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Simpson, Angus Ross; Elhay, Sylvan

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A FRAMEWORK FOR ALTERNATIVE FORMULATIONS OF THE PIPE NETWORK EQUATIONS

Angus R. Simpson[1] and Sylvan Elhay[2]

[1]School of Civil, Environmental and Mining Engineering, University of Adelaide, Adelaide SA 5005, Australia. email: asimpson@civeng.adelaide.edu.au
[2]School of Computer Science, University of Adelaide, Adelaide SA 5005, Australia. email: elhay@cs.adelaide.edu.au

Abstract

Since the late 1960s and early 1970s numerous papers have been written on the formulations and methods for solving the governing equations for flows and heads in water distribution systems. Many different names have been given to the various formulations and methods for solution used including the linear theory method, Newton's method, the Todini and Pilati method etc. The underlying equations are the flow continuity equations at nodes in the network, the head loss-flow relationships for individual pipes and finally the head losses around closed loops or for paths between fixed head nodes in the network. The set of equations is nonlinear and hence requires an iterative solution process.

The aim of this paper is to revisit the formulations of the equations for flows and heads in water distribution systems and provide clarity for a logical presentation of a framework for the different formulations. Five formulations are described including 1) flow equations where the equations are formulated only in terms of the unknown flows in a network 2) head equations 3) loop flow equations 4) flow and head equations and 5) the Todini and Pilati flow and head formulation. Graph theory is used to show how many unknowns are required to be solved for in each of the five formulations. A Newton solution method is derived for the flow formulation and the Todini and Pilati formulation.

INTRODUCTION

Pipes, pumps, tanks and valves are connected to form a water distribution system, often in a complex manner. A network for a city, small town or new subdivision is an example. Many or several loops or circuits are usually present in the network. There are usually entry points to the network (for example–tanks, storages and reservoirs) and also many withdrawal points (for example–homes, industries, parks and gardens, commercial buildings, etc.).

Nodes are defined as the end points of links in the network (pipes, pumps or valves) and are identified as either junction nodes with a variable head or fixed head nodes (for example–reservoirs). Flow may enter or leave the network at junction nodes.

There are a number of alternative formulations of the equations describing a water distribution system (Wood and Rayes (1981)–for the first three formulations; Todini and Pilati (1988) for the fifth formulation). Five formulations that are presented in detail in the paper include:

1. flow equations or *Q–equations* formulation in terms of unknown flows (*Q*s) in each link
2. head equations or *H–equations* formulation in terms of unknown heads or HGLs (*H*s) at each node
3. loop flow equations formulation or *LF–equations* in terms of unknown loop flows (*LF*s)
4. flow and head equations or *Q–H equations* formulation in terms of both the unknown flows and unknown heads

5. *Todini and Pilati Q–H equations* formulation in terms of unknown flows and unknown heads (Todini and Pilati 1988)

The objective of this paper is to provide clarity in the presentation of the five different possible formulations of the equations governing flow and head in a water distribution system network. The structure of the paper is as follows. Graph theory is briefly introduced to define the number of variables to be solved for in each formulation. The vectors and matrices required are then defined. Finally, details of the five formulations are given.

GRAPH THEORY FOR NETWORKS

Consider a network in the most general terms (based on the nomenclature of both Wood and Rayes 1981 and Boulos and Altman 1991). From graph theory it is shown that the above variables are related as follows (Boulos and Altman 1991)

$$NP = NJ + NL + (NF - NC) \tag{1}$$

where

- NP = number of links (including pipes, pumps and valves) in the network
- NJ = number of nodes in the network (excluding reservoirs or fixed head nodes))
- NL = number of closed simple loops in the network (loops that have no interior crossing links—also called non–overlapping or natural or primary loops)
- NF = number of reservoirs or fixed head nodes in the network
- NC = number of separate disconnected subnetworks. Usually NC = 1 but if a link closes in a network that results in separated parts it is possible to have $NC > 1$.

For networks containing two or more reservoirs (or other form of fixed head source), it is necessary to consider NF-NC required independent paths between nodes of fixed head in the analysis of a network. A path is defined as a non–intersecting series of links between any two fixed head nodes (e.g. reservoirs) that does not contain a closed simple loop. Consider an example network that is fully connected such that NC=1. For two reservoirs (NF=2) there is only one path between the two reservoirs (thus NF-NC= 2-1=1). For three reservoirs (NF=3) there are three possible paths between the reservoirs, however, there are only two independent paths required (NF-NC=3-1=2). It does not matter which of the two paths between the reservoirs are selected in the three reservoir case, as one path between two of the reservoirs is redundant and is not required.

Based on Eq. 1, it generally holds that there are more links NP in the network than nodes NJ such that

$$NP > NJ.$$

In addition there are usually more links NP or nodes NJ than closed simple loops NL plus required independent paths in the network (NF-NC) such that

$$NP > NL + (NF\text{-}NC).$$

Finally, for networks with at least one closed simple loop the number of nodes (NJ) exceeds the number of closed simple loops plus required independent paths

$$NJ > NL + (NF\text{-}NC).$$

As a result, the ordering from the minimum number of variables to the maximum to be solved for is shown in Table 1. A ten pipe example network is shown in Figure 1.

Table 1. Number of variables to be solved for each formulation

FORMULATION	NUMBER OF VARIABLES	NUMBER FOR TEN-PIPE NETWORK
Loop flows	$NL + (NF - NC)$	3
Heads	NJ	7
Flows	NP	10
Todini Q+H	$NP + NJ^{*}$	17/10
Flows+Heads	$NP + NJ$	17

*however, only a matrix solution of size *NJ* is required to solve for the heads (see later on in the paper)

Thus in summary, for networks that contain loops, the number of unknown loop flows (*LF*s) is always less than or equal to the number of unknown heads (*H*s) that in turn is always less than the number of unknown link flows (*Q*s).

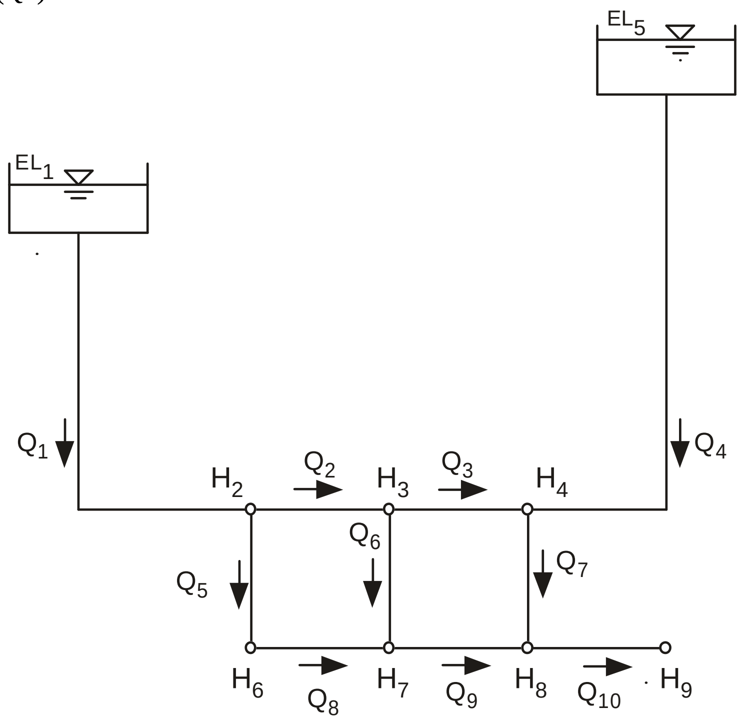

Figure 1. The unknown flows and heads in a ten-pipe network

VECTORS OF UNKNOWNS AND KNOWNS

Consider a water distribution network of pipes and junctions or nodes in which the system has *NP* pipes, *NJ* variable-head nodes, *NF* fixed-head nodes and *NL* closed loops. The number of required independent paths is (*NF*-*NC*). Assume the network is completely connected (thus *NC*=1). The general column vector of flows for *NP* pipes or links (expressed as the transpose of a row vector) for a water distribution system is

$$\mathbf{q} = [Q_1, Q_2, \ldots, Q_{NP}]^{\mathbf{T}}$$

The general column vector of heads for *NJ* nodes is

$$\mathbf{h} = [H_1, H_2, \ldots, H_{NJ}]^{\mathbf{T}}$$

The ten pipe network shown in Figure 1 has 10-unknown flows and 7-unknown heads.

The known column vector of demands for NJ nodes in a general network is

$$\mathbf{dm} = [DM_1, DM_2, \ldots, DM_{NJ}]^{\mathbf{T}}$$

In the third formulation of the equations for a water distribution system unknown loop flows will be solved for. A vector of unknowns is defined as

$$\mathbf{u} = [LF_1, LF_2, \ldots, LF_{NTL}]^{\mathbf{T}}$$

where the number of closed loops and required independent paths is $NTL = NL + (NF - NC)$.

TOPOLOGY MATRICES FOR NETWORKS

Two matrices are useful for describing the topology of a network (Todini and Pilati 1988) including:

- The unknown head node matrix = **A1** (the size of the matrix is NP x NJ). If a link (in row j of **A1**) enters the node i in the designated flow direction (in column i of the **A1** matrix) then the a_{ji} element is -1. If it leaves the node it is +1. If the node and link are not connected it is 0 (zero).
- The fixed head node matrix = **A2** (the size of matrix is NP x NF). If a link (in row j of matrix **A2**) enters the fixed head node f in the designated flow direction (in column f of matrix **A2**) then the a_{jf} element is -1. If it leaves the fixed head node it is +1. If the fixed head node and link are not connected it is 0 (zero).

THE BASIC NODE AND PIPE GOVERNING EQUATIONS

There are two types of governing equations for flow and head (HGL or pressure) in a network of pipes. These include

- continuity of flow at each node
- head loss–flow relationship for each individual pipe

The continuity equation at each of the variable-head nodes in the network is given by:

$$\sum_{j=1}^{NP} \left\langle a_{ij} Q_j \right\rangle + DM_i = 0 \ldots\ldots\ldots \text{for } i = 1, 2, \ldots, NJ \qquad (2)$$

where the a_{ij} elements (a zero, -1 or +1 value) are from the $\mathbf{A1}^{\mathbf{T}}$ matrix. Typically only 2 to 4 a_{ij} elements of a row of the $\mathbf{A1}^{\mathbf{T}}$ matrix will be non-zero. Thus only non-zero values of a_{ij} will contribute to the continuity equation for the pipes attached to node i. Note that flows out of the node will be positive while flows into the node will be negative.

The head loss equation (or energy equation) for the pipe p_j in the network connecting node i and node k is given by:

$$H_i - H_k = r_j Q_j \left| Q_j \right| \ldots\ldots\ldots \text{ for } j = 1, 2, \ldots, NP \qquad (3)$$

where r_j = resistance factor assuming say the Darcy Weisbach head loss equation based on the Darcy-Weisbach friction factor f is used (that is dependent on the Reynolds number and the relative roughness for the pipe). The modulus sign in Eq. 3 ensures that the sign of the flow matches the sign of the head difference on the left hand side of the equation. The resistance factor is given by:

$$r_j = \frac{8 f_j L_j}{\pi^2 g D_j^5} \text{ .... for } j = 1, 2, \ldots, NP \qquad (4)$$

where L_j = pipe length, g = gravitational acceleration, D_j = pipe internal diameter. The friction factor may be estimated by the Swamee and Jain (1976) equation as:

$$f_j = \frac{1.325}{\left[\mathbf{ln}\left(\frac{\varepsilon_j}{3.7D_j} + \frac{5.74}{\mathbf{Re}^{0.9}}\right)\right]^2} \text{ .... for } j = 1, 2, \ldots, NP \qquad (5)$$

where ε_j = roughness height, **Re** = Reynolds number for the flow in the pipe. A Hazen-Williams head loss equation could easily be also used to replace Eq. 4. The vector of pipe resistance factors is $\mathbf{r} = (r_1, r_2, \ldots, r_{NP})^T$.

FORMULATION 1: FLOW EQUATIONS OR THE Q–EQUATIONS FOR A NETWORK

The unknown flows Q_js are labeled on Figure 1 for the ten-pipe network. The direction of flow in each pipe has been selected based on an expected flow pattern in the network. The direction may have been selected incorrectly in some pipes but this does not matter. In the final solution, if a pipe flow turns out to be negative, then that flow was assumed to be in the wrong direction initially.

The flow equations are formed from both the continuity equations at each of the nodes (a total of *NJ* equations) and the energy equations for simple closed loops and required independent paths in the network (a total of *NL+NF-NC* equations). To implement this formulation the loops need to be defined via the loop matrix (Todini and Pilati 1988). In addition, required independent paths between nodes of fixed head also have to be defined.

For the Q-equations formulation, the set of continuity equations are given by Eq. 2.

Now consider the energy equations around closed simple loops or between fixed head nodes along required independent paths in a network. These equations are nonlinear. A direction (usually positive for clockwise) for each of the loops must be assumed. Upon traversing a closed simple loop in the network the sum of head loss around the loop must be zero. This is expressed as

$$\sum_{j \in S_k} r_j Q_j \left| Q_j \right| = 0 \text{ ....for } k = 1, \ldots, NL \qquad (6)$$

for each closed simple loop in the network where S_k = {indices of the pipes in loop k}.

In addition to closed simple loop energy equations, (*NF-NC*) required independent paths between reservoirs or fixed head nodes in the network must also be considered. Consider a series of pipes between two reservoirs designated as reservoir *m* and reservoir *q*. The head loss in the pipes between two reservoirs must be equal to the difference in elevation or HGL between the 2 reservoirs (EL_m and EL_q). Each path should be traversed in the same direction as the loop flow direction (usually clockwise). This may be expressed as

$$\sum_{j\in S_k} r_j Q_j \left|Q_j\right| - (EL_m - EL_q) = 0 \text{ ....for } k = 1,..., NF\text{-}NC \qquad (7)$$

where S_k = {indices of the pipes in path k}. Note that application of Eq. 7 may be a little tricky. It is best to traverse the loop in the direction of the loop flow arrow between the reservoirs–then make the right–hand side equal to the head loss difference between the reservoirs or fixed head nodes. Finally, move the right–hand side term to the left–hand side of the equation so that the function is then equal to zero. The same sign convention regarding the sign of the assumed head loss and the assumed flow direction in the pipe is assumed for required independent path energy equations as for closed simple loop energy equations. It is clear that the equations in Eq. 2 are linear while Eqs. 6 and 7 are non-linear.

Solving this set of non-linear equations for the flow equations formulation can be carried out by using a Newton iterative solution which requires computation of the Jacobian matrix. This approach provides a linearizing approximation that is amenable to an iterative solution process (details of the Newton solution process are given in Formulation 5). Newton's method is equally applicable to all formulations presented here. The characteristics of the Jacobian (for example- symmetry, positive definiteness, Stieltjes type) and the use of sparse solution methods, where applicable, will significantly affect the computational cost and hence the speed of the Newton solution technique for each of the formulations. The formulation with the minimum number of variables to be solved for does not necessarily lead to the most effective method in terms of speed of convergence. A detailed comparison of the numerical and convergence properties of the various formulations is beyond the scope of this paper.

FORMULATION 2: HEAD EQUATIONS OR THE H–EQUATIONS

The unknown heads H_is are labeled on Figure 1 for the ten-pipe network. The head equations are formed from the continuity equations (Eq. 2) and the link flow equations (Eq. 3) by eliminating the flow terms in Eq. 2 between both sets of equations. Solving for the flow from Eq. 3 gives for pipe p_j linking nodes i and k:

$$Q_j = \frac{(H_i - H_k)\left|H_i - H_k\right|^{(1/n)-1}}{r_j^{1/n}} \text{ ....for } j = 1,2,\ldots, NP \qquad (8)$$

The H-equations formulation thus becomes by substituting Eq. 8 into Eq. 2.

$$\sum_{i=1}^{NP} a_{ij} \frac{(H_i - H_k)\left|H_i - H_k\right|^{(1/n)-1}}{r_j^{1/n}} + DM_i = 0 \text{ ....for } i = 1,2,\ldots, NJ \qquad (9)$$

where the a_{ij} elements (a zero, -1 or +1 value) are from the $\mathbf{A1^T}$ matrix. If the node connected to node i for a particular pipe is a fixed head node then the H_k value is replaced by EL_m. Note that all equations in Eq. 9 are non-linear in contrast to the Q-equations formulation where the continuity equations (Eq. 2) are linear equations. The H-equations formulation is easy to implement in a computer code. It is not necessary to define the loops for this formulation (in contrast to the Q-equations formulation) and this is a huge advantage of the technique. Only the link connectivity information is needed including the node numbers at the end of each link, the length, diameter and roughness of each pipe and the properties of each node (elevation and demand) (Todini and Pilati 1988).

FORMULATION 3: LOOP FLOW OR LF–EQUATIONS

The loop flow or LF–equations are obtained by writing the energy equations (similar to Eq. 6 and 7) for the closed simple loops and all required independent paths in the network by incorporating appropriate loop flows or *LF* values. The LF values for the ten-pipe network are shown in Figure 2. As each closed simple loop is traversed the head loss along each link is added (or subtracted) according to the sign convention used. Flow directions must be assumed for each pipe. A direction (usually positive for clockwise) for each of the loop flow *LF*s must also be assumed.

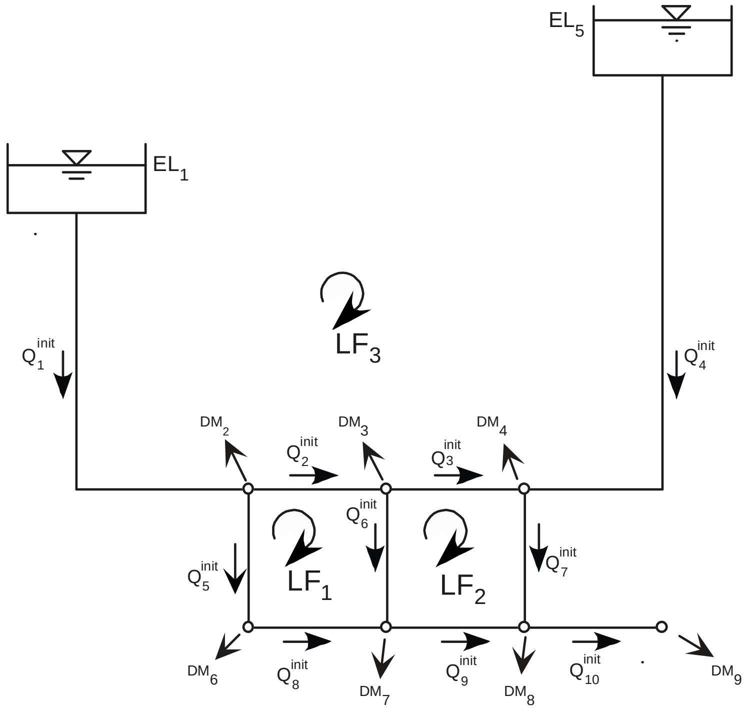

Figure 2. Unknown loop flows for the LF-equations formulation

A column vector of initial flows in the network must be chosen such that they satisfy continuity at each node as follows

$$\mathbf{q} = [Q_1^{init}, Q_2^{init}, ..., Q_{NP}^{init}]^{\mathbf{T}}. \tag{10}$$

Thus as shown in Figure 2, flow directions in each pipe must be chosen for each Q_j^{init} in each pipe. In Eq. 10, the Q_j^{init} values that satisfy continuity at each node in the network remain unchanged for all iterations. Note that this is quite different from the loop flow corrections formulation for the manual Hardy Cross method (1936) where the initially selected flows are updated in the loop immediately after the computation of each loop flow correction.

From Eq. 6, the energy equations in the LF-equations formulation for each of the *NL* closed loops in the network are

$$\sum_{j \in S_k} \left\langle r_j \left[Q_j^{init} + \sum_{s \in L_j} LF_s \right] \left| Q_j^{init} + \sum_{s \in L_j} LF_s \right| \right\rangle = 0 \ldots \text{for } k = 1,..., NL \tag{11}$$

where S_k = {indices of the pipes in closed loop *s*} and L_j = {indices of the loops associated with pipe *j*}.

From Eq. 7, the energy equations in the LF-equations formulation for each of the (*NF-NC*) required independent paths in the network are

$$\sum_{j\in S_k}\left\langle r_j\left[Q_j^{init}+\sum_{s\in L_j}LF_s\right]\left|Q_j^{init}+\sum_{s\in L_j}LF_s\right|\right\rangle-\left(EL_m-EL_q\right)=0\ \ldots\text{for } k=1,\ldots,(NF\text{-}NC) \quad (12)$$

where S_k = {indices of the pipes in closed loop *k*} and L_j = {indices of the loops associated with pipe *j*}. If the direction of the loop flow *LF* is in the same direction as the assumed link flow, the *LF* value is added to the initial flow assumed for the pipe or link. To the contrary, if the *LF* direction for the closed simple loop being considered is opposite to the assumed flow direction in the link, the *LF* value is subtracted from the pipe flow. The initial link flow is adjusted by the final converged value of *LF*s computed for each closed simple loop *s* of which the link is a member.

As seen in Table 1, the minimum number of equations results for the LF–equations formulation where the *NL* closed simple loop equations and the (*NF-NC*) required independent path equations are formulated in terms of a loop flows in each simple loop and path. All of the equations in the LF–equations formulation are nonlinear. In the LF-equations formulation, as for the Q–equations formulation, all loops and required independent paths need to be identified.

The continuity equations do not form part of the set of the LF–equations as long as the initial flow in each link satisfies the continuity at each node in the network.

The LF–equations in terms of the unknown loop flows in the network have in the past been recommended as the preferred formulation of the equations for a network due to the smaller size of the number of variables to be solved for in the equation formulation (Epp and Fowler 1970, Wood and Rayes 1981, Nielsen 1989). Ellis and Simpson (1998) indicated that the process for determining the preferred formulation is not straightforward. It not only depends on the number of equations but also the initial set up time prior to the equations being iteratively solved for (e.g. in determining a set of link flows that satisfy continuity) and also is quite dependent on the starting vector for the set of unknowns that is selected to commence the iterative solution process.

FORMULATION 4: THE Q-H EQUATIONS FORMULATION

The flow and head equations are formed from both the continuity equations (from Eq. 2) at each of the nodes and the head loss equations for each pipe expressed in terms of nodal heads at each end of the pipe (from Eq. 3). Thus there are a total of $NP + NJ$ unknowns to be solved for in the Q-H equation formulation.

The column vector of unknown flows and heads in the network is

$$\mathbf{m} = [Q_1, Q_2, \ldots Q_{NP}, H_1, H_2, \ldots H_{NJ}]^T.$$

Thus the Q-H equation formulation is made up of the continuity equations from Eq. 2 as:

$$\sum_{j=1}^{NP}\left\langle a_{ij}Q_j\right\rangle+DM_i=0\ \ldots\ldots \text{ for } i=1,2,\ldots,NJ \quad (13)$$

where the a_{ij} elements (a zero, -1 or +1 value) are from the $\mathbf{A1^T}$ matrix and the pipe head loss equations from Eq. 3 which are

$$H_i - H_k - r_j Q_j \left| Q_j \right| = 0 \qquad \text{for } j = 1,2,\ldots, NP \tag{14}$$

where the i and k indices refer to the node numbers at the end of each pipe j.

The number of equations in this formulation is clearly larger than the number of equations for the Q-equations formulation or the H-equations formulation. Thus the size of the matrices that need to be dealt with in the iterative solution process will consequently be larger. However, the form of the governing equations is simpler and computation of the Jacobian elements is considerably easier. In addition, it is not necessary to determine the loops in the network as for the Q-equation formulation.

FORMULATION 5: THE TODINI AND PILATI Q-H EQUATIONS FORMULATION

The Todini and Pilati (1988) formulation is effectively a type of Q–H equation formulation. The description below is based on material from Simpson and Elhay (2008). The formulation is based on two sets of equations solving for all unknown heads and flows simultaneously in a sequential iterative manner. The method has an extremely efficient approach to the inversion of the Jacobian matrix by partitioning the governing equations in a smart way. The advantage of the Todini and Pilati formulation is that the Jacobian matrices are symmetric. The Todini and Pilati (1988) formulation is used in EPANET (Rossman 1994) and a number of other commercially available hydraulic software packages for the simulation of water distribution systems.

Based on Eq. 2 the continuity equations for all the pipes in the network and the topology matrices introduced earlier in the paper, the following matrix form can be written:

$$\mathbf{A}1^{\mathbf{T}}\, \mathbf{q} + \mathbf{dm} = 0 = \mathbf{f}_2(\mathbf{q}, \mathbf{h}) \tag{15}$$

where the left-hand-side of Eq. 15 is denoted by the function $\mathbf{f}_2(\mathbf{q}, \mathbf{h})$. Now for the head loss equations for each pipe p_j connecting nodes i and k then Eq. 3 can be re-written as:

$$r_j Q_j \left| Q_j \right| - \left(H_i - H_k \right) = 0\,. \tag{16}$$

This can also be written in matrix form, but first a diagonal matrix $\mathbf{G}$ of size NP x NP is introduced where:

$$\mathbf{G} = \begin{pmatrix} r_1|Q_1| & 1 & \ldots & \ldots & 0 & 0 \\ 0 & r_2|Q_2| & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & r_{NP-1}|Q_{NP-1}| & 0 \\ 0 & 0 & \ldots & \ldots & 0 & r_{NP}|Q_{NP}| \end{pmatrix} \tag{17}$$

The non-linearity in the system arises because the matrix **G** depends on the unknown flows in **q**. The matrix form of Eq. 16 can be written as follows (taking into account that some nodes are fixed head nodes):

$$\mathbf{Gq} - \mathbf{A}1\mathbf{h} - \mathbf{A}2[\mathbf{el}] = 0 = \mathbf{f}_1(\mathbf{q},\mathbf{h}) \tag{18}$$

where **[el]** is a vector of the reservoir or fixed node heads. Note that Eq. 18 is denoted by the vector function $\mathbf{f}_1(\mathbf{q},\mathbf{h})$**.** The two sets of matrix equations in Eq. 15 and Eq. 18 may be written in the following block matrix form as:

$$\mathbf{f}(\mathbf{q},\mathbf{h}) = \begin{pmatrix} \mathbf{G} & | & -\mathbf{A}1 \\ -- & - & -- \\ -\mathbf{A}1^{\mathbf{T}} & | & 0 \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ -- \\ \mathbf{h} \end{pmatrix} - \begin{pmatrix} \mathbf{A}2[\mathbf{el}] \\ -- \\ \mathbf{dm} \end{pmatrix} = 0 \tag{19}$$

This system is really the set of equations for Formulation 4 – the Q-H equations, however, the partitioning in Eq. 19 will be exploited below to develop a much more efficient solution scheme. The first matrix on the left hand side of Eq. 19 shows the partitioning of a (*NP* + *NJ*)–square matrix into 2 block rows and 2 block columns. Note that the first block matrix on the left hand side of Eq. 19 has a special structure that may be exploited because it is symmetric. In addition the only non-constant values in this matrix are the diagonal elements of **G**. A Newton iterative solution to the set of non-linear equations in Eq. 19 can be formulated in terms of Taylor's series expansion and linearization as:

$$\mathbf{J}(\mathbf{q}^{(k)},\mathbf{h}^{(k)}) \begin{pmatrix} \delta\mathbf{q}^{(k+1)} \\ -- \\ \delta\mathbf{h}^{(k+1)} \end{pmatrix} = \begin{pmatrix} -\mathbf{f}_1^{(k)} \\ -- \\ -\mathbf{f}_2^{(k)} \end{pmatrix} \tag{20}$$

for the k+1st iteration where k = 0, 1, 2, etc. Decomposition methods that avoid the need for solving for the inverse of the Jacobian are applied to solve Eq. 20. An initial set of guesses of initial flows (Eq. 10) are required. A value corresponding to a velocity of 1.0 fps is often selected.

The Todini and Pilati (1988) method solves the equations describing the flows and nodal heads in a water distribution network by a reformulation of Eq. 19 which exploits the diagonal nature of **G** and which uses an explicit block-form of the inverse as shown below. This leads to a simplification and hence a significant improvement in the speed of the solution algorithm.

Now consider the derivative of the first matrix on the left side hand multiplied by the vector $(\mathbf{q}^{\mathbf{T}},\mathbf{h}^{\mathbf{T}})^{\mathbf{T}}$ of Eq. 19 so that we can form the Jacobian as shown in Eq. 20. The diagonal nature of **G** can be exploited with only the diagonal elements of the **Gq** matrix changing upon differentiation so computation of the Jacobian is very straightforward. Rewrite Eq. 19 as:

$$\mathbf{f}(\mathbf{q},\mathbf{h}) = \begin{pmatrix} \mathbf{Gq} & | & -\mathbf{A}1\mathbf{h} \\ -- & - & -- \\ -\mathbf{A}1^{\mathbf{T}}\mathbf{q} & | & 0 \end{pmatrix} - \begin{pmatrix} \mathbf{A}2[\mathbf{el}] \\ -- \\ \mathbf{dm} \end{pmatrix} = 0\,. \tag{21}$$

The derivatives of the diagonal elements of **Gq** assuming that r is constant (despite the fact that friction factors f actually depend on flow: the friction factors may be updated at the end of each iteration) are:

$$\frac{d}{dQ_j}\left(r_j Q_j \left|Q_j\right|^{n-1}\right) = r_j n \left|Q_j\right|^{n-1} \qquad \text{for } Q_j \neq 0 . \tag{22}$$

The terms in the second matrix (with **A2** etc.) on the left hand side of Eq. 21 do not depend on **q** and **h**. The Jacobian matrix for the system of equations in Eq. 22 becomes:

$$\mathbf{J} = \begin{pmatrix} n\mathbf{G} & | & -\mathbf{A}1 \\ -- & - & -- \\ -\mathbf{A}1^{\mathbf{T}} & | & 0 \end{pmatrix} \tag{23}$$

This matrix has some very nice properties of symmetry and sparseness. Todini and Pilati (1988) show an analytic expression for the block-form of the inverse of **J** where we denote the inverse of **G** to be **D** temporarily. This inverse is easy to compute for a diagonal matrix and is tractable as long as all flows Q_j are not zero. It has terms $1/(nr_j|Q_j|)$ at each location along the diagonal. Assume that the head loss exponent n is the same for each pipe. The Jacobian becomes:

$$\mathbf{J}^{-1} = \begin{pmatrix} \frac{1}{n}\mathbf{D} - \frac{1}{n}\mathbf{DA}1(\mathbf{A}1^{\mathbf{T}}\,\mathbf{DA}1)^{-1}\,\mathbf{A}1^{\mathbf{T}}\,\mathbf{D} & | & -\mathbf{DA}1(\mathbf{A}1^{\mathbf{T}}\,\mathbf{DA}1)^{-1} \\ ---------------- & - & ---------- \\ -(\mathbf{A}1^{\mathbf{T}}\,\mathbf{DA}1)^{-1}\,\mathbf{A}1^{\mathbf{T}}\,\mathbf{D} & | & -n(\mathbf{A}1^{\mathbf{T}}\,\mathbf{DA}1)^{-1} \end{pmatrix} \tag{24}$$

The reformulation allows the solution process for each iteration to be done in two stages: one for the flows and another for the heads. Substituting Eq. 24, Eq. 15 and Eq. 18 into Eq. 20 and simplifying gives the two-step Todini and Pilati algorithm for solving successively at each iteration for the heads and discharges as follows:

$$\mathbf{h}^{(k+1)} = (\mathbf{A}1^{\mathbf{T}}\,\mathbf{G}^{-1}\mathbf{A}1)^{-1}\left[\mathbf{A}1^{\mathbf{T}}\left\langle(1-n)\,\mathbf{q}^{(k)} - \mathbf{G}^{-1}\mathbf{A}\,2[\mathbf{el}]\right\rangle - n\,\mathbf{dm}\right] \tag{25}$$

and

$$\mathbf{q}^{(k+1)} = (1-\frac{1}{n})\mathbf{q}^{(k)} + \frac{1}{n}\mathbf{G}^{-1}(\mathbf{A}1\mathbf{h}^{(k+1)} + \mathbf{A}\,2[\mathbf{el}]) \tag{26}$$

where n = 2.0 for the Darcy Weisbach head loss formula or n = 1.852 for the Hazen-Williams head loss equation. Note that in the pair of equations above the only matrix inverse that is not trivial to compute occurs in Eq. 25 and involves the *Schur Complement* term of $\mathbf{A}1^{\mathbf{T}}\,\mathbf{G}^{-1}\mathbf{A}1$ which is the size of *NJ* by *NJ* (the number of nodes in the network). Thus although we are dealing with a total of (*NP* + *NJ*) unknowns the use of an analytical inverse of the Jacobian has reduced the matrix to be solved down to the size of *NJ*.

CONCLUSIONS

The objective of this paper has been to provide clarity in the presentation of the five different possible formulations of the equations governing flow and head is a water distribution system network. A set of non-linear equations arises for each of the formulations. These include: 1) flow equations 2) head equations 3) loop flow equations 4) flow and head equations and 5) the Todini and Pilati flow and head equations formulation. Details of each of these formulations have been presented in this paper. Graph theory has been used to show the relationship between the numbers of variables in each formulation.

The order of size of problem ranked from minimum to maximum number of variables for the five different formulations is: 1) loop flows 2) heads 3) Todini and Pilati Q-Hs 4) flows 5) flows and heads. The size of the problem does not necessarily determine which is the most effective formulation. Currently the Todini and Pilati Q-H equations formulation is commonly used in many commercial and Government hydraulic simulation packages. The main part of the iterative solution process in the Todini and Pilati algorithm has been reduced to same size of problem as the head equations formulation. This has been achieved by a smart reorganisation of the governing equations using an analytic expression for the inverse of the Jacobian using partitioning of the matrix. Further work is required to evaluate if the loop flow equations formulation can be competitive with the Todini and Pilati method is terms of computational speed.

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