FORMULATING THE WATER DISTRIBUTION SYSTEM EQUATIONS IN TERMS OF HEAD AND VELOCITY

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Abstract

The set of equations for solving for pressures and flows in water distribution systems are non-linear due to the head loss-velocity relationship for each of the pipes. The solution of these non-linear equations for the heads and flows is usually based on the Todini and Pilati method. The method is an elegant way of formulating the equations. A Newton solution method is used to solve the equations whereby the special structure of the Jacobian is exploited to minimize the computations and this leads to an extremely fast algorithm. Each iteration firstly solves for the heads and then solves for the flows. In the EPANET implementation of the Todini and Pilati algorithm an initial guess of the flows is based on an assumed velocity of 1.0 fps (0.305 m/s) in each pipe in the network. Each flow is then determined from the continuity equation by multiplying the assumed velocity by the area. Usually velocities in pipes are in the range of 0.5 to 1.5 m/s (and perhaps sometimes higher up to 3 or 4 m/s). Thus the velocities to be solved for are all of the same order of magnitude. In contrast, the range of discharges may be quite large in a system – ranging from below 10 L/s up to above 700 L/s – thus possibly three orders of magnitude of difference. As an alternative to the usual formulation of the Todini and Pilati method in terms of flows and heads, this paper recasts the Todini and Pilati formulation in terms of heads and velocities to attempt to improve the convergence properties. Results are compared for the two formulations for a range of networks from 553 to 10,354 pipes. Convergence criteria for stopping the iterative solution process are discussed. The impact of the initial guess of the velocities in each of the pipes in the network on the convergence behavior is also investigated. Statistics on mean flows and velocities in the network and the minimum and maximum velocities for each of the example networks are given and finally operation counts are also provided for these networks.

1. INTRODUCTION

There are a number of ways to choose the unknowns to be solved for in a water distribution system including 1) unknown flow equations (made up of continuity equations at nodes and loop and path energy equations) 2) unknown head equations (made up of the continuity equations at nodes substituted for the flows in terms of unknown nodal heads at each end of a pipe segment) 3) unknown loop flow corrections (made up of path and loop equations based on loop flow corrections for each loop and path in the network) and 4) combined unknown head and flow equations (made up of continuity equations for each node and a headloss equation for each pipe in terms of the unknown nodal heads at each end of the pipe related to the discharge through the pipe). Head loss equations for water flow in pipes are usually based on the Darcy-Weisbach equation or the Hazen-Williams equations. The Todini and Pilati solution algorithm is based on formulation #4 above. The hypothesis that led to this paper was that perhaps a velocity-head formulation based on the Todini and Pilati (1988) approach would have better and hence faster convergence characteristics due to the smaller range of velocities in pipe networks compared to the range of discharges.

The paper starts with the presentation of details of the Todini and Pilati (1988) method for the solution of flows and heads (Q-H) in a water distribution system. A derivation of the velocity-head (V-H) formulation is then given. Results are presented for the comparison of the convergence characteristics for the Todini-Pilati Q-H formulation and the V-H formulation for a range of different sized networks. In addition the sensitivity of the convergence to the initial guesses of velocity in the network to commence the solution to the non-linear equations is assessed.

2. THE TODINI AND PILATI Q-H FORMULATION

The ultimate goal in solving for conditions in water distribution systems is to solve for the flows in the pipes (and pumps and valves if they are present) and the heads at the nodes. Usually the velocities in each pipe and pressures at nodes are then calculated. In 1987, Todini and Pilati (1988) presented at a conference a break-through paper that changed the way water distribution networks would be solved. Prior to this there had been a number of formulations based on different configurations of the unknowns in the network. Four common formulations include:

1. flow equations or Q–equations formulation in terms of unknown flows (Qs) in each pipe
2. head equations or H–equations formulation in terms of unknown heads or HGLs (Hs) at each node
3. loop flow correction equations formulation or LFC–equations in terms of unknown loop flow corrections (LFCs) – this is similar to the manual Hardy Cross method for solving networks
4. flow and head equations or Q–H equations formulation in terms of unknown flows and unknown heads (Todini and Pilati 1988)

There are three types of governing equations for flow and head (HGL or pressure) in a network of pipes. These include:

1. continuity of flow at each node
2. head loss–flow relationships for each individual pipe
3. conservation of energy for each simple loop and the relevant paths (each comprising a number of pipes). A path is taken between two points (usually reservoirs) or nodes of fixed head - it is like an open loop.

Consider a water distribution network of pipes and junctions or nodes in which the system has *NP* pipes, *NJ* variable-head nodes and *NF* fixed-head nodes and also assume the network is completed connected. For simplicity pumps and valves will not be included in the analysis although they can be easily incorporated. The vectors of unknowns in the network include:

- **q** = $(Q_1,Q_2,\ldots,Q_{NP})^T$, where Q_j is the flow for the *j*-th pipe,
- **h** = $(H_1,H_2,\ldots,H_{NJ})^T$, where H_i is the head for the *i*-th node.

The demands and elevations at each node are known and are defined as:

- **dm** = $(DM_1, DM_2,\ldots, DM_{NJ})^T$, where is DM_i the demand at the *i*-th node.
- **el** = $(z_1, z_2,\ldots, z_{NJ})^T$, where z_i is the elevation at the *i*-th node.

The continuity equation at each of the variable-head nodes in the network is given by:

$$\sum_{i=1}^{NPJ_i} Q_i + DM_i = 0 \text{ ………for } i = 1,2,\ldots, NJ \qquad (1)$$

where NPJ_i is the number of pipes connected to node *i*.

The head loss equation (or energy equation) for each pipe in the network connecting node *i* and node *k* is given by:

$$H_i - H_k = r_j Q_j \left| Q_j \right| \qquad \text{for } j = 1,2,\ldots, NP \tag{2}$$

where r_j = resistance factor assuming the Darcy Weisbach head loss equation based on the Darcy-Weisbach friction factor f is used (that is dependent on the Reynolds number and the relative roughness for the pipe). The resistance factor is given by:

$$r_j = \frac{8 f_j L_j}{\pi^2 g D_j^5} \tag{3}$$

where L_j = pipe length, g = gravitational acceleration, D_j = pipe internal diameter. The friction factor may be estimated by the Swamee and Jain (1976) equation as:

$$f_j = \frac{1.325}{\left[\mathbf{ln}\left(\frac{\varepsilon_j}{3.7 D_j} + \frac{5.74}{\mathbf{Re}^{0.9}} \right) \right]^2} \tag{4}$$

where ε_j = roughness height, **Re** = Reynolds number for the flow in the pipe. A Hazen-Williams head loss equation could easily be also used to replace Eq. 2. The vector of pipe resistance factors is:

- $\mathbf{r} = (r_1, r_2, \ldots, r_{NP})^T$, r_j the resistance factor for the *j*-th pipe.

Todini and Pilati (1988) define two topology matrices for the network. The first is the unknown head node incidence matrix **A1** of dimension *NP* x *NJ* such that:

- $\mathbf{A1}(j, i) = -1$ if the flow in pipe *j* enters node *i*,
- $\mathbf{A1}(j, i) = 0$ if pipe *j* does not connect to node *i*,
- $\mathbf{A1}(j, i) = 1$ if the flow in pipe *j* leaves node *i*.

Note that this sign convention adopted here is the opposite sign convention to that adopted by Todini and Pilati (1988). The matrix is designated as **A1** and typically has the form:

$$\mathbf{A}1 = \begin{pmatrix} -1 & 1 & \ldots & \ldots & 0 & 0 \\ 0 & -1 & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & -1 & 0 \\ 0 & 0 & \ldots & \ldots & 1 & -1 \end{pmatrix} \tag{5}$$

Each row represents a pipe and has either 2 elements (a “-1” and a “+1”) representing the nodes at the ends of the pipe or one element (a “-1” or a “+1”) if the other end of the pipe is a fixed head node such as a reservoir. For example consider the first row of the matrix: the “-1” means that pipe 1 (the row) is connected to node 1 while the “+1” means that the other end of the pipe is connected to node 2. In row 2 (pipe 2) – this pipe is connected to node 2 and as all other elements in the row are zero then pipe 2 must also be connected to a fixed head node (row 1) and node 2 (row 2). All the other entries in column 2 must be zero.

The other topology matrix is the fixed head node incidence matrix of dimension *NP* x *NF* that has a -1 if flow from a pipe enters a fixed head node and +1 if flow from a pipe leaves a reservoir. The matrix is designated as **A2** and typically has the form:

$$\mathbf{A}2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \cdots & \cdots \\ \cdots & \cdots \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{6}$$

Each column must have at least one entry (a "+1" or a "-1"). Eq. 6 assumes that the network has two fixed head nodes. The entry at the top left hand corner of **A2** means that pipe 1 (row 1) is connected to reservoir (column 1) with a flow away from the reservoir. The entry of "+1" in column 2 means that pipe (*NP*-1) is connected to reservoir 2.

Based on Eq. 1, the continuity equations for all the pipes in the network can be written in matrix form as:

$$\mathbf{A}1^{\mathbf{T}}\,\mathbf{q} + \mathbf{dm} = 0 = \mathbf{f}_2(\mathbf{q},\mathbf{h}) \tag{7}$$

where we have denoted the left-hand-side of Eq. 7 by the function $\mathbf{f}_2(\mathbf{q},\mathbf{h})$**.** Now for the head loss equations for each pipe Eq. 2 can be re-written as:

$$r_j Q_j \left|Q_j\right| - \left(H_i - H_k\right) = 0 \tag{8}$$

This can also be written in matrix form but first a diagonal matrix **G** of size *NP* x *NP* is introduced where:

$$\mathbf{G} = \begin{pmatrix} r_1\left|Q_1\right| & 1 & \cdots & \cdots & 0 & 0 \\ 0 & r_2\left|Q_2\right| & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & r_{NP-1}\left|Q_{NP-1}\right| & 0 \\ 0 & 0 & \cdots & \cdots & 0 & r_{NP}\left|Q_{NP}\right| \end{pmatrix} \tag{9}$$

The non-linearity in the system arises because the matrix **G** depends on the unknown flows in **q**. The matrix form of Eq. 8 can be written as follows (taking into account that some nodes are fixed head nodes):

$$\mathbf{Gq} - \mathbf{A}1\mathbf{h} - \mathbf{A}2[\mathbf{el}] = 0 = \mathbf{f}_1(\mathbf{q},\mathbf{h}) \tag{10}$$

where **[el]** is a vector of the reservoir or fixed node heads. Note that Eq. 10 is denoted by the vector function $\mathbf{f}_1(\mathbf{q},\mathbf{h})$**.**The two sets of matrix equations in Eq. 7 and Eq. 10 may be written in the following block matrix form as:

$$\mathbf{f(q,h)} = \begin{pmatrix} \mathbf{G} & | & -\mathbf{A}1 \\ -- & - & -- \\ -\mathbf{A}1^{\mathbf{T}} & | & 0 \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ -- \\ \mathbf{h} \end{pmatrix} - \begin{pmatrix} \mathbf{A}2[\mathbf{el}] \\ -- \\ \mathbf{dm} \end{pmatrix} = 0 \qquad (11)$$

The first matrix on the left hand side of Eq. 11 shows the partitioning of a (*NP* + *NJ*)–square matrix into 2 block rows and 2 block columns. Note that the first block matrix on the left had hand side of Eq. 11 has a special structure that may be exploited because it is symmetric. Note that the only non-constant values in this matrix are the diagonal elements of **G**. A Newton iterative solution to the set of non-linear equations in Eq. 11 can be formulated in terms of Taylor's series expansion and linearization as:

$$\mathbf{J} \begin{pmatrix} \delta \mathbf{q} \\ -- \\ \delta \mathbf{h} \end{pmatrix} = \begin{pmatrix} -\mathbf{f}_1 \\ -- \\ -\mathbf{f}_2 \end{pmatrix}. \qquad (12)$$

Eq. 12 can be expressed as:

$$\begin{pmatrix} \delta \mathbf{q} \\ -- \\ \delta \mathbf{h} \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} -\mathbf{f}_1 \\ -- \\ -\mathbf{f}_2 \end{pmatrix} \qquad (13)$$

provided the Jacobian **J** is invertible. Normally we would not deal with the inverse of the Jacobian as it is computationally intensive and there are other much more efficient decomposition methods that avoid the need for solving for the inverse of the Jacobian.

The Todini and Pilati (1988) method solves the equations describing the flows and nodal heads in a water distribution network by a reformulation of Eq. 12 which exploits the diagonal nature of G and which uses an explicit block-form of the inverse as shown below. Thus, there is good reason in this instance to compute an analytical expression for the block inverse of the Jacobian as it leads to a simplification and hence a significant improvement in the speed of the solution algorithm.

Now consider the derivative of the first matrix on the left side hand multiplied by the vector $(\mathbf{q}^{\mathbf{T}},\mathbf{h}^{\mathbf{T}})^{\mathbf{T}}$ of Eq. 11 so that we can form the Jacobian as shown in Eq. 13. The diagonal nature of **G** can be exploited with only the diagonal elements of the **Gq** matrix changing upon differentiation so computation of the Jacobian is very straightforward. Rewriting Eq. 11 as:

$$\mathbf{f(q,h)} = \begin{pmatrix} \mathbf{Gq} & | & -\mathbf{A}1\mathbf{h} \\ -- & - & -- \\ -\mathbf{A}1^{\mathbf{T}}\mathbf{q} & | & 0 \end{pmatrix} - \begin{pmatrix} \mathbf{A}2[\mathbf{el}] \\ -- \\ \mathbf{dm} \end{pmatrix} = 0 \qquad (14)$$

The derivatives of the diagonal elements of **Gq** assuming that r is constant (despite the fact that friction factors *f* actually depend on flow: the friction factors are updated at the end of each iteration) are:

$$\frac{d}{dQ_j}\left(r_j Q_j \left| Q_j \right|^{n-1} \right) = r_j n \left| Q_j \right|^{n-1} \qquad \text{for } Q_j \neq 0. \qquad (15)$$

The terms in the second matrix (with **A2** etc.) on the left hand side of Eq. 14 do not depend on **q** and **h**. The Jacobian matrix for the system of equations in Eq. 14 becomes:

$$\mathbf{J} = \begin{pmatrix} n\mathbf{G} & | & -\mathbf{A}1 \\ -- & - & -- \\ -\mathbf{A}1^{\mathbf{T}} & | & 0 \end{pmatrix} \quad (16)$$

This matrix has some very nice properties of symmetry and sparseness. Todini and Pilati (1988) show an analytic expression for the block-form of the inverse of ***J*** where the inverse of n**G** which is called $\mathbf{D}^{-1}$ is easy to compute for a diagonal matrix. It has terms $1/(nr_j|Q_j|)$ at each location along the diagonal. Assume that the head loss exponent n is the same for each pipe. The Jacobian becomes:

$$\mathbf{J}^{-1} = \begin{pmatrix} \mathbf{D} - \mathbf{DA}1(\mathbf{A}1^{\mathbf{T}}\mathbf{DA}1)^{-1}\mathbf{A}1^{\mathbf{T}}\mathbf{D} & | & -\mathbf{DA}1(\mathbf{A}1^{\mathbf{T}}\mathbf{DA}1)^{-1} \\ ---------------- & - & ---------- \\ -(\mathbf{A}1^{\mathbf{T}}\mathbf{DA}1)^{-1}\mathbf{A}1^{\mathbf{T}}\mathbf{D} & | & -n(\mathbf{A}1^{\mathbf{T}}\mathbf{DA}1)^{-1} \end{pmatrix} \quad (17)$$

The reformulation allows the solution process for each iteration to be done in two stages: one for the flows and another for the heads. Substituting Eq. 17, Eq. 7 and Eq. 10 into Eq. 12 and simplifying gives the two-step Todini and Pilati algorithm for solving successively at each iteration for the heads and discharges as follows:

$$\mathbf{h}^{(k+1)} = (\mathbf{A}1 n\mathbf{G}^{-1}\mathbf{A}1^{\mathbf{T}})^{-1}\left[\mathbf{A}1^{\mathbf{T}} n\,(\mathbf{q}^{(k)} - \mathbf{G}^{-1}\mathbf{A}2[\mathbf{el}]) - (\mathbf{A}1^{\mathbf{T}}\mathbf{q}^{(k)} + \mathbf{dm})\right] \quad (18)$$

and

$$\mathbf{q}^{(k+1)} = (1 - \frac{1}{n})\mathbf{q}^{(k)} + \mathbf{G}^{-1}\frac{1}{n}(\mathbf{A}1\mathbf{h}^{(k+1)} + \mathbf{A}2[\mathbf{el}]) \quad (19)$$

where n = 2.0 for the Darcy Weisbach head loss formula and **N** is a diagonal matrix of the head loss exponent for each pipe. Note that the only matrix inverse involves the *Schur Complement* term of $(\mathbf{A}1^{\mathbf{T}}\mathbf{DA}1)^{-1}$ which is the size of *NJ* by *NJ* (the number of nodes in the network), and this matrix inverse only appears in the expression for the heads in Eq. 18. Thus although we are dealing with a total of (*NP* + *NJ*) unknowns the use of the analytical inverse of the Jacobian has reduced the matrix size down to the size of *NJ*. This inverse is not actually solved for because a decomposition method to solve the terms involving the Schur complement is much faster as mentioned previously. Eq. 18 depends on an initial guess of the flows – it is common to use 1.0 fps (0.3048 m/s) (Rossman 2000) to compute flows in all pipes (although the direction of flow will actually be incorrect in a number of pipes).

3. THE VELOCITY FORMULATION (V-H)

The velocity-head formulation based on the Todini and Pilati (1988) methodology is now presented in order to compare it to the discharge head formulation described in Section 2. The vectors of unknown velocities in the network are:

- $\mathbf{v} = (V_1, V_2, \ldots, V_{NP})^{\mathrm{T}}$, where V_j is the velocity for the j-th pipe,

The fluid flow in pipe j is related to the velocity by $Q_j = V_j A_j$,where V_j *is* the fluid velocity and $A_j > 0$ is the cross-sectional area of the pipe. The continuity equation (Eq. 1 modified) at each of the variable-head nodes in the network is now given by:

$$\sum_{j=1}^{NPJ_i} V_j A_j + DM_j = 0 \ldots\ldots\ldots \text{for } i = 1, NJ \qquad (20)$$

where NPJ_i is the number of pipes connected to node *i* and A_j is the area of the *j*-th pipe. The head loss equation from or energy equation (Eq. 8 modified) for each pipe in the network connecting node *i* and node *k* in terms of velocity is given by:

$$R_j V_j \left|V_j\right| - \left(H_i - H_k\right) = 0 \ \ldots\ldots\ldots \text{for } j = 1, NP \qquad (21)$$

where R_j = resistance factor (analogous to Eq. 3) assuming the Darcy Weisbach head loss equation and given by

$$R_j = \frac{f_j L_j}{2gD_j} \qquad (22)$$

First define the following diagonal matrix with size *NP* by *NP* of the velocities in each pipe **V** = diag{V_1, V_2, . . . , V_{NP}}. Now let **B** be a diagonal matrix *NP* by *NP* that has the pipe areas along the diagonal ***B***=diag{A_1, A_2, ..., A_{NP}}, and **R** be the *NP* by *NP* diagonal matrix **R**=diag(R_1, R_2, . . . , R_{NP}) of resistance terms. Then the diagonal matrix **G** from the Q-H formulation becomes $\mathbf{G} = \mathbf{R}\mathbf{B}^{n-1}\left|\mathbf{V}\right|^{n-1}$. Eq. 11 can be rewritten as:

$$\mathbf{F(q,h)} = \begin{pmatrix} \mathbf{RB}^{n-1}\left|\mathbf{V}\right|^{n-1} & | & -\mathbf{A}1 \\ --------- & - & -- \\ -\mathbf{A}1^{\mathbf{T}} & | & 0 \end{pmatrix} \begin{pmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ -- \\ \mathbf{h} \end{pmatrix} - \begin{pmatrix} \mathbf{A}2[\mathbf{el}] \\ -- \\ \mathbf{dm} \end{pmatrix} = 0 \,. \qquad (23)$$

We can symmetrize this relation by multiplying on the left by:

$$\mathbf{F(q,h)} = \begin{pmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{RB}^{n-1}\left|\mathbf{V}\right|^{n-1} & | & -\mathbf{A}1 \\ --------- & - & -- \\ -\mathbf{A}1^{\mathbf{T}} & | & 0 \end{pmatrix} \begin{pmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ -- \\ \mathbf{h} \end{pmatrix} - \begin{pmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{A}2[\mathbf{el}] \\ -- \\ \mathbf{dm} \end{pmatrix} = 0 \qquad (24)$$

and since diagonal matrices commute this gives:

$$\mathbf{F(q,h)} = \begin{pmatrix} \mathbf{RB}^{n+1}\left|\mathbf{V}\right|^{n-1} & | & -\mathbf{BA}1 \\ --------- & - & -- \\ -\mathbf{A}1^{\mathbf{T}}\,\mathbf{B} & | & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ -- \\ \mathbf{h} \end{pmatrix} - \begin{pmatrix} \mathbf{BA}2[\mathbf{el}] \\ -- \\ \mathbf{dm} \end{pmatrix} = 0 \,. \qquad (25)$$

Let us denote this new system by

$$\mathbf{F(q,h)} = \begin{pmatrix} \mathbf{M} & | & -\mathbf{A}3 \\ --------- & - & -- \\ -\mathbf{A}3^{\mathbf{T}} & | & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ -- \\ \mathbf{h} \end{pmatrix} - \begin{pmatrix} \mathbf{BA}2[\mathbf{el}] \\ -- \\ \mathbf{dm} \end{pmatrix} = 0 \qquad (26)$$

With the obvious definitions. Thus the system to be solved can be written as:

$$\mathbf{F(q,h)} = \begin{pmatrix} \mathbf{Mv} & | & -\mathbf{A}3\mathbf{h} \\ --------- & - & -- \\ -\mathbf{A}3^{\mathbf{T}}\,\mathbf{v} & | & 0 \end{pmatrix} - \begin{pmatrix} \mathbf{BA}2[\mathbf{el}] \\ -- \\ \mathbf{dm} \end{pmatrix} = 0 \qquad (27)$$

A Newton iterative solution to the set of non-linear equations in Eq. (27) can be formulated in terms of Taylor's series expansion and linearization (similar to Eq. 11) as:

$$\mathbf{J}\begin{pmatrix} \delta\,\mathbf{v} \\ -- \\ \delta\,\mathbf{h} \end{pmatrix} = \begin{pmatrix} -\mathbf{F}_1 \\ -- \\ -\mathbf{F}_2 \end{pmatrix}. \qquad (28)$$

The Jacobian becomes:

$$\mathbf{J} = \begin{pmatrix} n\mathbf{M} & | & -\mathbf{A}3 \\ -- & - & -- \\ -\mathbf{A}3^{\mathbf{T}} & | & 0 \end{pmatrix}. \qquad (29)$$

Again Eq. 16 holds if $n\mathbf{M}$ is taken as D^{-1} with the **A1** replaced by **A3**. The two-step solution algorithm for solving successively at each iteration for the heads and velocities is as follows:

$$\mathbf{h}^{(k+1)} = (\mathbf{A}3n\mathbf{M}^{-1}\mathbf{A}3^{\mathbf{T}})^{-1}\left[\mathbf{A}3^{\mathbf{T}}n\,(\mathbf{v}^{(k)} - \mathbf{M}^{-1}\mathbf{A}2[\mathbf{el}]) - (\mathbf{A}3^{\mathbf{T}}\,\mathbf{v}^{(\mathbf{k})} + \mathbf{dm})\right] \qquad (30)$$

and

$$\mathbf{v}^{(k+1)} = (\mathbf{I} - \mathbf{N}^{-1})\mathbf{v}^{(k)} + \mathbf{M}^{-1}\mathbf{N}^{-1}(\mathbf{A}3\mathbf{h}^{(k+1)} + \mathbf{A}2[\mathbf{el}]) \qquad (31)$$

4. CHECKING FOR CONVERGENCE

During the iterative process a check needs to be made to see if the solution has converged on a sufficiently accurate solution. For this study the L-infinity norm applied to the head changes at all nodes in the network has been used. The L-infinity norm designated as $\|\ldots\|_\infty$ is the absolute value of the maximum sized deviation of all head changes chosen from all nodes in the network. The L-infinity norm for the heads is computed as:

$$\| H_j^{(k+1)} - H_j^{(k)} \|_\infty \qquad (32)$$

The convergence based on the heads was chosen because of the physical relevance of the size of the head at a node. The convergence tolerance (or stopping criteria) was chosen such that the maximum change of head at any node in the network (that is the L-infinity head norm) for an iterative step needed to be less than 1 mm or 0.001 m.

An alternative to the head norm is the L-infinity norm for the discharges that is computed as:

$$\left\| Q_j^{(k+1)} - Q_j^{(k)} \right\|_\infty \tag{33}$$

The difficulty with the L-infinity flow norm is that it is difficult to assess a physically appropriate stopping criteria. Eq. 33 may be replaced by:

$$\left\| \frac{Q_j^{(k+1)} - Q_j^{(k)}}{Q_j^{(k+1)}} \right\|_\infty \tag{34}$$

However, this may lead to problems if any of the flows at convergence have a value that is close to zero. An adjustment may be made to Eq. 34 such that:

$$\left\| \frac{Q_j^{(k+1)} - Q_j^{(k)}}{1.0 + \left| Q_j^{(k+1)} \right|} \right\|_\infty . \tag{35}$$

Now the convergence criterion will not be affected by near zero flows.

The L-infinity norm for the velocities corresponding to Eq. 33 is computed as:

$$\left\| V_j^{(k+1)} - V_j^{(k)} \right\|_\infty \tag{36}$$

Again this has a physical relationship in that if the maximum velocity change is less than 1 mm/s or 0.001 m/s then it is reasonable to terminate the iterative solution process.

5. RESULTS

Comparison of convergence behavior

Both the flow and velocity formulation based on the Todini and Pilati procedure described in Sections 2 and 3 have been tested on five different sized networks ranging from 553 pipes to 10,354 pipes. The number of iterations for the two alternative formulations is shown in Table 1 for a stopping criteria of 10^{-3} for the L-infinity head norm of Eq. 32 (in other words the maximum change in head for any node in the network from one iteration to the next must be less than 1 mm). Properties of each of the networks in terms of pipe numbers, node numbers and reservoir numbers are also given in Table 1.

Thus from Table 1 it can be seen that the total number of iterations is the same for the flow and the velocity formulations when the L-infinity head norm is used as a stopping criteria.

Table 1. Comparison of Number of Iterations

No. of pipes	No. of nodes	No. of reservoirs	Iterations for flow formulation	Iterations for velocity formulation
553	290	4	9	9
1054	538	13	9	9
3217	1637	13	10	10
5187	2617	29	12	12
10354	5187	64	12	12

To compare the convergence characteristics the results in Table 2 are presented. As evident from Table 2, the L-infinity head norm for the velocity formulation is almost identical to the L-infinity head norm for the flow formulation thus provides no advantage over the discharge formulation.

Table 2. Iterative Solutions for the 553-Pipe Network

	Flow-head formulation		Velocity-head formulation	
Iteration	L-inf H norm Eq. 32	L-inf Q norm Eq. 33	L-inf head norm Eq. 32	L-inf vel norm Eq. 36
1	2.351E+02	5.597E-01	2.351E+02	8.660E+00
2	2.973E+01	2.523E-01	2.973E+01	3.890E+00
3	7.348E+00	1.015E-01	7.348E+00	2.288E+00
4	4.390E+00	3.037E-02	4.390E+00	1.124E+00
5	8.993E-01	8.037E-03	8.996E-01	4.354E-01
6	8.628E-02	1.968E-03	8.641E-02	1.037E-01
7	1.313E-02	2.957E-04	1.311E-02	3.642E-02
8	5.677E-03	8.057E-05	5.664E-03	1.023E-02
9	4.727E-04	1.304E-05	4.712E-04	1.655E-03

Note that the L-infinity velocity norm is larger than the L-infinity flow norm which is to be expected. If the change in velocity and change in flow were normalized by the velocity and discharge respectively (as per the form of Eq. 34) then the values would be identical.

Initial guess of velocity for the iterative solution process

Table 3 shows the variation of the number of iterations for various assumed starting velocities for the flow-head formulation based on the Todini and Pilati formulation.

Table 3. Impact of Varying the Initial Starting Guess of Velocity for the 553 Pipe Network

Initial Velocity (m/s)	Iterations
0.1	10
0.3	9
0.5	8
0.7	8
0.9	8
1.1	8
1.3	9
1.5	9
1.7	9
1.9	9

The results from Table 3 suggest that guesses between 0.5 and 1.1 m/s provide an improved guess compared with 0.3 m/s (approximately 1 fps). This is supported by the results in Table 4 which show the means of the absolute values of the flows and velocities in each of the example networks. The range of mean velocities is from 0.59 to 0.92 m/s. Thus guesses closer to the actual mean velocity in the network appear to have better convergence characteristics. The last two columns of Table 4 show that there are velocities that are very close to zero and that there are also some velocities that exceed 10 m/s.

Table 4. Mean Discharges and Velocities in the Example Networks for Final Solution

No. of Pipes	Mean of absolute values of Qs (m^3/s)	Mean of absolute values of Vs (m/s)	Absolute value of mininum velocity (m/s)	Absolute value of maximum velocity (m/s)
553	0.01986	0.588	0.000438	4.86
1054	0.02621	0.828	0.002250	7.87
3217	0.02886	0.918	0.001550	12.4
5187	0.02313	0.761	0.000120	10.2
10354	0.02515	0.779	0.000131	11.1

6. OPERATION COUNTS

The Todini and Pilati method has the following number of operations (multiplication and divide floating point operations - *flops*). For initialization the number of flops is:

$$NP(NF + 1) \tag{37}$$

where *NP* = number of pipes and *NF* = number of fixed node heads in the system (usually reservoirs or tanks). The per iteration cost in flops assuming that full matrix and not sparse matrix methods are used is:

$$\frac{1}{6}NJ^3 + \frac{11}{2}NJ^2 + \frac{1}{3}NJ + 15NP \tag{38}$$

where *NJ* = number of nodes in the water distribution system. Flop totals for each of the example networks in Table 1 are given in Table 5. The increase in the number of computations relative to the 533 pipe (or 290 node) network is given in Table 6.

Table 5. Operation Counts for Example Networks

Network	No. of pipes (NP)	Startup flops Eq. 37	Flops per iteration Eq. 38	Iterations	Total Ops. Count
1	553	6083	24855518	9	2.237057E+08
2	1054	21080	157320371	9	1.415904E+09
3	3217	64340	4401543647	10	4.401550E+10
4	5187	186732	17960724084	12	2.155289E+11
5	10354	735134	1.39704E+11	12	1.676450E+12

Thus from Table 6 in comparing the 553 pipe network (#4) and the 10,354 pipe network (#5) for an increase in number of nodes by a factor of 17.9 the increase in number of computation flops is by a factor of 7494.

Table 6. Comparison of Increase in Computation with Network Size Relative to the 553 Pipe Network

Network	**No. of nodes (NJ)**	**No. of nodes divided by 290**	**Relative no. of operations**
1	290	1.00	1.0
2	538	1.86	6.3
3	1637	5.64	196.8
4	2617	9.02	963.4
5	5187	17.89	7494.0

7. SUMMARY AND CONCLUSIONS

In this paper the equations for formulating the pipe network equations in terms of velocity and head based on the formulation of Todini and Pilati (1988) has been presented, building on the original derivation of the flow and head equations. Usually velocities are in the range of 0.5 to 10 m/s with only a few values being above or below this range. In this research it was hypothesized that a velocity-head formulation of the equations based on the original Todini and Pilati method may lead to a more effective iterative method and thus be quicker and solve in fewer iterations than the flow formulation due to the much smaller range of possible likely velocity values in contrast to the possible range of flows. Testing of the Todini and Pilati flow-head and the velocity-head formulations for a number of randomly generated networks of different sizes show that the results of the number of iterations to achieve convergence are almost identical thus the velocity-head formulation does not offer any improvement in convergence characteristics. In addition results of various starting values of velocity have been used to assess whether the Todini and Pilati flow-head formulation is sensitive to the initial guess of velocity for all pipes in the network. A value of 0.3 m/s may be too low as a starting guess and the results suggest that an initial guess of between 0.5 and 1.1 m/s is more appropriate.

8. REFERENCES

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