Multiple frequencies of synchronization in classical and quantum networks

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Complex systems consisting of vector or matrix oscillators can synchronize to a common state characterized by a frequency matrix with distinct eigenvalues, leading to multiple frequencies of synchronization. In quantum networked systems the synchronized state is a linear combination of states corresponding to different energy levels. Suitable symmetry-breaking network interactions, however, allow only one or more such frequencies to appear. A specific example in three dimensions, where all trajectories lie on the 2-sphere, is a model of interacting spin-1 quantum angular momentum states, where synchronization to a nontrivial frequency occurs despite the presence of zero-frequency modes of oscillation.

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I. INTRODUCTION

The phenomenon of self-synchronization on complex networks appears in many applications and has been widely investigated (see the general accounts [1–5]). We focus here on phase synchronization for complex networks of vector or matrix oscillators, in particular, models that generalize the standard Kuramoto model and are relevant to both classical and quantum networks. For such models the natural frequencies of oscillation at each node synchronize to one or more eigenvalues of a frequency matrix Λ that is independent of the node. Because Λ generally has unequal eigenvalues it is possible for the synchronized system to lie in a superposition of states of different synchronized frequencies, which can either evolve indefinitely in such a superposition or eventually be dominated by states corresponding to a single synchronized eigenvalue, depending on the initial values of the system and properties of the model. While multiple energy levels appear naturally in quantum systems, one might wish in classical applications to be able to restrict the system to one or more of the synchronized frequencies.

We show here how a particular mode of oscillation can be selected by means of suitable network interactions, for both quantum and classical networks, and take as a specific example quantum systems of spin-1 angular momentum interacting over a nonlinear quantum network. This system is defined by a set of vector equations that determine trajectories $\boldsymbol{x}_i(t)$, where each $\boldsymbol{x}_i$ is a unit 3-vector that remains confined to the 2-sphere for all $t > 0$. This model is of interest for both classical and quantum systems, since it may be regarded as a higher dimensional generalization of the widely studied Kuramoto model [6], for which all trajectories lie on the circle S^1, but can also be viewed as a system of spin-1 quantum oscillators with three-component wave functions at each node.

The possibility that synchronization over complex networks can occur in quantum systems is discussed in [7] and requires nonlinear network interactions for its operation. Nonlinear quantum mechanics has been investigated at various times (see, e.g., [8]), however, we do not specify here any mechanisms or environmental interactions by which these nonlinearities might appear, only that the wave function at each node evolves according to the Schrödinger equation, through a specific Hermitian Hamiltonian that is constructed from the wave functions at all nodes. When synchronized, all spin-1 wave functions of the quantum network are correlated both spatially and in frequency, i.e., the spins have almost-identical orientations and frequencies of oscillation. Multiple frequencies of synchronization correspond to different energy levels, but by means of suitable network interactions we ensure that all nodes have the same energy level.

These quantum models may also be regarded as classical complex systems, by parametrizing wave functions in terms of real variables. Conversely, however, classical systems do not generally correspond to quantum systems, which must satisfy more stringent requirements. First, every quantum system must evolve according to the time-dependent Schrödinger equation with an Hermitian Hamiltonian operator, which ensures that the amplitude of every wave function is constant in time, i.e., probability is conserved. Second, this Hamiltonian must be scale invariant, so that each wave function can be normalized to unity. Hence, the resulting differential equations are independent of the wave-function amplitudes. The Kuramoto model, Eq. (1), has this property and may, indeed, be formulated as a spin-0 quantum complex system. One can, however, devise models in which the amplitude, although constant in time, depends on the node; e.g., in the matrix models discussed in [9], one can write each variable U_i as the product of an amplitude matrix A_i and a unitary matrix, where each A_i is constant and, unless it is the identity matrix, appears explicitly in the resulting equations. Such classical models, and also those that do not preserve amplitudes, such as chaotic models, do not correspond to quantum systems.

The vector models of synchronization that we consider here are of interest in classical as well as quantum networks, because they allow for a much wider range of behaviors than do scalar systems such as the Kuramoto model and the generalizations described in [10] (Sec. V). Multiple frequencies of synchronization appear and can be controlled by

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means of coupling parameters as we describe in the following. We also discuss how to identify the frequency matrix Λ for a synchronized system, and hence whether multiple frequencies exist, and we analyze in detail the three-component spin-1 system.

II. VECTOR OSCILLATORS

The Kuramoto model describes a system of N oscillators parametrized by an angle θ_i at each node i, with equations given by

$$\dot{\theta}_i = \omega_i + \frac{\kappa}{N}\sum_{j=1}^{N} a_{ij}\sin(\theta_j - \theta_i) \quad (i = 1,\ldots,N) \qquad (1)$$

where ω_i is the natural frequency of the ith node, κ is a coupling constant, and (a_{ij}) is the connectivity matrix. As is well known [4,10,11], the trajectories of the unit 2-vectors $\boldsymbol{x}_i = (\cos\theta_i, \sin\theta_i)$ synchronize in phase provided that κ is sufficiently large, in the sense that $\theta_i(t) = \theta_i^0 + \lambda t$ is a limit cycle of the system, where $\lambda = \sum_i \omega_i/N$ is the phase-locked frequency and θ_i^0 denotes constant angles.

The Kuramoto model, (1), can be generalized to models in which matrix or vector variables at each node synchronize to a common frequency, depending on properties specific to the model [9]. Consider, in particular, the following equations for real m vectors $\boldsymbol{x}_i$ of unit length:

$$\dot{\boldsymbol{x}}_i = \Omega_i \boldsymbol{x}_i + \frac{\kappa}{N}\sum_{j=1}^{N} a_{ij}[\boldsymbol{x}_j - \boldsymbol{x}_i(\boldsymbol{x}_j\cdot\boldsymbol{x}_i)] + \frac{\kappa'}{N}\sum_{j=1}^{N} a_{ij}\, J\boldsymbol{x}_i(\boldsymbol{x}_j\cdot J\boldsymbol{x}_i), \qquad (2)$$

where Ω_i is a prescribed $m\times m$ antisymmetric real matrix, the eigenvalues of which are, in absolute value, the natural frequencies of oscillation at the ith node, J is an $m\times m$ antisymmetric real matrix independent of the node, and κ,κ' are positive coupling constants. Such models arise from both the matrix models and the quantum mechanical (vector) models considered in [7] and [9]; the last term in Eq. (2), e.g., appears in the quantum equations expressed in real form (see Eqs. (10) and (22) in [7]) and arises from an expansion of the Hamiltonian in powers of the wave functions (see Sec. 3.1 in [7]).

System (2) preserves the unit length of $\boldsymbol{x}_i$ as a constant of the motion. For $m = 2$ these equations reduce to the Kuramoto system, (1), if we set $\boldsymbol{x}_i = (\cos\theta_i, \sin\theta_i)$ and $\Omega_i = \omega_i J$, where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, with a coupling constant $\kappa + \kappa'$. For $m = 4$ and $\kappa' = 0$, Eqs. (2) arise from the matrix model discussed in [9] (Sec. VI) and, for sufficiently large κ, have properties such as phase locking that are similar to those of the Kuramoto model. All trajectories $\boldsymbol{x}_i(t)$ approach the limit cycle $e^{t\Lambda}\boldsymbol{n}_i$ arbitrarily closely, where $\boldsymbol{n}_i$ is a constant unit vector, and where the 4×4 real antisymmetric frequency matrix Λ commutes with Ω_i for all $i = 1,\ldots,N$. Since for this particular model the four eigenvalues of Λ are equal in absolute value, multiple frequencies of synchronization do not arise. Equations (2) for $m = 4$ and $\kappa = \kappa'$ arise in the model of quantum synchronization considered in [7], where $\boldsymbol{x}_i$ parametrizes spin-1/2 wave functions (qubits) that interact nonlinearly over a quantum network.

III. MULTIPLE FREQUENCIES

In general, multiple frequencies can occur for system (2) as follows: the frequency matrix Λ is an element of a subalgebra of the Lie algebra so_m of the rotation group $\mathrm{SO}(m)$; for $m = 4$, e.g., Ω_i belongs to one of the so_3 subalgebras of $\mathrm{so}_4 = \mathrm{so}_3 \oplus \mathrm{so}_3$, while Λ is an element of the other so_3. We require in general that $[\Lambda,\Omega_i] = 0$ in order that the limit cycle $e^{t\Lambda}\boldsymbol{n}_i$ should satisfy Eqs. (2) with $\kappa' = 0$, where the constant unit vectors $\boldsymbol{n}_i$ satisfy corresponding algebraic equations. The frequencies of synchronization of the system are given by the (real) eigenvalues of $i\Lambda$ and are generally distinct, and so multiple frequencies of synchronization can occur. The synchronized states, being close to the limit cycle, are a linear combination of eigenstates corresponding to these frequencies, with the actual combination depending on the initial values of the system.

We may select one of the multiple frequencies by allowing κ', in the last term in Eqs. (2), to take nonzero values and so act as a symmetry-breaking interaction. This last term, or possibly a sum of such terms, requires that $[\Lambda, J] = 0$ as well as $[\Lambda,\Omega_i] = 0$, in order that $e^{t\Lambda}\boldsymbol{n}_i$ should behave as a limit cycle. By making a suitable choice of J one can therefore restrict the form of Λ such that there exists only a single frequency of synchronization; i.e., by means of suitable network interactions one can restrict the frequencies to which the system synchronizes. For the $m = 4$ case of (2) with $\kappa' = 0$, as discussed in [9], Λ is an element of so_3 and hence has three independent frequency components. When the term with nonzero κ' is included, however [see Eq. (22) in [7], where $J = K_3$], we have $\Lambda = \lambda J$ and so there is now only a single frequency component, λ.

The additional term in Eq. (2) acts to restrict the frequencies of synchronization even when the scenario as outlined above does not eventuate. This can occur in two ways. First, if the matrices Ω_i in so_m are sufficiently general, then the limit cycle does not exist; i.e., there is no antisymmetric matrix Λ with the properties $[\Lambda,\Omega_i] = 0 = [\Lambda, J]$. Investigations for $m = 3$ as discussed below show, however, that synchronization still occurs in the sense that trajectories are closely correlated, although not phase locked. Second, even when Λ exists, being antisymmetric, for odd m it has at least one zero eigenvalue in addition to the nonzero eigenvalues. Depending on the initial trajectories, the system can indeed synchronize to this zero frequency, as we see for $m = 3$, which means that all trajectories approach a constant state, i.e., a fixed point on S^{m-1}. But for suitably chosen values of κ', the additional term in Eqs. (2) suppresses such states, and the system synchronizes to a nontrivial frequency determined by the nonzero eigenvalues of Λ. We quantify the level of synchronization by means of several measures.

IV. OSCILLATORS ON THE 2-SPHERE

In order to demonstrate these properties, we now consider explicitly the case $m = 3$. This is of interest in classical complex systems as an immediate generalization of the Kuramoto model from two to three real dimensions but is also relevant to quantum systems carrying spin angular momentum 1, where the quantum wave functions are distributed over a quantum network as discussed further below. The matrix Ω_i has the general form

$$\Omega_i = \begin{pmatrix} 0 & -\omega_i^3 & \omega_i^2 \\ \omega_i^3 & 0 & -\omega_i^1 \\ -\omega_i^2 & \omega_i^1 & 0 \end{pmatrix}, \tag{3}$$

and hence $\Omega_i \boldsymbol{x}_i = \boldsymbol{\omega}_i \times \boldsymbol{x}_i$, where $\boldsymbol{\omega}_i = (\omega_i^1, \omega_i^2, \omega_i^3)$ are prescribed vector frequencies, and $\boldsymbol{x}_i = (x_i, y_i, z_i)$. For general $\boldsymbol{\omega}_i$ the limit cycle does not exist, since there is no antisymmetric matrix Λ that commutes with all Ω_i, but it was found in [9] [taking $\kappa' = 0$ in Eqs. (2)], for sufficiently large κ, that trajectories on S^2 nevertheless synchronize spatially as measured by the order parameter $r = \|\boldsymbol{x}_{\rm CM}\|$, where $\boldsymbol{x}_{\rm CM} = \sum_{i=1}^N \boldsymbol{x}_i/N$ is the average, or center-of-mass, coordinate. The parameter r takes a value close to unity following the initial transient, which indicates that the trajectories, when plotted on a common 2-sphere, are bunched together. An alternative and related measure of spatial correlations is the average separation defined by

$$D_{\rm AvSep} = \frac{2}{N(N-1)} \sum_{i<j} \|\boldsymbol{x}_i - \boldsymbol{x}_j\|, \tag{4}$$

where the sum is over all $N(N-1)/2$ distinct nodes i, j. $D_{\rm AvSep}$ has the properties of a disorder parameter, since it is 0 for identical trajectories and large for uncorrelated trajectories. It was found numerically in [9] that as the system evolves, trajectories drift to one of the poles at $\pm\widehat{\boldsymbol{\omega}}$, where $\widehat{\boldsymbol{\omega}} = \boldsymbol{\omega}/\|\boldsymbol{\omega}\|$ and $\boldsymbol{\omega} = \sum_i \boldsymbol{\omega}_i/N$. This occurs because at $\kappa = 0$ the free equations $\dot{\boldsymbol{x}}_i = \boldsymbol{\omega}_i \times \boldsymbol{x}_i$ have the constant solution $\boldsymbol{x}_i = \widehat{\boldsymbol{\omega}}_i = \boldsymbol{\omega}_i/\|\boldsymbol{\omega}_i\|$, which is the zero-frequency mode of oscillation, and the nonlinear network interactions (for $\kappa' = 0$) evidently synchronize the system to this zero frequency, rather than to an average of the nontrivial frequencies $\|\boldsymbol{\omega}_i\|$ of Ω_i.

For large positive κ', however, we find numerically that the system synchronizes to a nontrivial frequency close to the average $\sum_i \|\boldsymbol{\omega}_i\|/N$. Equations (2) for nonzero κ' are covariant under global SO(3) rotations; i.e., if we replace $\boldsymbol{x}_i \to O\boldsymbol{x}_i$ where $O \in$ SO(3), then the equations retain the same form but with transformed frequency vectors $\boldsymbol{\omega}_i'$ and with $J \to J' = O^t J O$. By a suitable orientation of the whole system we may therefore choose J to be

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

In our numerical computations we chose $N = 50$, $\kappa = 2$, $\kappa' = 8$, with random frequencies $\boldsymbol{\omega}_i$ and random initial values for the unit vector $\boldsymbol{x}_i$, with all-to-all coupling. Following the initial transient, trajectories bunch together in a small cluster as measured by either r or $D_{\rm AvSep}$. Figure 1 shows, for a typical set of initial values, that r achieves a value close to unity following the initial transient but is not exactly constant in time. This is more evident from the plot of the average separation of trajectories $D_{\rm AvSep}$ which, although small in value, varies periodically. The unit center of mass trajectory $\widehat{\boldsymbol{x}}_{\rm CM} = \boldsymbol{x}_{\rm CM}/r$ is confined, even at very large times, to a narrow band at a fixed latitude on S^2 with respect to the poles $\pm\widehat{\boldsymbol{\omega}}$. This implies that all trajectories are synchronized to a nontrivial frequency, i.e. the network coupling involving κ' in (2) does indeed suppress the zero frequency modes of oscillation.

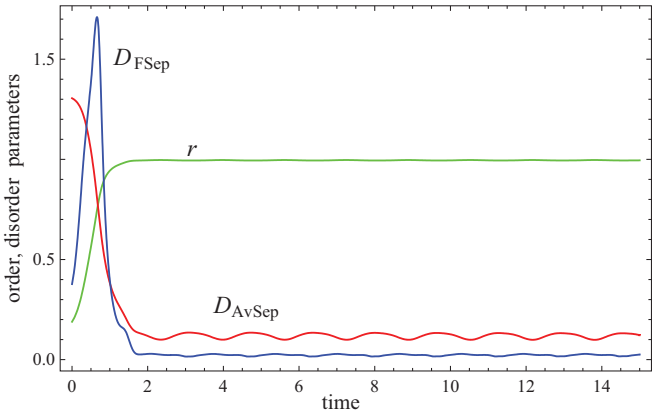

FIG. 1. (Color online) Spatial order and disorder parameters r and $D_{\rm AvSep}$, as well as the frequency separation $D_{\rm FSep}$, as functions of t for synchronized trajectories on S^2.

In order to measure frequency variations we define the average frequency separation $D_{\rm FSep}$ according to

$$D_{\rm FSep} = \frac{2}{N(N-1)} \sum_{i<j} \left| \frac{d}{dt}(\boldsymbol{x}_i \cdot \boldsymbol{x}_j) \right|, \tag{6}$$

which calculates the change in angle between any two trajectories, and hence the relative frequency, averaged over all pairs of trajectories. Figure 1 plots the frequency separation as a function of time, showing that $D_{\rm FSep}$ varies periodically following the initial transient and takes small but nonzero values, consistent with the fact that the distance $\|\boldsymbol{x}_i - \boldsymbol{x}_j\|$ between any two trajectories is not exactly constant. Frequencies of individual trajectories are therefore not locked to a single value. An approximate antisymmetric frequency matrix Λ may be determined at any fixed time by minimizing $\sum_i \|\dot{\boldsymbol{x}}_i - \Lambda \boldsymbol{x}_i\|^2$ with respect to Λ, which fits the synchronized trajectories $\boldsymbol{x}_i(t)$ to the form $e^{t\Lambda}\boldsymbol{n}_i$, from which we determine the synchronized frequencies as the eigenvalues of Λ, in absolute value. We find that the nonzero frequency is numerically close to $\sum_i \|\boldsymbol{\omega}_i\|/N$ and depends weakly on time. In summary, the system is spatially synchronized in the sense that each trajectory is approximately of the form $e^{t\Lambda}\boldsymbol{n}_i$, where the coordinates $\boldsymbol{n}_i$ are almost independent of i and are therefore each close to $\widehat{\boldsymbol{x}}_{\rm CM}$, as measured by r or $D_{\rm AvSep}$. The system is phase synchronized in the sense that Λ is independent of i with a nonzero eigenvalue close in absolute value to $\sum_i \|\boldsymbol{\omega}_i\|/N$, which indicates that the zero-frequency component in the synchronized system is small.

We next choose natural frequencies with $\omega_i^1 = \omega_i^2 = 0$, corresponding to commuting matrices $\Omega_i = \omega_i^3 J$. In this case the trajectories $\boldsymbol{x}_i(t) = e^{t\lambda J}\boldsymbol{n}_i$ have properties similar to those of a limit cycle, where λ is the synchronized frequency, however, the zero-frequency mode of oscillation still exists. As a numerical example we take $N = 200, \kappa = 2, \kappa' = 8$ with $\boldsymbol{\omega}_i = (0,0,\omega_i^3)$, chosen at random from a Gaussian distribution about $\omega = 0.5$, with a standard deviation of 0.2, and random initial values for $\boldsymbol{x}_i$. For $\kappa' = 0$ it was found in [9] that trajectories do indeed closely approach this "limit cycle" but eventually drift toward a pole at large times, implying that the vectors $\boldsymbol{n}_i$ in fact depend weakly on time, and so the system eventually synchronizes to the zero-frequency mode.

As with the case of general natural frequencies, however, we can suppress these zero modes by choosing $\kappa' > 0$ in Eqs. (2). If we first set $\kappa = 0$ and choose a sufficiently large κ', then we find, following the initial transient, that trajectories are approximately aligned along a common longitude on S^2 that varies with time, but with each node maintaining a constant latitude. But by allowing κ also to take a nonzero value ($\kappa = 2$) we find that all trajectories are quickly confined to a constant latitude, still bunched longitudinally, and are therefore spatially synchronized with a suppressed zero-frequency component. Figure 2 shows the trajectories $\boldsymbol{x}_i$ as they synchronize, with longitudinal bunching evident. All trajectories for $t > 3$ cluster tightly together and the unit center of mass $\widehat{\boldsymbol{x}}_{\mathrm{CM}}(t)$ circulates S^2 indefinitely at a fixed latitude. The constant frequency λ may be determined by minimizing $\sum_i \|\dot{\boldsymbol{x}}_i - \lambda J \boldsymbol{x}_i\|^2$ with respect to λ; we find that the minimum is numerically 0 and that $|\lambda|$ is close to but less than $\sum_i |\omega_i^3|/N$. The measures $1 - r$ and D_{AvSep} are small and constant following the initial transient, and D_{FSep} takes a constant value that is numerically 0. We may also define a disorder parameter that measures the departure of trajectories from the apparent limit cycle $\boldsymbol{x}_i(t) = e^{t\lambda J}\boldsymbol{n}_i$. Although very small, this parameter is not 0, which indicates that trajectories are not precisely aligned to the limit cycle, which is undoubtedly due to the existence of small zero-frequency components in the numerical solution.

Various numerical experiments may be performed in order to establish the effect of varying κ and κ' independently. Despite the observation above, that for $\kappa = 0$ and $\kappa' > 0$ there is longitudinal spatial synchronization, for general initial values with either $\kappa = 0$ or $\kappa' = 0$ the system usually eventually settles into the zero-frequency mode. This dependence on the initial values is evident for the case $\omega_i^1 = \omega_i^2 = 0$, for then an exact solution is $\boldsymbol{x}_i = (0,0,1)$; i.e., if the system begins with each node precisely in the zero-frequency mode, it remains in this mode indefinitely. On the other hand, if the last component z_i is initially 0 for all i, then z_i remains 0 for all later times and the equations reduce to the Kuramoto model, (1), with a coupling $\kappa + \kappa'$, and phase synchronization to the nontrivial frequency $\sum_i |\omega_i^3|/N$ occurs in the usual way. For random initial values we find that the system synchronizes to a nontrivial frequency when κ and κ' are of a similar magnitude.

V. SPIN-1 QUANTUM SYSTEM

Having outlined the properties of solutions to Eqs. (2), we now demonstrate how these equations may be viewed as a model of interacting quantum systems. Consider a quantum network where the wave function $|\psi_i\rangle$ at the ith quantum node is a spin-1 angular momentum eigenstate. We parametrize $|\psi_i\rangle$ in terms of real coordinates $\boldsymbol{x}_i = (x_i, y_i, z_i)$ according to

$$|\psi_i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -x_i - i y_i \\ \sqrt{2}\, z_i \\ x_i - i y_i \end{pmatrix}. \tag{7}$$

Although these components are the spherical harmonic functions for orbital angular momentum $\ell = 1$, with a zero relative phase between nodes, we view $\boldsymbol{x}_i$ merely as time-dependent parameters, not as the physical coordinates of the quantum oscillator at the ith node. Hence, $|\psi_i\rangle$ describes a spin-1 system, with three possible states with magnetic quantum numbers $m = 0, \pm 1$. We find that $|\psi_i\rangle$ maintains the form (7) as it evolves, provided it has this form initially, due to the restricted form of the local Hamiltonian H_i^0 as given below. One could in principle generalize this system by allowing $|\psi_i\rangle$ to be a three-component complex vector, and H_i^0 to be a general 3×3 Hermitian matrix, but we restrict our investigations to the present system. We normalize $|\psi_i\rangle$ in (7) to unity in the three-dimensional inner product, hence $\boldsymbol{x}_i$ is a unit vector and we find $\langle\psi_i|\psi_j\rangle = \boldsymbol{x}_i \cdot \boldsymbol{x}_j$. It is convenient to define

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 \\ i & 0 & i \\ 0 & \sqrt{2} & 0 \end{pmatrix};$$

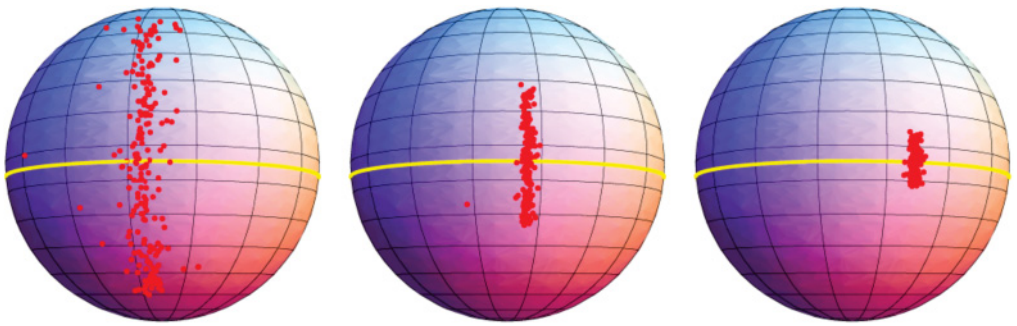

FIG. 2. (Color online) Trajectories $\boldsymbol{x}_i$ shown in red (dark-gray dots) on the unit sphere at times $t = 1$, 1.5, and 2; unit center-of-mass trajectory $\widehat{\boldsymbol{x}}_{\mathrm{CM}}(t)$ shown in yellow (light-gray line) for $t > 3$.

then U is unitary with $U|\psi_i\rangle = \boldsymbol{x}_i$. $|\psi_i\rangle$ evolves in time according to the Schrödinger equation, in units such that $\hbar = 1$:

$$i\frac{\partial}{\partial t}|\psi_i\rangle = \left(H_i^0 + H_i^{\rm int}\right)|\psi_i\rangle, \tag{8}$$

where $H_i^0 + H_i^{\rm int}$ is the total Hamiltonian and the 3×3 Hermitian matrix H_i^0, the local Hamiltonian, is defined by $H_i^0 = iU^\dagger\Omega_i U$, where Ω_i is given by (3). The eigenvalues of H_i^0 and hence the energy levels of the local quantum system are $\{0, \pm\|\boldsymbol{\omega}_i\|\}$, corresponding to magnetic quantum numbers $\{0, \pm 1\}$.

The interaction Hamiltonian $H_i^{\rm int}$, which is an Hermitian operator, depends explicitly on all wave functions and so gives rise to nonlinear network interactions. We choose

$$H_i^{\rm int} = \frac{i\kappa}{N}\sum_{j=0}^{N} a_{ij}(|\widehat{\psi}_j\rangle\langle\widehat{\psi}_i| - |\widehat{\psi}_i\rangle\langle\widehat{\psi}_j|) + \frac{i\kappa'}{N}\sum_{j=0}^{N} a_{ij}(J_3|\widehat{\psi}_j\rangle\langle\widehat{\psi}_i|J_3 - J_3|\widehat{\psi}_i\rangle\langle\widehat{\psi}_j|J_3), \tag{9}$$

where $|\widehat{\psi}_i\rangle = |\psi_i\rangle/\sqrt{\langle\psi_i|\psi_i\rangle}$ and $J_3 = -iU^\dagger JU = \mathrm{diag}[1,0,-1]$. Hamiltonians such as (9) arise from expansions of the interaction operators over the network in powers of the wave functions, as described in [7] (Sec. 3.1). Since $H_i^{\rm int}$ is invariant with respect to scale transformations $|\psi_i\rangle \longrightarrow \lambda_i|\psi_i\rangle$ for any $\lambda_i > 0$ we may normalize each wave function $|\psi_i\rangle$ to unity. The Schrödinger equation, (8), now reads

$$i\frac{\partial}{\partial t}|\psi_i\rangle = H_i^0|\psi_i\rangle + \frac{i\kappa}{N}\sum_{j=0}^{N} a_{ij}(|\psi_j\rangle - \langle\psi_j|\psi_i\rangle\,|\psi_i\rangle) - \frac{i\kappa'}{N}\sum_{j=0}^{N} a_{ij}\langle\psi_j|J_3|\psi_i\rangle\,J_3|\psi_i\rangle, \tag{10}$$

where we have used $\langle\psi_i|J_3|\psi_i\rangle = 0$. If we now premultiply (10) by U and use $\langle\psi_j|J_3|\psi_i\rangle = -i\,\boldsymbol{x}_j\cdot J\boldsymbol{x}_i$, then we recover Eqs. (2) for $m = 3$.

It is implicit in Eqs. (10) that the quantum network distributes wave functions between connected nodes. Realizations of quantum networks are discussed in [12], for example; however, it is necessary for our purposes only that the network preserves probabilities, i.e., that unitarity is preserved as the wave functions evolve. This is guaranteed by the Schrödinger equation (8), which controls the evolution of the complex system. Equations (10) should be understood in this context, specifically, that for each i,j there exists an operator T_{ij}, constant in time, which transports the wave function $|\psi_i\rangle$ at the ith node to the jth node; conversely, $T_{ji} = T_{ij}^\dagger$ transports $|\psi_j\rangle$ from the jth to the ith node, and both $T_{ij}T_{ji}$ and $T_{ji}T_{ij}$ are identity operators [7]. Hence $|\psi_j\rangle$, as it appears in (10), means the image of $|\psi_j\rangle$ at the ith node with respect to the operator T_{ji}. As is the case with standard models of synchronization, such as the Kuramoto system, (1), nonlinear interactions between $|\psi_i\rangle$ and $|\psi_j\rangle$ occur as the wave functions are distributed between nodes and are modeled by the nonlinear terms in (10). The wave function of the whole complex system is the direct product of individual wave functions and maintains this form as the system evolves.

Specific properties of the wave functions for the evolving system follow from those of the trajectories $\boldsymbol{x}_i(t)$, showing that the system synchronizes with all nodes having a common spin orientation and frequency of oscillation. The system settles into a linear combination of spin states with magnetic quantum numbers $m = 0, \pm 1$ where the zero-frequency mode corresponds to $m = 0$, but when synchronized with sufficiently large κ,κ', the states $m = 0$ are suppressed and the synchronized states correspond to either $m = 1$ or $m = -1$. Equations (2) are symmetric with respect to $x_i \to -x_i$ with y_i, z_i unchanged, corresponding to a reversal of sign of magnetic quantum number, as is evident if we premultiply (2) by the matrix $M = \mathrm{diag}[-1,1,1]$. We have $MJM = -J$ and $\Omega_i \to \widetilde{\Omega}_i = M\Omega_i M$, where $\widetilde{\Omega}_i$ is unchanged except that the signs of ω_i^2, ω_i^3 are reversed. Hence, whether the synchronized system has magnetic quantum number $m = 1$ or $m = -1$ depends on the initial values of $\boldsymbol{x}_i$. For $\omega_i^1 = \omega_i^2 = 0$ and sufficiently large κ,κ', the wave function approaches $|\psi_i\rangle = e^{it\lambda J_3}|\psi_i^0\rangle$, where $|\psi_i^0\rangle$ is constant in time, and λ is the nontrivial frequency of synchronization. We may regard λJ_3 as the Hamiltonian of the synchronized system, with energy levels $0, \pm\hbar\lambda$, with a suppressed zero energy.

The main difference between this system and the spin-1/2 model discussed in [7] is, first, that the latter has no zero modes and so the system settles into either a spin-up or a spin-down configuration and, second, that limit cycles exist and so frequencies are phase locked. Furthermore, by means of specific local unitary transformations the trajectories coincide exactly, and so synchronization is complete. By contrast, the spin-1 model discussed here is more difficult to analyze since, apart from the zero modes, it synchronizes only to the extent indicated by the measures r, $D_{\rm AvSep}$, and $D_{\rm FSep}$.

VI. SUMMARY

We have outlined how complex systems with vector oscillators at each node can synchronize to a linear combination of states corresponding to multiple synchronization frequencies and that, by means of suitable network interactions, one can select certain frequencies, or a single frequency of synchronization, by breaking the symmetry of the system. We have investigated in detail the $m = 3$ case for which trajectories are confined to the 2-sphere, since this is of interest, first, as a vector generalization of the Kuramoto model and, second, as a model of spin-1 quantum angular momentum states interacting nonlinearly over a quantum network. This example is complicated by the existence of zero modes that occur for all odd values of m, although in the quantum theory these are merely states with zero magnetic quantum number. The symmetry-breaking interaction nevertheless suppresses such modes of oscillation and so the system synchronizes to a nontrivial frequency, as measured by r and $D_{\rm AvSep}$ for spatial coherence and $D_{\rm FSep}$ for frequency correlations. Our numerical examples are restricted to all-to-all network couplings, but preliminary computations show that the synchronization properties are maintained even with nontrivial network topologies.

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