Business Cycle Dynamics

Sebastian Wende[*]
School of Economics
University of Adelaide
Ground Floor, Napier Building
Adelaide, SA 5005
Australia
(email: sebastian.wende@student.adelaide.edu.au)

Abstract: This paper attempts to simulate endogenous cyclical behaviour through variations on the standard real business models. This paper relaxes the perfect foresight assumption implied by the rational agent hypothesis. It is replaced by imperfect adaptive expectations. The model is extended with a delay between investment and capital accumulation. This paper also simulates a non-equilibrium time-differential wage adjustment in a model economy. The models show that the boom produced by a single positive technology shock can be followed by the equivalent of a recession. The models are solved using numerical methods for differential equations, which allow for non-linear dynamics, as opposed to the usual log linearisation.

I. INTRODUCTION

Historically periods of economic growth have eventually been followed by recessions. Periods of economic growth may be caused by technological progress and recessions may be explained by diminishing resource availability. But there may also exist inherent dynamics in an economy that cause periods of growth and recession without any reduction in resource availability. In light of this, I investigate the possibility of endogenously generating this business cycle behaviour in a model economy.

In the standard real business cycle (RBC) model changes in output are caused by exogenous changes in technology.[1] Although the results of advanced RBC models are consistent with the stylised facts observed in time series data for actual economies these still rely on exogenous shocks to produce cyclical behaviour.[2] If ones assumes cyclical behaviour is due to inherent

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1 For examples of early and standard RBC models see King, Plosser and Rebelo (1988), Lucas (1975), Hansen and Sargent (1980) and Lucas (1977)

2 For examples of a advanced RBC models see Weder (2003), Wen (1998) and Farmer and Guo (1994).

dynamics present in an economy then the models should be able to simulate cyclical behaviour endogenously. They should not rely on exogenous shocks.

RBC theory uses microeconomic general equilibrium theory to model whole economies. It models economies that are comprised of representative firms and agents. In these model economies agents optimise their utility over an infinite time-frame and firms maximise profits. These models are solved in a general equilibrium frame-work where consumers are considered to be fully rational. The solution of a typical model results in a set of non-linear dynamic equations. But in the standard solution, the models are generally solved using a Taylor approximation around a steady state. This results in a set of linear dynamic equations. I suggest that an economy cannot be modeled by linear dynamics even though recent advancements have meant the models are better able to describe certain economic phenomena. The linear approximation also restricts the ideas that can be included in the models.

In this paper I outline the standard model and a solution method that maintains the full dynamics of the economy. A model of adaptive expectations is then constructed and this is incorporated into a model economy. I incorporate these adaptive expectations in a model economy with a delay between investment and capital accumulation. Lastly an economy with differential wage adjustment is modeled. These model economies demonstrate the possibility of endogenous cyclical behaviour.

This paper tests whether these inherent cyclical dynamics may be caused by frictions or delays in adjustment or imperfect foresight by agents. I simulate a model where agents form their expectations based on past information instead of using their knowledge of the economy. This model is extended with a delay between investment and the accumulation of productive capital. I also test the effect of frictions in prices with an economy where wages adjust relative to excess demand.

There have been attempts to include near rational expectations and frictions in RBC models. Weder (2004) includes near rationality in a RBC model. The agents form their expectations which are equal to rational expectations plus some error. The error in expectations is due to some cost of forming expectations and is constant or stochastic. Chari, Kehoe and McGrattan (2007) attempt to introduce frictions into their model. But frictions may cause the economy to deviate from equilibrium and therefore cannot be solved in the standard RBC models. Chari et al. (2007) include frictions by introducing wedges which appear equivalent to taxes. Taxes would alter the steady state because they alter the return on capital received by agents. In theory frictions should not alter the steady state but would change the rate at which the economy moves back to steady state.

II. A NUMERICAL SOLUTION METHOD

In this section a basic Real Business Cycle model is outlined. In this basic RBC model a representative firm maximises profit in each time period and a consumer maximises utility over an infinite time period. There is perfect competition in this economy so the firm and the agent are both price takers and the market clears. The model is solved using numerical solution methods for differential equations. The solution method explained in this section is also used in the rest of the paper.

The representative agent is infinitely lived and maximises lifetime utility from consumption and leisure. The maximisation problem facing agents can be written as

$$\max_{C(t)\ L(t)} \int_0^\infty U(C(t)\ L(t)\ e^{-\rho t} \mathrm{d}t \qquad (1)$$

where $U(C(t), L(t)$ is the utility at time t which depends on the consumption $C(t)$ and labour $L(t)$. The utility is written in terms of labour because when agents work more then they have less time for leisure so the utility of leisure can also be measured as the disutility from the labour they supply. Agents value future utility less than current utility so the utility in period t is multiplied by the discount factor $e^{-\rho t}$. Here ρ is the discount rate and is the amount by which agents discount utility from one period to the next. For example if $\rho = 1$ then agents value utility in the next period approximately 10 per cent less than in the proceeding period.
The consumer is able to invest in capital $K(t)$ which is rented to the firm. Consumption, $C(t)$, and investment, $I(t)$, in each period are constrained to be less than or equal to the sum of income from labour supplied, income from capital rented to the firm and any other profit received from the firm $\pi(t)$. Hence

$$r(t)K(t) + w(t)L(t) + \Pi \geq C(t) + I(t) \qquad (2)$$

where $r(t)$ is the return on capital and $w(t)$ is the wage paid for labour.

The investment rate $I(t)$ determines the agents capital accumulation. Capital accumulation is equal to investment minus capital depreciation

$$\dot{K}(t) = I(t) - \delta K(t). \qquad (3)$$

where dot in $\dot{K}(t)$ indicates the first derivative with respect to time.

As mentioned earlier the firm maximises profit in each period. The firm sells an output which it produces subject to a production function $F(A(t), K(t), L(t)$ which uses both labour and capital and depends on the state of technology $A(t)$. The price of the output is normalised to one. The firm's costs are those of capital and labour used in production. So the firms profit maximisation problem is

$$\max_{K(t)L(t)} \Pi = \max_{K(t)L(t)} F(A(t), K(t), L(t)) - w(t)L(t) - r(t)K(t). \qquad (4)$$

From the first order conditions of the maximisation profit is maximised when

$$F_K(A(t), K(t), L(t) \qquad (5)$$

and

$$F_L(A(t), K(t), L(t). \qquad (6)$$

This implies that profit is maximised when the wage paid for labour is equal to the marginal product of labour and when the return on capital is equal to the marginal product of capital.

To be able to solve the consumer's optimal rate of consumption and capital accumulation, agent's expectations on future returns on capital and wages for labour must be taken into account. The agent is fully rational and therefore forms its expectations from the true structural model of the economy (Pesaran 1987). Rational agents know the production function and know that the return on capital at any point in time is given by the marginal product of capital. Likewise rational agents know the wage is equal to the marginal product of labour. Hence expectations of future wages and returns on capital depend on expectations of the future level of technology A(t). The problem faced by the rational agent is equivalent to the social planner problem.

I solve the agent's maximisation problem in continuous time using optimal control theory (Léonard 2002) as in the literature . A Hamiltonian *H(C(t), L(t), K(t), δ(t), t)* is defined as

$$H(C(t), L(t), K(t), \varphi(t), t) = U(C(t), L(t)) + \varphi(t)\,(F(K(t), L(t)) - \delta K(t) - C(t)). \tag{7}$$

The necessary conditions for optimisation are

$$U_C(C(t), L(t)) = \varphi(t), \tag{8}$$

$$U_L(C(t), L(t)) = -\varphi\, F_L(K(t), L(t)) \tag{9}$$

and

$$\dot{\varphi}(t) = \varphi(t)\,(\rho + \delta - F_K(K(t), L(t))). \tag{10}$$

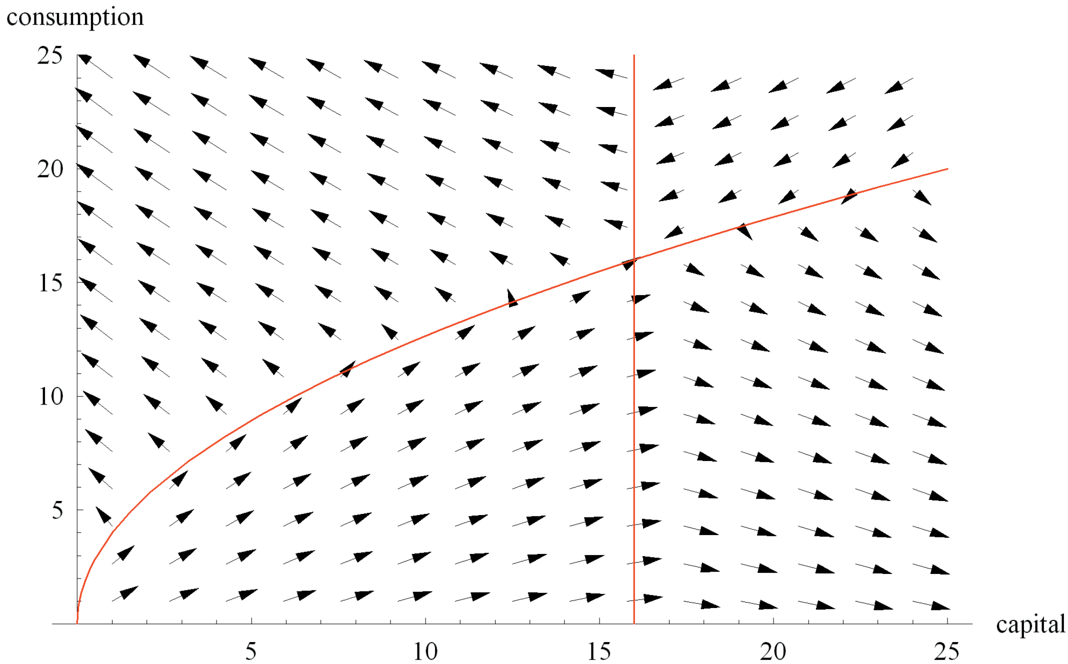
In the above equations $\varphi(t)$ is the shadow price of capital. The necessary conditions state that the optimal consumption occurs when the marginal benefit of consumption is equal to the shadow price of capital. The marginal utility of labour (negative) is equal to the negative of the shadow price of capital multiplied by the marginal product of labour.

The steady state of this economy is defined as the state where consumption, labour and capital remain constant. Equation implies that for consumption to be constant the shadow price of capital must also be constant. This means that the return on capital minus the depreciation rate must be equal to the discount factor, $\rho = F_K(K(t), L(t)) - \delta$, in the steady state. This is used to solve for the steady state values of consumption $\tilde{C}$, output $\tilde{Y}$, labour $\tilde{L}$ and capital $\tilde{K}$.

The dynamics of this economy are given by the agents optimality conditions in Equations (8), (9) and (10), and and capital accumulation in Equation . Capital accumulation depends on how much of the production remains after consumption. Both consumption and labour supply depend on the shadow price of capital. The shadow price of capital depends on the return on capital which depends on the level of capital. The inter-relationship in the dynamics between optimal consumption and capital are shown in the phase diagram in Figure 1. One can see there are many paths which satisfy the optimality conditions, but only those that lead to the steady state are optimal for the agent as explained below.

The optimality of steady state can be seen in the phase diagram in Figure 1. The arrows indicate the dynamics of the economy given by the optimality conditions. In the figure the steady state is where the two red lines meet; at this point the change in consumption and capital are both zero. At time t, I define C_t^* as the initial consumption that leads to the steady state. From the phase diagram one can see that if C_t is greater than C_t^* capital will eventually be equal to zero, conversely if C_t is below C_t^* consumption will eventually be equal to zero. Both zero consumption and zero capital are obviously bad outcomes for the agent and therefore the model predicts that the agent will always choose C_t^* and a consumption path that leads to the steady state. Obviously if the agent is at the steady state they will choose to remain there.

Figure 1: Phase diagram from Equations , and . The arrows indicate the dynamics of the economy and have been scaled.



Because the agent always remains at or chooses a path that returns to the steady state the model can be solved by Taylor approximating around the steady state. Taylor approximation is used in the standard solution, but this results in a set of linear dynamic equations.[3] In standard real business cycle models, as the equations are evaluated around the steady state, C_t^* does not have to be found explicitly as it is found by solving forward over future expected shocks.

In this paper the model is solved numerically but without linearisation around the steady state. Equations , and and determine the behaviour of the model and are solved using numerical methods for differential equations.[4] Numerical methods require starting values to calculate the paths of the variables. A starting value for capital is given but a starting value for consumption is not given. As explained above the agent will choose a starting value of consumption C_t^* such that they remain or move towards the steady state.

In the solution method, C_t^* is found by trial and error. When solving the model over a period starting at time equals t, a large value and small value of C_t are chosen, C_t^{upper} and C_t^{lower}. The average of C_t^{upper} and C_t^{lower} is the initial value of consumption used in a numerical evaluation of the Equations (8), (9), and (10) and (3). If at some chosen future time capital is moving toward zero, in the numerical evaluation, then C_0^{upper} is set equal to the average. If at the chosen future time capital is moving towards infinity then C_0^{lower} is set equal to the average. This gives a

3 This solution method is shown in discrete time in King, Plosser and Rebelo (2002) and in continuous time in Turnovsky (1995) and Wende (2008).

4 An example of such a numerical method is the Runge-Kutta method.

new set of values for C_0^{upper} and C_0^{lower}, using the new value the procedure is repeated. This is repeated until C_0^* is approximated to an acceptable accuracy.[5] This procedure is performed by the module **findy0** shown in Appendix A.

The model is solved in time steps of size m. Starting at time equals zero the model is solved over the period time equals zero to time equals m. Once the first period has been solved then the model can be solved over the period time equals m to time equals $2m$ and so on. At the start of each period the level of capital is given and C_0^* is as explained above. Equations (8), (9), (10) and (3), and and are solved using numerical methods for differential equations over that time period.

At the start of each period agents incorporate any new information into their decision, such as changes in the level of technology *A(t)*. Even though the model is evaluated over discrete periods the consumption and labour paths should be continuous if there are no exogenous changes in the economy. Capital must be continuous except under certain circumstances which are not relevant to this paper. The solution method satisfies these conditions as shown in the next section where the results of a model economy are given.

1. Results from a Model Economy

The method for solving a model economy given in the previous section is applied to a hypothetical economy. The model is solved over one time period step sizes. The economy modeled here is defined by the production function and utility functions which represent the agent and the firm. The production and utility functions are given by

$$F(K(t)) = A(t)K(t)^{.5}L(t)^{.5} \tag{11}$$

and

$$U(C(t)) = \ln(C(t)) + \ln(1 - L(t)). \tag{12}$$

The agent's discount rate is chosen as $\rho = .125$, and there is no depreciation, $\delta = 0$. The economy starts at the steady state where the level of capital is equal to $5\frac{1}{3}$, consumption and output are both equal to $1\frac{1}{3}$ and labour is equal to $\frac{1}{3}$.

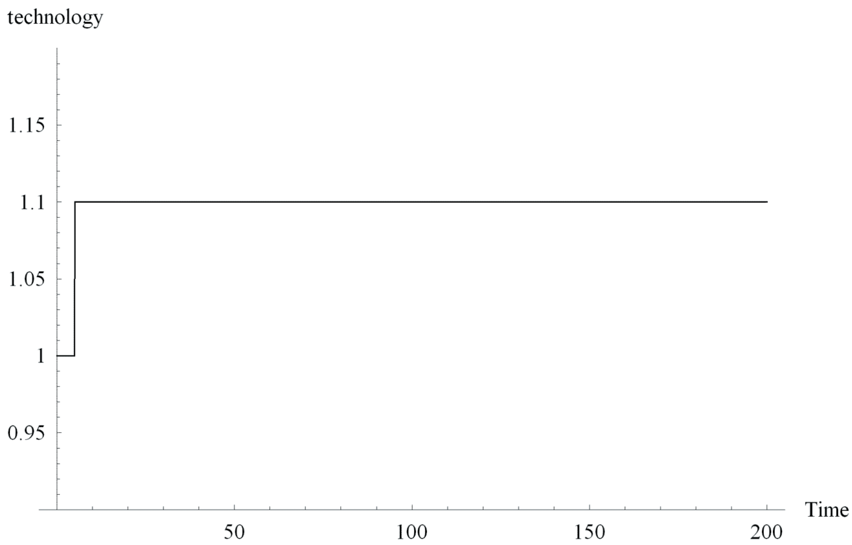
The economy's response to a one-off positive technology shock is tested. Initially the technology variable *A* is set equal to one, agents do not expect any future changes in this value. At time equal to 5, there is a permanent technology shock and technology jumps to A=1.1 as shown in *Figure 2*. This is a permanent shock and the agent does not expect another shock after the initial shock. The agent is fully rational.

The results of modeling the economy described above are shown in the graphs in *Figure 3*. Modeling this one-off technology shock is equivalent to modeling an impulse response.

One can see that consumption and capital adjust slowly after the technology shock and tend towards a new steady state as time progresses. There is an initial jump in output due to the increase in the technology variable. This is compounded by the increase in labour caused by the increase in the marginal productivity of labour. Initially after the technology

5 I choose to find this value to more the five decimal place accuracy.

Figure 2: A(t) the technology variable



shock the agent leaves a proportion of output unconsumed so that it is used for investment. This causes capital to increase until the agent nears the new steady state. As capital increases consumption increases such that investment decreases. As capital approaches the new steady state investment goes to zero so that the level of capital does not go above the new steady state. As the economy nears the steady state, the return on capital decreases. When the return on capital equals the discount rate then the shadow price of capital becomes constant and consumption becomes constant.

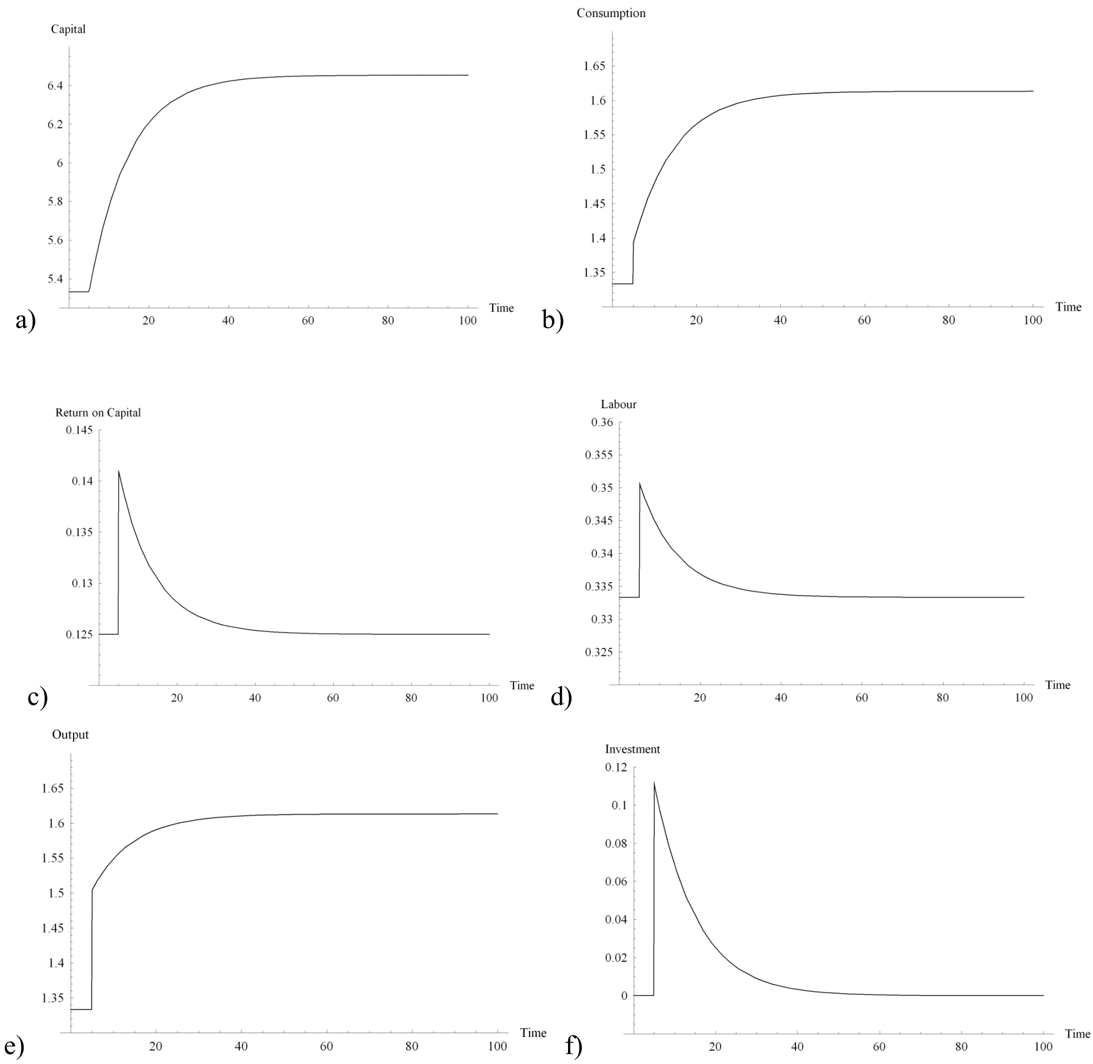
As consumption increases the shadow price of capital decreases which causes labour supply to fall. Although the marginal productivity of labour continues to increase, the effect of the decrease in the shadow price of capital is larger and labour decreases to the original steady state value with time. In this model economy the steady state level of labour is constant for all values of technology but this is a model specific property.

The relationship between the paths is described as causal where a change in one factor changes the others given by Equations (8), (9), (10) and (3). But it is important to remember that these paths of consumption and capital maximise the agents utility so changes over time are utility maximising changes and are the optimum response to the technology change.

The pace at which the agent moves towards the steady state depends on the agents rate of intertemporal substitution which is given by the agents utility function. Higher intertemporal substitution means the agent is willing to work more and sacrifice more consumption initially to reach the new steady state more rapidly.

In summary this section shows that the solution method outlined above can be used to solve standard real business cycle models. An economy's impulse response to a permanent positive technology shocks is simulated. An extension would be to compare these results with those from the standard log linearisation solution method. The results from the basic model allow for comparison with later models.

Figure 3: a) K(t) Capital, b) C(t) Consumption c) r(t) Return on Capital, d) L(t) Labour, e) F(K,L) Output, f) I(t) Investment



III. ADAPTIVE EXPECTATIONS

Real business cycle theory relies on the assumption of fully rational agents, but it is of interest to examine the behaviour of a model economy if this assumption is relaxed. Rational agents know the structure of the economy and the probability of different circumstances. In an economy without uncertainty this equates to perfect foresight. Adaptive agents use their perfect foresight when planning investment and consumption. Agents without an understanding of the structure of the economy will use their expectations of the returns on assets in determining their investment decision. These expectations will be formed adaptively using past and current information and can be referred to as imperfect foresight. An economy with adaptive agents who do not perfectly predict future returns will behave differently to an economy with rational agents. The magnitude of these differences will depend on the accuracy of agents' expectations and also on how often agents update their information and choices.

1. Beyond Rational Expectations

Muth first postulated the rational agent hypothesis, according to which rational agents form their expectations from the true structural model of the economy (Pesaran 1987). Specifically, rational agents know the production function and the dynamics of the economy. Rational agents correctly predict the likelihood of future technology shocks and understand the implications of their actions. Rational agents are able to calculate future changes in output given changes in capital and labour. Rational agents understand the relationship between output, inputs into production and the prices paid for these inputs. Because rational agents understand the structure of the economy, the marginal productivity of capital can substituted for the rate of return on capital when solving a rational agent's optimisation problem.

Agents may not be rational because they do not possess the cognitive ability required for them to be rational. The economy must be perfectly understood for agents to be able to calculate the exact implications of their decisions. Agents must know the production function to be able to calculate future price changes for inputs into production given changes in the quantity supplied of the inputs. This puts certain requirements on agents' cognitive abilities. Psychology suggests humans are not perfectly rational and suffer cognitive biases.

Psychology gives many examples of irrationality but also emphasises the importance of the way information is presented. Examples of cognitive biases are the availability and representative heuristic where people reach incorrect conclusions from the information presented. Gigerenzer (2004) explains that the representative heuristic can be largely eliminated if the information is presented in a way that is evolutionarily natural or representative. Evolutionarily natural refers to information being presented in a similar way to the way it was available during our evolution. Gigerenzer (2004) also suggests humans are able to observe patterns from representative data but are less able to when data is presented in an abstract manner. While the information which determines future output and rates of return on capital is not clearly presented it is easy to observe the current and past prices and rates of return.

It may be considered useful to use some of the results on human learning from game theory but most games have a result every period in which players win or lose and can update their strategy (Camerer 2003). In RBC models agents maximise over an infinite or extended period so they cannot determine the results of their optimisation strategy in any period. Despite this agents could learn whether outcomes match their expectations or not.

Agents with cognitive limitations are able to observe and interpret past output and rates of return and are able to learn from this information and form their expectations based upon the past information. Expectations of the return on capital are key in deciding agents' consumption and investment paths. If agents are not able to form expectations based on perfect foresight they may adopt simple rules to form their expectations.

Another reason why agents may adopt a simple rule for future rates of return is because they do not think their actions influence future rates of return. In an economy with many agents each has a negligible effect on total capital and therefore on the return on capital. But if conditions cause one agent to increases investment then it is likely that all other agents will also increase investment. So an action by one agent is likely to be mirrored by all agents and small individual changes in investment will have large effect on the total level of capital and therefore affect the rate of return. With multiple agents it becomes even more complicated to calculate changes in the return on capital. Hence agents may adopt a simple rule for forming expectations. But this situation should not be considered as a type of game theory equilibrium as agents are oblivious to each others strategies.

2. Adaptive Expectations of Returns

To be able to solve the agents maximisation problem the agents' expectations of future rates of return on capital and wages must be taken into account. It is assumed agents are not capable of forming rational expectations for reasons given in Section 3.1. I assume agents with imperfect foresight use a simple rule to form expectations. This rule incorporates current information and long-run trends in the return on capital. These non-rational expectations that adapt to current and past information are referred to as adaptive expectations.

Agents may have observed that on average the return on capital is equal to the steady state rate (the discount rate) and therefore expect this in the long run. It is also logical that agents expect that the rate of return in the near future is similar to the current rate. So a viable model of expectations of the rate of return on capital is one where the expected rate of return in the near future is equal to the current rate. In the long run it is equal to the steady state return. These expectations would be given by the expected return decaying from the current value to the steady state value. A decay in returns is similar to what is actually observed in *Figure 2* and is the reason I formulate expectations by a decay function.

I define the agents level of optimism as the rate at which they expect the return on capital to decay from the current value back to the steady state rate. When the current return is above the steady state, those agents who expect a quick return to steady state are referred to as pessimistic and those who expect current rate of return to last longer are referred to as optimistic. The opposite would be the case if the current rate is below the steady state rate but only positive technology shocks are investigated.

To be able to incorporate adaptive expectations in a model a functional form is given. In the steady state without growth or depreciation, the return on capital is equal to the discount factor ρ.[6] If it is assumed that agents are aware of this and they expect the economy to go the steady state in the long run, then $E_t R(\infty) = \rho$. At any time, τ, it is assumed agents are aware of the capital $K(\tau)$ and current rate of return on capital $r(\tau)$. At time τ adaptive agents expect return on capital in the distant future to be equal to the discount rate but would also expect returns in the near future to be close to the current return. So a function for expected return at time τ, $E_\tau\, r(t)$, would have the current rate of return decaying with time to the discount rate. Expected returns are given by

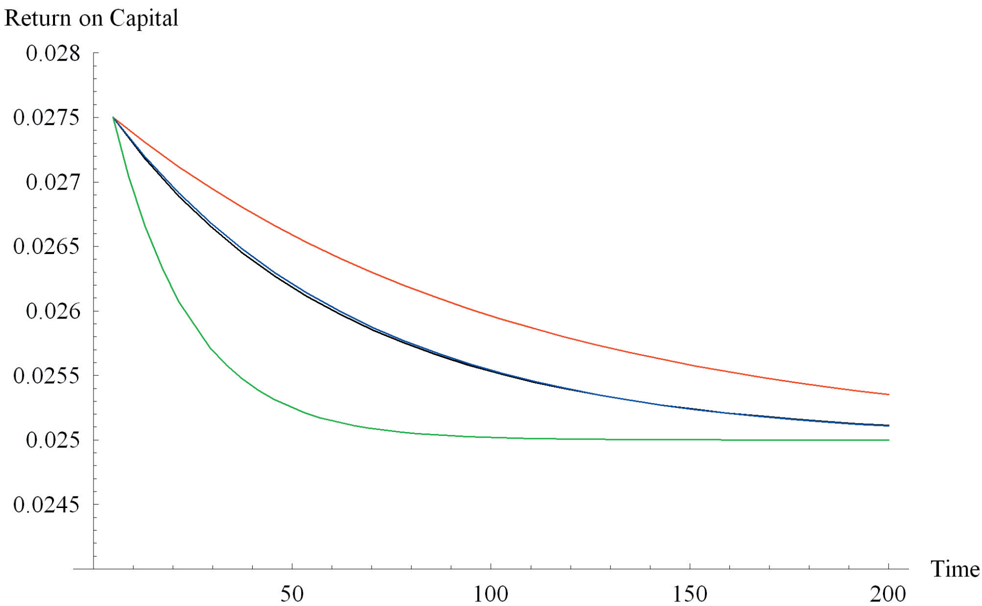
$$E_\tau (r(t)) = \rho + (r(\tau) - \rho)\, D^{t-\tau} \qquad 0 < D < 1 \tag{13}$$

6 If there is depreciation then steady state return equals $\delta + \rho$.

where D is the the agents level of optimism. The function in Equation goes to the discount rate, ρ, as time goes to infinity.

Agents optimism D determines whether the function for $E_\tau\ r(t)$ is a good approximation of what actually occurs in the rational agent scenario. In the graph in *Figure 3* one can see the similarity of the actual return on capital in the rational expectations case compared with the expected return for various values of D. I define $\check{D}$ as the value of D for which the adaptive expectations are almost identical to return observed in the rational agent. I refer to agents with $D > \check{D}$ as excessively optimistic and those with $D < \check{D}$ as excessively pessimistic.

Figure 4: The expected return on capital at time five, $E_5\ r(t)$ with D = .99 in red, D = *.985* in blue and D = *.95* in green. The return observed in the rational agent case are in black.



3. Incorporating Adaptive Expectations

The function for adaptive expectations given in Equation is incorporated in the agent's dynamic optimisation. The agent re-solves its optimisation problem at the start of every period as it updates its information. Consequently if the agent with imperfect foresight incorrectly predicts the rate of return, then its expectations will change at the start of the next period as it receives new information. At the start of every time period τ, the agent recalculates its optimisation problem given the new information and its newly formed expectations.

The Hamiltonian at time τ is given by

$$H_\tau = U(C(t), L(t) + \varphi\,(E_\tau(r(t)\ K(t) + w(t)\,L(t) - C(t) \qquad (14)$$

From the Hamiltonian the necessary conditions for optimisation at time τ, by adaptive agents, are given by

$$U_C(C(t), L(t)) = \varphi_\tau(t), \tag{15}$$

$$\dot{\varphi}_\tau(t) = \varphi_\tau(t)\,(\rho - E_\tau\; r(t)) \tag{16}$$

which can be written as

$$\dot{\varphi}_\tau(t) = \varphi_\tau(t)\,((r(\tau) - \rho)\; D^{t-\tau}). \tag{17}$$

The dynamics of an economy with an agent with imperfect foresight is determined by Equations (15) and (17) and plus the capital condition given in Equation (3).[7] In the model these equations are solved numerically as in Section 2.1.

It is important to distinguish between actual investment and capital accumulation from the agent's expectations. At the start of every period the agent updates its expectations and adjusts its optimal path of consumption. The agent chooses the path of optimal consumption and capital based on its expectations. In the model the agent's consumption is set equal to optimal consumption chosen from its expectations. Actual investment and capital accumulation may not be equal to that expected as actual rate of return on capital may not be equal to that expected. Actual investment and capital accumulation are determined by the agent's actual income minus the consumption chosen.

4. The Importance of the Rate of Intertemporal Substitution

The imperfect foresight agent's investment and consumption decisions depend not only on its level of optimism but also on the rate of intertemporal substitution. The rate of intertemporal substitution is given by the agent's utility function.

To see the effect of the intertemporal rate of substitution the agents planned consumption and investment for a model economy starting below steady state. The model economy is defined by the production and utility functions of the firm and the agent given by

$$F(K(t)) = AK(t)^{.5} \tag{18}$$

and

$$U(C(t)) = \ln(C(t)). \tag{19}$$

There is no labour input into production and firms make a profit. The agent receives this profit and therefore its income is equal to total output. As the agent with imperfect foresight is unaware of the form of the production function it predicts its future income from the expected rate of return on capital. The return on capital can be written as

$$r(t) = A\alpha\; K(t)^{\alpha - 1} \tag{20}$$

7 I have ignored the complication of also deriving expectations for wages. For consistency the only input into production is capital.

Therefore total output is equal to the return on capital multiplied by capital and divided by exponential coefficient on capital in the production function as seen in

$$F(K(t)) = \frac{r(t)\,K(t)}{\alpha} \tag{21}$$

This relationship is incorporated in the agent's optimisation problem when determining expected total income. This is used when finding C_0^* and when determining the agent's expected investment and capital accumulation. The agent's planned optimal consumption and capital paths are given by Equations (15) and (17) and plus the capital condition given in Equation (3).

The agent determines its optimal paths for an economy where the current capital is equal to 20 while the steady state value of capital for this economy is 25 where $\rho = 0$, $\delta = 0$, and $A = 1$. The agent's expectations depend on its level of optimism which is given by D. In this model $\breve{D} \approx .937$. In *Figure 5* one can see that for values of D higher than $\breve{D}$ agents expect higher returns for a longer period of time and so they expect to reach a level of capital above steady state. For D approximately equal to $\breve{D}$ expectations correspond roughly to the rational agent case. For values of D lower than $\breve{D}$ agents expect lower returns so they expect to reach a level of capital below the steady state of the rational agent scenario. The agent with a higher value of D expects a higher steady state level of capital than the rational agent because it is not aware of the production function. It is important to emphasise that these are the agent's expectations of the level of capital they will reach. When the model is evaluated the steady state will be the same for different values of D.

Figure 5: The expected return on capital at time 5, $E_5(r(t)$, with $D = .99$ in blue, $D = .985$ in green and $D = .95$ in red. The outcome with rational agents in black.

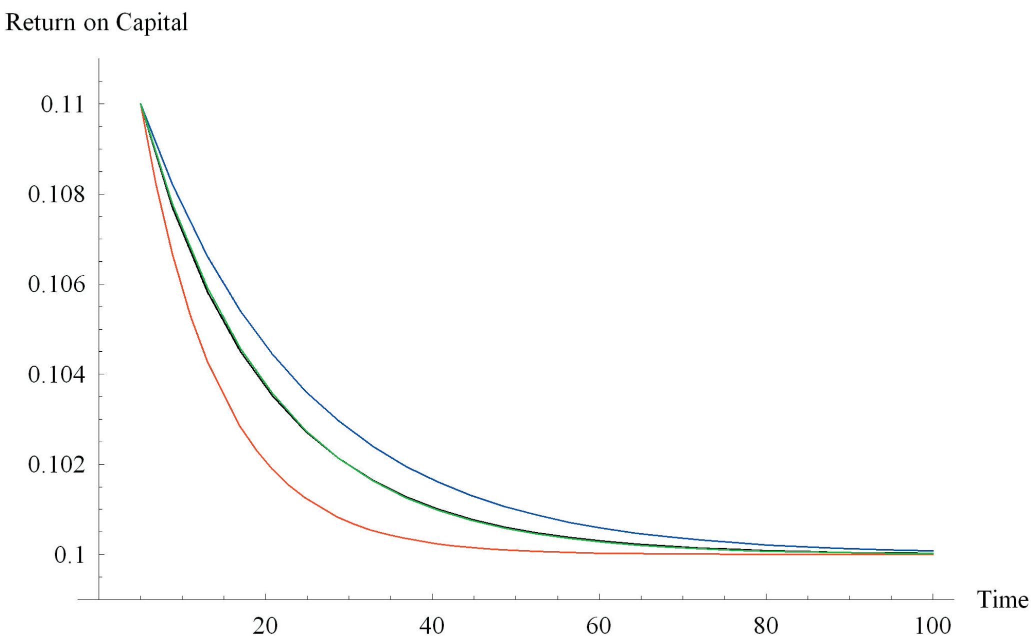
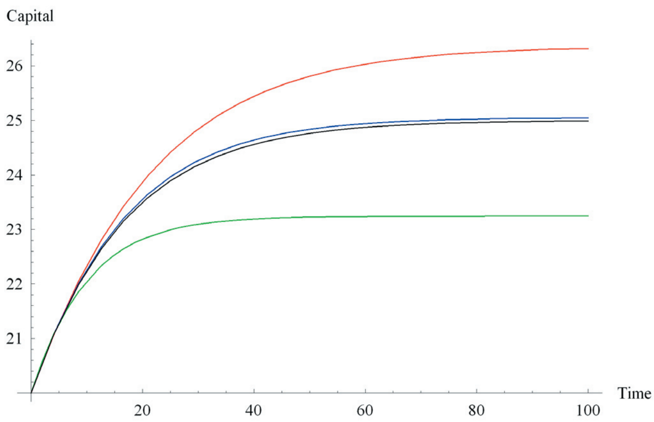
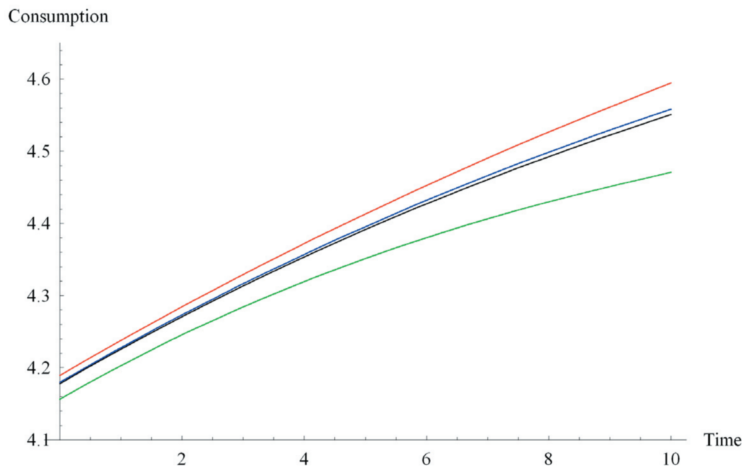


Figure 6: a) $E_0(K(t)$ expected capital, b) $E_0(C(t)$ expected consumption with $D = .95$ in red, $D = .937$ in blue and $D = .9$ in green. Rational agent solution in black.



a)



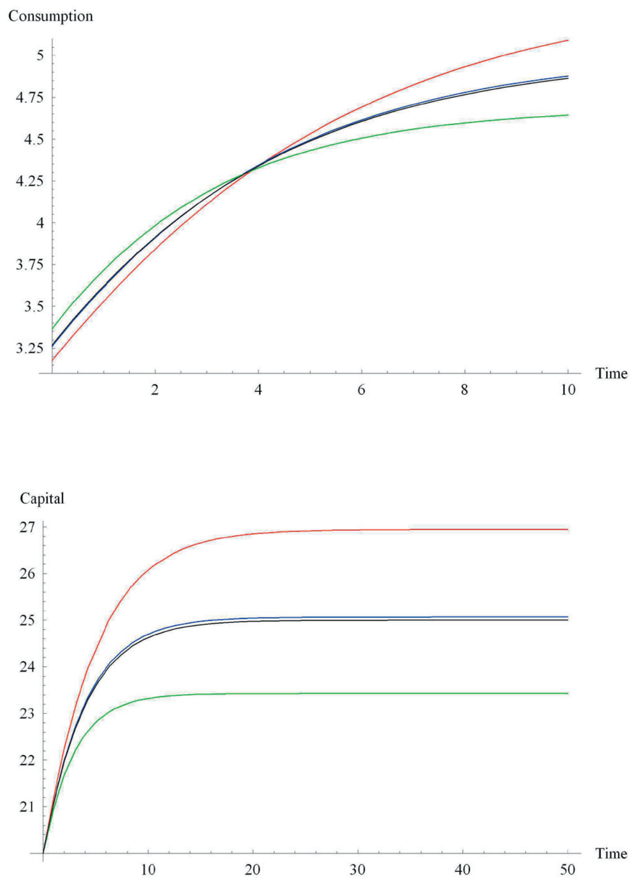
b)

In *Figure 6* we can see the expected optimal consumption paths for various values of D. A log based utility function means agents have in low rates of intertemporal substitution. The more optimistic agent with higher D and higher expected future returns has higher levels of consumption initially. Although the optimistic agent invests less initially it expects to reach a higher level of capital. This is due to the agent expecting returns, and therefore the investment it can afford, to remain high. This is a case of low intertemporal substitution where the agent is not willing to invest more initially to reach a higher level of output and consumes more initially. Optimistic agents with low intertemporal substitution will invest less but still expect a higher level of capital in the steady state. As this paper models excessively optimistic agents over investing I choose a utility function with a higher level of intertemporal substitution. This follows the intuition that one would expect the excessively optimistic agent to invest more and consume less initially. A higher rate of intertemporal substitution in the utility function is given by

$$U(C(t) = C(t)^{9}. \tag{22}$$

Figure 7: a) $E_0C(t)$ expected consumption, b) $E_0K(t)$ expected capital. *D* = *.8* in red, *D* = *.76* in blue and *D* = *.6* in green. Rational agent solution in black. Utility function given Equation (22).

a)
b)



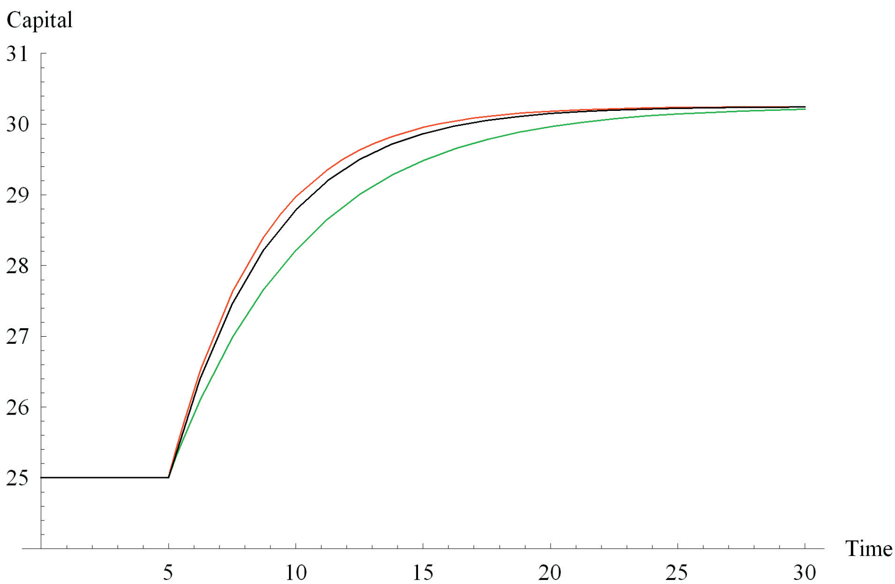
In *Figure 7* one can see with higher intertemporal substitution agents with higher expected future returns consume less initially. Although the excessively optimistic agent consumes less initially, after some time, its expected consumption is greater than that of the pessimistic as the optimistic agent expects a greater increase in income. The agents with *D* greater than $\hat{D}$ will investment more initially and expect to reach a higher level of capital as seen in *Figure 7*.

5. A Model Economy with Agents with Adaptive Expectations of Returns

I model the impulse response of an economy where the agent has expectations as were shown in the previous section. As the agent it not fully rational the expectations will not match what is actually observed in the economy. It is of interest to observe the different behaviour of the economy when the agent has different levels of optimism.

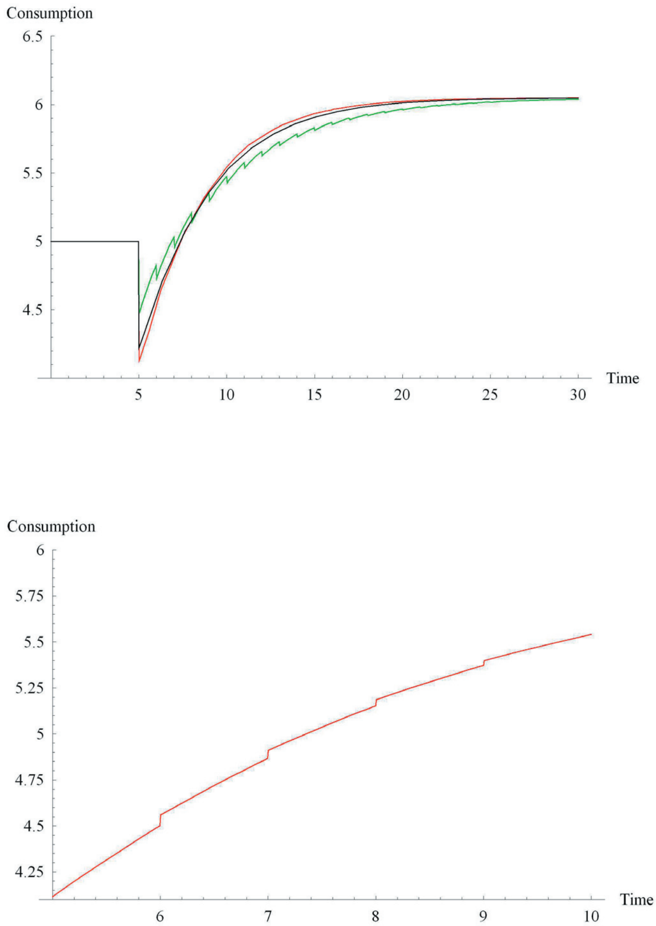
The model economy is described in Section 3.4 and the agents utility is given by Equation (22). The technology change that the economy responds to is as in Section 2.1 and is shown in *Figure 1*. The periods of time between the agent updating its information and its optimal paths are of length one. In the model actual consumption equals planned optimal consumption and investment equal output minus consumption. An optimistic agent with $D = .8$ and a pessimistic agent with $D = .6$ are modeled where $\hat{D} = .76$. The results are shown in *Figures 8* and *9*.

Figure 8: *K(t)* Capital if the consumer chooses consumption based on adaptive expectations with $D = .8$ in red, $D = .6$ in green and rational agent solution in black.



In *Figure 9* one can see that as the agent updates its information in every period it recalculates and adjusts its optimal consumption. The agent with $D > \hat{D}$ invests more than rational agents and hence accumulates capital more quickly. The return that the optimistic agent receives is lower than it expected, hence it adjusts its consumption path upwards each period. This faster capital accumulation means it moves toward the steady state at a faster rate, but capital does not go past the steady state level. As the agent approaches the steady state the agent receives information that the return on capital is close to the steady state value and the agent decreases its investment. The closer they are to the steady state the lower their investment becomes. This information feedback on investment and capital stops that agent over investing.

Figure 9: a) *C(t)* Consumption if the consumer chooses consumption based on adaptive expectations b) Magnified version of *C(t)* $D = .8$ in red, $D = .6$ in green and rational agent solution in black.



a) b)

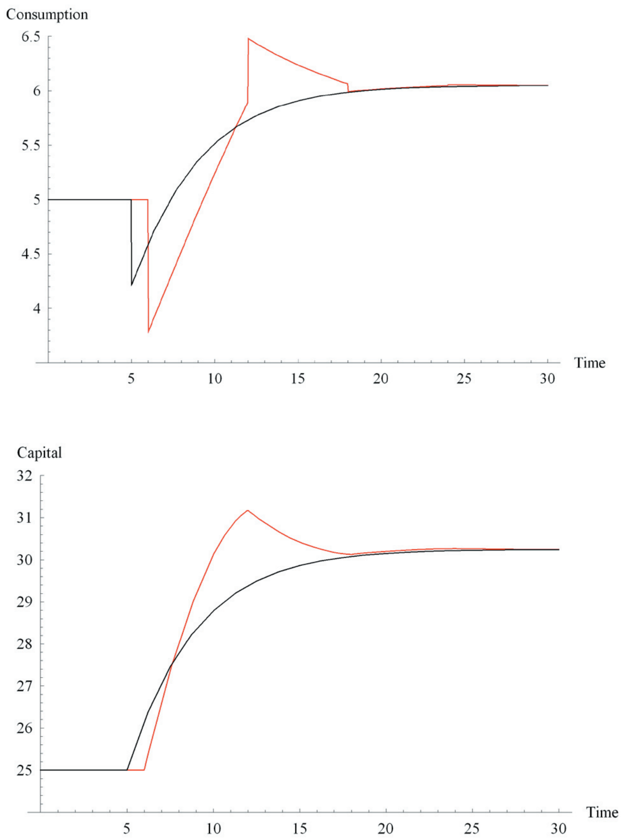
The pessimistic agent, with $D < \grave{D}$, invests less initially and consumes more. The return it receives is higher than it expects; hence it adjusts its consumption path downwards each period. The pessimistic agent's higher consumption and lower investment results in a slower return to the steady state. This slower adjustment could alleviate the weak propagation found in Real Business Cycle models by Cogley and Nason (1995).

In this model agents set their consumption path, but if agents set their investment path the results would be even more pronounced. When actual consumption is set to planned consumption, investment is lower than expected for optimistic agents. If the actual investment is equal to planned investment this would increase investment and a more rapid return to steady state would be observed for optimistic agents. Conversely an even slower return to the steady state would be observed for the pessimistic agents.

1. An Extreme Example

To show that adaptive expectations can result in over investment I model an extreme case. The agent is extremely optimistic with $D = .9$ which is far greater than $\hat{D} = .76$. The length of the period for which agents set their consumption before receiving new information is set to 6. The results of this are seen in *Figure 10*.

Figure 10: a) K(t) Capital b) C(t) Consumption for an agent that chooses consumption based on adaptive expectations with $D = .9$ and 6 period time delay in red time. Rational agent solution in black.



a)

b)

After an initial positive technology shock, the return on capital increases. This increases the agent's expectations of future returns on capital, and as such it increases their investment. As the agent consumes less and invests more than rational agents for an extended period of time their capital reaches a level above steady state. The agent does not realise that it is over investing because of the time delay between them updating their information. The inappropriate level of investment continues for a period of time and capital reaches a level above the steady state. When the agent becomes aware of the situation it sets a new path of consumption with the plan of reducing capital. Here its high value of D means it expects the low rate of return to continue for longer and consumes more to reduce the capital stock. Again, as the agent doesn't update for an extended period of time it over shoots the steady state. This process repeats but the overshoot become smaller and eventually capital converges to the steady state.

The behaviour observed in this extreme example shows a boom followed by a bust phenomena.[8] This model economy produces endogenous cyclical behaviour after a single positive productivity shock.

In a real economy this long information feedback may occur in cases where there is a long delay between commencing an investment project and it producing a yield. In the next section I model an economy with a delay between investment and productive capital coming *online*.

IV. AN ECONOMY WITH A DELAY BETWEEN INVESTMENT AND CAPITAL ACCUMULATION

This paper suggests that cyclical economic behaviour is in part due to delays in adjustment. A model economy with a delay between investment and productive capital being *brought online* may be able to simulate this behaviour.[9] For example, an increase in the demand for commercial real estate may generate a construction boom. As there is a period of time required for construction, new projects commence for a period of time without market feedback on this new potential supply and once construction is completed supply may actually be greater than demand. This excess supply of commercial real estate would then result in a downturn in real estate investment. A model economy that is able to incorporate this behaviour may be able to endogenously generate cyclical behaviour.

1. A Delay Between Investment and Capital Accumulation

Essentially a delay between investment and capital accumulation results in a delay in the feedback on the agent's investment decision. In the previous section a boom could be followed by a recession when there was a long period of time between the agent updating their expectations and adjusting their consumption path. When agents update their expectations they observe current information and are able to assess the appropriateness of their current level of investment. The boom-recession phenomenon was dependant on a large period of time between updating expectations, and adjusting consumption and investment. If there is a delay between investment and capital accumulation there is also a delay in the feedback of whether the current rate of investment is appropriate. Therefore this may produce over investment.

In a real economy there is a period of time required between buying new capital and it being added to the productive capital stock. An example would be buying and starting a new winery. It may be over three years between planting the vines and harvesting the first grapes. When the vines are planted agents may expect high prices for the grapes they produce, but as many agents expect this, large amounts of vines are planted and when all the vines become

8 Over investment may best be modeled by expectations of demand in an economy with multiple sectors, but this requires a more complicated model.

9 The delay between investment and productive capital being *brought online* it similar to the time to build in Kydland and Prescott (82)

productive the price falls significantly due to the large quantity produced. This would be an example of over investment.

A model economy with a delay between investment and capital accumulation is constructed and it is tested whether excessively optimistic adaptive expectations can results in over investment. Again the agent maximises utility. Agents form optimal consumption and investment decisions based on their expected return on capital. Capital accumulation is described by Equation where h is the time delay between investment occurring and the investment being added to the productive capital stock.

$$\dot{K}(t) = I(t-h) \tag{23}$$

I outline the necessary conditions for a maximisation problem when there is a delay between investment and capital accumulation as standard optimal control theory cannot be used. The proof of the necessary conditions is shown in Kamien and Schwartz (1991) and the proof is shown in expanded form in Wende (2008). For the problem of maximising $V(t,c)$ given by

$$V(t,c) = \int_{t_0}^{t_1} U(t,k(t),c(t))\mathrm{d}t \tag{24}$$

where $c(t)$ is the control variable and $k(t)$ is the state variable, subject to

$$\dot{k}(t) = g(t,k(t),k(t-h),c(t),c(t-h)) \tag{25}$$

Initially the state and control variables are

$$k(t) = k_0(t) \quad \text{for} \quad t_0 - h \le t \le t_0 \tag{26}$$

$$c(t) = c_0 \quad \text{for} \quad t_0 - h \le t < t_0. \tag{27}$$

The Hamiltonian, H, is defined as

$$\begin{aligned} &H(t,k(t)\, k(t-h)\, c(t)\, c(t-h)) \\ &= U(t,k(t)\, c(t)\,) + \lambda(t)\, g(t,k(t)\, k(t-h)\, c(t)\, c(t-h)). \end{aligned} \tag{28}$$

The necessary conditions for optimality in terms of the Hamiltonian are

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial K(t)} - \frac{\partial H}{\partial K(t-h)}\bigg|_{t+h} \quad t_0 \le t < t_1 - h \tag{29}$$

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial K(t)} \qquad t_1 - h < t \le t_1 \tag{30}$$

$$\frac{\partial H}{\partial c(t)} + \frac{\partial H}{\partial c(t-h)}\bigg|_{t+h} = 0 \qquad t_0 \le t < t_1 - h \tag{31}$$

$$\frac{\partial H}{\partial c(t)} = 0 \qquad t_1 - h < t \le t_1. \tag{32}$$

These optimality conditions can be applied to the problem of a utility maximising agent. The agent maximises utility from consumption over an infinite time frame as in Section 2 and is written as

$$\max_{C(t)} \int_0^\infty U(C)e^{-\rho t}\mathrm{d}t. \tag{33}$$

Again the agent's consumption is constrained by

$$r(t)K(t) \geq C(t) + I(t) \tag{34}$$

where consumption and investment are constrained to be less than income from capital. The change in capital now depends on past investment,

$$\dot{K}(t) = I(t-h) = r(t-h)K(t-h) - C(t-h). \tag{35}$$

The Hamiltonian for the optimisation is given by

$$H = U(C(t))e^{-\rho t} + \lambda(t)(r(t-h)K(t-h) - C(t-h)). \tag{36}$$

The necessary conditions from the Hamiltonian, in line with Equations (29) to (32), are

$$U_{C(t)}(C(t))\, e^{-\rho t} = \lambda(t+h) \tag{37}$$

and

$$\dot{\lambda}(t) = -\lambda(t+h)\, r(t). \tag{38}$$

As in the literature I define $\varphi(t)$ as

$$e^{\rho t}\lambda(t) = \varphi(t) \tag{39}$$

Differentiating $\varphi(t)$ with respect to time gives

$$\rho e^{\rho t}\lambda(t) + e^{\rho t}\dot{\lambda}(t) = \dot{\varphi}(t). \tag{40}$$

This is substituted into the necessary condition in Equation (38) to give

$$\dot{\varphi}(t) = \rho\varphi(t) - e^{-\rho h}\varphi(t+h)\, r(t) \tag{41}$$

where $\varphi(t)$ is the shadow price of capital. The optimality condition for consumption, Equation (37), can also be written as

$$U_{C(t)}(C(t)) = e^{-\rho h}\varphi(t+h). \tag{42}$$

Equations (35), (41) and (42) give the dynamic behaviour of the model economy. This economy is in steady state when consumption is constant and this occurs when $\dot{\varphi}(t) = 0$. Therefore $\varphi(t+h) = \varphi(t)$ and the steady state return on capital equals $r(t) = \rho e^{\rho h}$. The steady state return on capital is incorporated into the agent's expectations of returns.

When there is a delay between investment and capital accumulation the functional form of expectations is given by

$$E_\tau(r(t)) = \rho e^{\rho h} + (r(\tau) - \rho\, e^{\rho h})\, D^{t-\tau} \qquad 0 < D < 1. \tag{43}$$

This is similar to the standard adaptive expectations of agents with imperfect foresight derived in Section 3.2. The difference occurs due to the new steady state value for the rate of return on capital.

The equation for the expected return is substituted into the optimality conditions and the evolution the shadow price of capital, Equation , is written

$$\dot{\varphi}(t) = \rho\varphi(t) - (\rho + (r(\tau)\, e^{-\rho h} - \rho)\, D^{t-\tau})\varphi(t+h). \tag{44}$$

2. A Model Economy with Delay in Investment and Adaptive Expectations

An economy with a delay of time h between investment and capital accumulation, combined with adaptive expectations is simulated. As in Section 2 the model is solved using numerical methods for differential equations.[10] The dynamics of this economy is determined by the optimality conditions which are outlined in the previous sections and are given in Equations , and the capital accumulation constraint which is given in Equation .

The model economy is specified by the utility function of the agent and the production function of the firm. The agent's utility function is given by

$$U(C(t)) = C(t)^{.9} \tag{45}$$

and the agent's discount rate is set to $\rho = 0.1$. The firm's production function is given by

$$F(K(t)) = AK(t)^{.5} \tag{46}$$

There is no depreciation in the economy so $\delta = 0$. The delay between investment and capital accumulation is set to three periods, $h = 3$. The agent updates its expectations and adjusts its consumption path every half period.

The results of modeling this economy are shown in the graphs in *Figure 11* and *12*.

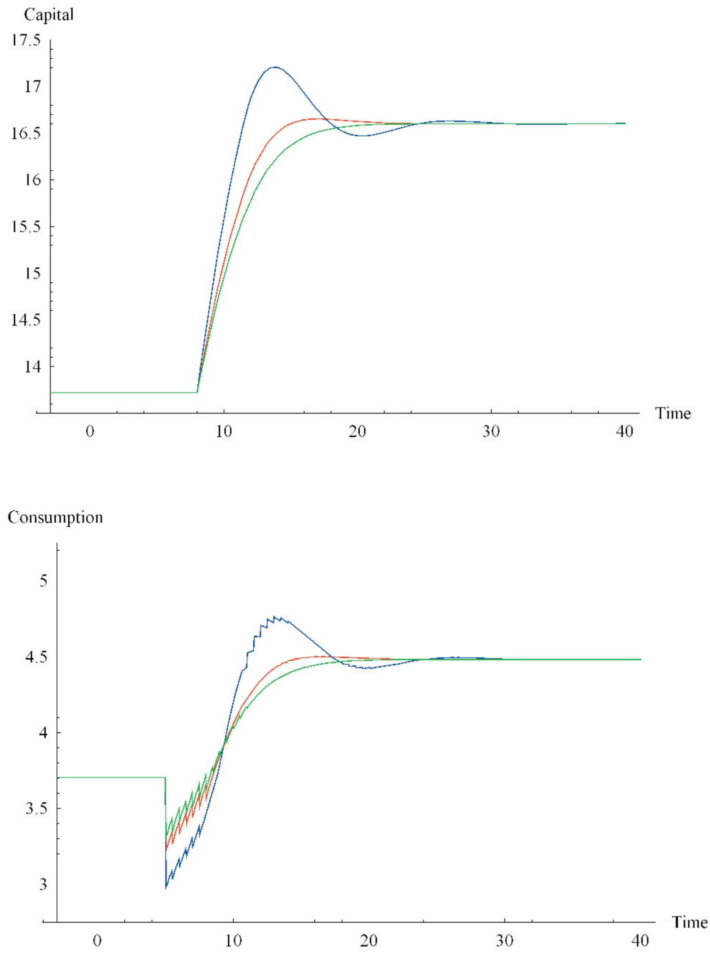
In the model economy excessively optimistic agents, those with values of $D > \hat{D}$, over invest. The initial growth in output is followed by a drop in output equivalent to a boom-recession phenomenon. In this case the result does not depend on large periods of time between agents updating their expectations. The time delay in capital accumulation means the agent is not able to update whether its current investment rate is actually optimal. The delay in capital accumulation causes a delay in the feedback on the effect of recent investment.

Once the overly optimistic agent, who invested heavily initially, reaches the steady state there are investments made earlier that are still waiting to come online. Although the agent stops investing capital continues to accumulated due to earlier investment. In response the

[10] Equation is not an ordinary differential equation because of the $\varphi(t + h)$ term. Therefore standard numerical methods for differential equations cannot be used. The iterative numerical method used is explained in Wende (2008).

agent begins to sell capital but it also takes time for sold capital to go offline. Hence the agent only stops selling once the steady state is reached but capital continues to decrease as previous sales take effect. This causes the cyclical behaviour observed in the graphs in *Figures 11* and *12*.[11]

Figure 11: a) K(t) capital in the economy b) C(t) consumption by agent. $D = 0.8$ in blue, $D = 0.6$ in red and $D = 0.5$ in green.

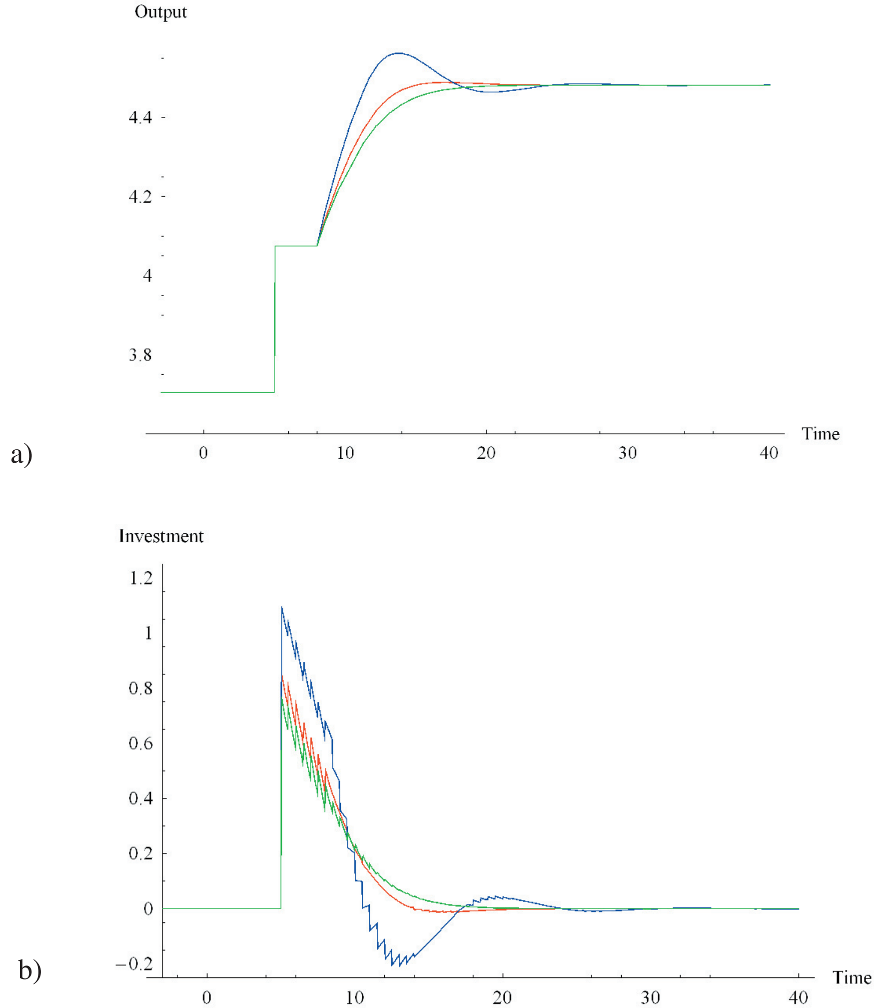


a)

b)

11 I realize it may be possible to change expectations to better incorporate the delay but overly optimistic agents would still over invest and the phenomena would not disappear.

Figure 12: a) F(k(t)) output of the economy b) I(t) investment by agent. $D = 0.8$ in blue, $D = 0.6$ in red and $D = 0.5$ in green.



V. DIFFERENTIAL WAGE ADJUSTMENT IN A NON-EQUILIBRIUM MODEL

In most business cycle models prices adjust perfectly or rely on so called Calvo price adjustment with the economy remaining in equilibrium. This paper suggests frictions in price changes may best be modeled using differential price adjustments. Under differential price adjustment the change in price with respect to time is a function of excess demand of a good *Z*. If prices take time to adjust, then the economy will be out of equilibrium.

When profitability of a firm increases it will employ more people initially. This demand for labour will force wages up, which will in turn, cause profitability and the demand for labour to decrease. If prices for labour do not adjust perfect or instantly employment may increase initially, then decreases to a new equilibrium level which would result in cyclical economic behaviour.

In this hypothetical economy frictions in prices are due the some wage setting department, for example a government fair-pay commission. This body does not adjust wages perfectly and as such the labour market is not always in equilibrium. It adjusts wages according to the excess demand or supply of labour. Z_L is defined as the excess demand of labour

$$Z_L = L_D - L_S \tag{47}$$

where the demand for labour by firms is given by L_D and supply of labour by agents is given by L_S.

I define the change in wages, W, with respect to time in a differential equation given by

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \varepsilon Z_L \tag{48}$$

where ϵ is a parameter that determines how fast the wages adjust to excess demand.

The demand and supply of labour are the utility and profit maximising quantities the firm and agent would chose given wage W, and are therefore functions of the wage set by the wage setting body. As the price set by the wage setting body may not be the equilibrium wage, supply may not equal demand. When the wage is such that the labour that firms demand is not equal to the labour agents would like to supply, the actual labour used, L_U, must be chosen between the supply and demand where $L_S \geq L_U \geq L_D$ for $L_S \geq L_D$. The choice of L_U depends on who is assigned the power in this economy. If the agent has the power it will work a utility maximising amount and then leave and the firm has to be satisfied with labour supplied. If the firm has the power then the agent works the amount demanded by firm or risks getting fired. The second example is similar to being offered a 40 hour a week job, you may only want to work 35.5 hours but if you say this to an employer you will not get the job so you work the 40 hours demanded. The first case would be equivalent to you leaving after 35.5 hours and your employer having to make do. Labour used may also be some value between the supplied or demanded amount. Actual labour used is a function of the bargaining power P of the firm compared to the agent where

$$L_U(t) = P\,L_D(t) + (1 - P)L_S(t) \tag{49}$$

It must be assumed that firms and agents accurately report their preferences. When the bargaining power is shared between agents and firms, $0 < P < 1$, there exists an incentive for each to misreport their preferences for labour so that the actual labour used is closer to their actual preferences. There is no way around this complication so it is assumed that both the agent and the firm are honest citizens and accurately report their supply and demand schedule or do not realise their power to manipulate the actual labour used.

While the agent honestly reports its preferences for labour it must also take the labour used into account when deciding it optimal consumption and labour paths. The maximisation problem face by the agent is given by

$$\max_{C(t)} \int_0^\infty U(C(t), L_U(t))\, e^{-\rho t} \mathrm{d}t \tag{50}$$

where consumption and investment are constrained by

$$W(t)L_U(t) + r(t)K(t)\pi \geq C(t) + I(t) \tag{51}$$

Change in capital is equal to investment

$$\dot{K}(t) = A(t)F(K(t), L_U(t)) - C(t) \tag{52}$$

where it is assumed there is no depreciation, $\delta = 0$

The necessary conditions for optimality are given by

$$U_C(C(t), L_U(t)) = \varphi(t) \tag{53}$$

and

$$\dot{\varphi}(t) = \varphi(t)\,(\rho - F_K(K(t), L_U(t))). \tag{54}$$

As discussed earlier the agent accurately reports its preferences for labour as if it was able to choose this quantity. The optimality condition for labour is given by

$$U_{L_S}(C(t), L_S(t)) = -\varphi(t)\,W(t) \tag{55}$$

This can be inverted to give the agents optimal labour supply

$$L_S(t) = U_L^{-1}(-\varphi(t)\,W(t)) \tag{56}$$

where U_L^{-1} is the inverse function of $U_L(C(t), L(t))$

From the firms profit maximisation

$$A(t)\,F_L(K(t), L(t)) = W(t) \tag{57}$$

Labour demanded is given by

$$L_D(t) = F_L^{-1}\left(\frac{W(t)}{A(t)}\right) \tag{58}$$

where F_L^{-1} is the inverse function of $F_L(K(t), L(t))$.

Substituting the labour supply and demand into the rate of change of the wage, Equation (48), gives

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \varepsilon\left(F_L(K(t))^{-1}\left(\frac{W(t)}{A(t)}\right) - U_L(C(t))^{-1}(-\varphi(t)\,W(t))\right) \tag{59}$$

Labour used depends not only on the wage, whose dynamics is given by the equation above, but also on the power of firms compared with agents P. Labour used is given by

$$L_U(t) = P\left(F_L^{-1}\left(\frac{W(t)}{A(t)}\right)\right) + (1 - P)\left(U_L^{-1}(-\varphi(t)\,W(t))\right) \tag{60}$$

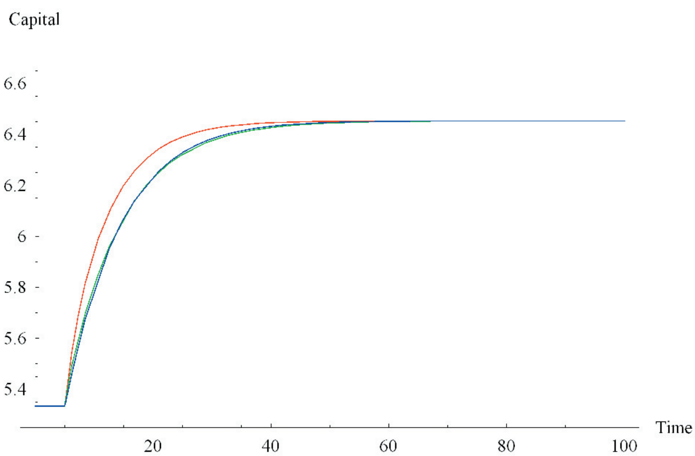
The amount of labour used determines the output of the economy

$$\text{Output}(t) = A(t)F(K(t)\ L_U(t)\ = A(t)K(t)^5\ L_U(t)^5\ . \tag{61}$$

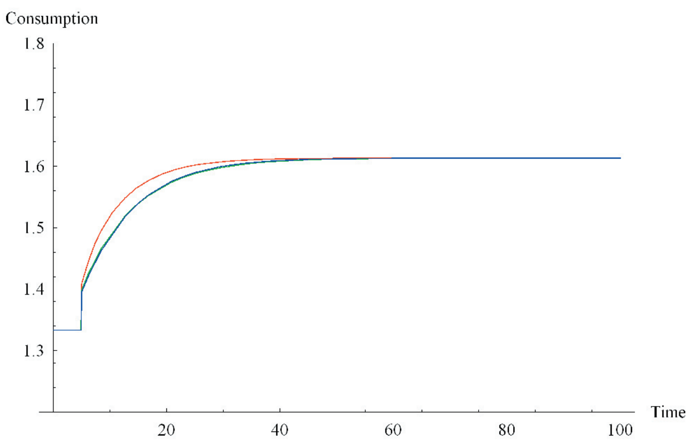
I simulated the response of various specifications of an economy with differential wage adjustment to a permanent technology shock. The utility and production function of the agent and firm are as in Section 2.1. Agents are assumed to be fully rational and aware of the wage setting condition The market for capital is efficient and in equilibrium. The discount rate is $\rho = 0.125$. The economy starts at steady state with the level of capital equal to $5 = \frac{1}{3}$ and the wage equal to 2. The technology variable is as in Section 2.1 and is shown in *Figure 2*. The specification simulated are P = 1 and ε = 5, and P = 1 and ε = 1 and $P = \frac{1}{2}$ and $\varepsilon = \frac{1}{10}$ and are shown the *Figures 12* and *13*.

Figure 13: a) $K(t)$ capital in the economy b) $C(t)$ consumption by agents. P = 1and ε = 1 in blue, P = 1 and ε = 5 in green and $P = \frac{1}{2}$ and $\varepsilon = \frac{1}{10}$ in red.

a)



b)



The results show that if there is no initial change in wages, after the technology shock, firms demand a high quantity of labour. The high labour demand causes an initial boom in output. As wages increase the labour demanded decreases and so does output. But then as capital increases, output also increases. The initial spike in output is followed by a downturn which is then followed by another upturn in output.

I realise the logical inconsistency in this model where agents and firms accurately report preferences and then optimise using a set quantity of labour. The problem may be avoided if unemployment is included so that the supply of labour is not necessarily equal to demand. In this extension there would be a natural rate of unemployment with actual employment moving around this according to supply and demand. This would remove to need for some system of power that decides actual labour used.

While it is not suggested that an economy actually behaves as in this model this technique shows the interesting features possible in a non-equilibrium model with time-differential price changes. This non-equilibrium dynamic model is not possible in the standard solution to RBC models. Further work exists in incorporating these effects with the models shown in earlier section. A model with money and differential price changes seems an obvious extension which would alleviate the necessity for Calvo style price changes.

VI. CONCLUSION

In this paper an alternative numerical solution method to solve standard real business cycle models is shown. The method maintains the full dynamics of the model and allows for the incorporation of adaptive expectations, delay in capital accumulation and non-equilibrium models.

It is shown that an agent with imperfect foresight may over-invest given a long period of time without updating information and choices. A realistic cause of the information feedback delay maybe a delay between investment and capital coming *online*. In cases with long time delays in feed back and with extremely optimistic agents, cyclical behaviour is observed after a single positive technology shock. The behaviour is equivalent to a boom-recession phenomenon observed as part of the business cycle.

It is shown that cyclical behaviour can be generated in a model with a time delay between investment and capital accumulation. The necessary conditions for optimisation with a delay in the capital accumulation is shown and are used to determine the behaviour of agents in the model. It is shown that with delayed capital accumulation, agents can be constantly updating their expectations but still over-invest given optimistic expectations. The model gives a more realistic example of over investment in a real economy.

Time-differential wage adjustment was incorporated in a non equilibrium model and it is shown that this can generate cyclical behaviour. The slow adjustment of wages cause a period of higher profitability for firms which is matched by higher output. The rise in wages causes output to fall but as capital increases output again increases.

This paper investigated a number of ideas not usually incorporated in Real Business Cycle models. The dynamics observed in these models, such as over investment and non-equilibrium markets are not usually observed in standard real business cycle models. The innovation in this paper is that the models generate endogenous cyclical behaviour.

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APPENDIX A

Mathematica Code

Here I include the mathematica code that solves the rational agent model. The code for the other model is, for the most part, very similar and therefore not include. To access all the mathematica code used in this paper please contact the author.

First some model parameters must be chosen. *a is* α is in the Cobb-Douglas productivity function. *p* is the discount factor ρ and *A*[*t*] is the technology function with respect to time.

```
a=.5;
p=.025;
Atime[t_]:=1+If[5≤ t,.1,0]
a=.5;
p=.025;
```

The module **findy0** finds the initial starting value of $\varphi^{*}(0)$ which in turn determines $C^{*}(0)$. This version of **findy0** is for the log utility function

```
findy0[up_, down_, time_, kintial_, it_, A_]:=Module[{sol, k, u, d, y0},
u = up;
d = down;
Do[y0 = (u+d)/2;
sol = NDSolve[{{k'[t]==Ak[t]^a − 1/y[t], k[0] == kintial}, {y'[t] == y[t](p − A ak[t]^(a−1)), y[0] == 1/y0}},
{k, y}, {t, 0, time}];
If[TrueQ[Head[(k[time]/.sol)[[1]]] == Real],
If[(k'[time]/.sol)[[1]] > 0,
d = y0, ];
If[(k'[time]/.sol)[[1]] < 0,
u = y0, ],
u = y0],
{it}];
y0]
```

findy0 is used to run the rational agent model economy with the unexpected technology shock given by **A[t]**.

```
ClearAll[K, t, y, k]
sol = {};
K[t_]:=0 + If[t==0, 400, 0]
Y[t_]:=0
Do[A = Atime[i];
kintial = K[i];
y0 = findy0[40, 0, 1000, kintial, 40, A];
n = i;
sol =
Append[sol,
NDSolve[{{k'[t]==Ak[t]^a − 1/y[t], k[n] == kintial}, {y'[t] == y[t](p − A ak[t]^(a−1)), y[n] == 1/y0}},
{k, y}, {t, i, i + 1}]];
K[s_] = K[s] + If[i < s ≤ i + 1, Evaluate[k[s]/.sol[[1 + i, 1, 1]]], 0];
Y[s_] = Y[s] + If[i < s ≤ i + 1, Evaluate[y[s]/.sol[[1 + i, 1, 2]]], 0],
{i, 0, 200}]
R[t_]:=Atime[t]aK[t]^−a
```

The result can be seen in

```
graph1 = Plot[K[t], {t, 0, 200}, PlotRange → {300, 700}];
```