30th. May 1946.

Dear Brück,

I am enclosing the method I spoke of, which may be perfectly well known in the literature, or which you may think unrealistic in some other way. The mathematical approach did, however, seem to me interesting.

Yours sincerely,

Relative to arbitrary orthogonal axes, the direction cosines of an object at two epochs are

$$p, q, r, \qquad \text{and} \qquad p', q', r',$$

Taking new orthogonal axes,

$$P_1, Q_1, R_1 \;;\; P_2, Q_2, R_2 \;;\; P_3, Q_3, R_3$$

the cosine of the angle between the first position, and the second referred to new axes, i.e. between

$$p, q, r$$

and $\quad p'P_1+q'Q_1+r'R_1 \,,\; p'P_2+q'Q_2+r'R_2 \,,\; p'P_3+q'Q_3+r'R_3$

is

$$\begin{aligned} & pp'P_1 + pq'Q_1 + pr'R_1 \\ + & qp'P_2 + qq'Q_2 + qr'R_2 \\ + & rp'P_3 + rq'Q_3 + rr'R_3 \end{aligned}$$

Summing for all objects, with or without weighting, ~~and allowing~~ for precession and allowance for variations in the time interval

$$\begin{aligned} & AP_1 + HQ_1 + GR_1 \\ + & H'P_2 + BQ_2 + FR_2 \\ + & G'P_3 + F'Q_3 + CR_3 \end{aligned}$$

is to be maximised for variations of the new system of axes, giving three equations of the form

$$G'P_2 + F'Q_2 + CR_2 = H'P_3 + BQ_3 + FR_3 \qquad (1)$$

If M stands for the matrix

$$\begin{matrix} A & H & G \\ H' & B & F \\ G' & F' & C \end{matrix}$$

and M' for the same interchanging rows and columns, then MM' is a symmetrical matrix having real latent roots all positive

$\lambda_i^2, \lambda_j^2, \lambda_k^2,$ found by solving a cubic

with associated orthogonal unit vectors

x_i, y_i, z_i found by solving simultaneous linear equations, such that

$$\left.\begin{aligned}
A^2 + H^2 + G^2 &= \lambda_i^2 x_i^2 + \lambda_j^2 x_j^2 + \lambda_k^2 x_k^2 \\
H'^2 + B^2 + F^2 &= \lambda_i^2 y_i^2 + \lambda_j^2 y_j^2 + \lambda_k^2 y_k^2 \\
G'^2 + F'^2 + C^2 &= \lambda_i^2 z_i^2 + \lambda_j^2 z_j^2 + \lambda_k^2 z_k^2 \\
\\
H'G' + BF' + FC &= \lambda_i^2 y_i z_i + \lambda_j^2 y_j z_j + \lambda_k^2 y_k z_k \\
G'A + F'H + CG &= \lambda_i^2 z_i x_i + \lambda_j^2 z_j x_j + \lambda_k^2 z_k x_k \\
AH' + HB + FG &= \lambda_i^2 x_i y_i + \lambda_j^2 x_j y_j + \lambda_k^2 x_k y_k
\end{aligned}\right\} \quad (2)$$

Making an appropriate choice of the signs of $\lambda_i, \lambda_j, \lambda_k$ solve the three sets of simultaneous equations

$$\begin{aligned}
\lambda_i x_i \xi_i + \lambda_j x_j \xi_j + \lambda_k x_k \xi_k &= A & \qquad \lambda_i x_i \eta_i + \lambda_j x_j \eta_j + \lambda_k x_k \eta_k &= H \\
\lambda_i y_i \xi_i + \lambda_j y_j \xi_j + \lambda_k y_k \xi_k &= H' & \qquad \lambda_i y_i \eta_i + \lambda_j y_j \eta_j + \lambda_k y_k \eta_k &= B \\
\lambda_i z_i \xi_i + \lambda_j z_j \xi_j + \lambda_k z_k \xi_k &= G' & \qquad \lambda_i z_i \eta_i + \lambda_j z_j \eta_j + \lambda_k z_k \eta_k &= F'
\end{aligned}$$

the solutions of which $(\xi, \eta, \zeta)_{i,j,k}$ must in view of (2) be unit orthogonal vectors, then the solution is

$$P_1 = x_i \xi_i + x_j \xi_j + x_k \xi_k \quad P_2 = y_i \xi_i + y_j \xi_j + y_k \xi_k \quad P_3 = z_i \xi_i + z_j \xi_j + z_k \xi_k$$
$$Q_1 = x_i \eta_i + x_j \eta_j + x_k \eta_k \quad Q_2 = y_i \eta_i + y_j \eta_j + y_k \eta_k \quad Q_3 = z_i \eta_i + z_j \eta_j + z_k \eta_k$$
$$R_1 = x_i \zeta_i + x_j \zeta_j + x_k \zeta_k \quad R_2 = y_i \zeta_i + y_j \zeta_j + y_k \zeta_k \quad R_3 = z_i \zeta_i + z_j \zeta_j + z_k \zeta_k$$

For $\quad P_1 \xi + Q_1 \eta + R_1 \zeta = x, \quad i, j, k$

$$P_2 \xi + Q_2 \eta + R_2 \zeta = y$$

$$P_3 \xi + Q_3 \eta + R_3 \zeta = z$$

whence

$$P_1 A + Q_1 H + R_1 G = \lambda_i x_i^2 + \lambda_j x_j^2 + \lambda_k x_k^2$$

$$\left.\begin{matrix} P_2 G' + Q_2 F' + R_2 C \\ P_3 H' + Q_3 B + R_3 F \end{matrix}\right\} = \lambda_i y_i z_i + \lambda_j y_j z_j + \lambda_k y_k z_k \quad \text{thus satisfying (1)}$$

The expression to be maximised then becomes $\lambda_i + \lambda_j + \lambda_k$, hence if $\lambda_i \lambda_j \lambda_k$ is positive all are to be taken positive, but if the product is negative the least is to be negative.

Relative to arbitrary orthogonal axes, the direction cosines of an object at two epochs are

$$p, q, r \quad \text{and} \quad p', q', r'$$

Taking new orthogonal axes

$$P_1, Q_1, R_1 \; ; \; P_2, Q_2, R_2 \; ; \; P_3, Q_3, R_3$$

the cosine of the angle between the first position, and the second referred to new axes, i.e. between

$$p, q, r$$

and $\quad p'P_1 + q'Q_1 + r'R_1 \; , \; p'P_2 + q'Q_2 + r'R_2 \; , \; p'P_3 + q'Q_3 + r'R_3$

is

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Summing for all objects, with or without weighting, and allowing for variation in the time interval

$$\begin{aligned} & AP_1 + HQ_1 + GR_1 \\ + \; & H'P_2 + BQ_2 + FR_2 \\ + \; & G'P_3 + F'Q_3 + CR_3 \end{aligned}$$

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If M stand for the matrix

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and M' for the same interchanging rows and columns, then MM' is a symmetrical matrix having real latent roots all positive

$$\lambda_i^2 \; , \lambda_j^2 \; , \lambda_k^2 \qquad \text{found by solving a cubic}$$

with associated orthogonal unit vectors $\quad x_i \, , y_i \, , z_i \qquad$ found by solving simultaneous linear equations

such that

$$\left.\begin{aligned} A^2 + H^2 + G^2 &= \lambda_i^2 x_i^2 + \lambda_j^2 x_j^2 + \lambda_k^2 x_k^2 \qquad \text{3 equations} \\ H'G' + BF' + FC &= \lambda_i y_i z_i + \lambda_j y_j z_j + \lambda_k y_k z_k \qquad \text{3 equations} \end{aligned}\right\} \qquad (2)$$

Making an appropriate choice of the signs of $\lambda_i, \lambda_j, \lambda_k$ solve the three sets of simultaneous equations

$$\begin{aligned} \lambda_i x_i \xi_i + \lambda_j x_j \xi_j + \lambda_k x_k \xi_k &= A & \qquad \lambda_i x_i \eta_i + \lambda_j x_j \eta_j + \lambda_k x_k \eta_k &= H \\ \lambda_i y_i \xi_i + \lambda_j y_j \xi_j + \lambda_k y_k \xi_k &= H' & &= B \\ \lambda_i z_i \xi_i + \lambda_j z_j \xi_j + \lambda_k z_k \xi_k &= G' & &= F' \end{aligned}$$

the solutions of which $(\xi, \eta, \zeta)_{i,j,k}$ must in view of (2) be unit orthogonal vectors, then the solution is

$$\begin{aligned} P_1 &= x_i \xi_i + x_j \xi_j + x_k \xi_k & \qquad P_2 &= y_i \xi_i + y_j \xi_j + y_k \xi_k & \qquad P_3 &= z_i \xi_i + z_j \xi_j + z_k \xi_k \\ Q_1 &= x_i \eta_i + x_j \eta_j + x_k \eta_k & Q_2 &= y_i \eta_i + y_j \eta_j + y_k \eta_k & Q_3 &= z_i \eta_i + z_j \eta_j + z_k \eta_k \\ R_1 &= x_i \zeta_i + x_j \zeta_j + x_k \zeta_k & R_2 &= y_i \zeta_i + y_j \zeta_j + y_k \zeta_k & R_3 &= z_i \zeta_i + z_j \zeta_j + z_k \zeta_k \end{aligned}$$

For these $\quad P_1 \xi + Q_1 \eta + R_1 \zeta = x \qquad i, j, k \qquad$ 3 sets of equations

whence

$$P_1 A + Q_1 H + R_1 G = \lambda_i x_i^2 + \lambda_j x_j^2 + \lambda_k x_k^2 \, . \qquad \text{6 equations, satisfying} \quad (1)$$

The conformity to be maximised then becomes $\lambda_i + \lambda_j + \lambda_k$, hence if $\lambda_i \lambda_j \lambda_k$ is positive all are to be taken positive, but if the product is negative the least is to be negative.