2 Sycamore Terrace,
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EDINBURGH.
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Dear Dr Fisher,

I do not know whether you still retain any interest in the problem of fitting polynomials to observations by least squares, or whether you are in the way of requiring applications of it in your work ; but I thought you might have a slight interest in a few ideas which I have had for some time past on the practical and numerical side. Recently, my curiosity aroused by some not altogether well-informed remarks in some American papers, I read right through the extant literature, from Tchebychef through Gram to Jordan and Chotimsky (who proves to be merely Tchebychef made more diffuse), and emerged pretty well surfeited on the theoretical side, and yet desiring something better on the practical. It appeared to me that we must at all costs avoid heavy multiplications, and cling to the advantages of your method of successive summations ; and I found two very simple methods, one non-symmetrical, with the origin at one end of the data, the other symmetrical, with the origin in the middle.

Miss Allan obtained a form of the Tchebychef polynomials containing central factorial polynomials ; but I find there are two such forms, containing alternately central and mean central factorials, in fact those which occur in the Stirling and Bessel formulae of interpolation, and like the latter, adapted to the cases where the number of data is odd or even. The numerical coefficients are simply those of the Legendre polynomials. The non-symmetrical form of the polynomials, in terms of descending factorials, has been given by Tchebychef, Gram, Jordan and others.

The whole point however lies in the type of summation chosen in order to obtain suitable factorial moments, descending, central or mean central, and this is where I have tried to innovate. I shall take a small example of 10 data, though a larger number of data merely involves longer columns in the summations.

EXAMPLE. $n = 10$.

x	0	1	2	3	4	5	6	7	8	9	Σ
u	17	40	47	49	52	69	111	123	127	115	750

* Notice the descent of a step each time.

x	u	$\Sigma = m_{(0)}$	$\Sigma^2 = m_{(1)}$	$\Sigma^3 = m_{(2)}$	$\Sigma^4 = m_{(3)}$	$\Sigma^5 = m_{(4)}$	u^2
0	17	750	*				289
1	40	733	4412				1600
2	47	693	3679	13140			2209
3	49	646	2986	9461	24244		2401
4	52	597	2340	6475	14783	29747	2704
5	69	545	1743	4135	8308	14964	4761
6	111	476	1198	2392	4173	6656	12321
7	123	365	722	1194	1781	2483	15129
8	124	242	357	472	687	702	16129
9	115	115	115	115	115	115	13225
							$\Sigma = 70768.$

I now take a certain triangular table of values, simply the terminal values of the T-polynomials in a first row, and their successive differences in later rows, and build the computation around it, as follows:

	.... a_r ...									
$m_{(r)}$	75	6.285	-0.01010	-0.1100	-0.07642	V_0	ΔV_0	$\Delta^2 V_0$	$\Delta^3 V_0$	$\Delta^4 V_0$
750	1	-9	36	-84	126	14.66				
4412		2	-24	112	-280		21.89			
13140			6	-70	315			-16.43		
24244				20	-210				1385	
29747					70					-5.35
$\Sigma T_r^2 =$	10	330	4752	34320	140140	$V_9 = 112.3.$ (for check)				

For example, $a_2 = (36 \times 750 - 24 \times 4412 + 6 \times 13140)/4752 = \frac{-48}{4752}$
$= -.01010.$

$$\Delta^2 V_0 = -0.0101 \times 6 - 0.1100 \times (-70) - 0.07642 \times 315 = -16.43.$$

$$V_9 = 75 + 6.285 \times 9 - 0.0101 \times 36 - 0.1100 \times 84 - 0.07642 \times 126 = 112.33.$$

I think the method of using the columns, ~~and~~ to find the coefficients a_i of T_i in the fitted polynomial, and then the rows of the same table, to find the terminal value and differences of the fitted data (from which all other values can be found by building up), is clear from the above ; and while engaged upon the a_i we find

$$a_0 = 750/10 = 75. \qquad a_0 \Sigma u = 75 \times 750 = 56250.$$
$$a_1 = 2074/330 = 6.285. \qquad a_1 \Sigma u T_1 = 6.285 \times 2074 = 13035.$$
$$a_2 = -48/4752 = -0.0101. \qquad a_2 \Sigma u T_2 = -0.0101 \times (-48) = 0.5,$$
$$a_3 = -3776/34320 = -0.1100. \qquad a_3 \Sigma u T_3 = -0.1100 \times (-3776) = 415,$$
$$a_4 = -10710/140140 = -0.07642. \qquad a_4 \Sigma u T_4 = -0.07642 \times (-10710) = 818,$$

which gives us the amounts to subtract at each step from Σu^2 in order to obtain the residual variance in the manner which is well known to you. We find indeed :

		9	8	7	6	5	n-r-1
	Σu^2						
	70768	-56250	-13035	0	-415	-818	
R^2		14518	1483	1483	1068	250 .	

By building up the difference table from terminal values we arrive at the quartic graduation

x	0	1	2	3	4	5	6	7	8	9	
u	17	40	47	49	52	69	111	123	127	115	
U	17.7	39.6	45.0	47.9	56.7	74.6	99.3	123.4	133.9	112.5	
|U-u|	0.7	0.4	2.0	1.1	4.7	5.6	11.7	0.4	6.9	2.5	R^2
$(U-u)^2$	0.5	0.2	4.0	1.2	22.1	31.4	136.9	0.2	47.6	6.3	= 250.

There are two useful checks. One is afforded by the agreement of the residual R^2, 250, with the former determination ; the other by the fact that the remote terminal value, U_9, as found from the first column of my table with plus signs instead of minus, agrees with that found by building up the difference table. Actually there are other checks, if one cares to apply them, namely that the remaining rows of the table also taken with plus signs, will give ΔU_8, $\Delta^2 U_7$, $\Delta^3 U_6$ and $\Delta^4 U_5$, that is, the terminal differences at the far end of the table.

These tables in triangular shape, which can be used in the manner above upon the successive sums (descending factorial moments), are very easily constructed for a particular n. All we have to is to juxtapose a fixed triangle of binomial coefficients, the one on the left below, and another (the one on the right) which depends on n, and consists simply of binomial coefficients of exponent n-1, n-2, n-3,.. put in rows below each other as shown. The example is for n = ~~56~~7.

n = 7.

1	-1	1	-1	1	-1		1	6	15	20	15	6
	2	-3	4	-5	6			1	5	10	10	5
		6	-10	15	-21				1	4	6	4
			20	-35	56	and				1	3	3
				70	-126						1	2
					252							1 .
					etc.							

By multiplying corresponding elements we have the desired triangle of terminal values and differences of the orthogonal polynomials, taken, I may say, with a convention as to numerical constant which differex from any hitherto employed, and which has the effect of giving exclusively integers, and smaller numbers. I define the unsymmetrical T-polynomial as

$$\text{(A)} \qquad T_r(x) = \Delta^r\,[x_{(r)} . (x-n)_{(r)}],$$

where $x_{(r)}$ means $x(x-1)\ldots(x-r+1)/r!$. I will not trouble you, however, with theoretical derivations, except to say what will, I think, please you, that all that heavy stuff of several Continental writers seems beside the mark, formulae like the above, and properties, tumbling out easily enough in a few lines.

If I change the x in (A) above to the centre as origin, and at the same time observe that $\Delta u_x = \delta u_{x+\frac{1}{2}}$, putting also $n = 2q$, I obtain central T-polynomials defined by

$$\hat{T}_r(x) = \frac{1}{(r!)^2}\,\delta^r\left\{(x^2-\overline{q+\tfrac{r}{2}}^{\,2})(x^2-\overline{q+\tfrac{r}{2}-1}^{\,2})\ldots(x^2-\overline{q-\tfrac{r}{2}}^{\,2})\right\},$$

and by expressing the operand on the right in terms of central factorials by means of a (hardly known) central analogue of Vandermondés theorem in binomial coefficients, I arrive once at at centred Tchebychef forms, which I have placed opposite the Legendre polynomials for comparison.

Legendre. $(-q, q)$.	Tchebychef.
1.	1.
$2x$.	$2x$.
$6x^2/2! - q^2$.	$6x^2/2! - (q^2-\frac{1}{4})$,
	or $6(x^2-\frac{1}{4})/2! - (q^2-1)$.
$20x^3/3! - 2q^2x$.	$20x(x^2-1)/3! - 2(q^2-\frac{9}{4})x$,
	or $20x(x^2-\frac{1}{4})/3! - 2(q^2-1)x$.
$70x^4/4! - 5q^2x^2/2! + \frac{1}{4}q^4$.	$70x^2(x^2-1)/4! - 5(q^2-\frac{9}{4})x^2/2! + \frac{1}{4}(q^2-\frac{1}{4})(q^2-\frac{9}{4})$,
	or $70(x^2-\frac{1}{4})(x^2-\frac{9}{4}) - 5(q^2-4)(x^2-\frac{1}{4})/2!$
	$\rightarrow + \frac{1}{4}(q^2-1)(q^2-4)$, etc.

Since the factorial polynomials which occur are central and mean central factorials, these expressions are their own central difference (Stirling or Bessel) interpolation formulae, and so the central or mean central (average of two middle) values and differences are just the coefficients of the factorials. Thus, for a given value of n = 2q we may construct triangular tables of central or mean central values and differences, much easier to work with than the former ones, since they will have zeros in every alternate place. We shall apply these triangles to central or mean central sums (factorial moments), and the work will appear as follows. Taking the same data as before we have a scheme for central or mean central sums, the latter, in brackets, involving only half of the last summand :

x	u	$\hat{\Sigma}$	$\mu\hat{\Sigma}^2$	$\hat{\Sigma}^3$	$\mu\hat{\Sigma}^4$	$\hat{\Sigma}^5$
$-\frac{9}{2}$	17	17	17	17	17	17
$-\frac{7}{2}$	40	57	74	91	108	125
$-\frac{5}{2}$	47	104	178	269	377 (677)	502
$-\frac{3}{2}$	49	153	331 (433.5)	600		
$-\frac{1}{2}$	52	205				
$\frac{1}{2}$	69	545				
$\frac{3}{2}$	111	476	(1440.5) 1198	2392		
$\frac{5}{2}$	123	365	722	1194	(2977) 1781	2483
$\frac{7}{2}$	127	242	357	472	587	702
$\frac{9}{2}$	115	115	115	115	115	115
	750					

u^2 as before & Σu^2.

Thus $m_{[0]} = 750$, $m_{[1]'} = 1440.5 - 433.5 = 1037$, $m_{[2]} = 2392 + 600 = 2992$,

$m_{[3]'} = 2977 - 677 = 2301$, $m_{[4]} = 2483 + 502 = 2985$.

Taking the appropriate triangle of central values and differences of the polynomials for n = 10 and building around it we have, with half the work required before :

$m_{[r]}$	75	6.285	-0.0101	-0.1100	-0.07642	μV_0	δW_0	$\mu\delta^2 V_0$	$\delta^3 W_0$	$\mu\delta^4 V_0$
750	1	.	-24	.	126	65.61				
1037		2	.	-48	.		17.85			
2992			6	.	-105			7.96		
2301				20	.				-2.20	
2985					40					-5.35
ΣT_r^2 : 10		330	4752	34320	140140					

$V_{\frac{1}{2}} = 65.61$,

$W_{\frac{1}{2}} = \frac{1}{2}(17.85) = 8.93$, $\delta^2 W_{\frac{1}{2}} = -1.10$.

From considerations of symmetry the values and differences obtained belong respectively to the even and odd parts, say V and W, of the fitted function U; we can therefore build two half tables for these separately, thus :

x	V	δ	δ^2	δ^3	W	δ	δ^2	δ^3
→		0		0	(−8.93)	17.85	(1.10)	−2.20
$\frac{1}{2}$	65.61		7.96		8.93		−1.10	
		7.96		−5.35		16.75		−2.20
$\frac{3}{2}$	73.57		2.61		25.68		−3.30	
		10.57		−10.70		13.45		−2.20
$\frac{5}{2}$	84.14		−8.09		39.13		−5.50	
		2.48		−16.05		7.95		−2.20
$\frac{7}{2}$	86.62		−24.14		47.08		−7.70	
		−21.66				.25		
$\frac{9}{2}$	64.96				47.33			

and then taking the sum and the difference of the V's and W's, for the positive and negative directions from the centre, we have

x	$-\frac{9}{2}$	$-\frac{7}{2}$	$-\frac{5}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$
U	17.6	39.5	45.0	47.9	56.7	74.5	99.3	123.3	133.7	112.3

in good agreement with the earlier results.

I do not know whether this central method, with its touches of sophistication in the mean central sums and the odd and even parts of U, would appeal to the laity as much as the blunt one-way method earlier described ; but I find it certainly twice as rapid. The procedure for an odd number of data is almost the same, but I may as well exemplify it. If we extrapolate an eleventh fitted datum, we get the value 35, approx. Hence if we throw this in with the unfitted data and proceed to graduate, we should reproduce the 35 and at the same time get the other results. So I take

x	−5	−4	−3	−2	−1	0	1	2	3	4	5	Σ
u	17	40	44	49	52	69	111	123	127	116	35	785

The calculation of central and mean central sums is then :

x	u	$\mu\hat{\Sigma}.$	$\hat{\Sigma}^2.$	$\mu\hat{\Sigma}^3.$	$\hat{\Sigma}^4.$	$\mu\hat{\Sigma}^5.$
−5	17	17	17	17	17	17
−4	40	57	74	91	108	125
−3	47	104	178	269	377	502
						(990.5)
−2	49	153	331	600	977	
				(666)		
−1	52	205	536			
		(230.5)				
→ 0	69					
		(545.5)				
1	111	511	1373			
				(2230.5)		
2	123	400	862	1544	2481	(2467.5)
3	127	277	462	682	937	1227
4	115	150	185	220	255	290
5	35	35	35	35	35	35
	785					

u^2 as before.

giving $m_{\{0\}'} = 785,\ m_{\{1\}} = 1373 - 536 = 837,\ m_{\{2\}'} = 2231.5 + 868 = 3098.5,$

$m_{\{3\}'} = 2481 - 977 = 1504,\quad m_{\{4\}'} = 2467.5 + 990.5 = 3458.$

Taking the central scheme for n = 11, we build round as before.

$m_{\{r\}}$	71.36	3.805	−0.6433	−0.2446	−0.07672	V_0	$\mu\delta W_0$	$\delta^2 V_0$	$\mu\delta^3 W_0$	$\delta^4 V_0$
785	1	·	−30	·	210	74.51				
837		2	·	−56	·		21.31			
3098.5			6	·	−140			6.89		
1504				20	·				−4.89	
3458					70					−5.37
ΣT_r^2	11	440	4722	16640	350350					

$\delta V_{\frac{1}{2}} = \frac{1}{2}(6.89-) = 3.44,$

$\delta^3 V_{\frac{1}{2}} = \frac{1}{2}(-5.37) = -2.69.$

The two half-tables of differences for V and W will then be :

x	V	δ	δ^2	δ^3	δ^4
0	74.51		6.89		−5.37
		3.44		−2.69	
1	77.95		4.20		−5.37
		7.64		−8.06	
2	85.59		−7.66		−5.37
		3.78		−13.43	
3	89.37		−17.29		−5.37
		−13.51		−18.80	
4	75.86		−36.09		
		−49.60			
5	26.26				

W	δ	δ^2	δ^3
0		0	−4.89
	21.31		
21.31		−4.89	−4.89
	16.42		
37.73		−9.78	−4.89
	6.64		
44.37		−14.67	−4.89
	−8.03		
36.74		−19.56	
	−27.59		
8.75			

,

and by addition and subtraction of the V's and W's we get

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
U	17.5	39.5	45.0	47.9	56.6	74.5	99.3	123.3	133.7	112.2	35.0

in good agreement with the earlier results.

I should have said that the divisors which I put at the foot of the columns in the triangular schemes are the values

$$\sum T_r^2(x;n) = n(n^2-1)(n^2-4)\cdots(n^2-r^2)\Big/\{(r!)^2(2r+1)\};$$

so $\sum T_0^2 = n$; $\displaystyle\sum_n T_r^2(x;n) = \frac{n+r}{n-r-1}\sum_{n-1} T_r^2(x;n-1) = \frac{(2r-1)(n^2-r^2)}{(2r+1)r^2}\sum_n T_{r-1}^2(x;n).$

Triangular schemes for the central cabe are easily constructed. If n = 2q is odd, we juxtapose

$$\begin{array}{cccccc}
1 & \cdot & -1 & \cdot & \frac{1}{4} & \\
 & 2 & \cdot & -2 & \cdot & 1 \\
 & & 6 & \cdot & -5 & \cdot \\
 & & & 20 & \cdot & -14 \\
 & & & & 70 & \cdot \\
 & & & & & 252
\end{array}
\qquad \text{and} \qquad
\begin{array}{cccccc}
1 & \cdot & q^2-\frac{1}{4} & , & (q^2-\frac{1}{4})(q^2-\frac{9}{4}) & \cdot \\
 & 1 & \cdot & q^2-\frac{1}{4} & \cdot & (q^2-\frac{9}{4})(q^2-\frac{25}{4}) \\
 & & 1 & \cdot & q^2-\frac{9}{4} & \cdot \\
 & & & 1 & \cdot & q^2-\frac{25}{4} \\
 & & & & 1 & \cdot \\
 & & & & & 1
\end{array}$$

and multiply corresponding elements. If n = 2q is even we use

$$\begin{array}{cccccc}
1 & \cdot & q^2-1 & \cdot & (q^2-1)(q^2-4) & \cdot \\
 & 1 & \cdot & q^2-1 & \cdot & (q^2-1)(q^2-4) \\
 & & 1 & \cdot & q^2-4 & \cdot \\
 & & & 1 & \cdot & q^2-4 \\
 & & & & 1 & \cdot \\
 & & & & & 1
\end{array}, \text{\&c.}$$

instead of the second triangle.

I do not know whether any of this will appeal to you ; but I have tried it out solidly on the machine, and can vouch for its efficiency. I have prepared a paper on this subject, with worked examples, together with a bibliography and commentary on all the papers I have read.

I am,
Yours sincerely,

A. C. Aitken.

I have computed tables of polynomials and differences, for terminal and central cases, with values of ΣT_r^2 , for n = 4 to 25. These took less than four hours to compute and check, and I can let you have a ms. copy for your personal possession, if you feel any interest in the matter. The tables are set out in triangular schemes, with an explanatory preface and description of how to compute and check for any special n.

A. C. A.