54 Braid Road,
Edinburgh 10.

February 5, 1939.

Dear Fisher,

I have just had a little time to look at your problem, and although what I have is incomplete, lacking one final step, I thought I would at once pass over the main ideas to you. You reach the distribution (p. 4 of your Notes)

$$\frac{\frac{n_1+n_2-2}{2}! \cdots \frac{n_1+n_2-p-1}{2}!}{\frac{n_1-2}{2}! \cdots \frac{n_1-p-1}{2}!\, \frac{n_2-2}{2}! \cdots \frac{n_2-p-1}{2}!} \frac{|a_{ij}|^{\frac{n_1-p-1}{2}} |a'_{ij}|^{\frac{n_2-p-1}{2}}}{|A_{ij}|^{\frac{n_1+n_2-p-1}{2}}} da_{11}\, da_{12} \cdots da_{p-1,p}\, da_{pp}.$$

In view of invariance under linear transformation, as you remark, we may take the matrix of the denominator determinant as the unit matrix I, and so we obtain a distribution which I would write in matrix notation

$$df = \text{(Gamma functions etc)}\ |A|^{\frac{1}{2}(n_1-p-1)}\ |I - A|^{\frac{1}{2}(n_2-p-1)}\ da_{11} \cdots da_{pp}.$$

Your $\theta_1, \theta_2, \ldots, \theta_p$ are the latent or characteristic roots of A, given by $|A - \theta I| = 0$. The relevant properties of these for our purpose will be that the elementary symmetric functions, say e_1, $e_2, \ldots, e_p$ of these roots are the sums of minors of $|A|$ symmetrically placed about the diagonal, principal minors. In fact

$$e_1 = \theta_1 + \theta_2 + \ldots + \theta_p = a_{11} + a_{22} + \ldots + a_{pp},$$

$$\sum \theta_i\theta_j = \sum |a_{ii}a_{jj}|,$$ where determinants are indicated by diagonal elements

$$\sum \theta_i\theta_j\theta_k = \sum |a_{ii}a_{jj}a_{kk}|,$$

$$\cdots\cdots\cdots\cdots$$

$$\theta_1\theta_2\theta_3\cdots\theta_p = |A| = |a_{11}a_{22}\cdots a_{pp}|.$$

What we wish to do is to transform your differential element df into a differential in $d\theta_1, d\theta_2, \ldots, d\theta_p$. To this end we can evaluate the Jacobians of the e_k with respect to the θ_k — this turns out with little difficulty to be the difference-product $(\theta_1-\theta_2)(\theta_1-\theta_3)\ldots(\theta_{p-1}-\theta_p)$ which figures in your conjecture,—and of the e_k, in the form of sums of principal minors, with respect to the a_{kk}. This Jacobian will appear to the power -1, and the only difficulty will be to integrate it with respect to the $\frac{1}{2}p(p-1)$ non-diagonal elements a_{ij}, over a range prescribed by given values of the θ's and the positive definiteness of A, i.e. all principal minors of every order must be positive.

As for your other conjectured factors, since the determinant of a matrix is the continued product of its latent roots, we have

$$|A| = \theta_1\theta_2\ldots\theta_p ,$$

while, putting $\theta = 1$ in the characteristic determinant $|A - \theta I|$, or $(-)^p(\theta-\theta_1)(\theta-\theta_2)\ldots(\theta-\theta_p)$, we obtain

$$|I - A| = (1-\theta_1)(1-\theta_2)\ldots(1-\theta_p) ,$$

so that the other factors of your conjectured result are fully accounted for.

Thus there remains only the final step, to show that the integral of the reciprocal of the Jacobian of the sums of principal minors with respect to the diagonal elements a_{kk}, when integrated out with respect to the non-diagonal elements, gives some suitable constant involving π and Gamma-functions. Doubtless this stage will be like the corresponding piece of work done by Ingham for Wishart and Bartlett in establishing the distribution of sampling variances and co-variances from a multivariate normal population. (This needed doing at the time, because Wishart's original paper of 1928 contained gaps, and I was never satisfied by his step from trivariate to multivariate variances by the use of quadratic co-ordinates and rather much analogy. I was actually engaged in attempting to rigorize the proof of that important result when I learned that Ingham had been called in by Wishart and Bartlett and had carried out the necessary integrations, involving a negative power of a positive definite symmetric determinant to be integrated in complex space over all the $\frac{1}{2}n(n+1)$ elements.) But Ingham's determinant was not a Jacobian of the present kind, a very peculiar kind, which I have not met before.

With kind regards,

Yours sincerely,

A.C. Aitken