the corresponding Euler character does not vanish, as it would be the case in $SU(N)$ [GKW05], but it will depend

on m^2 . The eigenvalue equation is then

$$\begin{aligned} \{\partial, \mathcal{D}[A]\}^{ab} \lambda_{(n)}^b[A] &\equiv \mathcal{M}_{\text{FP}}^{ab} \lambda_{(n)}^b[A] = \varepsilon_{(n)}^{ab}[A] \lambda_{(n)}^b[A], \\ \{\partial, \mathcal{D}[A]\}^{ab} \lambda_{(0)}^b[A] &\equiv \mathcal{M}_{\text{FP}}^{ab} \lambda_{(0)}^b[A] = 0. \end{aligned} \quad (10.10)$$

In this way, we will not worry of singularities once we will deal with the inverse of the Faddeev-Popov operator. The resulting partition function is

$$\begin{aligned} Z_{\text{mCF}}[A] &= \mathcal{C} \int \mathcal{D}\bar{c}_{(0)} \mathcal{D}c_{(0)} e^{\int im^2 \xi \bar{c}_{(0)}^a c_{(0)}^a + ig \frac{\xi}{2} \phi^a \varepsilon^{abc} \bar{c}_{(0)}^a c_{(0)}^b} \\ &\int \mathcal{D}g \mathcal{D}\bar{c}_{(n)} \mathcal{D}c_{(n)} \mathcal{D}b \mathcal{D}\phi e^{2\mathcal{L}_{\text{mCF}}^{(n)}}. \end{aligned} \quad (10.11)$$

Integrating out both ghost zero and non-zero modes, we obtain two functional determinants which both contain the auxiliary field ϕ , the standard feature in effective Meson theory [AvS01]

$$\begin{aligned} Z_{\text{mCF}}[A] &= \mathcal{C} \int \mathcal{D}g \mathcal{D}b \mathcal{D}\phi e^{-\int \frac{\xi}{2} b^a b^a + ib^a F^a + \frac{m^2}{2} (gA^a)^2 + \frac{\xi}{2} \phi^a \phi^a} \\ &\times \det \left\{ \int \left(\xi m^2 \delta^{ab} + g \frac{\xi}{2} \phi^c \varepsilon^{abc} \right) \right\}_{(0)} \\ &\times \det \left\{ - \int \left(\mathcal{M}_{\text{FP}}^{ab} - \xi m^2 \delta^{ab} - g \frac{\xi}{2} \phi^c \varepsilon^{abc} \right) \right\}_{(n)}. \end{aligned} \quad (10.12)$$

Using the formula $\det A = e^{\text{tr} \log A}$, we can write the effective action as a non-polynomial function in ϕ as

$$\begin{aligned} S_{\text{eff}}[\phi] &= \int \left(\frac{\xi}{2} \phi^a \phi^a - \text{tr} \log \left\{ \int \left(\xi m^2 \delta^{ab} + g \frac{\xi}{2} \phi^c \varepsilon^{abc} \right) \right\}_{(0)} \right. \\ &\left. - \text{tr} \log \left\{ - \int \left(\mathcal{M}_{\text{FP}}^{ab} - \xi m^2 \delta^{ab} - g \frac{\xi}{2} \phi^c \varepsilon^{abc} \right) \right\}_{(n)} \right). \end{aligned} \quad (10.13)$$

Following [AvS01], variation of the effective action $S_{\text{eff}}[\phi]$ (10.13) yields the Dyson-Schwinger equations in terms of the classical fields

$$\phi_{(0)}^m(x) = \frac{1}{\xi} \text{tr} \left\{ G_{\phi_{(0)}}^{ab}(x, x) g \frac{\xi}{2} \varepsilon^{abc} \delta^{cm} \right\} = \phi_{(0),\text{cl}}^m(x), \quad (10.14)$$

and

$$\phi_{(n)}^m(x) = \frac{1}{\xi} \text{tr} \left\{ G_{\phi_{(n)}}^{ab}(x, x) g \frac{\xi}{2} \varepsilon^{abc} \delta^{cm} \right\} = \phi_{(n),\text{cl}}^m(x). \quad (10.15)$$

The solution of these two equations determine the vacuum expectation value (VEV) of the boson fields $\phi_{(0)}$ and $\phi_{(n)}$. $G_{\phi_{(0)}}(x, x)$ and $G_{\phi_{(n)}}(x, x)$ are the ghost

propagators in the background respectively of the fields $\phi_{(0)}$ and $\phi_{(n)}$. They are defined, in matrix notation, as

$$\begin{aligned} G_{\phi_{(0)}}^{-1}(x, y) &= \left(\xi m^2 + g \frac{\xi}{2} [\phi_{(0)}, \cdot] \right) \delta(x - y), \\ G_{\phi_{(n)}}^{-1}(x, y) &= \left(\mathcal{M}_{\text{FP}} - \xi m^2 - g \frac{\xi}{2} [\phi_{(n)}, \cdot] \right) \delta(x - y). \end{aligned} \quad (10.16)$$

The non-zero mode effective potential $V_{\text{eff}}(\phi_{\text{cl}})$ for the space-time independent classical field ϕ_{cl} is obviously

$$\begin{aligned} V_{\text{eff}}[\phi_{\text{cl}}] &= \frac{\xi}{2} \phi \phi - \log \left\{ - \int \left(\mathcal{M}_{\text{FP}} - \xi m^2 - g \frac{\xi}{2} [\phi, \cdot] \right) \right\} \\ &= \frac{\xi}{2} \phi \phi - \log \left(G^{-1}[\phi_{\text{cl}}] g \frac{\xi}{2} \right) \end{aligned} \quad (10.17)$$

In this semiclassical approximation, the boson field is being shifted by $\phi \rightarrow \phi_{\text{cl}} + \tilde{\phi}$, such that the classical field coincides with the VEV $\phi_{\text{cl}} \equiv \langle \phi \rangle$ and the quantum fluctuation $\tilde{\phi}$ has a vanishing VEV. Assuming from (10.15) a non-vanishing VEV for ϕ , this would imply a non-vanishing ghost condensate: from the equations of motions of ϕ , we generate a gap equation for the ghosts as

$$\phi_{\text{cl}} \equiv \langle \phi^a \rangle = \frac{i}{2} g \langle \epsilon^{abc} \bar{c}^b c^c \rangle = M^2. \quad (10.18)$$

which can be solved by Fourier transform as

$$\frac{g}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 \Gamma^2(p^2) + m^2 \xi + g \frac{\xi}{2} M^2} = M^2. \quad (10.19)$$

With an ansatz for the Γ function as $\Gamma^2(p^2) = p^{2\kappa}$, the gap equation assumes the form

$$\int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^{2(1+\kappa)} + m^2 \xi + g \frac{\xi}{2} M^2} = \frac{1}{16\pi^2} \int_0^\Lambda dp \frac{p^3}{p^{2(1+\kappa)} + \Delta}, \quad (10.20)$$

with Δ being a mass function $\Delta = m^2 \xi + g \frac{\xi}{2} M^2$ and Λ a momentum cut-off. The solution to Eq. (10.20) is expressed in terms of the Lerch's Phi function, defined as $\Phi(z, s, a) = \sum_{j=0}^{\infty} \frac{z^j}{(a+j)^s}$

$$\begin{aligned} \langle \phi \rangle &= \frac{g}{32\pi^2} \int_0^\Lambda dp \frac{p^3}{p^{2(1+\kappa)} + \Delta} \\ &= \frac{g\Lambda^4}{32\pi^2(1+\kappa)\Delta} \Phi \left(-\frac{\Lambda^{2(1+\kappa)}}{\Delta}, 1, \frac{2}{1+\kappa} \right) \\ &\stackrel{m^2 \rightarrow 0}{=} \frac{\Lambda^4}{16\pi^2 \xi M^2} \left(\frac{1}{2} - \frac{2}{2+\kappa} \frac{\Lambda^{2(1+\kappa)}}{g \xi M^2} + O(g^{-1})^2 \right) \\ &= M^2. \end{aligned} \quad (10.21)$$

BRST formalism in Hubbard-Stratonovich transformations

After performing the Hubbard-Stratonovich transformation to linearize the quartic ghost interaction, the various BRST transformations of Eq. (10.2) will change. To generate the Lagrangian density of Eq. (10.8) as a double extended BRST variation we consider the following matrix transformations, (we employ from now on $SU(2)$ as a Lie group, whose generalisation to $SU(N)$ is obtained by substituting the structure constants ϵ^{abc} with f^{abc} in the exterior product \times)

$$\begin{aligned}
sA_\mu &= -\mathcal{D}_\mu c & \bar{s}A_\mu &= -\mathcal{D}_\mu \bar{c} \\
sc &= -\frac{g}{2}c \times c & \bar{s}\bar{c} &= -\frac{g}{2}\bar{c} \times \bar{c} \\
s\bar{c} &= b - \frac{g}{2}\bar{c} \times c & \bar{s}c &= -b - \frac{g}{2}\bar{c} \times c \\
sb &= im^2c - \frac{g}{2}c \times b & \bar{s}b &= im^2\bar{c} - \frac{g}{2}\bar{c} \times b \\
s\phi &= 2\Psi & \bar{s}\phi &= 2\bar{\Psi} \\
s\Psi &= -\frac{g}{2}\Psi \times \Psi & \bar{s}\bar{\Psi} &= -\frac{g}{2}\bar{\Psi} \times \bar{\Psi} \\
s\bar{\Psi} &= \phi - ig\bar{c} \times c & \bar{s}\Psi &= \phi - ig\bar{c} \times c
\end{aligned} \tag{10.22}$$

such that

$$\begin{aligned}
\mathcal{L}_{\text{mCF}} &= \frac{i}{2}s\bar{s} \left((gA^a)^2 - i\xi\bar{c}^a c^a - i\frac{\xi}{2}\phi^a\phi^a \right) \\
&\quad + \frac{m^2}{2} \left((gA^a)^2 - i\xi\bar{c}^a c^a \right) \\
&= \frac{\xi}{2}b^a b^a + ib^a F^a[gA] + \frac{m^2}{2}(gA^a)^2 + \frac{i}{2}\bar{c}^a \{\partial, \mathcal{D}\}^{ab} c^b \\
&\quad - im^2\xi\bar{c}^a c^a + \frac{\xi}{2}\phi^a\phi^a - ig\frac{\xi}{2}\phi^a \epsilon^{abc}\bar{c}^a c^b - \xi\bar{\Psi}^a \Psi^a.
\end{aligned} \tag{10.23}$$

It is worth noting that the transformations involving the Nakanishi-Lautrup field b^a change with respect to Eq. (10.2), in the way that the triple ghost term is no longer present in neither of sb^a nor $\bar{s}b^a$, respectively $-\frac{g^2}{8}(c \times c) \times \bar{c}$ and $+\frac{g^2}{8}(\bar{c} \times \bar{c}) \times c$. The auxiliary field ϕ has vanishing ghost number, $\text{Gh}(\phi) = 0$, whereas the two additional fermionic fields, introduced in (10.22) to generate the coupling of ϕ with the quadratic-ghost term in (10.1), Ψ and $\bar{\Psi}$ have respectively $\text{Gh}(\Psi) = 1$ and $\text{Gh}(\bar{\Psi}) = -1$. The field ϕ thus plays the role of an additional b -field in standard BRST, and Ψ and $\bar{\Psi}$ the role of c and \bar{c} . The appearance of the term $\xi\Psi^a\bar{\Psi}^a$ in (10.23) produces only a multiplicative overall factor, because

of Grassmann integration

$$\int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} e^{\int \xi \bar{\Psi}^a \Psi^a} = \det(\xi), \quad (10.24)$$

which will be absorbed into the overall constant $\mathcal{C} = (\det 2\pi\xi)^{-1/2}$. An other interesting aspect of these non-linear gauge BRST transformations is that the Lagrangian density (10.23) so generated is, at $m^2 = 0$ a true topological Lagrangian. In fact

$$Z_{\text{mCF}}[A] = \mathcal{C} \int \mathcal{D}g \mathcal{D}b \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}\phi e^{\frac{i}{2} s \bar{s} \int ((gA^a)^2 - i\xi \bar{c}^a c^a - i\frac{\xi}{2} \phi^a \phi^a)} \quad (10.25)$$

conserves its topological nature, which can be seen by rescaling the fields as

$$\begin{aligned} b &\rightarrow \frac{b}{\sqrt{\xi}} & \phi &\rightarrow \frac{\phi}{\sqrt{\xi}} & c &\rightarrow \frac{c}{4\sqrt{\xi}} & \bar{c} &\rightarrow \frac{\bar{c}}{4\sqrt{\xi}} \\ \Psi &\rightarrow \frac{\Psi}{\sqrt{\xi}} & \bar{\Psi} &\rightarrow \frac{\bar{\Psi}}{\sqrt{\xi}} \end{aligned} \quad (10.26)$$

and noticing that $Z_{\text{mCF}}[A]$ will remain unchanged. Furthermore, demanding the nil-potency of the BRST transformations we notice that

$$s^2 \bar{\Psi} = 2\Psi - igb \times c \neq 0 \quad \bar{s}^2 \Psi = 2\bar{\Psi} - ig\bar{c} \times b \neq 0. \quad (10.27)$$

Yet, the nil-potency is restored on-shell once we use the equations of motions of the ϕ -field. In fact, on-shell, $s\phi = ig\frac{\xi}{2}s(\bar{c} \times c) = ig\frac{\xi}{2}b \times c$, and therefore

$$s^2 \bar{\Psi} =_{\text{on-shell}} 2ig\frac{\xi}{2}b \times c - igb \times c = 0. \quad (10.28)$$

This is the reason why in the BRST transformations (10.22) we have a factor 2 upfront both $s\phi$ and $\bar{s}\phi$. Also the invariance of the Lagrangian density (10.23) under the transformations of Eq. (??) is preserved on-shell

$$\begin{aligned} s\mathcal{L}_{\text{mCF}} &= s \left(\frac{\xi}{2} \phi^a \phi^a - ig\frac{\xi}{2} \phi^a \epsilon^{abc} \bar{c}^a c^b - \xi \bar{\Psi}^a \Psi^a \right) \\ &= \xi \phi^a s\phi^a - ig\frac{\xi}{2} (s\phi^a) \epsilon^{abc} \bar{c}^a c^b - ig\frac{\xi}{2} \phi^a \epsilon^{abc} b^a c^b \\ &\quad - \xi (s\bar{\Psi}^a) \Psi^a + \xi \bar{\Psi}^a s(\Psi^a). \end{aligned} \quad (10.29)$$

Using the equations of motion for ϕ^a , $\phi^a = i\frac{\xi}{2}\epsilon^{abc}\bar{c}^a c^b$, and $\Psi^a = i\frac{\xi}{4}\epsilon^{abc}b^b c^c$, we restore the invariance. In this Appendix we will show how the BRST transformations (10.22) will change if we expand the convolution of the structure

constants, $\epsilon^{abc}\epsilon^{cmn}$ and $f^{abc}f^{cmn}$. Let's start with $SU(2)$: the convolution of the structure constants is very simple and gives

$$\epsilon^{abc}\epsilon^{cmn} = \delta^{am}\delta^{bn} - \delta^{an}\delta^{bm}, \quad (10.30)$$

which, inserted in Eq. (10.6) gives

$$\begin{aligned} \mathcal{L}_{\text{mCF}} = & -\frac{1}{2} \left\{ \frac{\xi}{2} (b^a)^2 + ib^a F^a[gA] + \frac{i}{2} \bar{c}^a \{\partial, \mathcal{D}\}^{ab} c^b \right. \\ & \left. + \frac{m^2}{2} (gA^a)^2 - im^2 \xi \bar{c}^a c^a + \frac{g^2}{8} (\bar{c}^a c^a)^2 \right\}. \end{aligned} \quad (10.31)$$

After performing the linearisation of the quartic term, we obtain

$$\begin{aligned} \mathcal{L}_{\text{mCF}} = & -\frac{1}{2} \left\{ \frac{\xi}{2} (b^a)^2 + ib^a F^a[gA] + \frac{i}{2} \bar{c}^a \{\partial, \mathcal{D}\}^{ab} c^b \right. \\ & \left. + \frac{m^2}{2} (gA^a)^2 - im^2 \xi \bar{c}^a c^a + \frac{\xi}{2} \phi \phi - ig \frac{\xi}{2} \phi (\bar{c}^a c^a) \right\}. \end{aligned} \quad (10.32)$$

We notice that now, the ϕ field carries no gauge index, due to the scalar nature of $(\bar{c}^a c^a)^2$, which implies that ϕ lives in the identity of $SU(2)$. It is worth noting that in this case, there is only one single ϕ field, whereas, in the case in which we do not employ the convolution of the structure constants there were as many fields as the generators of the algebra (ϕ^a , $a = 1 \dots N^2 - 1$) Consequently, the BRST transformations (10.22) will change accordingly as

$$\begin{aligned} sA_\mu &= -\mathcal{D}_\mu c & \bar{s}A_\mu &= -\mathcal{D}_\mu \bar{c} \\ sc &= -\frac{g}{2} c \times c & \bar{s}\bar{c} &= -\frac{g}{2} \bar{c} \times \bar{c} \\ s\bar{c} &= b - \frac{g}{2} \bar{c} \times c & \bar{s}c &= -b - \frac{g}{2} \bar{c} \times c \\ sb &= im^2 c - \frac{g}{2} c \times b & \bar{s}b &= im^2 \bar{c} - \frac{g}{2} \bar{c} \times b \\ s\phi &= 2\Psi & \bar{s}\phi &= 2\bar{\Psi} \\ s\Psi &= 0 & \bar{s}\bar{\Psi} &= 0 \\ s\bar{\Psi} &= \phi - ig\bar{c} \cdot c & \bar{s}\Psi &= \phi - ig\bar{c} \cdot c, \end{aligned} \quad (10.33)$$

where the two fermionic fields Ψ and $\bar{\Psi}$ transform trivially under s and \bar{s} because they live both in the identity of the group too, and so their exterior product vanishes. Thus, we see that the structure-constant convolution gives a $U(1)$ BRST theory in the additional fields. In $SU(N)$ the situation looks quite more complicated, because the convolution of f^{abc} gives

$$f^{abc}f^{cmn} = \frac{2}{N_c} (\delta^{am}\delta^{bn} - \delta^{an}\delta^{bm}) + d^{amc}d^{bnc} - d^{anc}d^{bmc}, \quad (10.34)$$

where d^{abc} comes from the commutation relations of the $su(N)$ generators

$$[X_a, X_b] = f_{ab}^c X_c \quad \{X_a, X_b\} = -\frac{1}{N_c} \delta_{ab} - i d_{ab}^c X_c. \quad (10.35)$$

The quartic ghost term will then be

$$\begin{aligned} g^2 \frac{\xi}{8} f^{abc} f^{cmn} \bar{c}^a c^b \bar{c}^m c^n &= g^2 \frac{\xi}{4N_c} (\bar{c}^a c^c)^2 \\ &+ g^2 \frac{\xi}{8} d^{anc} \bar{c}^a c^n d^{bmc} \bar{c}^b c^m \\ &- g^2 \frac{\xi}{8} d^{amc} \bar{c}^a \bar{c}^m d^{bnc} c^b c^n, \end{aligned} \quad (10.36)$$

which should determine three different couplings. Yet, the third vanishes because of the anti-commutativity of the ghost fields and the symmetry of the d symbols, *e.g.* $d^{abc} c^b c^c = -d^{acb} c^c c^b = d^{abc} c^c c^b = 0$, and the same thing for \bar{c} . Therefore, the $SU(N)$ Lagrangian density appears not so different from the $SU(2)$ case, specifically

$$\begin{aligned} \mathcal{L}_{\text{mCF}} &= -\frac{1}{2} \left\{ \frac{\xi}{2} (b^a)^2 + i b^a F^a [gA] + \frac{m^2}{2} (gA^a)^2 + \frac{i}{2} \bar{c}^a \{\partial, D\}^{ab} c^b \right. \\ &\quad - i m^2 \xi \bar{c}^a c^a + \frac{\xi}{2} \phi^2 - i g \frac{\xi}{\sqrt{2N_c}} \phi \bar{c}^a c^a \\ &\quad \left. + \frac{\xi}{2} \varphi^a \varphi^a - i g \frac{\xi}{2} \varphi^a d^{abc} \bar{c}^b c^c \right\} \end{aligned} \quad (10.37)$$

The corresponding partition function, expressed in terms of a double BRST variation is then a functional integral over the new additional fields as

$$\begin{aligned} \mathcal{Z}_{\text{mCF}}[A] &= \mathcal{C} \int \mathcal{D}g \mathcal{D}b \mathcal{D}\bar{c} \mathcal{D}c \mathcal{D}\phi \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \mathcal{D}\varphi \mathcal{D}\chi \mathcal{D}\bar{\chi} e^{-S\{\Phi, A\}} \\ &= \mathcal{C} \int \mathcal{D}\mu e^{i s \text{str} [f (gA)^2 - i \xi \bar{c} c - i \frac{\xi}{2} \varphi \varphi - i \frac{\xi}{2} \phi \phi]} \\ &\quad e^{-\frac{m^2}{2} \text{tr} [f (gA)^2 - i \xi \bar{c} c]} \end{aligned} \quad (10.38)$$

The corresponding $SU(N)$ BRST transformations are the following

$$\begin{aligned}
sA_\mu^a &= -\mathcal{D}_\mu^{ab}c^b & \bar{s}A_\mu^a &= -\mathcal{D}_\mu^{ab}\bar{c}^b \\
sc^a &= -\frac{g}{2}f^{abc}c^bc^c & \bar{s}\bar{c}^a &= -\frac{g}{2}f^{abc}\bar{c}^b\bar{c}^c \\
s\bar{c}^a &= b^a - \frac{g}{2}f^{abc}\bar{c}^bc^c & \bar{s}c^a &= -b^a - \frac{g}{2}f^{abc}\bar{c}^bc^c \\
sb^a &= im^2c^a - \frac{g}{2}f^{abc}c^bb^c & \bar{s}b^a &= im^2\bar{c}^a - \frac{g}{2}f^{abc}\bar{c}^bb^c \\
s\phi &= 2\Psi & \bar{s}\phi &= 2\bar{\Psi} \\
s\Psi &= 0 & \bar{s}\bar{\Psi} &= 0 \\
s\bar{\Psi} &= \varphi^a - ig\frac{1}{\sqrt{2N_c}}\bar{c}^ac^a & \bar{s}\Psi &= \varphi^a - ig\frac{1}{\sqrt{2N_c}}\bar{c}^ac^a \\
s\phi^a &= 2\chi^a & \bar{s}\phi^a &= 2\bar{\chi}^a \\
s\chi^a &= -\frac{g}{2}f^{abc}\chi^b\chi^c & \bar{s}\bar{\chi}^a &= -\frac{g}{2}f^{abc}\bar{\chi}^b\bar{\chi}^c \\
s\bar{\chi}^a &= \phi^a - igd^{abc}\bar{c}^bc^c & \bar{s}\chi^a &= \phi^a - igd^{abc}\bar{c}^bc^c.
\end{aligned} \tag{10.39}$$

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