P.S. I hope some of my work may prove useful to you. A.C.A.

2 Sycamore Terrace,
Corstorphine,
Edinburgh.
Oct. 10, 1932.

Dear Dr Fisher,

I can give you now a somewhat fuller account of what I have been doing in fitting data by the orthogonal polynomials of Least Squares.

On the theoretical side I think the easiest approach is as follows:

Let the data be $u_0, u_1, u_2, \ldots, u_{n-1}$,

and let $\sum_x$ be defined by $\sum_x u = u_0 + u_1 + \cdots + u_{x-1}$.

Then the operators $\sum$ and Δ are reciprocal.

We consider the minimizing of $\sum(V-u)^2$, where $V = V(x)$ is a polynomial of degree $\underline{k}$, say, and we find at once that it is advantageous to express V in terms of polynomials $T_r(x) \equiv T_r$ such that

$$\sum T_r^2 \neq 0, \quad \sum T_r T_s = 0, \quad r > s.$$

This enables us to find $T_r(x)$; for in the orthogonal relation $\sum T_r T_s = 0$ we can write any polynomial of degree s instead of T_s. If we write then $(x+1)(x+2)\cdots(x+s)/s!$, and apply summation by parts we obtain r relations of the form

$$c_1 \Sigma_1 T_r - c_2 \Sigma\Sigma T_r + c_3 \Sigma\Sigma\Sigma T_r - \cdots + (-)^s \Sigma^{s+1} T_r = 0,$$

$$s = 0, 1, 2, \ldots, r-1,$$

where the coefficients are certain factorials in $\underline{n}$. These $\underline{r}$ relations, examined from the lowest upwards, yield at once

$$\Sigma T_r = 0, \quad \Sigma^2 T_r = 0, \quad \cdots, \quad \Sigma^r T_r = 0.$$

This shows that, denoting for a moment $\sum_x{}^r T_r(x)$ by $G_{2r}(x)$, we have

$$T_r(x) = \Delta^r G_{2r}(x),$$

where $G_{2r}(x)$ is such that it itself and its first $r-1$ advancing differences vanish together at the lower limit of summation, of course, $x = 0$, and hence also at the upper limit, $x = n$. Hence $G_{2r}(x)$ must contain the factors $x, x-1, \cdots, x-r+1$, and also $x-n, x-n-1, \cdots x-n-r+1$, and so we may write

$$\text{(A)} \qquad T_r(x) = \Delta^r \{ x_{(r)} (x-n)_{(r)} \},$$

where $x_{(r)}$ denotes the binomial coefficient $x(x-1)\cdots(x-r+1)/r!$. This convention regarding the arbitrary constant multiplier differs from the four others already in existence (your own, Tchebychef's, Jordan's and Ghotimsky's), but it has the numerical advantage that it gives, for $r > 3$, smaller values than any of the other conventions, values which are moreover integers if x is an integer, since the polynomials emerge as differences of products of integer binomial coefficients.

Now the Gregory-Newton interpolation formula, applied to a factorial itself such as $(x-m)_{(r)}$, yields the very convenient form

$$(x-m)_{(r)} = x_{(r)} - x_{(r-1)} m + x_{(r-2)} (m+1)_{(2)} - \text{etc.},$$

so that, expressing $(x-n)_{(r)}$ in (A) above in terms of factorials in $x-r$, annexing the terms so obtained to $x_{(r)}$, and performing Δ^r on the result, using

$$\Delta\, x_{(r)} = x_{(r-1)},$$

we derive easily the Tchebychef-Gram-Jordan expression, as modified by the new convention; namely

$$T_r(x) = (2r)_{(r)}\, x_{(r)} - (2r-1)_{(r)}(n-r)\, x_{(r-1)} + (2r-2)_{(r)}(n-r+1)_{(2)}\, x_{(r-2)}$$

(B)

$$- \cdots + (-)^r (n-1)_{(r)}.$$

In this formula $\Delta^5 T_r(x)$ can be found by simply writing $x_{(i-5)}$ for $x_{(i)}$ throughout. Moreover since (B) as it stands is itself a Gregory-Newton formula of interpolation, the terminal values and differences $T_r(0)$, $\Delta T_r(0)$, $\Delta^2 T_r(0)$, ... are simply the coefficients of $1, x, x_{(2)}, \ldots$ in (B). This evident fact is highly valuable in computation.

The numerical coefficients in (B) can be found very simply in another way. The Legendre polynomial $P_r(x)$, when orthogonal in the range $(0, n)$, is, adopting our convention,

$$P_r(x) = \left(\frac{d}{dx}\right)^r \{x(x-n)\}^r \Big/ (r!)^2.$$

This is ~~in~~ perfect formal analogy with $T_r(x)$, and the numerical coefficients are the same as those of (B).

We also need $\sum T_r^2$. Applying summation by parts r times, and observing that, since $\sum^s T_r = 0$, $s = 1, 2, \cdots, r$, the partial integrate $= 0$ each time, we have

$$\sum T_r^2(x) = (-)^r \frac{(2r)!}{(r!)^2} \sum^{r+1} T_r(x)$$

$$= (-)^r \frac{(2r)!}{(r!)^2} \sum \{ x_{(r)} (x-n)_{(r)} \}.$$

Applying another r summations by parts, in the form

$$\sum_n u_x v_x = u_n v_n - u_0 v_0 - \sum_n (v_{x+1} \Delta u_x),$$

and noting that at every step the factorial coefficient in n of the partial integrate vanishes at $x = 0$ or $x = n$, we derive

$$\sum T_r^2 = (-)^{2r} \frac{(2r)!}{(r!)^2} \sum (x-n+r)_{(2r)}$$

$$= \frac{(2r)!}{(r!)^2} \Big[(x-n+r)_{(2r+1)} \Big]_0^n$$

(C) $$= n(n^2-1)(n^2-4) \cdots (n^2-r^2) / \{(2r+1)(r!)^2\},$$

the desired result.

I propose next to tabulate $T_r(0)$, $\Delta T_r(0)$, ... $r = 0, 1, 2, \ldots$, as well as $\sum T_r^2$, for various values of n, in a triangular scheme as follows, for future use.

$n = 7.$

	0	1	2	3	4	5
$T(0)$	1	-6	15	-20	15	-6
$\Delta T(0)$		2	-15	40	-50	30
$\Delta^2 T(0)$			6	-40	90	-84
$\Delta^3 T(0)$				20	-105	168
$\Delta^4 T(0)$					70	-252
$\Delta^5 T(0)$						252
ΣT_r^2	7	112	756	2400	3850	3024

Such a table is easily prepared by placing together two tables of binomial coefficients, the left fixed, the right depending on n; e.g. for $n = 7$,

1	-1	1	-1	1	-1
	2	-3	4	-5	6
		6	-10	15	-21
			20	-35	56
				70	-126
					252

and

1	6	15	20	15	6
	1	5	10	10	5
		1	4	6	4
			1	3	3
				1	2
					1

and simply multiplying corresponding elements. Also ΣT_r^2 can be computed from (C). I computed tables like the above for n up to 25, and other similar sets for the central case which I shall later describe, in some four hours. A single table can be computed in a few minutes.

Regarding the application of such tables I propose to modify your own method in various respects:

Let us denote by $m_{(r)}$ the r^{th} reduced factorial moment of u_x, i.e.

$$m_{(r)} \equiv \sum x_{(r)} u_x .$$

Then the formula for finding the coefficients in the expansion of V in orthogonal polynomials, viz.

$$a_r \sum T_r^2 = \sum u T_r ,$$

yields at once, with (B),

$$a_r \sum T_r^2 = (2r)_{(r)} m_{(r)} - (2r-1)_{(r)} (n-r) m_{(r-1)} + \cdots + (-)^r (n-1)_{(r)} m_{(0)} .$$

In particular, for the first few values of r, we have

$$\begin{aligned} n a_0 &= m_{(0)} , \\ a_1 \sum T_1^2 &= -(n-1) m_{(0)} + 2 m_{(1)} , \\ a_2 \sum T_2^2 &= (n-1)_{(2)} m_{(0)} - 3(n-2) m_{(1)} + 6 m_{(2)} , \text{ etc.} \end{aligned}$$

Now this triangular set of coefficients on the right, applied to the moments, is, in virtue of the remark made earlier about the formula (B), exactly the same as the table of terminal values and differences of T_r, only <u>turned through a right angle</u>, <u>rows</u> replacing <u>columns</u>. Consequently, having found (by simple <u>repeated summation</u>) the factorial moments $m_{(i)}$, we shall affect them by the <u>columns</u> of our triangular table, in order to find the <u>a</u>'s, and then, having found these, by the <u>rows</u>, in order to find the terminal values and differences of the graduated V's in the form

$$\begin{aligned} V_0 &= a_0 + a_1 T_1(0) + a_2 T_2(0) + \cdots \\ \Delta V_0 &= \qquad a_1 \Delta T_1(0) + a_2 \Delta T_2(0) + \cdots \\ \Delta^2 V_0 &= \qquad\qquad a_2 \Delta^2 T_2(0) + \cdots , \end{aligned}$$

whence finally the V's can be built up by summation. At the same time we have an easy way of calculating the residual variance R^2 in the familiar manner from the a's as they are found, and further, several very neat and useful checks, depending on the following easily proved results:

$$\begin{aligned} V_{n-1} &= a_0 - a_1 T_1(0) + a_2 T_2(0) - \cdots \\ \Delta V_{n-2} &= \qquad a_1 \Delta T_1(0) - a_2 \Delta T_2(0) + \cdots \\ \Delta^2 V_{n-3} &= \qquad\qquad a_2 \Delta^2 T_2(0) - \cdots , \end{aligned}$$

that is to say, we can check the <u>remote</u> terminal values and differences, if we please, by applying our triangle of multipliers, at the top of p. 5, with alternate signs changed, in fact <u>with positive signs exclusively</u>. One can at any rate obtain an almost certain check by computing V_{n-1} in this manner.

The reduced factorial moments $m_{(r)}$ are found by repeated summation in columns, neglecting one entry at the top each time. On a machine with plenty of accommodation one can do this two columns at a time, but in any case it is a simple and rapid process.

A small example will show all the preceding theory in operation.

x	u	$m_{(0)}$	$m_{(1)}$	$m_{(2)}$	$m_{(3)}$	$m_{(4)}$	u^2
0	17	<u>750</u>					289
1	40	733	<u>4412</u>				1600
2	47	693	3679	<u>13140</u>			2209
3	49	646	2986	9461	<u>24244</u>		2401
4	52	597	2340	6475	14783	<u>29747</u>	2704
5	69	545	1743	4135	8308	14964	4761
6	111	476	1198	2392	4173	6656	12321
7	123	365	722	1194	1781	2483	15129
8	127	242	357	472	587	702	16129
9	115	115	115	115	115	115	13225
							Σ = 70768

Now apply the triangle for $n = 10$.

	$a_{(r)}$: 75	6.285	−0.0101	−0.1100	−0.07662	V_0	ΔV_0	$\Delta^2 V_0$	$\Delta^3 V_0$	$\Delta^4 V_0$
$m_{(r)}$: 750	1	−9	36	−84	126	17.68				
4412		2	−24	112	−280		21.89			
13140			6	−70	315			−16.43		
24244				20	−210				13.85	
29747					70					−5.35
$\Sigma\pi_r^2$	10	330	4752	34320	140140					

$V_9 = 112.3.$

For example, $a_2 = [36 \times 750 - 24 \times 4412 + 6 \times 13140] / 4752 = -\frac{48}{4752}$

$= -0.01010.$

$\Delta^2 V_0 = -0.0101 \times 6 - 0.1100 \times 70 - 0.07662 \times 315 = -16.43,$

$V_9 = 75 + 6.285 \times 9 - 0.0101 \times 36 - 0.1100 \times 84 - 0.07662 \times 126$

$= 112.33.$

We thus have V_0 and its terminal differences. Building a table in the ordinary way we find

	0	1	2	3	4	5	6	7	8	9
V	17.7	39.6	45.0	47.9	56.7	74.6	99.3	123.6	133.9	112.5;

the last value checking with $V_9 = 112.3$ as found above.

While engaged on the $\bar{a}$'s we find the residual variance step by step in the usual manner

		$\Sigma u^2 = 70768.$	R^2
a_0	$= 750/10 = 75.$	$75 \times 750 = 56250.$	14518
a_1	$= 2074/330 = 6.285$	$6.285 \times 2074 = 13035$	1483
a_2	$= -48/4752 = -0.0101$	$-0.0101 \times -48 = 0.5$	1483
a_3	$= -3776/34320 = -0.1100$	$-0.1100 \times -3776 = 415$	1068
a_4	$= -10710/140140 = -0.07642$	$-0.07642 \times -10710 = 818$	250,

and may divide by the degrees of freedom.

All of the above work refers to the non-symmetrical case. I shall write in another letter about the symmetrical case, which is to my mind simpler and twice as fast. It involves central and mean central sums, which are decidedly smaller in numerical magnitude than end-to-end sums, while the triangular tables which I then employ have zeros in alternate places and so involve only half the work. The theory is unfortunately rooted in the idiom of the central difference calculus, which cannot be said to be as easy as that of advancing differences.

Yours sincerely,

A. C. Aitken.

2 Sycamore Terrace,
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Edinburgh.
Oct 10, 1932.

Dear Dr Fisher,

To continue my former letter, we derive central expressions for $T_r(x)$ in the following manner. Taking the formula (A) of my letter in the shape

$$\text{(D)} \qquad T_r(\xi) = \Delta^r \left\{ \xi_{(r)} (\xi - n)_{(r)} \right\},$$

altering the origin to the centre of the data by taking a new x such that $x = \xi - \frac{1}{2}(n-1)$, and at the same time transferring ordinary into central differences by $\Delta u_x = \delta u_{x+\frac{1}{2}}$, we derive (henceforth letting $T_r(x)$ be the polynomial in central shape), if $n = 2q$,

$$\text{(E)} \quad T_r(x) = \frac{1}{(r!)^2} \delta^r \left\{ \left(x^2 - \overline{q+\tfrac{r}{2}}^2\right)\left(x^2 - \overline{q+\tfrac{r}{2}-1}^2\right) \cdots \left(x^2 - \overline{q-\tfrac{r}{2}}^2\right) \right\}.$$

This is analogous to Rodrigues' formula for Legendre polynomials over the range $(-q, q)$, namely

$$P_r(x) = \frac{1}{(r!)^2} \left(\frac{d}{dx}\right)^r \left(x^2 - q^2\right)^r.$$

Now the operand on the right of (E) is not as it stands a pure central factorial polynomial, but if it is regarded as a function of q, it can be expressed in central factorials by one or other of the central interpolation formulas, which yield identities of the type, e.g. for $r = 2$,

$$(x^2-\overline{q+\tfrac12}^2)(x^2-\overline{q-\tfrac12}^2) = x^2(x^2-1) - 2(q^2-\tfrac14)(x^2-1) + (q^2-\tfrac14)(q^2-\tfrac94),$$

$$= (x^2-\tfrac14)(x^2-\tfrac94) - 2(q^2-1)(x^2-\tfrac14) + q^2(q^2-1),$$

or for $r = 3$,

$$(x^2-\overline{q+1}^2)(x^2-q^2)(x^2-\overline{q-1}^2) = x^2(x^2-1)(x^2-4) - 3(q^2-1)x^2(x^2-1)$$

$$+ 3q^2(q^2-1)(x^2-1) - q^2(q^2-1)(q^2-4)$$

$$= (x^2-\tfrac14)(x^2-\tfrac94)(x^2-\tfrac{25}{4}) - 3(q^2-\tfrac94)(x^2-\tfrac14)(x^2-\tfrac94)$$

$$+ 3(q^2-\tfrac14)(q^2-\tfrac94)(x^2-\tfrac94) - (q^2-\tfrac14)(q^2-\tfrac94)(q^2-\tfrac{25}{4}),$$

etc.

If then we substitute such expressions in (E), and operate with δ^r, using properties of central factorials, we derive centred Tchebychef polynomials, namely;

$$1.$$

$$2x.$$

$$6x^2/2! - (q^2-\tfrac14), \quad \text{or} \quad 6(x^2-\tfrac14)/2! - (q^2-1);$$

$$20x(x^2-1)/3! - 2(q^2-\tfrac94)x, \quad \text{or} \quad 20x(x^2-\tfrac14)/3! - 2(q^2-1)x;$$

$$70x^2(x^2-1)/4! - 5(q^2-\tfrac94)x^2/2! + \tfrac14(q^2-\tfrac14)(q^2-\tfrac94),$$

$$\text{or} \quad 70(x^2-\tfrac14)(x^2-\tfrac94)/4! - 5(q^2-4)(x^2-\tfrac14)/2! + \tfrac14(q^2-\tfrac14)(q^2-\tfrac94);$$

and so on, which have the <u>same numerical</u> coefficients as the <u>Legendre</u> polynomials <u>defined by Rodrigues</u>

formula, and can be written down from such Legendre polynomials in a very simple manner. The two forms are appropriate to the cases $\underline{n}$ odd or even.

Now these expressions are, as they stand, simply Newton-Stirling or Newton-Bessel central interpolation formulae. Hence the coefficients of the central factorials in x are just the central or <u>mean</u> central values and differences of $T_r(x)$ at the central origin $x = 0$, so that we can proceed as before to construct triangular schemes of central and mean central values and differences. Moreover, since now we are dealing with purely even and purely odd functions, zeros will appear in our tables in each <u>alternate</u> place. The tables look like this:

$n = 6.$

r	0	1	2	3	4
$\mu T(0)$	1	·	−8	·	10
$\delta T(0)$		2	·	−16	·
$\mu\delta^2 T(0)$			6	·	−25
$\delta^3 T(0)$				20	·
$\mu\delta^4 T(0)$					70
ΣT_r^2	6	70	336	720	700

$n = 7.$

r	0	1	2	3	4	5
$T(0)$	1	·	−12	·	30	·
$\mu\delta T(0)$		2	·	−20	·	30
$\delta^2 T(0)$			6	·	−50	·
$\mu\delta^3 T(0)$				20	·	−84
$\delta^4 T(0)$					70	·
$\mu\delta^5 T(0)$						252
ΣT_r^2	7	112	756	2400	3850	3024

The formation of such a table is simple. We place on the left a fixed triangular scheme containing the coefficients of the Legendre polynomials arranged in columns; on the right a variable triangle depending on q, thus, multiplying corresponding elements:

$$\begin{array}{cccccc} 1 & \cdot & -1 & \cdot & \frac{1}{4} & \cdot \\ & 2 & \cdot & -2 & \cdot & \frac{1}{2} \\ & & 6 & \cdot & -5 & \cdot \\ & & & 20 & \cdot & -14 \\ & & & & 70 & \cdot \\ & & & & & 252 \end{array} \quad \text{with} \quad \begin{array}{cccccc} 1 & \cdot & q^2-1 & \cdot & (q^2-1)(q^2-4) & \cdot \\ & 1 & \cdot & q^2-1 & \cdot & (q^2-1)(q^2-4) \\ & & 1 & \cdot & q^2-4 & \cdot \\ & & & 1 & \cdot & q^2-4 \\ & & & & 1 & \cdot \\ & & & & & 1 \end{array}$$

$$\text{or with} \quad \begin{array}{cccccc} 1 & \cdot & q^2-\frac{1}{4} & \cdot & (q^2-\frac{1}{4})(q^2-\frac{9}{4}) & \cdot \\ & 1 & \cdot & q^2-\frac{9}{4} & \cdot & (q^2-\frac{9}{4})(q^2-\frac{25}{4}) \\ & & 1 & \cdot & q^2-\frac{9}{4} & \cdot \\ & & & 1 & \cdot & q^2-\frac{9}{4} \\ & & & & 1 & \cdot \\ & & & & & 1 \end{array},$$

according as n, $= 2q$, is even or odd. The values of $\sum T_r^2$ are the same as before.

In order to use such triangular schemes analogously to the non-symmetrical procedure, we shall introduce central and mean central reduced factorial moments, defined by summing from $-q$ to q the values of u_x multiplied respectively by the central factorial polynomials which appear in the Newton-Stirling or in the Newton-Bessel interpolation formulae, as n is odd or even. We shall denote these reduced central moments by $m_{[i]}$ and $m_{[i]'}$, e.g. if n is odd,

$$m_{[4]'} = \sum_{-q}^{q} x^2(x^2-1)\, u_x / 4!\, ,$$

while if n is even,

$$m_{[4]} = \sum (x^2 - \tfrac{1}{4})(x^2 - \tfrac{9}{4}) u_x \,/\, 4! .$$

Such central factorial moments are skimmed over quickly even by such an authority as Steffensen, but they are very valuable and can be computed by repeated summation from the termini towards the centre in the manner I shall now describe by carrying out the former example.

$n = 10.$

x	u	$m_{[0]}$	$m_{[1]}$.	$m_{[2]}$	$m_{[3]}$	$m_{[4]}$.	u^2 as before.
$-\frac{9}{2}$	17	17	17	17	17	17	
$-\frac{7}{2}$	40	57	74	91	108	125	
$-\frac{5}{2}$	47	104	178	269	377	502	
$-\frac{3}{2}$	49	153	331	600	(677)		
$-\frac{1}{2}$	52	205	(433.5)				
$\frac{1}{2}$	69	545	(1470.5)				
$\frac{3}{2}$	111	476	1198	2392	(2977)		
$\frac{5}{2}$	123	365	722	1194	1781	2483	
$\frac{7}{2}$	127	242	357	472	587	702	
$\frac{9}{2}$	115	115	115	115	115	115	
	750						$\Sigma = 70768$

[Mean sums are bracketed, and are obtained by adding only half of the last summand in the preceding column.]

Then $m_{[0]} = 750$, $m_{[1]}' = 1470.5 - 433.5 = 1037$, $m_{[2]} = 2392 + 600$

$= 2992$, $m_{[3]}' = 2977 - 677 = 2300$, $m_{[4]} = 2483 + 502 = 2985$.

The triangular scheme for $n = 10$ is now brought into operation and used very much as before:

	a_i									
	75 6285		−.0401	−0.1100	−0.07602	μV_0	δV_0	$\mu\delta^2 V_0$	$\delta^3 V_0$	$\mu\delta^4 V_0$
7500	1	·	−24	·	126	65.61				
$^m[i]$ 1037		2	·	−48	·		17.85			
2992			6	·	−105			7.96		
2300				20	·				−2.20	
2985					70					−5.35
ΣT_r^2	10	330	4752	34320	140140					

I do not need to describe the steps, which are as in the former letter. Because of the blank places in the scheme, the work is much less. The $\underline{a}$'s of course turn out to be the same as before; they and the T's are numerically invariant, whatever origin we choose. The residual variance is also computed as before.

We have therefore arrived at mean central and central values and differences at the centre, and must from these build up the graduated V's. In my opinion the quickest way is to split up V into an <u>even</u> function V plus an <u>odd</u> function W; then the even central differences of V_0 will be those of V_0 also, while the odd - - - - - - - - - - - - - - - - - - - W_0. Hence, using the symmetry or alternation of V & W about the origin, we build from the given central differences two half tables, as shown on the next page, and by adding and subtracting the values of V and W we obtain V.

The procedure for an odd number of data is ~~for~~ similar, except that "mean central" and "central" change places in all that concerns sums and differences, and we must keep in mind that there is a <u>central</u> datum. I need not therefore give another example. $V_x = V_{-x}$. $W_x = -W_{-x}$.

x	V	δV	$\delta^2 V$	$\delta^3 V$	$\delta^4 V$	W	δW	$\delta^2 W$	$\delta^3 W$
		0		0			17.85		−2.20
$\frac{1}{2}$	65.61		7.96		−5.35	8.93		−1.10	
		7.96		−5.35			16.75		−2.20
$\frac{3}{2}$	73.57		2.61		−5.35	25.68		−3.30	
		10.57		−10.70			13.45		−2.20
$\frac{5}{2}$	84.14		−8.09		−5.35	39.13		−5.50	
		2.48		−16.05			7.95		−2.20
$\frac{7}{2}$	86.62		−24.14			47.08		−7.70	
		−21.66					0.25		
$\frac{9}{2}$	64.96					47.33			

Hence we have

x	$-\frac{9}{2}$	$-\frac{7}{2}$	$-\frac{5}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$
$U = V \pm W$	17.6	39.5	45.0	47.9	56.7	74.5	99.3	123.3	133.7	112.3,

practically as before.

Personally, I greatly prefer these central methods.

I have made some headway with weighted data, & non-equidistant data; and have a theoretical, though as yet not quite a practical solution, of the case of correlated equidistant data. The solution of the latter involves a generalised type of orthogonal polynomials, in which the correlation quadratic form takes the place of the sum of squares in $\sum T_r^2 \neq 0$, and the polarized quadratic (or symmetrical bilinear form) takes the place of $\sum T_r T_s = 0$; but the calculation of the polynomials is not simple. However I do not despair. I hope all this is of use and interest to you.

Yours sincerely,

A. C. Aitken.