St John's College
Cambridge
Aug 5.

Dear Fisher,

Bartlett has just sent me another paper on the Behrens-Fisher test, & I see you are having another go too. I sent Yates's paper on but have heard nothing further. I have never seen the full form of your function & should be interested to know whether it is the same as my methods give for the estimation problem. The significance one is in a Proc. Roy. Soc. paper on the comparison of means.

I should state the problem thus. x & y are two unknown location parameters, σ & τ two unknown standard errors, θ & θ' two sets of observations intended to estimate x & y. What, given θ & θ', is the probability distribution of $y-x$? We have (the four unknowns being supposed irrelevant)

$$P(dx\,dy\,d\sigma\,d\tau \mid H) \propto dx\,dy\,d\sigma\,d\tau/\sigma\tau$$

$$P(\theta\theta' \mid x\,y\,\sigma\,\tau\,H) \propto \sigma^{-m}\tau^{-n}\exp\left[-\frac{m}{2\sigma^2}\{(x-\bar{x})^2+s^2\}-\frac{n}{2\tau^2}\{(y-\bar{y})^2+t^2\}\right]$$

$$\therefore P(dx\,dy\,d\sigma\,d\tau \mid \theta\theta' H) \propto \sigma^{-m-1}\tau^{-n-1}\exp\left[\text{———}\right] dx\,dy\,d\sigma\,d\tau$$

Integrating with respect to σ & τ

$$P(dx\,dy \mid \theta\theta' H) \propto \left[1+\frac{(x-\bar{x})^2}{s^2}\right]^{-\frac{1}{2}m}\left[1+\frac{(y-\bar{y})^2}{t^2}\right]^{-\frac{1}{2}n} dx\,dy$$

& then putting $y = x+z$ & integrating for x

$$P(dz \mid \theta\theta' H) \propto dz\int_{-\infty}^{\infty}\frac{dx}{\left[1+\frac{(x-\bar{x})^2}{s^2}\right]^{\frac{1}{2}m}\left[1+\frac{(x+z-\bar{y})^2}{t^2}\right]^{\frac{1}{2}n}}$$

and it only remains to do the arithmetic. At all events it is short & no information has gone down the drain; & it is perfectly clear what has happened to σ and τ.

This is what I call an estimation, not a significance, problem. The distinction is in the form of the question, but also affects the answer. Suppose you get an estimated difference z within the acceptance limits; do you say that the true value is as likely to be $2z$ as 0, or that it is more likely to be 0? If the former it is what I call an estimation problem; if the latter, one of significance. It seems to me that most direct comparisons in agricultural experiments are estimation problems, but if you are testing whether a Mendelian ratio agrees with 3:1, it is a question of significance.

It seems to me that in your original fiducial argument you were speaking of the probability of the true value, given the data, in precisely the same sense as I should, & you and "Student" both make the point clearer in various places by emphasizing the uniqueness of the samples, thus saying what I say by using the prior probability that expresses previous ignorance. The trouble is that your language is not adequate to get into symbols what you are trying to say, and consequently you don't state all the steps that I filled in in my paper about "Student".

I notice a tendency for "Student's" real name to be used in papers now. I think this is a pity — it will mean a lot of trouble in reference, since his nom de plume is far the better known. I have put two entries in the index of my book: "W. L. Gosset, see Student"; "C. L. Dodgson, see Lewis Carroll." The printing of the book is well advanced — the Oxford Press are doing it — & I have struck in an appendix on the Γ function (or rather I call it $x!$) because all the books take twice as long over it as they need, & some considerably more. It was filling in the omitted steps in Whittaker & Watson's proof of Stirling that decided me that something had to be done about it. By the way Watson has a beautiful lemma in Bessel Functions, p. 236. I use the argument like this.

$x! = \int_0^\infty e^{-t} t^x dt$. Put $t = x(1+u)$. $x! = x^{x+1} e^{-x} \int [(1+u) e^{-u}]^x du$

Now put $(1+u)e^{-u} = e^{-\frac{1}{2}v^2}$ & invert the series; it leads to $\frac{du}{dv} = 1 + \alpha v + \beta v^2 + \ldots$ (I've forgotten the others now). Consider $\left|\frac{\frac{du}{dv} - 1 - \alpha v}{v^2}\right|$. This is finite for v small, & $\to 0$ for $v \to \pm\infty$. Hence $\exists\, M$ such that $\left|\frac{du}{dv} - 1 - \alpha v\right| < M v^2$ for all v. Then

$x! = x^{x+1} e^{-x} \int_{-\infty}^{\infty} e^{-\frac{1}{2}xv^2}(1 + \alpha v + \theta M v^2)\, dv = \sqrt{2\pi}\, x^{x+\frac{1}{2}} e^{-x}\left[1 + \frac{\theta M}{x}\right]$ where $\theta < 1$ & M is independent of x. Watson of course does it in full generality, but I think an easy specimen of how to use it is needed. It saves a lot of trouble to get the first term & use it for some of the main results, developing the rest later.

Yours

Harold Jeffreys.