



Mathematical Institute
16 Chambers Street
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Professor R. A. Fisher, Sc.D., F.R.S.
Cambridge University.

Dear Fisher, I have been delighted to read the award to you, by the Council of the Royal Society, of the Darwin Medal. Please accept my very sincere congratulations.

Last week I had recalled our pleasant correspondence of 1939 on the distribution of latent roots in statistical problems; and I chanced also to have in my hand Wishart's recent survey, in Biometrika, of papers on his product-moment-estimate distribution of 1928. Perusal left me still unsatisfied, for the distribution can be written in matrix notation as

$$d\mu = c_{n,k} \exp \left\{ -\frac{1}{2} \text{tr } n^{-1} V^{-1} \hat{V} \right\} |\hat{V}|^{\frac{1}{2}(n-k-2)} d\hat{V},$$

where "tr" means sum of diagonal elements in the product-matrix concerned, V is variance-matrix of population, \hat{V} its estimate from sample of n observations of k normal variates, and dV means volume element in the $\frac{1}{2}k(k+1)$ -dimensional space of the v_{ij} ; & $C_{n,k}$ involves gamma-functions and the determinant of V . Written in this matrix form the distribution exhibits a perfect analogy with

$$dp = c e^{-\frac{1}{2} \frac{S^2}{\sigma^2}} (S^2)^{-\frac{1}{2}(n-2)} dS^2$$

for one variate.

This set me thinking what was the analogue, in this space, of the gamma-function relation of Euler,

$$\int_0^\infty e^{-st} t^{n-1} dt = \Gamma(n) \cdot s^{-n}$$

in one variable; and a guess suggested:

$$\int e^{-\text{tr}(ST)} |T|^{n-\frac{1}{2}(k+1)} dt = \left(\Gamma\left(\frac{1}{2}\right)\right)^{\frac{1}{2}k(k-1)} \prod_{j=0}^{k-1} \Gamma\left(n-\frac{1}{2}j\right) |S|^{-n},$$

where S and T are positive definite, of k rows and columns. This useful lemma is provable in 4 or 5 lines. Any positive definite matrix can be factorized into XX' , where X is "lower triangular" (i.e.



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all elements above diagonal are zero), and X' is its transpose; & uniquely. But therefore $S = HH'$, $H' TH = U = XX'$, H and X ~~are~~ lower triangular. Then $\text{tr}(ST) = \text{tr}(H U H^{-1}) = \text{tr} U = \text{tr}(XX')$ = sum of squares of all x_{ij} . Also the Jacobian of the transformation $U \rightarrow X$ is lower triangular and seen at once to be $2^k x_{11}^k x_{22}^{k-1} \dots x_{kk}^1$. The Jacobian of $T \rightarrow U$ is by a known theorem in matrices $|H|^{-(k+1)} = |S|^{-\frac{1}{2}(k+1)}$. The integral resolves by these transformations into the product of $\frac{1}{2} k(k+1)$ separate integrals, giving in fact the Γ -functions in the enunciation. Q.E.D.

Now this lemma serves for the inversion of the Laplace transform or moment-generating function of the "Wishart" distribution. For the latter, by simple evaluation of the integral of $\exp \{ -\frac{1}{2} a$ ~~negative~~ positive definite quadratic form) comes out as,

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and has in fact for 17 years been known to be,

$$\left| I + \frac{2VS}{n-1} \right|^{-\frac{1}{2}(n-1)}$$

To get this from the lemma one would require the integrand

$$c \exp \left\{ -\text{tr} \frac{1}{n-1} \left[V^{-1} \left(I + \frac{2VS}{n-1} \right) \hat{V} \right] \right\} |V|^{-\frac{1}{2}(n-1)} \frac{n^{\frac{1}{2}(n-k-2)}}{|V|}$$

$$= c e^{-\text{tr}(S\hat{V})} \cdot e^{-\frac{1}{2}\text{tr}(\frac{1}{n-1} V^{-1} \hat{V})} |V|^{-\frac{1}{2}(n-1)} \frac{n^{\frac{1}{2}(n-k-2)}}{|V|},$$

and, taking away the $e^{-\text{tr}(S\hat{V})}$ that effects the Laplace multiple transform, we have Wishart's distribution displayed in the very neat and suggestive matrix garb, as required.

The Wishart-Bartlett effort of 1933 used not Laplace, but Fourier transforms in multiple complex space, and relied on Ingham's lemma for the reciprocation. It is clear from what I have written that Ingham's lemma admits of very simple proof, in a few lines, by factorizing a positive definite Hermitian matrix into XX' , X triangular. The analogy of the methods, reduced to one variable, is that of Euler's integral for $\Gamma(x)$ to Hankel's contour integral.

Yours ever sincerely,
A.C. Aitken

A.C.A.
22nd. November 1948.

My dear Aitken,

Thanks for your letter giving what at a short reading seems like a very beautiful method of handling multivariate analysis. I do hope you will print it somewhere, remembering to supply that amount of explanation which the weaker brethren require and which unfortunately is too rare in the London Mathematical Society's journal. What do you think of the Comptes Rendu, which is good for short notes and is read by a mathematically appreciative audience?

Yours sincerely,