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HIGHER TANNAKA DUALITY

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Abstract

In this thesis we prove a Tannaka duality theorem for $(\infty, 1)$-categories. Classical Tannaka duality is a duality between certain groups and certain monoidal categories endowed with particular structure. Higher Tannaka duality refers to a duality between certain derived group stacks and certain monoidal $(\infty, 1)$-categories endowed with particular structure. This higher duality theorem is defined over derived rings and subsumes the classical statement. We compare the higher Tannaka duality to the classical theory and pay particular attention to higher Tannaka duality over fields. In the later case this theory has a close relationship with the theory of schematic homotopy types of Toën. We also describe three applications of our theory: perfect complexes and that of both motives and its non-commutative analogue due to Kontsevich.

Résumé

Dans cette thèse, nous prouvons un théorème de dualité de Tannaka pour les $(\infty, 1)$-catégories. La dualité classique de Tannaka est une dualité entre certains groupes et catégories monoïdales munies d'une structure particulière. La dualité de Tannaka supérieure renvoie, elle, à une dualité entre certains champs en groupes dérivés et certaines $(\infty, 1)$-catégories monoïdales munies d'une structure particulière. Cette dualité supérieure est définie sur les anneaux dérivés et englobe la théorie de dualité classique.

D'un côté, la correspondance de la dualité supérieure décrit les catégories monoïdales symétriques supérieures. Nous présentons ici la théorie générale des (∞, n)-catégories $\mathcal{O}$-monoïdales qui contient les cas monoïdale et monoïdale symétrique. Les travaux de Toën et Vezzosi et ceux de Lurie présentent des notions correspondantes de $(\infty, 1)$-catégories cofibrées, des objets $\mathcal{O}$-monoïdes et des objets $\mathcal{O}$-modules dans une ∞-catégorie $\mathcal{O}$-monoïdale. Nous les étendons aux cas des (∞, n)-catégories et nous rappelons le prolongement naturel des catégories abéliennes (resp. des anneaux commutatifs) au domaine des $(\infty, 1)$-catégories sous la forme des $(\infty, 1)$-catégories stables (resp. des E_∞-anneaux). On construit alors la $(\infty, 2)$-catégorie large ambiante dans laquelle le théorème de Tannaka ici prouvé sera vérifié : il s'agit de l'$(\infty, 2)$-catégorie des $(\infty, 1)$-catégories monoïdales symétriques, R-linéaires, présentables et stables.

D'un autre côté, cette dualité décrit les champs en groupes dérivés, ou, plus généralement, les gerbes dérivées. Nous introduisons et étudions ces objets avec un intérêt particulier porté aux sites de R-algèbres, où R est un E_∞-anneau, dotées de topologies positives, plates et finies. Ceci conduit à une discussion sur les t-structures d'une $(\infty, 1)$-catégorie stable. Nous commençons alors l'étude du théorème de dualité en introduisant les $(\infty, 1)$-catégories rigides, les R-algèbres de Hopf et le champ de foncteurs fibres. Le théorème de dualité est prouvé dans trois cas distincts, s'appliquant à des topologies différentes. Dans chacun de ces cas, la preuve repose sur une conjecture concernant les endomorphismes lax sur la $(\infty, 1)$-catégorie des R-modules et des R-algèbres.

Nous comparons la dualité de Tannaka supérieure à la théorie de dualité de Tannaka classique et portons une attention particulière à la dualité de Tannaka sur les corps. Dans ce dernier cas, cette théorie a une relation étroite avec la théorie des types d'homotopie schématique de Toën. Nous décrivons également trois applications de la théorie : les complexes parfaits, les motifs et leur analogue non-commutatif dû à Kontsevich.

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♫: Shining: V (Halmstad), VI (Klagopsalmer) and VII (Född förlorare).

. . . To my father.

James Wallbridge
Toulouse 21/07/2011

Contents

1 Introduction

Classical Tannaka duality is a duality between certain groups and certain monoidal categories endowed with particular structure. Higher Tannaka duality refers to a duality between certain group stacks and certain monoidal ∞-categories endowed with particular structure. This duality theorem subsumes the classical case. Our starting point is the philosophy developed by Grothendieck which is to consider the fundamental groupoids (ie. 1-truncated homotopy types) arising in a given context as automorphism groupoids of certain "fiber" functors.

This philosophy began with Grothendieck's study of Galois theory axiomatically using purely categorical methods [SGA]. He introduced the notion of a *Galois category*, that is, a category C satisfying conditions that imply that it is equivalent to the category of representations of a profinite group, together with a "fiber functor" ω from this category to the category of finite sets. More precisely, let (C, ω) be a Galois category and define the fundamental group of C at the base point ω to be

$$\pi_1(C, \omega) := \mathrm{Aut}(\omega).$$

Then $\pi_1(C, \omega)$ is a profinite group and the functor

$$C \to \pi(C, \omega)\text{-FSet}$$

is an equivalence of categories where FSet is the category of finite sets. This is the *Galois duality* statement. By looking at the problem categorically Grothendieck was able to transfer the study of 1-truncated homotopy types to contexts where such a notion was previously difficult to define. In this way he defined a new topological invariant - the étale fundamental group.

An analogous notion in the case of compact topological groups was initiated much earlier by Tannaka [Ta] who showed that a compact group can be reconstructed from its category of representations. The group arises as the group of tensor preserving automorphisms of the forgetful fiber functor from the category of representations to its underlying category of vector spaces. In [Kr], Krein characterised those categories of the form $\mathrm{Rep}(G)$ which arise in this way.

The passage from Galois theory to Tannaka theory is the linearization process of replacing sets by vector spaces. Following the Galois philosophy of Grothendieck above, Saavedra developed a Tannaka duality theory for affine group schemes where the abstract category dual is termed a *neutralized Tannakian category* [Sa]. The *neutralized Tannaka duality* statement is then that the automorphism group of fiber functors is an affine group scheme and the Tannakian category is equivalent to the category of representations of this affine group scheme. More precisely, let k be a field. Then

$$\mathrm{Rep}_* : \mathrm{AffGp}_k^{op} \to (\mathrm{Tan}_k)_*$$

is an equivalence of categories where $(\mathrm{Tan}_k)_*$ is the category of pairs (T, ω) where T is a k-Tannakian category and ω is a fiber functor. The category AffGp_k on the right hand side is the category of affine k-group schemes. Let (T, ω) be a neutralized Tannakian category and define the algebraic fundamental group of T at the base point ω to be

$$\pi_1(T, \omega)^{alg} := \mathrm{Aut}^{\otimes}(\omega).$$

Then $\pi_1(T, \omega)^{alg}$ is an affine group scheme and the functor

$$T \to \mathrm{Rep}(\pi(T, \omega)^{alg})$$

is an equivalence of categories. These affine group schemes are considered as algebraic versions of 1-truncated homotopy types and lead to Deligne's definition of the algebraic fundamental group [D2].

More generally, Saavedra wrote a non-neutral Tannaka statement which characterises those categories T which are equivalent to the category of representations of the stack $\mathrm{Fib}(T)$ of fiber functors on T. In [D2], Deligne completed the proof that this stack is an affine gerbe (in the ffqc topology). More precisely they showed that

$$\mathrm{Rep} : \mathrm{Ger}^{ffqc}(k)^{op} \to \mathrm{Tan}_k$$

is an equivalence of categories where Tan_k is the category of k-Tannakian categories and $\mathrm{Ger}^{ffqc}(k)$ is the category of affine gerbes over $\mathrm{Spec}(k)$ in the ffqc topology. When $\mathrm{Fib}(T)$ is the neutral gerbe of G-torsors, for G an affine group scheme, we recover the neutral Tannaka statement. In [D2] the author also includes an internal characterisation of Tannakian categories (ie. without the extra data of a fiber functor) in characteristic 0. In [DR1], Doplicher and Roberts give an internal characterisation of the categories arising as the dual of a compact group. As a result they were able to deduce the existence of the compact gauge groups arising in quantum field theory by starting with a category satifying some physically motivated simple properties [DR2].

In order to study *higher* homotopy types it is necessary to move to a higher categorical generalisation of the above ideas. Work in this direction began in [T1] by Toën. It involves the use of ∞-categorical techniques recently developed in work by Joyal [Jo] and Lurie [Lu] and in the theory of "derived algebraic geometry" by Lurie [LI, LII, LIII] and Toën and Vezzosi [TVI, TVII]. Informally, the passage from categories to ∞-categories involves replacing the category of sets by the ∞-category of spaces (topological spaces, Kan complexes or one such equivalent model). Indeed, one possible (although not so convenient) model for an ∞-category is a simplical category. This forces generalisations of other familiar categorical concepts. The category of abelian groups is replaced by the ∞-category of spectra, an abelian category is replaced by a stable ∞-category, commutative rings are replaced by commutative ring spectra (called E_∞-rings) and a rigid category is replaced by a rigid ∞-category. More examples are discussed througout the text.

In the spirit of the above, we prove the following pointed or neutralized Tannaka duality statement for ∞-categories.

Theorem 1.0.1 (Neutralized ∞-Tannaka duality: see Theorem 5.3.13)**.** *Let τ be a subcanonical topology. Then the map*

$$\mathrm{Perf}_* : \mathrm{TGp}^\tau(R)^{op} \to (\mathrm{Tens}_R^{\mathrm{rig}})_*$$

is fully faithful. Moreover, the adjunction $\mathrm{Fib}_* \dashv \mathrm{Perf}_*$ induces the following:

1. *Let R be an E_∞-ring. Then (T, ω) is a pointed finite R-Tannakian ∞-category if and only if it is of the form $\mathrm{Perf}_*(G)$ for G a finite R-Tannakian group stack.*

2. *Let R be a connective E_∞-ring. Then (T, ω) is a pointed flat R-Tannakian ∞-category if it is of the form $\mathrm{Perf}_*(G)$ for G a flat R-Tannakian group stack.*

3. *Let R is a bounded connective E_∞-ring. Then (T, ω) is a pointed positive R-Tannakian ∞-category if it is of the form $\mathrm{Perf}_*(G)$ for G a positive R-Tannakian group stack.*

The category $\mathrm{TGp}^\tau(R)$ is the ∞-category of R-Tannakian group stacks. These are affine group stacks which are weakly rigid in an appropriate sense. The category $(\mathrm{Tens}_R^{\mathrm{rig}})_*$ is the ∞-category of pointed rigid R-tensor ∞-categories. The objects in this category are rigid stable R-linear symmetric monoidal ∞-categories together with an exact R-linear symmetric monoidal functor to the ∞-category of rigid R-modules.

We will introduce three topologies on the ∞-category of R-algebras for an E_∞-ring R called the finite, flat and positive topologies (denoted by fin, fl and ≥ 0 respectively). A rigid R-tensor ∞-category will be called pointed Tannakian with respect to one of these topologies if it is equipped with a fiber functor satisfying certain properties that reflect this topology (see Definition 5.3.10). A λ-R-Tannakian group stack for $\lambda \in \{fin, fl, \geq 0\}$ is an R-Tannakian group stack such that its associated Hopf R-algebra reflects the topology λ (see Definition 5.3.12). A key step in the proof of the theorem is Conjecture 3.6.10. This enables us to identify the ∞-category of lax monoidal endofunctors on Mod_R in a suitable $(\infty, 2)$-category with the ∞-category of R-algebras.

Let (T, ω) be a pointed λ-R-Tannakian ∞-category and define the algebraic homotopy type of T at the base point ω to be

$$\pi(T, \omega)^{alg} := \mathrm{Aut}^\otimes(\omega).$$

Then $\pi(T,\omega)^{alg}$ is a λ-R-Tannakian group stack and the functor

$$T \to \mathrm{Rep}(\pi(T,\omega)^{alg})$$

is an equivalence of ∞-categories.

We also have the more general neutral Tannaka duality statement for ∞-categories.

Theorem 1.0.2 (Neutral ∞-Tannaka duality: see Theorem 5.5.3)**.** *The adjunction* Fib $\dashv$ Perf *induces the following:*

1. *Let R be an E_∞-ring. Then T is a finite R-Tannakian ∞-category if and only if it is of the form* $\mathrm{Perf}(G)$ *for G a neutral finite R-Tannakian gerbe.*
2. *Let R be a connective E_∞-ring. Then T is a flat R-Tannakian ∞-category if it is of the form* $\mathrm{Perf}(G)$ *for G a neutral flat R-Tannakian gerbe.*
3. *Let R be a bounded connective E_∞-ring. If (T,ω_1) and (T,ω_2) are two pointed positive R-Tannakian ∞-categories then there exists a positive cover $R \to Q$ such that*

$$\omega_1 \otimes_R Q \to \omega_2 \otimes_R Q$$

is an equivalence.

An ∞-category is R-Tannakian with respect to a topology $\lambda \in \{fin, fl, \geq 0\}$ if it is a rigid R-tensor ∞-category such there exists a fiber functor with respect to λ (see Definition 5.5.1). A λ-Tannakian gerbe is a stack on the site of R-algebras with respect to λ which is locally equivalent to the classifying stack of a λ-R-Tannakian group stack. It is said to be a neutral if there exists a global point (see Definition 5.5.2). Since the positive topology is not subcanonical we rest with the weaker statement of (3). There may also exist a reasonable notion of *non-neutral* ∞-Tannaka duality where the duality holds over an extension of the base E_∞-ring but we will not consider this more general case here.

Overview

We include here a brief overview of the contents in this paper. See the beginning of each chapter for a more detailed account of the results in each section. In Chapter 2 we begin by recalling the basic theory of higher categories which for us will be that of (∞,n)-categories. Due to the foundational work of Simpson and Hirschowitz in [HS], there exists a Quillen model structure on the category of (∞,n)-precategories which is cartesian closed. The existence of this model structure facilitates the study of all the standard categorical notions as applied to (∞,n)-categories such as the theory of Kan extensions, limits and colimits. We also discuss the Quillen equivalence between the model category of (∞,n)-precategories and the model category of categories enriched over $(\infty,n-1)$-categories. This again simplifies certain higher categorical constructions by allowing us to choose strict models. There exits a dictionary between model categories and (∞,n)-categories obtained using the notion of localisation. We provide some localisation results in the context of (∞,n)-categories.

One side of the higher Tannaka duality describes particular symmetric monoidal $(\infty,1)$-categories. In Chapter 3 we introduce the general theory of $\mathcal{O}$-monoidal (∞,n)-categories which includes both the monoidal and symmetric monoidal cases. A key theorem, due to Toën and Vezzosi in [TV3], is an equivalence between $\mathcal{O}$-monoidal (∞,n)-categories considered either as a functor from $\mathcal{O}$ or as a cofibered (∞,n)-category over $\mathcal{O}$. We have corresponding notions of $\mathcal{O}$-monoid objects, $\mathcal{O}$-module objects and algebra objects in an $\mathcal{O}$-monoidal (∞,n)-category building on work of Lurie in [LII, LIII]. We recall the natural extensions of abelian categories and commutative rings to the $(\infty,1)$-categorical realm in the form of stable ∞-categories and E_∞-rings. We also consider t-structures on an $(\infty,1)$-category. We then construct the large ambient $(\infty,2)$-category where our Tannaka duality theorem will live: the $(\infty,2)$-category of stable, presentable R-linear symmetric monoidal $(\infty,1)$-categories.

The other side of the higher Tannaka duality describes derived group stacks, or more generally, derived gerbes. In Chapter 4 we introduce and study these objects with particular interest to those on the sites of R-algebras, for R an E_∞-ring, endowed with positive, flat and finite topologies (Definition 4.3).

We then embark on the study of the duality theorem itself in Chapter 5 by introducing rigid $(\infty, 1)$-categories (Definition 5.1.1), Hopf R-algebras (Definition 5.2.1) and the stack of fiber functors. The duality theorem is proven in three cases depending on the chosen topology. In all three cases the proof relies on a conjecture relating lax endomorphisms on the ∞-category of R-modules and R-algebras (see Section 3.6). We compare the higher Tannaka duality to the classical theory and pay particular attention to higher Tannaka duality over fields. In the later case this theory has a close relationship with the theory of schematic homotopy types of Toën [T2].

The classical Tannaka duality theory has had a large impact on the mathematical landscape since the pioneering paper [Ta] of Tannaka. Applications include differential Galois theory, Langlands duality, the theory of Picard-Vessiot, Hodge theory, quantum field theory and the theory of motives. We discuss two examples in the ∞-categorical context in Section 6 of this paper: perfect complexes and and that of both motives and its non-commutative analogue due to Kontsevich.

Relations to other work

The first article which discusses a theory of higher Tannaka duality is the paper [T1] by Toën. Here, the theory is motivated and many of the key ingredients are introduced. Several conjectures are then made. These ideas are then refined and the conjectures stated clearly in the authors influential habilitation memoir [T6]. Tannakian ∞-categories over fields are also discussed in *loc. cit.* and the present paper can be seen as one approach to answering the conjectures posed in this paper. In order for our proofs to be realised, we rely much on the foundational work on ∞-categories developed in [LI], [LII] and [LIII] by Lurie. We also mention the references [FI] and the very recent [LVIII] for other approaches to derived Tannaka duality.

1.1 Notation

We will assume basic familiarity with the theory of categories, enriched categories and model categories at the level of [MaC], [Ke] and [Ho] respectively. However, for the benefit of the reader, we include in the appendix a handful of results in the theory of enriched model categories referenced throughout the text. It follows from the axioms of a model category that the class of weak equivalences ($\mathscr{W}$) together with the class of cofibrations ($\mathscr{C}$) (resp. fibrations ($\mathscr{F}$)) determine the class of fibrations (resp. cofibrations) through a lifting property. Thus, when describing model structures in this paper we merely state one of the two classes ($\mathscr{C}$) or ($\mathscr{F}$). The initial object in an arbitrary category will be denoted $\emptyset$ and the final object by $*$.

Given categories C and D, an adjunction $F \dashv G$ between C and D will be often denoted $F : C \rightleftarrows D : G$. Given a diagram

$$C \xrightarrow{F} E \xleftarrow{G} D$$

in the category Cat of categories, recall that the comma category $(F \downarrow G)$ has as objects triples (c, d, f) where $c \in \mathrm{Ob}(C)$, $d \in \mathrm{Ob}(D)$ and $f : Fc \to Gd$ and as arrows $(c, d, f) \to (c', d', f')$ pairs (h, k) where $h : c \to c'$ and $k : d \to d'$ such that $f' \circ Fh = Gk \circ f$. We will denote the *overcategory* $(\mathrm{id} \downarrow e)$ by $E_{e/}$ and the *undercategory* $(e \downarrow \mathrm{id})$ by $E_{/e}$.

The common indexing categories used throughout the paper are as follows.

- Let Δ denote the category of non-empty finite ordinals and order preserving maps. This is equivalent to the category non-empty linearly ordered finite sets. We denote the ordinal $n + 1 = \{0, ..., n\}$ by $[n]$. An arrow $f : [n] \to [m]$ in Δ is said to be *inert* if it induces an isomorphism between $[n]$ and a convex subset of $[m]$.

- Let Δ_+ denote the category of (possibly empty) finite ordinals and order preserving maps. This is equivalent to the category of linearly ordered finite sets. We denote the ordinal n by $[n-1]$ so that the empty ordinal $0 = \emptyset$ is given by $[-1]$. The category Δ_+ is a monoidal category with monoidal structure $(\Delta_+, \oplus, [-1])$

- Let Γ denote the category of pointed finite ordinals and point preserving maps. This is equivalent to the category of all linearly ordered finite sets with a distinguished point $*$. We denote the pointed ordinal $n \coprod \{*\}$ by $[n]$. The category Γ is a monoidal category with monoidal structure $(\Gamma, \vee, [0])$. An arrow $f : [n] \to [m]$ in Γ is said to be *inert* (resp. *semi-inert*) if $f^{-1}\{j\} = \{i\}$ (resp. $f^{-1}\{j\} \in \{\emptyset, \{i\}\}$) for all $j \in [m]-*$. It is said to be *null* if $f(i) = *$ for all $i \in \langle n\rangle$. It is said to be *active* if $f^{-1}\{*\} = \{*\}$. Every arrow f in Γ admits a factorisation $f = f'' \circ f'$ by an inert arrow f' followed by an active arrow f''. This factorisation is unique up to (unique) isomorphism.

Let $\mathscr{M}$ be a cartesian closed model category and x and y two objects of $\mathscr{M}$. Then the internal Hom object $\underline{\mathrm{Hom}}(x, y)$ in $\mathrm{h}\mathscr{M}$ will be denoted by $\mathbb{R}\underline{\mathrm{Hom}}(x, y)$. An explicit model for $\mathbb{R}\underline{\mathrm{Hom}}(x, y)$ will depend on the context. For example, when all objects are cofibrant and the first variable is relatively simple, we will commonly define $\mathbb{R}\underline{\mathrm{Hom}}(x, y) := \mathrm{Hom}(x, Ry)$ where R is a fibrant replacement for y. Alternatively, the identification $\mathbb{R}\underline{\mathrm{Hom}}(x, y) := \mathrm{Hom}(Rx, Ry)$ will be used when it is more practical to have a composition that is functorial.

Let C be an ∞-category. The ∞-category $\mathbb{R}\underline{\mathrm{Hom}}(\Delta^{op}, C)$ of simplicial objects in C will be denoted sC. The ∞-category $\mathbb{R}\underline{\mathrm{Hom}}(\Delta_+, C)$ of augmented simplicial objects in C will be denoted s_+C. Similarly, we have the ∞-category of cosimplicial and augmented cosimplicial objects in C, denoted cC and c_+C respectively, given by replacing Δ^{op} by Δ and Δ_+^{op} by Δ_+. For convenience, the category $s\mathrm{Set}$ of simplicial sets will be denoted simply $\mathbf{S}$.

An (∞, n)-category is said to be *small* if the collection of objects form a set. Unless otherwise stated, we will neglect any kind of set theoretic issues and assume that our (∞, n)-categories are small when required.

2 Higher category theory

Let $m \leq n$ be a pair of non-negative integers which may also include ∞. An (n, m)-category is an n-category in which all k-morphisms are invertible for $m < k \leq n$. In this paper we will be primarily concerned with the case where $n = \infty$ and $m \in \{0, 1, 2\}$. However it is instructive to see, and in our case no less difficult to define, the general theory of (∞, n)-categories for arbitrary n. In Section 2.1 we introduce the theory of (∞, n)-categories. There are several different approaches to defining (∞, n)-categories (see the review article [Le] for a summary together with the more recent [Re]). Instead of seeing this as a burdon, we consider it a blessing since knowing that certain models are equivalent enables us to move between one model or another depending on the given context or calculation. The nature of the equivalence is a Quillen equivalence between certain model categories of (∞, n)-categories: we take the point of view that we are ultimately interested in the objects of the homotopy category. We will concentrate on two models which are known to be Quillen equivalent:

$$\mathrm{PC}(\mathscr{C}at_{(\infty,n-1)}) \rightleftarrows \mathrm{Cat}(\mathscr{C}at_{(\infty,n-1)})$$

where $\mathscr{C}at_{(\infty,n-1)}$ is a suitable model category of $(\infty, n-1)$-categories. The category on the left-hand side is the category of $\mathscr{C}at_{(\infty,n-1)}$-precategories and will play the principal role for our model category of (∞, n)-categories. The category on the right-hand side is the category of $\mathscr{C}at_{(\infty,n-1)}$-enriched categories and is often useful when one would like to choose a strict model. To simplify notation, we make the now common abuse of calling an $(\infty, 1)$-category simply an ∞-category.

Model categories provide a very powerful tool for proving results in the theory of (∞, n)-categories. Apart from being the natural setting to undertake comparison results as mentioned above, model categories themselves can be used to model (∞, n)-categories. In Section 2.2 we describe the construction taking a model category to an (∞, n)-category called *localisation*. In fact any (∞, n)-category which is presentable in an appropriately defined sense is equivalent to the localisation of a combinatorial $\mathscr{C}at_{(\infty,n-1)}$-enriched model category. See Proposition 2.3.20 for a precise statement in the ∞-categorical context. Moreover, by Proposition 2.2.13, any (∞, n)-category can be fully embedded into the localisation of a model category. This union between model categories and (∞, n)-categories is exploited to its full extent in Proposition 2.2.11 where we prove the (∞, n)-categorical Yoneda lemma.

In Section 2.3 we discuss adjoints in an (∞, n)-category and review the main results concerning limits and colimits in an ∞-category. These simpler limits and colimits are sufficient for the constructions in this paper. In particular, we discuss the important notion of ∞-category of ind-objects in an ∞-category. If C is an ∞-category then the ∞-category of prestacks on C is freely generated under small colimits by the image of the Yoneda embedding. The ∞-category of ind-objects of C is then the smallest full subcategory of this ∞-category of prestacks which contains the image of the Yoneda embedding and is stable under κ-filtered colimits. Thus it is freely generated under κ-filtered colimits by C. We then recall the adjoint funtor theorem (Proposition 2.3.21) for ∞-categories.

2.1 (∞, n)-categories

An (∞, n)-category is an ∞-category where all k-morphisms are invertible for $k > n$. The simplest way to formulate a definition of (∞, n)-category is by induction: one defines an (∞, n)-category as a category enriched over $(\infty, n-1)$-categories. Thus we begin by defining a convenient notion of $(\infty, 0)$-category. This will play a similar role in the theory of (∞, n)-categories as that of a set, in our notation a $(0, 0)$-category, in the theory of categories: every category is naturally enriched over the category Set of sets.

Recall that a map of simplicial sets $A \to B$ is said to be a *Kan fibration* if it has the right lifting property with respect to all horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 \leq i \leq n$. A simplicial set A is said to be a Kan complex if $A \to *$ is a Kan fibration.

Definition 2.1.1. An $(\infty, 0)$-category is a Kan complex.

There exists a model category structure on the category $\mathbf{S}$ of simplicial sets due to Quillen which we will call the *Kan model structure*, denoted $\mathbf{S}_{\mathscr{K}}$, whose fibrant objects are precisely the Kan complexes [GJ]. The geometric realisation and singular complex functors define a Quillen equivalence

$$|\bullet| : \mathbf{S}_{\mathscr{K}} \rightleftarrows \mathrm{Top} : \mathrm{Sing}$$

where Top is the category of topological spaces with the usual model structure. Thus one can equally think of an $(\infty, 0)$-category as a topological space. We could now define an $(\infty, 1)$-category as a Kan enriched category. That is, as a fibrant object in a certain model category $\mathrm{Cat}(\mathbf{S}_{\mathscr{K}})$ of categories enriched over the model category of simplicial sets with the Kan model structure. Although this definition leads to a reasonable definition of $(\infty, 1)$-category it suffers from a serious drawback: the model structure on $\mathbf{S}_{\mathscr{K}}$-enriched categories is not internal, ie. there does not exist a reasonable notion of an $\mathbf{S}_{\mathscr{K}}$-enriched category of functors between two $\mathbf{S}_{\mathscr{K}}$-enriched categories. This is analogous to the theory of model categories itself not being an internal theory. A solution to this problem is to work in a more general setting where composition in the definition of an $\mathbf{S}_{\mathscr{K}}$-enriched category is only defined up to equivalence. This leads to the definition of a *Segal category*. More generally, we can define the notion of a weak $\mathscr{M}$-category for $\mathscr{M}$ an arbitrary model category.

Notation 2.1.2. Let S be a set. We denote by Δ_S the category consisting of:

- An object of Δ_S is a pair $([n], c)$ where $[n] \in \Delta$ and $c : [n] \to S$ is an arbitrary map taking values in the set S. These objects will be written as strings of elements $(x_0 = c(0), ..., x_n = c(n))$ of S.
- Let $([n], c)$ and $([m], d)$ be two objects of Δ_S. An arrow from $([n], c)$ to $([m], d)$ is an element of the set $\Delta_S(([n], c), ([m], d)) = \{u \in \Delta([n], [m]) : c = d \circ u\}$.

Definition 2.1.3. Let $\mathscr{M}$ be a model category. An *$\mathscr{M}$-precategory* is a pair (S, A) where S is a set of *objects* and

$$A : \Delta_S^{op} \to \mathscr{M}$$

is a functor such that $A(x)$ is a final object of $\mathscr{M}$ for all $x \in S$.

A map $(S, A) \to (T, B)$ of $\mathscr{M}$-precategories is a pair (f, F) where $f : S \to T$ is a map of sets and $F : A \Rightarrow B \circ (f_*) : \Delta_S^{op} \to \mathscr{M}$ is a natural transformation where $f_* : \Delta_S^{op} \to \Delta_T^{op}$ denotes the natural map. Let $\mathrm{PC}(\mathscr{M})$ denote the category of $\mathscr{M}$-precategories.

We will commonly abuse notation by referring to an $\mathscr{M}$-precategory (S, A) as simply A and a map $(\alpha, F) : (S, A) \to (T, B)$ as simply $F : A \to B$. We will sometimes denote by $\mathrm{Ob}(A) := S$ the set of objects of A. We will abuse notation by writing $x \in A$ in place of $x \in \mathrm{Ob}(A)$. Thus $x, y \in A$ will mean $x, y \in S$ for an $\mathscr{M}$-precategory (S, A). For two objects $x, y \in A$, we will also utilise the notation $\mathrm{Map}_A(x, y)$ for the object $A(x, y)$ in $\mathscr{M}$ or simply $\mathrm{Map}(x, y)$ if the $\mathscr{M}$-precategory A is clear from the context. We remark that one of the main reasons for imposing the condition $A(x) = *$ in Definition 2.1.3 is to obtain a cartesian structure on the model category of $\mathscr{M}$-precategories. It effectively amounts to requiring strict units. See Section 19.3 of [S2] for further discussion.

Let $\mathscr{M}$ be a monoidal model category for the cartesian product. Then every $\mathscr{M}$-enriched category C is a $\mathscr{M}$-precategory (S, A) setting $S = \mathrm{Ob}(C)$ and

$$A(x_0, \ldots, x_n) = C(x_0, x_1) \times \ldots \times C(x_{n-1}, x_n)$$

for all $x_i \in C$. This induces a fully faithful functor

$$\mathfrak{G} : \mathrm{Cat}(\mathscr{M}) \to \mathrm{PC}(\mathscr{M}).$$

We will very often consider an $\mathscr{M}$-enriched category C as an $\mathscr{M}$-precategory by identifying C with $\mathfrak{G}(C)$.

Example 2.1.4. Let $\mathscr{M} = \mathrm{Set}$ be the category of sets with the trivial model structure, ie. the weak equivalences are the isomorphisms and all maps are both fibrations and cofibrations. Then the fully faithful functor $\mathfrak{G}$ is simply the nerve functor $\mathrm{Cat} \to \mathbf{S}$. We will often abuse notation by identifying a category as a simplicial set using this functor.

Definition 2.1.5. Let $\mathscr{M}$ be a model category whose weak equivalences are stable under finite products. A functor $A : \Delta_S^{op} \to \mathscr{M}$ is said to satisfy the *Segal condition* if for all $n \geq 2$ the map

$$A([n], c) \to \prod_{1 \leq i \leq n} A([1], c_i)$$

is a weak equivalence in $\mathscr{M}$ where $c_i([1]) := c(\{i-1, i\})$.

Definition 2.1.6. Let $\mathscr{M}$ be a model category with finite products. An $\mathscr{M}$-precategory is said to be a *weak $\mathscr{M}$-category* if it satisfies the Segal condition.

A weak $\mathbf{S}_{\mathscr{K}}$-category is also referred to as a 1-Segal category in the literature (see Section 2 of [HS]). We are now interested in defining a model structure on the category $\mathrm{PC}(\mathscr{M})$ of $\mathscr{M}$-precategories whose fibrant objects are weak $\mathscr{M}$-categories. This not only facilitates our study of the category of $\mathscr{M}$-precategories by making available all the tools inherent in the theory of model categories, like the existence of homotopy limits and colimits, but is also necessary for a rigorous comparison to other model structures. We will soon see that there exists a Quillen equivalence between a certain model category of $\mathscr{M}$-enriched categories and a model category of $\mathscr{M}$-precategories which we introduce presently.

There exists three model structures on the category of $\mathscr{M}$-precategories known as the *projective*, *Reedy* and *injective* model structures. All three have the same set of weak equivalences but differ in their choice of cofibrations. They are Quillen equivalent. The model structure most important for us is the Reedy model structure. This model structure, contrary to the other two, is known to be an *internal* model structure, ie. a model structure which is a monoidal model structure for the cartesian product monoidal structure. This enables us to define an (∞, n)-category through an inductive procedure. We start with some preliminaries.

Let $\mathrm{PC}(S, \mathscr{M})$ denote the category of $\mathscr{M}$-precategories with a fixed set S of objects, ie. $\mathrm{PC}(S, \mathscr{M}) := \underline{\mathrm{Hom}}(\Delta_S^{op}/S, \mathscr{M})$ where S on the right hand side is considered as a discrete subcategory of Δ_S in the obvious way. We make the following assumptions on the model category $\mathscr{M}$:

Definition 2.1.7. A model category $\mathscr{M}$ is said to be *cartesian excellent* if it satisfies the following conditions:

1. The model category $\mathscr{M}$ is combinatorial.
2. Every object in $\mathscr{M}$ is cofibrant.
3. The model category $\mathscr{M}$ is a monoidal model category for the cartesian product and the functor $x \times \bullet : \mathscr{M} \to \mathscr{M}$ preserves small colimits for all $x \in \mathscr{M}$.
4. Weak equivalences in $\mathscr{M}$ are stable under filtered colimits.

Every cartesian excellent model category is excellent by Definition 7.1.15. Thus by Proposition 7.1.17 and Remark 7.1.18 there exists a projective and injective model structure on $\mathrm{PC}(S, \mathscr{M})$ denoted by $\mathrm{PC}(S, \mathscr{M})_{\mathscr{P}}$ and $\mathrm{PC}(S, \mathscr{M})_{\mathscr{I}}$ respectively. A map $(f, F) : (S, A) \to (T, B)$ in $\mathrm{PC}(\mathscr{M})$ is said to be a *projective* (resp. *injective*) *cofibration* if $f : S \to T$ is an injective map of sets and $f_! A \to B$ is a cofibration in the projective (resp. injective) model structure on $\mathrm{PC}(T, \mathscr{M})$. By Proposition 14.3.1 of [S2] there also exists a Reedy model structure on $\mathrm{PC}(S, \mathscr{M})$ denoted $\mathrm{PC}(S, \mathscr{M})_{\mathscr{R}}$.

By the theory of left Bousfield localisation, see Proposition 7.1.23, we can obtain three other model structures on the underlying category $\mathrm{PC}(S, \mathscr{M})$ with the same set of cofibrations but where we enlarge the class of weak equivalences. By Theorem 14.1.1 of [S2] one can localise the projective and injective model structures on $\mathrm{PC}(S, \mathscr{M})$ at a set of maps which enforce the fibrant objects to be exactly those which are projectively and injectively fibrant respectively and satisfy the Segal condition of Definition 2.1.5. We denote these two model stuctures by $\mathrm{PC}(S, \mathscr{M})'_{\mathscr{P}}$ and $\mathrm{PC}(S, \mathscr{M})'_{\mathscr{I}}$ respectively. Similarly, by Theorem 14.3.2 of *loc. cit.* there exists a Bousfield localisation $\mathrm{PC}(S, \mathscr{M})'_{\mathscr{R}}$ of $\mathrm{PC}(S, \mathscr{M})_{\mathscr{R}}$ whose fibrant objects are the Reedy fibrant diagrams satisfying the Segal conditions.

If $\mathscr{M}$ is a model category, we will denote by $\mathrm{h}\mathscr{M}$ the *homotopy category* of $\mathscr{M}$ obtained from $\mathscr{M}$ by formally adjoining inverses to all weak equivalences. Let (S, C) be a weak $\mathscr{M}$-category. The *homotopy category* of (S, C), denoted $\mathrm{h}(S, C)$, is the $\mathrm{h}\mathscr{M}$-enriched category consisting of:

- $\mathrm{Ob}(\mathrm{h}(S, C)) = S$.
- For every $x, y \in S$, $\mathrm{Map}_{\mathrm{h}(S,C)}(x, y) = [C(x, y)]$ where $[\bullet] : \mathscr{M} \to \mathrm{h}\mathscr{M}$.
- For $x_0, \ldots, x_n \in S$, composition $\mathrm{Map}_{\mathrm{h}(S,C)}(x_0, x_1) \times \ldots \times \mathrm{Map}_{\mathrm{h}(S,C)}(x_{n-1}, x_n) \to \mathrm{Map}_{\mathrm{h}(S,C)}(x_0, x_n)$ is given by composing the inverse of the weak equivalence $C(x_0, \ldots, x_n) \to C(x_0, x_1) \times \ldots \times C(x_{n-1}, x_n)$ with the map $C(x_0, \ldots, x_n) \to C(x_0, x_n)$ and applying the functor $[\bullet]$.

We obtain in this way a functor $\mathrm{h} : \mathrm{PC}(\mathscr{M}) \to \mathrm{Cat}(\mathrm{h}\mathscr{M})$. A map $F : C \to D$ between $\mathscr{M}$-categories is said to be an equivalence if the induced functor $\mathrm{h}F : \mathrm{h}C \to \mathrm{h}D$ is an equivalence of $\mathrm{h}\mathscr{M}$-enriched categories, ie.

- For every $x, y \in C$, the map $C(x, y) \to D(F(x), F(y))$ is a weak equivalence in $\mathscr{M}$.
- Every $y \in D$ is equivalent to $F(x)$ in the homotopy category $\mathrm{h}D$ for some $x \in C$.

A functor between $\mathscr{M}$-categories satisfying these two conditions is said to be *fully faithful* and *essentially surjective* respectively. When referring to an $\mathscr{M}$-enriched category as a weak $\mathscr{M}$-category we will call it a *strict* $\mathscr{M}$-category. When C is an $\mathscr{M}$-enriched category, $\mathrm{h}C$ will refer to the $\mathrm{h}\mathscr{M}$-enriched category $\mathrm{h}\mathfrak{G}(C)$.

Definition 2.1.8. A map $(f, F) : (S, A) \to (T, B)$ of $\mathscr{M}$-precategories is said to be a *categorical equivalence* if there exists a commutative diagram

$$\begin{tikzcd}
(S,A) \arrow[r] \arrow[d, "\alpha"'] & (T,B) \arrow[d, "\beta"] \\
(S',A') \arrow[r, "\phi"] & (T',B')
\end{tikzcd}$$

where α and β are trivial cofibrations in $\mathrm{PC}(S, \mathscr{M})'_{\mathscr{P}}$ and $\mathrm{PC}(T, \mathscr{M})'_{\mathscr{P}}$ respectively such that the induced map $\mathrm{h}(S', A') \to \mathrm{h}(T', B')$ is an equivalence of $\mathrm{h}\mathscr{M}$-enriched categories.

If $\mathscr{M}$ is a cartesian excellent model category then there exists a projective (resp. injective) model structure on the category $\mathrm{PC}(\mathscr{M})$ of $\mathscr{M}$-precategories where the cofibrations are the projective (resp. injective) cofibrations and the weak equivalences are the categorical equivalences. These two model structures will be denoted by $\mathrm{PC}(\mathscr{M})_{\mathscr{P}}$ and $\mathrm{PC}(\mathscr{M})_{\mathscr{I}}$ respectively. An important observation is that the fibrant objects of $\mathrm{PC}(\mathscr{M})_{\mathscr{P}}$ are precisely the locally fibrant $\mathscr{M}$-categories, ie. those $\mathscr{M}$-precategories (S, A) for which $A(x_0, \ldots, x_n)$ is a fibrant object of $\mathscr{M}$ for all sets of objects $x_0 \ldots x_n$ and which satisfy the Segal condition.

The projective model structure on $\mathrm{PC}(\mathscr{M})$ is not cartesian closed. There does however exist a model structure on $\mathrm{PC}(\mathscr{M})$ which lies between the projective and injective model structures in a chain of left Quillen functors. We will introduce the relevant cofibrations. Let $\Delta_{S,n}$ denote the full subcategory of Δ_S spanned by objects $([m], c)$ with $m \leq n$. The inclusion $i_n : \Delta_{S,n} \to \Delta_S$ induces a natural restriction functor

$$i_n^* : \underline{\mathrm{Hom}}(\Delta_S^{op}, \mathscr{M}) \to \underline{\mathrm{Hom}}(\Delta_{S,n}^{op}, \mathscr{M})$$

which has a fully faithful left adjoint $(i_n)_!$ given by the left Kan extension along the inclusion i_n. The nth *skeleton functor* is given by

$$\mathrm{sk}_n := (i_n)_! \circ i_n^* : \underline{\mathrm{Hom}}(\Delta_S^{op}, \mathscr{M}) \to \underline{\mathrm{Hom}}(\Delta_S^{op}, \mathscr{M}).$$

For an explicit characterisation of $\mathrm{sk}_n(A)$ for a functor $A : \Delta_S^{op} \to \mathscr{M}$, we refer the reader to Section 15.1 of [S2]. It follows from Lemma 15.1.1 of [S2] that the skeleton functor can be naturally extended to a functor $\mathrm{sk}_n : \mathrm{PC}(\mathscr{M}) \to \mathrm{PC}(\mathscr{M})$ between the category of $\mathscr{M}$-precatgories. A map $F : A \to B$ in $\mathrm{PC}(\mathscr{M})$ is said to be a *Reedy cofibration* if the map

$$A \coprod_{\mathrm{sk}_n(A)} \mathrm{sk}_n(B) \to B$$

is an injective cofibration for all $n \geq 0$.

Theorem 2.1.9. *Let $\mathscr{M}$ be a cartesian excellent model category. There exists a left proper, combinatorial model structure on the category $\mathrm{PC}(\mathscr{M})$ of $\mathscr{M}$-precategories in which*

($\mathscr{C}$) *The cofibrations are the Reedy cofibrations.*

($\mathscr{W}$) *The weak equivalences are the categorical equivalences.*

The fibrant objects are those $\mathscr{M}$-precategories (S, A) that are Reedy fibrant in $\mathrm{PC}(S, \mathscr{M})_{\mathscr{R}}$ and satisfy the Segal condition.

Proof reference. See Theorem 21.2.1 of [S2]. The characterisation of fibrant objects follows from Proposition 21.4.1 of [S2]. $\square$

This model structure will be called the *Reedy model structure* on $\mathrm{PC}(\mathscr{M})$ and will be denoted $\mathrm{PC}(\mathscr{M})_{\mathscr{R}}$. An important remark is that when $\mathscr{M}$ is a presheaf category and the cofibrations are the monomorphisms in $\mathscr{M}$, then the Reedy model structure on $\mathrm{PC}(\mathscr{M})$ coincides with the injective model structure on $\mathrm{PC}(\mathscr{M})$ (see Proposition 15.7.2 of [S2]). This holds in particular for the model category of (∞, n)-precategories described below. There exists a chain of Quillen equivalences

$$\mathrm{PC}(\mathscr{M})_{\mathscr{P}} \xrightarrow{\mathrm{id}} \mathrm{PC}(\mathscr{M})_{\mathscr{R}} \xrightarrow{\mathrm{id}} \mathrm{PC}(\mathscr{M})_{\mathscr{I}}.$$

The model category $\mathrm{PC}(\mathscr{M})_{\mathscr{R}}$ is a cartesian closed model category which is moreover cartesian excellent. This follows from Theorem 21.3.2 of [S2]. The cartesian closed structure implies that for any two objects A and B of $\mathrm{PC}(\mathscr{M})_{\mathscr{R}}$, there exists an $\mathscr{M}$-precategory B^A together with an *evaluation map* $B^A \times A \to B$ such that

$$\mathrm{Hom}(C, B^A) \to \mathrm{Hom}(C \times A, B)$$

is bijective for every $C \in \mathrm{PC}(\mathscr{M})_{\mathscr{R}}$. The object B^A will be called the *internal Hom* object and will be denoted also by $\underline{\mathrm{Hom}}(A, B)$. The cartesian structure also implies that the homotopy category $\mathrm{hPC}(\mathscr{M})_{\mathscr{R}}$ is cartesian closed (see Theorem 4.3.2 of [Ho]). The internal Hom objects of $\mathrm{hPC}(\mathscr{M})_{\mathscr{R}}$ will be denoted $\mathbb{R}\underline{\mathrm{Hom}}(A, B)$, see Notation 1.1.

We are now in a position to define an (∞, n)-category by induction. Let $\mathrm{PC}^0(\mathscr{M})_{\mathscr{R}} := \mathscr{M}$ and for any $n \geq 1$

$$\mathrm{PC}^n(\mathscr{M})_{\mathscr{R}} := \mathrm{PC}(\mathrm{PC}^{n-1}(\mathscr{M})_{\mathscr{R}})_{\mathscr{R}}.$$

Note that since for a fibrant object C in $\mathrm{PC}^n(\mathscr{M})_{\mathscr{R}}$ the $\mathrm{PC}^{n-1}(\mathscr{M})$-precategory $C(x_0, \ldots, x_n)$ is fibrant in $\mathrm{PC}^{n-1}(\mathscr{M})_{\mathscr{R}}$ for any collection of objects $x_0, \ldots x_n$ in C, the object C satisfies the Segal condition iteratively on each sub-mapping space for all $1 \leq i \leq n$.

The category $\mathbf{S}_{\mathscr{K}}$ of simplicial sets with the Kan model structure is a cartesian excellent model category. We have defined $\mathbf{S}_{\mathscr{K}}$ to be the model category of $(\infty, 0)$-categories. Thus the model category of (∞, n)-categories is given by $\mathrm{PC}^n(\mathbf{S}_{\mathscr{K}})_{\mathscr{R}}$ and will be denoted by $\mathscr{C}at_{(\infty,n)}$. The model category $\mathscr{C}at_{(\infty,n)}$ will always be regarded as a $\mathscr{C}at_{(\infty,n)}$-enriched category unless otherwise stated. An (∞, n)-precategory will refer to an arbitrary object of $\mathscr{C}at_{(\infty,n)}$. We of course are particularly interested in the fibrant objects. A fibrant (∞, n)-precategory is a weak $\mathscr{C}at_{(\infty,n-1)}$-category satisfying a Reedy condition. This Reedy condition can often be ignored in applications since every fibrant (∞, n)-precategory is equivalent to a locally fibrant weak $\mathscr{C}at_{(\infty,n-1)}$-category (without the Reedy condition) by the equivalence to an object in the projective model structure on $\mathrm{PC}(\mathscr{M})$. We make the following definition:

Definition 2.1.10. Let $n \geq 1$. An (∞, n)-precategory C is said to be an (∞, n)-*category* if it is a weak $\mathscr{C}at_{(\infty,n-1)}$-category such that for any object x in C, the $(\infty, n-1)$-precategory $C(x)$ is an $(\infty, n-1)$-category.

Let C be an (∞, n)-category and $(\mathrm{h}D)'$ a subcategory of $\mathrm{h}C$. Then the homotopy pullback $D := C \times^h_{\mathrm{h}C} (\mathrm{h}D)'$ in $\mathscr{C}at_{(\infty,n)}$ will be called a *subcategory* of C (as opposed to a sub-(∞, n)-category for convenience of notation). The 1-morphisms in an (∞, n)-category will be simply called arrows. The category of (∞, n)-categories will be denoted $\mathrm{Cat}_{(\infty,n)}$. We will make the now standard abuse of referring to an $(\infty, 1)$-category as an ∞-category. Since these are the (∞, n)-categories most often used in this paper, this also lightens the notation.

Example 2.1.11. We have seen that every $\mathscr{C}at_{(\infty,n-1)}$-enriched category is naturally an (∞, n)-category. In particular, every category C can be thought of as an (∞, n)-category through the full embedding $\mathrm{Cat} \to \mathrm{Cat}_{(\infty,n)}$: the image of C has the same set of objects with $C(x_0, \dots x_n) := \mathrm{Hom}_C(x_0, x_1) \times \dots \times \mathrm{Hom}_C(x_{n-1}, x_n)$.

We briefly remark that there is a well defined way of passing from (∞, n)-precategories to $(\infty, n+m)$-precategories using an extension of the Poincaré n-groupoid construction of Tamsamani in [Ts2] which we will denote by $\prod_n$. In fact there exists a Quillen adjunction

$$\mathfrak{R}_n : \mathscr{C}at_{(\infty,n+m)} \rightleftarrows \mathscr{C}at_{(\infty,n)} : \prod\nolimits_m$$

where the left adjoint, called the *realisation*, formally adjoints inverses to all k-morphisms for $k > n$ (see Section 2 of [HS] for more details). There also exists a right adjoint which associates to an $(\infty, n+m)$-category C an (∞, n)-category which we denote by $\mathfrak{K}^n(C)$. This (∞, n)-category, called the *n-groupic interior* in [HS], can be informally regarded as the (∞, n)-category obtained from C by discarding all non-invertible k-morphisms for $k > n$. It satisfies the following universal property: for any (∞, n)-category D, the map

$$\mathbb{R}\underline{\mathrm{Hom}}(D, \mathfrak{K}^n(C)) \to \mathbb{R}\underline{\mathrm{Hom}}(D, C)$$

is an equivalence.

Proposition 2.1.12. *Let A be an (∞, n)-precategory. Then the following hold.*

1. *For every (∞, n)-category C, the (∞, n)-precategory $\mathbb{R}\underline{\mathrm{Hom}}(A, C)$ is an (∞, n)-category.*
2. *Let $C \to D$ be an equivalence of (∞, n)-categories. Then $\mathbb{R}\underline{\mathrm{Hom}}(A, C) \to \mathbb{R}\underline{\mathrm{Hom}}(A, D)$ is an equivalence of (∞, n)-categories.*
3. *Let C be an (∞, n)-category and $f : A \to B$ be an equivalence of (∞, n)-precategories. Then $\mathbb{R}\underline{\mathrm{Hom}}(B, C) \to \mathbb{R}\underline{\mathrm{Hom}}(A, C)$ is an equivalence of (∞, n)-categories.*

Proof. This is deduced from Theorem 10.1.1 of [S2]. □

We have found that the model category $\mathscr{C}at_{(\infty,n)}$ is enriched over itself and hence $\mathscr{C}at_{(\infty,n)}$ is a strict $\mathscr{C}at_{(\infty,n)}$-category and hence an $(\infty, n+1)$-precategory. Let $\mathscr{M}$ be a model category and $\mathscr{M}^\circ$ the full subcategory of $\mathscr{M}$ spanned by the fibrant-cofibrant objects. Then for two objects C and D in the $(\infty, n+1)$-precategory $(\mathscr{C}at_{(\infty,n)})^\circ$, the mapping space

$$\mathrm{Map}_{(\mathscr{C}at_{(\infty,n)})^\circ}(C, D) = \underline{\mathrm{Hom}}(C, D)$$

is a fibrant object of $\mathscr{C}at_{(\infty,n)}$ by Proposition 2.1.12. Hence $(\mathscr{C}at_{(\infty,n)})^\circ$ is a fibrant object in $\mathrm{PC}(\mathscr{C}at_{(\infty,n)})_{\mathscr{P}}$ (with the projective model structure). However, $(\mathscr{C}at_{(\infty,n)})^\circ$ is not a fibrant object of $\mathscr{C}at_{(\infty,n+1)}$: $(\mathscr{C}at_{(\infty,n)})^\circ$ is not Reedy fibrant in $\mathscr{C}at_{(\infty,n+1)}$.

For calculational convenience it is helpful to be able to pass from a $\mathscr{M}$-precategory to a strict model where composition is strictly defined. Let $\mathscr{M}$ be a cartesian excellent model category. There exists a left proper, combinatorial model structure on the category $\mathrm{Cat}(\mathscr{M})$ of $\mathscr{M}$-enriched categories in which the weak equivalences are the equivalences of strict $\mathscr{M}$-categories. This follows from Proposition A.3.2.4 of [Lu]. We will call this model structure the *enriched* model structure on $\mathrm{Cat}(\mathscr{M})$ and denote it by $\mathrm{Cat}(\mathscr{M})_{\mathscr{E}}$. It follows from Theorem A.3.2.24 of [Lu] that an object of $\mathrm{Cat}(\mathscr{M})_{\mathscr{E}}$ is fibrant with respect to the enriched model structure if and only if it is locally fibrant. The enriched model structure on $\mathrm{Cat}(\mathscr{M})$ is not cartesian. The cartesian product of two cofibrant $\mathscr{M}$-enriched categories is not necessarily cofibrant. This is one of the main advantages for considering the model category of $\mathscr{M}$-precategories where an $\mathscr{M}$-precategory of morphisms in the form of the internal Hom between two objects is available.

There exists a Quillen equivalence

$$\mathfrak{F} : \mathrm{PC}(\mathscr{M})_{\mathscr{P}} \rightleftarrows \mathrm{Cat}(\mathscr{M})_{\mathscr{E}} : \mathfrak{G}$$

between the category of $\mathscr{M}$-precategories with the projective model structure and the category of $\mathscr{M}$-enriched categories with the enriched model structure by Theorem 2.2.16 of [L2]. The left adjoint to the fully faithful inclusion $\mathfrak{G}$ is constructed as follows. Let S be a set and consider the following category $J_{x,y}(S)$ where:

- An object in $J_{x,y}(S)$ is a pair $(s,i)_{n,k}$ where $s = (s_0, \dots, s_n) \in S^{n+1}$ such that $s_0 = x$, $s_n = y$ and $i = \{0 = i_0 < \dots < i_k = n\} \in \mathbb{Z}^{k+1}$.
- An arrow $(s,i)_{n,k} \to (t,j)_{m,l}$ in $J_{x,y}(S)$ is a map $f : [m] \to [n]$ such that the following hold:
  - $f(0) = 0$ and $f(m) = n$.
  - For any $0 \leq p \leq m$, $s_{f(p)} = t_p$.
  - For each $0 \leq q \leq l$, there exists $0 \leq r \leq k$ such that $i_r \leq f(j_q) \leq f(j_{q+1}) \leq i_{r+1}$.

Given any $\mathscr{M}$-precategory (S, A) and $\{x, y\} \in S$, we define a functor $H^A_{x,y}$ given by

$$\begin{aligned} H^A_{x,y} &: J_{x,y}(S) \to \mathscr{M} \\ &(s,i)_{n,k} \mapsto A(s_0, \dots, s_{i_1}) \times A(s_{i_1}, \dots, s_{i_2}) \times \dots \times A(s_{i_{k-1}}, \dots, s_n). \end{aligned}$$

The left adjoint $\mathfrak{F}$ can now be defined as follows. Let (S, A) be a $\mathscr{M}$-precategory. The $\mathscr{M}$-enriched category $\mathfrak{F}(S, A)$ in $\mathrm{Cat}(\mathscr{M})$ consists of:

- $\mathrm{Ob}(\mathfrak{F}(S,A))$=$S$.
- For any $X, Y \in S$, the mapping space is given by $\mathrm{Map}_{\mathfrak{F}(S,A)}(x, y) := \mathrm{colim}\ H^A_{x,y}$.
- For any sequence of elements $(x_0, \dots, x_n)$ in S, the composition

  $$J_{x_0,x_1}(S) \times \dots \times J_{x_{n-1},x_n}(S) \to J_{x_0,x_n}(S) \xrightarrow{H^A_{x_0,x_n}} \mathscr{M}$$

  is canonically isomorphic to $\prod_{1 \leq i \leq n} H_{x_{i-1},x_i}$. The composition law for

  $$\mathrm{Map}_{\mathfrak{F}(S,A)}(x_0, x_1) \times \dots \times \mathrm{Map}_{\mathfrak{F}(S,A)}(x_{n-1}, x_n) \to \mathrm{Map}_{\mathfrak{F}(S,A)}(x_0, x_n)$$

  is then given by $\mathrm{colim}\ \prod_{1 \leq i \leq n} H^A_{x_{i-1},x_i} \to \mathrm{colim}\ H^A_{x_0,x_n}$.

It follows that a map $(f, F) : (S, A) \to (T, B)$ in $\mathrm{PC}(\mathscr{M})$ is a categorical equivalence if there exists a commutative diagram

$$\begin{tikzcd} (S,A) \arrow[r] \arrow[d, "\alpha"'] & (T,B) \arrow[d, "\beta"] \\ (S',A') \arrow[r, "\phi"] & (T',B') \end{tikzcd}$$

where α and β are trivial cofibrations in $\mathrm{PC}(S, \mathscr{M})'_{\mathscr{P}}$ and $\mathrm{PC}(T, \mathscr{M})'_{\mathscr{P}}$ respectively and such that the induced map $\mathfrak{F}(S', A') \to \mathfrak{F}(T', B')$ is an equivalence of $\mathscr{M}$-enriched categories.

We will now discuss further examples of cartesian excellent model categories. We have seen that the fibrant objects of $\mathrm{Cat}(\mathbf{S})$, where $\mathbf{S}$ is endowed with the Kan model structure, provide models for $(\infty, 1)$-categories. However, it is possible to construct other models of $(\infty, 1)$-categories by working directly in the category of simplicial sets itself. For example, one can ask for certain lifting properties to be satisfied in the spirit of the Kan lifting property or ask that a certain collection of 1-simplices be distinguished. The first of these leads to the notion of a weak Kan complex (also called quasicategories in the literature [Jo]) and the second to the notion of a marked simplicial set. We will discuss the model category of weak Kan complexes (and its equivalence to the model category of simplicial categories) presently and refer the reader to [Lu] for the theory of marked simplicial sets.

From a category, one can produce a simplicial category through the following construction:

Construction 2.1.13. Let Grph be the category of reflexive graphs (ie. one truncated simplicial sets). From the adjunction

$$U : \mathrm{Grph} \rightleftarrows \mathrm{Cat} : V,$$

where the right adjoint is the forgetful functor, we obtain the corresponding comonad $L = UV$ on Cat. The counit $\varepsilon : L \to I$ and comultiplication $\delta : L \to L^2$ can be used to construct "face" and "degeneracy" maps given by $d_i^n : L^i \varepsilon L^{n-i} : L^{n+1} \to L^n$ for $0 \leq i \leq n$ and $s_i^n : L^i \delta L^{n-i-1} : L^n \to L^{n+1}$ for $0 \leq i \leq n-1$ respectively. We can use these maps to construct a simplicial object $L_* C$ in Cat

$$\ldots \xrightarrow{(d_0, d_1, d_2)} L^2 C \xrightarrow{(d_0, d_1)} LC \xrightarrow{\varepsilon} C$$

given by $L_n C = L^{n+1} C$. Here LC is simply the free category on C where the morphisms are freely generated by the non-identity morphisms of C. Since the simplicial set $[n] \mapsto \mathrm{Ob}(L_n C)$ is constant (with value $\mathrm{Ob}(C)$), $L_* C$ can be viewed as a simplicial category

$$L_* : \mathrm{Cat} \to \mathrm{Cat}(\mathbf{S})$$
$$L_*(C)(x, y)_n = L^{n+1} C(x, y).$$

This is a convenient way to associate an $\mathbf{S}$-enriched category, and hence an ∞-category, to any ordinary category.

Example 2.1.14. Applying Construction 2.1.13 to the category $[n]$ we obtain the following simplicial category $L_*[n]$: the objects of $L_*[n]$ are the elements of $[n]$ and for $i, j \in [n]$ the mapping space is given by

$$L_*[n](i, j) = \begin{cases} \emptyset & \text{if } i > j, \\ \mathrm{N}(P_{i,j}) & \text{otherwise,} \end{cases}$$

where $P_{i,j}$ denotes the partially ordered set by inclusion $\{[m] \subseteq [n] : (i, j \in [m]) \wedge (\forall k \in [m])[i \leq k \leq j]\}$. For $i \leq j \leq k$ in $[n]$, composition $L_*[n](j, k) \times L_*[n](i, j) \to L_*[n](i, k)$ is induced by $([l], [m]) \mapsto [l] \cup [m]$.

From a simplicial category, one can produce a simplicial set through the *simplicial nerve* (also called the homotopy coherent nerve).

Definition 2.1.15. The *simplicial nerve* functor $\mathrm{N} : \mathrm{Cat}(\mathbf{S}) \to \mathbf{S}$ is given by

$$\mathrm{N}(C)_n = \mathrm{Hom}_{\mathrm{Cat}(\mathbf{S})}(L_*[n], C)$$

where $C \in \mathrm{Cat}(\mathbf{S})$.

From the equality $\mathrm{Hom}_{\mathrm{Cat}(\mathbf{S})}(L_*[n], C) = \mathrm{Hom}_{\mathbf{S}}(\Delta[n], \mathrm{N}(C))$, we construct a left adjoint to N,

$$\mathfrak{C} : \mathbf{S} \to \mathrm{Cat}(\mathbf{S}),$$

by left Kan extending L_* along the Yoneda functor $y : \Delta \to \mathbf{S}$. Thus

$$\mathfrak{C}(A) = (\mathrm{Lan}_h L_*)(A) = \mathrm{colim}(y_{/A} \to \Delta \xrightarrow{L_*} \mathrm{Cat}(\mathbf{S})) = \underset{\Delta[n]\to A}{\mathrm{colim}}\, L_*[n].$$

In the case when $C \in \mathrm{Cat} \subset \mathrm{Cat}(\mathbf{S})$, the simplicial nerve corresponds to the ordinary nerve.

A map $A \to B$ of simplicial sets is said to be a *categorical equivalence* if $\mathfrak{C}(A) \to \mathfrak{C}(B)$ is a weak equivalence of $\mathbf{S}$-enriched categories. The category $\mathbf{S}$ of simplicial sets admits a left proper, combinatorial model structure in which the cofibrations are the monomorphisms and the weak equivalences are the categorical equivalences. This was first proven by Joyal in [Jo] and later proven by Lurie in [Lu] as stated here.

We will call this model structure the *Joyal model structure* on $\mathbf{S}$ and denote it by $\mathbf{S}_{\mathscr{J}}$. It is a cartesian excellent model category. The fibrant objects of this model category can be explicitly described as follows. A map $A \to B$ of simplicial sets is said to be an *inner fibration* if it has the right lifting property with respect to all horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 < i < n$. A simplicial set A is called a *weak Kan complex* if $A \to *$ is an inner fibration. The fibrant objects in the Joyal model structure are precisely the weak Kan complexes (see Theorem 2.4.6.1 of [Lu]). Furthermore, there exists a Quillen equivalence

$$\mathfrak{C} : \mathbf{S}_{\mathscr{J}} \rightleftarrows \mathrm{Cat}(\mathbf{S}_{\mathscr{K}}) : \mathrm{N}$$

between the category $\mathbf{S}$ of simplicial sets with the Joyal model structure and the category $\mathrm{Cat}(\mathbf{S})$ of simplicial categories with the Kan model structure (Theorem 2.2.5.1 of [Lu]).

Thus weak Kan complexes provide an adequate theory of $(\infty, 1)$-categories. Every category is a weak Kan complex. In fact we have the following characterisation of those simplicial sets A which arise as (the nerve of) a category: there exists a small category C and an isomorphism $A \simeq C$ if and only if $A \to *$ has the *unique* right lifting property with respect to all horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 < i < n$. Unsurprisingly, there is also a very close relationship between the category of simplicial sets $\mathbf{S}_{\mathscr{J}}$ with the Joyal model structure and the category of simplicial sets $\mathbf{S}_{\mathscr{K}}$ with the Kan model structure: the category $\mathbf{S}_{\mathscr{K}}$ is the Bousfield localisation of $\mathbf{S}_{\mathscr{J}}$ with respect to the singleton set $S = \{* \to \Delta^1\}$.

In summary, we have the following chain of Quillen equivalences (where the arrows denote left Quillen functors):

$$\mathscr{C}at_{(\infty,1)} \to \mathrm{Cat}(\mathbf{S}_{\mathscr{K}}) \leftarrow \mathbf{S}_{\mathscr{J}}.$$

Thus an ∞-category may also refer to a fibrant object of the closed model category $\mathbf{S}_{\mathscr{J}}$. Throughout this paper, we will frequently use and cite results from [Lu],[LI],[LII] and [LIII] when using ∞-categories. In these references, the theory of weak Kan complexes is chosen for a model of $(\infty, 1)$-categories. Through the chain of Quillen equivalences above, we will be content in the understanding that all constructions in the $(\infty, 2)$-category of weak Kan complexes has a corresponding and equivalent statement in the $(\infty, 2)$-category of ∞-categories in our sense. Finally, if $\mathscr{M} \to \mathscr{N}$ is a Quillen equivalence between two model categories then $\mathrm{PC}(\mathscr{M}) \to \mathrm{PC}(\mathscr{N})$ is a Quillen equivalence so we have an equivalence

$$\mathscr{C}at_{(\infty,2)} \to \mathrm{PC}(\mathbf{S}_{\mathscr{J}})$$

between model categories of $(\infty, 2)$-precategories. Thus an $(\infty, 2)$-category may also refer to a fibrant object of the closed model category $\mathrm{PC}(\mathbf{S}_{\mathscr{J}})$.

2.2 From model categories to (∞, n)-categories

Definition 2.2.1. Let C be an (∞, n)-category and S a set of arrows in C. A *localisation* of C along S is a pair $(L_S C, l)$ where $L_S C$ is an (∞, n)-category and $l : C \to L_S C$ is a functor such that the following universal property is satisfied: for any (∞, n)-category D, the induced map

$$\mathbb{R}\underline{\mathrm{Hom}}(L_S C, D) \to \mathbb{R}\underline{\mathrm{Hom}}(C, D)$$

is fully faithful and its essential image consists of those functors $F : C \to D$ which send each arrow in S to an equivalence in D.

Here the localisation is taken in the $(\infty, n+1)$-category of (∞, n)-categories (see Definition 2.2.9). One may also consider localisations taken in certain subcategories of this $(\infty, n+1)$-category. See Section 2.3 for one example. We will often refer to a localisation $(L_S C, l)$ of C along S as simply $L_S C$. It follows from the universal property that the localisation commutes with finite products, ie. if S and T are sets of morphisms in C and D respectively, then the natural map $L_{S\times T}(C \times D) \to L_S C \times L_T D$ is an equivalence of (∞, n)-categories. The following proposition is an existence result for the localisation of an (∞, n)-category.

Proposition 2.2.2. *Let C be an (∞, n)-category. An explicit model for the localisation $(L_S C, l)$ is given by the homotopy pushout diagram*

$$\begin{tikzcd} S \times [1] \arrow[r] \arrow[d] & C \arrow[d] \\ S \times \widetilde{[1]} \arrow[r] & L_S(C) \end{tikzcd}$$

in $\mathscr{C}at_{(\infty,n)}$ where $\widetilde{[1]}$ is the groupoid generated by one isomorphism $\{0 \xrightarrow{\sim} 1\}$.

Proof. Firstly, note that we are only interested in the set S containing 1-morphisms. From Section 2.1 we can consider the associated statement in the model category $\mathrm{Cat}(\mathscr{C}at_{(\infty,n-1)})$ of $\mathscr{C}at_{(\infty,n-1)}$-enriched categories. The proposition now follows from Section 8.2 of [T3] after making the admissible replacement of dg-categories to $\mathscr{C}at_{(\infty,n-1)}$-enriched categories (or more generally categories enriched in any excellent model category). □

Note that since this is a homotopy pushout taken in the model category of (∞, n)-precategories, it is necessary in general to compose with a fibrant replacement functor to obtain an (∞, n)-category $L_S C$. It follows that

$$\mathrm{h}(L_S C) \to S^{-1}(\mathrm{h}C)$$

is an equivalence of categories where $S^{-1}(\mathrm{h}C)$ is the category obtained by formally inverting the elements of S.

Example 2.2.3. Let C be an (∞, n)-category with a zero object. For any subset $\{x_i\}_{i\in I}$ of objects of C we can construct the *quotient* (∞, n)-category $C/\langle x_i\rangle$ given by the localisation $L_S C$ of C at the set of maps $S = \{x_i \to 0\}_{i\in I}$. The essential image of $\mathbb{R}\underline{\mathrm{Hom}}(C/\langle x_i\rangle, D) \to \mathbb{R}\underline{\mathrm{Hom}}(C, D)$ consists of all arrows $F : C \to D$ such that $F(x_i) \simeq 0$ in D for all $i \in I$.

Every category C can be regarded as a (strict) ∞-category either by identifying C with its nerve (a weak Kan complex) or considering the set $C(x, y)$ for two objects $x, y \in C$ as a discrete simplicial set (a simplicial category and thus an ∞-category). More generally we may consider a pair (C, S) consisting of a category C together with a set of morphisms S of C and construct the localised category $S^{-1}C$. This procedure can be refined using the simplicial localisation construction of Dwyer and Kan [DK]. Let L_*C denote the simplicial category of Construction 2.1.13. The *simplicial localisation* of the pair (C, W) is the localisation $(L_*W)^{-1}(L_*C) := L_S^{DK}(C)$. It has the property that there exists a natural isomorphism $\pi_0 L_S^{DK}(C) \simeq S^{-1}C$ showing that in general, $L_S^{DK}(C)$ contains higher homotopical information not encoded in $S^{-1}C$. If C is a category, then $L_S C$ is an ∞-category and $L_S C \to L_S^{DK}(C)$ is an equivalence of ∞-categories [DK]. When $\mathscr{M}$ is a model category we will let $L\mathscr{M} := L_W\mathscr{M}$ be the localisation of $\mathscr{M}$ along the set of weak equivalences W of $\mathscr{M}$. Thus $\mathrm{h}(L\mathscr{M}) \to \mathrm{h}\mathscr{M}$ is an equivalence of categories.

If $\mathscr{M}$ be an excellent model category and $\mathscr{A}$ an $\mathscr{M}$-enriched model category, we will write $L\mathscr{A}$ for the localisation of $\mathscr{A}$ with respect to its set of weak equivalences W in the sense of enriched category theory, ie. if D is a category and C is a D-enriched category then the localisation of C along a set of arrows S of

C is a pair $(L_S C, l)$ where $L_S C$ is a D-enriched category and $l : C \to L_S C$ is a D-enriched functor such that for any D-enriched category E, the induced map

$$\mathrm{Hom}_{\mathrm{hCat}(D)}(L_S C, E) \to \mathrm{Hom}_{\mathrm{hCat}(D)}(C, E)$$

is fully faithful and its essential image consists of those D-enriched functors which send each arrow in S to an equivalence in E. When C is a D-enriched category, a localisation of C will always refer to a localisation in the enriched sense unless otherwise stated.

Let $\mathscr{M}$ be an excellent model category and $\mathscr{A}$ and $\mathscr{B}$ two $\mathscr{M}$-enriched model categories. Let $f : \mathscr{A} \to \mathscr{B}$ be a right Quillen functor. Since f is right Quillen, the restriction map $Rf : \mathscr{A}^f \to \mathscr{B}^f$ between fibrant objects preserves equivalences and thus induces a map $L\mathscr{A}^f \to L\mathscr{B}^f$ of $\mathscr{M}$-enriched categories. The existence of functorial fibrant replacement functors then ensures the existence of a diagram

$$L\mathscr{M} \xleftarrow{\sim} L\mathscr{M}^f \to L\mathscr{N}^f \xrightarrow{\sim} L\mathscr{N}$$

and thus a map $L\mathscr{M} \to L\mathscr{N}$ well defined in $\mathrm{hCat}(\mathscr{M})$. When $\mathscr{M} = \mathscr{C}at_{(\infty,n-1)}$, we have a well defined (up to homotopy) map of (∞, n)-categories. When $\mathscr{A} \to \mathscr{B}$ is an equivalence of $\mathscr{M}$-enriched model categories then clearly $L\mathscr{M} \to L\mathscr{N}$ is an equivalence of $\mathscr{M}$-enriched categories.

Example 2.2.4. When C is a simplicial category LC is equivalent to the simplicial nerve $\mathrm{N}C$ of C. More generally, when $\mathscr{M}$ is a simplicial model category we have an equivalence $L\mathscr{M} \to \mathrm{N}(\mathscr{M}^\circ)$ of ∞-categories.

When $\mathscr{A}$ is an enriched model category, we will denote by $\mathscr{A}^\circ$ the full subcategory of $\mathscr{A}$ spanned by the fibrant-cofibrant objects. This $\mathscr{M}$-enriched category is a fibrant object of $\mathrm{Cat}(\mathscr{M})$ and is in fact equivalent to the localisation of $\mathscr{A}$.

Lemma 2.2.5. *Let $\mathscr{M}$ be an excellent model category and $\mathscr{A}$ be an $\mathscr{M}$-enriched model category. Then the map*

$$L\mathscr{A} \to \mathscr{A}^\circ$$

is an equivalence of $\mathscr{M}$-enriched categories.

Proof. Let $\mathscr{A}^c$ be the subcategory of $\mathscr{A}$ spanned by the cofibrant objects. The natural equivalences $Q : \mathscr{A} \to \mathscr{A}^c$ and $R : \mathscr{A}^c \to \mathscr{A}^\circ$ induce a chain of equivalences $L(\mathscr{A}) \simeq L(\mathscr{A}^c) \simeq L(\mathscr{A}^\circ) \simeq \mathscr{A}^\circ$ between $\mathscr{M}$-enriched categories. ☐

Let $\mathscr{M}$ be an excellent model category. Then the symmetric monoidal structure on $\mathscr{M}$ induces a symmetric monoidal structure on the model category $\mathrm{Cat}(\mathscr{M})_{\mathscr{E}}$ of $\mathscr{M}$-enriched categories. Given two $\mathscr{M}$-enriched categories C and D, the objects of the $\mathscr{M}$-enriched category $C \otimes D$ are pairs (x, y) where $x \in C$ and $y \in D$ and the mapping space between two objects (x, y) and (x', y') is given by

$$\mathrm{Map}_{C\otimes D}((x, y), (x', y')) = \mathrm{Map}_C(x, x') \otimes \mathrm{Map}_D(y, y')$$

where the tensor product on the right hand side is taken in $\mathscr{M}$. We know that $\mathrm{Cat}(\mathscr{M})_{\mathscr{E}}$ is not an internal model category (the tensor product bifunctor is not left Quillen since it does not preserve cofibrant objects). However, the derived tensor product $\otimes^{\mathbb{L}}$ is: there exists an internal Hom object in $\mathrm{hCat}(\mathscr{M})_{\mathscr{E}}$ which we denote by $\mathbb{R}\underline{\mathrm{Hom}}(C, D)$.

Proposition 2.2.6. *Let $\mathscr{M}$ be an excellent model category and $\mathscr{A}$ a combinatorial $\mathscr{M}$-enriched model category. Let C be a $\mathscr{M}$-enriched category and $\mathscr{A}^C$ be endowed with the projective model structure. Then there exists an equivalence*

$$\mathbb{R}\underline{\mathrm{Hom}}(C, \mathscr{A}^\circ) \to (\mathscr{A}^C)^\circ$$

of $\mathscr{M}$-enriched categories.

Proof. The proof is simply a corollary of Lemma 6.2 of [T3] after making the admissible replacement of the monoidal model category of complexes of k-modules by an arbitrary excellent model category. Let $(\mathrm{h}(\mathscr{A}^C))^{\mathrm{iso}}$ be the set of isomorphism classes of objects of $\mathrm{h}(\mathscr{A}^C)$. By Lemma 6.2 of *loc.cit.*, the map $[C, \mathscr{A}^\circ] \to (\mathrm{h}(\mathscr{A}^C))^{\mathrm{iso}}$ is an isomorphism. Thus we have the following chain of isomorphisms

$$[D, \mathbb{R}\underline{\mathrm{Hom}}(C, \mathscr{A}^\circ)] \simeq [D \times C, \mathscr{A}^\circ] \simeq (\mathrm{h}(\mathscr{A}^{C\times D}))^{\mathrm{iso}} \simeq (\mathrm{h}((\mathscr{A}^C)^D))^{\mathrm{iso}} \simeq [D, (\mathscr{A}^C)^\circ].$$

Since the construction is functorial in D, the result follows. □

The following is the very useful *strictification theorem.*

Proposition 2.2.7. *Let $\mathscr{M}$ be an excellent model category and $\mathscr{A}$ a combinatorial $\mathscr{M}$-enriched model category. Let C be a $\mathscr{M}$-enriched category and $\mathscr{A}^C$ be endowed with the projective model structure. Then there exists an equivalence*

$$L(\mathscr{A}^C) \to \mathbb{R}\underline{\mathrm{Hom}}(C, L\mathscr{A})$$

of $\mathscr{M}$-enriched categories.

Proof. This follows from Proposition 2.2.6 and Lemma 2.2.5. □

Example 2.2.8. Let A be an (∞, n)-precategory, D a $\mathscr{C}at_{(\infty,n-1)}$-enriched category and $\mathfrak{F}(A) \to D$ an equivalence. Then the induced map

$$L((\mathscr{C}at_{(\infty,n)})^D) \to \mathbb{R}\underline{\mathrm{Hom}}(A, L\mathscr{C}at_{(\infty,n)})$$

is an equivalence of (∞, n)-categories.

We will now define the ∞-category and (∞, n)-category of (∞, n)-categories using the the localisation functor.

Definition 2.2.9. We denote by:

- $\underline{\mathrm{Cat}}_{(\infty,n)} := L\mathscr{C}at_{(\infty,n)}$ the $(\infty, n+1)$-category of (∞, n)-categories (where we view $\mathscr{C}at_{(\infty,n)}$ as a $\mathscr{C}at_{(\infty,n)}$-enriched model category). This is equivalent to $(\mathscr{C}at_{(\infty,n)})^\circ$.
- $\mathrm{Cat}^\infty_{(\infty,n)} := L\mathscr{C}at_{(\infty,n)}$ the ∞-category of (∞, n)-categories (where we view $\mathscr{C}at_{(\infty,n)}$ as a (Set-enriched) model category). We have equivalences $\mathrm{Cat}^\infty_{(\infty,n)} \simeq L_S\mathrm{Cat}^{\mathfrak{K}^0}_{(\infty,n)} \simeq \mathrm{N}(\mathrm{Cat}^{\mathfrak{K}^0}_{(\infty,n)})$ where $\mathrm{Cat}^{\mathfrak{K}^0}_{(\infty,n)}$ is the simplicial category whose objects are (∞, n)-categories and whose mapping space between any two objects C and D is the $(\infty, 0)$-category $\mathfrak{K}^0(\mathbb{R}\underline{\mathrm{Hom}}(C, D))$.

We denote by $\mathcal{K} := \underline{\mathrm{Cat}}_{(\infty,0)}$ the ∞-category of $(\infty, 0)$-categories. The ∞-category $\mathcal{K}$ will play an important role in the remainder of the text fulfilling an analogous role as the category of sets does in ordinary category theory. We have equivalences $\mathcal{K} \simeq (\mathbf{S}_{\mathscr{K}})^\circ \simeq \mathrm{N}(\mathrm{Kan})$ where Kan is the simplicial category of Kan complexes.

Definition 2.2.10. Let C be an (∞, n)-category and x and object of C. Consider the map $ev_0 : \mathbb{R}\underline{\mathrm{Hom}}([1], C) \to C$ given by evaluation at 0. Then the induced homotopy pullback

$$C_{x/} := \mathbb{R}\underline{\mathrm{Hom}}([1], C) \times^h_C \{x\}$$

will be called the *undercategory* of C with respect to the object x.

Likewise, consider the map $ev_1 : \mathbb{R}\underline{\mathrm{Hom}}([1], C) \to C$ given by evaluation at 1. Then the induced homotopy pullback

$$C_{/x} := \mathbb{R}\underline{\mathrm{Hom}}([1], C) \times^h_C \{x\}$$

will be called the *overcategory* of C with respect to the object x. When C is the $(\infty, n+1)$-category $\underline{\mathrm{Cat}}_{(\infty,n)}$ of (∞, n)-categories, the mapping space in $(\underline{\mathrm{Cat}}_{(\infty,n)})_{/X}$ between two objects $f : A \to X$ and $g : B \to X$ will be denoted by

$$\mathbb{R}\underline{\mathrm{Hom}}_X(A, B) := \mathbb{R}\underline{\mathrm{Hom}}(A, B) \times_{\mathbb{R}\underline{\mathrm{Hom}}(A,X)} \{f\}.$$

Let X be an (∞, n)-category and consider the functor

$$\begin{aligned} \mathrm{Pr}_X : \mathscr{C}at_{(\infty,n)} &\to \mathrm{Cat}_{(\infty,n)} \\ A &\mapsto \mathbb{R}\underline{\mathrm{Hom}}(A^{op}, X) \end{aligned}$$

between categories. The (∞, n)-category $\mathrm{Pr}_X(A)$ will be called the (∞, n)-category of *X-valued prestacks* on A. When A is an (∞, n)-precategory and X is the (∞, n)-category $\underline{\mathrm{Cat}}_{(\infty,n-1)}$ of $(\infty, n-1)$-categories, we write $\mathrm{Pr}(A)$ for $\mathrm{Pr}_{\underline{\mathrm{Cat}}_{(\infty,n-1)}}(A)$ and refer to $\mathrm{Pr}(A)$ as the (∞, n)-category of *prestacks* on A. This (∞, n)-category will also be denoted $A^\wedge$. Let A be an (∞, n)-precategory. Then we can replace A by a strict $\mathscr{C}at_{(\infty,n-1)}$-enriched category $C := \mathfrak{F}(A)$. Let $C^{op} \times C \to \mathscr{C}at_{(\infty,n-1)}$ be the natural $\mathscr{C}at_{(\infty,n-1)}$-enriched bifunctor. By adjunction this gives a map $C \to (\mathscr{C}at_{(\infty,n-1)})^{C^{op}}$ where the right hand side is equivalent to $A^\wedge$ by the strictification theorem. We will refer to the composition

$$A \simeq C \to (\mathscr{C}at_{(\infty,n-1)})^{C^{op}} \simeq A^\wedge,$$

which is well defined in $\mathrm{h}\mathscr{C}at_{(\infty,n)}$, as the *Yoneda embedding*.

Proposition 2.2.11 ((∞, n)-Yoneda lemma)**.** *Let $A \in \mathscr{C}at_{(\infty,n)}$ be an (∞, n)-precategory. Then the Yoneda embedding $A \to \mathrm{Pr}(A)$ is fully faithful.*

Proof. From Section 2.1, every (∞, n)-precategory A can be associated with a $\mathscr{C}at_{(\infty,n-1)}$-enriched category $C := \mathfrak{F}(A^{op})$. Let D be a fibrant replacement of C and $(\mathscr{C}at_{(\infty,n-1)})^D$ be endowed with the projective model structure. The Yoneda embedding can be written as the following composition of maps

$$A \xrightarrow{F} \mathfrak{G}(((\mathscr{C}at_{(\infty,n-1)})^D)^\circ) \xrightarrow{j} \mathbb{R}\underline{\mathrm{Hom}}(A^{op}, \mathfrak{G}(\mathscr{C}at_{(\infty,n-1)})^\circ) \xrightarrow{\sim} \mathbb{R}\underline{\mathrm{Hom}}(A^{op}, \underline{\mathrm{Cat}}_{(\infty,n-1)}).$$

Since $\mathfrak{G}(\mathscr{C}at_{(\infty,n-1)})^\circ$ is an (∞, n)-category, $\mathbb{R}\underline{\mathrm{Hom}}(A^{op}, \mathfrak{G}(\mathscr{C}at_{(\infty,n-1)})^\circ)$ can be identified with an exponential object $[\mathfrak{G}(\mathscr{C}at_{(\infty,n-1)})^\circ]^{[A^{op}]}$ in $\mathrm{h}\mathscr{C}at_{(\infty,n)}$. Using the equivalence $\mathrm{h}\mathscr{C}at_{(\infty,n)} \simeq \mathrm{hCat}(\mathscr{C}at_{(\infty,n-1)})$, the map j is an equivalence from Example 2.2.8. We then apply the adjoint to F and factor it as

$$\mathfrak{F}(A) \xrightarrow{\sim} D^{op} \to (\underline{\mathrm{Cat}}_{(\infty,n-1)})^{D^{op}}.$$

It remains to show that the second map of $\mathrm{Cat}_{(\infty,n-1)}$-enriched categories is fully faithful. This follows from the classical enriched Yoneda lemma [Ke]. ☐

We have the following characterisation of $\mathrm{Pr}(A)$ in terms of a universal property: for every (∞, n)-category C, there exists an equivalence

$$\mathbb{R}\underline{\mathrm{Hom}}(C, \mathrm{Pr}(A)) \to \mathbb{R}\underline{\mathrm{Hom}}(C \times A^{op}, \underline{\mathrm{Cat}}_{(\infty,n-1)})$$

of (∞, n)-categories.

Definition 2.2.12. Let C be an (∞, n)-category. A prestack $F \in \mathrm{Pr}(C)$ is said to be *representable* if it lies in the essential image of the Yoneda embedding $C \to \mathrm{Pr}(C)$. Similarly, a prestack $F \in \mathrm{Pr}(C^{op})$ is said to be *corepresentable* if it lies in the essential image of the Yoneda embedding $C^{op} \to \mathrm{Pr}(C^{op})$.

Equivalently, a presheaf F in $\mathrm{Pr}(C)$ is said to be representable if the $\mathrm{h}\mathscr{C}at_{(\infty,n-1)}$-enriched functor $\mathrm{h}F : \mathrm{h}C^{op} \to \mathrm{h}\underline{\mathrm{Cat}}_{(\infty,n-1)} \simeq \mathrm{h}\mathscr{C}at_{(\infty,n-1)}$ is representable.

Proposition 2.2.13. *Let C be an (∞,n)-category. Then there exists a $\mathscr{C}at_{(\infty,n-1)}$-enriched model category $\mathscr{A}$ and a fully faithful map*
$$C \to L\mathscr{A}$$
of (∞,n)-categories.

Proof. Let $D := \mathfrak{F}(C)$ be a strict model for C. Then the proposition follows from the composition

$$C \xrightarrow{y} \mathbb{R}\underline{\mathrm{Hom}}(C^{op}, L\mathscr{C}at_{(\infty,n-1)}) \xrightarrow{\sim} \mathbb{R}\underline{\mathrm{Hom}}(D^{op}, L(\mathscr{C}at_{(\infty,n-1)}) \xrightarrow{\sim} L((\mathscr{C}at_{(\infty,n-1)})^{D^{op}})$$

using the fully faithful (∞,n)-Yoneda lemma of Proposition 2.2.11 and the strictification theorem of Proposition 2.2.7. We conclude by setting $\mathscr{A} := (\mathscr{C}at_{(\infty,n-1)})^{D^{op}}$. □

In the next section, we will use this property to characterise (∞,n)-categories having special properties, for example (∞,n)-categories which we call presentable, by placing natural conditions on the model category $\mathscr{A}$ and asking that the fully faithful map $C \to L\mathscr{A}$ is an equivalence.

2.3 Adjoints, limits and colimits

Definition 2.3.1. Let C be an (∞,n)-category, $k \le n$ and $0 \le m \le k$. The *homotopy k-category* $\mathrm{h}_k C$ of C is given as follows:

- Let $F, G : X \to Y$ be a pair of $(k-1)$-morphisms in C. A k-morphism from F to G in $\mathrm{h}_k C$ is an isomorphism class of k-morphisms from F to G in C.
- The $(k-m)$-morphisms in $\mathrm{h}_k C$ are the $(k-m)$-morphisms in C.

By convention, a 0-morphism in C is an object of C.

Thus $\mathrm{h}_0 C$ is simply the set of isomorphism classes of objects in C. We will denote $\mathrm{h}_1 C$ by simply $\mathrm{h}C$. Composition is well defined since τ_0 commutes with finite products and thus we obtain a well defined functor
$$\mathrm{h}_k : \mathrm{Cat}_{(\infty,n)} \to \mathrm{Cat}_k$$
for $k \le n$.

Let C be a 2-category and $f : x \to y$ and $g : y \to x$ be 1-morphisms in C. Recall that an adjunction in C is a pair of 2-morphisms (α,β) where $\alpha : \mathrm{id}_x \to g \circ f$ and $\beta : f \circ g \to \mathrm{id}_y$ such that the following compositions
$$f \simeq f \circ \mathrm{id}_x \xrightarrow{\mathrm{id}_f \circ \alpha} f \circ g \circ f \xrightarrow{\beta \circ \mathrm{id}_f} \mathrm{id}_y \circ f \simeq f$$
$$g \simeq \mathrm{id}_x \circ g \xrightarrow{\alpha \circ \mathrm{id}_g} g \circ f \circ g \xrightarrow{\mathrm{id}_g \circ \beta} g \circ \mathrm{id}_y \simeq g$$
coincide with the identities id_f and id_g respectively. We write $(\alpha,\beta) : f \dashv g$ to denote that f is left adjoint to g in the adjunction (α,β).

Definition 2.3.2. Let C be an (∞,n)-category for $n \ge 2$. An *adjunction* in C is an adjunction in the 2-category $\mathrm{h}_2 C$.

If a map $f : x \to y$ in an (∞,n)-category C admits a right adjoint g then this right adjoint is uniquely determined up to homotopy.

Example 2.3.3. An adjunction between two (∞,n)-categories is an adjuction in the $(\infty,n+1)$-category $\underline{\mathrm{Cat}}_{(\infty,n)}$. An adjunction between F and G induces an adjunction $\mathrm{h}F \dashv \mathrm{h}G$ between homotopy categories. However, if F and G are functors such that the induced functor $\mathrm{h}F : \mathrm{h}C \to \mathrm{h}D$ of $\mathrm{h}\mathscr{C}at_{(\infty,n)}$-enriched categories admits a right adjoint $\mathrm{h}G$ then F and G may not be necessarily adjoint. The condition merely guarantees the existence of a right adjoint to F.

Example 2.3.4. If $F : C \rightleftarrows D : G$ is an adjunction between $\mathscr{C}at_{(\infty,n-1)}$-enriched categories then the induced diagram $F : LC \rightleftarrows LD : G$ is an adjunction between (∞, n)-categories. If $F : \mathscr{A} \rightleftarrows \mathscr{B} : G$ is a Quillen adjunction between $\mathscr{C}at_{(\infty,n-1)}$-enriched model categories then the induced diagram $F : L\mathscr{A} \rightleftarrows L\mathscr{B} : G$ is an adjunction between (∞, n)-categories. More generally, F induces a diagram

$$L(\mathscr{A}^C) \simeq \mathbb{R}\underline{\mathrm{Hom}}(C, L\mathscr{A}) \rightleftarrows \mathbb{R}\underline{\mathrm{Hom}}(C, L\mathscr{B}) \simeq L(\mathscr{B}^C)$$

using the strictification theorem.

In the definition of an adjunction in an (∞, n)-category C in Definition 2.3.2 we have considered the data $(\alpha, \beta) : f \dashv g$ as living in the 2-category h_2C. Here the unit and counit maps α and β are compatible in a strict 2-categorical sense, ie. $(\beta \otimes \mathrm{id}_f) \circ (\mathrm{id}_f \otimes \alpha) = \mathrm{id}_f$ and $(\mathrm{id}_g \otimes \beta) \circ (\alpha \otimes \mathrm{id}_g) = \mathrm{id}_g$. More generally we may consider the adjunction as living in the (∞, n)-category itself. In this case we would need to specify equivalences $u : (\beta \otimes \mathrm{id}_f) \circ (\mathrm{id}_f \otimes \alpha) \xrightarrow{\sim} \mathrm{id}_f$ and $v : (\mathrm{id}_g \otimes \beta) \circ (\alpha \otimes \mathrm{id}_g) \xrightarrow{\sim} \mathrm{id}_g$ together with higher dimensional equivalences which describe compatibility conditions between u and v, higher dimensional equivalences between these equivalences and so on. The object which includes all of this information is called an adjunction datum.

In this paper we will be interested in adjunction datum between two objects x and y in an arbitrary $(\infty, 2)$-category C. The collection of all adjunction data between these two objects form an ∞-category denoted by $\mathrm{ADat}_{x,y}(C)$. Let $C(x, y)_\dashv$ denote the subcategory of the ∞-category $C(x, y)$ spanned by those objects $f : x \to y$ which admit right adjoints and whose morphisms are equivalences of these maps. Then there exists an equivalence

$$\mathrm{ADat}_{x,y}(C) \to C(x, y)_\dashv$$

of ∞-categories. This important result, stating that every map between two objects in an $(\infty, 2)$-category which admits a left adjoint can be extended (uniquely up to a contractible space of choices) to an adjunction datum, will be proved in Proposition 7.2.6.

We will now recall the notions of limit and colimit in the higher categorical setting together with some related results. In this paper, limits and colimits in ∞-categories will suffice.

Definition 2.3.5. Let D be an ∞-precategory and $i : I \to J$ a full inclusion of ∞-precategories in $\mathbb{R}\underline{\mathrm{Hom}}_D(I, J)$. Let C be an ∞-category, $p : C \to D$ a map and $i^* : \mathbb{R}\underline{\mathrm{Hom}}_D(J, C) \to \mathbb{R}\underline{\mathrm{Hom}}_D(I, C)$ the restriction functor. Then the left adjoint to i^* (if it exists)

$$i_! : \mathbb{R}\underline{\mathrm{Hom}}_D(I, C) \to \mathbb{R}\underline{\mathrm{Hom}}_D(J, C)$$

is called the *p-left Kan extension functor* along i relative to D.

Dually, a right adjoint to i^* (if it exists) is called the *p-right Kan extension functor* along i. Let $J = D = *$. The projection $I \to *$ yields a well defined map

$$c : C \to \mathbb{R}\underline{\mathrm{Hom}}(I, C)$$

called the *constant diagram functor* which sends an object $x \in C$ to the constant functor $I \to \{x\} \subset C$. An ∞-category C is said to have *(co)limits* with respect to the ∞-precategory I if the constant diagram functor $c : C \to \mathbb{R}\underline{\mathrm{Hom}}(I, C)$ has a right (resp. left) adjoint. In this case the right adjoint is denoted by $\lim_I : \mathbb{R}\underline{\mathrm{Hom}}(I, C) \to C$ and the left adjoint by $\mathrm{colim}_I : \mathbb{R}\underline{\mathrm{Hom}}(I, C) \to C$. Note that although $\lim_I F$ and $\mathrm{colim}_I F$ are not uniquely determined by the diagram $F : I \to C$, they are unique up to a contractible $(\infty, 0)$-category of choices.

Example 2.3.6. Let C be an ∞-category. A *pullback* in C is a limit of the diagram $\{1 \to 0 \leftarrow 2\}$. Likewise, a *pushout* in C is a colimit of the diagram $\{1 \leftarrow 0 \to 2\}$.

Example 2.3.7. Let C be an ∞-category. If C admits all (small) colimits then C is tensored over $\mathcal{K}$. If C admits all (small) limits then C is cotensored over $\mathcal{K}$. See Section 3.2 for more details on the notions of tensored and cotensored ∞-categories.

An ∞-category admits all finite limits if and only if it admits pullbacks and a final object. A functor between ∞-categories preserves finite limits if and only if it preserves pullbacks and final objects. An analogous statement holds for finite colimits by passing to the opposite ∞-category. Let $F : C \to D$ be a functor between ∞-categories. If C admits finite limits then F is said to be *left exact* if it preserves finite limits. If C admits finite colimits then F is said to be *right exact* if it preserves finite colimits. It is said to be *exact* if it is both left and right exact. We denote by $\underline{\mathrm{Hom}}^{\mathrm{lex}}(C, D)$ (resp. $\underline{\mathrm{Hom}}^{\mathrm{rex}}(C, D)$) the full subcategory of $\underline{\mathrm{Hom}}(C, D)$ spanned by the left exact (resp. right exact) functors.

Recall that a cardinal κ is said to be *regular* if it cannot be given as the coproduct of fewer than κ cardinals of cardinality less than κ. If κ is a regular cardinal and C is an ∞-category with pullbacks and κ-small products then C admits κ-small limits. Moreover, if $F : C \to D$ is any functor into an arbitrary ∞-category D, then F preserves κ-small limits if and only if F preserves pullbacks and κ-small products. An analogous statement holds for κ-small colimits by passing to the opposite ∞-category. An ∞-category is said to be *complete* if it admits all (small) limits and *cocomplete* if it admits all (small) colimits. It is said to be *bicomplete* if it is both complete and cocomplete. Let $F : C \to D$ be a functor between ∞-categories. If C admits small limits then F is said to be *continuous* if it preserves small limits. If C admits small colimits then F is said to be *cocontinuous* if it preserves small colimits. It is said to be *bicontinuous* if it is both continuous and cocontinuous. We denote by $\underline{\mathrm{Hom}}^{\mathrm{ct}}(C, D)$ (resp. $\underline{\mathrm{Hom}}^{\mathrm{coct}}(C, D)$) the full subcategory of $\underline{\mathrm{Hom}}(C, D)$ spanned by the continuous (resp. cocontinuous) functors.

The following proposition shows that the existence of homotopy limits and colimits in a simplicial model category ensures the existence of limits and colimits in its associated ∞-category in the sense of Definition 2.3.5.

Proposition 2.3.8. *Let $\mathscr{A}$ be a combinatorial simplicial model category. Then $L(\mathscr{A})$ admits all limits and colimits.*

Proof. Let C be an ∞-category and $D := \mathfrak{F}(C)$ the simplicial category associated to C. Then the existence of homotopy colimits in $\mathscr{A}$ ensures the existence of the left Quillen functor $\mathrm{holim}_D : \mathscr{A}^D \to \mathscr{A}$ (left adjoint to the constant diagram functor) where $\mathscr{A}^D$ is endowed with the projective model structure. This induces the functor

$$\mathrm{colim}_C : \mathbb{R}\underline{\mathrm{Hom}}(C, L\mathscr{A}) \simeq \mathbb{R}\underline{\mathrm{Hom}}(D, L\mathscr{A}) \simeq L(\mathscr{A}^D) \to L\mathscr{A}$$

left adjoint to the constant diagram functor which we identify with the colimit functor. A similar analysis follows for the limit functor. □

Proposition 2.3.9. *Let C be an ∞-category. The Yoneda embedding $C \to \mathrm{Pr}(C)$ preserves small limits.*

Proof reference. See Proposition 5.1.3.2 of [Lu]. □

Proposition 2.3.10. *Let A be an ∞-precategory and C an ∞-category which admits small colimits. Then composition with the Yoneda embedding $A \to \mathrm{Pr}(A)$ induces an equivalence*

$$\mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{coct}}(\mathrm{Pr}(A), C) \to \mathbb{R}\underline{\mathrm{Hom}}(A, C)$$

of ∞-categories.

Proof reference. See Theorem 5.1.5.6 of [Lu]. □

Let $\mathcal{A}$ be a collection of ∞-precategories. An ∞-category C is said to admit $\mathcal{A}$-indexed colimits if it admits colimits along all diagrams indexed by elements in $\mathcal{A}$. A functor $F : C \to D$ is said to preserve $\mathcal{A}$-indexed colimits if it preserves colimits along all diagrams indexed by elements in $\mathcal{A}$. We denote by $\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{A}}(C, D)$ the full subcategory of $\mathbb{R}\underline{\mathrm{Hom}}(C, D)$ spanned by those functors which preserve $\mathcal{A}$-indexed colimits. If C is an ∞-category and D an ∞-category which admits $\mathcal{A}$-indexed colimits then any functor functor $F : C \to D$ can be extended, essentially uniquely, to an $\mathcal{A}$-colimit preserving functor $G : \mathrm{Pr}^{\mathcal{A}}(C) \to D$ where $\mathrm{Pr}^{\mathcal{A}}(C)$ is an ∞-category admitting $\mathcal{A}$-indexed colimits.

Proposition 2.3.11. *Let C be an ∞-category and $\mathcal{A}$ a collection of ∞-precategories. Then there exists an ∞-category $\mathrm{Pr}^{\mathcal{A}}(C)$ admitting $\mathcal{A}$-indexed colimits and a fully faithful functor $y : C \to \mathrm{Pr}^{\mathcal{A}}(C)$ satisfying the following universal property: for any ∞-category D admitting $\mathcal{A}$-indexed colimits, composition with y induces an equivalence*

$$\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathrm{Pr}^{\mathcal{A}}(C), D) \to \mathbb{R}\underline{\mathrm{Hom}}(C, D)$$

of ∞-categories.

Proof reference. This is a special case of Proposition 5.3.6.2 of [Lu]. □

When $\mathcal{A}$ is the collection of all ∞-precategories, the ∞-category $\mathrm{Pr}^{\mathcal{A}}(C)$ is identified with the ∞-category of prestacks by Proposition 2.3.10. Given an ∞-category C, we may also define other ∞-categories, thought of informally as the ∞-categories associated to C by formally adding colimits of type $\mathcal{A}$, using this universal property.

Definition 2.3.12. Let κ be a regular cardinal. An ∞-category C is said to be *κ-filtered* if for any functor $F : A \to C$ indexed by a κ-small ∞-category A, there exists a natural transformation from F to a constant functor in $\underline{\mathrm{Cat}}_{\infty}$.

A functor $F : C \to D$ is said to be *κ-filtered* if C is κ-filtered. An ∞-category is said to be *filtered* if it is ω-filtered and likewise for a functor. Every ∞-category with a final object is κ-filtered for every regular cardinal κ.

Definition 2.3.13. Let C be an ∞-category, κ a regular cardinal and $\mathcal{A}$ the class of all small κ-filtered simplicial sets. Then the ∞-category of *ind-objects* of C is given by $\mathrm{Ind}_{\kappa}(C) := \mathrm{Pr}^{\mathcal{A}}(C)$.

For $\kappa = \omega$, the ∞-category of ind-objects of C will be simply denoted $\mathrm{Ind}(C)$. Let $\mathcal{A}$ be the class of all small κ-filtered ∞-precategories. The ∞-category of ind-objects of C admits the following characterisation:

- The objects of $\mathrm{Ind}_{\kappa}(C)$ are functors $I \to C$ where $I \in \mathcal{A}$.
- Given two objects $F : I \to C$ and $G : J \to C$ in $\mathrm{Ind}_{\kappa}(C)$, the mapping space is given by

$$\mathrm{Map}(F, G) = \lim_{i \in I} \operatorname*{colim}_{j \in J} C(F(i), G(j)).$$

By Proposition 5.3.5.14 of [Lu], the Yoneda embedding $y : C \to \mathrm{Ind}_{\kappa}(C)$ taking x to the functor $x : * \to C$ preserves all κ-small colimits which exist in C. The essential image of y consists of objects satisfying the following:

Definition 2.3.14. Let C be an ∞-category which admits κ-filtered colimits. An object x in C is said to be *κ-compact* if the corepresentable functor

$$C(x, \bullet) : C \to \mathcal{K}$$

preserves κ-filtered colimits.

If C admits filtered colimits and x is ω-compact then x is said to be *compact*. Let C^{cpt} denote the full subcategory of C spanned by the compact objects. Let C and D be ∞-categories which admit κ-filtered colimits and $f : C \rightleftarrows D : g$ be an adjunction. If g preserves κ-filtered colimits then f preserves κ-compact objects.

If κ is a regular cardinal, then an ∞-category C is said to be *κ-closed* if every diagram in C indexed by a κ-small simplicial set admits a colimit in C. Clearly, an ∞-category C is equivalent to $\mathrm{Ind}_{\kappa}(D)$ for some small ∞-category D if and only if the ∞-category C is κ-closed and has a small subcategory D consisting of κ-compact objects such that every object of C is a κ-filtered colimit of objects of D. This motivates the following.

Definition 2.3.15. Let κ be a regular cardinal. An ∞-category C is said to be *κ-accessible* if there exists a small ∞-category D such that
$$\mathrm{Ind}_\kappa(D) \to C$$
is an equivalence of ∞-categories.

An ∞-category is said to be *accessible* if it is κ-accessible for some regular cardinal κ. If C is an accessible ∞-category then a functor $F : C \to D$ is said to be *accessible* if it preserves κ-filtered colimits for some regular cardinal κ. One can show that if C and D are accessible ∞-categories then a functor $F : C \to D$ is accessible if F admits a left or right adjoint. Let C be an ∞-category and κ a regular cardinal. Then C is κ-accessible if and only if C admits small κ-filtered colimits and is generated under small κ-filtered colimits by an essentially small full subcategory $D \subseteq C$ of κ-compact objects.

Example 2.3.16. For any small ∞-category C, the ∞-category of prestacks $\mathrm{Pr}(C)$ is accessible. In particular, the ∞-category $\mathcal{K}$ of spaces is accessible. More generally, for any accessible ∞-category C and any ∞-precategory A, the ∞-category $\mathbb{R}\underline{\mathrm{Hom}}(A, C)$ is accessible.

Definition 2.3.17. An ∞-category C is said to be *presentable* if it is accessible and admits small colimits.

Example 2.3.18. If C is a presentable ∞-category and A is an ∞-precategory then the ∞-category $\mathbb{R}\underline{\mathrm{Hom}}(A, C)$ is presentable. This follows from Example 2.3.16 and the fact that colimits are calculated pointwise in functor categories. In particular, the ∞-category of prestacks $\mathrm{Pr}(A)$ is presentable. Furthermore, if C and D are presentable ∞-categories then $\mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{coct}}(C, D)$ is presentable.

Let $\underline{\mathrm{Cat}}^p_\infty$ denote the full subcategory of $\underline{\mathrm{Cat}}_\infty$ spanned by the presentable ∞-categories and colimit preserving functors.

Example 2.3.19. Let C be a presentable ∞-category. Then the overcategory $C_{x/}$ and the undercategory $C_{/x}$ are presentable (Proposition 5.5.3.10 and Proposition 5.5.3.11 of [Lu]).

Let C be an ∞-category and S a set of arrows of S. In the setting of presentable ∞-categories, the theory of localisations of Section 2.2 has a simple characterisation: L_SC can be identified with a full subcategory of C. An object x in C is said to be *S-local* if for every arrow $f : y \to z$ in S, the induced map $C(z, x) \to C(y, z)$ is an equivalence in $\mathcal{K}$. An arrow $f : x \to y$ in C is said to be an *S-equivalence* if for every S-local object z in C, the induced map $C(y, z) \to C(x, z)$ is an equivalence in $\mathcal{K}$. Let C be a presentable ∞-category and (L_SC, l) a localisation of C in the $(\infty, 2)$-category $\underline{\mathrm{Cat}}^p_\infty$. Then an object x in C is S-local if and only if it belongs to L_SC. Furthermore, every element of S is an S-equivalence in C. A localisation in the setting of presentable ∞-categories will be called a *Bousfield localisation* to distinguish from the more general localisation of Definition 2.2.1.

Proposition 2.3.20. *Let C be an ∞-category. The following conditions are equivalent.*

1. *The ∞-category C is presentable.*
2. *There exists a combinatorial simplicial model category $\mathscr{M}$ and an equivalence*
$$C \simeq L\mathscr{M}.$$
3. *There exists a regular cardinal κ and an equivalence $C \to \mathrm{Ind}_\kappa(D)$ for a κ-cocomplete ∞-category D.*
4. *There exists a small ∞-category D such that C is an (accessible) localisation of* $\mathrm{Pr}(D)$.

Proof reference. The equivalence between (1) and (2) follow from Proposition A.3.7.6 of [Lu] together with the equivalence $\mathrm{N}(\mathscr{M}^\circ) \to L\mathscr{M}$. The others follow from Theorem 5.5.1.1 of *loc. cit.*. See also [S3]. □

Let C be a presentable ∞-category. By Proposition 5.5.2.2 of [Lu], a functor $F : C^{op} \to \mathcal{K}$ is representable if and only if it preserves small limits. Similarly, a functor $F : C \to \mathcal{K}$ is corepresentable if and only if it is accessible and preserves small limits (Proposition 5.5.2.7 of [Lu]). An important ramification of the representability statement is that a presentable ∞-category is bicomplete. We also have the following adjoint functor theorem:

Proposition 2.3.21. *Let C and D be presentable ∞-categories. A functor $F : C \to D$ admits a right adjoint if and only if it preserves small colimits. It admits a left adjoint if and only if it is accessible and preserves small limits.*

Proof reference. See Corollary 5.5.2.9 of [Lu]. □

Let C be an ∞-category. A full subcategory D of C is said to be *stable under colimits* if for all $I \in \mathscr{C}at_\infty$, the functor colim_I factors through D. Let C admit all small colimits and $\{x_\alpha\}$ be a set of objects of C. Then the $\{x_\alpha\}$ are said to *generate C under colimits* if for any full subcategory D of C containing $\{x_\alpha\}$ which is stable under colimits, we have $C = D$. A map $F : I \to C$ is said to *generate C under colimits* if the image $F(\mathrm{Ob}(I))$ generates C under colimits.

If κ is a regular cardinal, an ∞-category C is said to be *κ-compactly generated* if it is presentable and κ-accessible. An ∞-category which is ω-compactly generated will simply be called *compactly generated.* An ∞-category C is κ-compactly generated if and only if there exists a small ∞-category D which admits κ-small colimits and an equivalence $C \to \mathrm{Ind}_\kappa(D)$. For example, the ∞-categories $\mathcal{K}$ and $\mathrm{Cat}_\infty^\infty$ are compactly generated.

If C is an ∞-category which admits colimits along Δ^{op} then an object x in C is said to be *projective* if the functor $C(x, \bullet) : C \to \mathcal{K}$ corepresented by x preserves colimits along Δ^{op}. The collection of projective objects in an ∞-category is stable under finite coproducts. Let C be an ∞-category which admits small colimits and $\{x_\alpha\}$ a collection of objects of C. The collection $\{x_\alpha\}$ is said to be a set of *compact projective generators* for C if it satisfies the following conditions:

- Each element of $\{x_\alpha\}$ is both a projective and compact object of C.
- The full subcategory of C spanned by the elements of $\{x_\alpha\}$ is essentially small.
- The collection $\{x_\alpha\}$ generates C under small colimits.

If C is an ∞-category which admits small colimits then C is said to be *projectively generated* if there exists a set of compact projective generators for C. For example, the ∞-category $\mathcal{K}$ is projectively generated. Its set of compact projective generators is the set of spaces which are homotopy equivalent to finite sets.

3 Monoidal structures

In Section 3.1 we discuss the general theory of monoidal (∞, n)-categories. We will only need the special case of $n = 1$ and $n = 2$ but it is convenient and instructive to state our definitions for all n. The indexing category used throughout this section determines whether the monoidal structure is symmetric or merely associative. Thus we will generally speak of an $\mathcal{O}$-monoidal structure where $\mathcal{O} = \Gamma$ determines the symmetric case and $\mathcal{O} = \Delta^{op}$ determines the associative case (see Notation 1.1 for the definitions of the various indexing categories used here). We describe the general procedure of associating an $\mathcal{O}$-monoidal ∞-category to an $\mathcal{O}$-monoidal category refining the localisation process of Section 2.2. More generally, one can associate an $\mathcal{O}$-monoidal (∞, n)-category to a $\mathscr{C}at_{(\infty,n-1)}$-enriched $\mathcal{O}$-monoidal model category. An important result is then Proposition 3.1.7 which is an (∞, n)-categorical version of the Grothendieck construction which associates to a functor, taking values in the model category of (∞, n)-categories, a certain cofibered (∞, n)-category. This construction is very useful for endowing an (∞, n)-category with a monoidal structure. In this section we also define lax $\mathcal{O}$-monoidal functors and $\mathcal{O}$-monoid objects whilst describing several important examples.

In Section 3.2 we define the (∞, n)-category of $\mathcal{O}$-module objects in an $\mathcal{O}$-monoidal (∞, n)-category (passing to the opposite (∞, n)-categories defines the (∞, n)-category of $\mathcal{O}$-comodule objects). This utilises the general notion of an (∞, n)-category being $\mathcal{O}$-tensored of an $\mathcal{O}$-monoidal (∞, n)-category. We define what it means for an (∞, n)-category $\mathcal{O}$-tensored over a $\mathcal{O}$-monoidal (∞, n)-category to be enriched and cotensored. We also recall the proof giving conditions on a presentable ∞-category to be enriched and cotensored over an $\mathcal{O}$-monoidal ∞-category. We then recall from [LIII] the construction of the symmetric monoidal structure on the ∞-category of modules and construct an extension to the (∞, n)-categorical case. This enables us to define the (∞, n)-category of commutative algebra objects in a symmetric monoidal (∞, n)-category. Finally we describe the very useful result of the equivalence between the (∞, n)-category of commutative R-algebra objects in a symmetric monoidal (∞, n)-category and the (∞, n)-category of commutative monoid objects under the object R.

In Section 3.3 we introduce the notion of a stable ∞-category. A stable ∞-category can be described as a pointed ∞-category with finite limits whose loop space functor is an equivalence of ∞-categories (Proposition 3.3.2). Stable ∞-categories replace the role of abelian categories in the ∞-category realm. Indeed, after defining the ∞-category Sp of spectrum objects in the ∞-category of spaces, called spectra, together with its natural symmetric monoidal structure, we will see in Proposition 3.3.13 that every stable ∞-category is naturally enriched over spectra. Endowing Sp with its natural t-structure, Proposition 3.5.14 states that the heart of Sp is equivalent to the category of abelian groups.

The next step is to introduce our notion of ring in the ∞-categorical context. Since a commutative ring in classical algebra is simply a commutative monoid object in the category of abelian groups, it is natural to define a "generalised" commutative ring as a commutative monoid object in the ∞-category of spectra. These objects, called commutative ring spectra or more simply E_∞-rings, are introduced in Section 3.4. An important subcategory of E_∞-rings are those which are connective: objects whose nth homotopy group of underlying spectra vanishes for $n < 0$. We show in Proposition 3.5.15 that the subcategory of connective 0-truncated E_∞-rings is equivalent to the category of commutative rings.

In Section 3.5 we provide a brief review of t-structures in the ∞-categorical context. Our main examples are the canonical non-degenerate t-structures on the ∞-category of spectra and the ∞-category of modules over an E_∞-ring. We define the notions of t-exactness and what it means for a functor between stable ∞-categories to create a t-structure. We discuss the heart of a stable ∞-category admitting a t-structure and show that it admits a symmetric monoidal structure when the stable ∞-category is symmetric monoidal. Finally we prove an important lemma which provides conditions under which a functor between stable ∞-categories admitting non-degenerate t-structures commutes with limits of cosimplicial objects (Lemma 3.5.16).

In Section 3.6 we introduce the notion of an R-linear ∞-category for R a commutative monoid object in a symmetric monoidal ∞-category (Definition 3.6.1). When R is an E_∞-ring we extend this to define the $(\infty, 2)$-category $\mathrm{Tens}_R^{\mathrm{lax}}$ of R-tensor ∞-categories and lax symmetric monoidal functors (Definition 3.6.8) which makes use of a lax comma category construction. An R-tensor ∞-category is a symmetric monoidal

∞-category which is R-linear, stable and presentable. We give two examples of R-tensor ∞-categories arising from the theory of differential graded categories and spectral categories. An important ingredient in the Tannaka duality theorem for ∞-categories is Conjecture 3.6.10 which states that the ∞-category of commutative R-algebras is equivalent to the ∞-category of endomorphisms of the ∞-category Mod_R of R-modules in an appropriate subcategory of the $(\infty, 2)$-category of R-tensor ∞-categories and lax symmetric monoidal functors. Assuming this conjecture we find that the ∞-category of comonads on Mod_R in the $(\infty, 2)$-category $\mathrm{Tens}_R^{\mathrm{lax}}$ is equivalent to the ∞-category of comonoid objects in the ∞-category of R-algebras.

3.1 Monoidal (∞, n)-categories

Throughout this section we will let $\mathcal{O}$ denote an element of the two object set $\{\Delta^{op}, \Gamma\}$. For each $n \geq 1$ and $0 < i \leq n$, consider the n inclusions $p_i : [1] \to [n]$ in Δ given by $p_i(\{0,1\}) = \{i-1, i\}$ and the n pointed maps $p_i : [n] \to [1]$ in Γ given by $p_i(j) = \{j\}$ if $i = j$ and $p_i(j) = *$ otherwise.

Definition 3.1.1. Let $\mathscr{M}$ be a model category. An *$\mathcal{O}$-Segal monoid object* in $\mathscr{M}$ is a functor $A : \mathcal{O} \to \mathscr{M}$ such that for each $n \geq 0$, the map

$$A([n]) \xrightarrow{\prod_i A(p_i)} A([1])^n$$

is a weak equivalence in $\mathscr{M}$.

Let $\mathrm{SeMon}^{\mathcal{O}}(\mathscr{M})$ denote the full subcategory of $\underline{\mathrm{Hom}}(\mathcal{O}, \mathscr{M})$ spanned by the $\mathcal{O}$-Segal monoid objects. By convention, the map $A([0]) \to *$ is a weak equivalence in the definition of an $\mathcal{O}$-Segal monoid object.

Remark 3.1.2. Let $n \geq 1$, $0 < i \leq n$ and $p_i : [n] \to [1]$ be the map in Δ_+ given by $p_i(j) = \{j\}$ if $i = j$ and $\emptyset$ otherwise. A Segal monoid object in $\mathscr{M}$ can also be described as a functor $A : \Delta_+ \to \mathscr{M}$ such that

$$A([n]) \xrightarrow{\prod_i A(p_i)} A([1])^n$$

is a weak equivalence in $\mathscr{M}$.

Let $\mathscr{M}$ be an excellent model category and $\mathscr{A}$ an $\mathscr{M}$-enriched model category. It is well known that there exists a model structure on the $\mathscr{M}$-enriched category of functors $\mathscr{A}^{\mathcal{O}}$, given by a Bousfield localisation of the projective model structure, whose fibrant objects are precisely the $\mathcal{O}$-Segal monoid objects. We denote this $\mathscr{M}$-enriched model category by $\mathrm{SeMon}^{\mathcal{O}}(\mathscr{A})_{\mathscr{S}}$ and call it the *special* model structure. Let $\mathbb{R}\mathrm{SeMon}^{\mathcal{O}}(\mathscr{A})$ denote the full subcategory of $\mathbb{R}\underline{\mathrm{Hom}}(\mathcal{O}, \mathscr{A})$ spanned by the $\mathcal{O}$-Segal monoid objects. Then from Section 2.2 we have equivalences

$$L(\mathrm{SeMon}^{\mathcal{O}}(\mathscr{A})_{\mathscr{S}}) \simeq (\mathrm{SeMon}^{\mathcal{O}}(\mathscr{A})_{\mathscr{S}})^{\circ} \simeq \mathbb{R}\mathrm{SeMon}^{\mathcal{O}}(L\mathscr{A}) \simeq \mathbb{R}\mathrm{SeMon}^{\mathcal{O}}(\mathscr{A}^{\circ})$$

in $\mathrm{Cat}(\mathscr{M})$.

Definition 3.1.3. An *$\mathcal{O}$-monoidal (∞, n)-category* is an $\mathcal{O}$-Segal monoid object in the model category $\mathscr{C}at_{(\infty,n)}$ of (∞, n)-categories.

An *$\mathcal{O}$-monoidal functor* is simply a natural transformation of functors. A Δ^{op}-monoidal (∞, n)-category will be called a *monoidal* (∞, n)-category and a Γ-monoidal (∞, n)-category a *symmetric monoidal* (∞, n)-category. Likewise for $\mathcal{O}$-monoidal functors. Since $\mathscr{C}at_{(\infty,n)}$ is a $\mathscr{C}at_{(\infty,n)}$-enriched model category, the model category $\mathrm{SeMon}^{\mathcal{O}}(\mathscr{C}at_{(\infty,n)})_{\mathscr{S}}$ is the $\mathscr{C}at_{(\infty,n)}$-enriched model category of $\mathcal{O}$-premonoidal (∞, n)-precategories. Let $\underline{\mathrm{Cat}}^{\mathcal{O}}_{(\infty,n)} := \mathbb{R}\mathrm{SeMon}^{\mathcal{O}}(L\mathscr{C}at_{(\infty,n)}) = \mathbb{R}\mathrm{SeMon}^{\mathcal{O}}(\underline{\mathrm{Cat}}_{(\infty,n)})$ denote the $(\infty, n+1)$-category of $\mathcal{O}$-monoidal (∞, n)-categories. In particular we denote by:

- $\underline{\mathrm{Cat}}^{\mathrm{M}}_{(\infty,n)} := \mathbb{R}\mathrm{SeMon}^{\Delta^{op}}(\underline{\mathrm{Cat}}_{(\infty,n)})$ the $(\infty, n+1)$-category of monoidal (∞, n)-categories.
- $\underline{\mathrm{Cat}}^{\mathrm{sM}}_{(\infty,n)} := \mathbb{R}\mathrm{SeMon}^{\Gamma}(\underline{\mathrm{Cat}}_{(\infty,n)})$ the $(\infty, n+1)$-category of symmetric monoidal (∞, n)-categories.

Note by Proposition 2.2.6 that there exists an equivalence

$$\mathbb{R}\underline{\mathrm{Hom}}(\mathcal{O}, (\mathscr{C}at_{(\infty,n)})^{\circ}) \to ((\mathscr{C}at_{(\infty,n)})^{\mathcal{O}})^{\circ}$$

of $\mathscr{C}at_{(\infty,n)}$-enriched categories. This important strictification result enables us to consider the $(\infty, n+1)$-category of $\mathcal{O}$-monoidal (∞, n)-categories as ordinary functors into $\mathscr{C}at_{(\infty,n)}$ (as in the right hand side) as opposed to the much less explicit description of functors into some fibrant replacement of $\mathscr{C}at_{(\infty,n)}$.

Let A be an (∞, n)-category. The underlying (∞, n)-category of an $\mathcal{O}$-monoidal (∞, n)-category A is given by $A([1])$. More explicitly, an $\mathcal{O}$-monoidal (∞, n)-category A encodes the following structure:

- A unit object $1_A : A([0]) \to A([1])$ induced by the zero map $[0] \to [1]$ in $\mathcal{O}$.
- An $\mathcal{O}$-monoidal product $\otimes : A[1] \times A[1] \to A[1]$ given by the composition

$$A[1] \times A[1] \simeq A(\{0,1\}) \times A(\{1,2\}) \xleftarrow{\sim} A([2]) \to A(\{0,2\}) \simeq A[1]$$

  induced by the three inclusions $[1] \hookrightarrow [2]$ in $\mathcal{O}$.
- All higher homotopy coherences of the $\mathcal{O}$-monoidal product.

Let C be a $\mathcal{O}$-monoidal (∞, n)-category. Then the homotopy n-category $\mathrm{h}_n C$ inherits the structure of a $\mathcal{O}$-monoidal category. It is simply given by the composition

$$\mathcal{O} \xrightarrow{C} \mathscr{C}at_{(\infty,n)} \xrightarrow{\mathrm{h}_n} \mathrm{Cat}_n \hookrightarrow \mathscr{C}at_{(\infty,n)}$$

which satisfies the conditions to be an $\mathcal{O}$-Segal monoid object owing to the fact that the functor h_n commutes with finite products. In the opposite direction, let (C, S) be a pair consisting of a $\mathcal{O}$-monoidal category C together with a set of arrows S of C containing all the isomophisms and such that the tensor product bifunctor preserves maps in S. Then we can associate to (C, S) a symmetric monoidal ∞-category $L_S^{\mathcal{O}} C$ as follows. Let Fin denote the $\mathcal{O}$-monoidal category of finite sets and bijective maps. We first consider the functor

$$\begin{aligned} \mathrm{N}_{\mathcal{O}}(C) : \mathcal{O} &\to \mathrm{Cat} \\ [n] &\mapsto \underline{\mathrm{Hom}}^{\mathcal{O}}(\underline{\mathrm{Hom}}_0^{\mathcal{O}}([n], \mathrm{Fin}), C) \end{aligned}$$

where $\underline{\mathrm{Hom}}_0^{\mathcal{O}}([n], \mathrm{Fin})$ is the $\mathcal{O}$-monoidal category of functors sending 0 to 1. Composing $\mathrm{N}_{\mathcal{O}}(C)$ with the inclusion $\mathrm{Cat} \to (\mathrm{Cat}_\infty)^{\mathcal{O}}$, we obtain a functor

$$\mathrm{N}_{\mathcal{O}} : \mathrm{Cat}^{\mathcal{O}} \to \mathrm{Cat}_\infty^{\mathcal{O}}$$

where $\mathrm{N}_{\mathcal{O}}(C) : \mathcal{O} \to \mathscr{C}at_\infty$ is a symmetric monoidal ∞-category with the property that $\mathrm{N}_{\mathcal{O}}(S)_n = S_n$. We now set

$$\begin{aligned} L_S^{\mathcal{O}} C : \mathcal{O} &\to \mathscr{C}at_\infty \\ [n] &\mapsto L_{S_n} \mathrm{N}_{\mathcal{O}}(C) \end{aligned}$$

which is a $\mathcal{O}$-monoidal ∞-category owing to the fact that the localisation L preserves finite products. We also obtain the following universal property: for every $\mathcal{O}$-monoidal ∞-category D, the induced map

$$\mathbb{R}\underline{\mathrm{Hom}}^{\mathcal{O}}(L_S^{\mathcal{O}} C, D) \to \mathbb{R}\underline{\mathrm{Hom}}^{\mathcal{O}}(C, D)$$

is fully faithful and its essential image consists of those symmetric monoidal functors $F : C \to D$ which send each arrow of S in $C([1])$ to an equivalence in $D([1])$. When $\mathscr{M}$ is a symmetric monoidal model category we set $L^{\mathcal{O}} \mathscr{M} := L_W^{\mathcal{O}} \mathscr{M}^c$. The underlying $\mathcal{O}$-monoidal category $L^{\mathcal{O}} \mathscr{M}([1])$ is equivalent to $L\mathscr{M}$.

Lemma 3.1.4. *Let $\mathscr{M}$ be a monoidal model category and $\mathscr{A}$ a combinatorial $\mathcal{O}$-monoidal $\mathscr{M}$-enriched model category. Then the tensor product on $L^{\mathcal{O}}\mathscr{A}$ preserves (small) colimits separately in each variable.*

Proof. Since $\mathrm{h}\mathscr{A}$ is closed with respect to the $\mathcal{O}$-monoidal structure, the functor $x \otimes \bullet$ commutes with colimits since it is a left adjoint. □

Although the definition of an $\mathcal{O}$-monoidal (∞, n)-category as a $\mathcal{O}$-Segal monoid object is useful in a variety of applications, it is often easier to construct an $\mathcal{O}$-monoidal (∞, n)-category using the language of cofibered (∞, n)-categories which we introduce here. For the special case of $n = 1$ see Section 3.2 of [Lu] and Section 1.3 of [TV3]. Let I be a category and consider an object $p : A \to I$ in the category $(\mathscr{C}at_{(\infty,n)})_{/I}$. An arrow f in $A(a, b)$ is said to be *p-cocartesian* if for all $c \in A$, the induced morphism

$$A(b, c) \to A(a, c) \times_{I(p(a),p(c))} I(p(b), p(c))$$

is a weak equivalence in $\mathscr{C}at_{(\infty,n-1)}$.

Definition 3.1.5. An object $p : A \to I$ in the category $(\mathscr{C}at_{(\infty,n)})_{/I}$ is said to be a *cofibered* (∞, n)-category if for every arrow $u : i \to j$ in I and every object a in A with $p(a) = i$, there exists a p-cocartesian arrow f such that $p(f)$ is isomorphic to u in the undercategory $I_{i/}$. A morphism in the homotopy category of $(\mathscr{C}at_{(\infty,n)})_{/I}$ is said to be *cocartesian* if it preserves cocartesian arrows.

The (non-full) subcategory of $\mathrm{h}((\mathscr{C}at_{(\infty,n)})_{/I})$ consisting of cofibered objects and cocartesian morphisms will be denoted by $\mathrm{h}((\mathscr{C}at_{(\infty,n)})_{/I})^{cc}$. An important observation is that the condition to be cofibered is stable by equivalences in $\mathscr{C}at_{(\infty,n)}$. There exists a Quillen adjunction

$$\int_I : (\mathscr{C}at_{(\infty,n)})^I \rightleftarrows (\mathscr{C}at_{(\infty,n)})_{/I} : \mathrm{Sec}_I$$

which is defined as follows. The left adjoint $\int_I$ acts on a functor $F : I \to \mathscr{C}at_{(\infty,n)}$ to give the (∞, n)-precategory of *elements* of F, ie. the objects of $\int_I F$ are pairs (i, X) where $i \in I$, $X \in F(i)$ and

$$\left(\int_I F\right)((i_1, X_1), \ldots, (i_n, X_n)) := \coprod_{i_1 \to \ldots \to i_n} F(i_n)(X_1, \ldots, X_n)$$

for objects $\{(i_k, X_k)\}_{1 \le k \le n}$ where the X_k on the right hand side now denote the image of $X_k \in F(i_k)$ by the maps $F(i_k) \to F(i_n)$. The sum is over all diagrams in I of the form $i_1 \to \ldots \to i_n$. The right adjoint Sec_I acts on an object $A \to I$ of $(\mathscr{C}at_{(\infty,n)})_{/I}$ as

$$\mathrm{Sec}_I(A) := \underline{\mathrm{Hom}}_I(\bullet_{/I}, A) : I \to \mathscr{C}at_{(\infty,n)}.$$

Since $\int_I$ preserves weak equivalences between cofibrant objects and Sec_I preserves weak equivalences between fibrant objects we obtain a derived adjunction

$$\mathbb{L}\int_I : \mathrm{h}((\mathscr{C}at_{(\infty,n)})^I) \rightleftarrows \mathrm{h}((\mathscr{C}at_{(\infty,n)})_{/I}) : \mathbb{R}\mathrm{Sec}_I.$$

The left derived functor $\mathbb{L}\int_I$ factors through the subcategory $\mathrm{h}((\mathscr{C}at_{(\infty,n)})_{/I})^{cc}$. To see this let $p : A = \mathbb{L}\int_I F \to I$ be the image of the functor $F : I \to \mathscr{C}at_{(\infty,n)}$ by the map $\int_I$. We can suppose that p is a fibration of (∞, n)-categories since the property of being fibered is invariant by equivalences in $(\mathscr{C}at_{(\infty,n)})_{/I}$. Let $u : i \to j$ be an arrow in I and $a \in A$ with $p(a) \simeq i$. As p is a fibration, the isomorphism $p(a) \simeq i$ can be recovered from an equivalence $a \to a'$ in A with $\pi(a') = i$. Thus we can continue by assuming that $p(a) = i$. The object a is then of the form (i, X) with $X \in F(i)$. Let $Y = F(u)(X) \in F(j)$. Then the arrow u and the identity $\mathrm{id} : Y \to Y$ define an element

$$\alpha \in \left(\mathbb{L}\int_I F\right)((i, X), (j, Y)) := \coprod_{v : i \to j} F(j)(v(X), Y)$$

with $p(\alpha) = u$. Also, if $(k, z) \in A$, the following square

$$\begin{tikzcd}
\coprod_{v:j\to k} F(k)(v(y), z) \arrow[r, "\circ\alpha"] \arrow[d] & \coprod_{w:i\to k} F(k)(w(x), z) \arrow[d] \\
I(j,k) \arrow[r, "\circ p(\alpha)"'] & I(i,k)
\end{tikzcd}$$

is a pullback and hence a homotopy pullback since the map $\circ p(\alpha)$ is a map between discrete simplicial sets. This shows that α is cocartesian with $p(\alpha) = u$ and hence that p is fibered. It remains to show that if $f : F \to G$ is a morphism in $(\mathscr{C}at_{(\infty,n)})^I$, then the induced map $\mathbb{L}\int_I f : \mathbb{L}\int_I F \to \mathbb{L}\int_I G$ is cocartesian. Since the map $\mathbb{L}\int_I f$ sends an arrow (u, α) in $\mathbb{L}\int_I F$ to the arrow $(f(u), f(\alpha))$ in $\mathbb{L}\int_I G$, the result follows from the following lemma.

Lemma 3.1.6. *An arrow (u, α) in $(\int_I F)((i, X), (j, Y))$ is cocartesian if and only if α is an equivalence in $F(j)$.*

Proof. By definition the pair (u, α) consists of a map $u : i \to j$ together with a map α in $F(j)(u_*(X), Y)$. Let us assume firstly that (u, α) is cocartesian. Then there exists a homotopy pullback square

$$\begin{tikzcd}
\coprod_{\omega:j\to k} F(k)(\omega_*(Y), Z) \arrow[r, "\circ\alpha"] \arrow[d, "p"'] & \coprod_{\kappa:i\to k} F(k)(\kappa_*(X), Z) \arrow[d, "p"] \\
I(p(Y), p(Z)) \arrow[r, "\circ p(\alpha)"'] & I(p(X), p(Z)).
\end{tikzcd}$$

Since the square is a homotopy pullback, the homotopy fiber of p over an object v in $I(p(Y), p(Z))$ is equivalent to the homotopy fiber of p over $v \circ u$, ie. for all $Z \in F(k)$, the map

$$\circ\alpha : F(k)(v_*(Y), Z) \to F(k)(v_* u_*(X), Z)$$

is an equivalence. By the Yoneda lemma, this implies that α is an equivalence in $F(j)$. The converse follows similarly. □

Let $\mathbb{R}\mathrm{Sec}_I^{cc}$ denote the subfunctor of $\mathbb{R}\mathrm{Sec}_I$ formed by the cocartesian sections. This subfunctor acts on a fibrant object $p : A \to I$ of $(\mathscr{C}at_{(\infty,n)})_{/I}$ as

$$\mathbb{R}\mathrm{Sec}_I^{cc}(A)(i) = \underline{\mathrm{Hom}}_I^{cc}(i_{/I}, A)$$

where the right hand side is the full sub-(∞, n)-category spanned by the cocartesian maps $i_{/I} \to A$. The adjunction $\int_I \dashv \mathrm{Sec}_I$ induces, from the factorisation properties above, an adjunction $\mathbb{L}\int_I \dashv \mathbb{R}\mathrm{Sec}_I^{cc}$.

Proposition 3.1.7. *Let I be a category. Then the adjunction*

$$\mathbb{L}\int_I : \mathrm{h}((\mathscr{C}at_{(\infty,n)})^I) \to \mathrm{h}((\mathscr{C}at_{(\infty,n)})_{/I})^{cc} : \mathbb{R}\mathrm{Sec}_I^{cc}$$

is an equivalence of categories.

Proof reference. The proof follows an analogous argument to that of the $n = 1$ case in [TV3]. □

Let A be an (∞, n)-precategory and $A_{[n]}$ the fiber of the map $p : A \to \mathcal{O}$ at $[n] \in \mathcal{O}$. Proposition 3.1.7 states that a $\mathcal{O}$-monoidal (∞, n)-category may be described as a cofibered object $p : A \to \mathcal{O}$ in the category $\mathrm{h}((\mathscr{C}at_{(\infty,n)})_{/\mathcal{O}})$ such that

$$A_{[n]} \xrightarrow{\prod_i p_i^*} (A_{[1]})^n$$

is an equivalence of (∞, n)-categories for each $n \geq 0$ (here the p_i are the same maps in $\mathcal{O}$ given at the start of this section). We identify $A_{[1]}$ with the underlying (∞, n)-category of A. We will often abuse notation by referring to an $\mathcal{O}$-monoidal (∞, n)-category $p : A \to \mathcal{O}$ as simply A.

Let $p : A \to \mathcal{O}$ be an $\mathcal{O}$-monoidal (∞, n)-category. Then an arrow f in A is said to be *p-inert* if f is a cocartesian arrow in A such that $p(f)$ is inert in $\mathcal{O}$ (see Notation 1.1 for the definition of an inert arrow in $\mathcal{O}$).

Definition 3.1.8. Let $p : A \to \mathcal{O}$ and $q : B \to \mathcal{O}$ be two $\mathcal{O}$-monoidal (∞, n)-categories. A functor $F : A \to B$ is said to be *$\mathcal{O}$-monoidal* if the diagram

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & \mathcal{O} & \end{tikzcd}$$

commutes and F carries p-cocartesian arrows to q-cocartesian arrows. It is said to be *lax $\mathcal{O}$-monoidal* if F carries p-inert arrows to q-cocartesian arrows.

Let $\mathbb{R}\underline{\mathrm{Hom}}^{\otimes}_{\mathcal{O}}(A, B)$ and $\mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{lax}}_{\mathcal{O}}(A, B)$ denote respectively the full subcategory of the (∞, n)-category $\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{O}}(A, B)$ spanned by the $\mathcal{O}$-monoidal and lax $\mathcal{O}$-monoidal functors.

We have an equivalence between the $(\infty, n+1)$-category of $\mathcal{O}$-monoidal (∞, n)-categories and the $(\infty, n+1)$-category consisting of $\mathcal{O}$-monoidal (∞, n)-categories $p : A \to \mathcal{O}$ with mapping space between two objects $p : A \to \mathcal{O}$ and $q : B \to \mathcal{O}$ given by $\mathrm{Map}(A, B) := \mathbb{R}\underline{\mathrm{Hom}}^{\otimes}_{\mathcal{O}}(A, B)$. We abuse notation by also referring to this $(\infty, n+1)$-category as $\underline{\mathrm{Cat}}^{\mathcal{O}}_{(\infty,n)}$: it will be clear from the context if we are considering our $\mathcal{O}$-monoidal (∞, n)-categories as Segal monoid objects or as cofibered objects. Similarly, we denote by $\underline{\mathrm{Cat}}^{\mathcal{O},\mathrm{lax}}_{(\infty,n)}$ the $(\infty, n+1)$-category consisting of $\mathcal{O}$-monoidal (∞, n)-categories with mapping space between two objects $p : A \to \mathcal{O}$ and $q : B \to \mathcal{O}$ given by $\mathrm{Map}(A, B) := \mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{lax}}_{\mathcal{O}}(A, B)$.

Definition 3.1.9. Let $p : C \to \mathcal{O}$ be a $\mathcal{O}$-monoidal (∞, n)-category. A *$\mathcal{O}$-monoid object* in C is a lax $\mathcal{O}$-monoidal section of p (where the identity map $\mathcal{O} \to \mathcal{O}$ endows the trivial category $[0]$ with a $\mathcal{O}$-monoidal structure).

The (∞, n)-category of *$\mathcal{O}$-monoid objects* in C will be denoted $\mathrm{Mon}^{\mathcal{O}}(C) := \mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{lax}}_{\mathcal{O}}(\mathcal{O}, C)$. When $\mathcal{O} = \Delta^{op}$, the (∞, n)-category of $\mathcal{O}$-monoid objects, called *monoid objects*, will be denoted $\mathrm{Mon}(C)$. When $\mathcal{O} = \Gamma$, the (∞, n)-category of $\mathcal{O}$-monoid objects, called *commutative monoid objects*, will be denoted $\mathrm{CMon}(C)$. An *$\mathcal{O}$-comonoid object* in C is a $\mathcal{O}$-monoid object in C^{op}. Let $\mathrm{Comon}^{\mathcal{O}}(C)$ denote the (∞, n)-category of $\mathcal{O}$-comonoid objects in C, $\mathrm{Comon}(C)$ the (∞, n)-category of *comonoid* objects in C and $\mathrm{CComon}(C)$ the (∞, n)-category of *commutative comonoid* objects in C. For a $\mathcal{O}$-(co)monoid object A, we will sometimes use the notation $A_n := A([n])$.

Example 3.1.10. Let $p : C \to \mathcal{O}$ be an $\mathcal{O}$-monoidal (∞, n)-category and A an (∞, n)-precategory. Then the (∞, n)-category $\mathbb{R}\underline{\mathrm{Hom}}(A, C_{[1]})$ inherits the structure of a $\mathcal{O}$-monoidal (∞, n)-category $\widetilde{\mathbb{R}\underline{\mathrm{Hom}}}(A, C) \to \mathcal{O}$ called the *pointwise $\mathcal{O}$-monoidal structure* where we define

$$\widetilde{\mathbb{R}\underline{\mathrm{Hom}}}(A, C) := \mathbb{R}\underline{\mathrm{Hom}}(A, C) \times_{\mathbb{R}\underline{\mathrm{Hom}}(A,\mathcal{O})} \mathcal{O}$$

giving $\widetilde{\mathbb{R}\underline{\mathrm{Hom}}}(A, C)_{[n]} \simeq \mathbb{R}\underline{\mathrm{Hom}}(A, C_{[n]})$. As a result, there exists an equivalence

$$\mathrm{Mon}^{\mathcal{O}}(\widetilde{\mathbb{R}\underline{\mathrm{Hom}}}(A, C)) \to \mathbb{R}\underline{\mathrm{Hom}}(A, \mathrm{Mon}^{\mathcal{O}}(C))$$

of (∞, n)-categories.

Example 3.1.11. Let C be a $\mathcal{O}$-monoidal (∞, n)-category and $D_{[1]}$ a full subcategory of $C_{[1]}$. Assume that for every equivalence $x \to y$ in $C_{[1]}$, if $y \in D_{[1]}$ then $x \in D_{[1]}$. Define a subcategory D of C by letting an object $x \in C_{[n]}$ belong to D if and only if the image under $\prod_i (p_i)_* : C_{[n]} \to (C_{[1]})^n$ belongs to $(D_{[1]})^n$. Then it is clear that the restriction map $D \to \mathcal{O}$ is a $\mathcal{O}$-monoidal (∞, n)-category if D is closed under tensor products and contains the unit object of C.

Example 3.1.12. Let $p : C \to \Gamma$ be a symmetric monoidal ∞-category. Then by Example 1.8.20 of [LIII], there exists a symmetric monoidal structure on the ∞-category $\mathrm{CMon}(C)$ of commutative monoid objects in C, induced by the that on C, which we will denote by $\widetilde{\mathrm{CMon}}(C)$. Moreover, by Proposition 2.7.6 of *loc. cit.*, the tensor product of commutative monoid objects corresponds to the coproduct.

Example 3.1.13. Let C be a symmetric monoidal ∞-category. Then the ∞-category $\mathrm{Pr}(C)$ of prestacks on C admits a symmetric monoidal structure which is characterised, up to symmetric monoidal equivalence, by the properties that the tensor product bifunctor $\otimes : \mathrm{Pr}(C) \times \mathrm{Pr}(C) \to \mathrm{Pr}(C)$ preserves small colimits seperately in each variable and that the Yoneda embedding $C \to \mathrm{Pr}(C)$ can be extended to a symmetric monoidal functor. Similarly, if C admits finite colimits and the tensor product bifunctor preserves finite colimits seperately in each variable then the ∞-category $\mathrm{Ind}(C)$ of ind-objects of C admits a symmetric monoidal structure which is characterised, up to symmetric monoidal equivalence, by the properties that the tensor product bifunctor $\otimes : \mathrm{Ind}(C) \times \mathrm{Ind}(C) \to \mathrm{Ind}(C)$ preserves small colimits seperately in each variable and that the Yoneda embedding $C \to \mathrm{Ind}(C)$ can be extended to a symmetric monoidal functor. These two statements follow from a more general statement for a symmetric monoidal ∞-category C admitting colimits indexed by an arbitrary collection of simplicial sets. See Proposition 4.1.8 of [LIII] for a precise statement.

Example 3.1.14. Let C be an (∞, n)-category and x be an object of C. The $(\infty, n-1)$-category $\mathrm{End}(x) := C(x, x)$ of endomorphisms of X admits a monoidal structure given by composition where

$$\begin{aligned} \mathrm{End}(x) : \Delta^{op} &\to \mathscr{C}at_{(\infty, n-1)} \\ [n] &\mapsto C(x, \dots, x) \end{aligned}$$

is a functor satisfying the Segal condition. A *(co)monad* on an object x in an (∞, n)-category C is a (co)monoid object of $\mathrm{End}(x)$. Thus $\mathrm{Mon}(\mathrm{End}(x))$ and $\mathrm{Comon}(\mathrm{End}(x))$ denotes respectively the $(\infty, n-1)$-category of monads and comonads on an object x in an (∞, n)-category with respect to the composition monoidal structure.

Example 3.1.15. Let $\mathscr{M}$ be a combinatorial monoidal model category. It follows from Theorem 1.6.17 of [LII] that if every object of $\mathscr{M}$ is cofibrant then there exists an equivalence

$$L(\mathrm{Mon}(\mathscr{M})) \to \mathrm{Mon}(L(\mathscr{M}))$$

of ∞-categories. Equivalently, the statement holds if the cofibrant condition on objects is replaced by the conditions that the model category $\mathscr{M}$ is left proper, the class of cofibrations in $\mathscr{M}$ is generated by cofibrations between cofibrant objects, the monoidal structure on $\mathscr{M}$ is symmetric and $\mathscr{M}$ satisfies the monoid axiom (see Definition 3.3 of [SS2]). Similarly, by Theorem 4.3.22 of [LIII], if $\mathscr{M}$ is a combinatorial symmetric monoidal model category with the conditions that $\mathscr{M}$ is left proper, the class of cofibrations in $\mathscr{M}$ is generated by cofibrations between cofibrant objects, $\mathscr{M}$ satisfies the monoid axiom and every cofibration in $\mathscr{M}$ is a power cofibration (see Definition 4.3.17 of *loc. cit.*) then the map

$$L(\mathrm{CMon}(\mathscr{M})) \to \mathrm{CMon}(L(\mathscr{M}))$$

is an equivalence of ∞-categories.

Proposition 3.1.16. *Let $p : C \to \mathcal{O}$ be a monoidal (∞, n)-category. Then the (∞, n)-category $\mathrm{Mon}^{\mathcal{O}}(C)$ has an initial object A such that the unit map $1_{C_{[1]}} \to A([1])$ is an equivalence in $C_{[1]}$.*

Proof reference. The proof follows an analogous argument as the $(\infty,1)$-categorical statement as covered in Proposition 1.4.3 of [LII] (monoidal case) and Corollary 2.3.10 of [LIII] (symmetric monoidal case). $\square$

We will now show that $\mathcal{O}$-monoidal (∞,n)-categories also naturally arise from $\mathscr{C}at_{(\infty,n-1)}$-enriched categories equipped with a $\mathcal{O}$-monoidal structure. More generally, given an $\mathcal{O}$-monoidal $\mathscr{C}at_{(\infty,n-1)}$-enriched model category, one can construct a $\mathcal{O}$-monoidal (∞,n)-category.

Construction 3.1.17. Let $\mathscr{M}$ be a model category and C an $\mathscr{M}$-enriched category with a weakly compatible $\mathcal{O}$-monoidal structure. We define an $\mathscr{M}$-enriched category $\widetilde{C}$ as follows:

- An object is a pair $([n],(x_0,\dots,x_n))$ where each $x_i\in C$.
- The mapping space between two objects $([n],x_\bullet),([m],y_\bullet)$ is given by

$$\widetilde{C}(([n],x_\bullet),([m],y_\bullet)) = \coprod_{u:[n]\to[m]}\left(\prod_{j\in[m]\text{-}\{0\}} C(\bigotimes_{u(i)=j} x_i, y_j)\right).$$

Proposition 3.1.18. *Let C be a fibrant $\mathscr{C}at_{(\infty,n-1)}$-enriched category and $\widetilde{C}$ the $\mathscr{C}at_{(\infty,n-1)}$-enriched category of Construction 3.1.17 . Then the forgetful map $p:\widetilde{C}\to\mathcal{O}$ is an $\mathcal{O}$-monoidal (∞,n)-category.*

Proof. We first show that p is a cofibered (∞,n)-category. Let $u:[n]\to[m]$ be a map in $\mathcal{O}$ and $([n],x_\bullet)$ an object of $\widetilde{C}$ over $[n]$. We choose an arrow $f:([n],x_\bullet)\to([m],y_\bullet)$ in $\widetilde{C}$ where $p(f)$ is isomorphic to u in $\mathcal{O}_{[n]/}$ such that $f_j:\bigotimes_{u(i)=j}x_i\to y_j$ is an equivalence for each $1\le j\le m$. The arrow f is p-cocartesian by definition if

$$\widetilde{C}(([m],y_\bullet),([l],z_\bullet))\to\widetilde{C}(([n],x_\bullet),([l],z_\bullet))\times_{\mathcal{O}([n],[l])}\mathcal{O}([m],[l])$$

is an equivalence in $\mathscr{C}at_{(\infty,n-1)}$ which translates into the requirement that

$$C(\bigotimes_{v(j)=k} y_j,z_k)\to C(\bigotimes_{(v\circ u)(i)=k} x_i,z_k)$$

is an equivalence. Thus it suffices to show that the map $f':\bigotimes_{(v\circ u)(i)=k}x_i\to\bigotimes_{v(j)=k}y_j$ is an equivalence in C. This follows since $f'=\bigotimes_{v(j)=k}f_j$ where each f_j is an equivalence. Finally, the maps $p_i:[n]\to[1]$ in $\mathcal{O}$ induce an isomorphism $\widetilde{C}_{[n]}\simeq\widetilde{C}^n$ by observation. $\square$

Let $\mathscr{A}$ be an $\mathcal{O}$-monoidal $\mathscr{M}$-enriched model category for an aribitrary monoidal model category $\mathscr{M}$ (see Definition 7.1.10 of Section 7.1). Let $\widetilde{\mathscr{A}}$ be defined as in Construction 3.1.17. We will denote by $\widetilde{\mathscr{A}}^\circ$ the full subcategory of $\widetilde{\mathscr{A}}$ spanned by those objects $([n],(x_0,\dots,x_n))$ where each x_i is a fibrant-cofibrant object of $\mathscr{A}$.

Proposition 3.1.19. *Let $\mathscr{A}$ be an $\mathcal{O}$-monoidal $\mathscr{C}at_{(\infty,n-1)}$-enriched model category and let $\widetilde{\mathscr{A}}$ be defined as in Construction 3.1.17. Then the forgetful map $p:\widetilde{\mathscr{A}}^\circ\to\mathcal{O}$ is an $\mathcal{O}$-monoidal (∞,n)-category.*

Proof. We first show that p is a cofibered (∞,n)-category. Let $u:[n]\to[m]$ be a map in $\mathcal{O}$ and $([n],x_\bullet)$ an object of $\widetilde{\mathscr{A}}$ over $[n]$. We choose an arrow $f:([n],x_\bullet)\to([m],y_\bullet)$ in $\widetilde{\mathscr{A}}^\circ$ where $p(f)$ is isomorphic to u in $\mathcal{O}_{[n]/}$ such that $f_j:\bigotimes_{u(i)=j}x_i\to y_j$ is a trivial cofibration for each $1\le j\le m$. The arrow f is p-cocartesian by definition if

$$\widetilde{\mathscr{A}}(([m],y_\bullet),([l],z_\bullet))\to\widetilde{\mathscr{A}}(([n],x_\bullet),([l],z_\bullet))\times_{\mathcal{O}([n],[l])}\mathcal{O}([m],[l])$$

is an equivalence in $\mathscr{C}at_{(\infty,n-1)}$. This translates into the requirement that

$$\mathscr{A}(\bigotimes_{v(j)=k} y_j,z_k)\to\mathscr{A}(\bigotimes_{(v\circ u)(i)=k} x_i,z_k)$$

is an equivalence. Thus it suffices to show that the map $f' : \bigotimes_{(v\circ u)(i)=k} x_i \to \bigotimes_{v(j)=k} y_j$ is a weak equivalence in $\mathscr{A}$. This follows since $f' = \bigotimes_{v(j)=k} f_j$ where each f_j is a weak equivalence. Finally, the maps $p_i : [n] \to [1]$ in $\mathcal{O}$ induce an isomorphism $(\widetilde{\mathscr{A}^\circ})_{[n]} \simeq (\widetilde{\mathscr{A}^\circ})^n$ by observation. $\square$

Example 3.1.20. The model category $\mathscr{C}at_{(\infty,n)}$ of (∞,n)-categories is an $\mathcal{O}$-monoidal $\mathscr{C}at_{(\infty,n)}$-enriched model category for the cartesian product. Thus $\underline{\mathrm{Cat}}_{(\infty,n)} := L(\mathscr{C}at_{(\infty,n)})$ is a monoidal (∞,n)-category. Explicitly, the cofibered category $\widetilde{\underline{\mathrm{Cat}}}_{(\infty,n)} := (\widetilde{\mathscr{C}at}_{(\infty,n)})^\circ$ is given as follows:

- The objects are pairs $([n],(C_0,\ldots,C_n))$ where $[n]$ is an object of $\mathcal{O}$ and each C_i is a fibrant (∞,n)-precategory.
- A map between two objects $([n],C_\bullet)$ and $([m],D_\bullet)$ is a map $u : [n] \to [m]$ in $\mathcal{O}$ together with a collection of functors $\prod_{u(i)=j} C_i \to D_j$.

When the $(\infty,n+1)$-category $\underline{\mathrm{Cat}}_{(\infty,n)}$ is equipped with the cartesian monoidal structure then

$$\mathrm{Mon}^{\mathcal{O}}(\widetilde{\underline{\mathrm{Cat}}}_{(\infty,n)}) \to \underline{\mathrm{Cat}}^{\mathcal{O}}_{(\infty,n)}$$

is an equivalence of $(\infty,n+1)$-categories. This follows from the following more general equivalence. Let $p : C \to \mathcal{O}$ be an $\mathcal{O}$-monoidal (∞,n)-category and D an arbitrary (∞,n)-category. Let $u : [1] \to [n]$ denote the map $[1] \simeq \{0,n\} \to [n]$ in Δ and $u : [n] \to [1]$ denote an active arrow in Γ. A functor $F : C \to D$ is said to be a *lax cartesian structure* if for any pair $(x,f_i)_{1\le i\le n}$ where x is an object in $C_{[n]}$ and $f_i : x \to x_i$ denote the natural maps in C under $\prod_i (p_i)_* : C_{[n]} \to (C_{[1]})^n$, then $F(x) = \prod_{1\le i\le n} F(x_i)$ in D. It is said to be a *weak cartesian structure* if it is a lax cartesian structure such that for any p-cocartesian arrow $f : x \to y$ in C under $u_* : C_{[n]} \to C_{[1]}$, then $F(f)$ is an equivalence in D. It is said to be a *cartesian structure* if F induces an equivalence $C_{[1]} \to D$. Let $\mathbb{R}\underline{\mathrm{Hom}}^{\times,\mathrm{lax}}(C,D)$ denote the full subcategory of $\mathbb{R}\underline{\mathrm{Hom}}(C,D)$ spanned by the lax cartesian structures. Then for any cartesian structure $F : C \to D$, composition with F induces an equivalence

$$\mathrm{Mon}^{\mathcal{O}}(C) = \mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{lax}}_{\mathcal{O}}(\mathcal{O},C) \to \mathbb{R}\underline{\mathrm{Hom}}^{\times,\mathrm{lax}}(\mathcal{O},D) = \mathbb{R}\mathrm{SeMon}^{\mathcal{O}}(D)$$

of (∞,n)-categories. See Proposition 1.4.14 of [LIII] for a proof in the $(\infty,1)$-categorical case.

Proposition 3.1.21. *Let $F : C \to D$ be a $\mathcal{O}$-monoidal functor between $\mathcal{O}$-monoidal (∞,n)-categories. Then F is an equivalence if and only if F induces an equivalence of underlying (∞,n)-categories.*

Proof. Let us assume that $C_{[1]} \to D_{[1]}$ is a categorical equivalence. We will show that F is fully faithful and essentially surjective. Since $D_{[n]} \simeq (D_{[1]})^n$ then F is $\mathcal{O}$-monoidal if and only if for every p-cocartesian arrow f in C covering a map $u : [1] \to [n]$ in $\mathcal{O}$, the image $F(f)$ is a q-cocartesian arrow in D. Let $x,y \in C$ and let $\bar{u} : x' \to y$ be a p-cocartesian arrow of C lifting u. Then there exists a diagram of homotopy fiber sequences

$$\begin{tikzcd}
C_{[1]}(x,x') \arrow[rr] \arrow[d] & & D_{[1]}(F(x),F(x')) \arrow[d] \\
C(x,y) \arrow[rr] \arrow[dr] & & D(F(x),F(y)) \arrow[dl] \\
& \mathcal{O}([1],[n]). &
\end{tikzcd}$$

Since the top horizontal map is an equivalence by assumption, F is fully faithful. It is essentially surjective since any $z \in D$ is equivalent in $D_{[n]}$ to the image by F of some x in $C_{[n]}$. $\square$

Let C be a symmetric monoidal ∞-category. A symmetric monoidal structure on C is said to be *compatible with countable colimits* if for any simplicial set A with only countably many simplices, the ∞-category C admits A-indexed colimits and for any x in C, the functor $\bullet \otimes x : C \to C$ preserves these colimits. If C is compatible with countable colimits then the forgetful functor $\mathrm{CMon}(C) \to C$ admits a left adjoint

$$\mathrm{Fr} : C \to \mathrm{CMon}(C)$$

which we refer to as the *free functor*. A precise statement can be found in Proposition 2.6.8 of [LIII]. If C is equivalent to $L\mathscr{M}$ for $\mathscr{M}$ a symmetric monoidal model category then Fr is equivalent to a functor $\mathrm{Fr} : L\mathscr{M} \to L(\mathrm{CMon}(\mathscr{M}))$ by Example 3.1.15 so

$$\mathrm{Fr}(x) = \mathbb{L}\mathrm{Sym}(x)$$

where $\mathrm{Sym}(x) := \coprod_{n\geq 0} x^{\otimes n}/\Sigma_n$.

3.2 Modules and comodules

Definition 3.2.1. Let $p : D \to \mathcal{O}$ be a $\mathcal{O}$-monoidal (∞, n)-category. An (∞, n)-category C is said to be *$\mathcal{O}$-tensored* over D if there exists a map $F : C \to D$ in $\mathscr{C}at_{(\infty,n)}$ such that:

1. The composition $(p \circ F) : C \to \mathcal{O}$ is cofibered in $(\mathscr{C}at_{(\infty,n)})_{/\mathcal{O}}$.
2. The map F carries $(p \circ F)$-cocartesian arrows of C to p-cocartesian arrows of D.
3. For each $n \geq 0$, the inclusion $\{n\} \subseteq [n]$ induces an equivalence $C_{[n]} \to D_{[n]} \times C_{\{n\}}$ of (∞, n)-categories.

Let C be an (∞, n)-category $\mathcal{O}$-tensored over D. We will refer to the fiber $C_{[0]}$ as the underlying (∞, n)-category of C and by abuse, also denote it by C. We obtain a natural diagram

$$C_{\{0\}} \leftarrow C_{[1]} \xrightarrow{\sim} D_{[1]} \times C_{\{1\}}$$

which induces an $\mathcal{O}$-monoidal bifunctor $\otimes : D_{[1]} \times C_{[0]} \to C_{[0]}$, together with its higher $\mathcal{O}$-monoidal structure, which is well defined up to homotopy. An (∞, n)-precategory C which is Δ^{op}-tensored over D is said to be *left-tensored* over D. An (∞, n)-precategory C which is Γ-tensored over D is said to be simply *tensored* over D. One can similarly define an (∞, n)-precategory right tensored over a monoidal (∞, n)-category by replacing the inclusion $\{n\} \subseteq [n]$ in Definition 3.2.1 by the inclusion $\{0\} \subseteq [n]$.

Proposition 3.2.2. *Let C be a $\mathcal{O}$-monoidal (∞, n)-category. Then the $\mathcal{O}$-monoidal product $\otimes : C \times C \to C$ endows C with the structure of an (∞, n)-category $\mathcal{O}$-tensored over itself.*

Proof. The case of $n = 1$ and $\mathcal{O} = \Delta^{op}$ was given in Example 2.1.3 of [LII]. A similar argument can be used to prove this more general case. □

Definition 3.2.3. Let A be a monoidal (∞, n)-category and C an (∞, n)-category left-tensored over A. Let $a \in A$ and $x, y \in C$. If the functor

$$\begin{aligned} A^{op} &\to \underline{\mathrm{Cat}}_{(\infty,n-1)} \\ a &\mapsto C(a \otimes x, y) \end{aligned}$$

is representable, the representing object will be denoted $\mathrm{Mor}(x, y)$ and called the *morphism object* of x and y. The (∞, n)-category C is said to be *enriched* over A if the functor is representable for all $x, y \in C$. If the functor

$$\begin{aligned} C^{op} &\to \underline{\mathrm{Cat}}_{(\infty,n-1)} \\ x &\mapsto C(a \otimes x, y) \end{aligned}$$

is representable, the representing object will be denoted $\mathrm{Exp}(a, y)$ and called the *exponential object* of a and y. The (∞, n)-category C is said to be *cotensored* over A if the functor is representable for all $a \in A$ and $y \in C$.

It follows directly from Definition 3.2.3 that morphism objects are characterised by the following universal property: there exists a map $ev : \mathrm{Mor}(x,y) \otimes x \to y$ such that composition with ev yields an equivalence

$$A(a, \mathrm{Mor}(x,y)) \to C(a \otimes x, y)$$

of $(\infty, n-1)$-categories. Likewise, exponential objects are characterised by the following universal property: there exists a map $ev : a \otimes \mathrm{Exp}(a,y) \to y$ such that composition with ev yields an equivalence

$$C(x, \mathrm{Exp}(a,y)) \to C(a \otimes x, y)$$

of $(\infty, n-1)$-categories. The composition $\mathrm{Mor}(y,z) \otimes \mathrm{Mor}(x,y) \otimes x \xrightarrow{ev} \mathrm{Mor}(y,z) \otimes y \xrightarrow{ev} z$ yields a composition map

$$\mathrm{Mor}(y,z) \otimes \mathrm{Mor}(x,y) \to \mathrm{Mor}(x,z)$$

and the chain of equivalences $A(b, \mathrm{Mor}(a, \mathrm{Mor}(x,y))) \simeq A(b \otimes a, \mathrm{Mor}(x,y)) \simeq C(b \otimes a \otimes x, y) \simeq A(b, \mathrm{Mor}(a \otimes x, y))$ yields an equivalence

$$\mathrm{Mor}(a, \mathrm{Mor}(x,y)) \to \mathrm{Mor}(a \otimes x, y)$$

of morphism objects.

Example 3.2.4. When C is a monoidal (∞,n)-category then C is naturally left-tensored over itself. If it is furthermore enriched, then the morphism object $\mathrm{Mor}(c,d)$ is just the internal Hom object $\underline{\mathrm{Hom}}(c,d)$ in C.

Proposition 3.2.5. *Let A be a monoidal ∞-category and C an ∞-category left-tensored over A. Suppose further that C and A are presentable ∞-categories. Then the following hold:*

1. *The ∞-category C is enriched over A if the functor*

$$\bullet \otimes x : A \to C$$

   preserves small colimits for all $x \in C$.

2. *The ∞-category C is cotensored over A if the functor*

$$a \otimes \bullet : C \to C$$

   preserves small colimits for all $a \in A$.

Proof. Let $a = \mathrm{colim}_i a_i$ be an object in A. A prestack is representable if and only if it preserves small limits. Therefore, the ∞-category C is enriched over A if $C(\mathrm{colim}_i a_i \otimes x, y) \simeq \lim_i C(a_i \otimes x, y)$. By assumption, $C(\mathrm{colim}_i a_i \otimes x, y) \simeq C(\mathrm{colim}_i (a_i \otimes x), y)$ which is naturally equivalent to $\lim_i C(a_i \otimes x, y)$. The second statement follows similarly. □

Let C and D be (∞,n)-categories which are tensored over the (∞,n)-category $\underline{\mathrm{Cat}}_{(\infty,n-1)}$ and let $F : C \to D$ be a functor. Then there exists a natural map

$$A \otimes F(x) \to F(A \otimes x)$$

for any $(\infty, n-1)$-category A and any object x in C. The functor F is said to preserve the tensored structure if this map is an equivalence.

Proposition 3.2.6. *Let C and D be (∞,n)-categories which are tensored over the (∞,n)-category $\underline{\mathrm{Cat}}_{(\infty,n-1)}$ of $(\infty,n-1)$-categories. Suppose that $F : C \to D$ is a functor which preserves the tensored structure. Then F is an equivalence of (∞,n)-categories if and only if*

$$\mathfrak{K}^1 F : \mathfrak{K}^1 C \to \mathfrak{K}^1 D$$

is an equivalence of ∞-categories.

Proof. Assume $\mathfrak{K}^1 F$ is an equivalence. We will show that F is fully faithful and essentially surjective. Since $\tau_{\leq 1} C$ is equivalent to $\tau_{\leq 1}(\mathfrak{K}^1 C)$, essential surjectivity follows by assumption. Let $A \in \underline{\mathrm{Cat}}_{(\infty,n-1)}$ and x and y be objects of C. Then we have the following diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Cat}^{\infty}_{(\infty,n-1)}}(A, C(x,y)) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Cat}^{\infty}_{(\infty,n-1)}}(A, D(Fx,Fy)) \arrow[d] \\
\mathrm{Map}_{\mathfrak{K}^1 C}(A \otimes x, y) \arrow[r] & \mathrm{Map}_{\mathfrak{K}^1 D}(F(A \otimes x), F(y))
\end{tikzcd}$$

of Kan complexes. The lower horizontal arrow is an equivalence by assumption, the left vertical arrow is an equivalence by the tensored property of C and the right vertical arrow is an equivalence by the tensored property of D together with the supposition that F preserves the tensored structure. Thus the upper horizontal arrow is an equivalence and F is fully faithful. $\square$

Definition 3.2.7. Let $p : C \to \mathcal{O}$ be an $\mathcal{O}$-monoidal (∞, n)-category and $F : \overline{C} \to C$ exhibit C as $\mathcal{O}$-tensored over itself. An *$\mathcal{O}$-module object* of C is a functor $A : \mathcal{O} \to \overline{C}$ such that

1. If $u : [n] \to [m]$ is an inert map in $\mathcal{O}$ with $u(n) = m$ then $A(u)$ is a $p \circ F$-cocartesian arrow of $\overline{C}$.
2. The composition $F \circ A : \mathcal{O} \to C$ is a $\mathcal{O}$-monoid object in C. The map $p \circ F \circ A$ coincides with the identity on $\mathcal{O}$.

The information of Definition 3.2.7 is encoded pictorially in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O} \arrow[rr, "A"] & & \overline{C} \arrow[dl, "F"] \\
& C. \arrow[ul, "p"] &
\end{tikzcd}$$

Let $\mathrm{Mod}^{\mathcal{O}}(C)$ denote the full subcategory of $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{O}, \overline{C})$ spanned by the $\mathcal{O}$-module objects of C. When $\mathcal{O} = \Delta^{op}$, an $\mathcal{O}$-module object will be called a *left module object* and we will denote the (∞, n)-category of left module objects in C by $\mathrm{Mod}^L(C)$. When $\mathcal{O} = \Gamma$, an $\mathcal{O}$-module object will be called a *module object* and we will denote the (∞, n)-category of module objects in C by $\mathrm{Mod}(C)$.

Let $p : C \to \mathcal{O}$ be an $\mathcal{O}$-monoidal (∞, n)-category, $F : \overline{C} \to C$ exhibit C as $\mathcal{O}$-tensored over itself and A an $\mathcal{O}$-module object of C. Then the composition $F \circ A$ induces a functor $\theta : \mathrm{Mod}^{\mathcal{O}}(C) \to \mathrm{Mon}^{\mathcal{O}}(C)$.

Definition 3.2.8. Let $p : C \to \mathcal{O}$ be an $\mathcal{O}$-monoidal (∞, n)-category and R an $\mathcal{O}$-monoid object in C. Then the (∞, n)-category of *R-$\mathcal{O}$-modules* in C is given by the homotopy fiber $\theta^{-1}(R)$ of the forgetful map $\theta : \mathrm{Mod}^{\mathcal{O}}(C) \to \mathrm{Mon}^{\mathcal{O}}(C)$.

We will denote by $\mathrm{Mod}^{\mathcal{O}}_R(C)$ the (∞, n)-category of R-$\mathcal{O}$-modules in C. When $\mathcal{O} = \Delta^{op}$, an R-$\mathcal{O}$-module will be called a *left R-module* and we will denote the (∞, n)-category of left R-modules in C by $\mathrm{Mod}^L_R(C)$. When $\mathcal{O} = \Gamma$, an R-$\mathcal{O}$-module will be called an *R-module* and we will denote the (∞, n)-category of R-modules in C by $\mathrm{Mod}_R(C)$. When the (∞, n)-category C is clear from the context, the (∞, n)-categories $\mathrm{Mod}^L_R(C)$ and $\mathrm{Mod}_R(C)$ will be replaced by Mod^L_R and Mod_R respectively. Let C be an $\mathcal{O}$-monoidal (∞, n)-category and R an $\mathcal{O}$-comonoid object in C. By definition, the *(∞, n)-category of comodules* in C over R is

$$\mathrm{Comod}^{\mathcal{O}}_R(C) := \mathrm{Mod}^{\mathcal{O}}_R(C^{op})^{op}.$$

Example 3.2.9. Many examples of ∞-categories of modules arise from the localisation of model categories of modules. More precisely, let $\mathscr{M}$ be a combinatorial $\mathcal{O}$-monoidal model category and R an $\mathcal{O}$-monoid object of $\mathscr{M}$. Assume that R is cofibrant as an object of $\mathscr{M}$ (or in the symmetric case, that $\mathscr{M}$

satisfies the monoid axiom [SS2]). Then the category $\mathrm{Mod}_R(\mathscr{M})$ of R-modules admits a combinatorial model structure where a map is a fibration if and only if it is a fibration in $\mathscr{M}$ and a weak equivalence if and only if it is a weak equivalence in $\mathscr{M}$. If $\mathscr{M}$ is endowed with a simplicial model structure, then $\mathrm{Mod}_R(\mathscr{M})$ is a simplicial model category (Proposition 2.5.1 of [LII]). If we further assume that $\mathscr{M}$ satisfies the conditions of Example 3.1.15, furnishing an equivalence $s : L(\mathrm{Mon}^{\mathcal{O}}(\mathscr{M})) \to \mathrm{Mon}^{\mathcal{O}}(L(\mathscr{M}))$ of ∞-categories, then the natural map

$$L(\mathrm{Mod}^{\mathcal{O}}_R(\mathscr{M})) \to \mathrm{Mod}^{\mathcal{O}}_{s(R)}(L(\mathscr{M}))$$

is an equivalence of ∞-categories. This follows from Theorem 2.5.4 of [LII].

Notation 3.2.10. Let K^{si} denote the full subcategory of $\mathbb{R}\underline{\mathrm{Hom}}([1],\Gamma)$ spanned by the semi-inert arrows. Let K^{null} denote the full subcategory of K^{si} spanned by the null arrows. There are two natural maps $ev_i : K^\bullet \to \Gamma$ given by evaluation on $i \in \{0,1\}$. A morphism in $K^\bullet$ is said to be *inert* if its images under e_0 and e_1 are inert in Γ.

Let $p : C \to \Gamma$ be a symmetric monoidal (∞, n)-category. We define an (∞, n)-precategory $\mathfrak{M}(C)$ through the following universal property: for every (∞, n)-precategory A equipped with a map $A \to \Gamma$ the map

$$\mathbb{R}\underline{\mathrm{Hom}}_\Gamma(A, \mathfrak{M}(C)) \to \mathbb{R}\underline{\mathrm{Hom}}_{\mathbb{R}\underline{\mathrm{Hom}}(\{1\},\Gamma)}(A \times_{\mathbb{R}\underline{\mathrm{Hom}}(\{0\},\Gamma)} K^{si}, C)$$

is an equivalence. Thus

$$\mathfrak{M}(C)_{[n]} \to \mathbb{R}\underline{\mathrm{Hom}}_{\mathbb{R}\underline{\mathrm{Hom}}(\{1\},\Gamma)}(K^{si}_{[n]>}, C)$$

is an equivalence where $K^{si}_{[n]>}$ denotes the homotopy fiber of $K^{si} \to \mathbb{R}\underline{\mathrm{Hom}}(\{0\},\Gamma)$ at $[n]$. An object of $\mathfrak{M}(C)_{[1]}$ is then given by a commutative diagram

$$\begin{tikzcd} K^{si}_{[1]>} \arrow[rr, "F"] \arrow[dr, "ev_1"'] & & C \arrow[dl, "p"] \\ & \Gamma. & \end{tikzcd}$$

Let $\overline{\mathfrak{M}}(C)$ denote the full subcategory of $\mathfrak{M}(C)$ spanned by those vertices for which the functor F preserves inert morphisms.

Similarly we define an (∞, n)-precategory $\mathfrak{A}(C)$ through the following universal property: for every (∞, n)-precategory A equipped with a map $A \to \Gamma$ the map

$$\mathbb{R}\underline{\mathrm{Hom}}_\Gamma(A, \mathfrak{A}(C)) \to \mathbb{R}\underline{\mathrm{Hom}}_{\mathbb{R}\underline{\mathrm{Hom}}(\{1\},\Gamma)}(A \times_{\mathbb{R}\underline{\mathrm{Hom}}(\{0\},\Gamma)} K^{null}, C)$$

is an equivalence. Thus

$$\mathfrak{A}(C)_{[n]} \to \mathbb{R}\underline{\mathrm{Hom}}_{\mathbb{R}\underline{\mathrm{Hom}}(\{1\},\Gamma)}(K^{null}_{[n]>}, C)$$

is an equivalence where $K^{null}_{[n]>}$ denotes the homotopy fiber of $K^{null} \to \mathbb{R}\underline{\mathrm{Hom}}(\{0\},\Gamma)$ at $[n]$. An object of $\mathfrak{A}(C)_{[1]}$ is then given by a commutative diagram

$$\begin{tikzcd} K^{null}_{[1]>} \arrow[rr, "F"] \arrow[dr, "ev_1"'] & & C \arrow[dl, "p"] \\ & \Gamma. & \end{tikzcd}$$

Let $\overline{\mathfrak{A}}(C)$ denote the full subcategory of $\mathfrak{A}(C)$ spanned by those vertices for which the functor F preserves inert morphisms. We define

$$\widetilde{\mathrm{Mod}}_R(C) := \overline{\mathfrak{M}}(C) \times_{\overline{\mathfrak{A}}(C)} \{R\}.$$

Under some mild assumptions, the (∞, n)-category of R-modules inherits a symmetric monoidal structure.

Proposition 3.2.11. *Let C be a symmetric monoidal (∞, n)-category and R a commutative monoid object of C. If C admits colimits of simplicial objects such that for every object x in C the functor $\bullet \otimes x$ preserves these colimits, then the projection*

$$p : \widetilde{\mathrm{Mod}}_R(C) \to \Gamma$$

is a symmetric monoidal (∞, n)-category. The unit object of p is canonically equivalent to R.

Proof. The ∞-categorical statement is part (1) of Proposition 3.6.6. in [LIII]. □

The symmetric monoidal product offered by Proposition 3.2.11 is called the *relative tensor product* and for two R-modules M and N will be denoted $M \otimes_R N$. See Section 4.5 of [LII] for a detailed discussion of the relative tensor product functor. The above construction is functorial in p and hence we obtain a functor

$$\begin{aligned} \mathrm{Mod}(C) : \mathrm{CMon}(C) &\to \mathrm{Cat}^{\mathrm{sM}}_{(\infty,n)} \\ R &\mapsto \widetilde{\mathrm{Mod}}_R(C) \\ (f : R \to Q) &\mapsto \bullet \otimes_R Q \end{aligned}$$

where $\bullet \otimes_R Q$ is the symmetric monoidal base change functor left adjoint to the forgetful functor $\mathrm{Mod}_Q(C) \to \mathrm{Mod}_R(C)$.

Lemma 3.2.12. *Let C be a presentable ∞-category satisfying the conditions of Example 3.2.9. Then the bifunctor $\otimes : \mathrm{Mod}_R(C) \times \mathrm{Mod}_R(C) \to \mathrm{Mod}_R(C)$ preserves colimits seperately in each variable.*

Proof. This follows from Example 3.2.9 and Lemma 3.1.4. □

Thus by Proposition 3.2.5, the symmetric monoidal ∞-category $\widetilde{\mathrm{Mod}}^{\mathcal{O}}_R(C)$ is cotensored and enriched over itself.

Definition 3.2.13. Let C be a symmetric monoidal (∞, n)-category and R a commutative monoid object of C. Then an *commutative R-algebra object* in C is a commutative monoid object in the symmetric monoidal (∞, n)-category $\widetilde{\mathrm{Mod}}_R(C)$.

Let $\mathrm{CAlg}_R(C)$, or simply CAlg_R, denote the (∞, n)-category $\mathrm{CMon}(\widetilde{\mathrm{Mod}}_R(C))$ of commutative R-algebra objects in C. The (∞, n)-category $\mathrm{CAlg}_R(C)$ inherits the structure of a symmetric monoidal (∞, n)-category where the tensor product is given by the tensor product in C (see Example 3.1.12). Furthermore, this tensor product coincides with the coproduct in the (∞, n)-category of commutative R-algebras. We will denote this symmetric monoidal (∞, n)-category by $\widetilde{\mathrm{CAlg}}(C) := \widetilde{\mathrm{CMon}}(\widetilde{\mathrm{Mod}}_R(C))$.

Let C be a symmetric monoidal ∞-category such that the symmetric monoidal product preserves (small) colimits separately in each variable and the fiber $C_{[n]}$ is a presentable ∞-category for all $n > 0$. In this case we will say that C is a *presentable* symmetric monoidal ∞-category. If C is a presentable symmetric monoidal ∞-category then $\mathrm{CMon}(C)$ is a presentable ∞-category. This follows from Corollary 2.7.5 of [LIII]. Moreover, if C is a presentable symmetric monoidal ∞-category then the ∞-category $\widetilde{\mathrm{Mod}}_R(C)$ is a presentable symmetric monoidal ∞-category by Theorem 3.4.2 of [LIII]. Combining these two results, the ∞-category $\mathrm{CAlg}_R(C)$ is a presentable ∞-category.

Consider $\mathrm{CMon}(C)_{R/}$ as the following (outer) homotopy pullback

$$\begin{tikzcd} \mathrm{CMon}(C)_{R/} \arrow[r] \arrow[d] & \mathrm{CMon}(\mathrm{Mod}(C)) \arrow[r, "\theta"] \arrow[d] & \mathbb{R}\underline{\mathrm{Hom}}([1], \mathrm{CMon}(C)) \arrow[d, "ev_0"] \\ {[0]} \arrow[r] & \mathrm{CMon}(\mathrm{CMon}(C) \times \Gamma) \arrow[r, "\theta'"] & \mathbb{R}\underline{\mathrm{Hom}}(\{0\}, \mathrm{CMon}(C)) \end{tikzcd}$$

where R is given by the composition of the lower horizontal arrows and θ and θ' are constructed as in Section 3.2 of [LIII]. Then there exists a natural map $\theta_R : \mathrm{CMon}(\widetilde{\mathrm{Mod}}_R(C)) \to \mathrm{CMon}(C)_{R/}$ given by the universal property of the diagram.

Proposition 3.2.14. *Let C be a symmetric monoidal (∞, n)-category, R a commutative monoid object in C and $\mathrm{Mod}_R(C)$ the (∞, n)-category of R-modules in C. Then there exists an equivalence*

$$\theta_R : \mathrm{CAlg}_R(C) \to \mathrm{CMon}(C)_{R/}$$

of (∞, n)-categories.

Proof. The (∞, n)-categories $\mathrm{CAlg}_R(C)$ and $\mathrm{CMon}(C)_{R/}$ are tensored over $\underline{\mathrm{Cat}}_{(\infty,n-1)}$ and the map θ_R preserves this tensored structure. Thus from Proposition 3.2.6 it suffices to prove the analogous statement for ∞-categories. This follows from Corollary 3.2.7 of [LIII]. □

3.3 Stable ∞-categories

In this section we will review the basic theory of stable ∞-categories. A more detailed account can be found in [LI]. Let C be an ∞-category. We call an object which is both initial and terminal in C a *null object* and denote it by $0 \in C$. An ∞-category is said to be *pointed* if it contains a null object.

Definition 3.3.1. An ∞-category C is said to be *stable* if it is pointed, admits finite limits and colimits and pullback and pushout squares coincide.

Note that if a functor between stable ∞-categories is left or right exact it is automatically exact. Let $\underline{\mathrm{Cat}}^{\perp}_{\infty}$ denote the full subcategory of the $(\infty, 2)$-category $\underline{\mathrm{Cat}}_{\infty}$ of ∞-categories spanned by the stable ∞-categories and exact functors. Let C be a pointed ∞-category and $f : x \to y$ an arrow in C. A *kernel* of f is a pullback $x \times_y 0$ and a *cokernel* of f is a pushout $y \coprod_x 0$. They are uniquely determined up to equivalence in C. A full subcategory of a stable ∞-category is said to be a *stable subcategory* if it contains a zero object and is closed under the formation of kernels and cokernels.

The $(\infty, 2)$-category $\underline{\mathrm{Cat}}^{\perp}_{\infty}$ admits all (small) limits and all (small) filtered colimits. Since limits and filtered colimits in an $(\infty, 2)$-category are computed by taking limits and filtered colimits of the underlying ∞-categories, this result follows from Theorem 5.4 of [LI] and Proposition 5.6 of [LI]. The structure of a stable ∞-category induces a heavy simplification of the nature of its limits and colimits: if κ is a regular cardinal, then a stable ∞-category has all κ-small limits (resp. colimits) if and only if it has κ-small products (resp. coproducts). Furthermore, an exact functor between stable ∞-categories preserves κ-small limits (resp. colimits) if and only if it preserves κ-small products (resp. coproducts).

Let C be a pointed ∞-category with finite limits. The *loop functor* Ω of C is the endomorphism of C given by

$$\Omega : x \mapsto 0 \underset{x}{\times} 0.$$

This functor admits a left adjoint

$$\Sigma : x \mapsto 0 \coprod_x 0$$

called the *suspension functor*. The following proposition provides a useful equivalent definition of a stable ∞-category.

Proposition 3.3.2. *Let C be a pointed ∞-category. Then C is stable if and only if it admits finite limits and the loop functor $\Omega : C \to C$ is an equivalence of ∞-categories. Likewise, C is stable if and only if it admits finite colimits and the suspension functor $\Sigma : C \to C$ is an equivalence of ∞-categories.*

Proof. This follows from Proposition 4.4 of [LI] followed by Corollary 8.28 of [LI]. □

Example 3.3.3. If C is a stable ∞-category and A is an ∞-precategory then the ∞-category $\mathbb{R}\underline{\mathrm{Hom}}(A, C)$ is stable.

Example 3.3.4. Let C be a stable ∞-category and κ a regular cardinal. Then $\mathrm{Ind}_\kappa(C)$ is a stable ∞-category. See Proposition 4.5 of [LI].

Example 3.3.5. Let C be a stable, $\mathcal{O}$-monoidal ∞-category and R a $\mathcal{O}$-monoid object in C such that the map $x \mapsto R \otimes x$ is exact for all $x \in C$. Then $\mathrm{Mod}_R^{\mathcal{O}}(C)$ is a stable ∞-category. Moreover, the forgetful functor $\mathrm{Mod}_R^{\mathcal{O}}(C) \to C$ is exact. This is a special case of Proposition 4.4.3 of [LII].

Example 3.3.6. Let C be a stable ∞-category and $\mathbb{Z}_{\geq 0}$ the linearly ordered set of non-negative integers (thought of as a category in the obvious way). Then there exists an equivalence $sC \to \mathbb{R}\underline{\mathrm{Hom}}(\mathbb{Z}_{\geq 0}, C)$ of ∞-categories. The map assigns to a simplicial object X in C the filtered object F in $\mathbb{R}\underline{\mathrm{Hom}}(\mathbb{Z}_{\geq 0}, C)$ where $F(n)$ is the colimit of the n-skeleton of X. This is the *∞-categorical Dold-Kan correspondence.* See Theorem 12.8 of [LI].

Example 3.3.7. A pointed, closed model category $\mathscr{M}$ is said to be *stable* if the adjunction $\Sigma \dashv \Omega$ is an equivalence in the homotopy category $\mathrm{h}\mathscr{M}$. Thus for any stable model category $\mathscr{M}$, the ∞-category $L\mathscr{M}$ is stable. Moreover, if $\mathscr{M}$ is a cofibrantly generated, proper, stable, simplicial model category with a set P of compact generators, the authors in [SS1] prove an equivalence between $\mathscr{M}$ and the model category $\mathrm{Mod}_{\mathcal{E}(P)}$ of modules over a certain spectral endomorphism category $\mathcal{E}(P)$ (see Definition 3.7.5 of *loc. cit.*). We thus obtain an equivalence of stable ∞-categories $L(\mathscr{M}) \to L(\mathrm{Mod}_{\mathcal{E}(P)})$.

Example 3.3.8. Let A be an abelian category with enough projective objects and $C(A)$ the simplicial model category of chain complexes in A. Then the *derived* ∞-category $L(C(A))$ of A is a stable ∞-category. The homotopy category $\mathrm{h}L(C(A))$ can be identified with the derived category $D(A)$ of A. Likewise, one can define the bounded (resp. bounded above, bounded below) derived ∞-category of A. See Section 13 of [LI] for more details.

Note that, in the spirit of Section 2.1, it would be natural to define a stable ∞-category as a fibrant object of a model category $\mathrm{PC}(\mathbf{Sp})$ of precategories enriched over the monoidal model category $\mathbf{Sp}$ of symmetric spectra with the $\mathbb{S}$-model structure (see Example 7.1.14) and smash product monoidal structure. Proving the existence of a closed structure on this conjectured model category would lead to a reasonable definition of the $(\infty, n+1)$-category of stable (∞, n)-categories by iteration. However, since we do not wish to develop a general theory of precategories enriched over a general monoidal model category, Definition 3.3.1 will suffice for our purposes. Note that one could also define a stable (∞, n)-category using our definition by iteration by allowing a stable (∞, n)-category to be a $\mathrm{Cat}^{\perp}_{(\infty,n-1)}$-category where $\mathrm{Cat}^{\perp}_{(\infty,n-1)}$ is the category of stable $(\infty, n-1)$-categories (ie. with the trivial model structure).

Let C be an ∞-category with finite limits and $\mathbb{Z}$ the linearly ordered set of integers which we consider as a filtered category. Let T be an endofunctor on C. We construct the following endofunctor

$$\phi : \mathbb{R}\underline{\mathrm{Hom}}(\mathbb{Z}, C) \to \mathbb{R}\underline{\mathrm{Hom}}(\mathbb{Z}, C)$$

defined by $\phi(F)(n) := T(F(n+1))$.

Definition 3.3.9. Let C be an ∞-category with finite limits and T an endofunctor on C. A *T-spectrum object* of C is a functor $F : \mathbb{Z} \to C$ such that $F \to \phi(F)$ is an equivalence in $\mathbb{R}\underline{\mathrm{Hom}}(\mathbb{Z}, C)$.

The ∞-category of T-spectrum objects in C, denoted $\mathrm{Sp}_T(C)$, is given by the homotopy pullback

$$\begin{tikzcd}
\mathrm{Sp}_T(C) \arrow[r] \arrow[d] & \mathbb{R}\underline{\mathrm{Hom}}(\mathbb{Z}, C) \arrow[d, "(\phi,\mathrm{id})"] \\
\mathbb{R}\underline{\mathrm{Hom}}(\mathbb{Z}, C) \arrow[r, "d"] & \mathbb{R}\underline{\mathrm{Hom}}(\mathbb{Z}, C) \times \mathbb{R}\underline{\mathrm{Hom}}(\mathbb{Z}, C)
\end{tikzcd}$$

where d denotes the diagonal map. The equivalence $d(G) \simeq (\phi, \mathrm{id})(F)$ induces the equivalences $F \xleftarrow{\sim} G \xrightarrow{\sim} \phi(F)$ whose composition gives the equivalence required in Definition 3.3.9. The ∞-category of T-spectrum objects in C comes naturally equipped with an evaluation functor $\mathrm{Ev}_n : \mathrm{Sp}_T(C) \to C$ for every

$n \in \mathbb{Z}$ which acts on a spectrum F and picks out its n-th term $F(n)$. If C is a presentable ∞-category then this evaluation functor admits a left adjoint $\mathrm{Fr}_n : C \to \mathrm{Sp}_T(C)$.

We will be particularly interested in the case where the endofunctor T is the loop functor. In this case, if C is a pointed ∞-category with finite limits then the ∞-category $\mathrm{Sp}_\Omega(C)$ is a stable ∞-category (see Proposition 8.27 of [LI]). This defines a natural functor Sp_Ω from the $(\infty,2)$-category of pointed ∞-categories with finite limits and left exact functors to the $(\infty,2)$-category $\underline{\mathrm{Cat}}_\infty^\perp$ of stable ∞-categories whose right adjoint is the forgetful functor.

Let C_* denote the full subcategory of $\mathbb{R}\underline{\mathrm{Hom}}([1],C)$ spanned by those morphisms $x \to y$ for which x is a terminal object of C. We call C_* the ∞-category of *pointed objects* of C. If C is pointed, then the forgetful functor $C_* \to C$ is an equivalence of ∞-categories.

Definition 3.3.10. A *spectrum* is a Ω-spectrum object of the ∞-category $\mathcal{K}_*$ of pointed spaces.

Let $\mathrm{Sp} := \mathrm{Sp}_\Omega(\mathcal{K}_*)$ denote the ∞-category of spectra. The ∞-category Sp is stable and presentable. It follows from Definition 3.3.9 that the ∞-category Sp of spectra can be identified with the homotopy limit of the tower

$$\{\ldots \to \mathcal{K}_* \xrightarrow{\Omega} \mathcal{K}_* \xrightarrow{\Omega} \mathcal{K}_*\}.$$

Recall the homotopy group functor on spectra $\pi_n : \mathrm{Sp} \to \mathrm{Ab}$ which takes a spectrum A to the abelian group $\mathrm{Hom}_{\mathrm{hSp}}(\mathbb{S}[n], A)$. A map $f : A \to B$ of spectra is an equivalence if and only if it induces isomorphisms $\pi_n A \to \pi_n B$ for all $n \in \mathbb{Z}$. Let $\mathcal{K}^{fin}$ denote the smallest full subcategory of $\mathcal{K}$ which contains the final object and is stable under finite colimits. Then $\mathrm{Ind}(\mathcal{K}_*^{fin}) \to \mathcal{K}_*$ is an equivalence of ∞-categories and thus $\mathcal{K}_*$ is compactly generated. Moreover, let $\mathcal{K}_\infty^{fin}$ denote the colimit of the sequence

$$\{\mathcal{K}_*^{fin} \xrightarrow{\Sigma} \mathcal{K}_*^{fin} \xrightarrow{\Sigma} \ldots\}$$

in the $(\infty,2)$-category of ∞-categories and exact functors. Then the ∞-category of spectra is compactly generated where $\mathrm{Ind}(\mathcal{K}_\infty^{fin}) \to \mathrm{Sp}$ is an equivalence of ∞-categories.

Example 3.3.11. Let C be an ∞-category and X an ∞-category which admits finite limits. There exists an equivalence $\mathrm{Sp}(\mathrm{Pr}_X(C)) \to \mathrm{Pr}_{\mathrm{Sp}(X)}(C)$ of ∞-categories.

Let C be a presentable ∞-category. Then the ∞-category $\mathrm{Sp}_\Omega(C_*)$ is presentable and the natural functor $\mathrm{Ev}_n : \mathrm{Sp}_\Omega(C_*) \to C$ admits a left adjoint $\mathrm{Fr}_n : C \to \mathrm{Sp}_\Omega(C_*)$. Let $C = \mathcal{K}$ and $*$ be the final object of $\mathcal{K}$. The object $\mathrm{Fr}_0(*)$ of Sp will be called the *sphere spectrum* and will be denoted by $\mathbb{S}$.

Proposition 3.3.12. *The ∞-category* Sp *admits a symmetric monoidal structure which is uniquely characterized by the property that the unit object of* Sp *is the sphere spectrum* $\mathbb{S}$ *and the bifunctor* $\otimes : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ *preserves colimits seperately in each variable.*

Proof reference. This follows from Corollary 4.1.16 of [LIII]. □

This monoidal structure will be called the *smash product* monoidal structure on Sp. Let **Sp** be category of symmetric spectra endowed with the $\mathbb{S}$-model structure (see Example 7.1.14). The **Sp** can be lifted to a simplicial symmetric monoidal model category. By Corollary 4.1.16 of [LIII] there exists an equivalence $L\mathbf{Sp} \simeq \mathrm{N}(\mathbf{Sp}^\circ) \to \mathrm{Sp}$ of symmetric monoidal ∞-categories.

Let C be a stable ∞-category and x be any object of C. We use the following notation

$$x[n] := \begin{cases} \Sigma^n x & \text{if } n \geq 0, \\ \Omega^{-n} x & \text{if } n \leq 0, \end{cases}$$

for the object $x[n]$ of C given by taking the nth power of the suspension and loop functors. We use the same notation for the corresponding object in hC. There naturally exists a spectrum of maps between any two objects in C. For all $x, y \in C$, since $x \simeq 0 \times_{x[1]} 0$, the space $\mathrm{Map}_C(x,y)$ (pointed by the zero map) is the zeroth space of the spectrum

$$\ldots \xrightarrow{\Omega} \mathrm{Map}_C(x,y[2]) \xrightarrow{\Omega} \mathrm{Map}_C(x,y[1]) \xrightarrow{\Omega} \mathrm{Map}_C(x,y) \xrightarrow{\Omega} \mathrm{Map}_C(x[1],y) \xrightarrow{\Omega} \mathrm{Map}_C(x[2],y) \xrightarrow{\Omega} \ldots.$$

More precisely, let $\underline{\mathrm{Cat}}_\infty^p$ be the full subcategory of the $(\infty, 2)$-category of ∞-categories spanned by presentable objects and colimit preserving functors. Let $\underline{\mathrm{Cat}}_\infty^{\perp,p}$ denote the full subcategory spanned by stable, presentable objects. We have the following.

Proposition 3.3.13. *Let C be a stable, presentable ∞-category. Then C is tensored and enriched over the ∞-category* Sp *of spectra.*

Proof. By Proposition 4.2.7 of [LII] there exists an equivalence

$$\mathrm{Mod}_{\mathrm{Sp}}(\underline{\mathrm{Cat}}_\infty^p) \to \underline{\mathrm{Cat}}_\infty^{\perp,p}$$

of $(\infty, 2)$-categories. This follows since the $(\infty, 2)$-category $\underline{\mathrm{Cat}}_\infty^{\perp,p}$ is left tensored over the ∞-category Sp of spectra which is the unit object of $\underline{\mathrm{Cat}}_\infty^{\perp,p}$ and a presentable ∞-category which is tensored over Sp where the tensored structure preserves (small) colimits is automatically stable. Finally, since the functor $\bullet \otimes x : \mathrm{Sp} \to C$ preserves (small) colimits for all $x \in C$, the result follows from Proposition 3.2.5. $\square$

Let C be a stable ∞-category. Then we can guarantee that C is enriched over the ∞-category of spectra using the following criterion (see Corollary 15.2 of [LI]): a stable ∞-category C is presentable if and only if C admits small coproducts, the homotopy category $\mathrm{h}C$ is locally small and there exists a κ-compact generator x in C for a regular cardinal κ (ie. if the condition $\pi_0 \mathrm{Map}_C(x, y) \simeq *$ implies that y is a null object of C).

Recall that a triangulated category is an additive category A together with a translation functor $A \to A : x \to x[1]$ (an equivalence of categories) and a collection of *distinguished triangles* of the form $x \to y \to z \to x[1]$, which satisfy appropriate axioms. We refer the reader to [Ne] for the complete list of axioms. The homotopy category of a stable ∞-category C is a triangulated category. This can be seen as follows. By definition, C admits finite coproducts and hence $\mathrm{h}C$ does. Also, by the chain of equivalences

$$\mathrm{Hom}_{\mathrm{h}C}(x, y) \simeq \pi_0 \mathrm{Map}_C(x, y) \simeq \pi_1 \mathrm{Map}_C(\Omega x, y) \simeq \pi_2 \mathrm{Map}_C(\Omega^2 x, y),$$

where the base point is the zero map, the set $\mathrm{Hom}_{\mathrm{h}C}(x, y)$ is an abelian group. Thus $\mathrm{h}C$ is an additive category. We let the suspension functor $\Sigma : x \mapsto x[1]$ denote the translation functor which is an autoequivalence by Proposition 3.3.2. Finally, a triangle

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} x[1]$$

in $\mathrm{h}C$ will be called distinguished if there exists a double pushout diagram

$$\begin{tikzcd}
x \arrow[r, "\tilde{f}"] \arrow[d] & y \arrow[r] \arrow[d, "\tilde{g}"] & 0 \arrow[d] \\
0' \arrow[r] & z \arrow[r, "\tilde{h}"] & w
\end{tikzcd}$$

in C such that $0'$ and 0 are zero objects, the morphisms $\tilde{f}$ and $\tilde{g}$ represent f and g respectively and h is given by the composition $\phi \circ \tilde{h} : z \to x[1]$ where $\phi : w \xrightarrow{\sim} x[1]$ is the isomorphism determined by the outer pushout rectangle. It remains to show that this triangulated structure satisfies the axioms of a trianglulated category. This follows from Proposition 3.3.2 followed by Theorem 3.11 of [LI].

Definition 3.3.14. A spectrum A is said to be *connective* if $\pi_n A \simeq 0$ for all $n < 0$. It is said to be *discrete* if it is connective and 0-truncated.

We denote by Sp^c (resp. Sp^d) the full subcategory of Sp spanned by the connective (resp. discrete) spectra. The ∞-category of connective spectra is the smallest full subcategory of Sp closed under colimits and extensions which contains the sphere spectrum $\mathbb{S}$. It is projectively generated with the sphere spectrum being a compact projective generator. In Section 3.5 we will see that the ∞-category of discrete spectra is equivalent to the category of abelian groups.

3.4 Commutative ring spectra

Definition 3.4.1. A *commutative ring spectrum* is a commutative monoid object in the ∞-category Sp of spectra with respect to the smash product monoidal structure.

A commutative ring spectrum will be referred to as an E_∞-ring. The ∞-category of E_∞-rings will be denoted $\mathfrak{E} := \mathrm{CMon}(\mathrm{Sp})$. The ∞-category $\mathfrak{E}$ will play the role of our generalised theory of rings. If R is an E_∞-ring, then by Proposition 3.2.14, the ∞-category of *commutative R-algebras* $\mathrm{CAlg}_R := \mathrm{CMon}(\mathrm{Mod}_R)$ in Sp is equivalent to $\mathfrak{E}_{R/}$.

Example 3.4.2. Let S^n be the simplicial n-sphere. For any commutative ring k, one can associate an E_∞-ring spectrum Hk called the *Eilenberg Mac Lane* ring spectrum which is the sequence of simplicial abelian groups $k \otimes S^n$ where $(k \otimes S^n)_m$ is the free abelian group on the non-basepoint m-simplices of S^n. The basepoint is identified with 0.

For $R \in \mathfrak{E}$ and $n \in \mathbb{Z}$, let $\pi_n R$ denote the nth homotopy group of the underlying spectrum of R. We can identify $\pi_n R$ with the set $\pi_0 \mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}[n], R)$ where $\mathbb{S}$ denotes the sphere spectrum. The commutative structure on R endows the direct sum $\oplus_{n\in\mathbb{Z}}\pi_n(R)$ of the homotopy groups $\{\pi_n(R)\}_{n\in\mathbb{Z}}$ with the structure of a graded commutative ring, ie. for each $a \in \pi_n R$ and $b \in \pi_m(R)$ we have $ab = (-1)^{nm} ba$. In particular, $\pi_0 R$ is a discrete commutative ring and $\pi_n R$ has the natural structure of a $\pi_0(R)$-module.

An E_∞-ring R is said to be *connective* if $\pi_n R \simeq 0$ for all $n < 0$. The full subcategory of commutative ring spectra spanned by the connective objects, denoted $\mathfrak{E}^c$, is equivalent to the ∞-category $\mathrm{CMon}(\mathrm{Sp}^c)$ of commutative monoid objects in the ∞-category of connective spectra. We can think of connective E_∞-rings as simply spaces endowed with an addition and multiplication satisfying the axioms for a commutative ring up to coherent homotopy. More precisely:

Proposition 3.4.3. *Let* $\mathrm{Ev}_0 : \mathfrak{E}^c \to \mathcal{K}$ *denote the composition* $\mathfrak{E}^c \to \mathrm{Sp}^c \to \mathcal{K}$*. Then*

$$\theta : \mathfrak{E}^c \to \mathrm{Mod}_T(\mathcal{K})$$

is an equivalence of ∞-categories for the monad $T = \mathrm{Ev}_0 \circ \mathrm{Fr}_0$ *on* $\mathcal{K}$.

Proof. The functor $\mathrm{Ev}_0 : \mathfrak{E}^c \to \mathcal{K}$ is conservative, preserves finite colimits of simplicial objects and admits a left adjoint Fr_0. Beck's theorem of Proposition 5.2.13 then asserts that θ is an equivalence of ∞-categories. □

An E_∞-ring R is said to be *bounded* if $\pi_i(R) = 0$ for $i > n$ for some n. It is said to be *discrete* if it is connective and 0-truncated. We let $\mathfrak{E}^d$ denote the full subcategory of $\mathfrak{E}$ spanned by the discrete objects. A connective E_∞-ring R is discrete if and only if for all $n > 0$ the homotopy group $\pi_n R$ is trivial. The ∞-category $\mathfrak{E}^d$ is equivalent to the ∞-category $\mathrm{CMon}(\mathrm{Sp}^d)$ of commutative monoid objects in the ∞-category of discrete spectra. The ∞-category of commutative ring spectra contains, as a fully subcategory, the ordinary theory of commutative rings Rng. See Proposition 3.5.15 for a precise statement.

A convenient way to manipulate algebra in the context of ring spectra is to utilise its model categorical interpretation. Let $\mathbf{Sp}$ be endowed with its $\mathbb{S}$-model structure of Example 7.1.14. The category $\mathrm{CMon}(\mathbf{Sp})$ inherits the structure of a simplicial model category again by [Sh]. Let $\mathrm{CMon}(\mathbf{Sp})$ be endowed with this $\mathbb{S}$-model structure. Then by Example 3.1.15 there exists an equivalence

$$s : L\,\mathrm{CMon}(\mathbf{Sp}) \to \mathfrak{E}$$

of ∞-categories. Furthermore, by Example 3.2.9, there exists an equivalence

$$L\,\mathrm{Mod}_R(\mathbf{Sp}) \to \mathrm{Mod}_{s(R)}(\mathrm{Sp})$$

of ∞-categories.

Lemma 3.4.4. *Let R be an E_∞-ring. The ∞-category of modules* Mod_R *is a stable ∞-category.*

Proof. By the above discussion, the ∞-category Mod_R is equivalent to the localisation of a stable model category. The localisation of a stable model category is a stable ∞-category (Example 3.3.7). $\square$

Example 3.4.5. When $R \in \mathfrak{E}$ is discrete, ie. an ordinary commutative ring, the triangulated category hMod_R corresponds to the classical derived category of Mod_R.

Example 3.4.6. Let k be a field of characteristic zero and dgAlg_k the model category of commutative differential graded k-algebras where the weak equivalences are given by the quasi-isomorphisms (see Section 5 of [SS2]), ie. the model category of commutative monoid objects in the symmetric monoidal model category $C(k) := C(\mathrm{Mod}_k(\mathrm{Ab}))$ of chain complexes of k-modules. By Theorem 5.1.6 of [SS1] there exists a Quillen equivalence $\mathrm{Mod}_{Hk}(\mathbf{Sp}) \to C(k)$ where the model category $\mathbf{Sp}$ of symmetric spectra is endowed with the $\mathbb{S}$-model structure. Thus we have a diagram

$$\begin{tikzcd} \mathrm{CMon}(L\,\mathrm{Mod}_{Hk}(\mathbf{Sp})) \arrow[r, "\sim"] \arrow[d] & \mathrm{CMon}(LC(k)) \arrow[d] \\ \mathrm{CAlg}_{Hk}(\mathrm{Sp}) \arrow[r] & L\mathrm{dgAlg}_k \end{tikzcd}$$

of ∞-categories. The left vertical arrow is an equivalence by Example 3.2.9 and the right vertical arrow is an equivalence by Example 3.1.15. Therefore the ∞-category of Hk-algebras in Sp can be identified with the localisation of the model category of commutative differential k-algebras.

Let R be a connective E_∞-ring and $\mathrm{CAlg}^c_R := \mathrm{CMon}(\mathrm{Mod}^c_R)$ the ∞-category of *connective commutative R-algebras*. The ∞-category CAlg^c_R is projectively generated with the compact projective objects being identified with those connective commutative R-algebras that are retracts of a finitely generated free commutative R-algebra. If $(\mathrm{CAlg}^c_R)^{fp}$ denotes the smallest full subcategory of CAlg^c_R which contains all finitely generated free R-algebras and is stable under finite colimits then

$$\mathrm{Ind}((\mathrm{CAlg}^c_R)^{fp}) \to \mathrm{CAlg}^c_R$$

is an equivalence of ∞-categories. Moreover, the ∞-category CAlg^c_R is compactly generated with the compact objects being the finitely presented R-algebras.

Consider the ∞-category Mod_R of R-modules for R an E_∞-ring. If M is an R-module we will denote by $\pi_n M$ the homotopy group of its underlying spectrum. For any object $M \in \mathrm{Mod}_R$, the set $\oplus_{n\in\mathbb{Z}}\pi_n M$ forms a graded module over the graded commutative ring $\oplus_{n\in\mathbb{Z}}\pi_n R$. The ∞-category $(\mathrm{Mod}_R)_{\geq 0}$ is the smallest full subcategory of Mod_R which contains R and is stable under small colimits. A module M in Mod_R is said to be *connective* if $\pi_n M = 0$ for all $n < 0$ and we call $(\mathrm{Mod}_R)_{\geq 0}$ the ∞-category of *connective* R-modules. Likewise, a module M in Mod_R is said to be *anti-connective* if $\pi_n M = 0$ for all $n > 0$. An R-module is said to be *free* if it is equivalent to a coproduct of copies of R and *finitely generated* if it can be written as a finite coproduct of copies of R.

Let R be a connective E_∞-ring. An R-module M is said to be *projective* if it is a projective object of the ∞-category $(\mathrm{Mod}_R)_{\geq 0}$ of connective R-modules (note that the ∞-category Mod_R has no nonzero projective objects). The R-module M is projective if and only if there exists a free R-module N such that M is a retract of N. If N is moreover finitely generated, then M is a compact projective object of $(\mathrm{Mod}_R)_{\geq 0}$. This is equivalent to M being projective and $\pi_0 M$ being finitely generated as a $\pi_0 R$-module. The ∞-category of of connective modules over a connective E_∞-ring is projectively generated.

Let R be a connective E_∞-ring. Then the inclusion $\mathrm{CAlg}^c_R \hookrightarrow \mathrm{CAlg}_R$ commutes with colimits so there exists an adjunction

$$i : \mathrm{CAlg}^c_R \rightleftarrows \mathrm{CAlg}_R : (\bullet)^c$$

where the right adjoint is called the *connective cover*. Explicitly, a connective cover of an R-algebra A is a map $f : A \to A^c$ such that A^c is connective and for any connective R-algebra B, there exists an equivalence

$$\mathrm{Map}_{\mathrm{CAlg}_R}(B, A) \to \mathrm{Map}_{\mathrm{CAlg}^c_R}(B, A^c)$$

of $(\infty, 0)$-categories.

Let C be an ∞-category and $x, y \in C$. Then y is said to be a retract of x if it is a retract of x in $\mathrm{h}C$, ie. there exists a diagram $y \xrightarrow{i} x \xrightarrow{p} y$ which coincides with the identity id_y in $\mathrm{h}C$.

Definition 3.4.7. Let C be a stable ∞-category. Let C^{perf} denote the smallest stable subcategory of C which contains the unit object and is closed under retracts. An object $x \in C$ is said to be *perfect* if it is an object of C^{perf}.

Let R be an E_∞-ring. There exists equivalences

$$\mathrm{Mod}_R^{\mathrm{perf}} \simeq \mathrm{Mod}_R^{\mathrm{cpt}}$$

of ∞-categories. Furthermore if R is a connective E_∞-ring and $\mathrm{Mod}_R^{\mathrm{fgp}}$ denotes the smallest stable subcategory of Mod_R which contains all finitely generated projective modules then there exists an equivalence

$$\mathrm{Mod}_R^{\mathrm{fgp}} \to \mathrm{Mod}_R^{\mathrm{perf}}$$

of ∞-categories (see Remark 4.7.26 of [LII]).

3.5 t-structures

Let T be a triangulated category. Recall that a *t-structure* on T is a pair of full subcategories, $T_{\leq 0}$ and $T_{\geq 0}$, stable under isomorphism and which satisfy the following conditions:

1. $T_{\leq 0}[-1] \subseteq T_{\leq 0}$ and $T_{\geq 0}[1] \subseteq T_{\geq 0}$.
2. For all $x \in T$, there exists a distinguished triangle $y \to x \to z \to y[1]$ where $y \in T_{\geq 0}$ and $z \in T_{\leq 0}[-1]$.
3. If $x \in T_{\geq 0}$ and $y \in T_{\leq 0}[-1]$ then $T(x, y) = 0$.

The axioms imply that the distinguished triangle in condition (3) is unique up to isomorphism. Also, condition (1) implies that any subcategory in the pair determines the other. See [BBD] for further discussion. For ease of notation we write $T_{\leq n} := T_{\leq 0}[n]$ and $T_{\geq n} := T_{\geq 0}[n]$. Note that the indexing we use follows the homological convention. To pass to the cohomological indexing of *loc. cit.* we note that $T_{\leq n} = T^{\geq -n}$ and $T_{\geq n} = T^{\leq -n}$.

Definition 3.5.1. Let C be a stable ∞-category. Then C is said to admit a *t-structure* if there exists a t-structure on the homotopy category $\mathrm{h}C$.

Let C be a stable ∞-category. The full subcategory of C spanned by the objects of $(\mathrm{h}C)_{\leq n}$ and $(\mathrm{h}C)_{\geq n}$ will be denoted by $C_{\leq n}$ and $C_{\geq n}$ respectively.

Example 3.5.2. When C is a presentable stable ∞-category, any small collection of objects $\{x_\alpha\}$ determines a t-structure on C. The construction is as follows (see Proposition 16.1 of [LI]). One builds a subcategory C' of C as the smallest full subcategory of C containing $\{x_\alpha\}$ which is closed under small colimits and such that for every distinguished triangle

$$x \to y \to z \to x[1]$$

for which x and z are in C', then y is in C'. In this case, there exists a t-structure on C such that $C' = C_{\geq 0}$ and $C_{\geq 0}$ is presentable.

We call a t-structure on a presentable stable ∞-category C *accessible* if the subcategory $C_{\geq 0}$ is presentable. It follows that if C admits an accessible t-structure then $C_{\leq 0}$ is also presentable. The t-structures that we will be concerned with in this paper are the following accessible t-structures on the ∞-category of spectra and the ∞-category of R-modules for R a connective E_∞-ring.

Example 3.5.3. There exists an accessible t-structure on the ∞-category Sp of spectra given as follows:

- $\mathrm{Sp}_{\leq 0}$ is the full subcategory of Sp spanned by the objects $\{A \in \mathrm{Sp} | \forall n > 0, \pi_n A \simeq 0\}$.
- $\mathrm{Sp}_{\geq 0}$ is the full subcategory of Sp spanned by the objects $\{A \in \mathrm{Sp} | \forall n < 0, \pi_n A \simeq 0\}$.

Example 3.5.4. Let R be a connective E_∞-ring. Then there exists an accessible t-structure on the ∞-category Mod_R of R-modules where

- $(\mathrm{Mod}_R)_{\leq 0}$ is the full subcategory of Mod_R spanned by the objects $\{M \in \mathrm{Mod}_R | \forall n > 0, \pi_n M \simeq 0\}$.
- $(\mathrm{Mod}_R)_{\geq 0}$ is the full subcategory of Mod_R spanned by the objects $\{M \in \mathrm{Mod}_R | \forall n < 0, \pi_n M \simeq 0\}$.

This t-structure is left and right complete. With this t-structure, the heart of Mod_R is equivalent to the abelian category of discrete modules over the ring $\pi_0(R)$.

Definition 3.5.5. Let C be a stable ∞-category. A t-structure on C is said to be *non-degenerate* if for all objects x in C, if $\tau_{n,n}x = 0$ for all n then $x = 0$.

Let C be a stable ∞-category. A t-structure $(C_{\leq 0}, C_{\geq 0})$ on C is non-degenerate if and only if $\cup_i C_{\leq i} = \cup_i C_{\geq i} = C$ and $\cap_i C_{\leq i} = \cap_i C_{\geq i} = 0$. The t-structures of Example 3.5.3 and Example 3.5.4 are non-degenerate. The usefulness of this non-degeneracy is that it enables us to check equivalences in these ∞-categories on their corresponding truncations in the following sense. By Corollary 6.6 of [LI], the ∞-category $C_{\leq n}$ is stable under limits in C and the ∞-category $C_{\geq n}$ is stable under colimits in C. Hence there exists a left adjoint

$$\tau_{\leq n} : C \to C_{\leq n}$$

to the inclusion map $C_{\leq n} \hookrightarrow C$ and a right adjoint

$$\tau_{\geq n} : C \to C_{\geq n}$$

to the inclusion map $C_{\geq n} \hookrightarrow C$. By Proposition 6.10 of [LI] there exists an equivalence $\tau_{\leq m} \circ \tau_{\geq n} \simeq \tau_{\geq n} \circ \tau_{\leq m}$ of functors from C to $C_{\leq m} \cap C_{\geq n}$ which we will denote by $\tau_{n,m} : C \to C_{[n,m]}$.

Definition 3.5.6. Let C and D be stable ∞-categories admitting t-structures. A functor $F : C \to D$ is said to be *left* (resp. *right*) *t-exact* if it is exact and sends $C_{\leq 0}$ into $D_{\leq 0}$ (resp. sends $C_{\geq 0}$ into $D_{\geq 0}$). It is said to be *t-exact* if it is both left and right t-exact.

Lemma 3.5.7. *Let C and D be stable ∞-categories admitting t-structures and let $F \dashv G$ be an adjunction between them. Then F is right t-exact if and only if G is left t-exact.*

Proof. Let $F : C \to D$ be right t-exact. Then for any $x \in C_{\geq 0}$ we have $F(x) \in D_{\geq 0}$. Thus for any $y \in D_{\leq -1}$, $\mathrm{Hom}_{\mathrm{h}D}(F(x), y) \simeq \mathrm{Hom}_{\mathrm{h}C}(x, G(y)) = 0$. Therefore $G(y) \in C_{\leq -1}$. □

Another consequence of having a non-degenerate t-structure on a stable ∞-category is the following. We say that a stable ∞-category C endowed with a t-structure is *left t-complete* if the natural map

$$C \to \mathrm{holim}_n C_{\leq n} := \mathrm{holim}_n \{\ldots \to C_{\leq 2} \xrightarrow{\tau_{\leq 1}} C_{\leq 1} \xrightarrow{\tau_{\leq 0}} C_{\leq 0} C \xrightarrow{\tau_{\leq -1}} \ldots\}$$

is an equivalence of ∞-categories. By Proposition 7.3 of [LI], if a stable ∞-category with a t-structure admits countable products such that $C_{\geq 0}$ is stable under countable products, then C is left t-complete if and only if $\cap_i C_{\geq i} = 0$.

Definition 3.5.8. Let C be a stable ∞-category and D a stable ∞-category admitting a t-structure $(D_{\leq 0}, D_{\geq 0})$. A functor $F : C \to D$ is said to *create a t-structure* on C if $C_{\leq 0} := \{x \in C | F(x) \in D_{\leq 0}\}$ and $C_{\geq 0} := \{x \in C | F(x) \in D_{\geq 0}\}$ define a t-structure on C.

Definition 3.5.9. Let C be a stable ∞-category with a t-structure $(C_{\leq 0}, C_{\geq 0})$. The *heart* of C is the full subcategory $\mathcal{H}(C) := C_{\leq 0} \cap C_{\geq 0}$.

Let C be a stable ∞-category admitting a t-structure. Then for any object $x \in C$ and $n \geq -1$, the object x belongs to $C_{\leq n}$ if and only if the space $C(y, x)$ is n-truncated for all $y \in C_{\geq 0}$. Thus for x and y in $\mathcal{H}(C)$, the group $\pi_n C(x, y)$ vanishes for all $n > 0$ and so there exists an equivalence

$$\mathcal{H}(C) \to \mathrm{h}\mathcal{H}(C)$$

of ∞-categories.

Definition 3.5.10. Let C be a stable symmetric monoidal ∞-category. A t-structure on C is said to be *compatible* with the symmetric monoidal structure if for all $x \in C$ the functor $x \otimes \bullet$ is exact and $C_{\geq 0}$ is closed under tensor products and contains the unit object.

Example 3.5.11. The t-structure on the ∞-category Sp of spectra of Example 3.5.3 is compatible with the symmetric monoidal structure. As a result, by Proposition 3.1.11, the ∞-category Sp^c of connective spectra inherits the structure of a symmetric monoidal ∞-category. Let R be a connective E_∞-ring. By extension, the ∞-category Mod_R^c of connective R-modules and connective R-algebras $\mathrm{CAlg}_R^c := \mathrm{CMon}(\mathrm{Mod}_R^c)$ inherit symmetric monoidal structures.

Proposition 3.5.12. *Let C be a stable symmetric monoidal ∞-category which admits a t-structure that is compatible with the symmetric monoidal structure. Then the heart $\mathcal{H}(C)$ of C inherits the structure of a symmetric monoidal category.*

Proof. By Example 3.1.11, the ∞-category $C_{\geq 0}$ inherits the structure of a symmetric monoidal ∞-category. Then Proposition 1.3.12 of [LII] states that the truncation functor $\tau_{\leq n} : C_{\geq 0} \to (C_{\geq 0})_{\leq n}$ satisfies the following condition: if $f : x \to y$ is an arrow in $C_{\geq 0}$ and $g : x \otimes z \to y \otimes z$ the induced map for an object z in $C_{\geq 0}$, then $\tau_{\leq n} g$ is an equivalence if $\tau_{\leq n} f$ is an equivalence. By Proposition 1.31 of [LIII], the ∞-category $(C_{\geq 0})_{\leq n}$ inherits the structure of a symmetric monoidal ∞-category. Taking the case $n = 0$ the result follows. ☐

Definition 3.5.13. Let C be a stable ∞-category admitting a t-structure and let $n \in \mathbb{Z}$. We define a functor

$$\begin{aligned} \pi_n^t : C &\to \mathcal{H}(C) \\ x &\mapsto \tau_{0,0}(x[-n]). \end{aligned}$$

Let C be a stable ∞-category. It follows from Theorem 1.3.6 of [BBD] that the category $\mathcal{H}(C)$ is abelian. One can show that the heart of a presentable stable ∞-category equipped with an admissible t-struture is a presentable abelian category. Let Ab denote the category of abelian groups. When C is the ∞-category Sp of spectra we have the following result.

Proposition 3.5.14. *Let* Sp *be endowed with the t-structure given by Example 3.5.3. Then the functor*

$$\mathcal{H}(\mathrm{Sp}) \to \mathrm{Ab}$$

is an equivalence of ∞-categories.

Proof. This follows from Proposition 9.2 of [LI] so we only make a few comments. Let X be an object in Sp. Then X is an object of $\mathcal{H}(\mathrm{Sp})$ if and only if $X(n)$ is a pointed object of $\mathcal{K}$ which is both n-truncated and n-connective. A pointed object $* \to x$ in an ∞-category C is called an *Eilenberg-MacLane object* of degree n if it is both n-truncated and n-connective (see Definition 7.2.2.1 of [Lu]). Let $\mathrm{EM}_n(C)$ denotes the full subcategory of C_* spanned by the Eilenberg-MacLane objects of degree n. Thus

$$\mathcal{H}(\mathrm{Sp}) = \mathrm{holim}\{\ldots \xrightarrow{\Omega} \mathrm{EM}_1(\mathcal{K}) \xrightarrow{\Omega} \mathrm{EM}_0(\mathcal{K})\}$$

and one can show that this sequence stabalizes with value Ab after $n = 2$. ☐

Proposition 3.5.15. *The functor*

$$\pi_0^t : \mathfrak{E}^d \to \mathrm{Rng}$$

is an equivalence of ∞-categories.

Proof. This follows from Proposition 4.2.11 of [LIII] so again we will only sketch the proof. The map $\mathfrak{E}^d \to \mathrm{CMon}(\tau_{\leq 0}\mathrm{Sp}^c)$ is an equivalence of ∞-categories. There exists an equivalence $\mathrm{Sp}^d = \mathcal{H}(\mathrm{Sp}) \to \mathrm{Ab}$ of ∞-categories by Proposition 3.5.14 and the induced symmetric monoidal structure on $\mathcal{H}(\mathrm{Sp})$ coincides up to canonical equivalence with the symmetric monoidal structure on the category Ab owing to the properties that $\pi_0 S \simeq \mathbb{Z}$ and the induced tensor product on $\mathcal{H}(\mathrm{Sp})$ preserves colimits seperately in each variable. Thus $\mathrm{CMon}(\mathcal{H}(\mathrm{Sp})) \simeq \mathrm{CMon}(\mathrm{Ab}) =: \mathrm{Rng}$. □

Lemma 3.5.16. *Let C and D be stable ∞-categories admitting non-degenerate t-structures. Let $f : C \to D$ be a t-exact functor. If X is a cosimplicial object in C such that there exists $k \geq 0$ with $\pi_i^t(X_n) = 0$ for all $i > k$ and all n then*

$$f(\lim_{n\in\Delta} X_n) \to \lim_{n\in\Delta} f(X_n)$$

is an equivalence in D.

Proof. We can choose $k = 0$. Since the t-structure on D is non-degenerate, we can check the equivalence on the truncation

$$\tau_{\geq -N}(f(\lim_n X_n)) \to \tau_{\geq -N}(\lim_n f(X_n)).$$

The functor f is t-exact and the truncation commutes with limits so we are reduced to proving

$$f(\lim_n(\tau_{\geq -N} X_n)) \to \lim_n f(\tau_{\geq -N} X_n)).$$

The limits in $C_{[-N,0]}$ are considered as limits in $C_{\leq 0}$. Note that $C_{[-N,0]}$ is a subcategory of C where the mapping spaces are N-truncated. Any limit along Δ in an ∞-category whose mapping spaces are N-truncated is a finite limit. Since f is t-exact it commutes with finite limits and truncations so the induced functor $f : C_{[-N,0]} \to D_{[-N,0]}$ preserves finite limits and the result follows. □

3.6 Linear and R-tensor ∞-categories

In this section we will construct the $(\infty, 2)$-category of R-linear, stable, presentable, symmetric monoidal ∞-categories which we will call R-tensor ∞-categories. In Example 3.1.20 we saw an explicit construction of the cartesian monoidal structure on the $(\infty, 2)$-category $\underline{\mathrm{Cat}}_\infty$ of ∞-categories. From Proposition 4.1.7 of [LII] and Proposition 4.1.10 of [LIII], we deduce that there exists an $\mathcal{O}$-monoidal structure on the $(\infty, 2)$-category $\underline{\mathrm{Cat}}^p_\infty$ of presentable ∞-categories. This is a subcategory of $\widetilde{\underline{\mathrm{Cat}}}_\infty$ which can be explicitly described as follows:

- The objects are pairs $([n], (C_0, \ldots, C_n))$ where $[n]$ is an object of $\mathcal{O}$ and each C_i is a presentable fibrant $(\infty, 1)$-precategory.
- A map between two objects $([n], C_\bullet)$ and $([m], D_\bullet)$ is a map $u : [n] \to [m]$ in $\mathcal{O}$ together with a collection of functors $\prod_{u(i)=j} C_i \to D_j$ which preserve colimits seperately in each variable.

Let $\widetilde{\underline{\mathrm{Cat}}}^p_\infty$ denote this subcategory. One can show (see *loc. cit.*) that the unit object of $\underline{\mathrm{Cat}}^p_\infty$ with this $\mathcal{O}$-monoidal structure is the ∞-category $\mathcal{K}$ of spaces. Let $\underline{\mathrm{Cat}}^{p,\mathcal{O}}_\infty$ denote the subcategory of $\underline{\mathrm{Cat}}^{\mathcal{O}}_\infty$ spanned by presentable $\mathcal{O}$-monoidal ∞-categories whose monoidal bifunctor preserves colimits seperately in each variable and whose morphisms are colimit preserving $\mathcal{O}$-monoidal functors. Then we have an equivalence

$$\mathrm{Mon}^{\mathcal{O}}(\widetilde{\underline{\mathrm{Cat}}}^p_\infty) \to \underline{\mathrm{Cat}}^{p,\mathcal{O}}_\infty$$

of $(\infty,2)$-categories (see the discussion following Example 3.1.20). Thus an $\mathcal{O}$-monoidal ∞-category C belongs to $\mathrm{Mon}^{\mathcal{O}}(\widetilde{\underline{\mathrm{Cat}}}^p_\infty)$ if and only if C is presentable and the tensor product bifunctor $\otimes : C_{[1]} \times C_{[1]} \to C_{[1]}$ preserves (small) colimits seperately in each variable.

Let C be a presentable symmetric monoidal ∞-category. Then the ∞-category $\widetilde{\mathrm{Mod}}_R(C)$ of R-modules in C is a presentable symmetric monoidal ∞-category with a bicontinuous monoidal product. Thus $\widetilde{\mathrm{Mod}}_R(D)$ belongs to $\mathrm{CMon}(\widetilde{\underline{\mathrm{Cat}}}^p_\infty)$. We can then make the following definition.

Definition 3.6.1. Let D be a presentable symmetric monoidal ∞-category. A presentable ∞-category is said to be *R-linear* if it is endowed with the structure of a $\widetilde{\mathrm{Mod}}_R(D)$-module object in the symmetric monoidal $(\infty,2)$-category $\widetilde{\underline{\mathrm{Cat}}}^p_\infty$ of presentable ∞-categories.

The $(\infty,2)$-category of R-linear ∞-categories is given by $\mathrm{Mod}_{\mathrm{Mod}_R(D)}(\widetilde{\underline{\mathrm{Cat}}}^p_\infty)$. Note that an R-linear ∞-category is a presentable ∞-category C which is left-tensored over the ∞-category $\mathrm{Mod}_R(D)$. Then the functor $\bullet \otimes x : \mathrm{Mod}_R(D) \to C$ preserves colimits for all $x \in C$ owing to the monoidal structure on $\underline{\mathrm{Cat}}^p_\infty$. Thus by Proposition 2.3.7 the presentable ∞-category C is enriched over $\mathrm{Mod}_R(D)$ as expected. By Proposition 3.2.14, there exists an equivalence

$$\mathrm{CMon}(\widetilde{\mathrm{Mod}}_{\mathrm{Mod}_R}(\widetilde{\underline{\mathrm{Cat}}}^p_\infty)) \simeq \mathrm{CMon}(\widetilde{\underline{\mathrm{Cat}}}^p_\infty)_{\mathrm{Mod}_R\,/}.$$

of $(\infty,2)$-categories. The term on the left hand side is the $(\infty,2)$-category of R-linear presentable symmetric monoidal ∞-categories and R-linear symmetric monoidal functors.

Example 3.6.2. Let D be a presentable symmetric monoidal ∞-category. Then $\widetilde{\mathrm{Mod}}_R(D)$ is an R-linear ∞-category.

Example 3.6.3. Let k be a commutative ring and C a presentable k-linear symmetric monoidal category (ie. a presentable symmetric monoidal category with a $\mathrm{Mod}_k(\mathrm{Ab})$-module structure). Then $L(C)$ is a k-linear ∞-category, ie. if the map $\mathrm{Mod}_k \to C$ of categories endows C with a k-linear structure then the map of ∞-categories $L(\mathrm{Mod}_k) \to L(C)$ endows $L(C)$ with a k-linear structure. More generally, if $\mathscr{M}$ is a k-linear symmetric monoidal model category, ie. a $\mathrm{Mod}_k(\mathrm{Ab})$-enriched symmetric monoidal model category in the sense of Definition 7.1.6 where $\mathrm{Mod}_k(\mathrm{Ab})$ is endowed with the model structure of Example **??**, then $L(\mathscr{M})$ is a k-linear ∞-category.

Example 3.6.4. Let $\mathscr{M}$ be a combinatorial $\mathcal{O}$-monoidal model category. Then by Proposition 3.1.19, $\widetilde{\mathscr{M}}^\circ$ is an $\mathcal{O}$-monoidal ∞-category. Moreover, $\widetilde{\mathscr{M}}^\circ$ is presentable with a colimit preserving tensor product bifunctor so $\widetilde{\mathscr{M}}^\circ$ is an object in $\mathrm{Mon}^{\mathcal{O}}(\widetilde{\underline{\mathrm{Cat}}}^p_\infty)$. We can thus form the ∞-category of modules $\mathrm{Mod}_{\widetilde{\mathscr{M}}^\circ}(\widetilde{\underline{\mathrm{Cat}}}^p_\infty)$.

Now let $\mathscr{N}$ be an $\mathscr{M}$-enriched model category. Define $\widetilde{\mathscr{N}}^\circ$ as follows:

- An object is a pair $([n],(x_0,\dots,x_n))$ where for each $0 \le i \le n-1$, the x_i are fibrant-cofibrant objects of $\mathscr{M}$ and x_n is a fibrant-cofibrant object of $\mathscr{N}$.
- The mapping space between two objects $([n],x_\bullet),([m],y_\bullet)$ is given by

$$\widetilde{\mathscr{N}}(([n],x_\bullet),([m],y_\bullet)) = \coprod_{u:[n]\to[m]} \left(\prod_{j\in[m]\text{-}\{m\}} \mathscr{M}(\bigotimes_{u(i)=j} x_i, y_j) \times \mathscr{N}(\bigotimes_{u(i)=m} x_i, y_m) \right).$$

Then $\widetilde{\mathscr{N}}^\circ$ is an $\widetilde{\mathscr{M}}^\circ$-module object in $\mathrm{Mod}_{\widetilde{\mathscr{M}}^\circ}(\widetilde{\underline{\mathrm{Cat}}}^p_\infty)$.

Example 3.6.5. Let k be a commutative ring and $C(k)$ the symmetric monoidal category of complexes of k-modules. Recall that a *differential graded category* (or *dg-category*) is a $C(k)$-enriched category. Likewise, let **Sp** denote the category of symmetric spectra. Recall that a *spectral category* is a **Sp**-enriched category. Let **E** denote both $C(k)$ and **Sp** and let us speak of an **E**-enriched category to subsume both examples. Let $\mathscr{M}$ be a combinatorial $\mathbf{E}$-enriched model category (see Definition 7.1.6) and C an $\mathbf{E}$-enriched category. Then the category $\mathscr{M}^C$ can be endowed with the projective model structure where the fibrations and weak equivalences are taken objectwise. The model category $\mathscr{M}^C$ is an $\mathbf{E}$-enriched model category. Let $\mathbf{E}$ be endowed with its natural $\mathbf{E}$-enrichment. Then a *left C-module* is a $\mathbf{E}$-enriched functor $C \to \mathbf{E}$ (similarly we can define a right C-module by replacing C by C^{op}). Let Mod_C denote the category of left C-modules. We can consider $\mathbf{E}$ as a $\mathbf{E}$-enriched model category and so the category Mod_C is a $\mathbf{E}$-enriched model category. By Example 3.6.4, the ∞-category $\widetilde{\mathrm{Mod}}_C^{\circ}$ is a $\widetilde{\mathbf{E}}^{\circ}$-module object of $\widetilde{\underline{\mathrm{Cat}}}_{\infty}^{p}$. Thus when C is a dg-category, the ∞-category $\widetilde{\mathrm{Mod}}_C^{\circ}$ is a $C(k) \simeq \mathrm{Mod}_{Hk}(\mathbf{Sp})$-module object and hence a Hk-linear ∞-category. Likewise, when C is a spectral category $\widetilde{\mathrm{Mod}}_C^{\circ}$ is an $\mathbb{S}$-linear ∞-category.

We will now provide the tools needed to construct the $(\infty, 2)$-category of R-tensor ∞-categories. Let $\widetilde{\underline{\mathrm{Cat}}}_{\infty}^{p,\perp}$ denote the full subcategory of $\widetilde{\underline{\mathrm{Cat}}}_{\infty}^{p}$ spanned by presentable ∞-categories which are moreover *stable* ∞-categories. The projection $\widetilde{\underline{\mathrm{Cat}}}_{\infty}^{\perp,p} \to \mathcal{O}$ determines an $\mathcal{O}$-monoidal structure on the $(\infty, 2)$-category $\underline{\mathrm{Cat}}_{\infty}^{p,\perp}$ of stable, presentable ∞-categories (see Proposition 4.2.3 of [LII] and Proposition 4.1.14 of [LIII] for the closely related ∞-categorical case). One can show (see *loc. cit.*) that the unit object of $\underline{\mathrm{Cat}}_{\infty}^{p,\perp}$ with this $\mathcal{O}$-monoidal structure is the ∞-category Sp of spectra. Let $\underline{\mathrm{Cat}}_{\infty}^{p,\perp,\mathcal{O}}$ denote the subcategory of $\underline{\mathrm{Cat}}_{\infty}^{\mathcal{O}}$ spanned by stable, presentable $\mathcal{O}$-monoidal ∞-categories whose monoidal bifunctor preserves colimits seperately in each variable and whose morphisms are colimit preserving $\mathcal{O}$-monoidal functors. Then we have an equivalence

$$\mathrm{Mon}^{\mathcal{O}}(\widetilde{\underline{\mathrm{Cat}}}_{\infty}^{p,\perp}) \to \underline{\mathrm{Cat}}_{\infty}^{p,\perp,\mathcal{O}}$$

of $(\infty, 2)$-categories (see the discussion following Example 3.1.20). Thus an $\mathcal{O}$-monoidal ∞-category C belongs to $\mathrm{Mon}^{\mathcal{O}}(\widetilde{\underline{\mathrm{Cat}}}_{\infty}^{p,\perp})$ if and only if C is stable, presentable and the bifunctor $\otimes : C_{[1]} \times C_{[1]} \to C_{[1]}$ preserves colimits separately in each variable.

Let R be an E_∞-ring. Then the stable, presentable symmetric monoidal ∞-category $\widetilde{\mathrm{Mod}}_R$ belongs to $\mathrm{CMon}(\widetilde{\underline{\mathrm{Cat}}}_{\infty}^{p,\perp})$. Applying this observation to Proposition 3.2.14 gives the equivalence

$$\mathrm{CMon}(\widetilde{\mathrm{Mod}}_{\mathrm{Mod}_R}(\widetilde{\underline{\mathrm{Cat}}}_{\infty}^{p,\perp})) \simeq \mathrm{CMon}(\widetilde{\underline{\mathrm{Cat}}}_{\infty}^{p,\perp})_{\mathrm{Mod}_R\,/}.$$

of $(\infty, 2)$-categories. The term on the left hand side is the $(\infty, 2)$-category of R-linear, stable, presentable symmetric monoidal ∞-categories and R-linear symmetric monoidal functors. We make the following definition.

Definition 3.6.6. Let R be an E_∞-ring. A symmetric monoidal ∞-category is said to be a *tensor* ∞-category if it is stable and presentable. It is said to be an *R-tensor* ∞-category if it is R-linear, stable and presentable.

We will denote the $(\infty, 2)$-category of tensor ∞-categories by $\mathrm{Tens}^{\otimes} := \mathrm{CMon}(\widetilde{\underline{\mathrm{Cat}}}_{\infty}^{p,\perp})$ and the $(\infty, 2)$-category of R-tensor ∞-categories and R-linear symmetric monoidal functors by

$$\mathrm{Tens}_R^{\otimes} := \mathrm{CMon}(\widetilde{\underline{\mathrm{Cat}}}_{\infty}^{\perp,p})_{\mathrm{Mod}_R\,/}.$$

In the higher Tannaka duality theorem we will need to consider lax adjoints to R-linear symmetric monoidal functors so we to consider a more general $(\infty, 2)$-category than $\mathrm{Tens}_R^{\otimes}$ where the functors between R-tensor ∞-categories are R-linear lax symmetric monoidal. To do so, we introduce a notion of lax comma category.

Notation 3.6.7. Let $\mathrm{Tens}^{\mathrm{lax}}$ denote the following $(\infty, 2)$-category:

- The objects of $\mathrm{Tens}^{\mathrm{lax}}$ are tensor ∞-categories.

- Given two objects $p : C \to \Gamma$ and $q : D \to \Gamma$ in $\mathrm{Tens}^{\mathrm{lax}}$, the mapping space $\mathrm{Map}_{\mathrm{Tens}^{\mathrm{lax}}}(C, D)$ is given by the subcategory of $\mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{lax}}_{\Gamma}(C, D)$ spanned by lax symmetric monoidal functors such that the tensor product bifunctor preserves colimits in each variable.

Consider the strict 2-category I_1 consisting of three objects $\{0, 1, 2\}$, non-identity 1-morphisms $i : 0 \to 1$, $j : 1 \to 2$ and $k : 0 \to 2$ and single 2-morphism $h : k \circ j \to i$, ie.

$$I_1 := \begin{tikzcd} & 0 \arrow[dl, "i"'] \arrow[dr, "k"] \arrow[d, phantom, "\swarrow h"] & \\ 1 \arrow[rr, "j"] & & 2. \end{tikzcd}$$

We have two natural projection maps $I_1 \to [1]$ to i and k. We define I_n for all $n \geq 1$ by the pushout

$$I_n := I_1 \coprod_{[1]} I_1 \coprod_{[1]} \dots I_1 \coprod_{[1]} I_1$$

where by convention $I_0 := [1]$. Let C be an $(\infty, 2)$-category and x an object of C. Let $\mathrm{Hom}_x(I_0, C)$ denote the full subset of $\mathrm{Hom}([1], C)$ spanned by objects sending 0 to the object x in C. Likewise, define an $(\infty, 2)$-category $\underline{\mathrm{Hom}}_x(I_n, C)$ by the following homotopy pullback

$$\begin{tikzcd} \{x\} \arrow[r] & C \\ \underline{\mathrm{Hom}}_x(I_n, C) \arrow[u] \arrow[r] & \underline{\mathrm{Hom}}(I_n, C) \arrow[u, "ev_0"'] \end{tikzcd}$$

in $\mathscr{C}at_{(\infty,2)}$ where $ev_0 : \underline{\mathrm{Hom}}(I_n, C) \to C$ is the evaluation map at the object 0 in I_n. We then define an $(\infty, 2)$-category $\underline{\mathrm{Hom}}^*_x(I_n, C)$ for $n \geq 1$ given by the homotopy pullback

$$\begin{tikzcd} \underline{\mathrm{Hom}}_x(I_n, C) \arrow[r] & \underline{\mathrm{Hom}}_x(I_0, C)^{n+1} \\ \underline{\mathrm{Hom}}^*_x(I_n, C) \arrow[u] \arrow[r] & \mathrm{Hom}_x(I_0, C)^{n+1}. \arrow[u, "ev_0"'] \end{tikzcd}$$

Consider the following $(\infty, 2)$-category $C_{x/\!/}$:

$$\begin{aligned} C_{x/\!/} &: \Delta^{op} \to \mathscr{C}at_\infty \\ [n] &\mapsto \mathfrak{K}^1 \underline{\mathrm{Hom}}^*_x(I_n, C) \end{aligned}$$

where we define $\mathfrak{K}^1 \underline{\mathrm{Hom}}^*_x(I_0, C) := \mathrm{Hom}_x(I_0, C)$. This is indeed an $(\infty, 2)$-category since the map

$$\mathfrak{K}^1 \underline{\mathrm{Hom}}^*_x(I_n, C) = \mathfrak{K}^1 \underline{\mathrm{Hom}}^*_x(I_1 \coprod_{[1]} \dots \coprod_{[1]} I_1, C) \to \mathfrak{K}^1 \underline{\mathrm{Hom}}^*_x(I_1, C) \times \dots \times \mathfrak{K}^1 \underline{\mathrm{Hom}}^*_x(I_1, C).$$

is an equivalence for all $n \geq 2$ and so the Segal conditions are satisfied. Let $\phi : x \to a$ and $\psi : x \to b$ be two objects in $C_{x/\!/}$. Informally, a map $f : a \to b$ in $C_{x/\!/}$ is given by a diagram

$$\begin{tikzcd} & x \arrow[dl, "\phi"'] \arrow[dr, "\psi"] \arrow[d, phantom, "\swarrow h"] & \\ a \arrow[rr, "f"] & & b. \end{tikzcd}$$

in the $(\infty, 2)$-category C.

Let R be an E_∞-ring. In the theory of higher Tannaka duality we will think of our R-tensor ∞-categories as living in a subcategory of the large ambient $(\infty, 2)$-category $(\mathrm{Tens}^{\mathrm{lax}})_{\mathrm{Mod}_R /\!\!/}$.

Definition 3.6.8. We denote by $\mathrm{Tens}_R^{\mathrm{lax}}$ the (non-full) subcategory of $\mathrm{Tens}^{\mathrm{lax}}_{\mathrm{Mod}_R /\!\!/}$ satisfying the following conditions:

1. An object $\phi : \mathrm{Mod}_R \to T$ is a (strict) symmetric monoidal functor.
2. Given two objects $\phi : \mathrm{Mod}_R \to T$ and $\psi : \mathrm{Mod}_R \to U$ and a map $f : T \to U$, the natural composition
$$\psi(M) \otimes f(x) \to f\phi(M) \otimes f(x) \to f(\phi(M) \otimes x)$$
is an equivalence in U for all M in Mod_R and x in T.

Note that since $\mathrm{Tens}^{\otimes}$ is a (non-full) subcategory of $\mathrm{Tens}^{\mathrm{lax}}$ and $C_{x/}$ is a (non-full) subcategory of $C_{x/\!\!/}$ then the $(\infty, 2)$-category of R-tensor ∞-categories and R-linear symmetric monoidal functors $\mathrm{Tens}_R^{\otimes}$ is a (non-full) subcategory of $\mathrm{Tens}_R^{\mathrm{lax}}$.

Example 3.6.9. We now provide an extension of Example 3.6.5 where $\mathbf{E}$ again denotes the category of complexes of k-modules or symmetric spectra. When $(C, \otimes)$ is a symmetric monoidal $\mathbf{E}$-enriched category then there exists a unique monoidal structure $\otimes_!$ on Mod_C such that the objects of Mod_C are symmetric monoidal functors $C \to \mathbf{E}$ and the monoidal structure on C is weakly compatible with the $\mathbf{E}$-enrichment. This is given by the composition

$$\otimes_! : \mathrm{Mod}_C \times \mathrm{Mod}_C \xrightarrow{\boxtimes} \mathrm{Mod}_{C \otimes C} \xrightarrow{\mu_!} \mathrm{Mod}_C$$

where $\mu_!$ is the left-Kan extension along $\mu : C \otimes C \to C$. The tensor product $\otimes_!$ endows Mod_C with the structure of an $\mathbf{E}$-enriched monoidal model category. The natural left Quillen functor $\mathbf{E} \to \mathrm{Mod}_C$ between symmetric monoidal model categories sending an object E of $\mathbf{E}$ to $1 \otimes E$ extends to a symmetric monoidal functor $\widetilde{\mathbf{E}}^{\circ} \to \widetilde{\mathrm{Mod}}_C^{\circ}$ between ∞-categories (using the notation of Example 3.6.4). When C is a symmetric monoidal dg-category, this makes $\widetilde{\mathrm{Mod}}_C^{\circ}$ into a Hk-tensor ∞-category. When C is a spectral category, the ∞-category $\widetilde{\mathrm{Mod}}_C^{\circ}$ is then an $\mathbb{S}$-tensor ∞-category.

For any dg-category C, let $H^0(C)$ denote the category consisting of the same set of objects as C and whose morphism space $H^0(C)(x, y)$ between two objects x and y is given by $H^0(C(x, y))$. We obtain a well defined functor
$$H^0 : \mathrm{Cat}(C(k)) \to \mathrm{Cat}$$
from the category of dg-categories to the category of categories. A map $F : C \to D$ in $\mathrm{Cat}(C(k))$ is said to be a *quasi-equivalence* if for all $x, y \in C$, the map $C(x, y) \to D(Fx, Fy)$ is a quasi-isomorphism of complexes and the induced functor $H^0(F) : H^0(C) \to H^0(D)$ is essentially surjective. By [Tb1], the category $\mathrm{Cat}(C(k))$ can be endowed with a model structure $\mathrm{Cat}(C(k))_{\mathscr{T}}$ where the weak equivalences are the quasi-equivalences. There exists a Quillen equivalence between $\mathrm{Cat}(C(k))$ and the model category of $\mathrm{Mod}_{Hk}(\mathbf{Sp})$-enriched categories. Let $\mathrm{Cat}(C(k))^{\otimes}_{\mathscr{T}}$ denote the symmetric monoidal model category of dg-categories. We conjecture that the functor

$$\widetilde{\mathrm{Mod}}^{\circ} : L\mathrm{Cat}(C(k))^{\otimes}_{\mathscr{T}} \to \mathrm{Tens}^{\otimes}_{Hk}$$

between ∞-categories is fully faithful. For any spectral category C, let $[C]$ denote the category consisting of the same set of objects and whose morphism space $\mathrm{Hom}_{[C]}(x, y)$ between two objects x and y is given by $\mathrm{Hom}_{\mathrm{h}\mathbf{Sp}}(\mathbb{S}, C(x, y))$. We obtain a well defined functor

$$[\bullet] : \mathrm{Cat}(\mathbf{Sp}) \to \mathrm{Cat}.$$

A map $F : C \to D$ in $\mathrm{Cat}(\mathbf{Sp})$ is said to be a *quasi-equivalence* if for all $x, y \in C$, the map $C(x, y) \to D(Fx, Fy)$ is a stable equivalence in $\mathbf{Sp}$ and the induced functor $[F] : [C] \to [D]$ is essentially surjective.

By Theorem 5.10 of [Tb2], the category $\mathrm{Cat}(\mathbf{Sp})$ can be endowed with a model structure where the weak equivalences are the quasi-equivalences. If $\mathrm{Cat}(\mathbf{Sp})^{\otimes}_{\mathscr{T}}$ denotes the model category of symmetric monoidal spectral categories then we conjecture that there exists a full embedding

$$\widetilde{\mathrm{Mod}}^{\circ} : L\mathrm{Cat}(\mathbf{Sp})^{\otimes}_{\mathscr{T}} \to \mathrm{Tens}^{\otimes}_{\mathbb{S}}$$

of ∞-categories

By Proposition 3.1.16, the $(\infty, 2)$-category $\underline{\mathrm{Cat}}_{\infty}^{\perp,p,\mathcal{O}}$ has an initial object C such that the unit map $1 \to C_{[1]}$ is an equivalence in $\underline{\mathrm{Cat}}_{\infty}^{\perp,p}$. Therefore, the ∞-category Sp of spectra admits an $\mathcal{O}$-monoidal structure $\widetilde{\mathrm{Sp}}$ which is the initial $\mathcal{O}$-monoidal ∞-category in $\underline{\mathrm{Cat}}_{\infty}^{\perp,p,\mathcal{O}}$. This is the $\mathcal{O}$-monoidal structure characterised in Section 3.3. We have the following universal property: for two stable, presentable $\mathcal{O}$-monoidal ∞-categories C and D with a bicontinous monoidal product, there exists an equivalence

$$\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{O}}^{\otimes,\mathrm{ct}}(D, C) \to \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{O}}^{\otimes,\mathrm{ct}}(\widetilde{\mathrm{Sp}}, C)$$

in the $(\infty, 2)$-category $\underline{\mathrm{Cat}}_{\infty}^{\perp,p,\mathcal{O}}$. Here $\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{O}}^{\otimes,\mathrm{ct}}$ denotes functors which are colimit preserving $\mathcal{O}$-monoidal functors. An important ingredient in the proof of the higher Tannaka duality theorems is the following conjecture which identifies lax monoidal endomorphisms of Mod_R in $\mathrm{Tens}_R^{\mathrm{lax}}$ with R-algebras.

Conjecture 3.6.10. Let R be an E_∞-ring. Then there exists an equivalence

$$\mathrm{End}_{\mathrm{Tens}_R^{\mathrm{lax}}}(\mathrm{Mod}_R) \to \mathrm{CMon}(\mathrm{Mod}_R)$$
$$f \mapsto f(R)$$

of ∞-categories where the right adjoint takes a R-algebra A to the endomorphism $A \otimes \bullet$.

Comments related to the proof. One way to construct the proof is as follows. It consists of two parts. Let R be an E_∞-ring. In the first part, there should exists an equivalence

$$(*) \qquad \mathrm{End}_{\mathrm{Tens}_R^{\mathrm{lax}}}(\mathrm{Mod}_R) \to \mathrm{End}_{\mathrm{Tens}_{\mathbb{S}}^{\mathrm{lax}}}(\mathrm{Mod}_{\mathbb{S}})_{(\bullet \otimes R)/}$$

of ∞-categories. To demonstrate this, one proves that the unit and counit of an adjunction between these two ∞-categories are equivalences. Let $p_R := \bullet \otimes R$ and $i_R := \underline{\mathrm{Hom}}(R, \bullet)$. The left adjoint takes an endomorphism $f \in \mathrm{End}_{\mathrm{Tens}_R}(\mathrm{Mod}_R)$ to the map $p_R \to i_R \circ f \circ p_R$. The right adjoint takes a map $p_R \to g$ to the endomorphism $p_R \circ g \circ i_R$. The second part begins as follows. By the universal property of the symmetric monoidal ∞-category $\widetilde{\mathrm{Sp}}$, there exists an equivalence

$$\mathbb{R}\underline{\mathrm{Hom}}_{\Gamma}^{\otimes}(\widetilde{\mathrm{Sp}}, C) \to \mathbb{R}\underline{\mathrm{Hom}}_{\Gamma}^{\otimes}(\Gamma, C)$$

of ∞-categories for any symmetric monoidal ∞-category C. The difficult part of the conjecture states that this can be lifted to an equivalence

$$(**) \qquad \mathbb{R}\underline{\mathrm{Hom}}_{\Gamma}^{\mathrm{lax}}(\widetilde{\mathrm{Sp}}, C) \to \mathbb{R}\underline{\mathrm{Hom}}_{\Gamma}^{\mathrm{lax}}(\Gamma, C)$$

of ∞-categories of lax symmetric monoidal functors. When $C = \widetilde{\mathrm{Sp}}$ we obtain that the map

$$\mathrm{End}^{\mathrm{lax}}(\widetilde{\mathrm{Sp}}) \to \mathrm{CMon}(\widetilde{\mathrm{Sp}})$$

is an equivalence. Since $\mathrm{End}_{\mathrm{Tens}_{\mathbb{S}}^{\mathrm{lax}}}(\mathrm{Mod}_{\mathbb{S}})$ is nothing other than $\mathrm{End}^{\mathrm{lax}}(\widetilde{\mathrm{Sp}})$, combining statement (*) and statement (**) we obtain the the desired result. □

Remark 3.6.11. Taking comonoid objects of Conjecture 3.6.10 yields an equivalence

$$\mathrm{Comon}(\mathrm{End}_{\mathrm{Tens}_R^{\mathrm{lax}}}(\mathrm{Mod}_R)) \to \mathrm{Comon}(\mathrm{CAlg}_R)$$
$$L \mapsto L(R)$$

where the left hand side is the ∞-category of comonads on Mod_R in $\mathrm{Tens}_R^{\mathrm{lax}}$.

4 Stacks, gerbes and topologies

This chapter will be devoted to the group side of the correspondence. In Section 4.1 we discuss ∞-topoi and the most important example: the ∞-category of stacks on a site. More generally, we discuss the notion of an X-valued stack for X an arbitrary ∞-category with limits. In Proposition 4.1.14 we prove that the ∞-category of X-valued stacks on a site is tensored and enriched over both itself and X. In Proposition 4.1.15 we prove that the X-valued prestack of maps between two X-valued prestacks F and G is an X-valued stack when G is an X-valued stack. We will be particularly interested in the case where X is the ∞-category of ∞-categories. We provide in Proposition 4.1.19 sufficient conditions on a prestack valued in ∞-categories to be an ∞-category valued stack. We also introduce the notion of group object in an ∞-category and the important example of a group stack on a site.

Stacks on a site which are locally non-empty and locally connected are called gerbes. In Section 4.2 we discuss gerbes and provide in Proposition 4.2.4 the characterisation that gerbes are exactly those stacks which are locally equivalent to the classifying stack of a group stack. In Section 4.3 we describe three topologies on the ∞-category of R-algebras called the positive, flat and finite topologies. The flat and finite topologies are shown in Proposition 4.3.6 to be subcanonical. The main tool in the proof is the observation that the functor which takes an R-algebra, where R is an E_∞-ring, to its ∞-category of modules is a stack of ∞-categories on the site of R-algebras with respect to the flat and finite topologies (Proposition 4.3.5).

4.1 Stacks

Definition 4.1.1. An ∞-category is said to be an *∞-topos* if it is a left exact Bousfield localisation of $\mathrm{Pr}(C)$ for a small ∞-category C.

In other words, an ∞-category T is an ∞-topos if there exists a small ∞-category C and a fully faithful functor $T \to \mathrm{Pr}(C)$ which possesses a left exact left adjoint. There exist other more intrinsic characterisations of an ∞-topos in the literature. In particular there is a characterisation based on ∞-categorical analogues of Giraud's axioms (see for example Theorem 6.1.0.6 of [Lu]). Let T and U be ∞-topoi. A *geometric morphism* from T to U is a functor $f_* : T \to U$ which admits a left exact left adjoint. Let $\mathbb{R}\underline{\mathrm{Hom}}^{gm}(T,U)$ denote the full subcategory of $\mathbb{R}\underline{\mathrm{Hom}}(T,U)$ spanned by the geometric morphisms. The $(\infty,2)$-category of ∞-topoi and geometric morphisms will be denoted by Tpoi_∞.

The $(\infty,2)$-category Tpoi_∞ of ∞-topoi admits all (small) limits and colimits. Furthermore, it is tensored and cotensored over the $(\infty,2)$-category $\underline{\mathrm{Cat}}_\infty$ of ∞-categories, ie. for any $T \in \mathrm{Tpoi}_\infty$ and $C \in \underline{\mathrm{Cat}}_\infty$ there exists an object T^C in Tpoi_∞ and an object $T \otimes C$ in Tpoi_∞, both well defined up to equivalence, satisfying the property that for any $U \in \mathrm{Tpoi}_\infty$ the pair of maps

$$\mathbb{R}\underline{\mathrm{Hom}}^{gm}(T,U^C) \to \mathbb{R}\underline{\mathrm{Hom}}(C,\mathbb{R}\underline{\mathrm{Hom}}^{gm}(T,U)) \leftarrow \mathbb{R}\underline{\mathrm{Hom}}^{gm}(T \otimes C,U)$$

are equivalences of ∞-categories.

Definition 4.1.2. Let T be an ∞-topos and X an ∞-category. An X-valued prestack $F : T^{op} \to X$ on T is said to be an *X-valued stack* if it preserves limits.

Let $\mathrm{St}_X(T)$ denote the full subcategory of $\mathrm{Pr}_X(T)$ spanned by the X-valued stacks on T. We will denote the ∞-category $\mathrm{St}_{\mathcal{K}}(T)$ of $\mathcal{K}$-valued stacks by $\mathrm{St}(T)$ and refer to objects therein as simply stacks. Since a prestack on a presentable ∞-category is representable if and only if it preserves limits, stacks on an ∞-topos T correspond to representable prestacks on T. Thus there exists an equivalence $T \to \mathrm{St}(T)$ of ∞-categories from the Yoneda lemma of Proposition 2.2.11 and hence any ∞-topos can be identified with its ∞-category of stacks. Let T be an ∞-topos and X a presentable ∞-category. Then the inclusion functor $\mathrm{St}_X(T) \hookrightarrow \mathrm{Pr}_X(T)$ admits a left adjoint

$$a : \mathrm{Pr}_X(T) \to \mathrm{St}_X(T)$$

by Proposition 4.1.5 of [L1]. This left adjoint is called the *stackification* functor or the associated stack functor. It defines the following universal property: for any X-valued prestack P on T and any X-valued stack F on T, the map

$$\mathrm{Map}_{\mathrm{St}_X(T)}(a(P), F) \to \mathrm{Map}_{\mathrm{Pr}_X(C)}(P, F)$$

is an equivalence of $(\infty, 0)$-categories.

Remark 4.1.3. Let X be a fixed ∞-category. One can consider a generalisation of the $(\infty, 2)$-category Tpoi_∞ by considering the $(\infty, 2)$-category Tpoi_∞^X of X-*structured* ∞-topoi. An object of Tpoi_∞^X is a pair $(T, \mathcal{O}_T)$ where T is an (∞, n)-topos and $\mathcal{O}_T$ is an X-valued stack on T. A morphism $(T, \mathcal{O}_T) \to (U, \mathcal{O}_U)$ is a pair (f, ϕ) where $f : T \to U$ is a geometric morphism and $\phi : \mathcal{O}_U \to f_*\mathcal{O}_T$ is a map of X-valued stacks on U. For example, for an E_∞-ring R, one can consider the $(\infty, 2)$-category Tpoi_∞^R of $\mathrm{CAlg}_R(\mathrm{Sp})$-structured ∞-topoi as a foundation for the theory of *spectral schemes*. See [TV4] and [LVII] for more details.

We will now discuss in detail the main ∞-topos of interest: the ∞-topos of stacks on a site. Let C be an ∞-category. A *sieve* on an object x in C is a full subcategory $R \subseteq C_{/x}$ such that if $z \to y$ is any arrow in $C_{/x}$ and the object y belongs to R then the object z belongs to R. If $f : x \to y$ is an arrow in C and R is a sieve on y, then we define a sieve f^*R on x to be the unique sieve such that f^*R and R determine the same sieve on $C_{/f}$.

Definition 4.1.4. Let C be an ∞-category. A *topology* is a function τ which assigns to each object x of C a family $\tau(x)$ of sieves on x such that:

1. The sieve $C_{/x}$ is in $\tau(x)$.
2. If $f : x \to y$ is an arrow in C and R is a sieve in $\tau(y)$, then f^*R is a sieve in $\tau(x)$.
3. Let R be a sieve in $\tau(x)$ and R' an arbitrary sieve on x. If for any $f : y \to x$ in R we have f^*R' in $\tau(y)$, then R' is also in $\tau(x)$.

A topology on an ∞-category C is equivalent to a Grothendieck topology on the homotopy category $\mathrm{h}(C)$ (see Remark 6.2.2.3 of [Lu]). The pair (C, τ) will be called a site. Let C now be an ∞-category with pullbacks. To prove the existence of a topology on C it suffices to prove the existence of a pretopology on C. That is, a function cov_τ which assigns to each object x in C a collection $cov_\tau(x)$ of subsets of objects in $C_{/x}$ called *covering families* of x satisfying:

- Stability: If $f : y \to x$ is an equivalence in C then the singleton $\{f : y \to x\}$ is in $cov_\tau(x)$.
- Composition: If $\{f_i : y_i \to x\}_{i \in I}$ is in $cov_\tau(x)$ and if for each $i \in I$ one has a family $\{g_{ij} : z_{ij} \to y_i\}_{j \in J_i}$ in $cov_\tau(y_i)$ then the family $\{f_i \circ g_{ij} : z_{ij} \to x\}_{i \in I, j \in J_i}$ is in $cov_\tau(x)$.
- Base change: If $\{f_i : y_i \to x\}_{i \in I}$ is in $cov_\tau(x)$ then for any morphism $g : z \to x$, the pullbacks $z \times_x y_i$ exist and the family $\{z \times_x y_i \to z\}_{i \in I}$ is in $cov_\tau(z)$.

A pretopology on C determines a topology on C: a sieve R on an object x in C is in $\tau(x)$ if and only if there exists a covering family J in $cov_\tau(x)$ such that J is a subset of R.

Example 4.1.5. Let C be an ∞-category with pullbacks. For any $x \in C$ the covering families $\{y_i \to x\}_{i \in I}$ in $cov_\tau(x)$ for which each $y_i \to x$ is an equivalence and the set I is nonempty is called the *trivial topology* on C.

Example 4.1.6. Let $s\mathrm{Rng}$ be the category of simplicial commutative rings. Then $s\mathrm{Rng}$ admits a cofibrantly generated simplicial model structure where a map is a weak equivalence (resp. a fibration) if the map of underlying simplicial sets is one. A map $f : A \to B$ in $s\mathrm{Rng}$ is said to be étale if $\pi_0 A \to \pi_0 B$ is an étale map of ordinary commutative rings and if for each $i > 0$ the induced map $\pi_i A \otimes_{\pi_0 A} \pi_0 B \to \pi_i B$ is an isomorphism of abelian groups. The ∞-category $L(s\mathrm{Rng})$ admits a topology, called the *étale topology*, where the covering families $\{y_i \to x\}_{i \in I}$ consist of étale maps for which there exists a finite subset $J \subseteq I$ such that $\prod_{j \in J} y_j \to x$ is faithfully flat.

Example 4.1.7. The ∞-category of simplicial commutative rings is one of the possible choices for the base ∞-category to build the theory of derived algebraic geometry. Another choice, and the one we have chosen to adopt, is the ∞-category of commutative ring spectra of Section 3.4. In Definition 4.3.1 we will introduce the notions of *positive*, *flat* and *finite* topologies on the ∞-category $\mathfrak{E}$ of commutative ring spectra.

Let (C, τ) be a site and $\{u_i \to x\}_{i \in I}$ a covering family of $x \in C$. Let $u = \coprod_i u_i$. We will say that the map $u \to x$ is a *covering* of x. A *cover* of x (associated to u) is the simplicial prestack $u_* \in s\mathrm{Pr}(C)_{/x}$ given by

$$\begin{aligned} u_* : \Delta &\to \mathrm{Pr}(C) \\ [n] &\mapsto u \underset{x}{\times} \dots \underset{x}{\times} u. \end{aligned}$$

Definition 4.1.8. Let C be a site and X an ∞-category with limits. An X-valued prestack $F : C^{op} \to X$ on C is said to be an *X-valued stack* if for all $x \in C$ and all coverings u_* in $s\mathrm{Pr}(C)_{/x}$ the map

$$F(x) \to \lim_{\Delta} F(u_*)$$

is an equivalence in X.

The full subcategory of $\mathrm{Pr}_X(C)$ spanned by the X-valued stacks on the site (C, τ) will be denoted $\mathrm{St}^\tau_X(C)$. A $\mathcal{K}$-valued stack will simply be called a *stack* and we will denote the ∞-category of stacks $\mathrm{St}^\tau_{\mathcal{K}}(C)$ by $\mathrm{St}^\tau(C)$. The ∞-category $\mathrm{St}^\tau(C)$ is an ∞-topos by Lemma 6.2.2.7 of [Lu]. A topology τ on C is said to be *subcanonical* if every representable functor on C is a stack with respect to τ.

A map $f : C \to D$ between two sites (C, τ) and (D, η) is said to be *continuous* if the induced map $f^* : \mathrm{Pr}(D) \to \mathrm{Pr}(C)$ of prestacks preserves the full subcategory of stacks. As a result we have an adjunction

$$f_! : \mathrm{St}^\tau(C) \rightleftarrows \mathrm{St}^\eta(D) : f^*.$$

A continuous map of sites is a geometric morphism of ∞-topoi. Let $\mathfrak{S}$ denote the $(\infty, 2)$-category of sites together with continuous maps. We have a well defined functor

$$\begin{aligned} \mathrm{St}_{\mathcal{K}} : \mathfrak{S} &\to \mathrm{Tpoi}_\infty \\ (C, \tau) &\mapsto \mathrm{St}^\tau(C). \end{aligned}$$

By Proposition 9.0.9 of [L1], the ∞-category of stacks satisfies the following universal property: for any ∞-topos T, the map

$$\mathbb{R}\underline{\mathrm{Hom}}^{gm}(T, \mathrm{St}^\tau(C)) \to \mathbb{R}\underline{\mathrm{Hom}}^{gm}(T, \mathrm{Pr}(C))$$

is fully faithful and its essential image consists of those geometric morphisms $f : T \to \mathrm{Pr}(C)$ such that for all $x \in C$ and all covering sieves $R \hookrightarrow x$, the induced map $\coprod_{x' \in R} f^* x' \to f^* x$ is a surjection in T.

Remark 4.1.9. Note that the adjunction $\prod_1 : \mathscr{C}at_{(\infty,0)} \rightleftarrows \mathscr{C}at_\infty : \mathfrak{K}^0$ induces an adjunction

$$(\textstyle\prod_1)_* : \mathrm{St}^\tau_{\mathcal{K}}(C) \rightleftarrows \mathrm{St}^\tau_{\mathrm{Cat}^\otimes_\infty}(C) : (\mathfrak{K}^0)^*$$

where $((\mathfrak{K}^0)^* F)(x) = \mathfrak{K}^0 F(x)$ for a stack F in $\mathrm{St}^\tau_{\mathrm{Cat}^\otimes_\infty}(C)$.

Example 4.1.10. Let (C, τ) be a site for a subcanonical topology τ and h_x the representable prestack on an object x in C. Then there exists an equivalence $\mathrm{St}^\tau(C)_{/h_x} \to \mathrm{St}^\tau(C_{/x})$ of ∞-categories.

Let C be a symmetric monoidal ∞-category. We give the opposite ∞-category of commutative monoid objects in C the following special notation:

$$\mathrm{Aff}_C := \mathrm{CMon}(C)^{op}.$$

When C is the symmetric monoidal ∞-category $\widetilde{\mathrm{Mod}}_R(D)$ of R-modules in a symmetric monoidal ∞-category D, we will write $\mathrm{Aff}_R := \mathrm{Aff}_{\widetilde{\mathrm{Mod}}_R(D)}$. In other words, the ∞-category Aff_R is the opposite of the ∞-category of commutative R-algebras in D. The Yoneda embedding $\mathrm{Aff}_C \to (\mathrm{Aff}_C)^\wedge$ will be denoted by Spec.

Example 4.1.11. Let C be a symmetric monoidal ∞-category and R a commutative monoid object of C. Then we will denote by $\mathrm{St}^\tau(R) := \mathrm{St}^\tau(\mathrm{Aff}_R)$ the ∞-category of stacks with respect to the site (Aff_R, τ) of R-algebras in C. By Example 4.1.10 we have an equivalence $\mathrm{St}^\tau(R) \simeq \mathrm{St}^\tau(\mathrm{Aff}_C)_{/\operatorname{Spec} R}$ of ∞-categories.

The ∞-category of stacks on a site can be obtained by localisation (in the sense of Definition 2.2.1) of the ∞-category of prestacks. The following proposition gives two possible choices for the set of maps from which to localise.

Proposition 4.1.12. *Let (C, τ) be a site. The following classes of maps give the same localisation of* $\mathrm{Pr}(C)$*:*

1. *The set of all covering sieves $R \to x$.*
2. *The set of maps $u_* \to x$.*

Proof reference. We note that $\mathrm{Pr}(C)$ is a presentable ∞-category. Thus the localisation is a Bousfield localisation in the sense of Section 2.3. The result now follows from Proposition A1 of [DHI]. □

Let S denote the equivalent set of maps of Proposition 4.1.12, ie. $S = \{u_* \to x\}_{x \in C}$ or $S = \{R \to x\}_{R, x \in C}$. Then $\mathrm{St}^\tau(C) \simeq L_S(\mathrm{Pr}(C))$. Note that we also have the following model categorical interpretation of the ∞-category of stacks. Let $\mathfrak{F}(C)$ be the strict simplicial category associated to C and $\mathbf{S}_{\mathcal{K}}^{\mathfrak{F}(C)}$ the simplicial category of functors endowed with the projective model structure. Then there exists an equivalence $\mathrm{St}^\tau(C) \simeq L(L_S^B(\mathbf{S}_{\mathcal{K}}^{\mathfrak{F}(C)}))$ of ∞-categories.

Proposition 4.1.13. *Let (C, τ) be a site and X an ∞-category with limits. There exists an equivalence*

$$\mathrm{St}_X(\mathrm{St}_{\mathcal{K}}^\tau(C)) \to \mathrm{St}_X^\tau(C)$$

of ∞-categories.

Proof. This follows from the chain of equivalences

$$\mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{ct}}(\mathrm{St}^\tau(C), X) \simeq \mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{ct}}(L_S \mathrm{Pr}(C)^{op}, X) \simeq \mathbb{R}\underline{\mathrm{Hom}}_S^{\mathrm{ct}}(\mathrm{Pr}(C)^{op}, X) \simeq \mathbb{R}\underline{\mathrm{Hom}}_S^{\mathrm{ct}}(C^{op}, X) := \mathrm{St}_X^\tau(C)$$

where $\mathbb{R}\underline{\mathrm{Hom}}_S$ denotes the essential image of the fully faithful functor defining the universal property of the localisation and the last equivalence follows from (the dual of) Proposition 2.3.10. □

Proposition 4.1.14. *Let (C, τ) be a site and X a presentable $\mathcal{O}$-monoidal ∞-category whose tensor product preserves colimits seperately in each variable. Then the ∞-category $\mathrm{St}_X^\tau(C)$ is tensored and enriched over itself. Moreover, it is tensored and enriched over X.*

Proof. We first observe that the ∞-category $\mathrm{St}_X^\tau(C)$ of X-valued stacks on C is naturally an $\mathcal{O}$-monoidal ∞-category with the pointwise $\mathcal{O}$-monoidal structure of Example 3.1.10. Explicitly, $\widetilde{\mathrm{St}}_X(C)_{[n]} \simeq \mathrm{St}_{X_{[n]}}(C)$. To show that it is enriched over itself it remains to show that $\bullet \otimes F : \mathrm{St}_X^\tau(C) \to \mathrm{St}_X^\tau(C)$ preserves colimits (Proposition 3.2.5). This follows by the assumption that the tensor product on X preserves colimits seperately in each variable (colimits are calculated pointwise in functor categories).

To show that $\mathrm{St}_X^\tau(C)$ is tensored and enriched over X it suffices to show that there exists a colimit preserving $\mathcal{O}$-monoidal functor $X \to \widetilde{\mathrm{St}}_X(C)$. First consider the ∞-category $\widetilde{\mathrm{Pr}}_X(C)$ of X-valued prestacks endowed with its pointwise $\mathcal{O}$-monoidal structure. The constant prestack functor induces an $\mathcal{O}$-monoidal functor $X \simeq \widetilde{\mathrm{Pr}}_X(*) \to \widetilde{\mathrm{Pr}}_X(C)$. This functor preserves colimits seperately in each variable owing to the assumption that they do in X . Finally, the stackification functor L_S preserves colimits (it is a left adjoint) so the composition $X \to \widetilde{\mathrm{Pr}}_X(C) \to L_S\widetilde{\mathrm{Pr}}_X(C) \simeq \widetilde{\mathrm{St}}_X(C)$ preserves colimits seperately in each variable. □

Related to Proposition 4.1.14 is the property that ∞-category $\mathrm{St}^\tau_X(C)$ is left tensored over X where we regard $\mathrm{St}^\tau_X(C)$ as an X-module object in $\mathrm{Mod}_X(\widetilde{\underline{\mathrm{Cat}}}^p_\infty)$. Explicitly, the left-tensored structure is given by a functor $F : S \to X$ where an object of $S_{[n]}$ is a sequence of objects $(x_0, \ldots, x_n, f)$ where $x_i \in X$ and $f \in \mathrm{St}^\tau_X(C)$. Clearly the map F satisfies the conditions of Definition 3.2.1 where the inclusion $n \subseteq [n]$ induces the equivalence $S_{[n]} \xrightarrow{\sim} X_{[n]} \times \mathrm{St}^\tau_X(C)$. The monoidal product $\otimes : X \times \mathrm{St}^\tau_X(C) \to \mathrm{St}^\tau_X(C)$ is given by $(x \otimes F)(c) = x \otimes F(c)$ (well defined up to equivalence).

The internal Hom provided by Proposition 4.1.14, and more generally for X-valued prestacks, will be denoted by $\underline{\mathrm{Hom}}(F, G)$. We will now demonstrate that $\underline{\mathrm{Hom}}(F, G)$ is an X-valued stack under the weaker condition that F is only an X-valued prestack when the conditions of Proposition 4.1.14 are satisfied.

Proposition 4.1.15. *Let (C, τ) be a site, X a presentable $\mathcal{O}$-monoidal ∞-category whose tensor product preserves colimits separately in each variable and F and G two X-valued prestacks on C. If G is an X-valued stack then $\underline{\mathrm{Hom}}(F, G)$ is an X-valued stack on C.*

Proof. Let F be an X-valued prestack and G an X-valued stack. Then $\underline{\mathrm{Hom}}(F, G)$ is an X-valued stack if and only if the map

$$\mathrm{Mor}(x, \underline{\mathrm{Hom}}(F, G)) \to \mathrm{Mor}(\operatorname*{colim}_n u_*, \underline{\mathrm{Hom}}(F, G))$$

is an equivalence in X. This is equivalent to the condition that $\mathrm{Mor}(x \otimes F, G) \to \mathrm{Mor}(\mathrm{colim}_n(u_* \otimes F), G)$ is an equivalence in X and subsequently to $\mathrm{colim}_n(u_* \otimes F) \to x \otimes F$ being an equivalence of X-valued stacks. Let B be an object of X. Since G is an X-valued stack, the map $\mathrm{Map}(B, G(x)) \to \lim_n \mathrm{Map}(B, G(u_*))$ is an equivalence and so $\mathrm{Map}(x \otimes B, G) \to \mathrm{Map}(\mathrm{colim}_n(u_* \otimes B), G)$ is an equivalence for any $B \in X$. Now recall that any X-valued prestack can be written as a colimit given by $\mathrm{colim}_\alpha(v_\alpha \otimes B_\alpha)$ for v_α a set of prestacks and B_α a set of generators for the presentable ∞-category X. Therefore, $x \otimes F \simeq x \otimes \mathrm{colim}_\alpha(v_\alpha \otimes B_\alpha) \simeq \mathrm{colim}_\alpha(x \otimes v_\alpha) \otimes B_\alpha \simeq \mathrm{colim}_n(\mathrm{colim}_\alpha(u_* \otimes v_\alpha) \otimes B_\alpha) \simeq \mathrm{colim}_n(u_* \otimes \mathrm{colim}_\alpha(v_\alpha \otimes B_\alpha)) \simeq \mathrm{colim}_n(u_* \otimes F)$ and the result follows. □

Let $n \geq 0$ be an integer. Recall that a $(\infty, 0)$-category A is said to be *n-truncated* (resp. *n-connective*) if for every $i > n$ (resp. $i < n$)

$$\pi_i(A, a) \simeq *$$

for all objects $a \in A$. An $(\infty, 0)$-category which is 1-connective will be called *connected*. A map of $(\infty, 0)$-categories $f : A \to B$ is said to be n-truncated (resp. n-connective) if the homotopy fibers of f taken over any base point of B is n-truncated (resp. n-connective). A prestack $F : C^{op} \to \mathcal{K}$ is said to be n-truncated (resp. n-connective) if $F(x)$ is n-truncated (resp. n-connective) for all $x \in C$. A map of prestacks $F \to G$ is said to be n-truncated (resp. n-connective) if $F(x) \to G(x)$ is n-truncated (resp. n-connective) for all $x \in C$.

Definition 4.1.16. Let C be an ∞-category and $n \geq 0$ an integer. An object x in C is said to be *n-truncated* if the representable prestack $C(\bullet, x)$ is n-truncated. An arrow $f : x \to y$ in C is said to be *n-truncated* if the map of prestacks $C(\bullet, f)$ is n-truncated.

Let $\tau_{\leq n}C$ denote the full subcategory of C spanned by the n-truncated objects. There exists an equivalence

$$\tau_{\leq 0}C \to \mathrm{h}(\tau_{\leq 0}C)$$

of ∞-categories.

Example 4.1.17. Let (C, τ) be a site. The ∞-category $\tau_{\leq 0}\mathrm{St}^\tau(C)$ is equivalent to the ordinary ∞-topos $\mathrm{St}^\tau_{\mathrm{Set}}(\mathrm{h}C)$ of sheaves of sets on the homotopy category of C.

Note that objects in the ∞-category $\tau_{\leq 0}C$ are objects $x \in C$ such that for all $y \in C$, the $(\infty, 0)$-category $C(y, x)$ is homotopy equivalent to a discrete space, ie. $C(y, x) \to \pi_0 C(y, x)$ is a homotopy equivalence for all $y \in C$. We can construct a left adjoint to the inclusion functor $i : \tau_{\leq n}C \to C$ given by

$$t_n : C \to \tau_{\leq n}C.$$

Remark 4.1.18. An ∞-category C is said to be *t-complete* if for all $x \in C$ such that $t_n(x) \simeq *$ for all n implies that $x \simeq *$. In other words, an ∞-category C is t-complete if truncated objects detect isomorphisms in hC in that an arrow $u : x \to y$ is an isomorphism in hC if and only if $u^* : \mathrm{h}C(y,z) \to \mathrm{h}C(x,z)$ is bijective for any truncated object z in hC. The ∞-category $\mathrm{St}^\tau(C)$ of stacks is *not* a t-complete ∞-topos. However, there exists a stronger notion of a stack called a *hyperstack* which do form a t-complete ∞-topos. For all $n \geq 0$, let $\Delta_{\leq n}$ denote the full subcategory of Δ spanned by objects $[m]$ with $m \leq n$. Let C be a presentable ∞-category and sC the ∞-category of simplicial objects in C. The natural inclusion $i_n : \Delta_{\leq n} \to \Delta$ induces a restriction functor $i_n^* : sC \to C^{\Delta_{\leq n}^{op}}$ which has a fully faithful right adjoint $(i_n)_*$ and left adjoint $(i_n)_!$ given by the right and left Kan extensions along the inclusion i_n. The n^{th}*-skeleton functor*, $sk_n : sC \to sC$, is defined as $a_* \mapsto (i_n)_! i_n^*(a_*)$. The n^{th}*-coskeleton functor*, $cosk_n : sC \to sC$, is defined as $a_* \mapsto (i_n)_* i_n^*(a_*)$.

Let (C,τ) be a site. A simplicial object u_* in $s\mathrm{Pr}(C)$ is said to be a *hypercovering* in $\mathrm{Pr}(C)$ if for each $n \geq 0$ the map $u_n \to (cosk_{n-1}(u_*))_n$ is an effective epimorphism, ie. the induced map $\pi_0^\tau(F) \to \pi_0^\tau(G)$ is an epimorphism of sheaves. Equivalently, for every $x \in C$ and any map $h_x \to G$ in $\mathrm{Pr}(C)$ there exists a covering sieve R of x such that for any map $y \to x$ in R, there is a map $h_y \to G$ such that the diagram

$$\begin{tikzcd} h_y \arrow[r] \arrow[d] & h_x \arrow[d] \\ F \arrow[r] & G \end{tikzcd}$$

commutes (up to equivalence) in $\mathrm{Pr}(C)$. Let X be an ∞-category with limits. An X-valued prestack $F : C^{op} \to X$ on C is said to be an *X-valued hyperstack* on C if for all $x \in C$ and all hypercoverings u_* in $C_{/x}$

$$F(x) \to \lim_{n \in \Delta} F(u_*)$$

is an equivalence in X. If C is a small ∞-category then there exists a bijective correspondence between topologies on C and (equivalence classes of) t-complete left exact localisations of $\mathrm{Pr}(C)$ given by the rule $\tau \mapsto \mathrm{St}^\tau(C)$. For a discussion on the relative merits between stacks and hyperstacks we refer the reader to Section 6.5.4 of [Lu].

Proposition 4.1.19. *Let (C,τ) be a site and*

$$F : C^{op} \to \mathrm{Cat}_\infty^\infty$$

a prestack of ∞-categories satisfying the following conditions:

1. *For each object x in C, the ∞-category $F(x)$ admits limits.*
2. *For each object x in C and for any covering $\{u_i \to x\}_{i \in I}$ in $cov^\tau(x)$, the functor $F(x) \to F(u_i)$ preserves limits.*
3. *For each object x in C and for any covering $\{u_i \to x\}_{i \in I}$ in $cov^\tau(x)$, the functor $F(x) \to \prod_i F(u_i)$ is conservative.*
4. *For any map $f : y \to x$ in C, the functor $f^* := F(f) : F(x) \to F(y)$ admits a right adjoint $f_* : F(y) \to F(x)$.*
5. *For each pullback square*

$$\begin{tikzcd} y \arrow[r, "f"] & x \\ z \arrow[u, "p"] \arrow[r, "q"] & y' \arrow[u, "g"'] \end{tikzcd}$$

in C, the natural morphism $g^*f_* \Rightarrow q_*p^*$ is an equivalence in the ∞-category $\mathbb{R}\underline{\mathrm{Hom}}(F(y), F(y'))$.

Then F is a stack of ∞-categories.

Proof. We need to show that for any covering $u \to x$ in $cov^\tau(x)$ the map

$$F(x) \to \lim_{\Delta} F(u_n)$$

is an equivalence of ∞-categories. Consider the pullback diagram

$$\begin{tikzcd} u \arrow[r, "f"] & x \\ v \arrow[u, "p"] \arrow[r, "q"] & u \arrow[u, "f"'] \end{tikzcd}$$

in C where $f \in cov^\tau(x)$. Construct the section $\delta : u \to u \times_x u$ of the map q where $q \circ \delta = \mathrm{id}_u$. Taking the nerve of the maps f and q and using (1) we obtain a homotopy commutative diagram

$$\begin{tikzcd} F(x) \arrow[r, "A"] \arrow[d] & \lim_{\Delta} F(u_n) \arrow[d] \\ F(u) \arrow[r, "A'"] & \lim_{\Delta} F(v_n) \end{tikzcd}$$

in $\mathrm{Cat}_\infty^\infty$. Using (2), (4) and (5) we obtain an adjoint homotopy commutative diagram

$$\begin{tikzcd} F(x) \arrow[d] & \lim_{\Delta} F(u_n) \arrow[l, "B"'] \arrow[d] \\ F(u) & \lim_{\Delta} F(v_n) \arrow[l, "B'"'] \end{tikzcd}$$

in $\mathrm{Cat}_\infty^\infty$. To complete the proof it will suffice to show that the unit and counit of the adjunction $A \dashv B$ are equivalences. By (3) and the fact that both squares commute, we are able to check the corresponding statement for the adjunction $A' \dashv B'$. This adjunction is an equivalence owing to the fact that for any covering admitting a section, the prestack F satisfies descent. $\square$

Remark 4.1.20. There exists similar types of characterisations in the case of hyperstacks of ∞-categories. See the appendix of [T4] for details.

We now define the notion of a group object in an ∞-category. We start with the more general notion of a groupoid object. Let C be an ∞-category with pullbacks. A *category object* in C is a functor $F : \Delta^{op} \to C$ such that for all $n \geq 0$, the canonical map

$$F([n]) \to F([1]) \times_{F([0])} \times \ldots \times_{F([0])} F([1])$$

is an equivalence in C. Let $\mathrm{Ct}(C)$ denote the full subcategory of $\mathbb{R}\underline{\mathrm{Hom}}(\Delta^{op}, C)$ spanned by the category objects of C. A category object F in C is said to be a *groupoid object* in C if it takes every partition

$[2] = \{S \cup S' | S \cap S' = \{s\}, s \in S\}$ to a pullback square

$$\begin{tikzcd} F([2]) \arrow[r] \arrow[d] & F(S') \arrow[d] \\ F(S) \arrow[r] & F(\{s\}) \end{tikzcd}$$

in C. Let $\mathrm{Gpd}(C)$ denote the full subcategory of $\mathrm{Ct}(C)$ spanned by the groupoid objects of C. We have an adjoint pair

$$i : \mathrm{Gpd}(C) \rightleftarrows \mathrm{Ct}(C) : j$$

where $j(F)$ is the groupoid object of isomorphisms of a category object F in C.

Definition 4.1.21. Let C be an ∞-category and G a groupoid object in C. Then G is said to be a *group object* in C if $G([0])$ is a terminal object in C.

Let $\mathrm{Gp}(C)$ denote the full subcategory of $\mathrm{Gpd}(C)$ spanned by the group objects of C. The ∞-category $\mathrm{Gp}(\mathcal{K})$ will play the analogue of the category of groups in the ∞-categorical context. If C is an ∞-category then there exists an equivalence

$$\mathrm{Gp}(\mathrm{Pr}(C)) \to \mathbb{R}\underline{\mathrm{Hom}}(C^{op}, \mathrm{Gp}(\mathcal{K}))$$

of ∞-categories. This follows from the general fact that for an ∞-category D with limits, then $\mathrm{Gp}(\mathbb{R}\underline{\mathrm{Hom}}(C, D)) \simeq \mathbb{R}\underline{\mathrm{Hom}}(C, \mathrm{Gp}(D))$ since limits in functor categories a computed pointwise. If (C, τ) is a site then a $\mathrm{Gp}(\mathcal{K})$-valued stack will be called a *group stack* on C. The ∞-category $\mathrm{Gp}(\mathrm{St}^\tau(C))$ of group stacks on C will be denoted $\mathrm{Gp}^\tau(C)$.

Example 4.1.22. If C is the site (Aff_R, τ) of commutative R-algebras, we will denote the ∞-category $\mathrm{Gp}^\tau(\mathrm{Aff}_R)$ by $\mathrm{Gp}^\tau(R)$.

4.2 Gerbes

Let (C, τ) be a site. A stack F in $\mathrm{St}^\tau(C)$ is said to be *locally non-empty* if for all $x \in C$ there exists a τ-covering $u \to x$ such that $F(u)$ is non-empty. It is said to be *locally connected* if $t_0(F) \to *$ is an isomorphism (of sheaves of sets). A morphism of prestacks $\phi : F \to G$ is said to be a *local equivalence* if it is *fully faithful*, ie. $\phi_x : F(x) \to G(x)$ is fully faithful for all $x \in C$, and *locally essentially surjective*, ie. for all $x \in C$ and $a \in G(x)$ there exists a covering $\alpha : u \to x$ such that $\alpha^*(a)$ is equivalent to $\alpha^*(\phi_x(b))$ (ie. an isomorphism in $\mathrm{h}G(u)$) for some $b \in F(x)$. If F and G are stacks then ϕ is a local equivalence if and only if it is an equivalence of stacks.

Definition 4.2.1. Let (C, τ) be a site and F a stack in $\mathrm{St}^\tau(C)$. Then F is said to be a *gerbe* in $\mathrm{St}^\tau(C)$ if it is locally non-empty and locally connected.

The full subcategory of $\mathrm{St}^\tau(C)$ spanned by gerbes will be denoted by $\mathrm{Ger}^\tau(C)$. A gerbe G in $\mathrm{Ger}^\tau(C)$ is said to be *neutral* if there exists a morphism $* \to G$ in $\mathrm{Ger}^\tau(C)$.

Example 4.2.2. Let C be an $\mathcal{O}$-monoidal ∞-category and R an $\mathcal{O}$-monoid object of C. Then we will denote by $\mathrm{Ger}^\tau(R) := \mathrm{Ger}^\tau(\mathrm{Aff}_R)$ the ∞-category of stacks with respect to the site (Aff_R, τ) of R-algebras in C.

We now provide a requisite characterisation of a gerbe. First we will need a small lemma.

Lemma 4.2.3. *Let $f : C \to D$ be a map of $(\infty, 0)$-categories. Then the following are equivalent.*

1. *The map f is fully faithful, ie. for all $x, y \in C$, $x \times^h_C y = \mathrm{Map}_C(x, y) \xrightarrow{\sim} \mathrm{Map}_D(fx, fy) = fx \times^h_D fy$.*

2. *The map $\delta : C \to C \times^h_D C$ is an equivalence.*

3. *The map $\pi_0(C) \to \pi_0(D)$ is a monomorphism and for all $x \in C$ and $i > 0$, the map $\pi_i(C, x) \to \pi_i(D, f(x))$ is an equivalence.*

Proof. (1) $\Rightarrow$ (2). Since f is fully faithful, $C(x,y) \xrightarrow{\sim} C(x,y) \times^h_{D(fx,fy)} C(x,y)$ so $\delta : C \to C \times^h_D C$ is fully faithful. For essential surjectivity, we need to show that any object $(x, y, \alpha : fx \xrightarrow{\sim} fy)$ in $C \to C \times^h_D C$ is equivalent to an object $\delta(z) = (z, z, \mathrm{id}_z)$ for some $z \in C$. Let $z = x$. Since f is fully faithful, we set $\beta : x \xrightarrow{\sim} y$ and $\alpha = f(\beta)$. (2) $\Rightarrow$ (3). Let $d \in D$ and $(x, y) \in C \times^h_D C$ such that $d \simeq f(x) \simeq f(y)$. Then $\delta^{-1}(x, y)$ is equivalent to the path space between x and y in $f^{-1}(d)$. Thus $f^{-1}(d)$ is empty or contractible. Thus $\pi_0(C) \to \pi_0(D)$ is a monomorphism. (3) $\Rightarrow$ (1). This statement is clear. $\square$

Recall that the *classifying space* functor is given by

$$\begin{aligned} \mathrm{B} : \mathrm{Gp}(\mathcal{K}) &\to \mathcal{K} \\ G &\mapsto \mathrm{B}G : [n] \mapsto G_{n,n}. \end{aligned}$$

It admits a right adjoint Ω which sends an $(\infty, 0)$-category A to $\Omega(A) : [n] \mapsto A^{\Delta^n_*}$. Let (C, τ) be a site. We construct the following *classifying prestack functor*

$$\begin{aligned} \overline{\mathrm{B}} : \mathrm{Gp}(\mathrm{Pr}(C)) &\to \mathrm{Pr}(C) \\ G &\to \overline{\mathrm{B}}G : x \mapsto \mathrm{B}(G(x)). \end{aligned}$$

together with its right adjoint $\overline{\Omega}$, where $\overline{\Omega}(F) : x \mapsto \Omega(F(x))$. Finally, the *classifying stack functor*

$$\widetilde{\mathrm{B}} : \mathrm{Gp}^\tau(C) \to \mathrm{St}^\tau(C)$$

is the stackification of $\overline{\mathrm{B}}$ and admits the right adjoint

$$\widetilde{\Omega} : \mathrm{St}^\tau(C) \to \mathrm{Gp}^\tau(C).$$

Proposition 4.2.4. *Let (C, τ) be a site and F a stack on C. The following are equivalent:*

1. *The stack F is a gerbe.*

2. *The stack F is locally equivalent to $\widetilde{\mathrm{B}}G$ for G a group stack in $\mathrm{Gp}^\tau(C)$.*

Proof. Let Kan_0 be the category of Kan complexes with a single 0-simplex. We have a natural string of equivalences $\mathrm{Gp}(\mathcal{K}) \simeq L(\mathrm{Gp}(\mathrm{Kan})) \simeq L(\mathrm{Kan}_0) \simeq L(s\mathrm{Gp})$ of ∞-categories (see Corollary 6.4 of [GJ]). By Section 4 and 5 of Chapter 5 of *loc. cit* there exists an adjunction

$$W : s\,\mathrm{Gp} \rightleftarrows \mathrm{Kan}_0 : \Omega$$

where the construction WG is a model for $\mathrm{B}G$ where $G \in s\mathrm{Gp}$. Thus for a pointed stack F and a $\mathrm{Gp}(\mathcal{K})$-valued prestack G, we have an equivalence

$$\mathrm{Map}_{\mathrm{Pr}(C)_*}(\overline{\mathrm{B}}G, F) \to \mathrm{Map}_{\mathrm{Gp}(\mathrm{Pr}(C))}(G, \overline{\Omega}F)$$

of $(\infty, 0)$-categories where $\overline{\Omega}F := \mathrm{Aut}(s)$ for $s \in F(*)$. We claim that if F is a stack which is locally non-empty and locally connected with $F(*) \neq \emptyset$ then F is locally equivalent to $\widetilde{\mathrm{B}}G$ for $G = \mathrm{Aut}(s)$, $s \in F(*)$. By the equivalence above, the identity map $G \to \mathrm{Aut}(s)$ corresponds to a map of prestacks

$$\begin{aligned} \phi : \overline{\mathrm{B}}G &\to F \\ * &\mapsto s. \end{aligned}$$

But since F is a stack, the universal property of stackification implies that ϕ is actually a map of stacks $\phi : \widetilde{\mathrm{B}}G \to F$. It remains to show that ϕ is fully faithful and locally essentially surjective. By Lemma 4.2.3, fully faithfulness is equivalent to the condition that $\widetilde{\mathrm{B}}G \to \widetilde{\mathrm{B}}G \times_F \widetilde{\mathrm{B}}G$ is an equivalence of stacks. By the universal property if suffices to check it for a map of prestacks. By Lemma 4.2.3 again, it suffices to check the two conditions of Lemma 4.2.3 part (3). The first condition of (3) is clear. The second condition follows from the fact that for all $x \in C$ we have $\pi_i(\overline{\mathrm{B}}G(x), *) \simeq \pi_{i-1}(G(x), *) := \pi_{i-1}(\overline{\Omega}F(x), s) \simeq \pi_i(F(x), s)$. Finally, since F is locally non-empty and locally connected there always exists a τ-covering $\alpha : u \to x$ such that for $a \in F(x)$ the map $\alpha^*(a) \to \alpha^*(\phi_x(*))$ is an equivalence. □

4.3 The positive, flat and finite topologies

Recall that a module M over an ordinary ring R is said to be *flat* if the functor $\bullet \otimes_R M : \mathrm{Mod}_R(\mathrm{Ab}) \to \mathrm{Mod}_R(\mathrm{Ab})$ is exact (ie. preserves finite limits and colimits).

Definition 4.3.1. Let R be an E_∞-ring and A an R-algebra. An A-module M is said to be

1. *Positive* if the functor $\bullet \otimes_A M : \mathrm{Mod}_A \to \mathrm{Mod}_A$ preserves anti-connective objects.
2. *Flat* if the abelian group $\pi_0 M$ is flat as a module over the ordinary commutative algebra $\pi_0 A$ and for each $n \in \mathbb{Z}$, the map $\pi_n A \otimes_{\pi_0 A} \pi_0 M \to \pi_n M$ is an isomorphism of abelian groups.
3. *Finite* if the functor $\bullet \otimes_A M : \mathrm{Mod}_A \to \mathrm{Mod}_A$ preserves all (small) limits.

A map $A \to B$ of R-algebras is said to be *positive* (resp. *flat*, *finite*) if B is positive (resp. flat, finite) when considered as an A-module. If R is a connective E_∞-ring then every flat R-module is also connective. If R is a discrete E_∞-ring then every R-module M is flat if and only if M is discrete and $\pi_0(M)$ is flat over $\pi_0(R)$ in the classical sense. A module M is finite over an E_∞-ring if and only if it is perfect (see Proposition 5.1.7).

Let k be a commutative ring and M and N be k-modules. Recall the construction of the abelian groups $\mathrm{Tor}_n^k(M,N)$ (see for example [We]). Recall also that a k-module M is flat if and only if for any k-module N, the group $\mathrm{Tor}_1^k(M,N) = 0$. Let R be a discrete E_∞-ring and M and N be two discrete R-modules. Then the canonical map

$$\pi_n(M \otimes_R N) \to \mathrm{Tor}_n^{\pi_0 R}(\pi_0 M, \pi_0 N)$$

is an isomorphism.

Let $\mathrm{Mod}_R^{\geq 0}$ (resp. Mod_R^{fl}, Mod_R^{fin}) denote the full subcategory of Mod_R spanned by the positive (resp. flat, finite) R-modules. These full subcategories are closed under taking tensor products and contain the unit object R of Mod_R. Hence by Example 3.1.11, these ∞-categories inherit a symmetric monoidal structure. By Proposition 3.2.14 we deduce that the functor $\mathrm{CMon}(\mathrm{Mod}_R^{\geq 0}) \to \mathfrak{E}_{R/}$ is fully faithful and its essential image consists of positive maps $R \to R'$. Similarly statements hold for the flat and finite examples.

Lemma 4.3.2. *Let R be an E_∞-ring.*

1. *Maps of positive, flat and finite R-algebras are stable under composition.*
2. *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

be a pushout in $\mathfrak{E}_{R/}$ of R-algebras. If f is positive (resp. flat, finite) then g is positive (resp. flat, finite).

Proof. For part (1), let $A \to B \to C$ be two maps of R-algebras. For any A-module M, there exists a natural equivalence
$$C \otimes_B (B \otimes_A M) \simeq C \otimes_A M$$
showing that the functor $C \otimes_A \bullet$ is equivalent to the composition $C \otimes_B (B \otimes_A \bullet)$ of functors. Since the composition of two functors preserving connective objects is connective this proves the positive part. Since the composition of two exact functors is exact and the above equivalence is an isomorphism on π_0 objects, the flat case is satisfied. Finally, the composition of two functors preserving limits preserves limits which proves the finite case. To prove (2), observe that there exists a natural equivalence $D \simeq B \otimes_A C$. Thus for any C-module M, there exists an equivalence
$$D \otimes_C M \simeq (B \otimes_A C) \otimes_C M \simeq B \otimes_A M$$
of B-modules. Following the argument above, this shows that if f is positive (flat, finite) then g is also. $\square$

Let R be an E_∞-ring and $A \to B$ a map of R-algebras. Consider the base change functor
$$B \otimes_A \bullet : \mathrm{Mod}_A \to \mathrm{Mod}_B$$
$$M \mapsto B \otimes_A M.$$
A map of R-algebras $A \to B$ is said to be *conservative* if the base change functor $B \otimes_A \bullet$ is conservative, ie. $B \otimes_A M \simeq 0$ if and only if $M \simeq 0$.

Definition 4.3.3. Let R be an E_∞-ring. A finite family of maps $\{A \to B_i\}_{i \in I}$ of R-algebras is said to be a *positive* (resp. *flat*, *finite*) *covering* if $A \to B_i$ is positive (resp. flat, finite) and conservative for each $i \in I$.

Proposition 4.3.4. *Let R be an E_∞-ring. The positive, flat and finite coverings define a topology on the ∞-category* Aff_R.

Proof. The conservative property is clearly stable under composition and pushouts. Thus the three cases can be deduced from Lemma 4.3.2. $\square$

The positive, flat and finite topologies will be denoted by "≥ 0", "fl" and "fin" respectively. The most important example of a stack with respect to these topologies in our context is the stack of modules. We construct the following prestack with respect to a commutative ring spectrum R:
$$\mathrm{Mod} : \mathrm{Aff}_R^{op} \to \mathrm{Cat}_\infty^\infty$$
$$A \mapsto \mathrm{Mod}_A$$
$$(A \to B) \mapsto B \otimes_A \bullet$$

Proposition 4.3.5. *Let R be an E_∞-ring. The functor* Mod *is a stack of ∞-categories over the site* Aff_R *with respect to the flat and finite topologies.*

Proof. We begin with the finite topology. We will show that $\mathrm{Mod} : \mathrm{Aff}_R^{op} \to \mathrm{Cat}_\infty^\infty$ satisfies each of the conditions of Proposition 4.1.19. For any $A \in \mathrm{CAlg}_R$, the ∞-category Mod_A has limits since Mod_A is presentable (and presentable ∞-categories admit all limits). Given any $u : B \to A$ in Aff_R the base change functor $u^* : \mathrm{Mod}_A \to \mathrm{Mod}_B$ commutes with limits along Δ by virtue of the flat and finite topologies. Its right adjoint u_* is given by the conservative forgetful functor. For any pushout square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "v"'] & B \arrow[d, "u'"] \\ C \arrow[r, "v'"] & D \end{tikzcd}$$

in CAlg_R we have, for $M \in \mathrm{Mod}_B$,

$$(v')^* u'_*(M) = (v')^*(M \otimes_B D) \simeq (v')^*(M \otimes_A C) \simeq v_*(M \otimes_B A) = v_* u^*(M)$$

where the first equivalence follows from the natural equivalence $B \coprod C \simeq B \otimes_A C$ in CAlg_R of Example 3.1.12. The proof of the flat case can be extracted from Lemma 2.2.2.13 of [TVII] (the same arguments hold here). □

Proposition 4.3.6. *Let R be an E_∞-ring. The flat and finite topologies on Aff_R are subcanonical.*

Proof. This follows from the fact that if Mod is a stack on Aff_R for any topology τ then τ is subcanonical. This can be seen as follows. Assume Mod is a stack. Then by definition we have an equivalence $\mathrm{Mod}_A \to \lim_\Delta \mathrm{Mod}_{B_*}$ for any covering $B \to A$ of A. Thus for all $M \in \mathrm{Mod}_A$, the unit map $M \to \lim_\Delta(M \otimes_A B_*)$ is an equivalence. Take $M = A$. Then $A \to \lim_\Delta B_*$ is an equivalence and for all $C \in \mathrm{CAlg}_R$ the composition

$$\mathrm{Map}(C, A) \to \mathrm{Map}(C, \lim_\Delta B_*) \to \lim_\Delta \mathrm{Map}(C, B_*)$$

is an equivalence. Thus the representable prestack h_C is a stack. The result now follows from 4.3.5. □

Let $\tau \in \{\mathrm{fl}, \mathrm{fin}\}$. Since τ is subcanonical, we have a fully faithful morphism

$$\mathrm{Aff}_R \to \mathrm{St}^\tau(R)$$

of ∞-categories given by the Yoneda embedding. We denote a stack in the essential image of this functor by $\mathrm{Spec}\, A$ for an R-algebra A. A stack F in $\mathrm{St}^\tau(R)$ is said to be *affine* if $F \to \mathrm{Spec}\, A$ is an equivalence of stacks for some R-algebra A. An affine stack is called an *affine group stack* if the affine stack is a group stack.

5 Tannaka duality for ∞-categories

Tannaka duality for ∞-categories describes a correspondence between certain linear monoidal ∞-categories and certain stacks with respect to the ∞-category of symmetric ring spectra. More precisely, one studies the adjunction

$$\mathrm{Fib} : \mathrm{Tens}_R^{\mathrm{rig}} \rightleftarrows \mathrm{St}^\tau(R)^{op} : \mathrm{Perf}$$

where for a symmetric ring spectrum R, the category on the left is the $(\infty, 2)$-category of rigid R-tensor ∞-categories and the category on the right is the ∞-category of stacks on the site of R-algebras for various topologies τ. The duality theorem describes conditions on these categories for which this adjunction is an equivalence of ∞-categories.

In Section 5.1 we describe what it means for an ∞-category to be rigid. This amounts to every object being dualisable and is a strong condition which gives much of the Tannakian theory its flavour. The notion of a dual object in an ordinary category has its origins in the example of the category of vector spaces: a vector space admits a dual if and only if the vector space is finite dimensional. Thus the rigidification of an ∞-category, that is, discarding all objects that do not admit duals, can be thought of as the implementation of a "finiteness condition" on its objects. Most of the ∞-categories we work with are ind-rigid. That is, they are equivalent to the ∞-category of ind-objects of its full subcategory of dualizable objects. For example the ∞-category of modules over an E_∞-ring is ind-rigid: any module over an E_∞-ring is a given by a filtered colimit of rigid objects. In Proposition 5.1.6 we prove the very useful projection formula for ∞-categories and in Proposition 5.1.12 show that under certain conditions, the ∞-category of endomorphisms of a functor valued in R-modules is an affine group stack.

On one side of the higher Tannaka duality will be affine group stacks which correspond algebraically to Hopf algebras. In Section 5.2 we introduce the ∞-category of Hopf R-algebras for R a commutative monoid object in a symmetric monoidal ∞-category. We of course are interested in the case where R is an E_∞-ring and in Proposition 5.2.4 we prove that the Spec functor embeds fully the ∞-category of Hopf R-algebras into the ∞-category of group stacks with respect to a subcanonical topology. Informally, we have a diagram

$$\begin{tikzcd} \mathfrak{E}_{R/} \arrow[r, "\mathrm{Spec}"] \arrow[d] & \mathrm{St}^\tau(R) \arrow[d] \\ \mathrm{Hopf}_R \arrow[r, "\mathrm{Spec}"] & \mathrm{Gp}^\tau(R) \end{tikzcd}$$

of ∞-categories where the horizontal arrows take an algebraic object to its corresponding affine geometric object and the vertical arrows pass to the corresponding group objects.

In Section 5.3 we begin the main objective of this paper: to state and prove a Tannaka duality statement for ∞-categories. We state two duality theorems. One for the pointed (or neutralized) case and the other for the non-pointed (or neutral) case. We must first describe the adjunction in which the equivalence lives (Proposition 5.3.2). Restricting to pointed objects we obtain the corresponding pointed adjunction in Lemma 5.3.8. We define the notion of τ-fiber functor for each of our three topologies, the positive, flat and finite topologies, and define what it means for an ∞-category to be pointed R-Tannakian. The pointed ∞-Tannaka duality theorem is stated in Theorem 5.3.13 and proven in Section 5.4. In Section 5.5 we define (non-pointed) R-Tannakian ∞-categories and the ∞-category of neutral R-Tannakian gerbes with respect to our three topologies. The ∞-Tannaka duality theorem is stated in Theorem 5.5.3 and proven at the end of the section.

In Section 5.6 we begin the comparison results with the classical theory. In Proposition 5.6.1 we show that the category of stacks in the classical Tannaka theory, that is, stacks with respect to the ffqc topology, fully embeds into the ∞-category of stacks with respect to the flat topology. This proposition aids us in proving Corollary 5.6.4 which states that when k is a field, the category of k-Tannakian categories with respect to the ffqc topology is equivalent to the ∞-category of Hk-Tannakian ∞-categories with respect to the flat topology. The functor is given by taking a k-Tannakian category to the ∞-category given by the localisation of its category of bounded complexes. In Section 5.7 we discuss the relationship between pointed schematic homotopy types introduced in [T2] and neutralized positive Hk-Tannakian ∞-categories.

5.1 Rigid ∞-categories

Recall that an object y in a symmetric monoidal category C is said to be a *dual* of an object x in C if there exists maps $ev_x : x \otimes y \to 1$ and $coev_x : 1 \to y \otimes x$ such that the compositions

$$x \xrightarrow{id_x \otimes coev_x} x \otimes y \otimes x \xrightarrow{ev_x \otimes id_x} x$$

$$y \xrightarrow{coev_x \otimes id_y} y \otimes x \otimes y \xrightarrow{id_y \otimes ev_x} y$$

coincide with the identity maps of x and y. Let C now be a symmetric monoidal ∞-category. An object x in C is said to be *dualizable* if it admits a dual when considered as an object of the symmetric monoidal category $\mathrm{h}C$. We will denote the dual of an object x by $x^\vee$.

Definition 5.1.1. A symmetric monoidal ∞-category is said to be *rigid* if all objects are dualizable.

See also Proposition 2.6 of [TV3] for more equivalent characterisations of rigidity. Note how closely the definition of a rigid ∞-category resembles the definition of an ∞-category with adjoints (see Definition 2.3.2). The dictionary is as follows. A symmetric monoidal ∞-category C can be thought of as an $(\infty,2)$-category BC with a single object $*$. We identify $\mathrm{Map}_{BC}(*,*)$ with an ∞-category equivalent to C and with composition $\mathrm{Map}_{BC}(*,*) \times \mathrm{Map}_{BC}(*,*) \to \mathrm{Map}_{BC}(*,*)$ given by the tensor product in C (higher coherences take into account the complete monoidal structure on C). Then C is a rigid ∞-category if and only if the homotopy 2-category h_2BC has adjoints.

Let C be a symmetric monoidal ∞-category. Then the unit object 1_C is dualizable. Moreover, the dualizable objects are stable by isomorphism in $\mathrm{h}C$ and stable by the tensor product. We denote the full subcategory of C consisting of dualizable objects by C^{rig}. Let $\underline{\mathrm{Cat}}_\infty^{\mathrm{rig}}$ denote the full subcategory of $\underline{\mathrm{Cat}}_\infty^{\mathrm{sM}}$ spanned by the rigid ∞-categories. By Proposition 5.1.5 this is an ∞-category. By Theorem 2.10 and Lemma 2.11 of [TV3] we deduce that there exist adjunctions

$$(\mathrm{Cat}_\infty^\infty)^{\mathrm{sM}} \underset{i}{\overset{\mathrm{Fr}^{\mathrm{rig}}}{\rightleftarrows}} (\mathrm{Cat}_\infty^\infty)^{\mathrm{rig}} \underset{(\bullet)^{\mathrm{rig}}}{\overset{i}{\rightleftarrows}} (\mathrm{Cat}_\infty^\infty)^{\mathrm{sM}}$$

of ∞-categories where $\mathrm{Fr}^{\mathrm{rig}}(C)$ is the free rigid ∞-category generated by the symmetric monoidal ∞-category C. The right adjoint $(\bullet)^{\mathrm{rig}}$ will be called the *rigidification* functor and C^{rig} the rigidification of C.

Example 5.1.2. Let R be an E_∞-ring. Then there exists an equivalence $\mathrm{Mod}_R^{\mathrm{perf}} \simeq \mathrm{Mod}_R^{\mathrm{rig}}$ of ∞-categories.

Proposition 5.1.3. *Let C be a symmetric monoidal ∞-category. Then C is rigid if and only if it satisfies the following conditions:*

1. *The ∞-category C is enriched over itself.*
2. *The map $\underline{\mathrm{Hom}}(x,1) \otimes y \to \underline{\mathrm{Hom}}(x,y)$ is an equivalence in C for all $x, y \in C$.*

Proof. It is enough to prove this statement in $\mathrm{h}C$. The classical statement can then be found for example in Section 2 of [D2]. □

Let C be a rigid ∞-category. To any map $f : x \to y$ in C there corresponds a *transpose* map given by the composition

$${}^t f : x \xrightarrow{1 \otimes coev_x} y^\vee \otimes x \otimes x^\vee \xrightarrow{1 \otimes f \otimes 1} y^\vee \otimes y \otimes x^\vee \xrightarrow{ev_y \otimes 1} x^\vee.$$

Similarly, to any $f : y^\vee \to x^\vee$ we associate a composition map

$$f^t : x \xrightarrow{coev_y \otimes 1} y \otimes y^\vee \otimes x \xrightarrow{1 \otimes f \otimes 1} y \otimes x^\vee \otimes x \xrightarrow{y \otimes ev_x} y.$$

This induces an equivalence

$$C(x, y) \to C(y^\vee, x^\vee).$$

of $(\infty, 0)$-categories. The following Lemma follows straightforwardly from Proposition 5.1.3.

Lemma 5.1.4. *Let C be a rigid ∞-category.*

1. *For all $x, y, x', y' \in C$, the map $\underline{\mathrm{Hom}}(x, y) \otimes \underline{\mathrm{Hom}}(x', y') \to \underline{\mathrm{Hom}}(x \otimes x', y \otimes y')$ is an equivalence in C.*
2. *For all $x, y \in C$, the map $x \otimes y \to \underline{\mathrm{Hom}}(x^\vee, y)$ is an equivalence in C.*
3. *For all $x, y \in C$, the map $\underline{\mathrm{Hom}}(x, y)^\vee \to \underline{\mathrm{Hom}}(y, x)$ is an equivalence in C.*

Proof. Set $x^\vee = \underline{\mathrm{Hom}}(x, 1)$. For (1), we have a chain of equivalences $\underline{\mathrm{Hom}}(x, y) \otimes \underline{\mathrm{Hom}}(x', y') \simeq x^\vee \otimes (x')^\vee \otimes y \otimes y' \simeq \underline{\mathrm{Hom}}(x \otimes x', y \otimes y')$ since $x^\vee \otimes y^\vee$ is dual to $x \otimes y$. For (2), we have a chain of equivalences $C(z, x \otimes y) \simeq C(x^\vee \otimes y^\vee, z^\vee) \simeq C(x^\vee, \underline{\mathrm{Hom}}(y^\vee, z^\vee)) \simeq C(x^\vee, \underline{\mathrm{Hom}}(z, y)) \simeq C(z, \underline{\mathrm{Hom}}(x^\vee, y))$ which is functorial in z. Finally, for (3), we have a chain of equivalences $\underline{\mathrm{Hom}}(x, y)^\vee \simeq \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(x, y), 1) \simeq \underline{\mathrm{Hom}}(x^\vee \otimes y, 1) \simeq y^\vee \otimes x$ due to (2). Thus $\underline{\mathrm{Hom}}(x, y)^\vee \simeq \underline{\mathrm{Hom}}(y, x)$. $\square$

Proposition 5.1.5. *Let C and D be rigid ∞-categories and $F, G : C \to D$ be two symmetric monoidal functors. Then any map $\alpha : F \to G$ is an equivalence.*

Proof. As shown in [Sa], an explicit inverse to α is given by the map $\beta : G \to F$ making the following diagram

$$\begin{tikzcd}
F(x^\vee) \arrow[r, "\alpha_{x^\vee}"] \arrow[d] & G(x^\vee) \arrow[d] \\
F(x)^\vee \arrow[r, "{}^t\beta_x"] & G(x)^\vee
\end{tikzcd}$$

commute for all $x \in C$. $\square$

Proposition 5.1.6. *Let C and A be ∞-categories and $f : C \rightleftarrows A : g$ an adjunction in $\mathrm{Tens}_R^{\mathrm{lax}}$. Assume C is ind-rigid. Then for any object x in C and a in A, the map*

$$g(a) \otimes x \to g(a \otimes f(x))$$

is an equivalence.

Proof. Let x be a dualizable object of C and y be an arbitrary object of C. Then $C(y, ga \otimes x) \cong C(y \otimes x^\vee, ga) \cong A(f(y \otimes x^\vee), a) \cong A(fy \otimes fx^\vee, a) \cong A(fy \otimes (fx)^\vee, a) \cong A(fy, a \otimes fx) \cong C(y, g(a \otimes fx))$. Since any object in C is given by a colimit of dualizable objects by assumption and the above demonstration is functorial in x, the result follows. $\square$

The equivalence in Proposition 5.1.6 is often called the *projection formula*. Setting $a = 1$, $f(x) = b$ and applying f to the projection formula gives the equivalence $fg(1) \otimes b \simeq fg(b)$.

Proposition 5.1.7. *Let R be an E_∞-ring. If the functor $\bullet \otimes_R M : \mathrm{Mod}_R \to \mathrm{Mod}_R$ commutes with limits then M is dualizable.*

Proof. Let X and Y be R-modules. We can write any R-module as a colimit of perfect R-modules so we set $X = \mathrm{colim}_\alpha X_\alpha$. Assume that the functor $\bullet \otimes_R M$ commutes with limits and recall that the ∞-category $\mathrm{Mod}_R^{\mathrm{perf}}$ is equivalent to $\mathrm{Mod}_R^{\mathrm{rig}}$. We have that

$$\begin{aligned}\underline{\mathrm{Hom}}(X,Y) \underset{R}{\otimes} M &\simeq \lim_\alpha \underline{\mathrm{Hom}}(X_\alpha, Y) \otimes M \\ &\simeq \lim_\alpha (\underline{\mathrm{Hom}}(X_\alpha, Y) \otimes M) \\ &\simeq \lim_\alpha (X_\alpha^\vee \otimes Y \otimes M) \\ &\simeq \lim_\alpha \underline{\mathrm{Hom}}(X_\alpha, Y \otimes M) \simeq \underline{\mathrm{Hom}}(X, Y \otimes M).\end{aligned}$$

Setting $X = M$ and $Y = R$ we see that $M^\vee \otimes_R M \simeq \underline{\mathrm{Hom}}(M, M)$ so M is dualizable by Proposition 5.1.3. □

Definition 5.1.8. Let C be a stable, presentable symmetric monoidal ∞-category. Then C is said to be *ind-rigid* if $\mathrm{Ind}(C^{\mathrm{rig}}) \to C$ is an equivalence of ∞-categories.

Example 5.1.9. Let R be an E_∞-ring. Then the ∞-category Mod_R is ind-rigid. One can show that $\mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{perf}}) \to \mathrm{Mod}_R$ is an equivalence of ∞-categories so combining this with the equivalence between $\mathrm{Mod}_R^{\mathrm{perf}}$ and $\mathrm{Mod}_R^{\mathrm{rig}}$ we obtain the desired result. Let $G = F \circ y : \mathrm{Mod}_R^{\mathrm{rig}} \to \mathrm{Mod}_R$ denote the composition of $F : \mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{rig}}) \to \mathrm{Mod}_R$ with the Yoneda embedding. It follows from Proposition 5.3.5.11 of [Lu] that the set of objects $\{G(M)\}_{M \in \mathrm{Mod}_R^{\mathrm{rig}}}$ generate Mod_R under filtered colimits.

Lemma 5.1.10. *Let C and D be presentable symmetric monoidal ∞-categories. Assume that C is ind-rigid. Then there exists an equivalence*

$$\underline{\mathrm{Hom}}^{\otimes}(C, D) \to \underline{\mathrm{Hom}}^{\otimes}(C^{\mathrm{rig}}, D^{\mathrm{rig}})$$

of ∞-categories.

Proof. By the universal property of ind-objects, the map $\underline{\mathrm{Hom}}^{\otimes}(C, D) \to \underline{\mathrm{Hom}}^{\otimes}(C^{\mathrm{rig}}, D)$ is an equivalence. The result now follows from the fact that symmetric monoidal functors preserve rigid objects. □

Let $F : C \to D$ be a functor between ∞-categories. Then we denote by $\mathrm{End}(F)$ the mapping space $\mathrm{Map}(F, F)$ taken in the ∞-category $\mathbb{R}\underline{\mathrm{Hom}}(C, D)$. If C and D are symmetric monoidal ∞-categories we let $\mathrm{End}^{\otimes}(F)$ denote the mapping space $\mathrm{Map}(F, F)$ in $\mathbb{R}\underline{\mathrm{Hom}}_{\Gamma}^{\otimes}(C, D)$. We will now show that if C is ind-rigid then $\mathrm{End}(F)$ is representable. Let R be an E_∞-ring. First recall that given two rigid R-modules M and N, the mapping space $\mathrm{Map}_{\mathrm{Mod}_R}(M, N)$ as a functor on the ∞-category CAlg_R of R-algebras is given by

$$\mathrm{Map}(M, N)(A) := \mathrm{Map}(M \otimes_R A, N \otimes_R A).$$

This functor is representable by the chain of equivalences

$$\mathrm{Map}_{\mathrm{Mod}_R}(M \otimes_R A, N \otimes_R A) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(M \otimes_R A, N) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(M \otimes_R N^\vee, A) \simeq \mathrm{Map}_{\mathrm{CAlg}_R}(\mathrm{Fr}(M \otimes_R N^\vee), A)$$

where the second equivalence follows from Proposition 5.1.3 and the third follows from the equivalence

$$\mathrm{CAlg}_R(\mathrm{Fr}(M), \bullet) \simeq \mathrm{Mod}_R(M, \bullet)$$

arising from the adjunction $\mathrm{CAlg}_R \dashv \mathrm{Mod}_R$ (see Section 3.1).

Lemma 5.1.11. *Let C be a symmetric monoidal ∞-category and $F : C \to \mathrm{Mod}_R^{\mathrm{rig}}$ a symmetric monoidal functor. Then $\mathrm{End}^{\otimes}(F)$ is representable.*

Proof. Any symmetric monoidal ∞-category is of the form $\mathrm{hocolim}_\alpha C_\alpha$ where C_α is the free symmetric monoidal ∞-category over an ∞-graph G_α defined through the following universal property: for any symmetric monoidal ∞-category D, there exists an equivalence

$$\mathbb{R}\underline{\mathrm{Hom}}^{\otimes}(C_\alpha, D) \simeq \mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{grph}}(G_\alpha, D)$$

where $\mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{grph}}$ denotes the ∞-category of functors between ∞-graphs. Thus we have an equivalence

$$\mathbb{R}\underline{\mathrm{Hom}}^{\otimes}(C, \mathrm{Mod}_R^{\mathrm{rig}}) \to \operatorname*{holim}_{\alpha} \mathbb{R}\underline{\mathrm{Hom}}^{\mathrm{grph}}(G_\alpha, \mathrm{Mod}_R^{\mathrm{rig}})$$
$$F \mapsto F_\alpha$$

where F_α sends an object x in G_α to a rigid module M_x and the mapping space $G_\alpha(x, y)$ to the mapping space $\mathrm{Mod}_R^{\mathrm{rig}}(M_x, M_y)$. When G_α consists of a single object x, then $\mathrm{End}(F_\alpha) \simeq \mathrm{End}(M_x) = \mathrm{Spec}\,\mathrm{Fr}(E_x \otimes_R (E_x)^\vee)$. When G_α consists of two objects x and y and the simplicial set A of arrows between x and y then

$$\mathrm{End}(F_\alpha) = \mathrm{End}(M_x) \times^h_{\mathbb{R}\underline{\mathrm{Hom}}(M_x, M_y)^A} \mathrm{End}(M_y).$$

Since representable objects are stable under homotopy limits and any ∞-graph is generated under homotopy colimits by the above two simple graphs, the functor $\mathrm{holim}_\alpha \mathrm{End}^{\otimes}(F_\alpha) \simeq \mathrm{End}^{\otimes}(F)$ is representable. □

Proposition 5.1.12. *Let R be an E_∞-ring, C be a presentable ind-rigid symmetric monoidal ∞-category and $F : C \to \widetilde{\mathrm{Mod}}_R$ a symmetric monoidal functor. Then $\mathrm{End}^{\otimes}(F)$ is a representable $\mathrm{Gp}(\mathcal{K})$-valued prestack. Hence it is an affine group stack with respect to any subcanonical topology.*

Proof. Since the ∞-category C is ind-rigid, the map $\mathbb{R}\underline{\mathrm{Hom}}^{\otimes}(C, \mathrm{Mod}_R) \to \mathbb{R}\underline{\mathrm{Hom}}^{\otimes}(C^{\mathrm{rig}}, \mathrm{Mod}_R^{\mathrm{rig}})$ is an equivalence by Lemma 5.1.10. Thus we have an equivalence $\mathrm{End}^{\otimes}(F) \simeq \mathrm{End}^{\otimes}(F^{\mathrm{rig}})$. By Lemma 5.1.11, $\mathrm{End}(F^{\mathrm{rig}})$ is representable. Finally, $\mathrm{End}^{\otimes}(F) \simeq \mathbb{R}\mathrm{Aut}^{\otimes}(F)$ by Proposition 5.1.5 so $\mathrm{End}^{\otimes}(F)$ is in fact a representable $\mathrm{Gp}(\mathcal{K})$-valued prestack and hence an affine group stack for any subcanonical topology. □

5.2 Hopf algebras

Definition 5.2.1. Let C be a symmetric monoidal ∞-category and R a commutative monoid object in C. A *Hopf R-algebra* in C is a cogroup object B in the symmetric monoidal ∞-category $\widetilde{\mathrm{CAlg}}_R(C)$ of commutative R-algebras in C.

Let $\mathrm{Hopf}_R(C)$ denote the full subcategory of $\mathrm{Comon}(\widetilde{\mathrm{CAlg}}_R(C))$ spanned by the Hopf R-algebras in C. We will call $B_1 := B([1])$ the *underlying* R-algebra of B. We have a well defined functor

$$\mathrm{Hopf} : \mathrm{CMon}(C) \to \mathrm{Cat}_\infty^\infty$$
$$R \mapsto \mathrm{Hopf}_R(C).$$

When the ∞-category C is clear from the context we will simply write Hopf_R in place of $\mathrm{Hopf}_R(C)$. By Example 3.1.12 the relative tensor product monoidal structure on $\mathrm{CAlg}_R(C)$ coincides with the coproduct. Therefore, by Proposition 1.4.14 of [LIII], we deduce that there exists an equivalence

$$\mathrm{CAlg}_R(C) \to \mathrm{CSeMon}(\widetilde{\mathrm{Mod}}_R(C))$$

of ∞-categories. Thus a Hopf R-algebra in C may be described as a functor $B : \Delta \to \mathrm{CSeMon}(\widetilde{\mathrm{Mod}}_R(C))$ satisfying the following three conditions:

1. $B_0 = R$.

2. $B_1 \otimes_R \ldots \otimes_R B_1 \to B_n$ is an equivalence.

3. $B_1 \otimes_R B_1 \to B_2 : (x, y) \mapsto (x, xy)$ is an equivalence.

Definition 5.2.2. Let C be a symmetric monoidal ∞-category and (Aff_R, τ) a site. A Hopf R-algebra in C is said to be a *τ-Hopf R-algebra* if its underlying R-algebra is an element of $\tau(R)$.

We denote by $\mathrm{Hopf}^{\tau}_R(C)$ the ∞-category of τ-Hopf R-algebras in C. When R is an E_∞-ring, we will be primarily interested in the ∞-category of positive (resp. flat, finite) Hopf R-algebras in the ∞-category Sp of spectra. The Hopf R-algebras which arise in the higher Tannakian theory satisfy an additional property which we define in general.

Example 5.2.3. Let R be a discrete E_∞-ring. Then by Proposition 3.5.14 and Proposition 3.5.15, we have an equivalence $\mathrm{Hopf}^{fl}_R(\mathrm{Sp}^d) \simeq \mathrm{Hopf}^{fl}_{\pi_0 R}(\mathrm{Ab})$ which is the classical definition of the category of Hopf algebras.

Proposition 5.2.4. *Let C be a symmetric monoidal ∞-category, R a commutative monoid object of C and (Aff_R, τ) the site of R-algebras with respect to a subcanonical topology τ. Then the functor*

$$\mathrm{Spec} : \mathrm{Hopf}_R \to \mathrm{Gp}^{\tau}(R)$$

is fully faithful.

Proof. Since τ is subcanonical, the Yoneda embedding $\mathrm{Aff}_R \to \mathrm{Pr}(\mathrm{Aff}_R)$ factors through the subcategory of stacks and hence $\mathrm{Spec} : \mathrm{Aff}_R \to \mathrm{St}^{\tau}(R)$ is fully faithful. The tensor product in CAlg_R corresponds to the coproduct by Example 3.1.12 and the Yoneda lemma preserves limits by Proposition 2.3.9 so we have an induced fully faithful functor $\mathrm{Spec} : \mathrm{Comon}(\mathrm{CAlg}_R) \to \mathrm{Mon}(\mathrm{St}^{\tau}(R))$ on monoid objects. Restricting to group-like objects we find that $\mathrm{Spec} : \mathrm{Hopf}_R \to \mathrm{Gp}^{\tau}(R)$ is fully faithful. $\square$

Definition 5.2.5. Let R be an E_∞-ring. A group stack G in $\mathrm{Gp}^{\tau}(R)$ is said to be *affine* if it is of the form $\mathrm{Spec}\, B$ for B a Hopf R-algebra.

Let $\mathrm{Gp}^{\tau}(R)^{\mathrm{aff}}$ denote the full subcategory of $\mathrm{Gp}^{\tau}(R)$ spanned by the affine group stacks.

Definition 5.2.6. Let R be an E_∞-ring. A gerbe F in $\mathrm{Ger}^{\tau}(R)$ is said to be *algebraic* if it is locally equivalent to $\widetilde{\mathrm{B}}G$ for $G \simeq \mathrm{Spec}\, B$ where B is a Hopf R-algebra. It is said to be *τ-algebraic* if it is algebraic for B a τ-Hopf R-algebra.

Let $\mathrm{Ger}^{\tau}(R)^{\mathrm{alg}}$ denote the full subcategory of $\mathrm{Ger}^{\tau}(R)$ spanned by the τ-algebraic gerbes. Let C be an ∞-category with finite colimits and X a cosimplicial object in C. For any cosimplicial set A, we define a cosimplicial object in C given by

$$\begin{aligned} X \otimes A : \Delta &\to C \\ [n] &\mapsto \coprod_{A_n} X_n. \end{aligned}$$

Let $h_0, h_1 : X \rightrightarrows Y$ be two arrows in cC. A *homotopy* between h_0 and h_1 is a map $h : X \otimes \Delta^1 \to Y$ such that

$$h \circ (\mathrm{id}_X \times i_0) = h \circ (i_0 \times \mathrm{id}_X) = h_0$$
$$h \circ (\mathrm{id}_X \times i_1) = h \circ (i_1 \times \mathrm{id}_X) = h_1$$

where $i_0, i_1 : \Delta^0 \to \Delta^1$ denote the inclusion maps. A diagram

$$f : X \rightleftarrows Y : g$$

in cC is said to be a *homotopy equivalence* if there exists a map $k : X \otimes \Delta^1 \to X$ such that the two conditions

$$k \circ (\mathrm{id}_X \times i_0) = k \circ (i_0 \times \mathrm{id}_X) = k_0 := \mathrm{id} \qquad k \circ (\mathrm{id}_X \times i_1) = k \circ (i_1 \times \mathrm{id}_X) = k_1 := g \circ f$$

hold together with a map $l : Y \otimes \Delta^1 \to Y$ such that the two conditions

$$l \circ (\mathrm{id}_Y \times i_0) = l \circ (i_0 \times \mathrm{id}_Y) = l_0 := \mathrm{id} \qquad l \circ (\mathrm{id}_Y \times i_1) = l \circ (i_1 \times \mathrm{id}_Y) = l_1 := f \circ g$$

are also satisfied.

Lemma 5.2.7. *Let C be an ∞-category with finite limits. Then the functor $\mathrm{holim}_n : cC \to C$ takes homotopy equivalences in cC to equivalences in C.*

Proof. Let X be a cosimplicial object in C and A a cosimplicial set. It will suffice to show that $\mathrm{holim}_n(X \otimes A) \simeq \mathrm{holim}_n(X) \otimes A$. Let $\overline{X}$ be a constant cosimplicial object in C. We have that

$$\mathrm{holim}_n(A \otimes \overline{X}) \simeq \mathrm{holim}_n(A) \otimes \overline{X} \simeq A \otimes \overline{X} \simeq A \otimes \mathrm{holim}_n(\overline{X}).$$

Now let $X = \overline{X} \otimes B$ for B a simplicial set. We have that

$$\mathrm{holim}_n((\overline{X} \otimes B) \otimes A) \simeq \mathrm{holim}_n(\overline{X} \otimes (B \otimes A)) \simeq \mathrm{holim}_n(\overline{X}) \otimes (B \otimes A) \simeq \mathrm{holim}_n(\overline{X} \otimes B) \otimes A.$$

Finally, $\overline{X} \otimes \Delta^n$ generates the ∞-category cC by homotopy limits. Therefore, setting $X \simeq \mathrm{holim}_\alpha(\overline{X}_\alpha \otimes \Delta^{n_\alpha})$ we have

$$\begin{aligned} \mathrm{holim}_n(X \otimes A) &\simeq \mathrm{holim}_n(\mathrm{holim}_\alpha(\overline{X}_\alpha \otimes \Delta^{n_\alpha}) \otimes A) \\ &\simeq \mathrm{holim}_n(\mathrm{holim}_\alpha((\overline{X}_\alpha \otimes \Delta^{n_\alpha}) \otimes A)) \\ &\simeq \mathrm{holim}_\alpha(\mathrm{holim}_n(\overline{X}_\alpha \otimes \Delta^{n_\alpha}) \otimes A) \\ &\simeq \mathrm{holim}_\alpha(\mathrm{holim}_n(\overline{X}_\alpha \otimes \Delta^{n_\alpha})) \otimes A \simeq \mathrm{holim}_n(X) \otimes A. \end{aligned}$$

□

Notation 5.2.8. Let Δ_+ be the category of augmented simplicial sets (see Notation 1.1). We define a categrory $\Delta_{-\infty}$ given as follows:

- The set of objects is given by $\mathrm{Ob}(\Delta_{-\infty}) = \mathrm{Ob}(\Delta_+)$.
- The set of maps $\mathrm{Hom}_{\Delta_{-\infty}}([n], [m])$ is the set of order preserving maps $f : \{-\infty\} \cup [n] \to \{-\infty\} \cup [m]$ which preserve the base point $\{-\infty\}$ thought of as a least element of $\{-\infty\} \cup [p]$ for any $[p] \in \Delta_{-\infty}$.

We have the natural sequence of inclusions $\Delta \subseteq \Delta_+ \subseteq \Delta_{-\infty}$ where Δ_+ is identified with the full subcategory of $\Delta_{-\infty}$ with same set of objects and a where a map f in $\Delta_{-\infty}$ belongs to Δ_+ if and only if $f^{-1}(-\infty) = \{-\infty\}$. The forgetful functor $+ : \Delta_{-\infty} \to \Delta$ admits a left adjoint which we denote by Dec. Let C be an ∞-category and let $c_{-\infty}C$ denote the ∞-category $\mathbb{R}\underline{\mathrm{Hom}}(\Delta_{-\infty}, C)$. We have an induced adjunction

$$+ : c_{-\infty}C \rightleftarrows cC : \mathrm{Dec}$$

between ∞-categories. Let X be a cosimplicial object in C. We let $\mathrm{Dec}_+(X)$ denote the cosimplicial object given by the composition $+ \circ \mathrm{Dec}(X)$.

Proposition 5.2.9. *Let C be an ∞-category and X a cosimplicial object in C. Then there exists a homotopy equivalence $X_0 \to \mathrm{Dec}_+(X)$ between cosimplicial objects in C.*

Proof. See Proposition 1.4 of [Il]. □

Proposition 5.2.10. *Let C be a symmetric monoidal ∞-category and B a Hopf R-algebra in C. Then there exists an equivalence*

$$R \to \operatorname*{holim}_{n \in \Delta} B^{\otimes_R (n+1)}$$

in the ∞-category CAlg_R.

Proof. This follows from Lemma 5.2.7, Proposition 5.2.9 and the equivalence $\mathrm{Dec}_+(B)_n = B_{n+1} \simeq B^{\otimes_R(n+1)}$. □

Definition 5.2.11. Let C be an ∞-category. An augmented cosimplicial object $X : \Delta_+ \to C$ is said to be *split* if there exists a map $X \to \mathrm{Dec}(X)$. A cosimplicial object is said to be *split* if it extends to a split augmented cosimplicial object. Let $F : C \to D$ be a functor. A (augmented) cosimplicial object X of C is said to be F*-split* if $F \circ X$ is split as a (augmented) cosimplicial object of D.

Proposition 5.2.12. *Let C be an ∞-category and $X : \Delta_+ \to C$ a split augmented cosimplicial object in C. Then X is a limit diagram.*

Proof. This is essentially (the dual of) Lemma 6.1.3.16 of [Lu]. □

We now state the ∞-categorical Beck Theorem of Lurie. This is needed in our proof of the ∞-Tannaka duality theorems of Section 5.3 and Section 5.4. See Section 7.2.2 for an account of adjunction data in an arbitrary $(\infty, 2)$-category.

Proposition 5.2.13. *Let C and D be ∞-categories and $\mathfrak{a}$ an adjunction datum in $\mathrm{ADat}_{C,D}(\underline{\mathrm{Cat}}_\infty)$. Let $f : C \to D$ be the induced map, L the induced comonad on D and $\overline{\mathrm{ADat}}_{\mathfrak{a}} \simeq C$ the equivalence of Corollary 3.3.6 of [LII]. Then there exists an equivalence*

$$\psi : C \to \mathrm{Comod}_L(D)$$

of ∞-categories if and only if:

1. *The map f preserves f-split limits along Δ.*
2. *The functor f is conservative.*

Proof reference. See Theorem 3.5.1 of [LII]. □

5.3 Neutralized Tannaka duality for ∞-categories

We now introduce the stack of fiber functors and state our duality theorems for neutralized higher Tannaka duality. In the next section, we will describe the proofs. Let R be an E_∞-ring and $\tau \in \{fl, fin\}$. The stack Mod of modules of Proposition 4.3.5 can naturally be extended to act on the ∞-category of stacks $\mathrm{St}^\tau(R)$ as follows. The objects of $\mathrm{St}^\tau(R)$ can be considered as stacks associated to the functors

$$F : \mathrm{CAlg}_R \to \mathrm{Cat}_\infty^\infty$$

taking values in the ∞-category of ∞-categories using the inclusion of Remark 4.1.9. The action of Mod on the ∞-category of stacks $\mathrm{St}^\tau(R)$ is then given by

$$\begin{aligned} \mathrm{Mod} : \mathrm{St}^\tau(R)^{op} &\to \mathrm{Cat}_\infty^\infty \\ F &\to \mathrm{Mor}(F, \mathrm{Mod}) \end{aligned}$$

where Mor is the morphism object of $\mathrm{St}^\tau_{\mathrm{Cat}_\infty^\infty}(R)$ given by Proposition 4.1.14.

The ∞-category $\mathrm{Mor}(F, \mathrm{Mod})$ is naturally endowed with the structure of an R-tensor ∞-category. The tensor ∞-structure on $\mathrm{Mor}(F, \mathrm{Mod})$ is induced from that on Mod: it is presentable by Example 2.3.18 and stable by Example 3.3.3 and is given the pointwise symmetric monoidal structure

$$\widetilde{\mathrm{Mor}}(F, \mathrm{Mod}) := \mathrm{Mor}(F, \widetilde{\mathrm{Mod}}) \times_{\mathrm{Mor}(F,\Gamma)} \Gamma$$

of Example 3.1.10 where $\widetilde{\mathrm{Mod}}_{[n]}(A) := (\widetilde{\mathrm{Mod}}_A)_{[n]}$ for an R-algebra A and Γ is the constant prestack. The R-linear structure on $\mathrm{Mor}(F, \mathrm{Mod})$ is also induced from that on Mod through the composition

$$\mathrm{Mod}_R \xrightarrow{\psi} \mathrm{Mor}(F, \mathrm{Mod}_R) \to \mathrm{Mor}(F, \mathrm{Mod})$$

where Mod_R is the constant prestack and ψ is the natural constant map. Thus we obtain a functor

$$\mathrm{Mod} : \mathrm{St}^\tau(R)^{op} \to \mathrm{Tens}_R^\otimes.$$

Notation 5.3.1. Let $\mathrm{Tens}^{\mathrm{rig}}$ denote the ∞-category $\underline{\mathrm{Cat}}_{\infty}^{\perp,\mathrm{sM},\mathrm{rig}}$ of rigid, stable symmetric monoidal ∞-categories and exact symmetric monoidal functors. Note that $\mathrm{Tens}^{\mathrm{rig}}$ is indeed an ∞-category by Proposition 5.1.5. We let $\mathrm{Tens}_R^{\mathrm{rig}}$ denote the ∞-category $(\mathrm{Tens}^{\mathrm{rig}})_{\mathrm{Mod}_R^{\mathrm{rig}}/}$. We will call $\mathrm{Tens}_R^{\mathrm{rig}}$ the *∞-category of rigid R-tensor ∞-categories.*

Restricting the functor Mod to rigid objects we obtain a natural functor

$$\begin{aligned} \mathrm{Perf} : \mathrm{St}^{\tau}(R)^{op} &\to \mathrm{Tens}_R^{\mathrm{rig}} \\ F &\mapsto \mathrm{Mor}(F, \mathrm{Perf}) \end{aligned}$$

where the stack $\mathrm{Perf} : \mathrm{CAlg}_R \to \mathrm{Tens}_R^{\mathrm{rig}}$ on the right hand side sends a commutative R-algebra A to the ∞-category $\mathrm{Mod}_A^{\mathrm{rig}}$ of rigid A-modules.

Lemma 5.3.2. *The functor* Perf *admits a left adjoint.*

Proof. Let C be a rigid R-tensor ∞-category. We have the following chain of equivalences

$$\mathrm{Map}_{\mathrm{Tens}_R^{\mathrm{rig}}}(C, \mathrm{Perf}(F)) \simeq \mathrm{Map}_{\mathrm{St}^{\tau}(R)}(C \times F, \mathrm{Perf}) \simeq \mathrm{Map}_{\mathrm{St}^{\tau}(R)}(F, \underline{\mathrm{Hom}}(C, \mathrm{Perf})).$$

Here $\underline{\mathrm{Hom}}(C, \mathrm{Perf})$ is a stack by Proposition 4.1.15, since Perf is a stack, where we regard C as a constant prestack. □

The left adjoint to Perf of Lemma 5.3.2 will be denoted

$$\begin{aligned} \mathrm{Fib} : \mathrm{Tens}_R^{\mathrm{rig}} &\to \mathrm{St}^{\tau}(R)^{op} \\ C &\mapsto \underline{\mathrm{Hom}}(C, \mathrm{Perf}). \end{aligned}$$

where $\mathrm{Fib}(C)(A) := \mathrm{Map}_{\mathrm{Tens}_R^{\mathrm{rig}}}(C, \mathrm{Mod}_A^{\mathrm{rig}})$ for a commutative R-algebra A. We would now like to consider conditions on rigid R-tensor ∞-categories and stacks on certain sites of R-algebras for which the adjunction $\mathrm{Fib} \dashv \mathrm{Perf}$ is an equivalence. We begin with some preliminary definitions and results.

Definition 5.3.3. Let R be an E_∞-ring. The ∞-category of *Segal comodules* over a Hopf R-algebra B is given by the following limit

$$\mathrm{SeComod}_B := \lim_{n \in \Delta} \mathrm{Mod}_{B_n}$$

of ∞-categories.

We will often abuse terminology by calling a Segal comodule over a Hopf R-algebra simply a comodule over a Hopf R-algebra. Restricting to rigid objects we have an identification $\mathrm{SeComod}_B^{\mathrm{rig}} := \lim_n \mathrm{Mod}_{B_n}^{\mathrm{rig}}$. The forgetful functor $\mathrm{SeComod} \to \mathrm{Mod}_R$ is given by the evaluation map $ev_0 : \lim_n \mathrm{Mod}_{B_n} \to \mathrm{Mod}_{B_0} = \mathrm{Mod}_R$.

Example 5.3.4. The object $\mathrm{Dec}_+(B)$ is a B-comodule which is just B thought of as a comodule over itself. More precisely, a B-comodule, by definition, consists of objects $M_n \in \mathrm{Mod}_{B_n}$ for all $[n] \in \Delta$ and for every arrow $[n] \to [m]$ in Δ, an equivalence $M_n \otimes_{B_n} B_m \xrightarrow{\sim} M_m$ in Mod_{B_m}. We have that $\mathrm{Dec}_+(B)_n = B_{n+1}$ and $B_{n+1} \otimes_{B_n} B_m \simeq B^{\otimes_R n+1} \otimes_{B^{\otimes n}} B^{\otimes m} \simeq B^{\otimes m+1}$ where the first equivalence follows from Segal maps of the Hopf R-algebra structure on B. Thus $\mathrm{Dec}_+(B)$ is a B-comodule.

Proposition 5.3.5. *Let R be an E_∞-ring and B a Hopf R-algebra. Then there exists an equivalence*

$$\mathrm{SeComod}_B \simeq \mathrm{Mod}(\widetilde{\mathrm{B}}G)$$

of ∞-categories for an affine group stack $G = \mathrm{Spec}\, B$.

Proof. Firstly, $\widetilde{\mathrm{B}}G = |\widetilde{\mathrm{B}}G_\bullet|$ where $\widetilde{\mathrm{B}}G_\bullet : \Delta^{op} \to \mathrm{St}^\tau(C)$ takes $[n]$ to G_n so $\widetilde{\mathrm{B}}\,\mathrm{Spec}\,B = \mathrm{colim}_{n\in\Delta}\,\mathrm{Spec}\,B_n$. We thus have an equivalence

$$\mathrm{Mod}(\widetilde{\mathrm{B}}G) := \mathrm{Mor}(\widetilde{\mathrm{B}}\,\mathrm{Spec}\,B, \mathrm{Mod}) = \mathrm{Mor}(\mathrm{colim}_{n\in\Delta}\,\mathrm{Spec}\,B_n, \mathrm{Mod}) \simeq \lim_{n\in\Delta}\mathrm{Mod}_{B_n} =: \mathrm{SeComod}_B$$

given by the Yoneda Lemma. □

Let R be an E_∞-ring and B a rigid Hopf R-algebra. A corollary of Proposition 5.3.5 is that there exists an equivalence

$$\mathrm{SeComod}_B^{\mathrm{rig}} \simeq \mathrm{Perf}(\widetilde{\mathrm{B}}G)$$

of ∞-categories for an affine group stack $G = \mathrm{Spec}\,B$.

Remark 5.3.6. Recall that we defined the ∞-category of comodules $\mathrm{Comod}_B(C)$ over an R-coalgebra B in a symmetric monoidal ∞-category C as $\mathrm{Mod}_R(C^{op})^{op}$ in Section 3.2. One can show, with the aid of Proposition 2.6.2 of [LII], that there exists an equivalence

$$\mathrm{Comod}_B(C) \to \mathrm{SeComod}_B(C)$$

of ∞-categories. In this paper, we would like to consider our ∞-category of comodules as living in the $(\infty,2)$-category $\mathrm{Tens}_R^{\mathrm{lax}}$ of R-tensor ∞-categories. However, endowing $\mathrm{Comod}_B(C)$ with the structure of an R-tensor ∞-category and proving Proposition 5.3.5 is much more difficult than that of the ∞-category of Segal comodules. This is the reason why we have chosen to adopt Definition 5.3.3.

We call $\mathrm{Perf}(\widetilde{\mathrm{B}}G)$ the ∞-category of *representations* of G and denote it by

$$\mathrm{Rep}(G) := \mathrm{Perf}(\widetilde{\mathrm{B}}G).$$

for $G = \mathrm{Spec}\,B$ where B is a Hopf R-algebra in the ∞-category Sp of spectra.

Notation 5.3.7. Let $(\mathrm{Tens}_R^{\mathrm{rig}})_*$ denote the overcategory $(\mathrm{Tens}_R^{\mathrm{rig}})_{/\,\mathrm{Mod}_R^{\mathrm{rig}}}$. The objects of $(\mathrm{Tens}_R^{\mathrm{rig}})_*$ will be described as pairs (T,ω) where T is a rigid R-tensor ∞-category and $\omega : T \to \mathrm{Mod}_R^{\mathrm{rig}}$ is an R-tensor functor. They will be called pointed rigid R-tensor ∞-categories. Let $\mathrm{Fib}_* : (\mathrm{Tens}_R^{\mathrm{rig}})_* \to \mathrm{Gp}^\tau(R)^{op}$ be the functor defined by $\mathrm{Fib}_*(T,\omega) := \mathrm{End}^{\otimes}(\omega)$. Let $\mathrm{Perf}_* : \mathrm{Gp}^\tau(R)^{op} \to (\mathrm{Tens}_R^{\mathrm{rig}})_*$ be the functor defined by $\mathrm{Perf}_*(G) := (\mathrm{Perf}(\widetilde{\mathrm{B}}G), \nu)$ where $\nu := f^* : \mathrm{Perf}(\widetilde{\mathrm{B}}G) \to \mathrm{Mod}_R^{\mathrm{rig}}$ is the functor induced by the natural map $f : * \to \widetilde{\mathrm{B}}G$.

Lemma 5.3.8. *The maps of Notation 5.3.7 induce an adjunction*

$$\mathrm{Fib}_* : (\mathrm{Tens}_R^{\mathrm{rig}})_* \rightleftarrows \mathrm{Gp}^\tau(R)^{op} : \mathrm{Perf}_*$$

of ∞-categories.

Proof. Consider the homotopy pullback diagram

$$\begin{tikzcd}
\mathrm{Map}_*(\widetilde{\mathrm{B}}G, \mathrm{Fib}(T)) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{St}^\tau(R)}(\widetilde{\mathrm{B}}G, \mathrm{Fib}(T)) \arrow[d] \\
* \arrow[r, "\omega"] & \mathrm{Map}_{\mathrm{St}^\tau(R)}(*, \mathrm{Fib}(T))
\end{tikzcd}$$

and its corresponding adjoint diagram

$$\begin{tikzcd}
\mathrm{Map}_*(T, \mathrm{Perf}(\widetilde{\mathrm{B}}G)) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Tens}_R^{\mathrm{rig}}}(T, \mathrm{Perf}(\widetilde{\mathrm{B}}G)) \arrow[d] \\
* \arrow[r, "\omega"] & \mathrm{Map}_{\mathrm{Tens}_R^{\mathrm{rig}}}(T, \mathrm{Mod}_R^{\mathrm{rig}}).
\end{tikzcd}$$

using Lemma 5.3.2. Since the two diagrams are equivalent, the homotopy pullbacks are equivalent and we have a chain of equivalences $\mathrm{Map}(G, \mathrm{Fib}_*(T)) \simeq \mathrm{Map}_*(\widetilde{\mathrm{B}}G, \mathrm{Fib}(T)) \simeq \mathrm{Map}_*(T, \mathrm{Perf}(\widetilde{\mathrm{B}}G))$ using the adjunction $\widetilde{\mathrm{B}} \dashv \widetilde{\Omega}_*$. □

We now state the main results of the paper. They will be proven in the next section. We will begin with the pointed case, otherwise known as *neutralized* Tannaka duality for ∞-categories. We would like to study conditions on pointed rigid R-tensor ∞-categories and group stacks for which the adjunction of Lemma 5.3.8 is an equivalence of ∞-categories. We make use of the positive, flat and finite topologies introduced in Definition 4.3.1. We begin by defining the appropriate subcategory of pointed rigid R-tensor ∞-categories which we call *Tannakian*.

Definition 5.3.9. Let R be an E_∞-ring, C a rigid R-linear symmetric monoidal ∞-category and $\omega : C \to \mathrm{Mod}_R^{\mathrm{rig}}$ an R-linear symmetric monoidal functor. Denote $\mathrm{Ind}(\omega)$ by $\widehat{\omega}$. Then ω is said to be:

1. A *finite fiber functor* if $\widehat{\omega}$ is conservative and preserves (small) limits.

   Let R be a connective E_∞-ring. Then ω is said to be:

2. A *flat fiber functor* if $\widehat{\omega}$ is conservative, creates a t-structure on C, is exact and whose right adjoint is t-exact.

   Let R be a connective bounded E_∞-ring. Then ω is said to be:

3. A *positive fiber functor* if $\widehat{\omega}$ is conservative, creates a t-structure on C and is exact.

Note that a fiber functor is a functor in the $(\infty, 2)$-category $\mathrm{Tens}_R^{\otimes}$. Since $\widehat{\omega}$ is a presentable symmetric monoidal functor it commutes with colimits and hence by the adjoint functor theorem admits a right adjoint $\widehat{p}$ which is a lax symmetric monoidal functor. Thus the adjunction $\widehat{\omega} \dashv \widehat{p}$ lives in the $(\infty, 2)$-category $\mathrm{Tens}_R^{\mathrm{lax}}$. Also, we remark that since positive and flat fiber functors are conservative, t-exact and defined over a connective bounded base E_∞-ring, the t-structures created are non-degenerate.

Definition 5.3.10. Let R be an E_∞-ring. A *pointed R-Tannakian ∞-category* with respect to τ is a pair (T, ω) where T is a rigid R-tensor ∞-category and $\omega : T \to \mathrm{Mod}_R^{\mathrm{rig}}$ is a τ-fiber functor.

Let $(\mathrm{Tan}_R^{\tau})_*$ denote the full subcategory of $(\mathrm{Tens}_R^{\mathrm{rig}})_*$ spanned by pointed τ-R-Tannakian ∞-categories. We will often abuse terminology by referring to a pointed R-Tannakian ∞-category (T, ω) as simply T. A notion of rigidity manifests itself on the opposite side of the duality in the following sense. Let $\omega_G : \mathrm{Mod}(\widetilde{\mathrm{B}}G) \to \mathrm{Mod}_R$ be the forgetful functor.

Definition 5.3.11. Let R be an E_∞-ring. An affine group stack $G = \mathrm{Spec}\, B$ in $\mathrm{Gp}^{\tau}(R)$ is said to be

1. *Weakly rigid* if $\mathrm{End}^{\otimes}(\omega_G) \to \mathrm{End}^{\otimes}(\omega_G^{\mathrm{rig}})$ is an equivalence.
2. *Rigid* if the map $\mathrm{Comod}_B \to \mathrm{Ind}(\mathrm{Comod}_B^{\mathrm{rig}})$ is an equivalence of ∞-categories.

Clearly an affine group stack being rigid implies that it is weakly rigid. We will now define the objects on the group side of the correspondence.

Definition 5.3.12. Let R be an E_∞-ring and τ a subcanonical topology. A group stack G in $\mathrm{Gp}^{\tau}(R)$ is said to be *R-Tannakian* if it is of the form $\mathrm{Spec}\, B$ for B a Hopf R-algebra and is weakly rigid. It is said to be *λ-R-Tannakian* for a topology λ if it is R-Tannakian where B is a λ-Hopf R-algebra.

Let $\mathrm{TGp}^{\tau}(R)$ denote the full subcategory of $\mathrm{Gp}^{\tau}(R)$ spanned by the R-Tannakian group stacks.

Theorem 5.3.13 (Neutralized ∞-Tannaka duality)**.** *Let τ be a subcanonical topology. Then the map*

$$\mathrm{Perf}_* : \mathrm{TGp}^{\tau}(R)^{op} \to (\mathrm{Tens}_R^{\mathrm{rig}})_*$$

is fully faithful. Moreover, the adjunction $\mathrm{Fib}_* \dashv \mathrm{Perf}_*$ *induces the following:*

1. *Let R be an E_∞-ring. Then (T, ω) is a pointed finite R-Tannakian ∞-category if and only if it is of the form* $\mathrm{Perf}_*(G)$ *for G a finite R-Tannakian group stack.*

2. *Let R be a connective E_∞-ring. Then (T, ω) is a pointed flat R-Tannakian ∞-category if it is of the form* $\mathrm{Perf}_*(G)$ *for G a flat R-Tannakian group stack.*

3. *Let R is a bounded connective E_∞-ring. Then (T, ω) is a pointed positive R-Tannakian ∞-category if it is of the form* $\mathrm{Perf}_*(G)$ *for G a positive R-Tannakian group stack.*

5.4 Proof of the neutralized theorem

We will now embark on the proof of the higher Tannaka duality statement described at the end of the last section. For an R-linear tensor functor $f : C \to \mathrm{Mod}_R$, we will denote by f_A the composition

$$C \xrightarrow{f} \mathrm{Mod}_R \xrightarrow{\bullet \otimes_R A} \mathrm{Mod}_A$$

given by composing f with the base change functor.

Proposition 5.4.1. *Let R be an E_∞-ring, B a R-bialgebra and $\widehat{\omega} : \mathrm{SeComod}_B \to \mathrm{Mod}_R$ the forgetful functor. Then the map*

$$\phi : \mathrm{Spec}\, B \to \mathrm{End}^{\otimes}(\widehat{\omega})$$

is an equivalence.

Proof. We need to show there exists an equivalence $\mathrm{Map}_{\mathrm{CAlg}_R}(B, A) \to \mathrm{End}^{\otimes}(\widehat{\omega}_A)$ for all $A \in \mathrm{CAlg}_R$. Let $G = \mathrm{Spec}\, B$ and consider the homotopy pullback diagram

$$\begin{tikzcd}
* = \mathrm{Spec}\, R \arrow[r, "u"] & \widetilde{\mathrm{B}}G \\
G \arrow[u, "v"] \arrow[r, "v"] & * = \mathrm{Spec}\, R. \arrow[u, "u"']
\end{tikzcd}$$

Let

$$u_* : \mathrm{Mod}_R \to \mathrm{Mod}(\widetilde{\mathrm{B}}G) \qquad u^* : \mathrm{Mod}(\widetilde{\mathrm{B}}G) \to \mathrm{Mod}_R \qquad v^* : \mathrm{Mod}_R \to \mathrm{Mod}_B \qquad v_* : \mathrm{Mod}_B \to \mathrm{Mod}_R$$

be the induced functors between combinatorial, stable, $\mathrm{Mod}_R(\mathbf{Sp})$-enriched, symmetric monoidal model categories. We have the usual projection formula $u^* u_* \simeq v_* v^*$. By Section 9.1 of [Pr], the composition $v_* v^*$ is given by $v_* v^*(M) = B \otimes_R M$. Let $\widehat{p}$ be the right adjoint to $\widehat{\omega}$. Taking the localisation of the adjunction $u^* \dashv u_*$ corresponds to the adjunction $\widehat{\omega} \dashv \widehat{p}$ of ∞-categories so $\widehat{\omega}\widehat{p}$ is of the form $B \otimes_R \bullet$. Consider the commutative diagram (again choosing strict models)

$$\begin{tikzcd}
\mathrm{SeComod}_B \arrow[r, shift left, "\widehat{\omega}"] \arrow[dr, "\widehat{\omega}_A \dashv \widehat{p}_A"'] & \mathrm{Mod}_R \arrow[l, shift left, "\widehat{p}"] \arrow[d, "\bullet \otimes_R A"] \\
& \mathrm{Mod}_A
\end{tikzcd}$$

We have a chain of equivalences $\mathrm{End}^{\otimes}(\widehat{\omega}_A) \simeq \mathrm{End}^{\mathrm{lax}}(\widehat{p}_A) \simeq \mathrm{Map}_{\mathrm{Tens}_A^{\mathrm{lax}}}(\widehat{\omega}_A \widehat{p}_A, \mathrm{id})$ of $(\infty, 0)$-categories. Applying Conjecture 3.6.10 we obtain an equivalence

$$\mathrm{Map}_{\mathrm{Tens}_A^{\mathrm{lax}}}(\widehat{\omega}_A \widehat{p}_A, \mathrm{id}) \xrightarrow{ev_A} \mathrm{Map}_{\mathrm{CAlg}_A}(B \otimes_R A, A) \simeq \mathrm{Map}_{\mathrm{CAlg}_R}(B, A)$$

given by evaluation on the unit A. Thus ϕ is an equivalence. $\square$

Corollary 5.4.2. *Let R be an E_∞-ring. Then the counit map*

$$G \to \mathrm{Fib}_*(\mathrm{Perf}_*(G))$$

is an equivalence in $\mathrm{Gp}^\tau(R)$ *when G is of the form* $\operatorname{Spec} B$ *for B a Hopf R-algebra and is weakly rigid.*

Proof. We need to show that $\operatorname{Spec} B \to \mathrm{End}^\otimes(\omega)$ is an equivalence where $\omega : \mathrm{Perf}(\widetilde{\mathrm{B}}G) \to \mathrm{Mod}_R^{\mathrm{rig}}$ for G weakly rigid. We have a diagram

$$G = \operatorname{Spec} B \xrightarrow{\sim} \mathrm{End}(\widehat{\omega}) \xrightarrow{\sim} \mathrm{End}(\widehat{\omega}^{\mathrm{rig}})$$

where the first equivalence follows from Proposition 5.4.1 and the second equivalence follows from the assumption that G is weakly rigid. Thus $\operatorname{Spec} B$ is equivalent to $\mathrm{End}^\otimes(\omega)$. □

Corollary 5.4.2 states that we have a full embedding of the ∞-category of R-Tannakian group stacks into the ∞-category of pointed rigid R-tensor ∞-categories given by the rule $G \to \mathrm{Perf}_*(\widetilde{\mathrm{B}}G)$. This is the first statement of Theorem 5.3.13. We now prove the remainder of the neutralized ∞-Tannaka duality theorem.

Proof of Theorem 5.3.13. Let $\lambda \in \{\geq 0, fl, fin\}$ be a topology. We will first show that if ω is a λ-fiber functor then the R-algebra $B := \widehat{\omega}\widehat{p}(R)$ is a τ-R-algebra where $\widehat{\omega} \dashv \widehat{p}$ is an adjunction in $\mathrm{Tens}_R^{\mathrm{lax}}$. For all three cases we consider the projection formula $\widehat{\omega}\widehat{p}(M) \simeq B \otimes_R M$ of Proposition 5.1.6 for an R-module M. For the positive case, the functor $\widehat{\omega}$ is t-exact by definition so by Lemma 3.5.7, the right adjoint $\widehat{p}$ is left t-exact and so B is positive. For the flat case, R is connective and the functors $\widehat{\omega}$ and $\widehat{p}$ are t-exact so $B = \widehat{\omega}\widehat{p}(R)$ is a connective R-module. Also, the functor $\widehat{\omega}\widehat{p} \simeq B \otimes_R \bullet$ is left t-exact by Lemma 3.5.7. It then follows from Theorem 4.6.19 of [LII] that B is flat over the connective E_∞-ring R. The finite case follows simply since $\widehat{\omega}$ preserves limits by definition and $\widehat{p}$ preserves limits since it is a right adjoint. Thus the composition $\widehat{\omega}\widehat{p}$ preserves limits and by the projection formula we are done.

Let (T, ω) be a λ-R-Tannakian ∞-category. We will now show that the unit $(T, \omega) \to \mathrm{Perf}_*(\mathrm{Fib}_*(T, \omega))$ of the adjunction $\mathrm{Fib}_* \dashv \mathrm{Perf}_*$ is an equivalence when restricted to the subcategory $(\mathrm{Tan}_R^\lambda)_*$, ie. (T, ω) is equivalent to $(\mathrm{Perf}(\widetilde{\mathrm{B}}G), \nu)$ for $G = \mathrm{End}^\otimes(\omega)$ and $\nu : \mathrm{Perf}(\widetilde{\mathrm{B}}G) \to \mathrm{Mod}_R^{\mathrm{rig}}$ the forgetful functor. Note that G is affine by Proposition 5.1.12 and so $\operatorname{Spec} B := \mathrm{End}^\otimes(\omega) \simeq \mathrm{End}^\otimes(\widehat{\omega}) \simeq \mathrm{Map}_{\mathrm{Tens}_R^{\mathrm{lax}}}(\widehat{\omega}\widehat{p}, \mathrm{id}) \simeq \operatorname{Spec}(\widehat{\omega}\widehat{p}(R))$ where the last equivalence follows from Conjecture 3.6.10. We consider the corresponding map on ind-objects $\widehat{\phi} : \widehat{T} \to \mathrm{Mod}(\widetilde{\mathrm{B}}G)$. We have the commutativity $\widehat{\omega} \simeq \omega_G \circ \widehat{\phi}$ where $\omega_G : \mathrm{Mod}(\widetilde{\mathrm{B}}G) \to \mathrm{Mod}_R$ is the forgetful functor and we consider the following diagram

$$\begin{tikzcd}
\widehat{T} \arrow[rr, shift left, "\widehat{\phi}"] \arrow[dr, "\widehat{\omega} \dashv \widehat{p}"'] & & \mathrm{Mod}(\widetilde{\mathrm{B}}G) \arrow[ll, shift left, "\widehat{q}"] \arrow[dl, "\omega_G \dashv \alpha"] \\
& \mathrm{Mod}_R &
\end{tikzcd}$$

in $\mathrm{Tan}_R^{\mathrm{lax}}$ where $\widehat{q}$ is the right adjoint to $\widehat{\phi}$ owing to the fact that $\widehat{\phi}$ commutes with colimits (it is a map between presentable ∞-categories in $\mathrm{Tan}_R^{\mathrm{lax}}$). Now observe that $\widehat{\phi}$ is conservative since $\widehat{\omega}$ is conservative (by definition of a τ-fiber functor), ω_G is the conservative forgetful functor and $\widehat{\omega} \simeq \omega_G \circ \widehat{\phi}$. Therefore, we have the following:

(*) The map $\widehat{\phi}$ is an equivalence if and only if $\widehat{\phi} \circ \widehat{q} \to \mathrm{id}$ is an equivalence.

We treat the three different topology cases separately.

(1) *Finite case*: In other words, we assume that $\widehat{\omega}$ is conservative and preserves limits. We begin by proving that $\widehat{\phi} \circ \widehat{q}(E) \to E$ is an equivalence when E is of the form $\alpha(M)$ for $M \in \mathrm{Mod}_R$. We have that $\alpha(M) \simeq B \otimes_R M$ by Section 9.1 of [Pr]. Furthermore, by the commutativity of the diagram we have

$$\omega_G \circ \widehat{\phi} \circ \widehat{q}(E) \simeq \widehat{\omega} \circ \widehat{q} \circ \alpha(M) \simeq \widehat{\omega} \circ \widehat{p}(M).$$

By the projection formula of Proposition 5.1.6, we have the equivalence $\widehat{\omega} \circ \widehat{p}(M) \simeq M \otimes_R \widehat{\omega}\widehat{p}(R)$. Therefore $\omega_G \circ \widehat{\phi} \circ \widehat{q}(E) \simeq \omega_G(B \otimes_R M) \simeq \omega_G(E)$ and the result follows from the conservativity of ω_G. We now prove that $\widehat{\phi} \circ \widehat{q}(E) \to E$ is an equivalence for a general B-comodule E. The functor ω_G is conservative and preserves ω_G-split limits by the Beck theorem of Proposition 5.2.13. Thus by the dual of Proposition 3.5.5 of [LII] we have that for all $E \in \mathrm{SeComod}_B$, there exists an augmented cosimplicial object $E_\bullet : \Delta_+ \to \mathrm{Comod}_B$ given by $E_n \simeq (\alpha \circ \omega_G)^{n+1}E$ which is ω_G-split. It follows that there exists an equivalence

$$E \to \operatorname*{holim}_{n \in \Delta} ((E_0 \otimes_R B^{\otimes_R n}) \otimes_R B)$$

where $E_0 = \omega_G(E)$. Hence we can consider E to be of the form $\mathrm{holim}_n \alpha(M_n)$ for $M_n = (E_0 \otimes_R B^{\otimes_R n}) \in \mathrm{Mod}_R$. We have a diagram

$$\begin{tikzcd}
\widehat{\phi} \circ \widehat{q}(E) \arrow[r] \arrow[d] & E \arrow[d] \\
\operatorname*{holim}_n \widehat{\phi} \circ \widehat{q}(\alpha(M_n)) \arrow[r] & \operatorname*{holim}_n \alpha(M_n).
\end{tikzcd}$$

The lower horizontal arrow is an equivalence from the above case of $E = \alpha(M)$. Also, the left vertical arrow is an equivalence from the fact that $\widehat{\phi} \circ \widehat{q}$ commutes with limits: since $\widehat{q}$ is a right adjoint functor it preserves limits and $\widehat{\phi}$ preserves limits since $\widehat{\omega}$ preserves limits (being a finite fiber functor), ω_G preserves limits (because its the forgetful functor from $\mathrm{Mod}(\widetilde{\mathrm{B}}G)$ where G is finite) and $\widehat{\omega} \simeq \omega_G \circ \widehat{\phi}$. Thus the upper horizontal arrow is an equivalence. Finally, the equivalence $\widehat{T} \to \mathrm{Mod}(\widetilde{\mathrm{B}}G)$ induces an equivalence $T \to \mathrm{Perf}(\widetilde{\mathrm{B}}G)$ on rigid objects since monoidal functors preserve rigid objects. Combining this result with Proposition 5.4.1 we find that G is rigid and hence weakly rigid.

(2) *Positive case*: In other words, we assume that $\widehat{\omega}$ is conservative, creates a non-degenerate t-structure and is t-exact. We also assume that the base E_∞-ring R is bounded and connective. Let $E \in \mathrm{Mod}(\widetilde{\mathrm{B}}G)_{\leq n}$, where $\mathrm{Mod}(\widetilde{\mathrm{B}}G)_{\leq n}$ denotes the full subcategory of $\mathrm{Mod}(\widetilde{\mathrm{B}}G)$ spanned by objects which get mapped to $(\mathrm{Mod}_R)_{\leq n}$ under ω_G. Let R be a bounded connective E_∞-ring. We have that $E \xrightarrow{\sim} \mathrm{holim}_n((E_0 \otimes_R B^{\otimes_R n}) \otimes_R B)$ is an equivalence in $\mathrm{Mod}(\widetilde{\mathrm{B}}G)_{\leq n}$ as above with $E_0 = \omega_G E \in \mathrm{Mod}_{\leq n}$. Let $M_n = (E_0 \otimes_R B^{\otimes_R n}) \in (\mathrm{Mod}_R)_{\leq n}$. Now consider the diagram

$$\begin{tikzcd}
\widehat{\omega} \circ \widehat{q}(E) \arrow[r] \arrow[d] & \omega_G(E) \arrow[d] \\
\operatorname*{holim}_n \widehat{\omega} \circ \widehat{q}(M_n \otimes_R B) \arrow[r] & \operatorname*{holim}_n \omega_G(M_n \otimes_R B).
\end{tikzcd}$$

The bottom horizontal arrow is an equivalence since

$$\widehat{\omega} \circ \widehat{q}(M_n \otimes_R B) \simeq \widehat{\omega} \circ \widehat{q} \circ \alpha(M_n) \simeq \widehat{\omega} \circ \widehat{p}(M_n) \simeq \omega_G(M_n \otimes_R B).$$

The right vertical arrow is an equivalence since ω_G preserves ω_G-split limits by Beck's theorem of Proposition 5.2.13. The left vertical arrow is also an equivalence by the following. Note firstly that $\widehat{q}(M_n \otimes_R B) \simeq \widehat{q} \circ \alpha(M_n) \simeq \widehat{p}(M_n)$ and so $\widehat{q}(M_n \otimes_R B)$ is in $T_{\leq n}$ since $\widehat{p}$ is left t-exact. Therefore, the cosimplicial object

$$[k] \mapsto \widehat{q}(M_k \otimes B)$$

in T satisfies the property that $\pi_i^t(\widehat{q}(M_k \otimes_R B) = 0$ for all $i > n$ and all k. We can then apply Lemma 3.5.16 to the t-exact fiber functor $\widehat{\omega} : \widehat{T} \to \mathrm{Mod}_R$ to deduce that $\widehat{\omega}$ commutes with limits. Secondly, the functor $\widehat{q}$ preserves limits (it is a right adjoint) so the composition $\widehat{\omega} \circ \widehat{p}$ preserves limits and we are done. We then have that the map $\omega_G \circ \widehat{\phi} \circ \widehat{q}(E) \simeq \widehat{\omega} \circ \widehat{q}(E) \to \omega_G(E)$ is an equivalence. Thus by the conservativity of ω_G, we obtain the equivalence

$$\widehat{T}_{\leq n} \to \mathrm{Mod}(\widetilde{\mathrm{B}}G)_{\leq n}.$$

We deduce that the map $\cup_n \widehat{T}_{\leq n} \to \cup_n \mathrm{Mod}(\widetilde{\mathrm{B}}G)_{\leq n}$ between the unions over all n is an equivalence. Rigid modules over a bounded E_∞-ring R are bounded and so T is contained in $\cup_n \widehat{T}_{\leq n}$ (in fact $\widehat{T} = \cup_n \widehat{T}_{\leq n}$ since $\widehat{T}$ is left t-complete owing to its endowed non-degenerate t-structure) and $\mathrm{Perf}(\widetilde{\mathrm{B}}G)$ is contained in $\cup_n \mathrm{Mod}(\widetilde{\mathrm{B}}G)_{\leq n}$. Since $\widehat{\phi}$ is a symmetric monoidal functor it preserves rigid objects and so we obtain an induced fully faithful functor $\widehat{\phi} : T \to \mathrm{Perf}(\widetilde{\mathrm{B}}G)$. However, for any $E, F \in \mathrm{Perf}(\widetilde{\mathrm{B}}G)$, we have a diagram

$$\begin{tikzcd}
\widehat{\omega}(\widehat{q}(E) \otimes \widehat{q}(F)) \arrow[r, "\sim"] \arrow[d] & \omega_G(E) \otimes \omega_G(F) \arrow[d, "\sim"'] \\
\widehat{\omega} \circ \widehat{q}(E \otimes F) \arrow[r, "\sim"] & \omega_G(E \otimes F)
\end{tikzcd}$$

using the symmetric monoidal structures on $\widehat{\omega}$ and ω_G so the left vertical arrow of the diagram is an equivalence. Conseqently, the map $\widehat{q} : \mathrm{Perf}(\widetilde{\mathrm{B}}G) \to T$ is symmetric monoidal and thus preserves rigid objects leading to the equivalence

$$T \xrightarrow{\sim} \mathrm{Perf}(\widetilde{\mathrm{B}}G)$$

of ∞-categories. Finally, G is weakly rigid by combining this equivalence with Proposition 5.4.1.

(3) *Flat case*: In other words, we assume that $\widehat{\omega}$ is conservative, creates a non-degenerate t-structure, is t-exact and whose right adjoint is t-exact. The first part of the proof can be deduced directly from the positive case: it follows that $\widehat{T}_{\leq n} \to \mathrm{Mod}(\widetilde{\mathrm{B}}G)_{\leq n}$ is an equivalence. However, here the base E_∞-ring R is merely connective. In the flat case though, we actually have an equivalence

$$\widehat{T} \simeq \operatorname*{holim}_n \widehat{T}_{\leq n} \to \operatorname*{holim}_n \ \mathrm{Mod}(\widetilde{\mathrm{B}}G)_{\leq n} \simeq \mathrm{Mod}(\widetilde{\mathrm{B}}G)$$

of ∞-categories where the identification on the left hand side follows from the left t-exactness of $\widehat{T}$ and the identification on the right hand side follows from the fact that a non-degenerate t-structure is created on $\mathrm{Mod}(\widetilde{\mathrm{B}}G)$. Since monoidal functors preserve rigid objects we have an equivalence $T \to \mathrm{Perf}(\widetilde{\mathrm{B}}G)$ of ∞-categories. Combining this equivalence with Proposition 5.4.1 we find that G is rigid and hence weakly rigid. □

5.5 Neutral Tannaka duality for ∞-categories

We now consider the case where there simply exists a τ-fiber functor. This is the non-pointed case, otherwise known as *neutral* Tannaka duality for ∞-categories.

Definition 5.5.1. Let R be an E_∞-ring and $\tau \in \{fin, fl, \geq 0\}$ a topology. A rigid R-tensor ∞-category T is said to be a *R-Tannakian ∞-category* with respect to τ if there exists a τ-fiber functor $w : T \to \mathrm{Mod}_R^{\mathrm{rig}}$.

We denote the ∞-category of R-Tannakian ∞-categories with respect to τ by Tan_R^τ. The objects on the other side of the correspondence are described as follows.

Definition 5.5.2. Let R be an E_∞-ring. A stack F in $\mathrm{St}^\tau(R)$ is said to be a *τ-R-Tannakian gerbe* if it is locally equivalent to $\widetilde{\mathrm{B}}G$ for G a τ-R-Tannakian group stack. It is said to be a *neutral* τ-R-Tannakian gerbe if there exists a morphism of stacks $* \to F$.

Let $\mathrm{TGer}^\tau(R)$ denote the ∞-category of neutral τ-R-Tannakian gerbe's. We have natural inclusions $\mathrm{TGer}^\tau(R) \subseteq \mathrm{Ger}^\tau(R)^{\mathrm{alg}} \subseteq \mathrm{Ger}^\tau(R)$. We now state the Tannaka duality theorem for ∞-categories in the neutral setting. Note that we have a weaker statement in positive case owing to the fact that the positive topology is not subcanonical.

Theorem 5.5.3 (Neutral ∞-Tannaka duality)**.** *The adjunction* Fib $\dashv$ Perf *induces the following:*

1. *Let R be an E_∞-ring. Then T is a finite R-Tannakian ∞-category if and only if it is of the form* $\mathrm{Perf}(G)$ *for G a neutral finite R-Tannakian gerbe.*
2. *Let R be a connective E_∞-ring. Then T is a flat R-Tannakian ∞-category if it is of the form* $\mathrm{Perf}(G)$ *for G a neutral flat R-Tannakian gerbe.*
3. *Let R be a bounded connective E_∞-ring. If (T, ω_1) and (T, ω_2) are two pointed positive R-Tannakian ∞-categories then there exists a positive cover $R \to Q$ such that*
$$\omega_1 \otimes_R Q \to \omega_2 \otimes_R Q$$
is an equivalence.

To prove the neutral ∞-Tannaka duality statement of Theorem 5.5.3 it suffices to combine the neutralized statement of Theorem 5.3.13 with the demonstration that two fiber functors are equivalent after base change.

Proposition 5.5.4. *Let R be an E_∞-ring and $\tau \in \{\geq 0, fl, fin\}$. Given two τ-fiber functors ω and ν over R, there exists a τ-cover $R \to Q$ in $\mathfrak{E}$ such that ω and ν are equivalent over Q.*

Proof. Let (T, ω) and (T, ν) be two pointed R-Tannakian ∞-categories with respect to τ. By Proposition 5.1.5 this amounts to showing that there exists a τ-cover $R \to Q$ such that $\mathrm{Map}_{\mathrm{Tens}_R^{\mathrm{rig}}}(\omega^\vee, \nu^\vee)(Q) \neq \emptyset$. It suffices to prove that $\mathrm{Map}_{\mathrm{Tens}_R^{\mathrm{lax}}}(\widehat{\omega}^\vee, \widehat{\nu}^\vee)(Q) \neq \emptyset$. We have equivalences

$$\mathrm{Map}_{\mathrm{Tens}_R^{\mathrm{lax}}}(\widehat{\omega}^\vee, \widehat{\nu}^\vee) \simeq \mathrm{Map}_{\mathrm{Tens}_R^{\mathrm{lax}}}(\widehat{\nu}, \widehat{\omega}) \simeq \mathrm{Map}_{\mathrm{Tens}_R^{\mathrm{lax}}}(\widehat{\nu}\widehat{p}, \mathrm{id}) \simeq \mathrm{Spec}(\widehat{\nu}\widehat{p}(R))$$

where the last equivalence follows from Conjecture 3.6.10. Therefore, $\underline{\mathrm{Hom}}^{\otimes}(\widehat{\omega}^\vee, \widehat{\nu}^\vee)(Q) \simeq \underline{\mathrm{Hom}}_{\mathrm{CAlg}_R}(\widehat{\nu}(B), Q)$ and we set $Q := \widehat{\nu}(B)$ to consider the identity map. It remains to show that there exists a τ-cover $R \to \widehat{\nu}(B)$. Since $\widehat{\nu}$ is R-linear, $\widehat{\nu}(B) \otimes_R \bullet \simeq \widehat{\nu}(B \otimes_R \bullet)$ so $R \to \widehat{\nu}(B)$ is a τ-R-algebra since B is a τ-Hopf R-algebra and ν is a τ-fiber functor.

We will now show that $R \to \widehat{\nu}(B)$ is conservative, ie. given $M \in \mathrm{Mod}_R$ such that $\widehat{\nu}(B) \otimes_R M \simeq 0$ then $M \simeq 0$. Let $B' = B \otimes_R R'$. By the projection formula, we have the following statement:

(*) For all $R \to R'$, the map $\widehat{\nu}(B') \to \widehat{\nu}(B) \otimes_R R'$ is an equivalence.

By Proposition 5.2.10, the map $R' \to \mathrm{holim}_{n\in\Delta}((B')^{\otimes_R (n+1)})$ is an equivalence and since ν is a τ-fiber functor we have the following statement:

(**) The map $R' \to \mathrm{holim}_{n\in\Delta}(\widehat{\nu}(B')^{\otimes_R (n+1)})$ is an equivalence.

Set $R' = \mathrm{Sym}_R(M) := \coprod_{p\geq 0} M^{\otimes_R p}/\Sigma_p$ and assume that $\widehat{\nu}(B) \otimes_R M \simeq 0$. Therefore

$$\begin{aligned} R' &\simeq \operatorname*{holim}_{n\in\Delta}(\widehat{\nu}(B)^{\otimes_R n+1} \underset{R}{\otimes} \mathrm{Sym}_R(M)) \\ &\simeq \operatorname*{holim}_{n\in\Delta} \coprod_{p\geq 0} (\widehat{\nu}(B)^{\otimes n+1} \underset{R}{\otimes} M^{\otimes_R p}/\Sigma_p) \\ &\simeq R. \end{aligned}$$

The first line is an equivalence in CAlg_R and follows from $(**)$ and $(*)$. The second line is an equivalence in Mod_R and follows from the fact that the tensor product commutes with coproducts and the forgetful functor $\mathrm{CAlg}_R \to \mathrm{Mod}_R$ is conservative and commutes with limits (it is a right adjoint). The third line is a result of the only non-zero term being $p = 0$ by assumption followed by $(**)$ applied to R. Thus $M \simeq 0$. ☐

Proof of Theorem 5.5.3. This follows directly from Theorem 5.3.13 and Proposition 5.5.4. ☐

5.6 Comparison with the classical theory

The following series of comparison results shows that the Tannaka duality theorem for ∞-categories of Section 5.4 naturally generalises the classical theory. Let k be a field. In [Sa], Saavedra defined the notion of a neutral Tannakian category over k. That is, a rigid abelian k-linear symmetric monoidal category T for which there exists an exact k-linear symmetric monoidal functor $T \to \mathrm{Vect}_k$ (the fiber functor) taking values in the category of finite rank projective k-modules.

The collection of ordinary fiber functors form a stack $\mathrm{Fib}(T)$ over k in the faithfully flat quasi-compact topology, denoted $ffqc$. Recall that a finite family $\{A_i \to A\}_{i\in I}$ of arrows in Aff_k is an $ffqc$ cover if the morphism $\prod_{i\in I} A_i \to A$ is faithfully flat (ie. exact and conservative). Let $\mathrm{St}^{ffqc}(k)$ denote the ∞-topos of stacks on the classical site $(\mathrm{Aff}_k, ffqc)$.

Proposition 5.6.1. *Let k be a commutative ring and Hk its corresponding Eilenberg-Mac Lane ring spectrum. The inclusion*

$$i : \mathrm{St}^{ffqc}(k) \to \mathrm{St}^{fl}(Hk)$$

of ∞-categories is fully faithful.

Proof. We will show that there exists a composition of fully faithful maps

$$\mathrm{St}^{ffqc}(k) \hookrightarrow \mathrm{St}^{fl}_c(Hk) \hookrightarrow \mathrm{St}^{fl}(Hk)$$

where $\mathrm{St}^{fl}_c(Hk)$ is the ∞-category of stacks on the site of connective Hk-algebras. Firstly, there exist adjunctions

$$\mathrm{CAlg}_k \underset{\tau_{\leq 0}}{\overset{i}{\rightleftarrows}} \mathrm{CAlg}^c_{Hk} \underset{(\bullet)^c}{\overset{j}{\rightleftarrows}} \mathrm{CAlg}_{Hk}$$

by Section 3.5 and Section 3.4 respectively. We then left Kan extend these adjunctions to obtain the adjunctions

$$\mathbb{R}\underline{\mathrm{Hom}}(\mathrm{CAlg}_k, \mathcal{K}) \underset{\tau^*_{\leq 0}}{\overset{i_!}{\rightleftarrows}} \mathbb{R}\underline{\mathrm{Hom}}(\mathrm{CAlg}^c_{Hk}, \mathcal{K}) \underset{((\bullet)^c)^*}{\overset{j_!}{\rightleftarrows}} \mathbb{R}\underline{\mathrm{Hom}}(\mathrm{CAlg}_{Hk}, \mathcal{K})$$

This chain of adjunctions induces adjunctions on the subcategories of stacks such that $i_!$ and $j_!$ are fully faithful by Section 2.2.4 and 2.3.5.1 of [TVII]. $\square$

The right adjoint to the functor i of Proposition 5.6.1

$$\tau_{\leq 0} : \mathrm{St}^{fl}(Hk) \to \mathrm{St}^{ffqc}(k)$$

is explicitly given by $\tau_{\leq 0}(F)(k') := F(Hk')$ for a stack $F \in \mathrm{St}^{fl}(Hk)$. If F is a neutral affine gerbe in the sense of [D2] then $i(F) \in \mathrm{St}^{fl}(Hk)$ is a neutral flat Hk-Tannakian gerbe. Thus neutral affine gerbes as defined in the classical Tannakian theory form a full subcategory of neutral flat Hk-Tannakian gerbes in our sense.

For the following comparison results, we will need to introduce a finite condition on some of our categories in order for the statements to make sense. The notion is that of finite cohomological dimension of an arbitrary abelian category.

Definition 5.6.2. Let C be an abelian category. Then C is said to be of *finite cohomological dimension* if there exists n such that for every object x in C, the group $\mathrm{Ext}^i(y, x) = 0$ for all $i > n$ and $y \in C$.

Let T be an abelian category and $C(T)$ (resp. $C^b(T)$) be the category of unbounded (resp. bounded) complexes in T. We denote by $LC^b(T)$ the ∞-category given by localising $C^b(T)$ at the set of quasi-isomorphisms. If T is a Tannakian ∞-category (resp. Tannakian category) we will denote $\mathrm{Fib}^\tau_*(T)$ the group stack (resp. group scheme) of fiber functors on C with respect to the topology τ.

Proposition 5.6.3. *Let k be a field and T a k-Tannakian category. Assume that T is of finite cohomological dimension. Then the ∞-category $LC^b(T)$ is a flat Hk-Tannakian ∞-category and there exists an equivalence*

$$i(\mathrm{Fib}_*^{ffqc}(T)) \to \mathrm{Fib}_*^{fl}(LC^b(T))$$

of stacks in $\mathrm{St}^{fl}(Hk)$.

Proof. Since T is a k-Tannakian category there exists a natural k-tensor functor $\mathrm{Mod}_k^{\mathrm{rig}} \to T$ defined by $M \mapsto M \otimes 1_T$. This exists since for any $x, y \in T$, the k-linear structure on T induces an equivalence

$$\mathrm{Hom}_{\mathrm{Mod}_k^{\mathrm{rig}}}(M, \mathrm{Hom}_T(x,y)) \simeq \mathrm{Hom}_T(M \otimes x, y).$$

Using the equivalence $LC^b(\mathrm{Mod}_k^{\mathrm{rig}}(\mathrm{Ab})) \simeq \mathrm{Mod}_{Hk}^{\mathrm{rig}}(\mathrm{Sp})$ we obtain a map

$$\mathrm{Mod}_{Hk}^{\mathrm{rig}} \to LC^b(T)$$

of symmetric monoidal ∞-categories. This map commutes with colimits and induces the Hk-linear structure. Furthermore, the ∞-category $LC^b(T)$ is stable by Example 3.3.8 which makes $LC^b(T)$ a rigid Hk-tensor ∞-category. The fiber functor $\omega : LC^b(T) \to \mathrm{Mod}_{Hk}^{\mathrm{rig}}$ induced from the fiber functor on T is clearly flat: the forgetful functor $\widehat{\omega} : LC(T) \to C(k)$ is conservative and creates a t-structure for which $\widehat{\omega}$ and its right adjoint are t-exact.

By the classical Tannaka duality theorem there exists an equivalence $T \to \mathrm{Rep}(H)$ for an affine group scheme H. Thus $\mathrm{Fib}_*^{ffqc}(T)$ is naturally equivalent to $\widetilde{\mathrm{B}}H$. We also have that $i(\widetilde{\mathrm{B}}H) \simeq \widetilde{\mathrm{B}}(iH)$ is an equivalence using the fully faithful functor i of Proposition 5.6.1. It follows from Proposition 5.4.1 that $\mathrm{Fib}_*^{fl}(LC^b(T)) = G$ for $G = \mathrm{End}^{\otimes}(\omega)$ a flat affine group stack is also naturally equivalent to $\widetilde{\mathrm{B}}(iH)$. ☐

Corollary 5.6.4. *Let k be a field and T a k-Tannakian category. The functor*

$$LC^b : \mathrm{Tan}_k^{ffqc} \to \mathrm{Tan}_{Hk}^{fl}$$

is an equivalence of ∞-categories.

Proof. This follows directly from Proposition 5.6.3, Proposition 5.6.1 and the flat ∞-Tannaka duality Theorem 5.3.13. ☐

The simplification of the theory when working with Tannakian ∞-categories over fields is exemplified in the following proposition which states that any positive Hk-Tannakian ∞-category can be realised as the ∞-category of modules in the localisation of complexes in an ordinary k-Tannakian category. We first observe that when k is a field and T is a positive Hk-Tannakian ∞-category then $\mathcal{H}(T)$ is a k-Tannakian category. The category $\mathcal{H}(T)$ is clearly rigid, it is abelian by Theorem 1.3.6 of [BBD] and symmetric monoidal by Proposition 3.5.12. Furthermore, the heart of a positive fiber functor over Hk is also exact where $\mathcal{H}(\mathrm{Mod}_{Hk}^{\mathrm{rig}}) = \mathrm{Vect}_k$.

Proposition 5.6.5. *Let k be a field and T a positive Hk-Tannakian ∞-category. Then there exists a commutative monoid R in $LC(\mathrm{Ind}(\mathcal{H}(T)))$ such that there exists an equivalence*

$$T \to \mathrm{Mod}_R(LC(\mathrm{Ind}(\mathcal{H}(T))))$$

of ∞-categories.

5.7 Tannakian ∞-categories over fields

We will fix a field k throughout this section. In [T2], Toën introduced the notion of a schematic homotopy type. Given two pointed stacks F and G, let $\mathrm{Map}_*(F, G)$ denote the $(\infty, 0)$-category of pointed maps between them.

Definition 5.7.1. Let F and G be two pointed stacks in $\mathrm{St}^{ffqc}(k)_*$. A map $F \to G$ is said to be a *P-equivalence* if the induced map

$$\mathrm{Perf}(F) \to \mathrm{Perf}(G)$$

is an equivalence of ∞-categories. A pointed stack F in $\mathrm{St}^{ffqc}(k)_*$ is said to be *P-local* if for any P-equivalence $H_1 \to H_2$, the induced map

$$\mathrm{Map}_*(H_2, F) \to \mathrm{Map}_*(H_1, F)$$

is an equivalence.

Lemma 5.7.2. *Let $\tau \in \{\geq 0, fl, fin\}$ and T be a pointed k-Tannakian ∞-category with respect to τ. Then $\widetilde{\mathrm{B}}\mathrm{Fib}_*(T)$ is P-local.*

Proof. Let $H_1 \to H_2$ be a P-equivalence. From the pointed adjunction of Lemma 5.3.8 we have a chain of equivalences

$$\mathrm{Map}_*(H_2, \widetilde{\mathrm{B}}\mathrm{Fib}_*(T)) \simeq \mathrm{Map}_*(T, \mathrm{Perf}(H_2)) \simeq \mathrm{Map}_*(T, \mathrm{Perf}(H_1)) \simeq \mathrm{Map}_*(H_1, \widetilde{\mathrm{B}}\mathrm{Fib}_*(T))$$

of $(\infty, 0)$-categories. □

Let $A \in c\mathrm{CAlg}_k$ be a cosimplicial k-algebra in the category Ab of abelian groups. We define the following prestack

$$\begin{aligned} \mathrm{Spec}\, A : \mathrm{CAlg}_k &\to \mathcal{K} \\ B &\mapsto \mathrm{Map}(A, B) \end{aligned}$$

where $\mathrm{Map}(A, B)_n := \mathrm{Hom}(A_n, B)$. This prestack can be shown to be a stack for the *ffqc*-topology and thus defines a natural functor

$$\mathrm{Spec} : c\mathrm{CAlg}_k \to \mathrm{St}^{ffqc}(k)$$

between ∞-categories. We are interested in objects which lie in the essential image of the Spec functor.

Definition 5.7.3. Let k be a commutative ring. An *affine stack* over k is a stack in $\mathrm{St}^{ffqc}(k)$ which is equivalent to an object of the form $\mathrm{Spec}\, A$ for A a cosimplicial k-algebra.

Let $s : * \to F$ be a pointed stack in $\mathrm{St}^{ffqc}(k)$. We define the prestack $\Omega_* F$ of loops at s by the formula

$$\begin{aligned} \Omega_* F : \mathrm{CAlg}_k &\to \mathcal{K} \\ x &\mapsto \Omega_{s(*)} F(x) \end{aligned}$$

where $\Omega_{s(*)}F(x)$ is the subsimplicial set of $\mathrm{Map}(\Delta^1, F(x))$ which sends the endpoints $\{0, 1\}$ of Δ^1 to $s(*)$.

Definition 5.7.4. Let k be a field. A pointed stack F on the site $(\mathrm{Aff}_k, ffqc)$ is said to be a *pointed schematic homotopy type* over k if it is P-local, connected and $\Omega_* F$ is an affine stack over k.

Let $(\mathrm{SHT}_k)_*$ denote the full subcategory of $\mathrm{St}^{ffqc}(k)$ spanned by the schematic homotopy types. Every pointed connected affine stack is naturally a pointed schematic homotopy type. Furthermore, a stack F is a pointed schematic homotopy type if and only if F is a pointed connected stack in $\mathrm{St}^{ffqc}(k)$ such that the sheaf $\pi_1(F, x)$ is represented by an affine group scheme and for any $i > 1$, the sheaf $\pi_i(F, x)$ is represented by a unipotent affine group scheme (see Section 3.2 of [T2]).

Let k be a field and F a pointed schematic homotopy type over k. We will say that F is of finite cohomological dimension if the abelian category $\mathcal{H}(\mathrm{Perf}(i(F)))$ is of finite cohomological dimension.

Proposition 5.7.5. *Let k be a field and F a pointed schematic homotopy type over k. Assume that F is of finite cohomological dimension. Then $\mathrm{Perf}_*(i(F))$ is a pointed positive Hk-Tannakian ∞-category.*

Proof. The stack iF is clearly a positive Hk-Tannakian group stack in $\mathrm{TGp}^{fl}(R)^{\geq 0}$. The result then follows from the positive ∞-Tannaka duality Theorem 5.3.13. □

6 Applications

Tannakian ∞-categories arise in a number of mathematical applications. We have seen in Section 5.6 that the Tannaka duality statement for ∞-categories with the flat topology subsumes the classical statement. It furthermore extends the 1-categorical duality by allowing arbitrary commutative rings as opposed to just fields. We will discuss how two applications in the classical theory can be generalised to the ∞-categorical context.

In Section 6.1 we will discuss the ∞-category of perfect complexes on a topological space X. The ∞-category of perfect complexes on X is a k-Tannakian ∞-category with respect to the positive topology for k a field. When k is a field of characteristic zero, then the Tannakian dual of the ∞-category of perfect complexes on X is precisely the schematization of X introduced by Toën in [T2].

In Section 6.2 will briefly discuss motives and non-commutative motives from a higher Tannakian point of view. We will introduce the stable $\mathbb{A}^1$ ∞-category (Definition 6.2.3) and the ∞-category of non-commutative motives (Definition 6.2.9).

6.1 Perfect complexes and schematization

In this section we will discuss an important schematic homotopy type which we define through the following universal property.

Definition 6.1.1. Let k be a field and X a fibrant simplicial set considered as a constant prestack. Then the *schematization* of X over k is a schematic homotopy type $X_k^{\sim}$ over k together with a map $f : X \to X_k^{\sim}$ satisfying the following universal property: for any stack F in $\mathrm{St}^{ffqc}(k)$, composition with f induces an equivalence

$$\mathrm{Map}_{\mathrm{St}^{ffqc}(k)}(X_k^{\sim}, F) \to \mathrm{Map}_{\mathrm{St}^{ffqc}(k)}(X, F)$$

of $(\infty, 0)$-categories.

The main result in the theory of schematizations is Theorem 3.3.4 of [T2] which states that for a pointed, connected simplicial set (X, x), a schematization always exists. The schematization of a pointed space admits a number of fundamental properties which follow from Definition 6.1.1 together with this existence result. Firstly, the affine group scheme $\pi_1(X_k^{\sim}, x)$ is naturally isomorphic to the pro-algebraic completion of the discrete group $\pi_1(X, k)$ (thought of as a constant group scheme over k). When X is finite and simply connected, then for $i > 1$, the group scheme $\pi_i(X_k^{\sim}, x)$ is naturally isomorphic to the pro-unipotent completion of the discrete group $\pi_i(X, x)$. Finally, if V is a local system of finite dimensional k-vector spaces on X, then V corresponds to a linear representation of the affine group scheme $\pi_1(X_k^{\sim}, x)$. This induces a local system $\mathcal{V}$ on the schematization $X_k^{\sim}$ (a sheaf of abelian groups $\mathcal{V}$ on $(\mathrm{Aff}_k, ffqc)$ together with an action of $\pi_1(X_k^{\sim})$ on $\mathcal{V}$) such that the map $X \to X_k^{\sim}$ furnishes an isomorphism

$$H^*(X_k^{\sim}, \mathcal{V}) \to H^*(X, V)$$

in cohomology with local coefficients. See [T2] for further discussion.

Let k be a commutative ring. The category $C(k)$ of (unbounded) complexes of k-modules admits a cofibrantly generated model structure where the fibrations are degree-wise surjective morphisms and the equivalences are the quasi-isomorphisms (those maps inducing isomorphisms on homology groups) (see Theorem 2.3.11 of [Ho]). Then for a topological space X, the category $C(X, k)$ of complexes of sheaves of k-modules on X is a $C(k)$-enriched model category. Here $C(k)$ is endowed with its usual monoidal structure and $C(X, k)$ is naturally tensored over $C(k)$ since the category of sheaves of k-modules is naturally tensored over the category of k-modules (if F is a sheaf of k-modules on X and M is a k-module then $M \otimes F$ is defined to be the sheaf associated to the presheaf $U \mapsto M \otimes F(U)$). The model category structure on $C(X, k)$ arises from a more general result of a model structure on the category $C(A)$ of complexes in any Grothendieck category A (see [H2]).

We define $\mathrm{Perf}(X, k) := (L^{\otimes}C(X, k))^{\mathrm{rig}}$. The objects of $\mathrm{Perf}(X, k)$ are complexes of presheaves of k-modules which are, locally on X, quasi-isomorphic to a constant complex of presheaves associated with a bounded complex of projective k-modules of finite type. Let $x : * \to X$ be a point of X and define $\mathrm{Perf}(k) := \mathrm{Perf}(*, k)$. There exists a k-linear structure on the symmetric monoidal ∞-category $\mathrm{Perf}(X, k)$ from the symmetric monoidal functor $\mathrm{Perf}(k) \to \mathrm{Perf}(X, k)$ and a k-linear symmetric monoidal functor

$$\omega_x : \mathrm{Perf}(X, k) \to \mathrm{Perf}(k)$$

induced by the point x.

Proposition 6.1.2. *Let k be a field and X a finite CW complex. Then $(\mathrm{Perf}(X, k), \omega_x)$ is a pointed k-Tannakian ∞-category with respect to the positive topology.*

By the pointed ∞-Tannaka duality of Theorem 5.3.13, there exists a positive affine group stack dual to the positive Tannakian ∞-category $\mathrm{Perf}(X, k)$ of Proposition 6.1.2. This dual is the schematization of X when k is a field of characteristic zero.

6.2 Motives and non-commutative motives

One of the most important open questions in the theory of algebraic varieties is the construction of a universal cohomology theory: this is the theory of motives envisioned by Grothendieck in the 1960's [Gr]. Such a theory was thought to arise following the observation that not only were an unexpected number of cohomology theories found to exist but that in many cases they encoded the same information. Although this field of study has seen immense progress over the past two decades, much of the theory still remains conjectural. Let S be a Noetherian scheme and Sm_S the category of smooth proper S-schemes. Ultimately, and informally, we would like to produce a bijection

$$\mathrm{Hom}^{\mathscr{R}}(\mathrm{MM}_S^{\mathrm{rig}}, \mathrm{Vect}_K) \simeq \mathrm{Hom}^{\mathscr{H}}(\mathrm{Sm}_S, \mathrm{Vect}_K)$$

where $\mathrm{MM}_S^{\mathrm{rig}}$ is the conjectural K-Tannakian category of (mixed) motives, $\mathrm{Hom}^{\mathscr{R}}$ denotes the set of realisation functors and $\mathrm{Hom}^{\mathscr{H}}$ is the set of (mixed) Weil cohomology functors taking values in the category Vect_K of finite dimensional K-vector spaces over a coefficient field K. Thus every well behaved cohomology theory should factor through the category of motives. Although the Tannakian category MM_S has not been constructed, a candidate for the derived category $\mathrm{DM}_S = \mathrm{Ind}(\mathrm{DM}_S^{gm})$ of mixed motives has, where DM_S^{gm} is the triangulated category of geometric motives [V1].

Let k be a field. The problematic passage from the rigid monoidal triangulated category DM_k^{gm} to the abelian category $\mathrm{MM}_k^{\mathrm{rig}}$ rests on the following conjecture:

Conjecture 6.2.1. A t-structure exists on the category DM_k^{gm} with $\mathbb{Q}$-coefficients satisfying the following conditions:

1. The functor $H : \mathrm{DM}_k^{gm} \to \mathrm{MM}_k^{\mathrm{rig}} := \mathcal{H}(\mathrm{DM}_k^{gm})$ is conservative.
2. The t-structure is compatible with the tensor product and realisation functors $R : \mathrm{DM}_k^{gm} \to \mathrm{D}^b(\mathrm{Vect}_{\mathbb{Q}})$.

Assuming this conjecture, Beilinson proved the following:

Proposition 6.2.2 ([Be])**.** *Assume conjecture 6.2.1. Then $(\mathrm{MM}_k^{\mathrm{rig}}, R)$ is a $\mathbb{Q}$-Tannakian category with the restriction $R : \mathrm{MM}_k^{\mathrm{rig}} \to \mathrm{Vect}_{\mathbb{Q}}$ the corresponding fiber functor.*

We would now like to prove a similar statement in the ∞-categorical context using our results in higher Tannaka duality. We begin by constructing the stable $\mathbb{A}^1$ ∞-category. Let X be a Noetherian scheme and Sm_X the category of smooth and proper X-schemes. Consider the natural sequence

$$\mathrm{Sm}_X \to \mathrm{Pr}_{\mathbf{S}}(\mathrm{Sm}_X) \to \mathrm{Pr}_{\mathbf{S}}(\mathrm{Sm}_X)_* \to L_{Nis}\mathrm{Pr}_{\mathbf{S}}(\mathrm{Sm}_X)_* \to L_{Nis,\mathbb{A}^1}\mathrm{Pr}_{\mathbf{S}}(\mathrm{Sm}_X)_*$$

where $L_{Nis,\mathbb{A}^1}\mathrm{Pr}_{\mathbf{S}}(\mathrm{Sm}_X)_*$ is the ∞-category given by the localisation of the simplicial model category of pointed simplicial presheaves on Sm_X with respect to the Nisnevich topology and $\mathbb{A}^1$-equivalences. We denote this ∞-category by S_X.

Definition 6.2.3. Let X be a Noetherian scheme. The *stable* $\mathbb{A}^1$ ∞*-category* is given by

$$\mathrm{SM}_X := \mathrm{Sp}_{\mathbb{P}^1}(\mathrm{S}_X)$$

where $\mathbb{P}^1 \simeq S^1_t \wedge S^1_s$.

Remark 6.2.4. One can invert an arbitrary object in an ∞-category through the following universal construction: let $p : C \to \Gamma$ be a symmetric monoidal ∞-category and x an object of C. Let $\mathrm{id} : \Gamma \to \Gamma$ denote the symmetric monoidal structure on $*$ and $\overline{\mathrm{id}} : \overline{\Gamma} \to \Gamma$ its group object. Then the symmetric monoidal ∞-category $p : C[x^{-1}] \to \Gamma$ given by inverting the object x in C is given by the homotopy pushout of the diagram

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d, "x"'] & \overline{\Gamma} \arrow[d] \\ C \arrow[r] & C[x^{-1}]. \end{tikzcd}$$

There exists an equivalence

$$\mathrm{SM}_X \to \mathrm{S}_X[(\mathbb{P}^1)^{-1}]$$

of ∞-categories.

The natural map $\mathrm{Pr}_{\mathbf{S}}(\mathrm{Sm}_{\mathbb{C}}) \to \mathrm{Pr}_{\mathbf{S}}(*) = \mathbf{S}$ induces a natural map $\widehat{\omega} : \mathrm{SM}_{\mathbb{C}} \to \mathrm{Sp}$.

Proposition 6.2.5. *Assume that* $\widehat{\omega} : \mathrm{SM}_{\mathbb{C}} \to \mathrm{Sp}$ *is conservative and creates a t-structure. Then* $(\mathrm{SM}_{\mathbb{C}}, \omega)$ *is a positive Tannakian* ∞*-category over the sphere spectrum* $\mathbb{S}$*.*

Therefore there exists an equivalence

$$\mathrm{SM}^{\mathrm{rig}}_{\mathbb{C}} \to \mathrm{Perf}(\widetilde{\mathrm{B}}G)$$

of ∞-categories where $G = \mathrm{End}^{\otimes}(\omega)$ is a positive affine group stack. We also have a well defined functor

$$\mathrm{h}(\mathrm{SM}_{\mathbb{C}}) \to \mathrm{DM}_{\mathbb{C}}$$

of triangulated categories.

We will now describe a higher Tannakian result in the context of the theory of non-commutative motives [Ko][CT]. Let C be a small dg-category and $C(k)$ be endowed with its natural $C(k)$-enrichment. Let Mod_C be the $C(k)$-enriched model category of C-modules of Example 3.6.5. We denote by $D(C) := \mathrm{h}\,\mathrm{Mod}_C$ the *derived category* of C. In Example 3.6.9 we also briefly described a model structure on the category $\mathrm{Cat}(C(k))$ of dg-categories denoted by $\mathrm{Cat}(C(k))_{\mathscr{T}}$ where the weak equivalences were given by the quasi-equivalences. There exists another cofibrantly generated model structure on the category of dg-categories called the *Morita* model structure where for two dg-categories C and D, the weak equivalences are given by $C(k)$-enriched functors $f : C \to D$ such that $f^* : D(D) \to D(C)$ is an equivalence of triangulated categories [Tb3]. This model category will be denoted $\mathrm{Cat}(C(k))_{\mathscr{M}}$. We consider the associated ∞-category $L\mathrm{Cat}(C(k))_{\mathscr{M}}$ and its corresponding rigidification $(L\mathrm{Cat}(C(k))_{\mathscr{M}})^{\mathrm{rig}}$. The objects in this ∞-category will be called *dualizable* dg-categories.

Remark 6.2.6. Let C be a small dg-category. Then the fully faithful $C(k)$-enriched yoneda embedding $y : C \to \mathrm{Mod}_{C^{op}}$ reduces to a quasi-fully faithful dg-functor $y : C \to (\mathrm{Mod}_{C^{op}})^{\circ}$ since $y(x)$ is both a fibrant and cofibrant C^{op}-module for all x in C. An object in the essential image of the fully faithful functor $y : H^0(C) \to D(C^{op})$ (induced by passage to the homotopy category) will be called *quasi-representable*. A dg-category C is said to be *triangulated* if every compact object in $D(C^{op})$ is quasi-representable. The full subcategory of the homotopy category of $\mathrm{Cat}(C(k))_{\mathscr{T}}$ spanned by the triangulated dg-categories is equivalent to the homotopy category of $\mathrm{Cat}(C(k))_{\mathscr{M}}$.

The natural closed symmetric monoidal structure on $\mathrm{Cat}(C(k))$ does not endow $\mathrm{Cat}(C(k))_{\mathscr{M}}$ with the structure of a symmetric monoidal model category and hence the ∞-category of dualizable dg-categories does not inherit the structure of a symmetric monoidal ∞-category (see Section 4 of [T3]). The problem is that the tensor product bifunctor is not left Quillen since it does not preserve cofibrant objects. As a result, there does not exist internal Hom objects in $\mathrm{hCat}(C(k))_{\mathscr{M}}$. A remedy for this is to consider the following derived tensor product

$$\begin{aligned} \bullet \otimes^{\mathbb{L}} \bullet : \mathrm{Cat}(C(k))_{\mathscr{M}} \times \mathrm{Cat}(C(k))_{\mathscr{M}} &\to \mathrm{Cat}(C(k))_{\mathscr{M}} \\ (C, D) &\mapsto Q(C) \otimes D \end{aligned}$$

where Q is the cofibrant replacement functor in $\mathrm{Cat}(C(k))_{\mathscr{M}}$.

It follows from Theorem 4.8 of [CT] that dualizable dg-categories can be characterized as follows. Let C be a small dg-category. A C-module is said to be *perfect* if it is a compact object of the derived category $D(C)$. A small dg-category C is said to be smooth if the dg-module

$$C(\bullet, \bullet) : C \otimes^{\mathbb{L}} C^{op} \to C(k)$$

is perfect. It is called *proper* if for any two objects x and y in C, the complex $C(x, y)$ is perfect. Let $\mathrm{Cat}(C(k))^{\mathrm{sat}}_{\mathscr{M}}$ denote the full subcategory of $\mathrm{Cat}(C(k))_{\mathscr{M}}$ spanned by the smooth and proper dg-categories (called *saturated* dg-categories in the literature). Then there exists an equivalence

$$L\mathrm{Cat}(C(k))^{\mathrm{rig}}_{\mathscr{M}} \to L\mathrm{Cat}(C(k))^{\mathrm{sat}}_{\mathscr{M}}$$

of ∞-categories.

We denote by $\mathrm{Mod}^{\mathfrak{c}}_{C^{op}}$ the full subcategory of $\mathrm{Mod}_{C^{op}}$ spanned by the C^{op}-modules which are cofibrant and compact (in $D(C^{op})$). Then $\mathrm{Mod}^{\mathfrak{c}}_{C^{op}}$ is an exact complicial category in the sense of [Sch]: the cofibrations are the cofibrations for the projective model structure on $\mathrm{Mod}_{C^{op}}$ and the equivalences are the quasi-isomorphisms. Therefore, one can define a (non-connective) spectrum $\mathbb{K}(C)$ using the construction of Schlichting in Chapter 3 of *loc. cit.*.

Definition 6.2.7. Let C be a dg-category. Then the *K-theory spectrum* $\mathbb{K}(C)$ of C is the spectrum $\mathbb{K}(\mathrm{Mod}^{\mathfrak{c}}_{C^{op}})$.

Notation 6.2.8. Let KPM_k denote the following spectral category:

- The objects are dualizable dg-categories.
- Let C and D be two dualizable dg-categories. Then $\mathrm{Map}_{\mathrm{KPM}_k}(C, D) := \mathbb{K}(C^{op} \otimes^{\mathbb{L}} D)$.
- Composition is given by the derived tensor product of bimodules.

Let C be a spectral category. Then we have at our disposal the **Sp**-enriched yoneda embedding $C \to \mathrm{Mod}_{C^{op}}$ which induces a fully faithful morphism $y : [C] \to D(C^{op})$. An object in the essential image of this functor, called *quasi-representable*, is always compact. A spectral category C is said to be *triangulated* if every compact object in $D(C^{op})$ is quasi-representable, ie. there exists an equivalence

$$[C] \to D(C^{op})^{\mathrm{cpt}}$$

of categories. We denote the full subcategory of $\mathrm{hCat(Sp)}$ spanned by the triangulated spectral categories by $\mathrm{hCat(Sp)}^{\mathrm{tri}}$. By Theorem 5.1.4 of [Tb4] there exists a left adjoint

$$(\bullet)^{\mathrm{tri}} : \mathrm{hCat(Sp)} \to \mathrm{hCat(Sp)}^{\mathrm{tri}}$$

to the inclusion $\mathrm{hCat(Sp)}^{\mathrm{tri}} \hookrightarrow \mathrm{hCat(Sp)}$ called the *triangulated hull*. We donote by KMM_k the spectral category obtained from $\mathrm{KPM}^{\mathrm{tri}}_k$ formed by formally adding direct summands for projectors. By Proposition 8.5 of [CT], the spectral category KMM_k is endowed with a compatible symmetric monoidal structure.

Definition 6.2.9. The ∞-category of *non-commutative motives* is given by $L^{\otimes}\mathrm{KMM}_k$.

We will denote this symmetric monoidal ∞-category by Mot_k^{nc}. Let BU be the topological K-theory spectrum (see [Ka] for further discussion).

Proposition 6.2.10. *Assume that* $\omega : \mathrm{Mot}_{\mathbb{C}}^{nc} \to \mathrm{Mod}_{BU}$ *is conservative and creates a t-structure. Then* $(\mathrm{Mot}_{\mathbb{C}}^{nc}, \omega)$ *is a positive* $\mathbb{S}$*-Tannakian* ∞*-category.*

7 Appendix

In Section 7.1 we state without proof some results in the theory of model categories, in particular, to enriched model category theory. Particularly important is the (enriched) projective model structure (Proposition 7.1.17). Section 7.2 begins by describing the notion of adjunction data in an $(\infty, 2)$-category. This is motivated by the fact that an adjunction in an $(\infty, 2)$-category does not determine a monad like in the 2-categorical case. An adjunction datum between two objects in an $(\infty, 2)$-category is a structure which determines an adjunction between the two objects but also the natural monad provided by the data making up the adjunction. We sketch a proof in Proposition 7.2.6 stating that any functor between two objects in an $(\infty, 2)$-category admitting a right adjoint can be promoted in an essentially unique way to an adjunction datum.

7.1 Enriched monoidal model categories

In this appendix we gather together some results in the theory of model categories referred to in the text. We refer the reader to [Ho], [Hi] and the appendix of [Lu] for further details.

Definition 7.1.1. Let $\mathscr{M}$, $\mathscr{N}$ and $\mathscr{P}$ be model categories. A functor $F : \mathscr{M} \times \mathscr{N} \to \mathscr{P}$ is said to be a *left Quillen bifunctor* if it satisfies the following:

1. Let $i : X \to X'$ and $j : Y \to Y'$ be cofibrations in $\mathscr{M}$ and $\mathscr{N}$, respectively. Then the induced map
$$i \wedge j : F(X', Y) \coprod_{F(X,Y)} F(X, Y') \to F(X', Y')$$
is a cofibration in $\mathscr{P}$. Moreover, if either i or j is a trivial cofibration, then $i \wedge j$ is also.
2. The functor F preserves small colimits seperately in each variable.

Definition 7.1.2. Let $\mathscr{M}$ be a monoidal category. Then $\mathscr{M}$ is said to be a *(symmetric) monoidal model category* if it is equipped with a model structure such that

1. The tensor product functor $\otimes : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ is a left Quillen bifunctor.
2. The (symmetric) monoidal structure on $\mathscr{M}$ is closed.
3. The unit object $1_{\mathscr{M}}$ of $\mathscr{M}$ is cofibrant.

Example 7.1.3. A model category is said to be *cartesian* if it is a symmetric monoidal model category with respect to the cartesian product.

Definition 7.1.4. Let $(A, \otimes)$ be a monoidal category. An *A-enriched category C* consists of the following data:

1. A set of objects $\mathrm{Ob}(C)$.
2. For each pair of objects $x, y \in C$, a mapping object $\mathrm{Map}_C(x, y)$ of A.
3. For every triple of objects $x, y, z \in C$, a composition map $\mathrm{Map}_C(y, z) \otimes \mathrm{Map}_C(x, y) \to \mathrm{Map}_C(x, z)$ which is associative.
4. For every object $x \in C$, a unit map $j_x : 1_A \to \mathrm{Map}_C(x, x)$ such that the following compositions
$$\mathrm{Map}_C(x, y) \otimes 1_A \xrightarrow{\mathrm{id} \otimes j_x} \mathrm{Map}_C(x, y) \otimes \mathrm{Map}_C(x, x) \xrightarrow{\circ} \mathrm{Map}_C(x, y)$$
$$1_A \otimes \mathrm{Map}_C(x, y) \xrightarrow{j_y \otimes \mathrm{id}} \mathrm{Map}_C(y, y) \otimes \mathrm{Map}_C(x, y) \xrightarrow{\circ} \mathrm{Map}_C(x, y)$$
coincide with the left and right unit maps of the monoidal structure on A.

Definition 7.1.5. Let $(A, \otimes)$ be a monoidal category and C and D be A-enriched categories. An *A-enriched functor* $F : C \to D$ consists of the following data:

1. A map $F : \mathrm{Ob}(C) \to \mathrm{Ob}(D)$.
2. For each pair of objects $x, y \in C$, a map $F_{x,y} : \mathrm{Map}_C(x, y) \to \mathrm{Map}_D(Fx, Fy)$ such that for every $z \in C$, the diagram

$$\begin{tikzcd} \mathrm{Map}_C(y,z) \otimes \mathrm{Map}_C(x,y) \arrow[r, "\circ"] \arrow[d, "F_{y,z} \otimes F_{x,y}"'] & \mathrm{Map}_C(x,z) \arrow[d, "F_{x,z}"] \\ \mathrm{Map}_D(Fy,Fz) \otimes \mathrm{Map}_D(Fx,Fy) \arrow[r, "\circ"] & \mathrm{Map}_D(Fx,Fz) \end{tikzcd}$$

commutes and for every $x \in C$, the composition

$$1_A \xrightarrow{j_x} \mathrm{Map}_C(x, x) \xrightarrow{F_{x,x}} \mathrm{Map}_D(Fx, Fx)$$

coincides with the unit map $j_{Fx,Fx}$ for $F(x) \in D$.

Definition 7.1.6. Let $\mathscr{M}$ be a monoidal model category. An $\mathscr{M}$-enriched category $\mathscr{A}$ is said to be a *$\mathscr{M}$-enriched model category* if it is equipped with a model structure such that

1. The category $\mathscr{A}$ is tensored and cotensored over $\mathscr{M}$.
2. The tensor product functor $\otimes : \mathscr{M} \times \mathscr{A} \to \mathscr{A}$ is a left Quillen bifunctor.

Example 7.1.7. A *simplicial model category* is an $\mathbf{S}$-enriched model category where $\mathbf{S}$ is endowed with the cartesian monoidal structure and the Kan model structure.

Example 7.1.8. Let k be a commutative ring. The category $C(k)$ of (unbounded) complexes of k-modules admits a cofibrantly generated model structure where the fibrations are degree-wise surjective morphisms and the equivalences are the quasi-isomorphisms (those maps inducing isomorphisms on homology groups) (see Theorem 2.3.11 of [Ho]).

Definition 7.1.9. Let A be a monoidal category and C an A-enriched category. A (symmetric) monoidal structure on C is said to be *weakly compatible* with the A-enriched structure on C if the bifunctor $\otimes : C \times C \to C$ admits the structure of an A-enriched functor which is compatible with the (commutativity and) associativity and unit constraints of $(C, \otimes)$.

Definition 7.1.10. Let $\mathscr{M}$ be a monoidal model category. A (symmetric) monoidal category $\mathscr{A}$ is said to be a *(symmetric) monoidal $\mathscr{M}$-enriched model category* if $\mathscr{A}$ is an $\mathscr{M}$-enriched model category with a weakly compatible monoidal structure and such that the natural maps

$$\mathrm{Map}_{\mathscr{A}}(y, {}^{x}z) \to \mathrm{Map}_{\mathscr{A}}(x \otimes y, z) \leftarrow \mathrm{Map}_{\mathscr{A}}(x, z^{y})$$

are bijections in $\mathrm{h}\mathscr{M}$ for all $x, y, z \in \mathscr{A}$.

Definition 7.1.11. Let $\mathscr{M}$ be a model category. Then $\mathscr{M}$ is said to be *left proper* if for any cofibration $f : z \to x$ and weak equivalence $g : z \to y$ in $\mathscr{M}$, the map $x \to x \coprod_z y$ is a weak equivalence. Similarly, $\mathscr{M}$ is said to be *right proper* if for any fibration $f : x \to z$ and weak equivalence $g : y \to z$ in $\mathscr{M}$, the map $x \times_z y \to x$ is a weak equivalence.

Example 7.1.12. The category of simplicial sets with the Kan model structure $\mathbf{S}_{\mathscr{K}}$ is left and right proper. However, the category of simplicial sets with the Joyal model structure $\mathbf{S}_{\mathscr{J}}$ is only left proper.

Definition 7.1.13. A model category $\mathscr{M}$ is said to be *combinatorial* if

1. The category $\mathscr{M}$ is presentable.

2. There exists a set I, called the set of *generating cofibrations*, such that the collection of all cofibrations in $\mathscr{M}$ is the smallest weakly saturated class of morphisms containing I.

3. There exists a set J, called the set of *generating trivial cofibrations*, such that the collection of all trivial cofibrations in $\mathscr{M}$ is the smallest weakly saturated class of morphisms containing J.

Condition (3) pertaining to Definition 7.1.13 can be replaced by the following often more useful condition: there exists a regular cardinal κ such that the collection W of weak equivalences in $\mathscr{M}$ determines a full subcategory of $\mathscr{M}^{[1]}$ which is stable under κ-filtered colimits and for which there exists a (small) set of objects of W which generates W under κ-filtered colimits.

Example 7.1.14. An important example of combinatorial, **S**-enriched monoidal model category is the category **Sp** of symmetric spectra with the smash product monoidal structure and $\mathbb{S}$-model structure [HSS][Sh]. Let M be the class of monomorphisms in the category of symmetric sequences. The cofibrations in the $\mathbb{S}$-model structure are the maps with the left lifting property with respect to those maps with the right lifting property with respect to the maps in $\mathbb{S}\otimes M$. The weak equivalences are the stable equivalences: those maps $f : X \to Y$ such that

$$\mathrm{Map}_{\mathbf{Sp}}(Y, A) \to \mathrm{Map}_{\mathbf{Sp}}(X, A)$$

is a weak equivalence for every fibrant Ω-spectrum A with respect to the injective model structure on **Sp**. Using the $\mathbb{S}$-model structure, one can construct a model structure on the category of commutative R-algebras for R a commutative symmetric ring spectrum. This is called the R-model structure. A fibration is a map with the right lifting property with respect to those maps which are stable equivalences, having the left lifting property with respect to those maps of R-modules with the right lifting property with respect to the maps in $R \otimes M$ and which are isomorphisms in level zero.

Definition 7.1.15. A model category $\mathscr{M}$ is said to be *excellent* if it is endowed with a symmetric monoidal structure and satisfies the following conditions:

1. The model category $\mathscr{M}$ is combinatorial.

2. Every monomorphism in $\mathscr{M}$ is a cofibration and the collection of cofibrations is stable under products.

3. The tensor product bifunctor $\otimes : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ is a left Quillen bifunctor.

4. The collection of weak equivalences in $\mathscr{M}$ is stable under filtered colimits.

5. For every $\mathscr{M}$-enriched category C containing an equivalence f, the map $C \to C[f^{-1}]$ is a weak equivalence of $\mathscr{M}$-enriched categories.

Example 7.1.16. The model categories $\mathbf{S}_{\mathscr{K}}$ and $\mathbf{S}_{\mathscr{J}}$ are excellent model categories. The model categories $C(k)$ and **Sp** are not.

Proposition 7.1.17. *Let $\mathscr{M}$ be an excellent model category, $\mathscr{A}$ a combinatorial $\mathscr{M}$-enriched model category and C a $\mathscr{M}$-enriched category. Then there exists a combinatorial model structure on the category $\mathscr{A}^C$ of $\mathscr{M}$-enriched functors from C to $\mathscr{A}$ consisting of*

($\mathscr{F}$) The fibrations are the maps $F \to G$ such that $F(x) \to G(x)$ is a fibration in $\mathscr{A}$ for every $x \in C$.

($\mathscr{W}$) The weak equivalences are the maps $F \to G$ such that $F(x) \to G(x)$ is a weak equivalence in $\mathscr{A}$ for every $x \in C$.

Proof reference. See Proposition A.3.3.2. of [Lu]. □

This is called the *projective* model structure on $\mathscr{A}^C$.

Remark 7.1.18. There also exists an analogous $\mathscr{M}$-enriched combinatorial model structure on $\mathscr{A}^C$ called the *injective* model structure where the cofibrations and weak equivalences are defined objectwise.

Proposition 7.1.19. *Let $\mathscr{M}$ be an excellent model category, $\mathscr{A}$ a combinatorial $\mathscr{M}$-enriched model category and C a $\mathscr{M}$-enriched category. Then $\mathscr{A}^C$ is an $\mathscr{M}$-enriched model category with respect to the projective model structure.*

Proof reference. See Proposition A.3.3.2 and Remark A.3.3.4 of [Lu]. □

Proposition 7.1.20. *Let $\mathscr{M}$ be an excellent model category, C an $\mathscr{M}$-enriched category and $F : \mathscr{A} \rightleftarrows \mathscr{B} : G$ a Quillen equivalence between $\mathscr{M}$-enriched model categories. Then composition with F and G determines a Quillen equivalence*

$$F^C : \mathscr{A}^C \rightleftarrows \mathscr{B}^C : G^C$$

between $\mathscr{M}$-enriched model categories with respect to the projective model structure.

Proof reference. See Proposition A.3.3.6 of [Lu]. □

Proposition 7.1.21. *Let $\mathscr{M}$ be an excellent model category, $\mathscr{A}$ an $\mathscr{M}$-enriched model category and $F : C \to D$ an equivalence between $\mathscr{M}$-enriched categories. Let $F^* : \mathscr{A}^D \to \mathscr{A}^C$ be given by composition with F and $F_!$ its left adjoint. Then there exists a Quillen equivalence*

$$F_! : \mathscr{A}^C \rightleftarrows \mathscr{A}^D : F^*$$

between $\mathscr{M}$-enriched model categories with respect to the projective model structure.

Proof reference. See Proposition A.3.3.8 of [Lu]. □

Remark 7.1.22. Analogous results can be formed with respect to the injective model structure.

The theory of localisations presented in Section 2.1 has an analogue in the setting of model categories. Let $\mathscr{M}$ be a simplicial model category and S a set of arrows in $\mathrm{h}\mathscr{M}$. An object z in $\mathrm{h}\mathscr{M}$ is said to be *S-local* if, for every arrow $f : x \to y$ in S, the induced map

$$\mathrm{Map}_{\mathrm{h}\mathscr{M}}(y, z) \to \mathrm{Map}_{\mathrm{h}\mathscr{M}}(x, z)$$

is a homotopy equivalence. An object z' in $\mathscr{M}$ is said to be S-local if its image in $\mathrm{h}\mathscr{M}$ is S-local. An arrow $f : x \to y$ in $\mathrm{h}\mathscr{M}$ is said to be an *S-equivalence* if, for every S-local object z in $\mathrm{h}\mathscr{M}$, the induced map

$$\mathrm{Map}_{\mathrm{h}\mathscr{M}}(y, z) \to \mathrm{Map}_{\mathrm{h}\mathscr{M}}(x, z)$$

is a homotopy equivalence. An arrow f' in $\mathscr{M}$ is said to be an S-equivalence if its image in $\mathrm{h}\mathscr{M}$ is an S-equivalence.

Proposition 7.1.23. *Let $\mathscr{M}$ be a left-proper combinatorial simplicial model category and S a (small) set of cofibrations in $\mathscr{M}$. Then there exists another left-proper combinatorial simplicial model structure on the underlying category of $\mathscr{M}$ where:*

($\mathscr{C}$) A cofibration is a map which is a cofibration when regarded as a morphism of $\mathscr{M}$.

($\mathscr{W}$) The weak equivalences are the S-equivalences.

The fibrant objects coincide with objects of $\mathscr{M}$ which are both S-local and fibrant in $\mathscr{M}$.

Proof reference. See Proposition A.3.7.3 of [Lu]. □

The model structure is called the *Bousfield localisation* of $\mathscr{M}$ with respect to S and will be denoted $L_S^B\mathscr{M}$.

Proposition 7.1.24. *Let $\mathscr{M}$ and $\mathscr{N}$ be left proper combinatorial simplicial model categories and S a (small) set of cofibrations in $\mathscr{M}$. If $F : \mathscr{M} \rightleftarrows \mathscr{N} : G$ is a Quillen equivalence then the induced Quillen adjunction*

$$F : L_S^B \mathscr{M} \rightleftarrows L^B_{\mathbb{L}F(S)}(\mathscr{N}) : G$$

is a Quillen equivalence.

Proof. This follows from Theorem 3.3.20 of [Hi]. □

Remark 7.1.25. By a result of Dugger [Du], every combinatorial model category is Quillen equivalent to a simplicial model category which is left proper and in which every object is cofibrant. This simplicial model category is given by the Bousfield localisation of the model category of simplicial presheaves on a small category.

7.2 Adjunction data in an $(\infty, 2)$-category

A monad in an ordinary category C is simply a monoid in the category $\underline{\mathrm{Hom}}(C, C)$ of endofunctors on C provided with the composition monoidal structure. Every adjunction $(\alpha, \beta) : F \dashv G$ in a 2-category determines a monad: the composite $M = G \circ F$ is the underlying endofunctor, the unit α of the adjunction is the unit map for the monoid and the monoid product is given by $G \circ \beta \circ F : M^2 \to M$. The most common example is the monad determined by the adjunction in the 2-category $\underline{\mathrm{Cat}}$ of categories.

Our aim in this section is to formulate an $(\infty, 2)$-categorical analogue of this construction. However, an adjunction in an $(\infty, 2)$-category in the sense of Definition 2.3.2 does not determine a monad as in the 2-categorical case. To do so we need to construct another equivalent notion of an adjunction in an $(\infty, 2)$-category where we specify (non-identity) 2-morphisms $(\beta \otimes \mathrm{id}_F) \circ (\mathrm{id}_F \otimes \alpha) \Rightarrow \mathrm{id}_F$ and $(\mathrm{id}_G \otimes \beta) \circ (\alpha \otimes \mathrm{id}_G) \Rightarrow \mathrm{id}_G$ together with all their higher dimensional homotopies. All this data is contained in an object which is called the adjunction datum of an $(\infty, 2)$-category.

Definition 7.2.1. Let $S = \{x, y\}$ where x and y are two objects in an arbitrary $(\infty, 2)$-category. An arrow $u : ([n], c) \to ([m], d)$ in Δ_S is said to be *x-inert* if, whenever $k \in [m]$ satisfies $d(k) = x$ and $u(i) \leq k \leq u(i')$ for suitably chosen $i, i' \in [n]$, there exists a unique $j \in [n]$ such that $u(j) = k$.

Let (T, C) be an $(\infty, 2)$-category and $S := \{x, y\} \subseteq T$. The full subcategory of C spanned by the objects S will be denoted by C_S, ie. C_S is a $\mathscr{C}at_\infty$-precategory $C_S : \Delta_S^{op} \to \mathscr{C}at_\infty$ satisfying the Segal condition. We denote by $p : \int_{\Delta_S^{op}} C_S \to \Delta_S^{op}$ its corresponding integral. Note that $\int_{\Delta_S^{op}} * = \Delta_S^{op}$.

Definition 7.2.2. Let C be an $(\infty, 2)$-category and $S = \{x, y\}$ two objects of C. The ∞-category of *adjunction data* between x and y is the full subcategory of $\mathbb{R}\underline{\mathrm{Hom}}_{\Delta_S^{op}}(\Delta_S^{op}, \int_{\Delta_S^{op}} C_S)$ which carries every x-inert arrow in Δ_S to a p-cocartesian arrow in $\int_{\Delta_S^{op}} C_S$.

The ∞-category of adjunction data between x and y in the set $S = \{x, y\}$ in C will be denoted $\mathrm{ADat}_S(C)$ or simply ADat_S if the $(\infty, 2)$-category C is clear from the context. Observe that there exists a natural restriction map $\Delta_{\{x,y\}} \to \Delta_{\{x\}}$ where $\Delta_{\{x\}}$ is the full subcategory of $\Delta_{\{x,y\}}$ spanned by the objects $([n], c)$ such that $c : [n] \to \{x\} \subset \{x, y\}$ only takes values in the object x. Applying this restriction map to the functors $*$ and C_S determines a map $\mathrm{ADat}_{x,y}(C) \to \mathrm{ADat}_x(C)$ where $\mathrm{ADat}_x(C)$ is canonically isomorphic to the ∞-category $\mathrm{Mon}(\mathrm{End}(x))$. Thus every adjunction datum determines in this way a monad on x (see Example 3.1.14 and note that $\Delta_{\{x\}} \simeq \Delta$).

Each adjunction datum $\mathfrak{a}$ in $\mathrm{ADat}_S(C)$ determines (as a small subset) the following data:

- A functor $f : x \to y$ determined by $\mathfrak{a}(x, y)$.

- A functor $g : y \to x$ determined by $\mathfrak{a}(y, x)$.

- A functor $T : x \to x$ together with a natural transformation $\mathrm{id}_x \to T$ determined by the pair $(\mathfrak{a}(x,x), \mathfrak{a}(u))$ where $u : (x,x) \to (x)$. By the preceding discussion, this is the unit map for a monad T on x (note that the product $T \circ T \to T$ for the monad is induced from the three inclusions $(x,x) \to (x,x,x)$ two of which are x-inert). Moreover, the map $\mathfrak{a}$ acting on the x-inert inclusions of (x,y), (y,x) and (x,x) into (x,y,x) determine equivalences $f' \to f$, $g' \to g$ and $g' \circ f' \to T$ respectively. Hence we obtain a natural transformation $\mathrm{id}_x \to g \circ f$ (well defined up to homotopy) for a monad T of the form $g \circ f$.
- A functor $U : y \to y$ together with a natural equivalence $\mathrm{id}_y \to U$ determined by the pair $(\mathfrak{a}(y,y), \mathfrak{a}(v))$ for the x-convex map $v : (y,y) \to (y)$. Moreover, the map $\mathfrak{a}$ acting on the x-inert inclusions of (x,y) and (y,x) into (y,x,y) determine equivalences $f'' \to f$ and $g'' \to g$ respectively. Combining this result with the map $f'' \circ g'' \to U$ determined by the action of the map $\mathfrak{a}$ on the inclusion of (y,y) into (y,x,y) determines a natural transformation $f \circ g \to U$. Hence we obtain a natural transformation $f \circ g \to \mathrm{id}_y$ (well defined up to homotopy).

We will show that this subset determines an adjunction between x and y in the $(\infty,2)$-category C. The proof is similar to the special case of $C = \underline{\mathrm{Cat}}_\infty$ exposed in Lemma 3.2.9 of [LII].

Lemma 7.2.3. *Let C be an $(\infty,2)$-category and x and y two objects of C. Let*

$$f : x \to y \qquad \alpha : \mathrm{id}_x \to g \circ f \qquad \beta : f \circ g \to \mathrm{id}_y \qquad g : y \to x$$

denote the maps obtained from an adjunction datum $\mathfrak{a}$ in $\mathrm{ADat}_{x,y}(C)$. Then $(\alpha, \beta) : f \dashv g$ is an adjunction between f and g.

Proof. We need to show that the adjunction identities

$$(\beta \circ \mathrm{id}_f)(\mathrm{id}_f \circ \alpha) = \mathrm{id}_f \qquad (\mathrm{id}_g \circ \beta)(\alpha \circ \mathrm{id}_g) = \mathrm{id}_g$$

hold in $\mathrm{h}_2(C)$. For the first identity, consider the following commutative diagram in $\Delta_{\{x,y\}}$:

$$\begin{tikzcd}
 & (x,y) \arrow[r] \arrow[d] & (x,x,y) \arrow[r] \arrow[d] & (x,y) \\
(x,y) & (x,y,y) \arrow[l] \arrow[r] & (x,y,x,y) &
\end{tikzcd}$$

where the maps preserve both the first and last object. Applying $\mathfrak{a}$ we obtain a diagram in the 2-category $\mathrm{h}_2(C)$ equivalent to

$$\begin{tikzcd}
 & f & f \circ (g \circ f) \arrow[l, "\gamma"'] & f \arrow[l, "\mathrm{id}_f \circ \alpha"'] \\
f \arrow[r, "\mathrm{id}"] & f \arrow[u, "\mathrm{id}"] & (f \circ g) \circ f. \arrow[l, "\beta \circ f"'] \arrow[u, "\mathrm{id}"'] &
\end{tikzcd}$$

Since the square commutes (up to homotopy) and $\gamma \circ (\mathrm{id}_f \circ \alpha)$ is (homotopic to) the identity, this diagram gives the first identity. The second identity follows from exchanging x and y in the diagrams above. Thus f admits a right adjoint which can be identified (up to homotopy) with g. □

Let $\mathrm{Cat}^{**}_{(\infty,2)} := (\mathrm{Cat}_{(\infty,2)})_{* \coprod */}$ denote the category of $(\infty,2)$-categories with a fixed set of two objects. We have a well defined functor

$$F : \mathrm{Cat}^{**}_{(\infty,2)} \to \mathscr{C}at_\infty$$
$$(S, C) \mapsto \mathrm{ADat}_S(C)$$

between ∞-categories. Let Set^{**} denote the set of isomorphism classes of objects of $\mathrm{Cat}^{\infty}_{(\infty,2)}$ with two objects. We define $\mathrm{ADat}(C)$ to be the pullback

$$\mathrm{ADat}(C) := (\int_{\mathrm{Cat}^{**}_{(\infty,2)}} F) \times_{\mathrm{Set}^{**}} \{C\}$$

in $\mathrm{Cat}^{\infty}_{\infty}$. In other words, $\mathrm{ADat}(C)$ is the cofibered ∞-category $\int_{**_C} F_C \to **_C$ where $**_C$ is the category whose objects are pairs S of objects of C and whose arrows between two objects S and S' is a map f in C such that $fS = S'$, and F_C is the functor $F_C : **_C \to \mathscr{C}at_\infty$ which sends $S \mapsto \mathrm{ADat}_S(C)$. Note that we have an equivalence

$$\mathrm{ADat}_S(C) \simeq \mathrm{ADat}(C) \times_{C \times C} \{S\}$$

of ∞-categories. We obtain a well defined functor

$$\begin{aligned} \mathcal{A} : \mathrm{Cat}^{\infty}_{(\infty,2)} &\to \mathcal{K} \\ C &\mapsto \mathfrak{K}^1(\mathrm{ADat}(C)) \end{aligned}$$

between ∞-categories. Note that we only consider the largest sub-$(\infty,0)$-category contained in the ∞-category $\mathrm{ADat}(C)$ since we are not interested in non-invertible maps between adjunctions.

Proposition 7.2.4. *The functor $\mathcal{A}$ is corepresentable.*

Proof. It suffices to prove that $\mathcal{A}$ is accessible and preserves (small) limits. We first prove the accessibility of $\mathcal{A}$. Since $\mathrm{Cat}^{\infty}_{(\infty,2)} := L(\mathscr{C}at_{(\infty,2)})$ is the localisation of a model category, it follows from Proposition 2.3.20 that $\mathrm{Cat}^{\infty}_{(\infty,2)}$ is presentable. It remains to show that $\mathcal{A}$ preserves filtered colimits. Let I be a filtered $(\infty,2)$-category and set $C := \mathrm{colim}_{i\in I} C_i$. Then $\mathcal{A}(C) := \int F_C \simeq F_C$. We evaluate on an object S in $**_C$ to find $F_C(S) = \mathrm{ADat}_S(C) = \mathbb{R}\underline{\mathrm{Hom}}^{in}_{\Delta^{op}_S}(\Delta^{op}_S, \int_{\Delta^{op}_S} C_S)$. Since Δ^{op}_S is countable we have an equivalence

$$\mathbb{R}\underline{\mathrm{Hom}}_{\Delta^{op}_S}(\Delta^{op}_S, \int_{\Delta^{op}_S} C_S) \simeq \operatorname*{colim}_{i\in I} \mathbb{R}\underline{\mathrm{Hom}}_{\Delta^{op}_S}(\Delta^{op}_S, \int_{\Delta^{op}_S} (C_i)_S)$$

since I in γ-filtered for $\gamma > \omega$. Adding the inert conditions preserves this equivalence and so $\mathcal{A}(C) \simeq \mathrm{colim}_{i\in I}\mathcal{A}(C_i)$. It is left to prove that $\mathcal{A}$ preserves limits. $\square$

The next main result shows that every functor between objects in an $(\infty,2)$-category which admits a right adjoint can be extended (uniquely up to a contractible space of choices) to an adjunction datum. First we will need a small lemma. Let C and D be (∞,n)-categories. A functor $f : C \to D$ is said to be *conservative* if given an arrow u in C such that $f(u)$ is an equivalence in D then u is an equivalence in C.

Lemma 7.2.5. *Let C and D be (∞,n)-categories and $f : C \to D$ a functor. Consider the induced map*

$$f_* : \mathbb{R}\underline{\mathrm{Hom}}(A,C) \to \mathbb{R}\underline{\mathrm{Hom}}(A,D)$$

of (∞,n)-categories for any (∞,n)-precategory A.

1. *If the functor f is fully faithful then the induced map f_* is fully faithful.*
2. *If the functor f is conservative then the induced map f_* is conservative.*

Proof. For (1) we factor $f : C \to D$ as $C \to C' \to D$ where C' is the essential image of f in D. Since f is fully faithful we have a cartesian square

$$\begin{array}{ccc} \mathrm{Map}([1]\times A, C') & \longrightarrow & \mathrm{Map}([1]\times A, D) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Map}(\partial[1]\times A, C') & \longrightarrow & \mathrm{Map}(\partial[1]\times A, D). \end{array}$$

Since the right vertical map is a fibration, this cartesian square is equivalent to the homotopy cartesian square

$$\begin{tikzcd}
\mathrm{Map}([1], C^A) \arrow[r] \arrow[d, "ev_{0,1}"'] & \mathrm{Map}([1], D^A) \arrow[d, "ev_{0,1}"] \\
\mathrm{Map}(*, C^A \times C^A) \arrow[r] & \mathrm{Map}(*, D^A \times D^A).
\end{tikzcd}$$

The result now follows from the fact that the fibers of a homotopy cartesian square are equivalent. Now let $f : C \to D$ be conservative. Then (2) follows simply from the following commutative diagram

$$\begin{tikzcd}
\mathbb{R}\underline{\mathrm{Hom}}(A, C) \arrow[r] \arrow[d, "\{ev_a\}_{a\in A}"'] \arrow[dr, "\alpha"] & \mathbb{R}\underline{\mathrm{Hom}}(A, D) \arrow[d, "\{ev_a\}_{a\in A}"] \\
C \arrow[r, "F"] & D
\end{tikzcd}$$

where the evaluation maps $\{ev_a\}$ form a conservative family and $\alpha(h)(a) = f(h(a))$. $\square$

Let C be an $(\infty, 2)$-category. Denote by $\mathfrak{K}^1 C(x, y)_\dashv$ the subcategory of the ∞-category $C(x, y)$ spanned by those objects $f : x \to y$ which admit right adjoints and whose morphisms are equivalences of these maps.

Proposition 7.2.6. *Let C be an $(\infty, 2)$-category, $S = \{x, y\}$ two objects in C and $ev : \mathrm{ADat}_S(C) \to C(x, y)$ be the evaluation map at $(x, y) \in \Delta_S$. Then ev induces an equivalence*

$$\mathrm{ADat}_S(C) \to \mathfrak{K}^1 C(x, y)_\dashv$$

of ∞-categories.

Sketch of the proof. Let $\mathrm{Map}([1], C)_\dashv$ denote the ∞-category of maps in C which admit right adjoints. We first prove that $\mathrm{ADat}(C^\wedge) \to \mathrm{Map}([1], C^\wedge)_\dashv$ is an equivalence where $C^\wedge$ is the $(\infty, 2)$-category of prestacks on C. From Proposition 7.2.4, we have an equivalence $\mathrm{ADat}(C^\wedge) \simeq \mathrm{Map}(\mathcal{A}, C^\wedge)$. We denote by $j, i : [1] \to \mathcal{A}$ the inclusion and consider the natural diagram

$$\begin{tikzcd}
\mathrm{Map}(\mathcal{A}, C^\wedge) \arrow[r, shift left, "j^*"] \arrow[d, "\{\circ ev_x\}_{x\in C}"'] & \mathrm{Map}([1], C^\wedge) \arrow[l, shift left, "j_!"] \arrow[d, "\{\circ ev_x\}_{x\in C}"] \\
\mathrm{Map}(\mathcal{A}, \underline{\mathrm{Cat}}_\infty) \arrow[r, shift left, "i^*"] & \mathrm{Map}([1], \underline{\mathrm{Cat}}_\infty) \arrow[l, shift left, "i_!"]
\end{tikzcd}$$

where the adjunction $i^* \dashv i_!$ is made up of the restriction map i^* and $i_! : f \mapsto \mathrm{Lan}_i f$ is the left Kan extension functor. The same follows for the adjunction $j^* \dashv j_!$. These adjunctions exists owing to the fact that the category $[1]$ is small and $\underline{\mathrm{Cat}}_\infty$ and $C^\wedge$ are cocomplete. By Lemma 7.2.5, the vertical evaluation maps are conservative. Furthermore, both squares commute since clearly $ev_x \circ i^*(f) \simeq i^* \circ ev_x(f)$ and

$$ev_x \circ (i_! g)(a) = ev_x \circ (\operatorname*{colim}_{i(z)\to a} g(j)) \simeq \operatorname*{colim}_{i(j)\to a} (ev_x \circ g(j)) = i_! \circ (ev_x g)(a)$$

for all $x \in C, a \in \mathcal{A}$ and $z \in [1]$ since $ev_x : C^\wedge \to \underline{\mathrm{Cat}}_\infty$ is cocontinous. By Proposition 3.2.10 of [LII], the map $\mathrm{ADat}(\underline{\mathrm{Cat}}_\infty) \to \mathrm{Map}([1], \underline{\mathrm{Cat}}_\infty)_\dashv$ is an equivalence (a map is an equivalence if it induces an equivalence on all fibers). Thus it follows that $\mathrm{ADat}(C^\wedge) \to \mathrm{Map}([1], C^\wedge)_\dashv$ is an equivalence if the following property is satisfied: a map $f \to g$ in $C^\wedge$ admits an adjoint if and only if $f(x) \to g(x)$ admits an adjoint for all $x \in C$. Let us assume this statement. Now consider the following diagram:

$$\begin{tikzcd}
\mathrm{Map}(\mathcal{A}, C) \arrow[r, shift left, "k^*"] \arrow[d, "\circ h"'] & \mathrm{Map}([1], C)_\dashv \arrow[l, shift left, "k_!"] \arrow[d, "\circ h"] \\
\mathrm{Map}(\mathcal{A}, C^\wedge) \arrow[r, shift left, "j^*"] & \mathrm{Map}([1], C^\wedge)_\dashv \arrow[l, shift left, "j_!"]
\end{tikzcd}$$

where h is the $(\infty, 2)$-yoneda embedding of Proposition 2.2.11. The map k^* is fully faithful so it remains to prove that it is essentially surjective which can be shown using the commutativity of the diagram. $\square$

Thus by Proposition 7.2.6, if an arrow $f : x \to y$ in an $(\infty, 2)$-category C admits a right adjoint then it can be extended to an adjoint pair together with a monad on x through the following diagram:

$$C(x, y)_\dashv \xleftarrow{\sim} \mathrm{ADat}_{x,y} \to \mathrm{Mon}(\mathrm{End}(x))$$

of ∞-categories.

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