34 Braid Road,
Edinburgh 10.

Feb. 21, 1939.

Professor R. A. Fisher.

Dear Fisher,

By different routes we had come to the same final point, that whatever function remained after the various integrations was a function of p, the number of variates, only. This is how I reached it, but some points of detail remain.

The sums of principal minors of a matrix A, let us say $sp_k A$ for "sum of principal minors of order k", are the respective elementary symmetric functions of the latent roots θ. I expressed the Jacobian of the diagonal elements a_{kk} with respect to the roots θ as the product of two Jacobians

$$\left\{ \left| \frac{\partial\, sp_k}{\partial\, a_{kk}} \right| \right\}^{-1} \left| \frac{\partial\, sp_k}{\partial\, \theta} \right| .$$

The second factor can be seen at once to be the difference-product of the roots θ, which gives the required factor in your distribution function. It remains to integrate the first Jacobian factor, which works out in detail as

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ sp_1|A_{11}| & sp_1|A_{22}| & sp_1|A_{33}| & \dots & sp_1|A_{pp}| \\ sp_2|A_{11}| & sp_2|A_{22}| & sp_2|A_{33}| & \dots & sp_2|A_{pp}| \\ \vdots & \vdots & \vdots & & \vdots \\ |A_{11}| & |A_{22}| & |A_{33}| & \dots & |A_{pp}| \end{vmatrix}^{-1} \qquad \left(|A_{kk}| \equiv \underline{\text{cofactor}} \text{ of } a_{kk} \text{ in } |A| . \right)$$

with respect to nondiagonal elements a_{hk} of A, over a range given by assigned values of the θ's, and to show that this result is a function of p only. Now the determinant (which is new to me) is easily seen to alternate in sign when the h^{th} and k^{th} rows and columns of A are interchanged. Hence its square is a symmetric function of principal minors of A, and therefore of sums of principal minors of A, and so by the relations expressing these sums as elementary symmetric functions of θ's we can remove the elements a_{kk} and express this function in terms of a symmetric function in the θ's (I am practically sure that this part is the squared difference-product of the θ's) and residual terms involving non-diagonal elements a_{hk} only. Now integrating this reciprocal of a $\sqrt{(\text{symmetric function})}$ with respect to the a_{hk}, $\frac{1}{2}p(p-1)$ of them, over fixed θ-ranges, we must surely obtain a function of p only. You arrive at the same conclusion by orthogonal transformation of A, your e_{ij} being elements of an orthogonal matrix E such that $E'E = I$ It is known that an orthogonal matrix of order p has $\frac{1}{2}p(p-1)$ degrees of arbitrariness.

Both approaches are bound, I think, to come up against this squared Jacobian which I have mentioned. It may be that it does not require to be evaluated, but it would be of interest to know its value in terms of the roots θ and the non-diagonal elements a_{hk}. I have been unable to get down to this (having been for the last week the butt of various local importuners) and I do not even know the result for p = 3, which would give a clear clue. For p = 2 the squared Jacobian is $(\theta_1 - \theta_2)^2 - 4a_{12}^2$, equivalent to the $p^2 - 4q - 4b^2$ of your first letter of Jan. 29.

It is curious that this interesting Jacobian has not received notice before. (It may have, of course, but in a moderately wide reading I have not met it.) One would have thought that the Jacobian of sums $sp_k A$ of principal minors, those fundamental invariants of a matrix A, with respect to principal elements a_{kk}, would have played some important rôle in the study of positive definiteness, for example. Perhaps it may yet, and if so I shall have been indebted to you for bringing the matter, however indirectly, to my notice.

Yours sincerely,

A. C. Aitken